HEAT CONDUCTION IN MULTICONNECTED COMPOSITE REGIONS WITH RADIATION BOUNDARY CONDITIONS

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ABSTRACT

The boundary point least squares method is used to determine the approximate steady state temperature distribution in composite regions which involve heat transfer by the nonlinear fourth power radiation law. The nonlinear boundary equations are solved by way of a modified successive approximation scheme. The details of the solution technique are developed and several example problems solved.

1. INTRODUCTION

The boundary point least squares method has by now been applied to a wide variety of problems involving second-, fourth-, and eighth-order elliptic equations. No attempt will be made to present a complete list of papers even for the second-order Laplace equation. Typical examples of papers include for the second order equation References [1-5]; for the fourth order equation References [6-8]; and for the eighth order equation in spherical shell theory References [9,10].

Since the boundary point least squares method depends on developing the solution in terms of a truncated form of the general series solution of the differential equation, the method has been restricted to linear differential equations. In addition, problems involving only linear boundary equations have been considered. This second restriction will be removed here and the BPLS method will be applied to the solution of heat conduction problems involving the nonlinear radiation boundary condition which is of fourth order in the temperature. Problems involving composite regions in which the subregions may have different heat capacities and thermal conductivities will also be treated. Provision is made for considering radiative heat transfer between two adjoining regions whose boundary temperatures may be initially unknown.

An iterative solution procedure is used for problems involving radiative heat transfer. The problem solved at each iteration step is the linear boundary value problem obtained by replacing the radiation condition by a linear convective heat transfer equation with an effective heat transfer coefficient. The values of this coefficient depend on a cubic function of the temperature at the previous iteration and thus depend on position.

The present paper first describes the equations for the heat conduction boundary value problem and the development of the iterative procedure for solving the non-linear problem. Numerical results obtained with a computer program developed to apply the method are then described. Problems described include composite annuli with radiation boundary conditions and multiholed composite regions with both linear and radiation boundary conditions. The development of the computer program included the investigation of convergence rates for the iteration procedure and ways of improving the convergence rates. This investigation is also described.

2. ANALYSIS

Governing Equations and Assumptions

We will consider only the plane steady state temperature boundary value problem governed by the Laplace equation

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad . \tag{1}$$

It is assumed that the temperature satisfies boundary conditions of one of the two types:

$$G \frac{\partial T}{\partial n} + HT + K = 0 \tag{2}$$

$$G' \frac{\partial T}{\partial n} + H'T^4 + K = 0 \ , \tag{3}$$

where n denotes the outer normal to the boundary and G, H, K, G', H', and K' are known functions of position.

We shall consider composite regions which may involve three different interface conditions:

Solid contact

$$K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n}$$

$$T_1 = T_2 \ . \tag{4-a}$$

Convective heat transfer between noncontacting boundaries

$$K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n}$$

$$K_1 \frac{\partial T_1}{\partial n} + H_1 \left(T_2 - T_1\right) = 0 \ . \tag{4-b}$$

Radiative heat transfer between non-contacting boundaries

$$K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n}$$

$$K_1 \frac{\partial T_1}{\partial n} + H_2 \left(T_1^4 - T_2^4\right) = 0 , \qquad (4\text{-c})$$

where T_i and K_i denote the temperature and conductivity of the i-th region, H_1 is the convective heat transfer coefficient, and H_2 is the effective radiation coefficient for the radiating surfaces. In cases (4-b) and (4-c), it is assumed that the surfaces, while not in contact, are sufficiently close together that the distance between the surfaces is everywhere small with respect to the radius of curvature of either surface.

Development of the Temperature Series

The basic idea behind the boundary point least squares technique is to fit, in the least squares sense, a finite series solution of the governing differential equation to the boundary conditions of the problem region. When executed, this process yields an exact solution of the differential equation which approximately satisfies the prescribed boundary conditions.

The efficient implementation of the method requires an awareness of certain properties of the solution. In particular, the locations of any singularities in the solution should be known. For regions containing circular holes, the solution has a singularity at the center of each hole. Thus, trial functions having the required singularities must be chosen. This is accomplished by including in the total solution series terms which have singularities at the desired locations.

Because the solution so formed is in essence a series expansion about the singularities, these points are referred to as expansion centers. Of course the complete set of expansion centers includes any points about which the solution is expanded. For example, problems involving finite simply-connected domains have the solution expanded about the origin of coordinates.

The solutions of many temperature problems exhibit either symmetry or antisymmetry about various lines. This phenomenon is a consequence of the boundary conditions and the geometry of the problem region. The smallest subregion of the total problem region bounded by symmetry lines is called a symmetry element.

The symmetry lines which define the symmetry element are determined by the symmetry of the boundary conditions as well as the geometric symmetry. Thus the solution of a square region having a constant temperature T_1 on one pair of opposite sides and a constant temperature T_2 on the remaining sides would exhibit two-fold symmetry and the symmetry element would be one quadrant of the plate.

Other examples of symmetry elements are afforded by problems involving symmetric rings of holes. Let a set of m holes be located with their centers uniformly spaced on a circle of radius b. It is assumed that one of the holes has its center on the x-axis. If the boundary of each hole is maintained at a constant temperature, one of the symmetry elements would be the wedge bounded by the lines $\theta = 0$, $\theta = \pi/2m$, and the circular arc formed by the top half of the hole centered on the x-axis.

The concept of a symmetry element is pertinent only if the trial functions can be constructed with the same periodicity. One of the best known and most useful solutions to the homogeneous harmonic equation is the classic polar series given below.

$$\begin{aligned} T(r, \theta) = {} & A_{0,1} \log r + A_{0,6} + A_{0,7}\, x + A_{0,8}\, y \\ & + A_{1,3}\, r^{-1} \cos\theta \\ & + A_{1,7}\, r^{-1} \sin\theta \\ & + \sum_{n=2}^{\infty} (A_{n,1}\, r^{n} + A_{n,3}\, r^{-n}) \cos n\theta \\ & + \sum_{n=2}^{\infty} (A_{n,5}\, r^{n} + A_{n,7}\, r^{-n}) \sin n\theta \end{aligned} \qquad (5)$$

This series contains all of the terms necessary for the solution of simply or doubly-connected regions for which the symmetry element contains only the origin as an expansion center.

When there is more than one expansion center in the problem region, the solution is obtained by developing trial series at the additional singularities from the terms of the following series.

$$\begin{aligned} T_i (r_i, \theta_i) = {} & A^{i}_{0,1} \log r_i \\ & + A^{i}_{1,3}\, r_i^{-1} \cos\theta \\ & + A^{i}_{1,7}\, r_i^{-1} \sin\theta_i \\ & + \sum_{n=2}^{\infty} (A^{i}_{n,3}\, r_i^{-n}) \cos n\theta_i \\ & + \sum_{n=2}^{\infty} (A^{i}_{n,7}\, r_i^{-n}) \sin n\theta_i \end{aligned} \qquad (6)$$

In eq. (6), (r_i, θ_i) are the polar coordinates of the i^{th} expansion center. The solution is obtained by summing the individual series so that

$$T = \sum_{i=0}^{m-1} T_i \ (r_i, \ \theta_i)$$

where m denotes the total number of expansion centers.

When the expansion centers are symmetrically arranged, it is possible to obtain relations between terms of the different series T_i (r_i, θ_i) so that the resulting combined series automatically satisfied the symmetry conditions across the symmetry lines. Without loss of generality it can be assumed that the x-axis ($\theta = 0$) is one of the symmetry lines. With this assumption the sine terms in the T_i (r_i, θ_i) series can be dropped since they are antisymmetric about the axis. In terms of the angle $\psi_i = \theta_i - \pi i/m$, the symmetric temperature function is then written in the form:

$$T = A_{0,1} \sum_{i=0}^{m-1} \log r_i + A_{1,3} \sum_{i=0}^{m-1} (r_i^{-1} \cos \psi_i) + \sum_{n=2}^{\infty} \left[A_{n,3} \sum_{i=0}^{m-1} r_i^{-n} \cos n \psi_i \right] \qquad (7)$$

Composite Regions

The solution of problems involving composite regions by the boundary point least squares method is realized by developing a separate solution series applicable to each region. The various solution series are interrelated through appropriate interface conditions across common boundaries such as the three sets of conditions given by eq. (4-a), (4-b), and (4-c). These equations are used to relate the unknown coefficients in the two series by evaluating them at points along the common boundaries.

3. SOLUTION OF PROBLEMS INVOLVING THE RADIATION BOUNDARY CONDITION

Once an appropriate trial series has been chosen, the unknown coefficients are determined by first evaluating the boundary and interface equations, in terms of the trial series, at a selected set of points on these curves. More equations than unknowns are written and the resulting system of equations is solved in the least squares sense. When some of the boundary conditions involve the fourth power radiation condition, the boundary equation set contains nonlinear equations. For such problems, the approach used here is to solve the equations by a modified successive approximations scheme.

The fourth power law given by eq. (4-c) can be written in the form

$$\frac{K_1}{H_2} \frac{\partial T_1}{\partial n} + \left[\left(T_1 + T_2\right)\left(T_1^2 + T_2^2\right) \right] \left(T_1 - T_2\right) = 0 \qquad (8)$$

Let the quantity in square brackets be denoted by $\beta(T_1, T_2)$ and a superscript n be used to denote the iteration number. Than an iterative procedure can be set up for which boundary equations of the type (8) are replaced by

$$\frac{K_1}{H_2} \frac{\partial T_1^{(n+1)}}{\partial n} + \beta\left(T_1^{(n)}, T_2^{(n)}\right)\left(T_1^{(n+1)} - T_2^{(n+1)}\right) = 0 \tag{9}$$

at the (n + 1)-th iteration step. The solution for $T^{(n+1)}$ is obtained by the boundary point least squares method. When applied to relatively complex problems, the iterates defined by eq. (9) converged too slowly for practicality. The scheme was then modified to become

$$\frac{K_1}{H_2} \frac{\partial T_1^{(n+1)}}{\partial n} + \bar{\beta}^{(n)}\left(T_1^{(n+1)} - T_2^{(n+1)}\right) = 0 \tag{10}$$

$$\bar{\beta}^{(n)} = \left[\frac{1}{2}\left(\bar{\beta}^{(n-1)}\right)^{1/3} + \frac{1}{2}\left(\beta^{(n)}\right)^{1/3}\right]^3 , \tag{11}$$

where $\beta^{(n)} = \beta\left(T_1^{(n)}, T_2^{(n)}\right)$ as defined by eq. (9).

The rationale behind the iterative scheme given by eq. (10) and (11) was based largely on observation. The cube root in eq. (11) was chosen because the function β is cubic in T_1 and T_2. The averaging was done because it was observed that successive values of β as obtained from eq. (9) oscillated about the correct value of β. Clearly, if the iterates were monotone, the averaging technique would have, in fact, decelerated the rate of convergence.

The procedure of first writing the nonlinear equations in the successive approximations form and then solving the resulting linear equations in the least squares sense did not yield a true least squares solution of the original system of equations. To see this, let the system of least squares equations resulting from eq. (10) and (11) be denoted by

$$A^T [Ax - b] = 0 , \tag{12}$$

where A is the coefficient matrix, x is the vector of unknown coefficients, and b is the vector of nonhomogeneous terms.

If on the other hand the sum of the squares of the residuals of the linear and nonlinear equations is minimized and then the successive approximation scheme applied, the following system of equations is obtained:

$$B^T [Ax - b] = 0 , \tag{13}$$

where the matrix B is given by

$$B = A + C .$$

The matrix C has entries of the form

$$\frac{\partial\beta}{\partial x_i}(T_1 - T_2)$$

in the rows corresponding to the nonlinear equations, and zeros elsewhere. As a result, the solution of eq. (12) will differ from the solution of eq. (13) unless $C^T Ax = C^T b$ at the "common" solution. This will be the case if the equations $Ax = b$ have an exact solution. Similarly, the two solutions will be practically identical if the residuals $Ax - b$ are sufficiently small. Thus, if the problem under consideration has a solution representable by the trial functions chosen, the method will yield the least squares solution for all practical purposes. For this reason and because the approximate scheme requires less computation time than the true least squares approach, the simpler scheme was used to solve the problems discussed below.

4. NUMERICAL RESULTS

In order to test the iterative solution technique and insure that the program was working properly, a simple radially symmetric problem with a known exact solution was considered. The problem consisted of concentric cylinders as shown in Figure 1. The inner annulus is bounded by the curves $r = 0.5$ and $r = 1.0$ while the outer annulus is bounded by the curves $r = 1.0$ and $r = 2.0$. The temperature along the curve $r = 0.5$ was prescribed to be $T_1 = 1.693$ and along the curve $r = 2.0$ was prescribed to be $T_2 = -0.693$. The conditions prescribed by eq. (4-c) were assumed to hold along the curve $r = 1.0$. It was assumed further that $K_1 = K_2 = 1.0$ and $H_2 = 1.0$. The solution of this problem is given by

$$\begin{aligned} T_1 &= A_1 + A_2 \ln r \\ T_2 &= B_1 + B_2 \ln r , \end{aligned} \qquad (14)$$

where $A_1 = 1.0$, $B_1 = 0.0$, and $A_2 = B_2 = -1.0$.

Using the assumed series given by eq. (14) boundary equations were written along the three boundary curves every 9° from $\theta = 0$ to $\theta = \pi/4$. The problem was solved by the iterative techniques given by eq. (9) and by eq. (10) and (11). Table 1 compares the values of β, A_1, and B_1 as a function of the iteration number for both iterative schemes. In both cases the initial guess β_o was taken to be 0.2. Note that the scheme given by eq. (9) produces values of β, A_1, and B_1 which oscillate about their true values of 1.0, 1.0, and 0.0, respectively. By contrast, the modified scheme produces a monotone sequence of iterates and, more importantly, converges at a considerably faster rate. Although this example problem is quite simple, the behavior of the iterates and the comparative rates of convergence were observed to hold in the case of considerably more complex problems.

The remaining numerical results all pertain to the configuration shown in Figure 2. The main annulus (region 1) is bounded by the curves $r = .9335$ cm. and $r = 2.936$ cm. The ring of small holes (region 2) have their centers on the curve $r = 1.935$ and have radii of .6015. Each hole has twelve point sources of strength .6 located at the positions

(2.135, 0.0), (2.035, ± 0.173), (1.835, ± 0.173), (1.735, 0.0), (2.2814, ± 0.2), (1.935, ± 0.4) and (1.5886, ± 0.2) for the hole centered on the x-axis. The locations of the sources in the remaining holes can be obtained by successive rotations of 45°. The conductivities of the two regions were taken to be .2W/cm°C and the emissivity was taken to be .8. The outer boundary was maintained at a temperature of 937° C while the inner boundary was maintained at 977°C.

The problem exhibits eight fold symmetry and the symmetry element is shown in Figure 2.* The set of expansion centers for region 1 consists of the origin and the center of the hole (1.935, 0.0) while the expansion centers for region 2 are the hole center and the locations of the heat sources. The set of trial functions chosen were:

For region 1 at the origin

$$T_1 = A^{(1)}_{0,1} \ln r + A^{(1)}_{0,6} + \sum_{n=1}^{3} \left(A^{(1)}_{8n,1} r^{8n} + A^{1}_{8n,3} r^{-8n} \right) \cos 8n\theta, \tag{15}$$

for region 1 at (1.935, 0.0)

$$T_2 = A^{(1)}_{0,1} \ln r + \sum_{n=1}^{14} A^{(2)}_{n,3} r^{-n} \cos n\theta; \tag{16}$$

for region 2 at (1.935, 0.0)

$$T_3 = A^{(3)}_{0,6} + A^{(3)}_{0,7} x + \sum_{n=2}^{14} A^{(3)}_{n,1} r^{n} \cos n\theta; \tag{17}$$

and for region 2 at the source locations

$$T_4 = .6 \sum_{i=1}^{12} \ln r_i \ . \tag{18}$$

The series T_2 for region 1 is symmetric with respect to the x-axis. The required 8-fold symmetry was obtained by utilizing series of the form (7). Note that the series given by eq. (17) contains no singular terms since this point is not a singularity for region 2.

Using these trial functions the temperatures and gradients were determined separately for each of the three conditions given by eq. (4-a), (4-b), and (4-c) prescribed along the interface between regions 1 and 2. In the case of eq. (4-b) (convective heat transfer between noncontacting boundaries) it was assumed that a "gap" of .012 cm existed between the boundaries of region 2 and region 1 and that the gap was filled with helium gas. The

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* This figure represents a model of the 8-hole teledial fuel pin being considered for the OECD High Temperature Reactor Project.

conductivity of the gas was taken to be .516 x 10^{-2} W/cm°C.

Figure 3 is a graph of the temperature around the interface for the three cases. Of significance is the high temperature of the radiating surface which results from the nonlinear equations, eq. (4-c). The temperatures along the region 1 boundary for the convective boundary conditions are not shown because the maximum temperature drop across the gap was 10°C. This temperature ranged from 946°C at $\theta = 0$ to 974°C at $\theta = \pi$.

The residual errors in meeting the boundary conditions for all three problems were quite reasonable. For the problem in which direct contact was assumed, the prescribed boundary temperatures were satisfied to within .3°C on the inner annular surface and to within .0006°C on the outer surface. The interface equations were evaluated every 5° from $\theta = 0$ to $\theta = \pi$ and the maximum residuals were .17°C for the temperatures and .9 x 10^{-3}°C/cm for the gradients. Comparable residuals were obtained on the convective heat transfer problem. The maximum residual calculated for the nonlinear radiation condition was .0032 W/cm^2 which is about three significant digits for this problem. It is believed that the magnitudes of these residuals are encouraging evidence of the applicability of the BPLS method to problems involving nonlinear boundary conditions.

ACKNOWLEDGMENT

The authors wish to express their gratitude to Mr. M. Tikson, Associate Manager of the Computation Systems and Applications Section, for the use of Battelle's computational facilities. We would also like to acknowledge the guidance of Mr. A. N. Kinkead of the OECD High Temperature Reactor Project, Dragon Project Office, who sponsored this research.

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TABLE I. COMPARISON OF THE ITERATIVE METHODS

Iteration Number	Ordinary Successive Approximations			Modified Method		
	β	A_1	B_1	$\bar{\beta}$	A_1	B_1
	$\beta_o = .2$			$\beta_o = .2$		
1	2.2450	1.4341	-.4341	2.2450	1.4341	-.4341
2	.6683	.7901	.2098	1.3020	.7901	.2099
3	1.2670	1.1193	-.1193	1.0670	.9254	.0746
4	.8747	.9328	.0672	1.0140	.9813	.0187
5	1.0820	1.0392	-.0392	1.0030	.9959	.0041
6	.9557	.9774	.0226	1.0010	.9991	.0009
7	1.0270	1.0132	-.0132	1.0000	.9998	.0002
8	.9848	.9923	.0076	1.0000	.9999	.0001

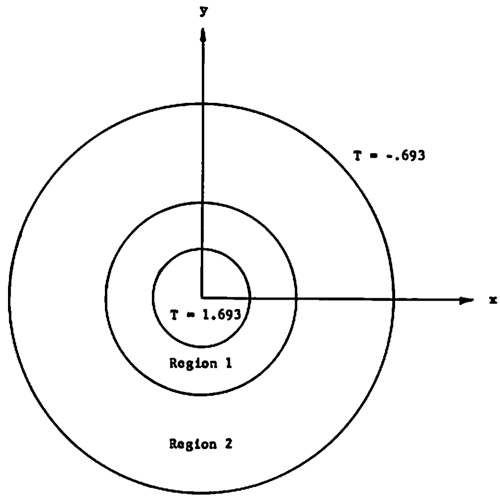


FIGURE 1. CONCENTRIC ANNULI WITH HEAT TRANSFER BY RADIATION

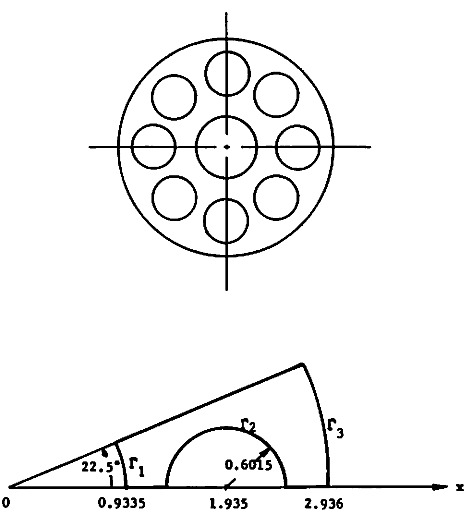
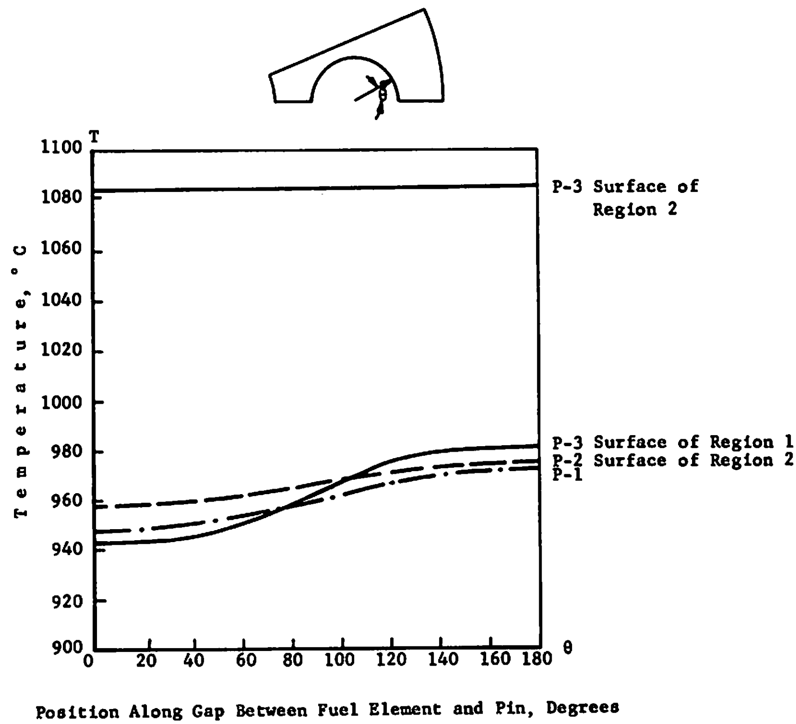


FIGURE 2. ANNULUS WITH RING OF HEAT GENERATING CYLINDRICAL INCLUSIONS



Conductivities Expressed in W/cm° C

P-1 $T_1 = T_2$

$$K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n}; \; K_1 = .2, \; K_2 = .2$$

P-2 $K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n}$

$$K_1 \frac{\partial T_1}{\partial n} + H_1 (T_1 - T_2) = 0; \; K_1 = .2, \; K_2 = .2, \; H_1 = .43$$

P-3 $K_1 \frac{\partial T_1}{\partial n} = K_2 \frac{\partial T_2}{\partial n}$

$$K_1 \frac{\partial T_1}{\partial n} + H_2 (T_1^4 - T_2^4) = 0; \; K_1 = .2, \; K_2 = .2, \; H_2 = 4.5 \times 10^{-12}$$

FIGURE 3. TEMPERATURES AROUND THE HOLE BOUNDARY FOR THREE DIFFERENT BOUNDARY CONDITIONS

DISCUSSION

Q A. N. KINKEAD, U. K.

In order to arrive at the most accurate temperature distribution within the fuel compact of a fuel pin such as the second example in Dr. Hulbert's paper it is necessary to consider a dual mode of heat transfer across the gas filled gap between the fuel matrix and the graphite structure, which takes into account both direct conduction through the gas and radiative heat transfer across the gap. I understand that Dr. Hulbert has recently amended his analysis to take this dual mode into account and I would like him to explain briefly how this has been accomplished.

A point about the TACOR code which has not been highlighted in the paper examples is that although the solution is distinctly different from other methods which use finite element mesh, the resulting temperature distribution can be arranged to be output for all node points of any selected finite element mesh which fits the geometry of the problem. Perhaps Dr. Hulbert could enlarge a little on this facility.

A L. E. HULBERT, U. S. A.

I would like to thank Mr. Kinkead for mentioning these additional features of the method.

a) The solution to problems involving combined radiation and convection heat transfer is easily carried out by adding the convection transfer coefficient to the effective heat transfer coefficient $H_2 \bar{\beta}^{(n)}$ in the linearized radiation Equation (10). Thus for the n-th iteration the boundary point least squares solution is obtained with the following approximate combined radiation-convection heat transfer condition:

$$K_1 \left(\frac{\delta T_1{}^n}{\delta n}\right) + (H_1 + H_2 \bar{\beta}_n)(T_1^n - T_2^n) = 0.$$

b) Since the BPLS method depends on series solution to the differential equation, the obtained solution is valid at every point of the problem region including its boundary. Thus, when the series coefficients have been calculated, the temperatures or heat fluxes at any interior or boundary point may be determined directly from the solution series. This makes it very convenient to use the series solution to generate temperatures at any set of node points as input to a finite element program for calculating thermal stresses. Further, the thermal code has been coupled with a boundary point least squares code for calculating thermal stresses in two-dimensional composite regions.

Q J. A. SWANSON, U. S. A.

1. Is the convergence monotonic or oscillatory in your first and in your improved convergence technique ?
2. What is the first approximation for β ?

A L. E. HULBERT, U. S. A.

1. The convergence is oscillatory in the first approximation and monotonic but much more rapid in the second approximation. With experience, it would probably be possible to choose a weighted average that would increase the convergence rate even more.
2. The first approximation can be any reasonable non-zero value. One good guess would be to take $\beta = H_2 T_1^3$.

Q L. WOLF, Germany

You have mentioned that it is sufficient to have a doubly-redundant system of equations to solve. Do you have any numerical experience on this fact ?

A L. E. HULBERT, U. S. A.

Many years ago we ran a large number of experiments in which we varied the number of boundary conditions from 1.0 to 5.0 times the number of coefficients. We found then and in most of the problems that we have since run that the number of boundary points may be safety chosen to be 1.5 to 2.0 times the number of equations.
The boundary point spacing may be chosen to be fairly uniform except in the neighbourhood of an irregularity in the boundary conditions or boundary shape which leads to large temperature gradients. In such regions it is necessary to space boundary points more closely. If there is a singularity in the temperature function, it should be represented in the solution series or the BPLS method will not converge. Such behaviour is quite equivalent to the classical approaches to solving such problems.

Q F. C. WEILER, U. S. A.

Would you please explain how this "boundary point least squares method" compares with finite difference or finite element methods on accuracy, computer run times for obtaining solutions, etc. ?

A L. E. HULBERT, U. S. A.

In all cases that we have examined the boundary point least squares method is far more accurate and easy to use than the finite element method. This is because the solution satisfies the differential equation of the problem. Thus the approximations to the exact solution are made on the boundary and in addition it is necessary to specify in the input data only data points on the boundary, in addition to specifying the solution series. Not having to calculate the coordinates of interior points enormously simplifies the labor of making up input data.
Running times are comparable to the best finite element methods. However, if fewer series terms were taken so that accuracies of the two methods are comparable, BPLS is much faster.

Q J. DONEA, JRC Ispra, Italy

How can you determine the gas-gap thickness without coupling the thermal problems with the mechanical one ? At Ispra we use the same method as explained by Dr. Hulbert for both temperature and stress fields. This enables the hot gap between fuel and canning materials to be found. Moreover, the temperature and stress fields are found simultaneously (without using finite elements).

A L. E. HULBERT, U. S. A.

It is not possible to exactly evaluate the gas-gap thickness without calculating the thermal strains. We too have a computer program (as noted in an earlier comment) for calculating the thermal stresses and strains with the boundary point least squares methods at the same time the temperatures are calculated. This program can be used in an iterative way to evaluate the change in gas-gap and the change in radiation heat transfer because of thermal strains.