ON SHAKEDOWN DESIGN OF STRUCTURES WITH TEMPERATURE DEPENDING ELASTIC MODULI

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ABSTRACT

An extension of the Melan shakedown theorem for the case of temperature dependent elastic moduli is presented in the paper. A new formulation of the appropriate shakedown theorem requires modified methods of solving problems. An explicit relation between stresses and plastic deformation is used in order to determine possible ranges of elastic response of structures.

Examples of calculations of clamped bars, plates, and thick-walled shells show that the influence of the Young modulus variation usually does net exceed a few per-cent.

List of symbols

x - material point

t - time

θ - temperature

$\underset{\sim}{\sigma}$ - stress tensor

σ_{ij} - stress tensor conponents

$\underset{\sim}{\varepsilon}$ - strain tensor

ε_{ij} - strain tensor components

$\underset{\sim}{\sigma}^E$ - elastic stress tensor

$\underset{\sim}{\rho}$ - actual residual stress tensor

$\underset{\sim}{\bar{\rho}}$ - fictive residual stress tensor

$\varphi(\underset{\sim}{\sigma})$ - yield function

σ - axial stress

ε - axial strain

A_{ijkl} - elastic moduli

E - Young's modulus

ν - Poisson ratio

σ_0 - yield stress

α - coefficient of thermal linear expansion

e - material constant

^oC - centigrade

ϵ - small parameter

W_p - integral plastic work

m - safety factor

1. Introduction

Whenever loads or other factors causing stresses in an elastic-plastic structure vary arbitrarily within sufficiently broad limits,the structure may collapse by "inadaptation". New plastic deformations appear then on every subsequent cycle of loading even if the yield point load of the structure has not yet been reached for any mono-parametric loading program. After a sufficient number of loading cycles inadaptation results either in a low-cycle fatigue or in an incremental collapse. The first type of failure is associated with "alternating plasticity",and occurs when the subsequent increments of plastic deformations are of opposite sign. In an incremental collapse plastic strains grow up on every cycle of loading to an unbounded increase of displacements.

A study of such problems - the shakedown theory - constitutes a natural complement to the theory of limit analysis for the case of variable loads. This is the reason why the shakedown analysis has raised more interest in the last few years though the theory had been initiated in the early thirties by BLEICH [1] and MELAN [2].

Two fundamental theorems regarding the shakedown are due to MELAN [2] and KOITER [3] ; the theorems are essentially appropriate generalizations of the well known theorems of limit analysis [4].

An evaluation of the shakedown capacity is becoming a routine procedure for various structures.

In many cases of practical importance a structure is acted upon not only by variable mechanical actions but also by a temperature field.

For an elastic-plastic structure the following situations may occur

1^o thermal strains affect the stress field ;

2^o yield stress alters with temperature ;

3^o elastic moduli vary with temperature.

The first two effects have been already studied in the theory of limit analysis and in the shakedown analysis by PRAGER [5] as well as by other authors [6 - 12]. The present note concerns the third effect.

The fundamental theorems of the shakedown theory need appropriate modifications whenever any of the above mentioned effects is to be accounted for.

The presence of a thermal strain does not change the wording of Melan's theorem.It is to be noted, however, that the presence of such a strain is to be accounted for in terms describing "elastic stress". In Koiter's theorem an additional term appears in the expression for the internal work [7], [8].

The fact that the yield stress alters with temperature does not affect the original formulation of the theorems yet all relations appearing in them must hold for any temperature within the temperature variation range admitted by the loading program.

As far as the variation of elastic moduli is concerned, the author is unaware of any study regarding Koiter's theorem. The theorem of Melan must be restated and an appropriate formulation was presented in [13].

Under specific conditions each of the three phenomena listed above may become the most important. In the case of mild steel the yield stress decreases significantly within the temperature range of ~~from 300 to 600~~ centigrades but remains constant within range 0 - 200 centigrades. For a very wide temperature range the Young modulus E changes by 5-10 per cent per every 100 centigrades according to the following linear relation

$$E(\theta) = E(0)\left[1 - e\theta\right] \qquad /1/$$

where θ stands for the temperature, e is a material constant, Fig 1.

The aim of this paper is to show that in shakedown analysis of elastic plastic structures the dependence of elastic moduli on temperature may play also a nonnegligible rôle.

In the present study we shall neglect all visco-elastic effects restricting the analysis to an ideally elastic-plastic behaviour.

2. Assumptions

The strain tensor in an elastic-plastic medium subjected to thermal actions may be presented in the form

$$\varepsilon_{ij} = \varepsilon^{E}_{ij} + \varepsilon^{P}_{ij} + \varepsilon^{T}_{ij} \qquad /2/$$

where $\varepsilon^{E}_{ij} = A_{ijkl}\,\sigma_{kl}$ and A_{ijkl} is the elastic moduli tensor, ε^{T}_{ij} stands for the thermal strain and ε^{P}_{ij} denotes the plastic strain.

By the elastic stress σ^{E}_{ij} we shall understand the solution of the actual boundary value problem obtained under the condition $\varepsilon^{P}_{ij} = 0$. The tensor σ^{*}_{ij} - which, in general, differs from the tensor σ^{E}_{ij} concerns the solution for an elastic perfectly plastic solid, where the constitutive relations are given by a yield condition and by a flow rule.

The difference

$$\rho_{ij} = \sigma_{ij} - \sigma^{E}_{ij} \qquad /3/$$

is a residual stress state satisfying the zero stress boundary conditions.

It is important to observe that whenever the tensor of elastic moduli varies with temperature, the plastic deformation field ε^{P}_{ij} does not specify uniquely the residual stress field ρ_{ij}. The field ρ_{ij} depends now also on the actual temperature and is susceptible to vary with temperature even for an invariable field ε^{P}_{ij}.

Let us assume now that the mechanical loading and temperature vary arbitrarily within the prescribed limits. We neglect the inertia forces and any thermoelastic an thermoplastic coupling and assume that the displacements remain sufficiently small so that the principle of virtual works holds true. Then the following theorem may be formulated.

3. Theorem

If there exists a real number $m > 1$ and a time-independent plastic strain field $\bar{\varepsilon}^P_{ij}$ such that the associated residual stress $\bar{\varrho}_{ij}$ satisfy the inequality

$$\varphi \left\{ m \left[\sigma^E_{ij}(x,t,\theta) + \bar{\varrho}_{ij}(x,\theta) \right] \right\} \leqslant k(x,\theta) \qquad /4/$$

for all loads and temperatures admitted by the loading program, then the structure will shake down to the program. The relation $\varphi[\sigma_{ij}] = k(x,\theta)$ is yield condition, m plays the role of a safety factor. Certain quantities appearing in /4/ are visualized in Fig 2 .

For a case $\partial A_{ijkl}/\partial\theta = 0$ the theorem reduces to the theorem of Melan and may be formulated solely in terms of elastic and residual stresses as a steady plastic deformation is equivalent to a constant residual stress in this case.

Proof. In order to prove the theorem we generalize the proof of Melan's theorem as given by Koiter [14]. An asymptotic stability of plastic deformations will be demonstrated by using the appropriately generalized positive functional of residual stress and temperature :

$$Y = \frac{1}{2}\int_V A_{ijkl}(\varrho_{ij} - \bar{\varrho}_{ij})(\varrho_{kl} - \bar{\varrho}_{kl})\, dV + \frac{1}{2}\int_0^t\int_V \dot{A}_{ijkl}(\varrho_{ij} - \bar{\varrho}_{ij}) \cdot (\varrho_{kl} - \bar{\varrho}_{kl})\, dV\, dt + \frac{1}{2}\int_V \bar{A}_\theta \cdot R^2 \cdot \bar{\theta}\, dV \qquad /5/$$

The first integral coincides with the original functional as introduced by Koiter [14]. The second one is introduced because of elastic moduli variation. The third is to keep the whole sum positive.

Here dot denotes differentiation with respect to time, comma - with respect to the preceding variable, ϱ_{ij} stands for the actual residual stress. Moreover the following denotations are introduced

$$|\underset{\sim}{\alpha}| = \sqrt{\alpha_{ij}\,\alpha_{ij}}\ ; \quad R = \sup_{\{\theta\}}\sup_{\{\varphi(\underset{\sim}{\sigma})=k\}}|\underset{\sim}{\sigma}| + \sup_{\{\theta\}}|\underset{\sim}{\bar{\varrho}}| + \sup_{\{t\}}|\underset{\sim}{\sigma}^E|$$

$$\bar{A}_\theta = \sup_{\{\theta\}}\sup_{\{|\underset{\sim}{\sigma}|=1\}}|A_{ijkl,\theta}\,\sigma_{ij}\,\sigma_{kl}|\ ; \quad \bar{\theta} = \sup_{\{t\}}|\theta|$$

$$\bar{A} = \sup_{\{\theta\}}\sup_{\{|\underset{\sim}{\sigma}|=1\}} A_{ijkl}\,\sigma_{ij}\,\sigma_{kl}\ ; \quad r = \sup_{\{\theta\}}|\underset{\sim}{\bar{\varrho}}| \qquad /6/$$

The symbol $\sup_{\{\theta\}}[\ldots]$ denotes the upper bound of the quantity in brackets when θ varies arbitrarily within the limits prescribed by the loading program. Appropriately the symbols $\sup_{\{\varphi(\underset{\sim}{\sigma})=k\}}[\ldots]$ and $\sup_{\{|\underset{\sim}{\sigma}|=1\}}[\ldots]$ denote the respective upper bounds for cases when the stress tensor may become any tensor satisfying the condition $\varphi[\underset{\sim}{\sigma}] = k$ or $|\underset{\sim}{\sigma}| = 1$, respectively. The symbol $\sup_{\{t\}}[\ldots]$ denotes the upper bound of the expression in brackets over all positive times.

It may be shown that the third integral in /5/ is always greater than the second one. We have namely

$$\int_0^t\int_V A_{ijkl}(\varrho_{ij}-\bar{\varrho}_{ij})(\varrho_{kl}-\bar{\varrho}_{kl})\,dV\,dt = \int_0^t\int_V A_{ijkl,\theta}(\varrho_{ij}-\bar{\varrho}_{ij})\cdot$$
$$\cdot(\varrho_{kl}-\varrho_{kl})\,\dot{\theta}\,dV\,dt \leqslant \int_0^t\int_V \bar{A}_\theta\,|\varrho-\bar{\varrho}|^2\,\dot{\theta}\,dV\,dt \leqslant \int_0^t\int_V A_\theta\,R^2\,\dot{\theta}\,dV\,dt \leqslant$$
$$\leqslant \int_V A_\theta\,R^2\,\bar{\theta}\,dV$$

Hence

$$Y \geqslant 0 \qquad /7/$$

By differentiation with respect to time we obtain

$$\dot{Y} = \int_V A_{ijkl}(\varrho_{ij}-\bar{\varrho}_{ij})(\dot{\varrho}_{kl}-\dot{\bar{\varrho}}_{kl})\,dV + $$
$$+\int_V \dot{A}_{ijkl}(\varrho_{ij}-\bar{\varrho}_{ij})(\varrho_{kl}-\bar{\varrho}_{kl})\,dV \qquad /8/$$

From the principle of virtual work it follows that

$$\int_V(\varrho_{ij}-\bar{\varrho}_{ij})\Big[\dot{A}_{ijkl}(\varrho_{kl}-\bar{\varrho}_{kl}) + A_{ijkl}(\dot{\varrho}_{kl}-\dot{\bar{\varrho}}_{kl}) +$$
$$+\dot{\varepsilon}^P_{ij} - \dot{\bar{\varepsilon}}^P_{ij}\Big]\,dV = 0 \qquad /9/$$

Hence, from /8/ and since, moreover, $\dot{\bar{\varepsilon}}^P_{ij} = 0$ we have

$$\dot{Y} = -\int_V(\varrho_{ij}-\bar{\varrho}_{ij})\,\dot{\varepsilon}^P_{ij}\,dV = -\int_V\left[(\sigma^E_{ij}+\varrho_{ij}) - (\sigma^E_{ij}+\varrho_{ij})\right]\dot{\varepsilon}^P_{ij}\,dV \leq$$
$$\leq 0 \qquad /10/$$

The last inequality is derived from the Drucker postulate and from the assumptions of the theorem considered. Hence, for $t > 0$

$$Y(t) \leq Y(0) \quad \text{and} \quad \dot{Y} \to 0,\ Y \to \text{const} \quad \text{for} \quad t \to \infty \qquad /11/$$

Repeating the procedure adopted by KOITER [14] it one can conclude that if $\varphi\left[m(\sigma^E_{ij}+\bar{\varrho}_{ij})\right] \leq k$ at each point of the structure then the total rate of plastic work dissipated during an arbitrary process within the given loading program may be estimated as follows

$$\dot{W}_p = \int_V \sigma_{ij}\,\dot{\varepsilon}^P_{ij}\,dV \leq \frac{m}{m-1}\int_V(\varrho_{ij}-\bar{\varrho}_{ij})\,\dot{\varepsilon}^P_{ij}\,dV \leq \frac{m}{m-1}(-\dot{Y}) \qquad /12/$$

By integrating with respect to time we obtain finally an estimation of the total plastic work dissipated during an arbitrary process

$$W_p = \int_0^\infty \int_V \sigma_{ij} \, \dot{\varepsilon}^P_{ij} \, dV \, dt = \frac{m}{m-1} Y(0) = \frac{m}{2(m-1)} \int_V \left[\bar{A} \, r^2 + A_\theta R^2 \bar{\theta} \right] dV \qquad /13/$$

As the plastic work is bounded, the theorem is proved and therefore the adaptation is assured.

4. Examples

Example 1

Let us consider a bar clamped on both its ends, heated /cooled/ within the limits $0 < \theta < \theta_0$. For a sufficiently high value θ_0 the structure will collapse after a sufficient number of cycles owing to the low-cycle fatigue.

The highest value of θ_0 allowing for perfectly elastic response of the bar, after some initial plastic yielding for the steady Young's modulus is

$$\max \theta_0 = 2 \, \sigma_0 / E \alpha \qquad /14/$$

where σ_0 - yield stress, E - Young modulus, α - coefficient of thermal linear expansion.

In our case the relation /2/ may be rewritten as

$$\varepsilon = \varepsilon^E + \varepsilon^P + \varepsilon^T = \frac{\sigma}{E} + \alpha\theta + \varepsilon^P$$

ε and σ denoting the strain and stress component along the axis of the bar. From the boundary conditions on the bar ends $\varepsilon = 0$. Thus

$$\sigma = -E\left[\alpha\theta + \varepsilon^P\right]$$

Yield condition for uniaxial stress state is

$$-\sigma_0 \le \sigma \le \sigma_0$$

By substituting we obtain

$$-\sigma_0/E \le \alpha\theta + \varepsilon^P \le \sigma_0/E \quad \text{or} \quad -\sigma_0/E - \alpha\theta \le \varepsilon^P \le \sigma_0/E - \alpha\theta$$

According to the theorem the existence of a time-independent plastic strain field $\varepsilon^P = \bar{\varepsilon}$ such that the above inequality is satisfied for all temperature changes prescribed by the load program is a necessary condition of adaptation for safety factor $m = 1$. It will be so if the lower bound of the right-hand side expression in the above inequality is not less than the upper bound of the left-hand side expression. Thus, if Young's modulus is a monotonic function of temperature for $0 < \theta < \theta_0$, we obtain

$$-\frac{\sigma_0}{E(0)} \le \frac{\sigma_0}{E(\theta_0)} - \alpha\theta_0$$

and the condition of adaptation will be now as follows

$$\max \theta_0 = \frac{\sigma_0}{\alpha}\left[\frac{1}{E(0)} + \frac{1}{E(\theta_0)}\right] \qquad /15/$$

and the constant plastic strain is $\bar{\varepsilon} = - \sigma_0/E(0)$.

Assuming $E(\theta_0) = E(0) = E = \text{const}$ we obtain the result /14/ which may be obtained from the original Melan theorem as generalized by PRAGER [5] and ROZENBLUM [6] by putting $\bar{\rho} = \sigma_0$, where $\bar{\rho}$ is the constant residual stress.

We see that if $\partial E/\partial \theta \leq 0$ the formula /15/ gives a higher value of θ_0 than that given by /14/ .

In the case of mild steel, Fig 1, both formulas give $\theta_0 \sim 200\ ^\circ C$. In the range $0 - 200\ ^\circ C$ the yield stress remains constant yet Young's modulus drops down by about 15 per-cent. In such a case the formula /15/ will give a result by about 10 per-cent higher showing that the effect considered broadens the range of temperature variation.

For other values of the material constant e , the comparison of the formulas /14/ and /15/ is given in Fig 3 .

Example 2

A clamped plate of arbitrary shape is subjected to the following temperature variation

$$\theta = \Phi(t) + \zeta \Psi(t) \qquad \zeta = z/H \qquad /16/$$

as considered by ROZENBLUM [7] , 2H - thickness of the plate, z - coordinate in the direction normal to the middle plane, Φ and Ψ being arbitrary functions of time prescribed within the limits

$$0 < \Phi(t) < \Phi_0 \quad ; \quad 0 < \Psi(t) < \Psi_0 \qquad /17/$$

For the Young modulus and the yield stress constant ROZENBLUM [7] obtained the following shakedown condition /dashed line in Fig 4/

$$\Phi_0 + \Psi_0 = 2(1 - \nu)\sigma_0/E\alpha \qquad /18/$$

ν being Poisson's ratio. The solution has been obtained under the assumption of plane stress and isotropy. All the stresses and strains depend solely on the thickness coordinate.

If the Young modulus follows the rule /1/ we obtain from /2/ the following expressions for principal strains

$$\varepsilon_1(z) = \varepsilon_2(z) = \varepsilon(z) = \frac{(1-\nu)\,\sigma(z)}{E_0[1 - e\theta]} + \alpha\theta + \varepsilon^P \qquad /19/$$

In view of /16/ and because the boundary conditions result in = 0 the stress is found to be

$$\sigma_1(z) = \sigma_2(z) = -\frac{E_0\alpha}{1-\nu}\Big[\varepsilon^P/\alpha + (\Phi + \zeta\Psi)(1 - e\,\varepsilon^P/\alpha) + - e(\Phi + \zeta\Psi)^2 \Big] \qquad /20/$$

A residual stress associated with any given, invariable plastic strain field $\bar{\varepsilon}(z)$ has the form

$$\varrho(z) = \sigma(z) - \sigma^E(z) = -\bar{\varepsilon}(z)\frac{E_0}{1-\nu}\left[1 - e(\Phi + \zeta\Psi)\right] \qquad /21/$$

because, by putting $\varepsilon^P = 0$ in /20/ we arrive at the following formula for the elastic stress $\sigma^E(z)$

$$\sigma^E(z) = -\frac{E_0}{1-\nu}(\Phi + \zeta\Psi)\left[1 - e(\Phi + \zeta\Psi)\right] \qquad /22/$$

From the shakedown condition /4/ one concludes the following :
for all the ζ such that $|\zeta| \le 1$ and for all the Φ and Ψ satisfying /8/ there must exist such a constant $\bar{\varepsilon}(z)$ that

$$-\sigma_0 \le \frac{E_0}{1-\nu}(\Phi + \zeta\Psi)\left[1 - e(\Phi + \zeta\Psi)\right] + \bar{\varepsilon}\frac{E_0}{1-\nu}\left[1 - e(\Phi + \zeta\Psi)\right] \le \sigma_0 \qquad /23/$$

The result will hold for any form of yield condition because in the case considered $\sigma_1 = \sigma_2 = \sigma$, $\sigma_3 = 0$.
Transforming the inequalities, we obtain

$$-\frac{\sigma_0(1-\nu)}{E_0\alpha\left[1 - e(\Phi + \zeta\Psi)\right]} - (\Phi + \zeta\Psi) \le \frac{\bar{\varepsilon}(z)}{-\alpha} \le \frac{\sigma_0(1-\nu)}{E_0\alpha\left[1 - e(\Phi + \zeta\Psi)\right]} + - (\Phi + \zeta\Psi) \qquad /24/$$

For the Φ and Ψ varying within the limits /17/ the inequalities /24/ will hold everywhere and an appropriate $\bar{\varepsilon}(z)$ will exist if for every apex of the rectangle /17/ in the (Φ, Ψ) plane the same $\bar{\varepsilon}(z)$ will fit. This condition leads to the four families of one-parametric simultaneous inequalities; ζ being the parameter

$$-\frac{\sigma_0(1-\nu)}{E_0\left[1 - e\Phi_0\right]} - \Phi_0 \le \frac{\bar{\varepsilon}(z)}{\alpha} \le \frac{\sigma_0(1-\nu)}{E_0\alpha\left[1 - e\Phi_0\right]} - \Phi_0$$

$$-\frac{\sigma_0(1-\nu)}{E_0\left[1 - e\zeta\Psi_0\right]} - \zeta\Psi_0 \le \frac{\bar{\varepsilon}(z)}{\alpha} \le \frac{\sigma_0\, 1 -}{E_0\alpha\left[1 - e\zeta\Psi_0\right]} - \zeta\Psi_0$$

etc for the remaining two apexes.

It is quite involving to establish which one of the above inequalities is the most stringent one for the given particular value of ζ and which one is the value of ζ rendering the final shakedown condition on Φ_0 and Ψ_0. But if we take the advantage from the fact that e is a small parameter then the analysis may be done exactly as only the cases $\zeta = -1$ and $\zeta = 1$ need then to be taken into account. We obtain at last

$$\Phi_0 + \Psi_0 \le \frac{\sigma_0(1-\nu}{E_0\alpha}\left[\frac{1}{1 + e\Psi_0} + \frac{1}{1 - e\Phi_0}\right] \qquad /25/$$

/solid line in Fig 4/. Putting $e = 0$ the result /18/ due to ROZENBLUM [6] follows.

From /25/ we conclude that :

1° if the effect of uniform heating, described by Φ_0 is stronger, then the admissible value of $\Phi_0 + \Psi_0$ is higher than the corresponding value obtained from the formula /18/ ;

2° if the "bending" heating prevails then /25/ is more stringent than /18/.

The results /18/ and /25/ for mild steel are presented in Fig 4 illustrating the above given conclusions.

Example 3

A thick-walled circular cylinder is subjected to cycles of internal pressure and temperature varying within prescribed limits. In the case of steady Young modulus the problem of adaptation has been solved approximately by HOCHFELD and YERMAKOV [9] also when accounting for changes of yield stress with temperature.

If the cycles of temperature are sufficiently slow, so that the stationary solution of the heat conduction equation can be assumed, then the temperature field becomes :

$$\theta(r) = \theta(a) + \left[\theta(b) - \theta(a)\right] \frac{\ln r/a}{\ln b/a} \qquad /26/$$

here a , b , r stand for the internal,external and current radius respectively.We shall consider a particular case when the pressure may be neglected.

The equilibrium equation

$$\frac{d\,\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0 \qquad /27/$$

the compatibility condition

$$\frac{d\,\varepsilon_\varphi}{dr} + \frac{\varepsilon_\varphi - \varepsilon_r}{r} = 0 \qquad /28/$$

and the constitutive law

$$\varepsilon_r = \frac{1}{E(\theta)}\left[\sigma_r - \nu\sigma_\varphi\right] + \alpha\,\theta + \varepsilon_r^P$$

$$\varepsilon_\varphi = \frac{1}{E(\theta)}\left[\sigma_\varphi - \nu\,\sigma_r\right] + \alpha\,\theta + \varepsilon_\varphi^P \qquad /29/$$

after rearrangements result in the following differential equation

$$\frac{r^2}{E}\frac{d^2\sigma_r}{dr^2} + \frac{r}{E}\frac{d\,\sigma_r}{dr}\left[3 - \frac{1}{E}\frac{dE}{d\theta}\frac{d\theta}{dr}\right] - \frac{1-\nu}{E^2}\frac{dE}{d\theta}\frac{d\theta}{dr}\sigma_r +$$
$$+ r\,\alpha\frac{d\theta}{dr} + r\frac{d\,\varepsilon_\varphi^P}{dr} + \varepsilon_\varphi^P - \varepsilon_r^P = 0 \qquad /30/$$

here σ_r, σ_φ denote stresses, ε_r, ε_φ are strains, a radial and a circumferential respectively ; θ is temperature, α - thermal expansion coefficient, the superscript "P" denotes the plastic part of the respective strain component.

If the expressions /1/ , /26/ are substituted into /30/ and $\theta(b) = 0$ then the following equation is arrived at

$$\frac{r^2 d^2\sigma_r}{dr^2} + r\frac{d\sigma_r}{dr}\left[3 - \frac{e\,\theta(a)}{r\ln(b/a)\left[1 - e\,\theta(a)\ln(b/r)/\ln(b/a)\right]} + \right.$$

$$- \frac{(1-\nu)\; e\,\theta(a)/\ln(b/r)\cdot\sigma_r/r}{E_o\left[1 - e\,\theta(a)\ln(b/r)/\ln(b/a)\right]} - E_o\left[1 - e\,\theta(a)\ln(b/r)/\ln(b/a)\right.$$

$$\left.\alpha\,\theta(a)\right]/\ln(b/a) + E_o\left[1 - e\,\theta(a)\ln(b/r)/\ln(b/a)\right]\left[r\frac{d\,\varepsilon_\varphi^P}{dr} + \varepsilon_\varphi^P + \right.$$

$$\left. - \varepsilon_r^P\right] = 0 \qquad /31/$$

The third term may be neglected as e is a small parameter and σ_r /for sufficiently small ratio b/a/ it is small in comparison with $d\sigma_r/dr$.

Thus, the equation is reduced to the form

$$r^2 d^2\sigma_r/dr^2 + (3r - \epsilon)\, d\sigma_r/dr = E_o\alpha\,\theta(a)/\ln\frac{b}{a}\,(1 - \ln\frac{b}{r}) - \xi(r) \qquad /32/$$

where $\epsilon = e\,\theta(a)/\ln\frac{b}{a}$ is a small parameter and

$$\xi(r) = E_o\left[r\frac{d\,\varepsilon_\varphi^P}{dr} + \varepsilon_\varphi^P - \varepsilon_r^P\right](1 - e\,\theta(a)\ln\frac{b}{r}/\ln\frac{b}{a})$$

The equation /32/ may be rewritten as follows

$$r^2\, d\tau/dr + \tau\,(3r - \epsilon) = E_o\alpha\,\theta(a)/\ln\frac{b}{a}\,(1 - \epsilon\ln\frac{b}{r}) - \xi(r)\;;$$

$$\tau = d\sigma_r/dr \qquad /33/$$

and this equation may be solved approximately by means of the small parameter method. The solution is then expressed in terms of some integrals of $\xi(r)$. We assume namely that

$$\tau(r) = f_o(r) + \epsilon f_1(r) \qquad /34/$$

for an arbitrary value of ϵ . Neglecting the higher powers of we obtain

$$f_o(r) = C_1/r^3 + E_o\alpha\,\theta(a)/2r\ln\frac{b}{a} - \frac{1}{r^3}\int r\,\xi(r)\,dr$$

$$f_1(r) = \frac{1}{r^3}\int r\,f_o(r)\,dr + D_1/r^3 + E_o\alpha\,\theta(a)/2r\ln\frac{b}{a}\left(\ln\frac{r}{b} - \frac{1}{2}\right) \qquad /35/$$

The integration constants are to be determined from the stress boundary conditions. If the pressure influence on the stress state may be neglected, i.e. $\sigma_r(a) = \sigma_r(b) = 0$, then we obtain

$$C_1 = -2a^2b^2\left[E_o\alpha\,\theta(a) + \mathcal{J}(a) - \mathcal{J}(b)\right]/(b^2 - a^2)$$

$$C_2 = \mathcal{F}(a) - 2b^2 \left[E_0 \alpha \, \Theta(a) + \mathcal{F}(a) - \mathcal{F}(b) \right]$$

$$D_1 = -2a^2b^2 \Big\{ \mathcal{G}(a) - \mathcal{G}(b) + E_0 \alpha \, \Theta(a)/\ln\frac{b}{a} \Big[\frac{b-a}{2ab} + \frac{1}{2}\ln\frac{a}{b}\Big(\frac{1}{2} + - \ln a\Big) + \frac{2\ln\frac{b}{a}\,(b^2 + ab + a^2)}{3ab\,(b+a)} \Big] \Big\} \Big/ (b^2 - a^2)$$

$$D_2 = D_1/2a^2 + \mathcal{G}(a) - E_0 \alpha \, \Theta(a)/\ln\frac{b}{a} \Big[-\frac{1}{2a} - \frac{2b^2 \ln b/a}{3a(b^2 - a^2)} + + \frac{1}{2}\ln a \,\Big(\ln\frac{a}{b} - \frac{1}{2}\Big) \Big]$$

where

$$\mathcal{F}(r) = \int \frac{dr}{r^3} \int r\, \xi(r)\, dr$$

$$\mathcal{G}(r) = \int \frac{dr}{r^3} \int r \frac{d\mathcal{F}(r)}{dr}\, dr - \frac{2a^2b^2[\mathcal{F}(a) - \mathcal{F}(b)]}{b^2 - a^2} \int \frac{dr}{r^4} \qquad /36/$$

and

$$\sigma_r(r) = \int f_0(r)\, dr + C_2 + \epsilon \int f_1(r)\, dr + \epsilon D_2$$

$$\sigma_\varphi(r) = \int f_0(r)\, dr + C_2 + r f_0(r) + \epsilon \int f_1(r)\, dr + \epsilon D_2 + + \epsilon r f_1(r) \qquad /37/$$

this way, as in the examples 1 and 2 , we have obtained the formulas expressing the stress components by a temperature field and by plastic deformations.

According to the theorem of section 3 the adaptation condition requires, in the case of the Tresca yield condition, the following relations to be satisfied

$$|\sigma_r - \sigma_\varphi| \le \sigma_0 \; ; \; |\sigma_r| \le \sigma_0 \; ; \; |\sigma_\varphi| \le \sigma_0 \qquad /38/$$

for all temperatures admitted by the program $0 < \Theta(a) < \Theta_0$ with the same constant plastic deformations.

In our case it suffices to consider only two states : for $\Theta(a) = 0$ and for $\Theta(a) = \Theta_0$.

Writing all these inequalities we obtain subsequently

$$-\sigma_0 \le \frac{C_1}{r^2}\Big|_{\Theta(a)=0} - \frac{1}{r^2}\int r\, \xi(r)\, dr + \epsilon \Big\{ -\frac{1}{r^2}\int r \frac{d\mathcal{F}(r)}{dr} dr + + \frac{2a^2b^2[\mathcal{F}(a) - \mathcal{F}(b)]}{r^3(b^3 - a^3)} + \frac{D_1}{r^2}\Big|_{\Theta(a)=0} \Big\} \le \sigma_0$$

etc /39/

By combining these inequalities we eliminate the terms containing the plastic deformations and we obtain a family of inequalities with the parameter r e.g.

$$-2\sigma_0 \leq E_0\alpha\theta_0/2\ln\frac{b}{a} - 2a^2b^2E_0\alpha\theta_0/r^2(b^2-a^2) + \epsilon\left\{\frac{E_0\alpha\theta_0}{\ln b/a}\left[\frac{1}{2r} + \frac{2a^2b^2\ln b/a}{r^3(b^2-a^2)}\right] + E_0\alpha\theta_0/2r\ln\frac{b}{a}\left(\ln\frac{r}{b} + -\frac{1}{2}\right) - \frac{2a^2b^2E_0\alpha\theta_0}{\ln\frac{b}{a}(b^2-a^2)r^2}\left[\frac{b-a}{2ab} + \frac{1}{2}\ln\frac{a}{b}\left(\frac{1}{2} - \ln a\right) + \frac{2\ln\frac{b}{a}(b^2+ab+a^2)}{3ab(b+a)}\right]\right\} \leq 2\sigma_0 \qquad /40/$$

A general analysis of the above system of inequalities is practically impossible. Yet for specified values of a, b, E_0, σ_0, α, e it may usually be done numerically.

For the material parameters as in the previous sections and the upper bound for the temperature amplitude θ_0 of 100 centigrades the influence of Young's modulus variation will approximate 5 per-cent.

An anlogous analysis may be performed for a thick-walled spherical shell - as all basic equations are the same or nearly the same.

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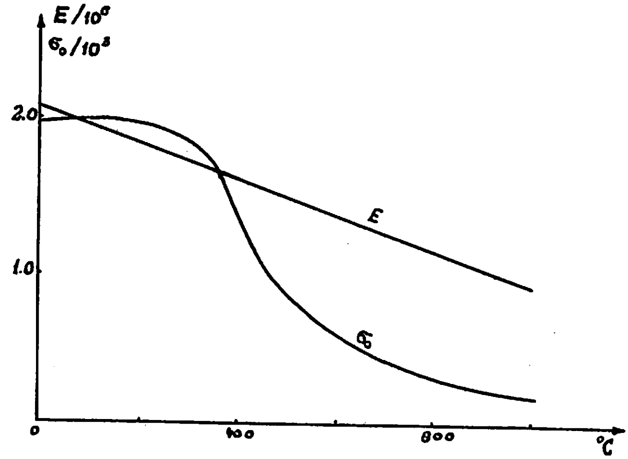

Fig 1

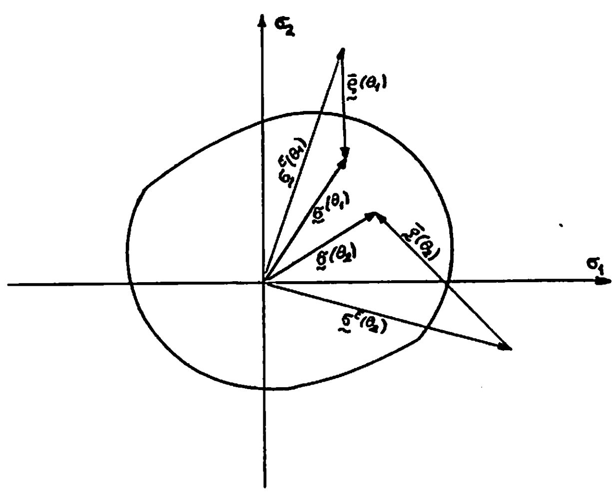

Fig 2

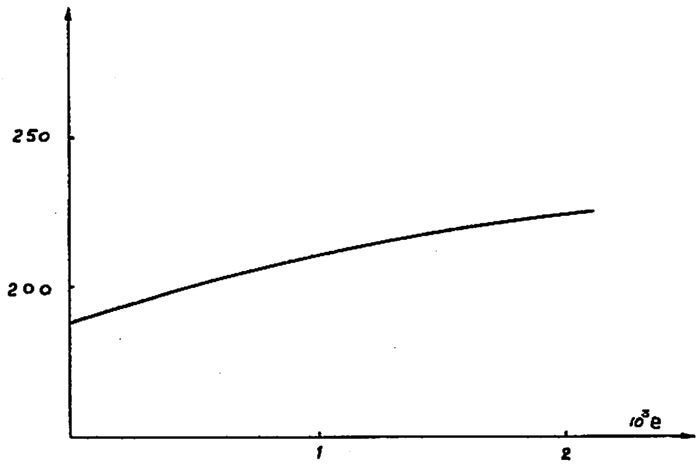
250
200
$10^3 e$
1
2

Fig 3

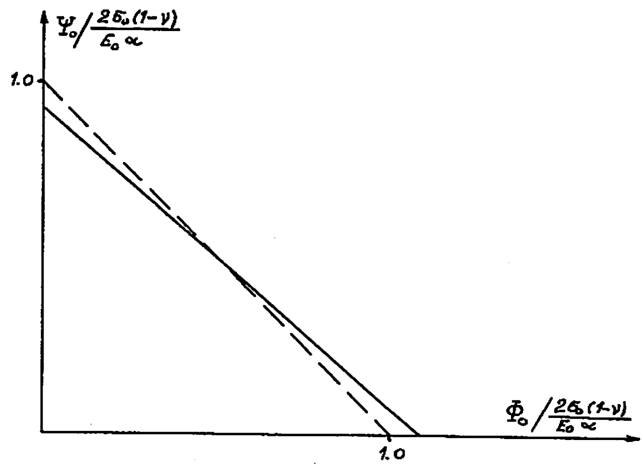
$\Psi_o \Big/ \frac{2\sigma_o(1-\nu)}{E_o \alpha}$
1.0
$\Phi_o \Big/ \frac{2\sigma_o(1-\nu)}{E_o \alpha}$
1.0

Fig 4