Applications of Mixed Galerkin and Petrov-Galerkin
Finite Element Methods to Creep Analysis

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INTRODUCTION

By perturbing the Galerkin approximation of Hellinger-Reissner principle Loula, Hughes, Franca and Miranda (1987) developed a mixed Petrov-Galerkin finite element method, for constrained problems, consisting in adding to the classical formulation least square residuals of the governing equations in the interior of the elements. It is a variationally consistent formulation with enhanced stability properties and Brezzi's theorem (Brezzi, 1974) can be used to prove existence, uniqueness and convergence in a mesh-dependent norm.

Internal constraints, like incompressibility, are usually present in inelastic problems like in creep or plastic flows. In this case, due to the nonlinearity Brezzi's theorem on mixed methods is not applicable. More general results on the analysis of saddle point problems can be found in Scheurer (Scheurer, 1977) where two versions of Brezzi's theorem in Banach spaces are presented covering nonlinear variational equations or inequations under linear constraints.

In (Loula and Guerreiro, 1987, Guerreiro and Loula, 1988 and Guerreiro, 1988) we studied steady state and transient creep problems using the classical Galerkin and the above mentioned Petrov-Galerkin finite element methods. For the steady problem only few combinations of stress and velocity interpolations are Galerkin stable and convergent. Unstable Galerkin approximations exhibit non-unique solutions with spurious pressure oscillations. The transient problem does not present explicitly the incompressibility constraint and consequently Galerkin approach in this case gives always a unique solution. However, this solution does not describe the behavior of the transient creep problem when unstable approximations are used due to the fact that in the limit as time goes to infinity the rate of deformation is fully inelastic and incompressible. On the other hand we have proved that with the Petrov-Galerkin formulation, stability, convergence and the proper asymptotic behavior of the transient solution are recovered for a large combination of interpolations. In this paper we present the finite element approximations of the steady state and the transient creep problems illustrating their behavior with a simple example.

THE STEADY STATE CREEP PROBLEM

A variational formulation for the steady state creep problem we consider here, based upon the Hellinger-Reissner principle, consists in

Problem $H^0$: Given $f \in V^*$, dual space of $V$, find $(\sigma^0, u^0) \in U_0 \times V$ such that

$$
(A(\sigma^0), \tau) = b(\tau, u^0) \quad \forall \tau \in U_0,
$$

$$
b(\sigma^0, v) = f(v) \quad \forall v \in V,
$$

$$
(1)
$$

$$
(2)
$$
with

\[
(A(\sigma), \tau) = \iint_{\Omega} A(S) \cdot \tau \, d\Omega \quad \forall \sigma, \tau \in U_0, \tag{3}
\]

\[
b(\tau, v) = \iint_{\Omega} \tau \cdot Bv \, d\Omega \quad \forall \tau \in U_0, \forall v \in V, \tag{4}
\]

\[
f(v) = \iint_{\Omega} f \cdot v \, d\Omega \quad \forall v \in V, \quad f \in V^*, \tag{5}
\]

where \( \Omega \subset \mathbb{R}^2 \) is the region occupied by the body, \( Bu=(\nabla u + \nabla u^T)/2 \) is the creep strain rate, \( A(\sigma) \) is in general a nonlinear function of the deviatoric part of the stress tensor

\[
S = \sigma - \frac{1}{3} \text{tr}\sigma I = \sigma - pI \tag{6}
\]

with \( p \) being the hydrostatic pressure. Normally, \( A \) is subjected to the constraint \( \text{tr} A = A \cdot I = 0 \), which means that the creep strain rate must satisfy the incompressibility condition \( \text{div} \, u = 0 \) in \( \Omega \), responsible for the main difficulties in constructing finite element approximations for this type of problems.

\( U_0 = \{ \tau \in U, \iint_{\Omega} \tau \, d\Omega = 0 \} \) and the complete characterization of the spaces \( U \) and \( V \) will depend on the creep constitutive equation defined by the operator \( A : U \to U^* \), with \( U^* \) denoting the dual space of \( U \). In general \( A \) is a nonlinear operator of monotone type, which makes Banach spaces the natural choice for this kind of variational problem. For linear creep laws, Problem \( H^0 \) becomes formally identical to the linear incompressible elasticity problem. In this case \( U \) and \( V \) are Hilbert spaces obtained by products of \( L^2(\Omega) \) and \( H^1_0(\Omega) \), respectively, and Brezzi's theorem on mixed methods can be applied to analyze of this problem. In (Guerreiro, 1988) a general result on existence and uniqueness of solution of Problem \( H^0 \) in Hilbert spaces is proved, based on appropriate ellipticity and continuity of the constitutive operator \( A(\sigma) \).

**THE TRANSIENT CREEP PROBLEM**

We also consider here the following elasto-creep problem. **Problem H**: Find \( (\sigma(t), u(t)) \in U \times V \) satisfying, for each time \( t \in [t_0, t_1] \),

\[
c(\dot{\sigma}, \tau) + A(\sigma), \tau) = b(\tau, u) \quad \forall \tau \in U, \tag{7}
\]

\[
b(\sigma, v) = f(v) \quad \forall v \in V, \tag{8}
\]

and the initial condition

\[
(\sigma(t_0), \tau) = (\sigma^0, \tau) \quad \forall \tau \in U, \tag{9}
\]

with \( \sigma^0 \) being the stress field which satisfies the associated elastic problem. The forms \( (A(\sigma), \tau), b(\tau, v) \) and \( f(v) \) were already defined in (3), (4) and (5), respectively, and \( c(\hat{\sigma}, \tau) \) is given by

\[
c(\hat{\sigma}, \tau) = \iint_{\Omega} C \hat{\sigma} \cdot \tau \, d\Omega \quad \forall \hat{\sigma}, \tau \in U. \tag{10}
\]

For isotropic and homogeneous elastic constitutive equation,
\[ c(\tilde{\sigma}, \tau) = \frac{1+\nu}{E} \int_{\Omega} (\tilde{\sigma} \cdot \tau - \frac{\nu}{1+\nu} \text{tr} \tilde{\sigma} \text{tr} \tau) \, d\Omega \quad \forall \tilde{\sigma}, \tau \in U, \] (11)

where \( E \) is the Young modulus and \( \nu \) is the Poisson ratio. In (Guerreiro, 1988) it is proved that the transient solution of the elasto-creep problem tends asymptotically to the solution of the steady state creep problem as \( t \to \infty \).

### FINITE ELEMENT APPROXIMATIONS

Let \( Q_h^0(\Omega) \) be the space of \( C^1 \) piecewise polynomial finite element interpolations of degree \( \ell \), and let \( S_h^0(\Omega) = Q_h^0(\Omega) \cap H^1_0(\Omega) \) be the space of \( C^0 \) piecewise polynomial finite element interpolations of degree \( k \), with zero value on the boundary \( \Gamma \).

Here \( h = \max h_e \), \( e = 1, 2, \ldots, n_{e \ell} \), denotes the mesh parameter with \( h_e \) being the diameter of element \( e \). We define \( U_h^0 = (Q_h^0)^k \subset U \) and \( V_h^0 = (S_h^0)^k \subset V \) and introduce the subspaces: \( Q_h^0 \cap U_h^0 \cap U_0 \), \( V_h^0 \cap U_0 \) and \( U_h^0 \cap V_h^0 \). To overcome the well known limitations of the classical Galerkin formulation for constrained problems we approximate Problem \( M_0 \), corresponding to steady state creep, by the following Petrov-Galerkin method:

**Problem \( PG_h^0 \):** Find \( (\sigma_h^0, u_h^0) \in U_h^0 \times V_h^0 \), such that

\[
(A_h(\sigma_h^0), \tau_h) = b(\tau_h, u_h^0) - g_h(\tau_h) \quad \forall \tau_h \in U_h^0, \]
(12)

\[
b(\sigma_h^0, \nu_h) = f(\nu_h) \quad \forall \nu_h \in V_h^0, \]
(13)

where,

\[
(A_h(\sigma_h), \tau_h) = (A(\sigma_h), \tau_h) + \frac{\delta h^2}{\theta} \left( \text{div} \sigma_h, \text{div} \tau_h \right)_h \quad \forall \sigma_h, \tau_h \in U_h^0, \]
(14)

\[
g_h(\tau_h) = \frac{\delta h^2}{\theta} \left( f, \text{div} \tau_h \right)_h \quad \forall \tau_h \in U_h^0, \]
(15)

with \( \delta \) being a positive scalar, \( \theta \) a viscosity parameter, and \( (\cdot, \cdot)_h \) defined as

\[
(\mu_h, \phi_h)_h = \sum_{e=1}^{n_{e \ell}} \int_{\Omega_e} \mu_e \cdot \phi_e \, d\Omega \quad \forall \mu_h, \phi_h \in (Q_h^0)^k, \]
(16)

with

\[
\| \mu_h \|_h^2 = (\mu_h, \mu_h)_h \quad \forall \mu_h \in (Q_h^0)^k, \]
(17)

where \( \mu_e \) and \( \phi_e \) are the restrictions of \( \mu_h \) and \( \phi_h \) to element \( e \).

Note that with \( \delta = 0 \) Problem \( PG_h^0 \) recovers Galerkin's formulation which is stable only for very few combinations of stress and velocity interpolations. With this Petrov-Galerkin formulation \( (\delta > 0) \) stability and convergence can be proved for a large choice of combinations of interpolations including those with equal order.

For the elasto-creep problem we define the following Petrov-Galerkin finite element approximation:

**Problem \( PG_h^\ell \):** For each \( t \in \{ t_0, t_1 \} \), find \( (\sigma_h, u_h) \in U_h^\ell \times V_h^k \), such that

\[
c(\tilde{\sigma}, \tau) + (A_h(\sigma_h), \tau_h) = b(\tau_h, u_h) - g_h(\tau_h) \quad \forall \tau_h \in U_h^\ell, \]
(18)

\[
b(\sigma_h, \nu_h) = f(\nu_h) \quad \forall \nu_h \in V_h^k, \]
(19)

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with
\[ (\sigma_h(t_0), \tau_h) = (\sigma^e_h, \tau^e_h) \quad \forall t_h \in U_h^e. \] (20)

The asymptotic behavior of the approximate solution of the transient creep problem is also proved (Loula and Guerreiro, 1987 and Guerreiro, 1988), that is:
\[ \lim_{t \to \infty} \sigma_h(t) = \sigma_h^0 + c_h l, \] (21)
\[ \lim_{t \to \infty} u_h(t) = u_h^0, \] (22)

with \( c_h \in \mathbb{R} \), representing a constant hydrostatic pressure. We also observe that transient Galerkin formulations have always unique solution with asymptotic behavior but their limits do not approximate properly the steady state creep problem unless stable Galerkin approximations are adopted.

**NUMERICAL RESULTS**

To solve Problem \( PG^0 \) we used a Uzawa's type algorithm. For the transient problem we combined this algorithm with a first order implicit finite difference discretization in time. As a model constitutive equation we consider Odquist-Norton power law governing incompressible creeping flow of metal or non-newtonian fluid, in which \( A \) is given by
\[ A(\sigma) = \mu |\sigma|^{p-2} \sigma, \] (23)

with \( \sigma \) being the deviatoric stress tensor, and \( \mu \) the viscosity coefficient. The example considered is a homogeneous and isotropic rectangular block with Young modulus \( E=1 \), Poisson ratio \( \nu=0.3 \), \( \mu=1 \) and \( p=4 \), modelled as a plane strain problem completely clamped on the borders \( x=0, x=2 \) and \( y=0 \), and subjected to a partially distributed load on the border \( y=1 \) with intensity \( \bar{a}=32 \), as shown in Fig. 1.

In this analysis we adopted equal order biquadratic interpolations for both stress and velocity fields, solving steady state and transient (with the time step integration \( \Delta t=1. \) problems) with Galerkin and Petrov-Galerkin (\( \delta=1. \)) methods. We present here only the results obtained for the post-processed pressure fields. As we can see in Fig. 2, the Galerkin solutions exhibit spurious oscillations and the limit of the transient solution differs from the steady state solution. With the Petrov-Galerkin formulation, Fig. 3, stability is recovered, no spurious oscillations are present and the limit of the transient solution is identical to the steady state solution.

Fig. 1 - Sample problem. Rectangular block simulated as a plane strain problem.
Fig. 2 - Post-processed pressures obtained with Galerkin's method. 
(a) steady-state solution (b) Transient limit solution.

\[ y = 0.5625 \quad \text{and} \quad y = 0.9325 \]

Fig. 3 - Post-processed pressures obtained with Petrov-Galerkin's method. 
(a) steady-state solution (b) Transient limit solution.

\[ y = 0.5625 \quad \text{and} \quad y = 0.0325 \]

REFERENCES


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