Inelastic Analyses of Axisymmetrically Loaded Circular Plates

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1 INTRODUCTION

We consider in this paper the deformation of inelastic circular plates subjected to axisymmetric loading. It is assumed that the material of the plate deforms elasto–viscoplastically. Further, it is presupposed that the total strains remain small throughout the entire deformation process. However, the deflections and strains are physically nonlinear and history dependent. We assume that the viscoplastic part of the deformation is governed by a constitutive model with internal state variables (e.g. Anand 1985, Hart 1976, Miller 1976a, 1976b, Walker 1981).

A comprehensive study of inelastically deformed square and triangular plates using a boundary element formulation is based on Kirchhoff's plate theory and Hart's inelastic model (Morjarja and Mukherjee 1980). However, only pure loading of the plates is considered. Hold times or unloading are not included. Further an explicit time integration algorithm is used (Kumar et al. 1977) which is only conditionally stable. Therefore, one equation of Hart's model had to be replaced by the so called viscoplastic limit (Kumar and Mukherjee 1977). In this paper a circular disk loaded axisymmetricall is considered. A numerical example is based on Hart's constitutive model without using this viscoplastic approximation.

2 ANALYSIS

The following basic assumptions for the underlying constitutive model are introduced. The total strain rate tensor can be decomposed additively into a purely elastic and into an inelastic part.

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^{e}_{ij} + \dot{\epsilon}^{n}_{ij} \tag{1}$$

For the inelastic strain rate tensor exist evolution equations of the form

$$\dot{\epsilon}^{n}_{ij} = \dot{\epsilon}^{n}_{ij}\left[\sigma_{kl}, q^{(k)}_{kl}, T\right] \tag{2}$$

Here σ_{kl} are the components of the stress tensor, $q^{(k)}_{kl}$ is a set of suitably selected but otherwise unspecified state variables and T is the absolute temperature. It is generally accepted that the inelastic deformation is incompressible and therefore

$$\dot{\epsilon}^{n}_{kk} = 0 \tag{3}$$

It is further stipulated that the internal state variables obey evolution equations

$$\dot{q}^{(k)}_{ij} = \dot{q}^{(k)}_{ij}\left[\sigma_{kl}, q^{(n)}_{kl}, T\right] \tag{4}$$

It can be shown that the constitutive models cited in section 1 fit into the mathematical frame of equations (1) to (4). It is further assumed that information on the values of the inelastic strains and of the state variables at time t = 0 is available. The loading history from the unloaded state begins at t = 0.

We assume a circular plate of uniform thickness h. We introduce a cylindrical coordinate system with radius r and axial coordinate z. We presuppose that the plate thickness h is small in comparison to the radial extensions of the plate (thin plate theory). Further, the load is assumed as axisymmetric.

The total radial and azimuthal strains ϵ_{rr} and ϵ_{zz} are principal strains and the same conclusion holds for the respective stresses. We denote the axial deflection of the plate by w. We suppose that the assumptions of the classical Kirchhoff plate theory are valid. Then we can apply the following kinematic relations (Timoshenko and Woinowsky–Krieger 1959)

$$\epsilon_{rr} = -z\frac{d^2w}{dr^2} \tag{5}$$

$$\epsilon_{\varphi\varphi} = -\frac{z}{r}\frac{dw}{dr} \tag{6}$$

Due to the thin plate assumption ($\sigma_{zz} = 0$), a state of plane stress can be presupposed. Then Hooke's law takes the form

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[\epsilon_{rr} + \nu\,\epsilon_{\varphi\varphi} - (\epsilon^n_{rr} + \nu\,\epsilon^n_{\varphi\varphi})\right] \tag{7}$$

$$\sigma_{\varphi\varphi} = \frac{E}{1-\nu^2}\left[\epsilon_{\varphi\varphi} + \nu\,\epsilon_{rr} - (\epsilon^n_{\varphi\varphi} + \nu\,\epsilon^n_{rr})\right] \tag{8}$$

Here E denotes Young's modulus and ν Poisson's ratio.

We introduce the plate bending moments

$$M_{rr} = \int_{-h/2}^{h/2} \sigma_{zz} z\,dz \tag{9}$$

$$M_{\varphi\varphi} = \int_{-h/2}^{h/2} \sigma_{\varphi\varphi} z\,dz \tag{10}$$

Combining equations (9), (10), (5), (6), (7) and (8) yields

$$M_{rr} = -D\left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right] - \frac{E}{1-\nu^2}\int_{-h/2}^{h/2} (\epsilon^n_{rr} + \nu\,\epsilon^n_{\varphi\varphi}) z\,dz \tag{11}$$

and

$$M_{\varphi\varphi} = -D\left[\frac{1}{r}\frac{dw}{dr} + \nu\frac{dw^2}{dr^2}\right] - \frac{E}{1-\nu^2}\int_{-h/2}^{h/2}(\epsilon^n_{\varphi\varphi} + \nu\,\epsilon^n_{rr})zdz \tag{12}$$

Here D denotes the plate bending stiffness

$$D := \frac{1}{12}\frac{Eh^3}{1-\nu^2} \tag{13}$$

We assume that the plate is loaded by an axisymmetric line force Q(r) and take the following equation of equilibrium (Timoshenko and Woinowsky–Krieger 1959)

$$\frac{dM_{rr}}{dr} + \frac{1}{r}(M_{rr} - M_{\varphi\varphi}) + Q(r) = 0 \tag{14}$$

We further assume that the plate is loaded by an axisymmetric surface load p(r). Then the shearing force Q(r) is given by

$$Q(r) = \frac{1}{r}\int_{r_i}^{r} p(\rho)\rho d\rho \tag{15}$$

Next, the expressions (11) and (12) are inserted into equ.(14).

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} = \frac{Q(r)}{D} + \frac{f(r)}{r} - \frac{dg(r)}{dr} \tag{16}$$

Here the inelastic functions f(r) and g(r) are defined as

$$f(r) := \frac{12(1-\nu)}{h^3}\int_{-h/2}^{h/2}(\epsilon^n_{\varphi\varphi} - \epsilon^n_{rr})\,zdz \tag{17}$$

$$g(r) := \frac{12}{h^3}\int_{-h/2}^{h/2}(\epsilon^n_{rr} + \nu\,\epsilon^n_{\varphi\varphi})\,zdz \tag{18}$$

If the inelastic functions f(r) and g(r) are set equal to zero, the governing equation of the classical Kirchhoff theory is recovered from equ.(16).

The general solution of the linear differential equation (16) takes the following form

$$w(r) = w_n(r) + w_p(r) + w_n(r) \tag{19}$$

Here $w_n(r)$ is the general solution of the homogeneous part of equ.(16).

$$w_n(r) = A_1 + A_2\,r^2 + A_3\ln\frac{r}{r_o} \tag{20}$$

A_1, A_2 and A_3 are integration constants to be determined from the boundary conditions. A particular solution $w_p(r)$ can easily be derived as

$$w_p(r) = \frac{pr^4}{64D} - \frac{pr^2\ r_i^2}{8D}\left[\ln\frac{r}{r_o} - 1\right] \tag{21}$$

A particular solution $w_n(r)$ can be found by the method of variation of parameters in analogy to the problem of an inelastically deformed cylindrical shell (Kollmann and Mukherjee 1984). For the inelastic plate the following particular solution can be derived

$$w_n(r) = F_1(r) + F_2(2)\, r^2 + F_3(r) \ln\frac{r}{r_o} \tag{22}$$

The functions $F_1(r)$, $F_2(r)$ and $F_3(r)$ are given as

$$F_1(r) = -\frac{1}{4}\left\{\int_{r_i}^{r}\left[(1 - 2\ln\frac{\rho}{r_o}\, f(\rho) - 4g(\rho)\ln\frac{\rho}{r_o}\right]\rho d\rho \right.$$
$$\left. -g(r)r^2(1 - 2\ln\frac{r}{r_o} + g(r_i)r_i^2(1 - 2\ln\beta)\right\} \tag{23}$$

$$F_2(r) = \frac{1}{4}\left[\int_{r_i}^{r}\frac{f(\rho)}{\rho}\,d\rho - g(r) + g(r_i)\right] \tag{24}$$

$$F_3(r) = -\frac{1}{2}\left\{\int_{r_i}^{r}\left[f(\rho) + 2g(\rho)\right]\rho d\rho - r^2\, g(r) + r_i^2\, g(r_i)\right\} \tag{25}$$

where

$$\beta : = \frac{r_i}{r_o} \tag{26}$$

In this paper we adjust our solution only to one set of boundary conditions. We assume an annular plate with clamped outer and free inner boundary. Then the following boundary conditions have to be met

$$w(r_o) \quad = \quad 0 \tag{27}$$

$$\left.\frac{dw}{dr}\right|_{r_o} = \quad 0 \tag{28}$$

$$M_{rr}(r_i) \; = \; 0 \tag{29}$$

These boundary conditions yield the following integration constants

$$A_1 = \frac{1}{N}\left\{ \frac{p r_o^4}{16D}\left[1-\nu+2(1-\nu)\beta^2+(1+3\nu)\beta^4-4(1+\nu)\beta^4 \ln\beta\right] + 2(1-\nu) r_o^2 F_1(r_o) + (1-\nu) F_2(r_o) + r_o^2 \beta^2 g(r_i) \right\} - \frac{p r_o^4}{64D} - \frac{p r_o^4 \beta^2}{8D} - r_o^2 F_1(r_o) - F_3(r_o) \tag{30}$$

$$A_2 = -\frac{1}{N}\left\{ \frac{p r_o^2}{16D}\left[(1-\nu)+(1+3\nu)\beta^4+2(1-\nu)\beta^2-4(1+\nu)\beta^4 \ln\beta\right] + 2(1-\nu) F_1(r_o) + \frac{1-\nu}{r_o^2} F_2(r_o) + \beta^2 g(r_i) \right\} \tag{31}$$

$$A_3 = -\frac{1}{N}\left\{ \frac{p r_o^4}{16D}\left[2(1+\nu)\beta^2+8(1+\nu)\beta^4 \ln\beta+2(1-\nu)\beta^4\right] + 4(1+\nu)\beta^2 r_o^2 F_1(r_o) + 2(1+\nu)\beta^2 F_2(r_o) - 2\beta^2 r_o^2 g(r_i) \right\} \tag{32}$$

where

$$N := 2(1-\nu) + 2(1+\nu)\beta^2 \tag{33}$$

3 TIME INTEGRATION AND CONSTITUTIVE MODEL

Equation (19) in combination with eqns.(21), (22), (23), (24, (25), (30), (31) and (32) is the formal solution of a genuinely nonlinear boundary value problem. The nonlinearity stems from the functions f(r) and g(r) which depend on the accumulated inelastic strains due to eqns.(17) and (18). The evolution eqns.(2) and (3) constitute, in combination with suitably prescribed initial values, an initial value problem. Therefore, an initial–boundary–value problem has to be solved. It proves useful to include all physical nonlinearities into an "extended" initial value problem (Kumar and Mukherjee 1977). For this purpose eqns.(7), (8), (16), (17) and (18) are differentiated formally with respect to time. Then the general solution (19) depends on the pressure rate $\dot{p}$ and the inelastic strain rates $\dot{\epsilon}^n_{rr}$ and $\dot{\epsilon}^n_{\varphi\varphi}$.

We apply the following general solution strategy for the inelastic initial–boundary–value problem (Mukherjee 1982). Given the loading history p(t), the first step is the calculation of the purely elastic solution at zero time. The stresses at zero time (together with the internal state variables if any) are used in the inelastic constitutive equations to compute the initial nonelastic strain rates and the rates of the internal state variables. Then the initial time derivatives of the displacement and the stresses can be computed from the differentiated eqns.(19), (7) and (8). Next, these derivatives and the rates of the inelastic strains and internal state variables are used to obtain the relevant integrated variables throughout the plate after a small time increment Δt. This process is continued until the time histories of all relevant variables are determined throughout the plate as functions of position and time. The parameter integrals in eqns.(23), (24) and (25) are evaluated numerically by the trapezoidal rule.

The time integration has to be performed numerically due to the nonlinearity of the constitutive model. Explicit time integration schemes (Banthia and Mukherjee 1985) are simple to implement. But they suffer from the drawback that they are only conditionally stable. For the present work an implicit time integration algorithm is applied (Cordts and Kollmann 1986).

In this study viscoplastic deformation of the plate is described by Hart's constitutive model. Since this model has been presented repeatedly in the literature (e.g. Hart 1976, Mukherjee 1982, Kollmann 1987) it is omitted here for the sake of brevity. In our work we assumed a plate of stainless steel 304 at a homogeneous temperature of 400° C. The material parameters for Hart's model have been reported elsewhere (Mukherjee 1982, Kollmann 1987). We assumed that the initial values of all inelastic strains are zero and that the plate has a homogeneous initial hardness of $\sigma^* = 117.26$ MPa.

4 RESULTS AND DISCUSSION

We consider a circular plate with 0.25 m outer radius, 0.05 m inner radius and 0.013 m thickness. The pressure p increases linearly with time in 0.2 s from zero to the maximal pressure of 1.014 MPa. Next a hold time of 10 s is added. Finally, the solution is traced under zero pressure until a total elapsed time of 15.0 s. This loading history is depicted schematically in Fig. 1.

Fig. 1 shows the distribution of the deflection w over the radius s for different times. For comparison the purely elastic solution under maximal pressure is also plotted (which for the sake of brevity will be called the "elastic" solution in the sequel). It can be seen clearly that at the end of the loading time (t = 0.2 s) the deflection of the inelastic plate at the inner radius is more then twice of the elastic one. Further it can be observed that substantial viscoplastic flow occurs under constant load (compare curves for t = 0.2 and t = 10.2 s). Analogously viscoplastic deformation continues between t = 10.4 s and t = 15.0 s where the plate is completely unloaded. The deflection of the unloaded plate is at t = 15.0 s still larger than the elastic one.

Fig. 2 shows the radial dependence of the hoop stress at a height of z = –6.06 mm for different times in comparison with the elastic solution. The hoop stress takes its maximal value for all times at the inner edge of the plate. Remarkably the maximal elastic stress is larger then the corresponding viscoplastic stress (t = 0.2). During the hold time a redistribution of the stresses can be noticed. Finally, the residual stress in the completely unloaded plate (t = 15.0 s) is plotted. In Fig. 3 the distribution of the hoop stress over the plate thickness at the inner edge (r = 50 mm) is presented. Again for comparison the elastic solution is depicted. The nonlinear effects and the redistribution of the stress field due to viscoplastic effects are distinct. Also the residual stress field at t = 15.0 s is depicted. It has to be emphasized that the residual stress field is obtained as a part of the solution by tracing the loading history until the plate is unloaded. No yield criterion nor a loading–unloading condition has to be considered in Harts model.

5 CONCLUSIONS

A semianalytic solution has been derived for the analysis of viscoplastic circular plates under axisymmetric loading. For the general solution it is assumed that the total strain rate tensor can be decomposed additively into an elastic and an inelastic part. The inelastic deformation is governed by a set of evolution equations using internal state variables. For a numerical example Hart's constitutive model is used. From the computations the dominating influence of the viscoplastic effects over the elastic ones can be clearly seen. We feel that our results can be used as a benchmark for viscoplastic finite element computations.

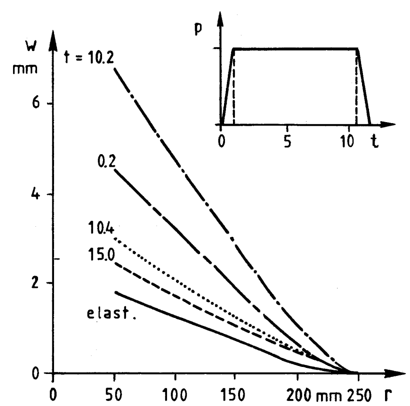

Figure 1: Radial distribution of deflection for different times

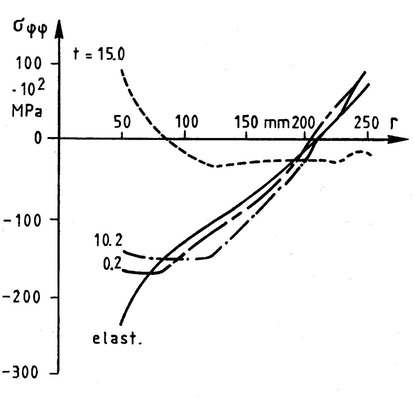

Figure 2: Radial distribution of hoop stress at z = – 6.06 mm for different times

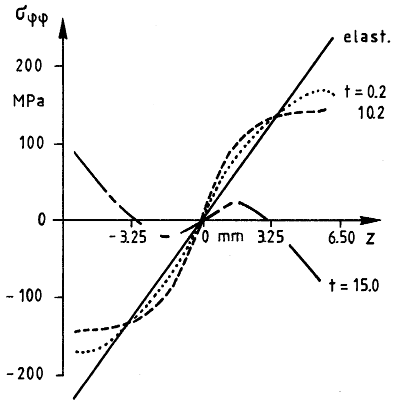

Figure 3: Distribution of hoop stress over the plate thicknesses at bore of the plate

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