Bending Analysis of Shallow Spherical Shells by BEM

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INTRODUCTION

The problem of investigation of the stress-strain state of shells and plates is a problem of importance for structures. This problem has been established by finite element method, asymptotical analysis, etc. In the present paper the boundary element technique (Tosaka and Miyake, I983) is used as a tool for numerical analysis of elastic shell bending problem.

Some authors (Tottenham, I968; Newton and Tottenham, I968) have used the Vekua's theory of ellyptic partial differential equations for determination of the solution and corresponding integral equations formulation to apply it to shallow shells.

Using well known linear shallow shell theory for determination of normal displacement w and the membrane stress function f (or their complex combination - the function ψ , the inverse formulation for single fundamental equation and the fundamental solution for a weighting function) the new coupled set of integral equations is applied to the determination of the functions w and f by BEM.

SYMMETRIC BENDING OF SHALLOW SPHERICAL SHELLS

We follow Timoshenko and Woinowsky-Krieger (I963).

Let the midle surface of considered shell (Fig.I) is given by equation:

$$z = \sqrt{a^2-r^2} - (a-z_0) \qquad (I)$$

and let

$$\frac{dz}{dr} \sim -\frac{r}{a}$$

The equilibrium equations are:

$$\frac{d(rN_r)}{dr} - N_\theta - \frac{r}{a} Q_r + rp_r = 0 \qquad (2)$$

$$\frac{d(rQ_r)}{dr} - \frac{r}{a}(N_r+N_\theta) + rp = 0 \tag{3}$$

$$\frac{d(rM_r)}{dr} - M_\theta - rQ_r = 0 \tag{4}$$

where p and p_r are the intensitet of the loading in the normal and meridional directions; N_r, N_θ, M_r, M_θ, Q_r are the stresses, bemding moments and the shear force of the shell, respectively; a is the radius of curvature of shell surface. The following relations between the stresses and strains are valid:

$$\begin{aligned} \varepsilon_r &= \frac{1}{Eh}(N_r - \nu N_\theta) = \frac{dv}{dr} - \frac{w}{a} \\ \varepsilon_\theta &= \frac{1}{Eh}(N_\theta - \nu N_r) = \frac{v}{r} - \frac{w}{a} \end{aligned} \tag{5}$$

$$\begin{aligned} M_r &= -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) \\ M_\theta &= -D\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) \end{aligned} \tag{6}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \quad ; \tag{7}$$

u,v are the displacements in the shell midle surface; h is the thickness of the shell; E and ν are Young's modulus and Poisson's ratio, respectively.

Introdicing the membrane stress function f and the normal displacement w the equilibrium equations take the following form:

$$(\Delta\Delta\psi - i\varepsilon^2\Delta\psi)\frac{Eh}{a^2} = \varepsilon^4 p \tag{8}$$

where

$$\psi = w - \lambda f \quad ; \quad \varepsilon^2 = \frac{\sqrt{12(1-\nu^2)}}{ah}$$

Now, let complex function $\tilde{\psi}$ be:

$$\tilde{\psi} = \tilde{w} + i\tilde{f} \tag{9}$$

where

$$\tilde{w} = w\frac{Eh}{a^2} \quad ; \quad \tilde{f} = -\frac{\sqrt{12(1-\nu^2)}}{a^2h} f$$

This leads to following complex equation:

$$\nabla^2\nabla^2\tilde{\psi} - i\,\varepsilon^2\nabla^2\tilde{\psi} - \varepsilon^4 p = 0 \qquad (10)$$

The corresponding inverse method of weighted residual statement w^* gives:

$$\int_\Omega \left[\nabla^2(\nabla^2\tilde{\psi} - i\,\varepsilon^2\tilde{\psi}) - \varepsilon^4 p\right] w^* d\Omega = 0 \qquad (11)$$

With the aid of Green's theorem we obtain:

$$\int_\Omega (\nabla^2\nabla^2 w^* - i\,\varepsilon^2\nabla^2 w^*)\tilde{\psi}\, d\Omega - \varepsilon^4\int_\Omega p w^* d\Omega +$$
$$+\int_\Gamma \left[w^*(\nabla^2\tilde{\psi})_{,\nu} - \tilde{\psi}(\nabla^2 w^*)_{,\nu}\right] d\Gamma + \int_\Gamma \left[(\nabla^2 w^*)\tilde{\psi}_{,\nu} - (\nabla^2\tilde{\psi}) w^*_{,\nu}\right] d\Gamma -$$
$$- i\,\varepsilon^2\int_\Gamma \left[w^*\tilde{\psi}_{,\nu} - \tilde{\psi} w^*_{,\nu}\right] d\Gamma = 0 \qquad (12)$$

The function w^* is in fact the fundamental solution of the following equation:

$$\nabla^2\nabla^2 w^* - i\varepsilon^2\nabla^2 w^* = \delta(\vec{x} - \vec{x}_i) \qquad (13)$$

where $\delta(\vec{x}-\vec{x}_i)$ is two-dimensional delta function of Dirac. As usual w^* is a function of two points: the source point $\vec{x}_i$ where delta function has a singularity and the field point $\vec{x}$ which is the independent variable for the considered equation.

Applying Fourie transform to equation (13) we obtain the fundamental solution as:

$$w^* = \frac{i}{2\pi\varepsilon^2}\left[\ln(\varepsilon r) + K_0(\sqrt{i}\,\varepsilon r)\right] = \theta^* + i\chi^* \qquad (14)$$

Here K_0 is the modified second order Bessel function. Finally, for the real and imaginary parts of w^* we have:

$$\theta^* = -\frac{1}{2\pi\varepsilon^2}\,\mathrm{kei}(\varepsilon r)$$
$$\chi^* = \frac{1}{2\pi\varepsilon^2}\left[\ln(\varepsilon r) + \mathrm{ker}(\varepsilon r)\right] \qquad (15)$$

Putting equation (13) into equation (12) and using the property of delta function we obtain the following boundary integral equation for the complex function $\tilde{\psi}$ and the fundamental solution:

$$c\tilde{\psi} = \varepsilon^4\int_\Omega p w^* d\Omega + i\varepsilon^2\int_\Gamma (w^*\tilde{\psi}_{,\nu} - \tilde{\psi} w^*_{,\nu})\, d\Gamma +$$
$$+\int_\Gamma \left[\tilde{\psi}(\nabla^2 w^*)_{,\nu} - w^*(\nabla^2\tilde{\psi})_{,\nu} + (\nabla^2\tilde{\psi}) w^*_{,\nu} - (\nabla^2 w^*)\tilde{\psi}_{,\nu}\right] d\Gamma \qquad (16)$$

where coefficient c depends on the geometry of the boundary

The second equation is derived using the differentiation of (I6) with respect to the normal ν_o at the source point on the boundary of shell. We have:

$$c\tilde{\psi}_{,\nu_o} = \varepsilon^4\int_\Omega p w^*_{,\nu_o} d\Omega + i\varepsilon^2\int_\Gamma (w^*_{,\nu_o}\tilde{\psi}_{,\nu} - \tilde{\psi} w^*_{,\nu\nu_o}) d\Gamma + \int_\Gamma \left[\tilde{\psi}(\nabla^2 w^*)_{,\nu\nu_o} - w^*_{,\nu_o}(\nabla^2\tilde{\psi})_{,\nu} + (\nabla^2\tilde{\psi}) w^*_{,\nu\nu_o} - (\nabla^2 w^*)_{,\nu_o}\tilde{\psi}_{,\nu}\right] d\Gamma \qquad (I7)$$

where $w^* = \theta^* + i\chi^*$; $\tilde{\psi} = \tilde{w} + i\tilde{f}$.

We suppose that $\tilde{w}$ and $\tilde{f}$ satisfy the following boundary conditions:

$$\tilde{w} = \tilde{f} = 0 \quad ; \quad \nabla^2\tilde{w} = \nabla^2\tilde{f} = 0$$

Then the coupled system of boundary integral equations takes the form:

$$\varepsilon^4\int_\Omega p\theta^* d\Omega - \varepsilon^2\int_\Gamma \chi^*\tilde{w}_{,\nu} d\Gamma - \varepsilon^2\int_\Gamma \theta^*\tilde{f}_{,\nu} d\Gamma - \int_\Gamma \theta^*(\nabla^2\tilde{w})_{,\nu} d\Gamma + \int_\Gamma \chi^*(\nabla^2\tilde{f})_{,\nu} d\Gamma - \int_\Gamma (\nabla^2\theta^*)\tilde{w}_{,\nu} d\Gamma + \int_\Gamma (\nabla^2\chi^*)\tilde{f}_{,\nu} d\Gamma = 0$$

$$\varepsilon^4\int_\Gamma p\chi^* d\Omega + \varepsilon^2\int_\Gamma \theta^*\tilde{w}_{,\nu} d\Gamma - \varepsilon^2\int_\Gamma \chi^*\tilde{f}_{,\nu} d\Gamma - \int_\Gamma \chi^*(\nabla^2\tilde{w})_{,\nu} d\Gamma - \int_\Gamma \theta^*(\nabla^2\tilde{f})_{,\nu} d\Gamma - \int_\Gamma (\nabla^2\theta^*)\tilde{f}_{,\nu} d\Gamma - \int_\Gamma (\nabla^2\chi^*)\tilde{w}_{,\nu} d\Gamma = 0 \qquad (I8)$$

$$c\tilde{w}_{,\nu_o} - \varepsilon^4\int_\Omega p\theta^*_{,\nu_o} d\Omega + \varepsilon^2\int_\Gamma \chi^*_{,\nu_o}\tilde{w}_{,\nu} d\Gamma + \varepsilon^2\int_\Gamma \theta^*_{,\nu_o}\tilde{f}_{,\nu} d\Gamma + \int_\Gamma \theta^*_{,\nu_o}(\nabla^2\tilde{w})_{,\nu} d\Gamma - \int_\Gamma \chi^*_{,\nu_o}(\nabla^2\tilde{f})_{,\nu} d\Gamma + \int_\Gamma (\nabla^2\theta^*)_{,\nu_o}\tilde{w}_{,\nu} d\Gamma - \int_\Gamma (\nabla^2\chi^*)_{,\nu_o}\tilde{f}_{,\nu} d\Gamma = 0$$

$$c\tilde{f}_{,\nu_o} - \varepsilon^4\int_\Omega p\chi^*_{,\nu_o} d\Omega - \varepsilon^2\int_\Gamma \theta^*_{,\nu_o}\tilde{w}_{,\nu} d\Gamma + \varepsilon^2\int_\Gamma \chi^*_{,\nu_o}\tilde{f}_{,\nu} d\Gamma + \int_\Gamma \chi^*_{,\nu_o}(\nabla^2\tilde{w})_{,\nu} d\Gamma + \int_\Gamma \theta^*_{,\nu_o}(\nabla^2\tilde{f})_{,\nu} d\Gamma + \int_\Gamma (\nabla^2\chi^*)_{,\nu_o}\tilde{w}_{,\nu} d\Gamma + \int_\Gamma (\nabla^2\theta^*)_{,\nu_o}\tilde{f}_{,\nu} d\Gamma = 0$$

We introduce the following expressions for the matrix coefficients at the unknown variables $\tilde{f}_{,\nu}$; $\tilde{w}_{,\nu}$; $(\nabla^2\tilde{f})_{,\nu}$; $(\nabla^2\tilde{w})_{,\nu}$:

$$[\theta^1_{ij}] = \int_\Gamma \theta^* d\Gamma \ ; \ [\theta^2_{ij}] = \int_\Gamma \theta^*_{,\nu_o} d\Gamma \ ; \ [o^3_{ij}] = \int_\Gamma \nabla^2\theta^* d\Gamma \ ; \ [\theta^4_{ij}] = \int_\Gamma (\nabla^2\theta^*)_{,\nu_o} d\Gamma$$

$$[\chi^1_{ij}] = \int_\Gamma \chi^* d\Gamma \ ; \ [\chi^2_{ij}] = \int_\Gamma \chi^*_{,\nu_o} d\Gamma \ ; \ [\chi^3_{ij}] = \int_\Gamma \nabla^2\chi^* d\Gamma \ ; \ [\chi^4_{ij}] = \int_\Gamma (\nabla^2\chi^*)_{,\nu_o} d\Gamma$$

$$[Q^1] = \int_\Omega \varepsilon^4 p\theta^* d\Omega \ ; \ [Q^2] = \int_\Omega \varepsilon^4 p\chi^* d\Omega \ ; \ [Q^3] = \int_\Omega \varepsilon^4 p\theta^*_{,\nu_o} d\Omega \ ; \ [Q^4] = \int_\Omega \varepsilon^4 p\chi^*_{,\nu_o} d\Omega$$

Finally, the linear system for determination of unknown variables at nodes is obtained:

$$
\begin{aligned}
&(\varepsilon^2[\chi^1_{ij}]+[\theta^3_{ij}])\{\tilde{w}_{,\nu}\}_j + (\varepsilon^2[\theta^1_{ij}]-[\chi^3_{ij}])\{\tilde{f}_{,\nu}\}_j + \\
&\qquad + [\theta^1_{ij}]\{(\nabla^2\tilde{w})_{,\nu}\}_j - [\chi^1_{ij}]\{(\nabla^2\tilde{f})_{,\nu}\}_j = [Q^1]_i \\
&([\chi^3_{ij}]-\varepsilon^2[\theta_{ij}])\{\tilde{w}_{,\nu}\}_j + (\varepsilon^2[\chi^1_{ij}]+[\theta^3_{ij}])\{\tilde{f}_{,\nu}\}_j + \\
&\qquad + [\chi^1_{ij}]\{(\nabla^2\tilde{w})_{,\nu}\}_j + [\theta^1_{ij}]\{(\nabla^2\tilde{f})_{,\nu}\}_j = [Q^2]_i \\
&\{c\tilde{w}_{,\nu}\}_i + (\varepsilon^2[\chi^2_{ij}]+[\theta^4_{ij}])\{\tilde{w}_{,\nu}\}_j + (\varepsilon^2[\theta^2_{ij}]-[\chi^4_{ij}])\{\tilde{f}_{,\nu}\}_j + \\
&\qquad + [\theta^2_{ij}]\{(\nabla^2\tilde{w})_{,\nu}\}_j - [\chi^2_{ij}]\{(\nabla^2\tilde{f})_{,\nu}\}_j = [Q^3]_i \\
&\{c\tilde{f}_{,\nu}\}_i + ([\chi^4_{ij}]-\varepsilon^2[\theta^2_{ij}])\{\tilde{w}_{,\nu}\}_j + (\varepsilon^2[\chi^2_{ij}]+[\theta^4_{ij}])\{\tilde{f}_{,\nu}\}_j + \\
&\qquad + [\chi^2_{ij}]\{(\nabla^2\tilde{w})_{,\nu}\}_j + [\theta^2_{ij}]\{(\nabla^2\tilde{f})_{,\nu}\}_j = [Q^4]_i
\end{aligned}
\qquad (20)
$$

We note here that we adopt the linear approximation for the variables. The integrals are calculated analytically using the finite series for the fundamental solutions. The case of singular kernel is calculated by means of Cauchy principal value.

NUMERICAL EXAMPLE

We consider a shallow spherical shell with circular base with radius R=4 m (Fig.2), radius of curvature a=25, 50, 100 m, thickness h=0.15 m, clamped on the ends. The following values are used: $E=2.1\times10^6$ t/m^2 ; ν=0; 0.3 ; p=0.1 t/m^2.

The midle surface of the considered shell is divided into 63 internal cells and 18 boundary elements.

The distribution of the normal displacement w and the value of stress function f on the axis of symmetry of shell are given on Fig.3 and Table I, respectively.

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Tosaka, N. and Miyake, S. (1983). A Boundary Integral Equation Formulation for Elastic Shallow Shell Bending Problems, Proc. 5th Int. Conf. on Boundary Elements, Hiroshima, Japan, pp. 527-538.

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Table I

f / R [m]	a=25 m		a=50 m		a=I00 m	
	ν=0	ν=0.3	ν=0	ν=0.3	ν=0	ν=0.3
4	0.0	0.0	0.0	0.0	0.0	0.0
3	-5.03	-5.42	-2.69	0.39	-I33.0	-I69.0
2	-7.82	-8.3I	-0.93	3.85	-I96.0	-250.0
I	-I0.2	-I0.5	I.02	6.59	-222.0	-283.0
0	-II.2	-II.3	I.77	7.56	-223	-293

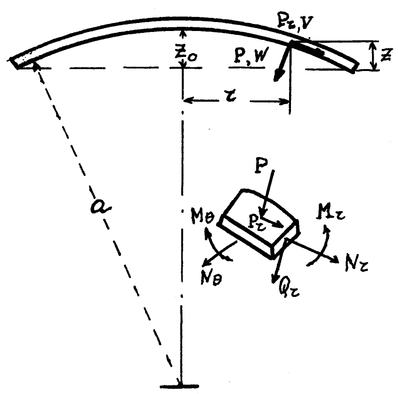

Fig.I

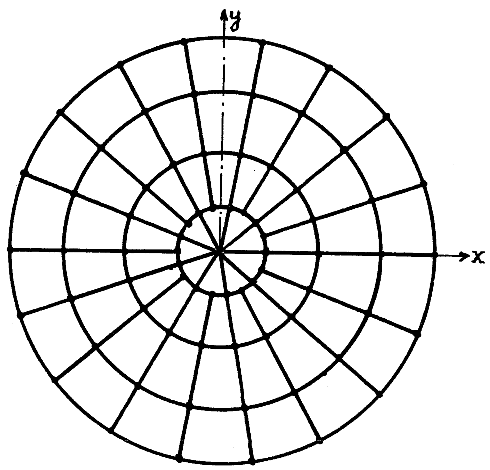

Fig.2

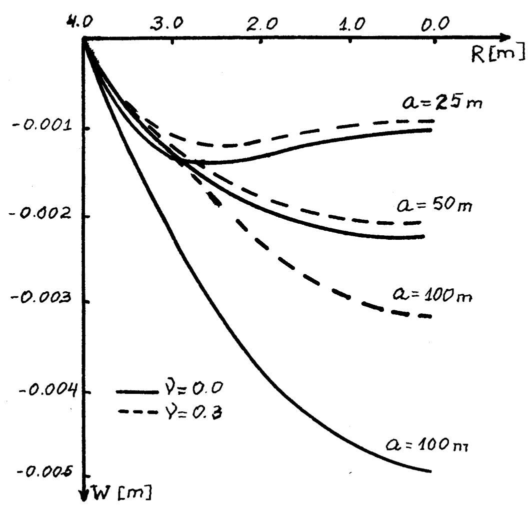

Fig.3