An Implicit Elastoplastic Beam Element for the Study of Frames Subjected to Dynamic Loads

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1. Introduction

The elastoplastic beam element that is presented here is intended primarily for modeling inelastic effects in reinforced concrete columns for buildings, with particular emphasis on three-dimensional behavior. This inelastic behavior is defined for the cross section of the beam as a whole, rather than for individual fibers ("lumped plasticity")

The main characteristics of this element are :

- The elastic beam properties are defined by an axial stiffness, two flexural stifnesses, a torsional stiffness and two shear rigidities (if shear deformation is taken into account)

- Inelastic behavior is confined to zero-length plastic hinges at the element ends. The part of the beam between the potential hinges is assumed to remain linearly elastic.

- Each hinge is initially rigid, so that the stiffness of the complete element is the stiffness of the elastic beam. As the moments and forces at the element ends increase, the hinges can yield, leading to a stiffness reduction in the element.

- At a hinge, the bending moments and the axial force interact with each other to produce an initial three-dimensional yield surface. Interaction between the torque and the shear forces are neglected.

- Tangent stiffness relationships between the actions and the deformations at a yielding hinge are established using a plasticity theory.

- Plastic flow is assumed to take place normal to the yield surface of the hinges.

- After first yield, the yield surfaces of the hinges are assumed to follow a kinematic hardening rule.

- If the actions at a hinge decrease, the hinge becomes rigid again and the beam unloads.

- Only small displacements are allowed, second order (P-Δ) theory is not considered.

Hereafter is described the way to obtain the elastoplastic stiffness matrix in the case of a static or a dynamic problem.

2. Yield surface of a reinforced concrete column

The global approach of the plasticity for a reinforced concrete section needs the definition of a yield surface in terms of limit internal forces. The most general formulation of a yield surface can be written :

$$f(N1,T1,T2,Mt,M1,M2)=0$$

where N is the axial force
Mt is the torque
T1, T2 are the shear forces along the principal axes
M1, M2 are the bending moments about the principal axes

In the case of a rectangular concrete column symetrically reinforced on the four sides, torque and shear forces are generally neglected and a very good approximation of the yield surface is obtained by (fig. 1) :

$$\sqrt{\left(\frac{M1}{M1\ max}\right)^2 + \left(\frac{M2}{M2\ max}\right)^2} + \left(\frac{2N - N^+ - N^-}{N^+ - N^-}\right)^2 - 1 = 0$$

where M1 max, M2 max are the maximum bending moments
N^+ is the maximum positive axial force (traction)
N^- is the maximum negative axial force (compression)

3. Plastic flow - Normality rule

Plastic flow is assumed to develop along the normal to the yield surface of the hinge (associated flow rule : see fig. 2)

$$dq_{p1} = d\alpha_1\ a_1 \qquad (3.1)$$
$$dq_{p2} = d\alpha_2\ a_2 \qquad (3.2)$$

where dq_{p1}, dq_{p2} are the plastic displacements
a_1 , a_2 is the direction of the outward normal to the yield surface
$d\alpha_1$, $d\alpha_2$ are positive scalar factors of proportionality which are nonzero only when plastic deformations occur
The form of the factors $d\alpha$ is discussed below.
Indexes 1 and 2 are relative to the hinge 1 and 2

After first yield, the yield surfaces of the hinges are assumed to translate in stress space without any change of shape or size (kinematic hardening rule : fig. 3), what can be expressed by :

$$ds_{01} = B_1\ dq_{p1} \qquad (3.3)$$
$$ds_{02} = B_2\ dq_{p2} \qquad (3.4)$$

where s_{01}, s_{02} are the coordinates of the center of the yield surface in stress space
B_1, B_2 are the work-hardening constants

4. Elastoplastic stiffness

4.1. Static analysis

For a beam, the incremental formulation of the elasticity law is written :

$$ds = K\ dq_e + ds_I \qquad (4.1)$$

where ds is the vector of the forces increments
K is the elastic stiffness matrix
dq_e is the vector of the elastic displacement increments
ds_I is the vector of the nodal forces increments due to the implicit forces on the beam

According to the fundamental elastoplastic theory, the total displacement q is decomposed into an elastic component q_e and an elastoplastic component $q_p = q_{p1} + q_{p2}$, leading with (4.1) to :

$$ds = K\ (dq - dq_{p1} - dq_{p2}) + ds_I \qquad (4.2)$$

To compute the elastoplastic stiffness matrix, it is assumed that the two ends of the beam are initially in a plastic state of stress (a point on the yield surface).

The consistency condition requires that the final state of stress remains on the yield surface :

$$df_1 = 0 \quad \text{or} \quad a_1^T\ (ds - ds_{01}) = 0 \qquad (4.3)$$
$$df_2 = 0 \quad \text{or} \quad a_2^T\ (ds - ds_{02}) = 0 \qquad (4.4)$$

Replacing (4.2) into (4.3) and (4.4), and taking account of (3.1), (3.2), (3.3), (3.4) implies that :

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{Bmatrix} d\alpha_1 \\ d\alpha_2 \end{Bmatrix} = \begin{Bmatrix} a_1^T\ (K\ dq + ds_I) \\ a_2^T\ (K\ dq + ds_I) \end{Bmatrix} \qquad \begin{matrix} (4.5) \\ (4.6) \end{matrix}$$

with :
$$\Omega_{11} = a_1^T\ (K + B_1)\ a_1$$
$$\Omega_{22} = a_2^T\ (K + B_2)\ a_2$$
$$\Omega_{12} = a_1^T\ K\ a_2 = a_1^T\ d_2$$
$$\Omega_{21} = a_2^T\ K\ a_1 = a_2^T\ d_1$$

Solving equations (4.5) and (4.6) leads to :

$$d\alpha_1 = (\mu_{11}\ a_1^T + \mu_{12}\ a_2^T)\ (K\ dq + ds_I) \qquad (4.7)$$
$$d\alpha_2 = (\mu_{22}\ a_2^T + \mu_{21}\ a_1^T)\ (K\ dq + ds_I) \qquad (4.8)$$

with :
$$\mu_{11} = +\ \Omega_{22}\ /\ (\Omega_{11}\ \Omega_{22} - \Omega_{12}\ \Omega_{21})$$
$$\mu_{22} = +\ \Omega_{11}\ /\ (\Omega_{11}\ \Omega_{22} - \Omega_{12}\ \Omega_{21})$$
$$\mu_{12} = -\ \Omega_{12}\ /\ (\Omega_{11}\ \Omega_{22} - \Omega_{12}\ \Omega_{21})$$
$$\mu_{21} = -\ \Omega_{21}\ /\ (\Omega_{11}\ \Omega_{22} - \Omega_{12}\ \Omega_{21})$$

With (4.7) and (4.8), the normality rule (3.1,3.2) becomes :

$$dq_{p1} = a_1\ (\mu_{11}\ a_1^T + \mu_{12}\ a_2^T)\ (K\ dq + ds_I) \qquad (4.9\ )$$
$$dq_{p2} = a_2\ (\mu_{22}\ a_2^T + \mu_{21}\ a_1^T)\ (K\ dq + ds_I) \qquad (4.10)$$

By introducing (4.9) and (4.10) into (4.2), the incremental elastoplastic law is finally obtained :

$$ds = K_p\, dq + T\, ds_I \qquad (4.11)$$

where the elastoplastic stiffness matrix is :

$$K_p = K - \mu_{11}\, d_1\, d_1^T - \mu_{22}\, d_2\, d_2^T - \mu_{12}\, d_1\, d_2^T - \mu_{21}\, d_2\, d_1^T$$

and the redistribution matrix is :

$$T = I - \mu_{11}\, d_1\, a_1^T - \mu_{22}\, d_2\, a_2^T - \mu_{12}\, d_1\, a_2^T - \mu_{21}\, d_2\, a_1^T$$

I is the unit matrix.

4.2. Dynamic analysis

For a transient response of a non linear structure, the dynamic equilibrium equation is :

$$r(q) = M\,(\ddot{q}(t) + \ddot{q}_G(t)) + g(q,\dot{q}) - Fext(t) = 0 \qquad (4.12)$$

where
- M is the consistent mass matrix
- $g(q,\dot{q})$ are the internally resisting forces
- $Fext(t)$ are the external forces
- $\ddot{q}_G(t)$ is the soil acceleration
- $q, \dot{q}, \ddot{q}$ are respectively the displacements, the velocities and the accelerations
- $r(q)$ is the unbalanced load vector

The term $g(q,\dot{q})$ describing the internal forces include a contribution corresponding to the elastic restoring forces and another one representing the internal dissipation of energy. If linearity is assumed with respect to displacements and velocities, the internal forces take the classical linear form :

$$g(q,\dot{q}) = K\, q(t) + C\, \dot{q}(t)$$

with K = the stiffness matrix
C = the viscous damping matrix

Equations (4.12) are solved for discrete times t_n.
Assuming that the equilibrium state (characterized by q_n, $\dot{q}_n$, $\ddot{q}_n$) is known at time t_n, the solution is sought at time t_{n+1}.
In Newmark's method, accelerations and velocities becomes functions of the displacements q_{n+1} at time t_{n+1} (4.13,4.14)

$$\dot{q}_{n+1} = \alpha/(\beta\ \delta t)\ (q_{n+1} - q_n) + (1 - \alpha/\beta)\ \dot{q}_n + (1 - 2\alpha/\beta)\ \delta t\ \ddot{q}_n$$

$$\ddot{q}_{n+1} = 1/(\beta\ \delta t^2)\ (q_{n+1} - q_n) - 1/(\beta\ \delta t)\ \dot{q}_n + (1 - 1/(2\beta))\ \ddot{q}_n$$

where α et β are the Newmark's constants
δt is the time step

After substitution the residual vector becomes an implicit function of q_{n+1} only : $r(q_{n+1}) = 0$

These equations are generally non linear and an equilibrium iteration sequence is required.

At time t_{n+1}, the unbalanced load vector is computed by :

$$r(q_{n+1}) = Fext(t_{n+1}) - s(q_{n+1})$$

where the external forces at time t_{n+1} are :

$$Fext(t_{n+1}) = Fext(t_n) + \delta Fext$$

the internal forces at time t_{n+1} are :

$$s(q_{n+1}) = s(q_n) + M\ \delta\ddot{q}_e + C\ \delta\dot{q}_e + K\ \delta q_e + M\ \delta\ddot{q}_G + \delta s_I$$

with : $\delta\ddot{q}_e = (\ddot{q}_{n+1} - \ddot{q}_n) - (\ddot{q}_{n+1} - \ddot{q}_n)_p$

$\delta\dot{q}_e = (\dot{q}_{n+1} - \dot{q}_n) - (\dot{q}_{n+1} - \dot{q}_n)_p$

$\delta\ddot{q}_G$ = the soil acceleration increment

If it is assumed that the dependance of $\dot{q}_{n+1p}$ and $\ddot{q}_{n+1p}$ with respect to $\dot{q}_{np}$ and $\ddot{q}_{np}$ is the same as (4.13) and (4.14), the internal forces at time t_{n+1} can be expressed by :

$$s(q_{n+1}) = s(q_n) + K^*\ (\delta q - \delta q_p) + \delta s_I^* \qquad (4.15)$$

where the dynamic elastic stiffness matrix is :

$K^* = K + 1/(\beta\ \delta t^2)\ M + \alpha/(\beta\ \delta t)\ C$

the vector of the incremental implicit forces is :

$\delta s_I^* = \delta s_I + M\ \delta\ddot{q}_G + k(\dot{q}_n, \ddot{q}_n, \dot{q}_{np}, \ddot{q}_{np})$

$k(\dot{q}_n, \ddot{q}_n, \dot{q}_{np}, \ddot{q}_{np})$ depends only on the initial state at time t_n

If equation (4.15) is compared with (4.2), it appears now that a dynamic problem can be considered just like a static one for which the matrix K and the vector ds_I are replaced respectively by K^* and δs_I^*. The resulting incremental elastoplastic law becomes :

$$ds = K_p^*\ dq + T^*\ ds_I^*$$

where the dynamic elastoplastic stiffness matrix is :

$K_p^* = K^* - \Sigma\ \mu_{ij}^*\ d_i^*\ d_j^{T*} \quad (i,j = 1,2)$

the dynamic redistribution matrix is :

$T^* = I - \Sigma\ \mu_{ij}^*\ d_i^*\ a_j^T \quad (i,j = 1,2)$

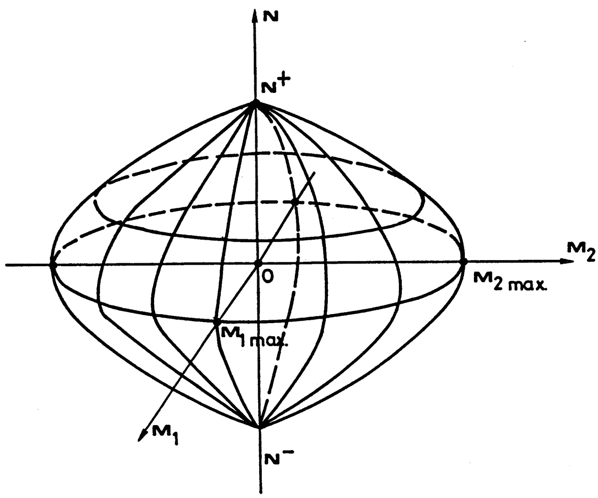
with $d_i^* = K^*\ a_i$

5. Applications

This elastoplastic beam element has been implanted in the non linear finite element program LAGADYN (ref. 1).
It has been successfully used to study the inelastic behavior of three-dimensional frames subjected to earthquakes and blast loads.

6. References

[1] LAGADYN : an implicit finite element program for the study frames subjected to dynamic loads
University of Liege - MSM - Belgium



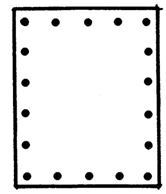


FIG. 1 YIELD SURFACE OF A REINFORCED CONCRETE COLUMN

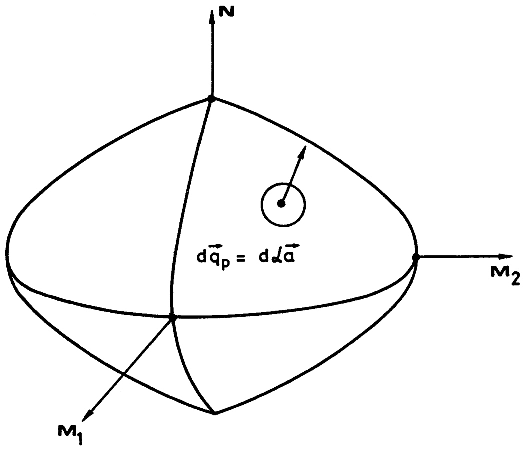


FIG. 2 ASSOCIATED FLOW RULE

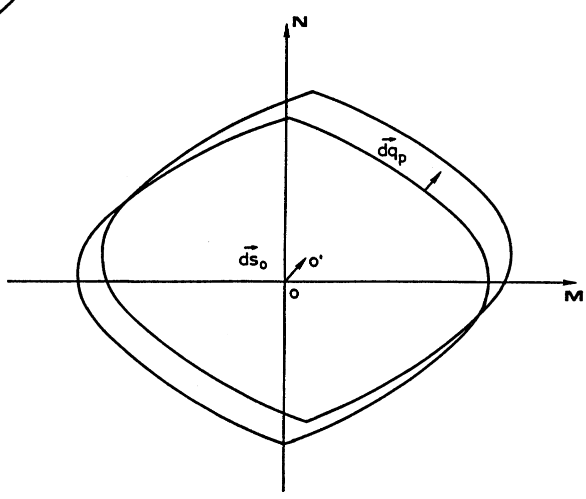


Fig. 3 KINEMATIC HARDENING