A METHOD TO GENERATE BOTH UPPER AND LOWER BOUNDS TO PLATE EIGENVALUES BY CONFORMING DISPLACEMENT FINITE ELEMENTS

P.P. LYNN,
Department of Civil Engineering,
University of Colorado, Boulder, Colorado,

G.E. RAMEY,
Department of Civil Engineering,
Auburn University, Auburn, Alabama, U.S.A.

ABSTRACT

Due to the approximate nature of the finite element method, the generation of both upper and lower bounds to the exact solution is of great value in ascertaining the accuracy of the approximation. This is particularly true for eigenvalue solutions. In this paper the authors present a method, using only the conforming displacement elements, for determining both bounds. The first solution bound is obtained immediately from the finite element numerical solution. The second bound is generated by using the first bound numerical results for two or more mesh sizes. For large value of n (n is a measure of the number of elements used) the method is mathematically defensible. For small values of n certain assumptions have to be made. The bounding of the exact solution can be proved by assuming the validity of these assumptions.

The method has been extensively tested on both free vibration and buckling plate problems to assure its workability. Results of these tests reveal that within practical element subdivisions the technique generally guarantees the generation of the other bound; but for extremely coarse finite element approximations, a caution has to be exercised. For the latter case, a supplementary scheme to increase the effectiveness of the method is provided. The method works extremely well in static problems, and the attainment of the other bound for these problems can be shown mathematically.
1. INTRODUCTION

As in other approximate solution methods, the important questions of convergence and solution bounds arise in the finite element method. In the linear-elastic structural eigenvalue problems, it is a well known fact that the Rayleigh-Ritz's approximate method yields upper bound solutions while Trefftz's method gives lower bound solutions[1].

Direct application of these two methods in the finite element analysis has been considered by Veubeke[2],[3]. Static bending of a rectangular plate under a concentrated load is alternately analyzed by a displacement element and an equilibrium element. In this manner, upper and lower bounds to the plate deflection under the load are obtained. This approach, however, requires the formulation of two finite element models. In general, the force method requires, for efficient utilization, automated methods of selection of redundancies and generation of the self-equilibrating force system[4]. Primarily for this reason and because of the greater simplicity in matrix operations in the displacement method, the force method is not widely used in finite elements.

From a practical point of view, it would be highly desirable that both upper and lower bound solutions be obtained through the use of one finite element model - displacement or equilibrium element. In this paper the authors propose a simple practical scheme to generate the error bounds through the application of a conforming displacement finite element in two or more mesh sizes. The method is applicable to eigenvalue and static problems with no stress singularities, and it considers only the discretization error of the finite element approximations.

The study begins with a brief convergence proof of the upper bound eigenvalue solutions employing conforming displacement plate elements. Conditions for monotonically convergent upper bound eigenvalue solutions are then established. Next, a theoretical analysis of the element convergence properties for conforming elements is undertaken. From the above theoretical investigations the asymptotic convergence rate of the quantity under consideration, with respect to the use of finer element subdivisions, is estimated. Assuming the validity of this asymptotic rate of convergence, a practical method of generating the lower bound solution (the other bound) is described. Even though the theoretical validity of the method applied to the eigenvalue problems is still lacking, the extensive numerical studies reveal that the technique generally guarantees the lower bound solutions for practical element subdivisions. However, application of the method to static cases can be shown mathematically as well as numerically to yield the other bound.

2. BRIEF PROOF ON THE FINITE ELEMENT EIGENVALUE CONVERGENCE

Convergence proofs on finite element solutions have been centered around static cases where mathematical proofs have been constructed primarily through the use of the minimum potential energy principle[5],[6],[7],[8]. Recently, Lynn and Dhillon[9] extended the theoretical convergence proof to
eigenvalue problems by employing conforming displacement plate elements. This was accomplished by transforming plate eigenvalue-eigenfunction problems into corresponding isoperimetric variational problems. From this transformation, the \( r \)th eigenvalue \( \lambda_r \) calculated from a finite element analysis is shown to be the minimum of the plate strain energy with respect to the approximate \( r \)th set of eigenfunctions under the normalization and orthogonality conditions. The variational maximum-minimum principle was used to show the upper bound character of the finite element eigenvalue solutions. To put this study in proper perspective, it is helpful to consider first a brief review of the eigenvalue convergence proof as applied to thin plate finite elements based on Kirchhoff's thin plate theory.

In the small deflection theory of thin plates, the plate displacement fields \( u, v, \) and \( w \) in the directions of the \( x, y, \) and \( z \) coordinates, respectively, are assumed to be

\[
(u, v, w) = (-zw, x, -zw, y, w)
\]

where \( w \) is the lateral deflection of the middle-plane of the plate, and the comma denotes differentiation. The equation of free vibration of the plate after removal of the time variable is

\[
Dv^2w^2 - \lambda \rho hw = 0
\]

where \( D \) is the plate flexural rigidity, \( v^2 \) is the Laplace operator in \( (x, y) \), \( \rho \) is the mass per unit volume, \( h \) is the plate thickness, \( \lambda \) represents the eigenvalue corresponding to the eigenfunction \( w \). Let one of the following pairs of boundary conditions be prescribed along the plate boundary \( \Gamma \):

1) \( w = 0, \quad w_n = 0 \)
2) \( w = 0, \quad M(w) = 0 \)
3) \( M(w) = 0, \quad V(w) = 0 \)

in which \( M(w) \) and \( V(w) \) are the plate internal moment and Kirchhoff shear resultants, and \( n \) is the outer-normal to the boundary \( \Gamma \).

The eigenvalue problem of Eqs. (2) and (3) may be alternatively formulated as an isoperimetric variational problem[9],[10]. Thus instead of solving Eqs. (2) and (3) we may seek the \( r \)th eigenfunction \( w_r \) such that it minimizes the plate strain energy

\[
I = \iint_A \left[ \frac{D}{2} \left( \frac{w_{xx} + w_{yy}}{2} - 2(1-\nu)(w_{xx}w_{yy} - w_{xy}^2) \right) \right] \, dx \, dy
\]

under the subsidiary condition

\[
J = \iint_A w^2 \, dx \, dy = C \quad \text{(a constant)}
\]

and the \((r-1)\) orthogonality conditions

\[
\iint_A w_r w_j \, dx \, dy = 0 \quad \text{for } j = 1, 2, \ldots, (r-1)
\]
where \( w_j \) represents a previous eigenfunction. It should be noted that the constant \( C \) in Eq. (5) may be taken as unity (Normalization) without loss of generality.

By considering the \( r \)th eigenvalue \( \lambda_r \) as the Lagrange multiplier, and minimizing the functional

\[
F(w_r) = I(w_r) - \lambda_r J(w_r)
\]  

(7)

it can be shown that \( \lambda_r \) equals the so called Rayleigh's quotient\[9],[10]

\[
\lambda_r = \min. \frac{I}{C} \quad \text{or} \quad \lambda_r = \min. I, \text{ with } C = 1
\]  

(8)

under the subsidiary conditions described earlier. For the finite element approximating system, which replaces the real plate, similar conclusions may be reached, i.e.,

\[
\lambda_r^* = I(w_r^*)
\]  

(9)

subject to the conditions

\[
J(w_r^*) = 1 \quad \text{and} \quad \int_A w_j^* \, dx \, dy = 0, \quad j = 1, 2, \ldots, (r-1)
\]  

(10)

where \( w_r^* \) is the \( r \)th minimizing finite element approximate eigenfunction defining the \( r \)th approximate eigenvalue \( \lambda_r^* \).

In order to show the upper bound nature of the displacement finite element eigenvalue solutions, i.e., \( \lambda_r^* \geq \lambda_r \), we first note that the \( r \)th finite element approximate eigenfunction \( w_r^* \) is not orthogonal, but rather approximately orthogonal to the \( (r-1) \) real eigenfunctions, \( w_1, w_2, \ldots, w_{r-1} \).

Secondly, the discretization of the finite element system, in effect, constrains or reduces the class of functions admissible in the variational problem. Accordingly, the violation of the original orthogonality conditions together with the constrained finite element approximate system requires careful deduction of the previous theoretical variational properties.

We let \( \lambda_r^* \) and \( w_r^* \) be the successive eigenvalues and eigenfunctions of the constrained finite element system. Correspondingly, we let \( \lambda_r \) and \( w_r \) be the real sequences of eigenvalues and eigenfunctions. Since the first eigenvalue \( \lambda_1 \) belongs to a free variational problem (without any orthogonality condition), \( \lambda_1 \) is an absolute minimum. Hence, it follows that \( \lambda_1^* \geq \lambda_1 \). To show the increasing nature of the higher finite element eigenvalues, \( \lambda_r^* \) with \( r > 1 \), we construct an intermediate auxiliary system. Let \( \lambda_r^*_1 \) denote the minimum of \( I(\tilde{\bar{w}}) \) for the function \( \tilde{\bar{w}} \) belonging to the constrained finite element system and subjected to the original orthogonality conditions

\[
\int_A \tilde{\bar{w}} \, w_j \, dx \, dy = 0, \quad j = 1, 2, \ldots, (r-1)
\]  

(11)

For the finite element and auxiliary systems, we have according to the maximum-minimum principle of variational calculus\[9],[11] that
\[ \lambda_r^* \geq \bar{\lambda}_r \]  (12)

This is because Eq. (11) represents \((r-1)\) orthogonality conditions other than the natural orthogonality conditions for the finite element system (Eq. (10)). On the other hand, if we compare the real and the auxiliary system, we obtain

\[ \bar{\lambda}_r \geq \lambda_r \]  (13)

This is due to the fact that both systems in the minimizing problems defining \(\lambda_r\) and \(\bar{\lambda}_r\) possess the same orthogonality conditions (Eqs. (6) and (11)), but the auxiliary system is subjected to a smaller class of admissible functions \([9],[11]\). Accordingly, from Eqs. (12) and (13), we arrive at the conclusion

\[ \lambda_r^* \geq \bar{\lambda}_r \geq \lambda_r \]  (14)

Consequently, for any conforming finite element approximating function \(w_r^*\) with corresponding eigenvalues \(\lambda_r^*\) satisfying \(J(w_r^*) = 1\), we have

\[ I(w_r^*) \leq I(w_r) \]

or

\[ \lambda_r^* - \lambda_r = I(w_r^*) - I(w_r) \geq 0 \]  (15)

Hence, for the convergence of \(\lambda_r^*\) to \(\lambda_r\) in the finite element free vibration analysis, it is only necessary to estimate the upper bound of the right hand side of Eq. (15) in terms of the finite element size or the total number of element subdivisions and show that it can be made as small as desirable.

This estimation procedure is closely related to the particular conforming finite element model used. Thus, we shall demonstrate the procedure by means of a 16-degree of freedom rectangular thin plate element.

3. ASYMPTOTIC CONVERGENCE RATES OF FINITE ELEMENT EIGENVALUES

Consider a conforming 16-degree of freedom thin rectangular element\([12],[13]\) for which the ith element trial displacement \((w_r^*)_i\) is taken as

\[ (w_r^*)_i = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2y + a_9xy^2 \]

\[ + a_{10}y^3 + a_{11}x^3y + a_{12}xy^3 + a_{13}x^2y^2 + a_{14}x^3y^2 + a_{15}x^2y^3 + a_{16}xy^3 \]  (16)

It should be noted that \((w_r^*)_i\) is a complete cubic polynomial in \(x\) and \(y\). If \((w_r^*)_i\) is defined only in the ith element and zero elsewhere, we may define the finite element approximating function for the entire plate domain as

\[ w_r^* = \sum_{i=1}^{N} (w_r^*)_i \]  (17)

where \(N\) is the total number of element subdivisions.

We now introduce a comparison function in the form of a third degree finite Taylor expansion (in \(x\) and \(y\)) of the exact solution \(w_r\) about a point
within the ith element

\[(\tilde{w}_r)_i = w_r|_0 + (x-x_0) w_{r,x}|_0 + (y-y_0) w_{r,y}|_0 + \ldots + \frac{1}{3!}(y-y_0)^3 w_{r,yyy}|_0\]  

(18)

Let \((R)_i\) be the remainder term of the expansion, i.e.,

\[(R)_i = \frac{1}{4!} \left[ (x-x_0)^4 w_{r,xxxx}|_0 + \ldots + (y-y_0)^4 w_{r,yyyy}|_0 \right]\]  

(19)

The notations \(|_0\) and \(|_0\) designate that the quantities to the left of the signs are evaluated at points 0 and 0 respectively. The point 0 is a point on the line joining points \((x_0, y_0)\) and \((x, y)\). Accordingly, the third degree Taylor series comparison function for the entire plate domain is

\[\tilde{w}_r = \sum_{i=1}^{N} (\tilde{w}_r)_i\]  

(20)

where \((\tilde{w}_r)_i\) is defined in the ith element and zero elsewhere.

Next we note that among all the third degree polynomials in \(x\) and \(y\), i.e., \(w_r^*\) and \(\tilde{w}_r\), \(w_r^*\) minimizes the integral I under the subsidiary conditions. Hence in view of Eq. (15), we have

\[I(\tilde{w}_r) \geq I(w_r^*) \geq I(w_r)\]

or

\[I(\tilde{w}_r) - I(w_r) \geq \lambda_r^* - \lambda_r \geq 0\]  

(21)

When \(\tilde{w}_r\) in Eq. (21) is replaced by

\[\tilde{w}_r = w_r - \sum_{i=1}^{N} (R)_i\]  

(22)

the left hand side of the equation becomes[8],[9]

\[I(\tilde{w}_r) - I(w_r) = \sum_{i=1}^{N} \left\{ \frac{D}{A_i} \left[ (1-\nu)(R_{,xx}R_{,yy} - R_{,xy})^2 - 2(1-\nu)(R_{,xx}R_{,yy} - R_{,xy}) \right] \right\} dx \]  

\[+ \sum_{i=1}^{N} \left\{ \frac{D}{A_i} \left[ (1-\nu)(R_{,xx}^2 + R_{,yy}^2) + \nu(R_{,xx} + R_{,yy})^2 + 2(1-\nu)R_{,xy}^2 \right] \right\} \leq 0\]  

(23)

where for simplicity the subscript \(i\), which denotes the quantities belonging to the ith element, on the remainder \(R\) has been taken out of the parentheses, and \(A_i\) denotes the area of the ith element. By means of the Schwarz and triangular inequalities of norms of functions we find

\[0 \leq I(\tilde{w}_r) - I(w_r) \leq \sum_{i=1}^{N} \left\{ \frac{D}{A_i} \left[ \| R_{,xx} \|^2 + \| R_{,yy} \|^2 + 2\nu \| R_{,xx} \| \| R_{,yy} \| \right. \right\} \]  

\[+ 2(1-\nu) \| R_{,xy} \|^2 \right\} \]  

(24)
On recalling the expression of (R)\_i and assuming all the fourth order partial derivatives of the exact solution w\_i are bounded (no stress singularities), we can estimate the right hand side of Eq. (24) in terms of the largest element size (a, b) in the domain. Hence, we find[8],[9]

\[ 0 \leq \lambda^*_r - \lambda_r \leq I(\bar{w}_r) - I(w_r) \leq K(a+b)^4 \]  

(25)

Letting a = b = n^{-1} for the case of a square plate and redefining the constant K, we have

\[ 0 \leq (\lambda^*_r)_n - \lambda_r \leq K n^{-4} \]  

(26)

where n is a measure of the number of elements along one side (or mesh size). Accordingly, the asymptotic convergence rate of the eigenvalue obtained from this element is n^{-4}. This is in agreement with the value cited by Lindberg and Olson[14].

Though the development presented here is in terms of a thin plate vibration problem, the method has been extended to studies of thin plate buckling, thick plate vibrations, and static elasticity problems. In static problems, however, upper and lower bounds of the total potential energies or the strain energies of the system must be considered as the measures of approximations. Analyses similar to that of the rectangular thin plate element were performed on various conforming displacement finite elements. Those considered were ones for which numerical results on eigenvalues problems could be found. For the sake of brevity, these analyses will not be repeated here; but the results are summarized in Table I. Similar analyses have been performed for static cases by other investigators[18],[6].

4. MONOTONIC CONVERGENCE

Monotonic nature of convergence, with respect to the mesh size n, is not always guaranteed by the use of conforming elements as was shown numerically by Lindberg and Olson[14] and also by Carson and Newton[16]. However, when the mesh sizes are taken in a sub-sequence of n = 2, 4, 8, 16 or 3, 6, 12, 24, etc., such that the nth mesh contains all the nodal points of the previous mesh sizes, there is a theoretical assurance of monotonic convergence [9],[17].

To show this, we let (w^*_r)_n be the finite element approximating displacement field which corresponds to the nth mesh size, and let the associated eigenvalue be (\lambda^*_r)_n. Recalling that (w^*_r)_n contains all the previous displacement fields, we now construct a set of displacement fields belonging to the nth family of approximating displacement fields. This is done by writing

\[ (w^*_r)_n + \epsilon [(w^*_r)_n/2 - (w^*_r)_n] \]  

(27)

in which \epsilon varies from 0 to 1. From the fact that among all the functions belonging to the nth family of displacement fields the one which minimizes
the integral \( I \) is \( (w_1^*)_n \), we have

\[
I \left\{ (w_1^*)_n + \left[ (w_1^*)_{n/2} - (w_1^*)_n \right] \right\} \geq I \left[ (w_1^*)_n \right]
\] (28)

In view of the continuous nature of the integral \( I \), we let \( \epsilon \to 1 \) and thus obtain

\[
I \left[ (w_1^*)_{n/2} \right] \geq I \left[ (w_1^*)_n \right]
\] (29)

or equivalently

\[
(w_1^*)_{n/2} \geq (w_1^*)_n \quad \text{for all } n.
\] (30)

5. GENERATION OF OTHER BOUND

The objective of an error estimate in the finite element method is to obtain an estimate of the maximum error in the nth approximation. This information in turn may be used to determine the necessity for further mesh sub-
division. The method of bounding the exact solution proposed in this paper yields a practical means of determining a realistic error estimate. The
method should be used only with conforming elements; hence, the first solution
bound is obtained directly from the finite element numerical analysis.

To generate the other bound, we first observe the convergence character
of conforming finite element numerical solutions. For example, in eigenvalue
problems utilizing the thin plate element, Eq. (26) states that \( (\lambda_n^*) \) converges from above to the exact solution \( \lambda_r \), as \( n \to \infty \), at the rate of \( n^{-4} \). Furthermore, if \( n \) takes on a sub-sequence of \( n = 2, 4, 8, \) or \( 3, 6, 12, \) the convergence is monotonic (Eq. (30)). Graphically, the convergence of \( (\lambda_n^*) \) to \( \lambda_r \) can be illustrated as in Fig. 1, in which the asymptotic \( K \) \( n^{-p} \) (with \( p = 4 \)) curve is bounding the \( (\lambda_n^*) \) curve from above. It should be noted that the abscissa of Fig. 1 is a \( n^{-1} \) scale.

In general, the asymptotic convergence rates of \( (\lambda_n^*) \) may be regarded as \( K \) \( n^{-p} \) with \( p \geq 2 \). This can be seen in Eq. (24) where the partial derivatives of the remainder term \( (R)_4 \) under the integral sign have to be at least of the order \( O(n^{-1}) \), hence \( O(n^{-2}) \) for the norms of these derivatives, such that

convergence of \( (\lambda_n^*) \) to \( \lambda_r \) may be attained as \( n \to \infty \). Exceptions to this conclusion arise when there are unbounded higher order derivatives of the exact
solution which are exhibited by stress singularities.

If these exceptional cases are excluded, we may state that all the
curves \( K \) \( n^{-p} \) with \( p \geq 2 \) are concave upward on the \( n^{-1} \) scale (see Fig. 1), and

bound the \( (\lambda_n^*) \) curves closely from above. For large value of \( n \), the finite

element solutions \( (\lambda_n^*) \) should follow the bounding curves \( (K \) \( n^{-p} \) quite

closely, and it may be assumed that the \( (\lambda_n^*) \) curves also present the upward concavity as shown in Fig. 1. This assumption will be justified by extensive

numerical investigations which are to follow. In static cases the concavity
of the curves representing the finite element solutions can be shown mathe-
matically; however, the measure of errors must be based on the total potential
energies or the strain energies of the systems.

Next we consider the \( (\lambda^*_r)_n \) curve as shown in Fig. 2, where three consecutive approximate eigenvalues to \( \lambda_r \) are given as \( (\lambda^*_r)_2 \), \( (\lambda^*_r)_4 \), and \( (\lambda^*_r)_8 \). We now construct the lower bound \( (\lambda^L_r)_n \) by subtracting the difference of \( (\lambda^*_r)_j \) and \( (\lambda^*_r)_{2j} \) from \( (\lambda^*_r)_2j \) for all \( j \). Mathematically, we have

\[
(\lambda^L_r)_4 = \left(\lambda^*_r\right)_4 - \left[\left(\lambda^*_r\right)_2 - \left(\lambda^*_r\right)_4\right] \leq \left(\lambda^*_r\right)_4
\]

\[
(\lambda^L_r)_8 = \left(\lambda^*_r\right)_8 - \left[\left(\lambda^*_r\right)_4 - \left(\lambda^*_r\right)_8\right] \leq \left(\lambda^*_r\right)_8
\]

(31) \hspace{1cm} (32)

The difference of \( (\lambda^L_r)_8 \) and \( (\lambda^L_r)_4 \) is

\[
(\lambda^L_r)_8 - (\lambda^L_r)_4 = 2\left(\lambda^*_r\right)_8 - 3\left(\lambda^*_r\right)_4 + \left(\lambda^*_r\right)_2
\]

\[
- \left[\left(\lambda^*_r\right)_2 - \left(\lambda^*_r\right)_8\right] - 3\left[\left(\lambda^*_r\right)_4 - \left(\lambda^*_r\right)_8\right] \geq 0
\]

(33)

The inequality (Eq. (33)) can be shown on the basis of the upward concavity of the \( (\lambda^*_r)_n \) curve. On referring to the dotted similar triangles of Fig. 2, we note that

\[
\left(\lambda^*_r\right)_2 - \left(\lambda^*_r\right)_8 = 3\Delta
\]

where \( \Delta \) is the length of the line segment indicated in Fig. 2. The upward concavity of the \( (\lambda^*_r)_n \) curve implies that

\[
\Delta \geq (\lambda^*_r)_4 - (\lambda^*_r)_8
\]

hence we have the inequality of Eq. (33).

Eqs. (32) and (33) implies that the generated lower bound \( (\lambda^L_r)_n \) curve is monotonically increasing and it can never be be higher than the upper bound \( (\lambda^*_r)_n \) curve. Since the finite element upper bound solutions \( (\lambda^*_r)_n \) converge monotonically from above, the \( (\lambda^L_r)_n \) values obtained in this manner are indeed the corresponding lower bound solutions.

It must be emphasized that the above discussion about the generation of the lower bound to the eigenvalue \( \lambda_r \) is based on the assumption that the \( (\lambda^*_r)_n \) curves are concave upward. Numerically, it is observed that for \( n \) values within practical mesh sizes the finite element solutions are asymptotic to the \( K n^{-p} \) curves, hence the concavity of the \( (\lambda^*_r)_n \) curves is ensured. However, for extremely small \( n \), the \( (\lambda^*_r)_n \) curves exhibit more deviation from the \( K n^{-p} \) curves. To ensure the upward concavity of the finite element solution curves \( (\lambda^*_r)_n \) numerically, it is proposed that at least three finite element solutions (rather than two) be determined in the \( n \) sub-sequence stated earlier. Upon obtainment of three consecutive numerical \( (\lambda^*_r)_n \) values showing concave upward convergence, it is assumed that further increment in \( n \) will continue to show the concave upward convergence. The assumption
should be valid since we are now working with sufficiently large \( n \) values
where the \( (\lambda_{P}^{*})_{n} \) curves follow the \( K_{n+1} \) curves closely. The necessity for a
fourth numerical \( (\lambda_{P}^{*})_{n} \) value to obtain concave upward convergence has rarely
been encountered in all the numerical studies undertaken.

On the other hand, in static problems the concavity of the finite element
solution curves can be guaranteed mathematically and this will be presented
later. Hence, only two approximate finite element solutions are needed to
generate the other bound.

The method has been tested on both plate eigenvalue and static problems
to assure its workability. Although the method assumes the concavity of the
\( (\lambda_{P}^{*})_{n} \) curves for eigenvalue problems, extensive numerical tests have failed
to yield a case where the method will not work. Additionally, the resulting
bounds of the exact solution is usually very tight. Results of some numeri-
cal studies are presented in the next section.

6. RESULTS FROM NUMERICAL STUDIES

General Comments. Numerical results for four conforming displacement
finite elements were studied. The elements were selected based on the avail-
ability of numerical results for eigenvalue problems. The elements consider-
ed were the following: (1) 16-degree of freedom rectangular thin plate ele-
ment, (2) Clough and Felippa's thin plate quadrilateral element, (3) Cowper
et al. thin plate triangular element, and (4) Lynn and Dhillon's thick plate
triangular element. The theoretical convergence rates of these elements are
listed in Table I. Graphs showing numerical verification of theoretical con-
vergence rates for the 16-degree of freedom rectangular thin plate element
and Cowper et al. thin plate triangular element are given in references [14],
[15], and [18].

The proposed method of bounding has been applied to the numerical data
available for these elements (these data may be found in references [9],[13],
[15],[16],[18],[19], and [20]). The results of all applications indicate the
workability of the method. Some example figures are included in this section
to give a visual picture of the results of application of the method to a
broad class of problems. In these figures, solid lines represent the finite
element solutions and dotted lines the generated other bounds. Drawings de-
fining the mesh size designations are shown in Fig. 3.

The method has been applied to many problems for which insufficient nu-
merical data points were available to give the recommended three-point
sequence. The results of these applications were again favorable. Also, the
method has been applied to static problems where strain energy data is avail-
able, and favorable results were obtained. Some example figures of the latter
two applications are included for completeness.

A study of the figures indicates that it is desirable to have a three-
point \( n \) sequence such that the middle point gives a good approximation to the
exact solution. This will yield a "tight" bracket of the solution. The
rapidly converging elements usually accomplish this with the minimum 2-4-8 \( n \)
sequence. For slower converging elements, the 3-6-12 n sequence will probably be more efficient if a "tight" bracket is required. This is because a value of n as high as six may be required to get a good approximation; and, if this is the case, then the 6-12 combination will yield a "tight" bracket.

16-Degree of Freedom Rectangular Thin Plate Element. Numerical results, employing this element, are available ([13],[15],[16],[18],[19]) for plate free vibration frequencies and buckling coefficients. The results are for square or rectangular plates utilizing five different boundary support conditions. Example plots of some of these quantities are given in Figs. 4 through 11. In specifying the free vibration frequencies $\lambda_{ij}^*$, the subscript $i$ indicates the number of half sine waves perpendicular to the simple supports and $j$ denotes those in the opposite direction. As seen in Fig. 9, the 2-4-8 sequence for the particular problem is one of the rare cases which requires a fourth point, i.e., $n = 16$. From the plot, we can see that the 4-8-16 sequence will give concave upward convergence, indicating that $\lambda_{18}^*$ and $\lambda_{14}^*$ should give the other bound which it does indeed. Fig. 10 is a plot for this same problem using a partial 3-6-12 sequence ($\lambda_{12}^*$ is missing), and we can observe the workability of this sequence. Hence, use of $\lambda_{18}^*$ and $\lambda_{23}^*$ should and does give the other bound. This problem represents a case where it would be more efficient to use the 3-6-12 sequence. Figure 11 shows a plot of the potential energy of the system which is the quantity bounded for static problems.

Clough and Felippa's Thin Plate Quadrilateral Element. Numerical results obtained from applications of this element are available[20] for several plate buckling problems. Some example plots of these results are given in Figs. 12 and 13. Comparing Fig. 13 with Fig. 8 of the previous section, we get a direct comparison of the relative efficiency of the two finite element models since the plots are for the same problem. We can observe the faster convergence of the 16-degree of freedom rectangular element. This is as expected since it has a theoretical convergence rate of $K n^{-4}$ versus $K n^{-2}$ for the CFQ element.

Cowper et al. Thin Plate Triangular Element. This is a high degree of freedom element with very rapid convergence characteristics. Because of this fast convergence, unfortunately, only numerical results for $n = 6$ and less are available[15] for eigenvalue problems. Hence, the three-point n sequence cannot be obtained; however, much data is available for plates with commonly used boundary conditions for 2-4 and 3-6 sequences. Using these data yields the other bound in every case. An example plot of these data is given in Fig. 14.

Data in the three-point n sequence for this element are available[15] for some static problems. In each of these cases the workability of the method was verified. Example plots are given in Figs. 15 through 17. Note in these figures the quantity under consideration is the plate strain energy.

Lynn and Dhillon's Thick Plate Triangular Element. Numerical results obtained from application of this element are available[9] for simply-supported
and clamped square plate free vibration problems. Analytical eigenvalues from Mindlin's thick plate theory are known for the simply-supported boundaries but not for the clamped condition. Tests of these data again verify the workability of the proposed bounding method. Some example plots are given in Figs. 18 and 19. In these figures, we can observe the relatively "tight" bracket of the solution that results from application of the proposed bounding method.

7. CONCLUSIONS

The theoretical and numerical results of the study provide basic information on convergence properties and error estimates of finite element solutions. A theoretical analysis of the convergence rate \( n^{-p} \) is undertaken to establish the value of \( p \), and hence be assured of convergence. Based primarily on the value of \( p \), a decision is made on the finite element mesh sequence to be used in the analysis. The criteria is that the middle approximate solution give a good answer, such that a "tight" bracket on the exact solution can be obtained.

Extensive numerical studies have failed to reveal a case where the proposed bounding method will not work, and this gives confidence in the workability of the method. If one had both finite element models (conforming displacement and equilibrium), it would be preferable to run both models at a given mesh size and thus obtain guaranteed bounds to the exact solution. However, the possession of both models is not likely at the present time; and, if this is the case, the method proposed here may be employed. It provides a practical means of determining a realistic error estimate and lends itself well to automated mesh generation. In static problems there is a theoretical assurance of generating the other energy bound. In these cases, only two-point \( n \) sequences are needed for the method.

ADDENDUM

The purpose of this addendum is to present the theoretical proof of finite element energy bounds for static problems. By means of conforming finite elements, upper bound values of the total potential energy \( \pi \) are obtained immediately. The other bound (lower bound) of \( \pi \) can be generated by the method described in the main paper. That method assumes upward concavity of the approximate \( \pi^*_n \) curves (see Fig. A-1), we now verify that assumption mathematically for static cases.

Consider a plate subject to transverse load \( q(x,y) \), edge moment \( m(s) \), and edge shear \( v(s) \), where \( (n,s) \) denote normal and tangential axes at a point \( s \) on the boundary \( \Gamma \) bounding plate region \( A \). The total potential energy \( \pi(w) \) of the plate is[8],

\[
\pi(w) = \frac{1}{2} \int_A \left[ \left( w^2 \right)^2 - 2(1-\nu)\left( w_{xx} w_{yy} - (w_{xy})^2 \right) \right] \, dA
- \int_A q w dA + \int_{\Gamma} m(s) w_\nu ds - \int_{\Gamma} v(s) w ds
\]  
(A-1)
By introducing the admissible comparison function \( \xi \) and seeking the increment of \( \pi \), we find[8]

\[
\Delta \pi = \pi(w + \xi) - \pi(w) = \delta \pi + \frac{1}{2} \delta^2 \pi \geq 0
\]  

(A-2)

where

\[
\delta \pi(w) = \iint_A \left( D \varepsilon^2 + q - q \right) dA + \int_I \left( M(w) + m(s) \right) \xi_n ds
\]

\[
- \int_I \left( V(w) - v(s) \right) \xi ds
\]

(A-3)

and

\[
\delta^2 \pi(\xi) = D \iint_A \left( (\varepsilon^2 \xi)^2 - 2(1-\nu) \left[ \xi_{xx} \xi + \xi_{yy} - (\xi_{xy})^2 \right] \right) dA \geq 0
\]

(A-4)

in which \( M(w) \) and \( V(w) \) are the plate internal moment and the Kirchhoff's shear resultant in terms of \( w \).

When the finite element approximation function \( W_n^* \) is used to minimize \( \pi \), in essence, the structural stiffness equation implies the satisfaction of \( \delta \pi(W_n^*) = 0 \). Let \( w_2^* \), \( w_4^* \), and \( w_8^* \) be three successive finite element approximations such that

\[
\delta \pi(w_2^*) = 0, \quad \delta \pi(w_4^*) = 0, \quad \delta \pi(w_8^*) = 0
\]

(A-5)

Next we construct an admissible function \( W_8 \) belonging to the family of functions defined for mesh size 8 as

\[
W_8 = w_8^* + \frac{\epsilon}{2} \left[ 9 w_4^* - 8 w_8^* - w_2^* + \epsilon(3 w_2^* - 9 w_4^* + 6 w_8^*) \right]
\]

(A-6)

It can be noted that this function degenerates to \( W_8^* \) for \( \epsilon = 0 \), \( w_4^* \) for \( \epsilon = 1/3 \), and \( w_2^* \) for \( \epsilon = 1 \). We rewrite Eq. (A-6) as

\[
W_8 = w_8^* + \frac{\epsilon}{2} (\eta + \epsilon \zeta) = w_8^* + \xi
\]

(A-7)

where \( \xi = \frac{\epsilon}{2} (\eta + \epsilon \zeta) \)

and \( \eta = 9 w_4^* - 8 w_8^* - w_2^* \), \( \zeta = 3 w_2^* - 9 w_4^* + 6 w_8^* \)

We form \( \Delta \pi = \pi(W_8) - \pi(w_8^*) \), and let it be denoted by \( \hat{\pi} \) (see Eqs. (A-2), (A-3), and (A-4))

\[
\hat{\pi} = \pi(W_8) - \pi(w_8^*) = \delta \pi(w_8^*) + \frac{1}{2} \delta^2 \pi(\xi) = \frac{1}{2} \delta^2 \pi(\xi) \geq 0
\]

(A-8)

where the quantity \( \hat{\pi} \) is indicated graphically in Fig. A-1. Substitution of \( \xi \), defined in Eq. (A-7), into Eq. (A-4) yields

\[
\hat{\pi} = \frac{\epsilon^2}{8} \hat{\pi}
\]

(A-9)

where
\[ \sigma = \int_A \left[ \frac{\nu^2}{\nu + \epsilon \zeta} \left( \frac{\nu^2}{\nu + \epsilon \zeta} \right)^2 - 2(1-\nu) \left( (\nu + \epsilon \zeta)_{xx} (\nu + \epsilon \zeta)_{yy} - (\nu + \epsilon \zeta)_{xy}^2 \right) \right] \, dA \geq 0 \]  

(A-10)

For upward concavity of the convergence curve \( \pi^*_n \) with \( n = 2, 4, 8 \), we want to show that the first and second derivatives of \( \sigma \) with respect to \( \epsilon \) are positive. Thus, we consider
\[ \frac{\partial \sigma}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \left( \frac{\nu^2}{\nu + \epsilon \zeta} \right) \]  

(A-11)

and
\[ \frac{\partial^2 \sigma}{\partial \epsilon^2} = \frac{1}{2} \frac{\partial^2}{\partial \epsilon^2} \left( \frac{\nu^2}{\nu + \epsilon \zeta} \right) + \frac{\partial}{\partial \epsilon} \left( \frac{\nu^2}{\nu + \epsilon \zeta} \right) \]  

(A-12)

in which
\[ \frac{\partial \sigma}{\partial \epsilon} = 2 \int_A D \left( \frac{\nu^2}{\nu + \epsilon \zeta} \right)_{xx} \zeta_{xx} - (1-\nu) \left( (\nu + \epsilon \zeta)_{xx} \zeta_{yy} \right. \]
\[ \left. + (\nu + \epsilon \zeta)_{yy} \zeta_{xx} - 2(\nu + \epsilon \zeta)_{xy} \zeta_{xy} \right) \, dA \]  

(A-13)

\[ \frac{\partial^2 \sigma}{\partial \epsilon^2} = 2 \epsilon^2 \pi(\zeta) \geq 0 \]  

(A-14)

On referring to Eqs. (A-11) and (A-12), we know that \( \epsilon \geq 0 \), \( \sigma \geq 0 \), and \( \sigma_{\epsilon \epsilon} \geq 0 \); thus it is only necessary to show \( \frac{\partial \sigma}{\partial \epsilon} \geq 0 \). Furthermore, since \( \frac{\partial \sigma}{\partial \epsilon} \geq 0 \), we have only to show \( \frac{\partial \sigma}{\partial \epsilon} = 0 \) at \( \epsilon = 0 \).

Integration by parts and use of Green's formula transforms Eq. (A-13), evaluated at \( \epsilon = 0 \), into
\[ \frac{\partial \sigma}{\partial \epsilon} \bigg|_{\epsilon=0} = 2 \int_A \int_{\Gamma} \left( \nabla^2 \frac{\nu^2}{\nu + \epsilon \zeta} \right) \zeta_{\nu} \, dA + \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds - \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds \]  

(A-15)

On adding and subtracting \( q \), \( m(s) \), and \( v(s) \) in the first, second, and third integrands of Eq. (A-15) respectively; and substituting the expression of \( \sigma \), we find
\[ \frac{\partial \sigma}{\partial \epsilon} \bigg|_{\epsilon=0} = 2 \int_A \int_{\Gamma} \left( \nabla^2 \frac{\nu^2}{\nu + \epsilon \zeta} \right) \zeta_{\nu} \, dA + \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds - \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds \]
\[ - q \int_A \int_{\Gamma} \left( \nabla^2 \frac{\nu^2}{\nu + \epsilon \zeta} \right) \zeta_{\nu} \, dA + \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds - \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds \]
\[ - v(s) \int_A \int_{\Gamma} \left( \nabla^2 \frac{\nu^2}{\nu + \epsilon \zeta} \right) \zeta_{\nu} \, dA + \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds - \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds \]
\[ = 2q \int_A \int_{\Gamma} \left( \nabla^2 \frac{\nu^2}{\nu + \epsilon \zeta} \right) \zeta_{\nu} \, dA + \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds - \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds \]
\[ - 2v(s) \int_A \int_{\Gamma} \left( \nabla^2 \frac{\nu^2}{\nu + \epsilon \zeta} \right) \zeta_{\nu} \, dA + \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds - \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds \]
\[ = 2q \left( \int_A \int_{\Gamma} \left( \nabla^2 \frac{\nu^2}{\nu + \epsilon \zeta} \right) \zeta_{\nu} \, dA + \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds - \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds \right) \]
\[ - 2v(s) \left( \int_A \int_{\Gamma} \left( \nabla^2 \frac{\nu^2}{\nu + \epsilon \zeta} \right) \zeta_{\nu} \, dA + \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds - \int_{\Gamma} \left[ \nabla(\nu) \cdot \zeta \right]_{\nu} \, ds \right) \]
\[ = 2q \delta \pi(\nu) - 2v(s) \delta \pi(\nu) \]  

(A-16)

By virtue of Eq. (A-5), we conclude that Eq. (A-16) is identically zero. Accordingly \( \frac{\partial \sigma}{\partial \epsilon} \geq 0 \) for all \( \epsilon \geq 0 \). This in turn implies that Eqs. (A-11) and (A-12) are always positive for the values of \( \epsilon \) under consideration. Hence,
we have shown the upward concavity of the convergence curve \( \pi_n^* \) with \( n = 2, 4, 8 \). The upward concavity of the entire \( \pi_n^* \) curve can be shown by repeated applications of the above procedure.

It must be pointed out, however, that the proof excludes all the cases where stress singularities exist such that the strain energies become improper integrals. The concavity of the approximate strain energy convergence curves can be shown in a similar manner.

---

**TABLE I**

Conforming Displacement Plate Finite Element Energy and Eigenvalue Convergence Rates

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Type Element</th>
<th>Element Shape</th>
<th>Element Degr. of Freedom</th>
<th>Theoretical Conver. Rate</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>R16</td>
<td>Thin Plate</td>
<td>Rectangle</td>
<td>16</td>
<td>( K n^{-4} )</td>
<td>12, 13, 15, 16, 18, 19</td>
</tr>
<tr>
<td>CFQ</td>
<td>Thin Plate</td>
<td>Quadrilateral</td>
<td>19</td>
<td>( K n^{-2} )</td>
<td>20</td>
</tr>
<tr>
<td>T18</td>
<td>Thin Plate</td>
<td>Triangle</td>
<td>18</td>
<td>( K n^{-6} )</td>
<td>15</td>
</tr>
<tr>
<td>LDT</td>
<td>Thick Plate</td>
<td>Triangle</td>
<td>9</td>
<td>( K n^{-2} )</td>
<td>9</td>
</tr>
</tbody>
</table>
REFERENCES


Figure 1. Convergence of Finite Element Solutions

Figure 2. Convergence Curve for $\lambda^*_r$ for n = 2, 4, 8
Rectangular and Quadrilateral Elements

2 x 2 or n = 2

4 x 4 or n = 4

... ETC.

Triangular Elements

2 x 2 or n = 2

4 x 4 or n = 4

... ETC.

Triangular Elements and Plate

n = 3

n = 6 and n = 12

Figure 3. Mesh Size Designation
Figure 4. Frequency Parameter - Mode 3 for a Clamped Square Plate Employing R16 Elements[19]

Figure 5. Frequency Parameter - $\lambda_{11}$ for a S. S. Square Plate Employing R16 Elements[15]
Figure 6. Frequency Parameter - $\lambda_{12}$ for a S. S. Square Plate Employing R16 Elements[18]

Figure 7. Buckling Coefficient - k for a Square Plate S. S. on Three Sides and Free on Other under Uniaxial Compression Employing R16 Elements[18]
Figure 8. Buckling Coefficient - k for a S. S. Square Plate under Uniaxial Compression Employing R16 Elements[16]

Figure 9. Buckling Coefficient - k for a Clamped Square Plate under Biaxial Compression Employing R16 Elements[16] (using 2-4-8 Data)
Figure 10. Buckling Coefficient $-k$ for a Clamped Square Plate under Biaxial Compression Employing R16 Elements[16] (using 3-6 Data)

Figure 11. Potential Energy $-\pi$ for a Clamped Square Plate under Uniform Load Employing R16 Elements[13]
Figure 12. Buckling Coefficient - k for a S. S. Rectangular Plate under Pure Shear Employing CFQ Elements [20]

Figure 13. Buckling Coefficient - k for a S. S. Square Plate under Uniaxial Compression Employing CFQ Elements [20]
Figure 14. Frequency Parameter - $\lambda_{11}$ for a S. S. Square Plate Employing T18 Elements[15]

Figure 15. Strain Energy - I for a S. S. Square Plate under Uniform Load Employing T18 Elements[15]
Figure 16. Strain Energy - I for a Clamped Square Plate under Uniform Load Employing T18 Elements[15]

Figure 17. Strain Energy - I for a S. S. Triangular Plate under Uniform Load Employing T18 Elements[15]
Figure 18. Frequency Parameter $\lambda_{33}$ for a S. S. Square Thick Plate Employing LDT Elements[9]

Figure 19. Frequency Parameter $\lambda_{13}$ for a Clamped Square Thick Plate Employing LDT Elements[9]
Figure A-1. Convergence Curve for $\pi_n^*$
A. K. RAO, India

Dr. Lynn has made a very valuable contribution. I am pleased to see his presentation as we, at Bangalore, have also been seeking simple methods of generating both upper and lower kernels from a single formulation for the analysis. Over a period of years, we have successfully pursued this line of thought in respect of both analytical and finite element solutions for local values such as stresses and for global quantities like eigenvalues. In the finite element analysis for eigenvalues, we achieve changes in converging sequence (from one type to the other) by introducing a simple geometric parameter into the element description and perturbing it suitably. A part of this work is to appear in the International Journal of Numerical Methods in Engineering (Bounds for eigenvalues in some vibration and stability problems, by G. V. Rao, A. V. Krishna Murty and A. K. Rao).