THERMAL AND CREEP ANALYSIS OF CYLINDRICAL SHELLS UNDER INTERNAL PRESSURE

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ABSTRACT

A simultaneous thermal and nonlinear viscoelastic analysis of circular cylindrical shell belongs to the group of problems in structural mechanics of reactor technology which have the most important practical significance.

An interesting point of the analysis is the form of distributions of moments and internal forces due to nonlinearity of the shell material . This point has not been clarified satisfactorily in previous papers on the subject because of fargoing simplifications made . Furthermore , the influence of temperature field is not taken into account .

It is the aim of this paper to present a more adequate approach in investigating thermal effects and creep of circular cylindrical shells under internal pressure,on the basis of interaction of internal forces . The analysis is done by applying a theory of nonlinear viscoelasticity given by Bychawski and Fox [8] , [9],generalized here by accounting the influence of temperature . Two variants of the theory are considered , one that of small nonlinearity and the other of the Odqvist creep theory which follows as a particular case .

It is pointed out the difference between fundamental equations derived compared to those of other papers . In particular , some resulting equations are solved in exact manner and solutions discussed .

1. INTRODUCTION

Creep bending in the presence of temperature field of circular cylindrical shells belongs to the group of creep problems in structural mechanics of reactor technology which have the most important practical significance .

As far as I know there is no creep solution of this problem in the literature which could be recognized as satisfactory from theoretical and practical point of view . It is clear that such a situation must be related to the considerable difficulties of the problem we meet in deriving and looking for solution of its fundamental equations . Since some of the available results in the literature are founded on fargoing simplifications , it is obvious that they should be regarded cautiously as conclusions are concerned .

Long circular cylindrical shells in creep state , but without the influence of temperature , under internal pressure has been treated by Onat and Yuksel [1] , Bieniek and Freudenthal [2] , Rabotnov [3] , Gemma [4] , Murakami and Iwatsuki [5] , and other authors .

In [1] a simplified model for a real shell is investigated in the form of a sandwich-walled construction . In analysis the criterion of maximum shear stress is applied . A system of two differential equations of the problem is rather simple and solved for a special case .

In [2] a full-walled shell is examined on the basis of the Odqvist creep theory . Since it is assumed that the effects of bending moment and normal forces are independent , the stresses are separated completly and expressed analogously as in one-dimensional case . Although , such an assumption allows to avoid the main difficulty of the problem,which consists in integrating expressions for internal forces , nevertheless , it seems to be questionable . The same can be said about the assumption of deflection shape analogously to linear solution . This form is then used in calculating a free parameter by application of a variational principle .

A profound review of the two mentioned papers is also given by Odqvist [6] , and general aspects of the problem are discussed by Olszak and Sawczuk [10].In [3] are derived fundamental equations on the basis of variational methods and boundary effects are discussed . The results obtained are approximate .

In [5] transient creep analysis is given by using strain-hardening and time-hardening theories in the presence of elastic strains . Difference method is applied and resulting set of linear equations is integrated numerically .

Some related questions of the problem has been treated by Calladine [7] and in [4] . In the latter a shell equation is obtained in the form analogous as in [2] .

The most interesting point of the problem is the question of distribution form of bending moment and normal force along shell axis in an axially symmetrical case . This form is connected with boundary effects in dependence on the assumed boundary conditions . Due to nonlinearity of physical properties of shell material , these effects cannot be considered separately , with respect to qualitatively different causes,since there exists an interaction of internal forces as expressed by effective quantities . Only on that basis is possible to reveal a real picture of shell behavior . It may be guessed that it differs considerably , not only quantitatively, but also qualitatively,from the known results for linear shell materials .

It is the aim of the present paper to give a more adequate approach to the behavior of a cylindrical circular shell under internal pressure in creep and , in particular , in the presence of temperature field .

The analysis is based on the assumption of interaction of internal forces as expressed by effective strain , strain rate or stress . On the other hand , the analysis is founded on the theory of nonlinear viscoelasticity as formulated by Bychawski and Fox [8],[9] which is here generalized by introducing thermal term . Two variants of the theory are considered as particular cases . One that of small nonlinearity, which allows to linearize the solution , and the other of the Odqvist creep theory, which results from general equations given in [9] . As it will be proved , in the last case it is possible to simplify effective strain rate considerably , so that integration of internal forces can be effectively carried out . It is obvious , that the resulting equation is then completly different(qualitatively)from that obtained in [2] . The form of the latter is given explicitly in [6] .

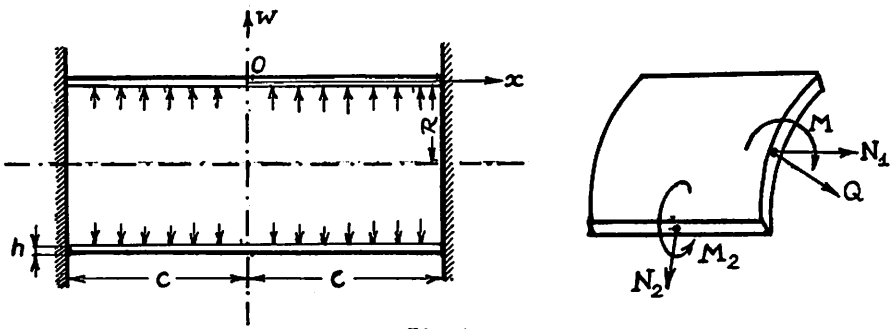

Fig.1

2. FUNDAMENTAL EQUATIONS FOR NONLINEAR VISCOELASTIC SHELL

To establish fundamental equations of the problem I consider an element of the shell shown in Fig.1 . Radius of the cylinder is R and its wall thickness h. The shell is loaded by a constant pressure p . There will be considered two cases of boundary conditions . One that of clamped edges and the other of simply supported edges .

Since the symmetry of shell loading with respect to its axis takes place , from equilibrium consideration of the shell element,we find the following set of equations relating bending moment M , shear force Q and normal forces N_1 , N_2 , in longitudinal and circumferential directions , respectively ,

$$DQ + \frac{c}{R} N_2 = cp \quad , \quad DM + cQ = 0 \quad , \quad DN_1 = 0 \quad . \tag{1}$$

Here , c is one half of shell length and $D = \partial/\partial x$ is derivative with respect to longitudinal non-dimensional variable $x = \bar{x}/c$.

The last of eqs.(1)indicates that N_1 does not depend on x . In particular , it can be assumed equal zero . However , such an assumption cannot be done in statically undetermined cases with the influence of temperature .

Combining Eqs.1 , we have ,

$$D^2M - \frac{c^2}{R} N_2 = -c^2p \quad , \qquad N_1 = \text{const.} \tag{2}$$

where the internal forces are expressed through integrals taken throughout the thickness of shell h. If $z = 2/h\,\bar{z}$ is non-dimensional variable and σ_i $(i = 1, 2)$ denotes normal stresses in longitudinal and circumferential directions, respectively, then, with $\varkappa = 1/2\,(h/c)^2$, we can write,

$$M = \frac{1}{2}c^2\varkappa\int_{-1}^{1}\sigma_1 z\,dz \quad , \quad N_1 = h^{-1}c^2\varkappa\int_{-1}^{1}\sigma_1 dz \quad , \quad N_2 = h^{-1}c^2\varkappa\int_{-1}^{1}\sigma_2 dz \quad . \tag{3}$$

The set of eqs.(2), although obtained on the basis of geometrically linear theory, is in reality nonlinear since we apply nonlinear constitutive equation of the shell material.

3. CONSTITUTIVE EQUATIONS

According to the assumption stated in par.1, I use the constitutive relation formulated in [8], and generalized here by accounting the influence of temperature, in the form relating strain tensor e_{ij} with stress deviator s_{ij},

$$e_{ij} = Ls_{ij} + \alpha T\delta_{ij} \quad , \tag{4}$$

valid for a nonlinear viscoelastic incompressible material. Here,

$$Ls_{ij} = (2G)^{-1}s_{ij} - \int_{t_0}^{t} s_{ij}\partial_\tau H d\tau \quad , \tag{5}$$

and

$$s_{ij} = s_{ij}[t, T(t)] \; , \; H = H\{t, t, s[t, T(t)]\} \; , \; H\Big|_{\tau=t} = 0 \quad , \tag{6}$$

where H is generalized creep function and s effective stress

$$s = \left(\frac{3}{2}\,s_{ij}s_{ij}\right)^{\frac{1}{2}} \quad . \tag{7}$$

Furthermore, G is elastic modulus, t time, t_0 initial instant, α thermal constant, T temperature and δ_{ij} the Kronecker tensor.

In particular, it is possible to reduce eq.(5) to a simpler form by assuming

$$\partial_\tau H = \overline{\Psi}(s)\,\partial_\tau C(t,\tau) \quad , \quad C(t,t) = 0 \quad , \tag{8}$$

where C (usual creep function) is here called creep factor.

On the other hand, if C is a linear function of $(t - \tau)$ then, by neglecting elastic term in eq.(5), we obtain

$$Ls_{ij} = \int_{t_0}^{t}\Psi(s)\,s_{ij}\,d\tau \quad , \tag{9}$$

where Ψ differs from $\overline{\Psi}$ by a multiplicative constant. Furthermore, if we assume

$$s = \frac{3}{2}\,Es^{n-1} \quad , \tag{10}$$

where B and n are creep constants , we arrive at the Odqvist creep theory .

Let us now assume that according to [8] we consider the range of small nonlinearity, and at $t = t_o$ the shell material begins to creep nonlinearly . Then , in a small interval $[t_o, t]$ we can assume in eq.(8)

$$\overline{\Psi}(s) = 1 + \beta s \quad , \quad \overline{\Psi}\Big|_{t=t_o} = 1 + \beta s_o \quad , \tag{11}$$

where β is a small constant of physical significance . It is seen that , if β disappears , eq.(5) becomes a constitutive relation for a linear visco-elastic material with creep function C .

In order to carry out effectively integration of eqs.(3) is necessary to dispose of inverted form of eq.(4), i.e. , to have stress tensor σ_{ij} expressed through strain tensor or strain rate tensor . Taking into account the relation

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma\delta_{ij} \quad , \quad \sigma = \sigma_{kk} \quad , \tag{12}$$

we rewrite eq.(4) as follows

$$e_{ij} = L\left(\sigma_{ij} - \frac{1}{3}\sigma\delta_{ij}\right) + \alpha T\delta_{ij} \quad , \tag{13}$$

and from there we formally find

$$\sigma_{ij} = \frac{1}{3}\sigma\delta_{ij} + L^{*}\left(e_{ij} - \alpha T\delta_{ij}\right) . \tag{14}$$

Here , L^{*} denotes an inverse operator . The form of the latter can easily be found in particular cases of eqs.(8),(9),(10) and (11) , if certain additional assumptions are made .

For example , if we consider alternative eqs.(9) and (10), then accounting the relation existing between effective strain and effective stress

$$e = B\int_{t_o}^{t} s^n dt \quad , \tag{15}$$

we rewrite eq.(4) in terms of strain rates $\dot{e}_{ij}$,

$$\dot{e}_{ij} = \frac{3}{2} B s^{n-1} s_{ij} + \alpha\dot{T}\delta_{ij} \quad , \tag{16}$$

where dot denotes differentiation with respect to time . Its inverted form can be then written as follows

$$s_{ij} = \frac{2}{3}\overline{B}\,\dot{e}^{\,\overline{n}-1}\dot{e}_{ij} - \alpha\dot{T}\delta_{ij} \quad , \tag{17}$$

with $\overline{B} = B^{-\overline{n}}$, $\overline{n} = n^{-1}$ and

$$\dot{e} = \left(\frac{2}{3}\dot{e}_{ij}\dot{e}_{ij}\right)^{\frac{1}{2}} \quad , \tag{18}$$

the last value representing effective strain rate .

On the other hand , if eq. (14) is taken into account, as regards its appearence , eq. (17) may be rewritten ,

$$\sigma_{ij} = \frac{1}{3}\,\sigma\,\delta_{ij} + \dot{L}^{*}(\dot{e}_{ij} - \alpha\dot{T}\delta_{ij}) \quad , \tag{19}$$

where

$$\dot{L}^{*}.. = \frac{2}{3}\,B\,\dot{e}^{\,\overline{n}-1}_{..} \quad . \tag{20}$$

Here the dot over the operator is symbolic .

Now we shall consider the applicability of the second variant of the theory , i.e. , small nonlinearity . In order to do that we develop into power series of β all quantities contained in eq. (4) and (14), the operators included . Thus , in the vicinity of t_o, all the said quantities are represented in the scheme forms,

$$f = \sum_{k=o}^{\infty} f_k \beta^k = \sum_{k=o}^{\infty} f^k \beta^k \cong f_o + \beta f_1 = f^o + \beta f^1 \quad , \tag{21}$$

the two last expressions being a sufficient approximation for a small time-interval considered .

Doing so in eq. (4), we obtain as coefficients of eqs. (21) by β to powers zero and one , respectively ,

$$e^o_{ij} = L_o s^o_{ij} + \alpha T_o \delta_{ij} \quad , \tag{22}$$

$$e^1_{ij} = L_o s^1_{ij} + L_1 s^o_{ij} + \alpha T_1 \delta_{ij} \quad , \tag{23}$$

In a similar way , we find on the basis of eq. (14) , respectively ,

$$\sigma^o_{ij} = \frac{1}{3}\,\sigma^o \delta_{ij} + L^*_o e^o_{ij} - \alpha T_o \delta_{ij} \quad , \tag{24}$$

$$\sigma^1_{ij} = \frac{1}{3}\,\sigma^1 \delta_{ij} + L^*_o e^1_{ij} + L^*_1 e^o_{ij} + (L^*_o \alpha T_1 + L^*_1 \alpha T_o)\,\delta_{ij} \quad . \tag{25}$$

the relations representing the inverted form we need in our further calculations .

It is clear that in accordance with our assumption of approximation taken , eq. (11) is included into operational forms in the second form , i.e. , that of initial condition . Therefore , the problem is linearized in the vicinity of t_o .

The developed forms of the operator L can easily be found by identifying eq. (22) and eq. (23) with the form given by eq. (4) . Thus , we find

$$L_o.. = (2G)^{-1}.. - \int_{t_o}^{t} ..\partial_\tau C d\tau, \quad L_1.. = - \int_{t_o}^{t} ..s_o \partial_\tau C d\tau \quad . \tag{26}$$

On the basis of eqs. (24) and (25) it is then possible to find the forms of inverted operators L_0^* and L_1^*, containing relaxation functions corresponding to creep functions C . We shall point out this problem later .

4. VISCOELASTIC SHELL WITH SMALL NONLINEARITY

Let us consider at first deformation and stress states of shell by taking into account the range of small nonlinearity for the shell material .

Main strains found on the basis of geometrically linear theory are (we drop double index)

$$e_1 = Du - z\varkappa D^2 w \quad , \quad e_2 = \varkappa_0 w \quad , \tag{27}$$

where $u = \bar{u}/c$ is non-dimensional longitudinal displacement component and $\varkappa_0 = h/R$. We develop eqs. (27) in our further calculations according to the sheme of eq. (21) . In eq. (27), $w = \bar{w}/h$ denotes non-dimensional deflection .

On the other hand , main stresses calculeted from eqs. (24) and (25) are

$$\sigma_1^o = L_0^*(2e_1^o + e_2^o - 3\alpha T_o) \ , \quad \sigma_1^1 = L_0^*(2e_1^1 + e_2^1 - 3\alpha T_1 + L_1^*(2e_1^o + e_2^o - 3\alpha T_o) \ , \tag{28}$$

$$\sigma_2^o = L_0^*(2e_2^o + e_1^o - 3\alpha T_o) \ , \quad \sigma_2^1 = L_0^*(2e_2^1 + e_1^1 - 3\alpha T_1) + L_1^*(2e_2^o + e_1^o - 3\alpha T_o) \ , \tag{29}$$

Introducing the above equations into the developed eqs. (3) and accounting developed strains in eqs. (27), we express internal forces through displacement components u and w . Carrying out the integration indicated in eq. (3) we find , successively ,

$$M_o = -\tfrac{2}{3}c^2\varkappa^2 L_0^* D^2 w_o \ , M_1 = -\tfrac{2}{3}c^2\varkappa^2(L_0^* D^2 w_1 + L_1^* D^2 w_o) \ , \tag{30}$$

$$\begin{aligned} N_1^o &= 2h^{-1}c^2\varkappa\, L_0^*(2Du_o + \varkappa_o w_o - 3\alpha T_o) \ , \\ N_1^1 &= 2h^{-1}c^2\varkappa[L_0^*(2Du_1 + \varkappa_o w_1 - 3\alpha T_1) + L_1^*(2Du_o + \varkappa_o w_o - 3\alpha T_o)] \ , \end{aligned} \tag{31}$$

$$\begin{aligned} N_2^o &= 2h^{-1}c^2\varkappa\, L_0^*(Du_o + 2\varkappa_o w_o - 3\alpha T_o) \ , \\ N_2^1 &= 2h^{-1}c^2\varkappa[L_0^*(Du_1 + 2\varkappa_o w_1 - 3\alpha T_1) + L_1^*(Du_o + 2\varkappa_o w_o - 3\alpha T_o)] \ . \end{aligned} \tag{32}$$

The internal forces given by eqs. (30), (32), we put into condition of equilibrium (2) and , simoultaneously , we consider internal pressure p as time-variable quantity in the vicinity of t_o . Doing so , we obtain, in accordance with assumed approximation, a set of two recurrent integro-differential equations, containing unknown functions w and u in the developed form,

$$L_0^* D^4 w_o + 4\bar{a}^{-4} L_0^* w_o = bp_o - \bar{a} L_0^*(Du_o - 3\alpha T_o) \quad , \tag{33}$$

$$L_0^* D^4 w_1 + 4\bar{a}^{-4} L_0^* w_1 = bp_1 - \bar{a}[L_0^*(Du_1 - 3\alpha T_1) + L_1^*(Du_o - 3\alpha T_o)] \ .+ \tag{34}$$

$$- L_1^* L_o \left[bp_o - dL_o^* (Du_o - 3\alpha T_o) \right] \quad .$$

Here denote

$$\bar{a}^4 = \frac{3}{4} (\varkappa_o \varkappa^{-1})^2 , \quad b = \frac{3}{2} \varkappa^{-2} , \quad \bar{d} = \frac{3}{2} \varkappa_o \varkappa^{-2} \quad . \tag{35}$$

Further , we take into account that, according to the assumed boundaries, the longitudinal displacement satisfies the condition,

$$\int_0^1 e_{o1} dx = \int_0^1 Du dx = u(1) - u(0) = 0 \quad , \tag{36}$$

where $e_{o1} = e_1 \big|_{z=o}$ is longitudinal membrane strain , and ,

$$Du = L\left[\frac{1}{3}(2\sigma_{o1} - \sigma_{o2})\right] + \alpha T \tag{37}$$

σ_{o1} , σ_{o2} denoting main membrane stresses .

On the other hand , circumferential membrane strain is ,

$$e_{o2} = e_2 \big|_{z=o} = \varkappa_o w = L\left[\frac{1}{3}(2\sigma_{o2} - \sigma_{o1})\right] + \alpha T \quad . \tag{38}$$

By solving the set of eqs.(36) and (38) with respect to membrane stresses, we can express them through deflection w . Doing so for expanded set, we obtain , successively ,

$$\sigma_{o1}^{o} = L_o^* (\omega_o - 3\alpha T_o) \ , \quad \sigma_{o1}^{1} = L_o^* (\omega_1 - 3\alpha T_1) + L_1^* (\omega_o - 3\alpha T_o) \ , \tag{39}$$

$$\sigma_{o2}^{o} = L_o^* \left[\frac{1}{2}(\varkappa_o \cdot 3w_o + \varkappa_o \cdot \omega_o) - 3\alpha T_o\right] \quad ,$$

$$\sigma_{o2}^{1} = L_o^* \left[\frac{1}{2}(\varkappa_o \cdot 3w_1 + \varkappa_o \cdot \omega_1) - 3\alpha T_1\right] + L_1^* \left[\frac{1}{2}(\varkappa_o \cdot 3w_o + \varkappa_o \cdot \omega_o) - 3\alpha T_o\right] \ , \tag{40}$$

where

$$\omega_o = \int_0^1 w_o dx \quad , \quad \omega_1 = \int_0^1 w_1 dx \quad . \tag{41}$$

Taking into account the equality,

$$\sigma_{o2} = L^* (Du + 2\varkappa_o w - 3\alpha T) \ , \tag{42}$$

and eq.(40), we reduce the set of eqs.(33),(34) to the form ,

$$L_o^* D^4 w_o + 4a^4 L_o^* w_o = bp_o - dL_o^* (\omega_o - kT_o) = \bar{A}_o \quad , \tag{43}$$

$$L_o^* D^4 w_1 + 4a^4 L_o^* w_1 = bp_1 - d\left[L_o^* (\omega_1 - kT_1) + L_1^* (\omega_o - kT_o)\right] -$$

$$- L_1^* L_o \left[bp_o - dL_o^* (\omega_o - kT_o)\right] = \bar{A}_1 \quad , \tag{44}$$

where we put,

$$a^4 = \left(\frac{3}{2}\bar{\varkappa}_1\right)^2 \quad , \quad d = 3\bar{\varkappa}_1^2 \quad , \quad k = 6\alpha\varkappa_o^{-1} \quad , \bar{\varkappa}_1 = \varkappa_1^{-1} = \frac{c^2}{Rh} \quad . \qquad (45)$$

The general solution of the system of eqs.(43) , (44), can be written as follows,

$$L_o^* w_o = A_o + f_o^s(ax) + f_o^a(ax) \quad , \quad L_o^* w_1 = A_1 + f_1^s(ax) + f_1^a(ax) \quad , \qquad (46)$$

if by f_i^s , f_i^a , $(i = o,1)$, we denote symmetric and antisymmetric parts of the solutions , respectively ,

$$f_i^s(ax) = C_1^i ch(ax)\cos(ax) + C_2^i sh(ax)\sin(ax) \; , \qquad (47)$$

$$(i = o,1) \; ,$$

$$f^a(ax) = C_3^i ch(ax)\sin(ax) + C_4^i sh(ax)\cos(ax) \; , \qquad (48)$$

and

$$A_o = \lambda\bar{A}_o \quad , \quad A_1 = \lambda\bar{A}_1 \quad , \quad \lambda = \left(4a^4\right)^{-1} \quad . \qquad (49)$$

The integration constants C^i, in eqs.(47) and (48), are found from the boundary conditions which we assume in two alternative forms . The first is that of simply supported edges . Here , we have,

$$L_o^* w_o\Big|_{\substack{x=1\\x=-1}} = L_o^* w_1\Big|_{\substack{x=1\\x=-1}} = 0 \quad , \qquad (5o)$$

$$D^2 L_o^* w_o\Big|_{\substack{x=1\\x=-1}} = D^2 L_o^* w_1\Big|_{\substack{x=1\\x=-1}} = 0 \quad . \qquad (51)$$

On the other hand , for the second case of clamped edges, we state the occurence of eq.(5o) and , additionaly ,

$$DL_o^* w_o\Big|_{\substack{x=1\\x=-1}} = DL_o^* w_1\Big|_{\substack{x=1\\x=-1}} = 0 \quad . \qquad (52)$$

Furthermore , since we assume the coordinate system, as shown in Fig.1 , there is an additional condition related to symmetry of the problem considered, which implies that,

$$DL_o^* w_o\Big|_{x=o} = DL_o^* w_1\Big|_{x=o} = 0 \quad . \qquad (53)$$

The latter reduces the number of independent boundary conditions, and also the number of integration constants. Therefore , the effective solutions are simplified considerably .

Thus , in considering the first set of boundary conditions and eq.(53), we find the solution for a shell with simply supported edges to be,

$$L_o^* w_o = A_o + f_o^s(ax) \ , \quad L_o^* w_1 = A_1 + f_1^s(ax) \ , \tag{54}$$

where the integration constants are : for simply supported edges,

$$C_1^o = -\gamma A_o \ , \ C_2^o = -\gamma A_o \mathrm{th}(a)\mathrm{tg}(a), \gamma = \{[1+\mathrm{th}^2(a)\mathrm{tg}^2(a)]\ \mathrm{ch}(a)\cos(a)\}^{-1}, \tag{55}$$
$$C_1^1 = -\gamma A_1 \ , \ C_2^1 = -\gamma A_1 \mathrm{th}(a)\mathrm{tg}(a) \ ,$$

and for clamped edges,

$$\overline{C}_1^o = -\overline{C}_2^o = -\delta A_o \quad , \ \delta = [\mathrm{ch}(a)\cos(a) - \mathrm{sh}(a)\sin(a)]^{-1}, \tag{56}$$
$$\overline{C}_1^1 = -\overline{C}_2^1 = -\delta A_1 \ .$$

Finally , on the basis of eqs.(54) and (55) we obtain non-dimensional deflection for simply supported shell in the form,

$$w = w_o + \beta w_1 = f(ax) L_o(A_o + \beta A_1) \ , \tag{57}$$

where we put,

$$f(ax) = 1 - \gamma[\mathrm{ch}(ax)\cos(ax) + \mathrm{th}(a)\mathrm{tg}(a)\mathrm{sh}(ax)\sin(ax)] \ . \tag{58}$$

The analogous solution for a shell with clamped edges,on the basis of eqs. (54) and (56), becomes

$$w^* = w_o^* + \beta w_1^* = g(ax) L_o(A_o + \beta A_1) \ , \tag{59}$$

if we put,

$$g(ax) = 1 - \delta[\mathrm{ch}(ax)\cos(ax) - \mathrm{sh}(ax)\sin(ax)] \ . \tag{60}$$

It is seen from the solutions obtained above that,for a shell material, characterized by small nonlinearity of creep properties,the successive approximations can be obtained in an exact manner . Furthermore , in accordance with the linearization of the problem performed,they are expressed in the form analogous to linear elastic solution for an incompressible material . In fact , if we write initial conditions for eqs.(57) and (59), then we find, respectively ,

$$w|_{t=t_o} = (2G)^{-1} A_o f(ax) \ , \tag{61}$$

$$w^*|_{t=t_o} = (2G)^{-1} A_o g(ax) \ , \tag{62}$$

which represent the solutions for an elastic shell .

It should be noticed that , in principle , there is a set of eqs.(54) and (39), (40) to be solved simultanously . Thus , by introducing eqs.(54) into eqs.(39) and (40),we find the normal stresses created by temperature field.

In order to complete the given solutions, it is necessary to give the forms of inverted operators appearing in our calculations . However , their particular forms can be found only then , when we dispose of concrete assumed forms of creep functions C . As regards the operator L_o^* , its shape may be easily written for an arbitrary C,

$$L_o^* \ldots = 2G \left[1+2G \int_{t_o}^{t} \partial_t C d\tau\right]^{-1} \ldots \quad , \qquad (54)$$

if the operation is performed on a constant or , in general ,

$$L_o^* \ldots = 2G \ldots + \int_{t_o}^{t} \ldots \partial_t R^* d\tau \quad , \qquad (55)$$

where R^* denotes relaxation function of the material .

5. CREEPING SHELL WITH HIGH NONLINEARITY

Now, we shall consider our problem by using the alternative form of constitutive equation (16) . The main strain rates in this case are,

$$\dot{e}_1 = D\dot{u} - z\,\varkappa D^2\dot{w} \quad , \quad \dot{e}_2 = \varkappa_o \dot{w} \quad , \qquad (56)$$

and main stresses calculated on the basis of inverted form of eq. (17) become

$$\sigma_1 = \frac{2}{3} \bar{B} \dot{e}^{-(1-\bar{n})} (2\dot{e}_1+\dot{e}_2-3\alpha\dot{T}) \; , \qquad (57)$$

$$\sigma_2 = \frac{2}{3} \bar{B} \dot{e}^{-(1-\bar{n})} (2\dot{e}_2+\dot{e}_1-3\alpha\dot{T}) \; , \qquad (58)$$

where effective strain rate is given by eq. (18).

At first we find the value of normal force N_1 on the basis of eq. (3) to be

$$N_1 = \frac{2}{3} h^{-1} c^2 \varkappa \bar{B} \int_{-1}^{1} \dot{e}^{-(1-\bar{n})} \left[2(D\dot{u}-z\varkappa D^2\dot{w}) + \varkappa_o\dot{w} - 3\alpha\dot{T}\right] dz = \sigma_{o1}^* h \; . \qquad (59)$$

By applying the theorem of mean value to eq. (59) we obtain (membrane stress) mean stress

$$\sigma_{o1}^* = \frac{4}{3} c^2 \varkappa \bar{B} \; \dot{e}^{-(1-\bar{n})}\Big|_{z=\xi} \cdot \left[2\left(D\dot{u} - \varkappa D^2\dot{w}\cdot\xi\right) + \varkappa_o\dot{w} - 3\alpha\dot{T}\right] \quad , \qquad (60)$$

where in our further investigations ξ is considered as a small quantity .

The value of σ_{o1} (which is constant with respect to x) can be calculated from a system of equations based on conditions analogous to those given by eqs. (36) and (38). Here , this related to boundaries is,

$$\frac{1}{2} B \int_{0}^{1} s_o^{n-1} (2\sigma_{o1}-\sigma_{o2}) dx + \alpha\dot{T} = 0 \quad , \qquad (61)$$

and that relating physical and geometrical strain rate becomes,

$$s_o^{n-1}(2\sigma_{o2}-\sigma_{o1}) = 2B^*(\varkappa_o\dot{w}-\alpha\dot{T}) \quad , \quad B^* = B^{-1} \ . \tag{62}$$

By solving the system of eqs.(61) and (62) with respect to membrane stresses σ_{o1} and σ_{o2} , we find

$$\sigma_{o1} = \frac{2}{3}\dot{\Omega}^{-1}B(\varkappa_o\dot{\omega} - 3\alpha\dot{T}) \quad , \tag{63}$$

$$\sigma_{o2} = \frac{1}{3}\dot{\Omega}^{-1}B(\varkappa_o\dot{\omega} - 3\alpha\dot{T}) + \bar{B}\dot{e}_o^{-(1-\bar{n})}(\varkappa_o\dot{w}-\alpha\dot{T}) \quad , \tag{64}$$

where

$$\dot{e}_o = Bs_o^n \quad , \quad \dot{\Omega} = \int_0^1 s_o^{n-1}dx = B^{\bar{n}-1}\int_0^1 \dot{e}_o^{1-\bar{n}}dx \quad , \quad \dot{\omega} = \int_0^1 \dot{w}\,dx \ . \tag{65}$$

On the other hand , we calculate the value of normal force in circumferential direction N_2, by applying the same procedure as in eq.(60). Doing so, we get on the basis of eq.(3),

$$\sigma_{o2}^* = \frac{4}{3}c^2\varkappa\bar{B}\,\dot{e}^{-(1-\bar{n})}\Big|_{z=\zeta}(D\dot{u}-\varkappa D^2\dot{w}.\zeta + 2\varkappa_o\dot{w} - 3\alpha\dot{T}) \ , \tag{66}$$

where , as before , ζ is assumed as small quantity .

Approximate values of membrane stresses, we obtain, if we put,

$$\sigma_{o1} \cong \sigma_{o1}^*\Big|_{\xi=0} = \frac{4}{3}c^2\varkappa\bar{B}\,\dot{e}_o^{-(1-\bar{n})}(2D\dot{u} + \varkappa_o\dot{w} - 3\alpha\dot{T}) \ , \tag{67}$$

$$\sigma_{o2} \cong \sigma_{o2}^*\Big|_{\zeta=0} = \frac{4}{3}c^2\varkappa\bar{B}\,\dot{e}^{-(1-\bar{n})}(D\dot{u} + 2\varkappa_o\dot{w} - 3\alpha\dot{T}) \ , \tag{68}$$

where

$$\dot{e}_o = \dot{e}\Big|_{\xi=0} = \dot{e}\Big|_{\zeta=0} = \left\{\frac{4}{3}\left[(D\dot{u})^2 + \varkappa_o\dot{w}(D\dot{u} + \varkappa_o\dot{w})\right]\right\}^{\frac{1}{2}} \ . \tag{69}$$

Let us assume now a particular case of our problem in which we neglect the influence of temperature field, and assume in eq.(2), $N_1 = 0$. Then on the basis of eq.(67) we obtain,

$$D\dot{u} = -\frac{1}{2}\varkappa_o\dot{w} \quad , \tag{70}$$

and , thus , the first of eqs.(56) becomes,

$$\dot{e}_1 = -\frac{1}{2}\varkappa_o(\dot{w} + \varkappa_1 z\, D^2\dot{w}) \ . \tag{71}$$

On the other hand , from eq.(16), we find effective strain rate

$$\dot{e} = \varkappa_o\dot{w}\left[1 + \frac{1}{3}\varkappa_1^2 z^2\dot{w}^{-2}(D^2\dot{w})^2\right]^{\frac{1}{2}} \ . \tag{72}$$

It should be taken into consideration that $\dot{e}$, as a physical quantity, cannot assume singular values . Therefore , it must be either constant or zero for x tending to zero or to ± 1 , the result depending on the assumed boundaries.

Further , by raising eq.(72) to the power $(\bar{n}-1)$, we notice that the second term in the brackets may be assumed as a small quantity with respect to unity and , even , may be arbitrarily small for a long shell , i.e. , for $\varkappa_1$ tending to zero (see eq.(45)). Thus , the approximate value of eq.(72) gives

$$\dot{e}^{\,\bar{n}-1} = \left(\varkappa_0 \dot{w}\right)^{\bar{n}-1}\left[1 - \bar{\beta}\varkappa_1^2\, \dot{w}^{-2} \cdot \left(D^2\dot{w}\right)^2\right], \qquad \bar{\beta} = \tfrac{1}{3}(1 - \bar{n}) \quad . \tag{73}$$

Having at disposal eq.(73), we easily find stresses on the basis of eqs. (57) and (58) (for $\dot{T} = 0$), and then we calculate internal forces from eq.(3). Thus , we get

$$N_2 = h\bar{B}\,\varkappa_0^{\bar{n}}\dot{w}^{\bar{n}}\left[1 - \tfrac{1}{3}\bar{\beta}\varkappa_1^2\, \dot{w}^{-2}\left(D^2\dot{w}\right)^2\right] , \tag{74}$$

$$M = -\tfrac{1}{9}\, h^2\bar{B}\varkappa_0^{\bar{n}}\varkappa_1 \dot{w}^{\bar{n}-1} \cdot D^2\dot{w}\left[1 - \tfrac{1}{5}\bar{\beta}\varkappa_1^2\, \dot{w}^{-2}\left(D^2\dot{w}\right)^2\right] . \tag{75}$$

By introducing eq.(74) and eq.(75) into the condition of equilibrium of eq.(2), we obtain a differential equation of the problem valid for a sufficiently long shell , i.e. , for a sufficiently small $\varkappa_1$. Such an equation has been recently considered by Bychawski [11] in approximate formulation of the creep problem for a short shell .

If we further simplify eqs.(73),(74) and (75), by neglecting terms containing $\varkappa_1^2$, what can be done in the case of a long shell , we arrive at the following differential equation of the problem which results from eq.(2),

$$D^2\, \dot{w}^{\bar{n}-1} D^2\dot{w} \;+ b\dot{w}^{\bar{n}} = a \quad . \tag{76}$$

Here , we put

$$a = 9\bar{B}^{\bar{n}} p\, \varkappa_0^{-(1+\bar{n})} \cdot \varkappa_1^2 \quad , \quad b = 9\,\varkappa_1^2 \quad . \tag{77}$$

The equation (76) simplifies to a linear differential equation for elastic shell by assuming $\bar{n} = 1$, and identifying B with elastic constant 2G , and $\dot{w}$ with deflection w . Thus , we have

$$D^4 w + bw = 6(2G)p\left(\varkappa_0 \varkappa_1\right)^{-2} . \tag{78}$$

It is curious to note that the equation of the problem obtained in [2] and explicitly given by Odqvist [6] has qualitatively a completly different form

$$D^2\left[\left(D^2\dot{w}\right)^{\bar{n}}\right] + \bar{b}\dot{w}^{\,\bar{n}} = \bar{a} \quad , \tag{79}$$

where $\bar{a}$ and $\bar{b}$ are certain constants . Although , it is possible to obtain on its basis eq.(78) (see Odqvist [6]), it does not give an answer in the case of n tending to infinity , i.e. , for a rigid plastic shell . This problem has been discussed by Bychawski [11] . The analogous, to eq.(79), result is obtained by Gemma [4] .

In order to solve eq.(76) we make a substitution of the form

$$\dot{w}^{\overline{n}} = F + q = F^{*} \quad , \tag{80}$$

so that

$$\dot{w} = (F + q)^{n} = F^{*\,n} \quad , \tag{81}$$

where q is a constant to be determined and n, here and further, is uneven natural number . Further , we assume

$$F = \sum_{i=o}^{\infty} F_i x^i \quad , \tag{82}$$

and , thus , on the basis of eq.(81) , we have,

$$F^{*\,n} = \sum_{i=o}^{\infty} R_i x^i \quad , \quad R_i = \sum_{j=o}^{n} \binom{n}{j} q^{n-j} F_i^{(j)} \quad , \quad R_o = (F_o + q)^n \quad , \tag{83}$$

$$F_o^{(j)} = F_o^{\,j} \quad , \quad F_i^{(j)} = (iF_o)^{-1} \sum_{k=1}^{i} (kn-i+k) \; F_k F_{i-k}^{(j)} \quad . \tag{84}$$

The equation (76), by applying the substitution of eq.(80), transforms into ,

$$D^2\left[F^{*\,(1-n)} D^2 F^{*\,n}\right] + bF^{*} = a \quad , \tag{85}$$

or ,

$$D^2\left[(F+q)^{(1-n)} D^2 (F+q)^n\right] + bF = 0 \quad , \quad q = ab^{-1} \quad , \tag{86}$$

the latter being homogeneous .

In the equation (85) we have

$$M^{*} = F^{*\,(1-n)} D^2 F^{*\,n} = R_o^{-1} \sum_{i=o}^{\infty} \omega_i x^i \; , \; \omega_i + R_o^{-1} \sum_{k=1}^{i} \omega_{i-k} R_k - \Omega_i = 0 \; , \; \omega_o = \Omega_o \; , \tag{87}$$

where ,

$$\Omega_i = qQ_i + P_i \quad , \quad Q_i = \prod_{m=1}^{2} (i+m) \sum_{j=o}^{n} \binom{n}{j} q^{n-j} F_{i+2}^{(j)} \quad , \; P_i = \sum_{k=o}^{i} F_k Q_{i-k} \quad , \tag{88}$$

and ,

$$D^2 M^{*} = R_o^{-1} \sum_{i=o}^{\infty} \prod_{m=1}^{2} (i+m) \, \omega_{i+2} x^i \; . \tag{89}$$

By introducing eqs. (80) , (82) and (89) into eq. (85), we obtain a condition which must be satisfied for every x taken from the interval considered $x \in [-1, 1]$,

$$\sum_{i=o}^{\infty} \left[\prod_{m=1}^{2} (i+m) \omega_{i+2} + R_o b \, F_i \right] \cdot x^i = 0 \quad . \tag{90}$$

On the basis of the above condition we are able to calculate the coefficients of the expansion assumed in eq.(82) which will be uniquely determined, if we additionally consider appropriate boundary conditions . These are, in the first case of simply supported edges,

$$F^{*(1-n)} D^2 F^{*\,n} \Big|_{\substack{x=1 \\ x=-1}} = 0 \quad , \qquad F^{*\,n} \Big|_{\substack{x=1 \\ x=-1}} = 0 \quad . \tag{91}$$

From the above conditions we find that,

$$q = -\sum_{2i}^{\infty} F_{2i} \quad , \quad \sum_{2i}^{\infty} \omega_{2i} = 0 \quad , (i = o,1,2,....) \; , \tag{92}$$

the results being related to the symmetry of the problem .

On the other hand , for the second case of clamped edges we have conditions,

$$DF^{*\,n} \Big|_{\substack{x=1 \\ x=-1}} = 0 \quad , \qquad F^{*\,n} \Big|_{\substack{x=1 \\ x=-1}} = 0 \quad , \tag{93}$$

from which results the first of eqs. (92), and also ,

$$\sum_{2i}^{\infty} i \cdot R_{2i} = 0 \quad , \quad (i = o,1,2,.....) \quad . \tag{94}$$

The proof of the convergence of the series assumed in eq. (82) can be founded on the Abel theorem of convergence . It is clearly seen from eq. (92) that, for x = 1 , the series of coefficients of eq. (82) is summed and , therefore , the series in the interval considered is convergent .

It is not possible to present in a short way the solution for the problem if temperature should be taken into account for highly nonlinear shell.. It is not even easy to write down an appropriate system of equations because of extreme difficulties in integrating the expressions of internal forces . However , it seems that the method indicated at the beginning of this paragraph, which consists in applying the theorem of mean value , can be, in a way, useful as , for example , for perturbation purposes .

6. CONCLUSIONS

It was the aim of this paper to present solutions in the extreme cases of creeping cylidrical shells under internal pressure . In the first case of small creep nonlinearity it was possible to linearize the problem and , therefore , to obtain exact solutions of successive approximations which , according to our opinion , are sufficient to describe the first stage of the creep process . They also indicate the influence of a temperature field considered . In the second case of high creep nonlinearity a method of obtaining exact solution of the resulting nonlinear differential equation was given . It was also indicated the difference in qualitative form of equation derived with respect to those treated by other authors . Although , temperature was neglected , the way of its accounting was pointed out .

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