TEMPERATURE AND DEFORMATIONS IN RODS AND PLATES MELTING UNDER INTERNAL HEAT GENERATION*

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ABSTRACT

Some basic problems concerning melting plates under conditions which may be expected in nuclear reactors are considered. These include the development of an exact analytical solution for a plate melting under the action of internal heat generation, an outline of an approximate solution for the same problem, an estimate of the time required for total melting and the construction of a lower bound for that time. The deformations of the melting plate are then treated, and the accuracy of the use of a purely elastic analysis is assessed.

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1. INTRODUCTION

Although one of the important critical conditions in the design of solid fuel nuclear reactors is the one concerning melt-down (e.g. Refs.[1,2]), comparatively little research has been carried out regarding the pertinent methods of analysis. Some published works (e.g.[3]) have studied the conditions at complete core melt-down, but the extensive literature on problems of heat conduction with change of phase (as indicated for example by the bibliographies in [4,5,6]) has little direct application to nuclear reactors. The principal reason for this is that melting due to internal heat generation has not been treated in the literature, while considerable effort has been expended in the analysis of melting and solidification problems under a variety of surface heating conditions for various geometries. Accordingly, it is the purpose of the present paper to perform some preliminary studies of this type of problem so as to provide some initial useful results and at the same time indicate some general methods of attack.

More specifically, Section 2 of the paper presents a basic short-time exact analytical solution of a typical melting plate problem with heat generation, in which some general conclusions and some useful similarities with the analogous surface-heating problems are noted. Section 3 indicates an approximate analysis of the above melting problem including an estimate and an lower bound to one important practical parameter, namely the total meltdown time. Section 4 contains an approximate analysis of the deformations of a plate during the change-of-phase process, which may be useful in assessing the consequent possible variations in coolant channel dimensions [1,2]. A discussion of the relative accuracy of analyses performed on the basis of purely elastic and of elasto-plastic material behavior is included.

It should be noted that, although referred specifically to plates, the developments of this paper are valid for rods as well, with no modification in Sections 2 and 3, and with Poisson's ratio omitted in Section 4.

2. MELTING UNDER HEAT GENERATION

Consider a plate ($-h<x<h$) initially ($t=0$) solid at $T = 0$, under a distributed heat generation $H(x,t)$, symmetrical about $x = 0$, whose surfaces $x = \pm h$ are maintained at the same prescribed temperature or heat input. Then $T(x,t) = T(-x,t)$ and we assume that the maximum temperature occurs always at $x = 0$ and reaches at $t = t_m$ the melting value T_m. The solution for the premelting period $t \leq t_m$ is obtained by standard methods [4]; that for the post-melting period will now be constructed by the embedding technique [5,7-13].

Let the portion $|x| < s(t)$ be liquid with temperature $T_L(x,t)$ and the portion $s(t) < |x| < h$ be solid with temperature $T_S(x,t)$ for $t_m < t < t_h$, where

$$s(t_m) = 0 \quad ; s(t_h) = h \tag{2.1}$$

Then the temperatures T_S and T_L must satisfy the heat conduction equations

$$k_{S,L}\frac{\partial^2 T_{S,L}}{\partial x^2} - \rho c_{S,L}\frac{\partial T_{S,L}}{\partial t} + H(x,t) = 0 \tag{2.2}$$

where k is the thermal conductivity, ρ the density, and c the specific heat of the solid or liquid as indicated by the subscript. At $t = t_m$ the temperature $T_S(x,t_m)$ is known from the premelting solution, and

$$T_L(0,t_m) = T_S(0,t_m) = T_m . \tag{2.3}$$

We must furthermore satisfy the conditions

$$k_L \frac{\partial T_L}{\partial x}(0,t) = 0 \tag{2.4}$$

and the required ones on T_L at $|x| = h$; for the short time solutions desired here, however, the latter are immaterial [8] and need not be stated explicitly. The interface $[x=s(t)]$ conditions are

$$T_S[s(t),t] = T_L[s(t),t] = T_m \tag{2.5a}$$

$$k_S \frac{\partial T_S}{\partial x} - k_L \frac{\partial T_L}{\partial x} = \rho \ell \frac{ds}{dt} \qquad \text{at } x = s(t) \tag{2.5b}$$

Except for conditions (2.3), it is evident that the initial liquid temperature $T_L(0,t_m) = T_m \theta(x)$ and the heat flux on the solid at $x = 0$, i.e. $Q'(t) = - k_S[\partial T_S(0,t)/\partial x]$, are perfectly arbitrary. Hence, if we mathematically embed both phases in the region $|x| < h$ for all time, and similarly extend the region of definition of the functions T_S and T_L, we may then adjust the three unknowns $s(t)$, $Q'(t)$ and $\theta(x)$ so as to satisfy the three interface conditions (2.5a,b).

When the embedding technique just outlined is employed in the usual manner (as in the previously cited references), it leads to three integro-differential equations for the three unknowns, and these are then solved by means of an asymptotic short-time analysis and in terms of series or numerically for longer times. On the basis of the experience gained from previous analyses, however, we may circumvent the more rigorous procedure just outlined, and assume immediately the form of the three functions in terms of the following series:

$$\xi(y) \equiv \frac{s}{2\sqrt{\kappa_S t_m}} = \sum_{i=1}^{\infty} \xi_i y^{(i+2)/3} \tag{2.6a}$$

$$\frac{Q'(y)}{Q_o} = \sum_{i=1}^{\infty} q_i' y^{i/2} \tag{2.6b}$$

$$\theta(X) = 1 + \sum_{i=1}^{\infty} \theta_i X^i \tag{2.6c}$$

The unknown constants ξ_i, q_i' and θ_i are then determined by direct substitution of the temperatures T_S and T_L, consistent with eqs. (2.6), into eqs. (2.5). The correctness of the assumed form of the unknowns is verified a posteriori by noting that all terms are uniquely determined and that all problem conditions are satisfied. The following dimensionless notation is found convenient:

$$y = \frac{t-t_m}{t_m}; \quad X = \frac{x}{2\sqrt{K_S t_m}}; \quad m = \frac{\sqrt{\pi}\, c_S T_m}{2\ell}; \quad D = \frac{\kappa_S}{\kappa_L}; \quad K = \frac{k_S}{k_L} \qquad (2.7)$$

where κ is the thermal diffusivity and where the reference heat input Q_o will be taken, as in [8], to be $Q_o = \sqrt{\pi}\, k_S T_m/(2\sqrt{\kappa_S t_m})$.

The solutions of the heat conduction equation corresponding to (2.6b,c) are

$$T_S(X,y) = T_m\sqrt{\pi}\sum_{i=1}^{\infty} q_i' y^{\frac{i+1}{2}} \frac{i^{i+1}\mathrm{erfc}(X/\sqrt{y})}{i^{i}\mathrm{erfc}\ 0} + T_{S,p}(X,y) \qquad (2.8a)$$

$$T_L(X,y) = T_m[1+\theta_1(X+\sqrt{\tfrac{y}{D}}\ \mathrm{ierfc}\ X\sqrt{\tfrac{D}{y}}) + \theta_2(\tfrac{y}{2D}+X^2) + \ldots] + T_{L,p}(X,y) \qquad (2.8b)$$

where however $\theta_1 = 0$ from eq. (2.4), and where the particular solution $T_{S,p}$ can be taken, as usual, as the analytic continuation (denoted by T^*) of the premelting solution into the post-melting regime, or [8]:

$$T^*(X,y) = T_m \sum_{n=o}^{\infty}\sum_{i=o}^{n} a_{ni} y^i X^{n-i} \qquad (2.9)$$

where

$$a_{ni} = \frac{1}{i!(n-i)!}\left[\frac{\partial^n (T/T_m)}{\partial y^i \partial X^{n-i}}\right]_{\substack{X = 0 \\ y = 0+}} \qquad (2.9a)$$

so that, for example

$$a_{10} = 1; \quad a_{ni} = 0 \text{ for } n-i = 1; \quad a_{20} - 2a_{11} = 2H_o, \text{ etc.} \qquad (2.9b)$$

The last expression given results from the heat-conduction equation with H constant and with

$$H_o = Ht_m/(\rho c_S T_m) \qquad (2.9c)$$

The particular solution $T_{L,p}$ must be found from eqs. (2.2), and, for H constant, is

$$T_{L,p}(X,y) = T_m H_o (D/K) y \qquad (2.10)$$

Hence comparison of (2.8b), (2.10) and the second of (2.5a) gives

$$\theta_2 = -2KH_o \tag{2.10a}$$

while substitution of (2.9) and (2.8a) into the first of (2.5a) gives

$$q_1' = -\frac{4a_{11}}{\pi}; \quad q_2' = 0; \quad q_3' = -\frac{16}{3\pi}\left(\frac{16m\,a_{11}^2}{3\pi^{3/2}} + a_{22}\right); \quad \text{etc.} \tag{2.11}$$

after expansion of T_S, with the aid of (2.6a), in powers of $\sqrt{y}$. Finally, substitution of the above results into (2.5b) yields, after integration with respect to time:

$$\xi_1 = \frac{4m}{3\pi}a_{11}; \quad \xi_2 = \frac{m\,a_{21}}{4\sqrt{\pi}}; \quad \xi_3 = \frac{16m}{15\pi}\left(a_{22} + \frac{16m\,a_{11}^2}{3\pi^{3/2}}\right); \quad \text{etc.} \tag{2.12}$$

The first few terms of the exact solution for the problem under consideration have thus been obtained. The following comments about this solution may be useful:

(a) The solution for this case is of the same type as that obtained in the references cited earlier for melting under a prescribed finite heat input, i.e. the leading term of the series for $s(t)$ is proportional to $(t-t_m)^{3/2}$ rather than to $(t-t_m)^{1/2}$ as in the classical Neumann solution corresponding to sudden surface temperature variations (cf., for example [8]).

(b) The coefficients ξ_i given above are identical with the corresponding entries in Table 2 of Ref. [8], for the case of instantaneous removal of the liquid portion (i.e. as if the heat were applied directly on the melt interface).

(c) It follows from remark (b) that the presence of the liquid is not explicitly observed in the first few ξ_i coefficients, just as in the analogous surface heating problem [7,8]. It must be noted, however, that in the calculation of ξ_3 the term θ_2 is used, but happens, for the present type of heat generation, to cancel when the last of (2.9b) is employed.

(d) Because of the above similarities with the surface-heating problem, most of the conclusions pertaining to that problem as regards temperature-dependence of properties, start of melting, comparison theorems, ablation of the liquid, etc. for short times are still valid here. The reader is therefore referred to the works referenced above for further details.

3. APPROXIMATE ANALYSIS

As a further consequence of the preceeding developments, the various approximate methods of heat-conduction analysis which have been found useful in the surface-heating problem [5,14-17] are again applicable here. Consider, for example, the problem of the plate of Section 2, under a prescribed temperature $T_o < T_m$ at $|x| = h$ and initially; then the heat-balance method, with an assumed quadratic distribution (which was found sufficiently accurate in [18]) and for constant H, gives the premelting solution as:

$$T = T_o + \frac{H}{2k_S}(h^2-x^2)(1-e^{-3\kappa_S t/h^2}) \; ; \qquad t \leq t_m \tag{3.1}$$

and therefore

$$t_m = -\frac{h^2}{3\kappa_S}\log[1-\frac{2(T_m-T_o)k_S}{Hh^2}] \tag{3.1a}$$

For the post-melting solution, let both the liquid and solid temperature distributions be assumed to be quadratic polynomicls in x, or (after the known boundary conditions are satisfied):

$$T_S(x,t) = T_m + a(x-s) - (\frac{x-s}{h-s})^2[(T_m-T_o) + a(h-s)]; \qquad s(t) < x < h \tag{3.2a}$$

$$T_L(x,t) = T_m + b(x^2-s^2); \qquad 0 < x < s(t) \tag{3.2b}$$

where the unknown functions $a(t)$, $b(t)$ must be chosen so as to satisfy eqs. (2.2) in some approximate way. If the heat-balance integral is used for this purpose, the result is

$$2[(h-s)\frac{ds}{dt} - 6\kappa_S][T_m-T_o+a(h-s)] - (h-s)^2\frac{da}{dt} + \frac{H(h-s)^2}{\rho c_S} = 0 \tag{3.3a}$$

$$6b\kappa_L + 2s^2\frac{db}{dt} + 6sb\frac{ds}{dt} + \frac{3H}{\rho c_L} = 0 \tag{3.3b}$$

Eq.(2.5b) yields

$$k_S a - 2k_L bs = \rho\ell\frac{ds}{dt} \tag{3.4}$$

The desired approximate solution is then obtained by simultaneous solution of the last three equations under the initial conditions $s(t_m) = 0$, $a(t_m) = 0$ so as to match T_S at $t = t_m$ with that given by eq. (3.1), and $b(t_m) = -H/(2k_L)$ from (3.3b). This solution can be obtained by forward numerical integration, and use of the exact short-time solution of the preceding section facilitates the start of the procedure.

Even the numerical approximate solution just outlined can however be rather cumbersome, and it is therefore useful to obtain a rapid estimate of at least one of the practically important parameters, namely the time required for total melting, i.e. t_h as defined by eqs. (2.1). To this end, a total energy balance is set up, as follows, for the entire plate in the entire period $0 < t < t_h$. The total energy supplied is

$$\int_0^{t_h}\int_0^h H(x,t)dxdt + \int_0^{t_h}[Q_1(t)-Q_h(t)]dt \tag{3.5a}$$

where Q_1 is the heat input at $x = 0$ and Q_h the heat output at $x = h$. This heats the solid from its initial temperature $T_S(x,0)$ to T_m, and the liquid from T_m to its final temperature $T_L(x,t_h)$; hence the quantity (3.5a) must equal the quantity

$$\rho c_S[T_m h-\int_0^h T_S(x,0)dx] + \rho c_L[\int_0^h T_L(x,t_h)dx-hT_m] + \rho \ell h \tag{3.5b}$$

Thus for the present problem, in which $Q_1 = T_S(x,0) = 0$,

$$\rho(c_S-c_L)T_m h+ \rho c_L\int_0^h T_L(x,t_h)dx + \rho\ell h = \int_0^{t_h}\int_0^h Hdxdt - \int_0^{t_h} Q_h dt \tag{3.6}$$

The quantity $T_L(x,t_h)$ is unknown, but its integral can be estimated by using for it the temperature that would result in the liquid, under the prescribed internal heat generation, starting at $t = t_m$. This is then, from eq. (3.1), for constant H,

$$T_L(x,t_h) = T_m + \frac{H}{2k_L}(h^2-x^2)[1-e^{-3\kappa_L(t_h-t_m)/h^2} \tag{3.7}$$

and therefore eq. (3.6) gives

$$t_h = \left[\frac{\rho(\ell+c_S T_m)}{H} + \frac{h^2}{3\kappa_L}\right]\frac{1}{1-[Q_{h,tot}/(Hht_h)]} \geq \frac{\rho(\ell+c_S T_m)}{H} + \frac{h^2}{3\kappa_L} \tag{3.8}$$

where $Q_{h,tot}$ stands for the last integral in (3.6); note that $0 < Q_{h,tot} < Hht_h$. Eq. (3.8) gives lower bound to t_h, and is therefore a conservative result.

Should a closer estimate be desired, one may calculate $Q_{h,tot}$ from eq. (3.1), i.e.

$$Q_{h,tot} = Hh\left[t_h - \frac{h^2}{3\kappa_S}(1-e^{-3\kappa_S t_h/h^2})\right] \tag{3.9}$$

Eq. (3.6) now gives

$$De^{-3\kappa_L(t_h-t_m)/h^2} - e^{-3\kappa_S t_h/h^2} = \frac{3\rho(c_S T_m+\ell)}{Hh^2} - 1 + D \tag{3.10}$$

If, for example, $\kappa_L = \kappa_S (D=1)$, then

$$t_h = \frac{h^2}{3\kappa}\log\frac{Hh^2(e^{3\kappa t_m/h^2}-1)}{3\rho\kappa(c_S T_m+\ell)} = \frac{h^2}{3\kappa}\log\frac{2T_m c}{3(cT_m+\ell)(1-\frac{2T_m k}{Hh^2})} \quad (3.10a)$$

where, in order to obtain the last expression, the value of t_m from eq. (3.1a) has been introduced.

4. DEFORMATIONS OF A MELTING PLATE

The deformations of melting plates were treated in Ref. [18], for surface-heating problems, on the assumption that the material was elastic-perfectly plastic with a temperature-dependent yield stress. It was further noted there that calculations based on the assumption of elastic behavior at all times yielded deformations in good agreement with the inelastic ones, and on the conservative side (i.e. larger-then-actual deflections). The validity of similar conclusions for the present heat-generation case will now be examined.

For an elastic-perfectly plastic material, the stress σ in the plane of the plate $s < x < h$ is [19]

$$\sigma(x) = g\frac{E}{1-\nu}[-\alpha T + f_1 + (x - \frac{h+s}{2})f_2] + (1-g)(\mathrm{sgn}\ \sigma)Y(T) \quad (4.1)$$

where $g(x) = 0$ if the point x is in a plastic region and $g(x) = 1$ if x is in an elastic region, and where $\mathrm{sgn}\ \sigma = \pm 1$ respectively for tension and compression. The functions f_1 and f_2 are calculated from the conditions

$$\int_s^h \sigma dx = N \quad ; \quad \int_s^h \sigma x_1 dx = M \quad (4.2)$$

where

$$x_1 = x - \frac{h+s}{2} \quad (4.2a)$$

If the plate is entirely elastic, then $g \equiv 1$ and

$$\sigma_e = \frac{E}{1-\nu}(-\alpha T + f_{1e} + f_{2e}x_1) \quad (4.3a)$$

$$f_{1e} = \frac{(1-\nu)N+N_T}{(h-s)E} \qquad f_{2e} = \frac{12[(1-\nu)M+M_T]}{(h-s)^3 E} \quad (4.3b)$$

where

$$N_T = \int_s^h \alpha ET dx \quad ; \quad M_T = \int_s^h \alpha ETx_1 dx \quad (4.4)$$

The first plastic zone may be assumed to form near $x = s$, because that is the most highly stressed point, and at the same time the temperature is there the highest and therefore the yield stress can be expected to be the lowest. Let therefore the region $s < x < s + h_p$ be plastic; then conditions (4.2) give

$$f_1 = f_{1e} + \frac{I_1}{(h-s)} \qquad ;f_2 = f_{2e} + \frac{12\, I_2}{(h-s)^3} \tag{4.5}$$

where

$$I_{1;2} = \int_s^{s+h_p} [-(\mathrm{sgn}\sigma)\frac{Y(1-\nu)}{E} - \alpha T + f_1 + x_1 f_2](1;x_1)dx \tag{4.5a}$$

The quantities f_1 and f_2 are proportional respectively to the strains in the plane of the plate and to the curvature of the plate [19, p.381] and are thus directly related to the deflections of the plate. For example, for a free plate of arbitrary planform, the displacements in the plane of the plate are given by

$$\frac{u}{x} = \frac{v}{y} = f_1 \tag{4.6a}$$

and the transverse deflection is

$$w = -\frac{f_2}{2}(x^2+y^2) \tag{4.6b}$$

exclusive of rigid-body motions [19,p401]. A comparison of between the values of $f_{1,2}$ and $f_{1,2e}$ is therefore equivalent to a comparison of the corresponding deflections, and will now be carried out.

Consider eq. (2.2) for the solid: at any point, a portion of the heat energy $H \geq 0$ supplied produces a temperature rise at the rate $\partial T/\partial t \geq 0$, while the remainder is removed by conduction. Hence in general

$$-\kappa\frac{\partial^2 T}{\partial x^2} = \frac{H}{\rho c} - \frac{\partial T}{\partial t} \geq 0 \tag{4.7}$$

and we can indeed verify that is true in any of the available solution [such as that of eq. (3.1)]. This means that the temperature distribution is of the type shown in Fig. 1, with the stresses as indicated there. Thus $\cdot\mathrm{sgn}\sigma = +1$ in the plastic zone, and the stress which would occur there if the material remained elastic must exceed Y, or

$$\frac{E}{1-\nu}(-\alpha T + f_1 + x_1 f_2) \geq Y \tag{4.7a}$$

the equal sign holding at $x = s + h_p$. The integrand of I_1 is therefore positive, and that of I_2 negative (since $x_1 < 0$ in the plastic zone). However $N_T > 0$ and $M_T < 0$, so that $f_1 > 0$ and $f_2 < 0$; hence

$$f_1 - f_{1e} > 0 \quad ; \quad |f_2| - |f_{2e}| > 0 \tag{4.8}$$

so that in either case use of the elastic solution will yield too large, and therefore non-conservative, absolute values of the deformations. It may be noted that, in the analogous surface-heating problem, inequalities (4.7) and (4.8) are reversed, which $\operatorname{sgn}\sigma = -1$ in the plastic zone, and hence (es was indeed found in[18]) in that case the elastic analysis leads to conservative estimates for the deflections.

We may obtain a more precise estimate for the plastic deformations by noting that the quantity is the bracket in the integrands of I_1 and I_2 varies from zero at the upper limit [cf. eq.(4.7a)] to $[-\alpha T_m + f_1 - (h-s)f_2/2] = \epsilon$ at the lower limit, on the reasonable assumption [18,20] that the temperature-dependent yield stress Y is zero at $T = T_m$. Assume the variation just referred to be linear, in order to get approximate values of I_1 and I_2; then

$$I_{1;2} = \epsilon \int_s^{s+h_p} \left(\frac{s+h_p-x}{h_p}\right)(1;x_1)dx = \frac{\epsilon h_p}{2} ; \frac{\epsilon h_p}{12}(3s - 3h + 2h_p) \tag{4.9}$$

Thus, from eqs. (4.5) and with $R = h_p/(h-s)$,

$$f_1 = f_{1e} + \epsilon R/2 \; ; \; f_2 = f_{2e} + \epsilon R(2R-3)/(h-s) \tag{4.10}$$

Substituting the definition of ϵ one obtains two simultaneous equations for f_1 and f_2, namely

$$(2-R)f_1 + (R/2)f_2(h-s) = 2f_{1e} - \alpha T_m R \tag{4.11}$$

$$2R(3-2R)f_1 + [2-R(3-2R)]f_2(h-s) = 2f_{2e}(h-s) + 2\alpha T_m R(3-2R) \tag{4.11a}$$

If $R = 0$, $f_1 = f_{1e}$ and $f_2 = f_{2e}$, while for other values of R eqa. (4.11) are easily solved for f_1 and f_2. If, for example, the temperature distributions is quadratic, with $T = 0$ at $x = h$ and with $f_{1e} = 2\alpha T_m/3$ and $f_{2e} = -\alpha T_m/(h-s)$, eqs. 4.11 become

$$\begin{aligned} &4(2-R)\left(\frac{f_1}{f_{ie}}\right) - 3R\left(\frac{f_2}{f_{2e}}\right) = 2(4-3R) \\ &\frac{4R(3-2R)}{3}\left(\frac{f_1}{f_{1e}}\right) - [2-R(3-2R)]\left(\frac{f_2}{f_{2e}}\right) = -2 + 2R(3-2R) \end{aligned} \tag{4.12}$$

A plot of (f_1/f_{1e}) and (f_2/f_{2e}) against R, given in Fig. 2, was obtained from eqs. (4.12); it shows that these quantities are indeed larger than unity, as predicted by inequalities (4.8), that more accuracy can be expected, when plasticity is ignored, for small values of R (as is reasonable because then the plastic effect is small) and for in-plane rather than for transverse deflections. Of course, as R becomes larger, further plastic zones will usually arise; these are not considered here, but it can be expected that the accuracy of the purely elastic solution (which can still be estimated by the methods here introduced) will not be worse than that shown in Fig. 2.

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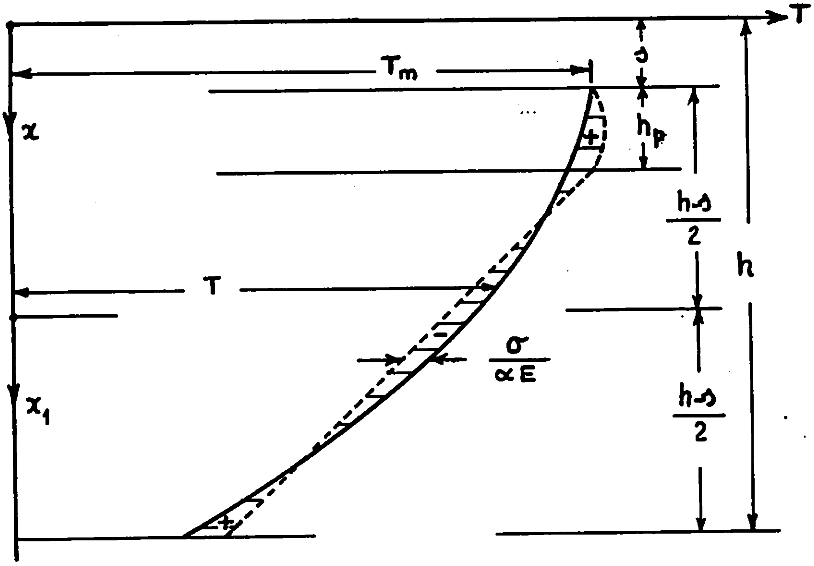

Fig. 1. Temperature and stress distribution in an elasto-plastic melting plate. Yield stress assumed zero at melting temperature.

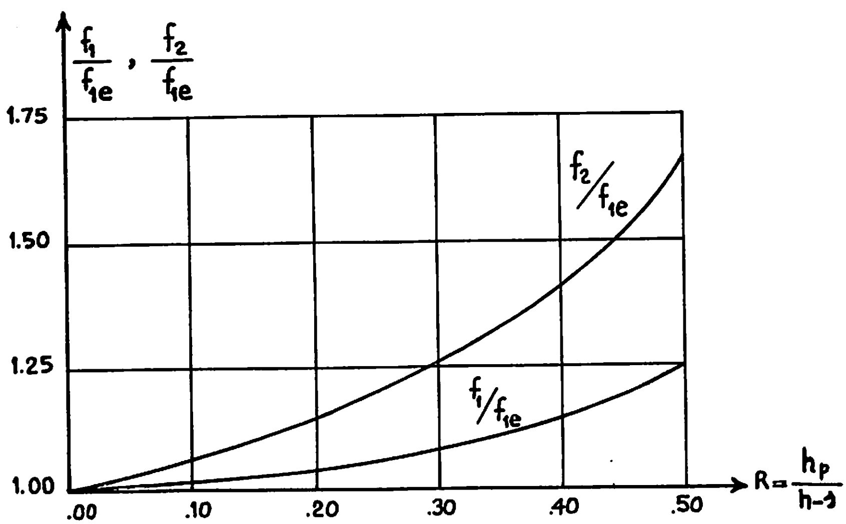

Fig. 2. Errors in in-plane and transverse deflection functions due to neglect of plastic effect.

DISCUSSION

Q C. F. BONILLA, U. S. A.

The illustration employed for this method is a one-dimensional case, which I would presume can be fairly readily solved by a single line of nodes, calculating the progress of temperature and melting by finite time increments. What are the relative advantages and disadvantages of the analytical method in this case ?

A B. A. BOLEY, U. S. A.

While in principle the method proposed by Prof. Bonilla is quite possible, probably in conjunction with an iterative technique (to account for the fact that the position of the solid-liquid interface is not a priori known), in practice I believe it would be difficult to obtain sufficient accuracy without a large number of iterations and nodes. Fortunately, in this case, some alternative procedures are available, such as the approximate analytical solutions referred to in the paper, or numerical solutions obtained by solving the integral equations of the embedding technique (Cf. either Refs. (11) or (12) of the paper).

Q G. MELESE-d'HOSPITAL, U. S. A.

1. Is your technique applicable to other one-dimensional geometries than plates, for instance to cylindrical fuel elements ?
2. Could you include time-dependent boundary conditions, as would occur in fuel elements during rapid flow variations (loss of coolant for instance with constant internal heat generation) ?

A B. A. BOLEY, U. S. A.

The answer is "yes" to both questions. My general approach has been applied to cylindrical elements, for example, in Ref. (12) of the paper, with results quite similar to those for a plate. The inclusion of time-dependent boundary conditions present no real new difficulties, and I believe also no substantially increased numerical effort for obtaining the solution.