INTRODUCTION

The design of efficient structural systems is of fundamental interest to both structural and control engineers. A knowledge of the vibrational characteristics of cylindrical shells is important for the nuclear applications of these structures. In connection with the latter the natural frequencies of the cylindrical shell must be known in order to avoid the destructive effects of resonance with nearby rotating or oscillating equipment or other dynamic excitations such as earthquakes. The designing of a structure to isolate it from the destructive effects of resonant vibration rather than strengthening it to withstand the vibration, is a recognized fact. This is achieved by elevating the natural frequencies of the structure excitations in the vicinity. The problem concerning the minimum volume and wall shape of the cylindrical shell for a given minimum allowable frequency received a small attention (Thambiratnam et al. 1988). After some theoretical considerations concerning the optimum cylindrical shells in stationary regime using optimal control theory and the Pontryagin's maximum principle, in the second part it is considered a finite element analysis of radial and axial natural vibrations for multiple situations: cylinder empty, cylinder with step fluid filling, cylinder with constant and variable thickness and different bearings.

OPTIMUM DESIGN OF CYLINDRICAL SHELLS CONCERNING THE STATIONARY VIBRATIONS

It is considered the cylinder vibrations equation:

\[
\frac{\partial^2}{\partial x^2} \left( D \frac{\partial^2 w}{\partial x^2} \right) + \frac{E h}{R^2} w + \rho h \frac{\partial^2 w}{\partial t^2} = 0
\]

(1)

where: \( D = \frac{E h^3}{12(1-\nu^2)} \), \( w \) - the radial displacement, \( h \) - thickness of the shell. After using the substitution

\[ w = y(x) \cos ft \]

it obtains the differential equation:
\[
\frac{d^2}{dx^2} (h^3 \frac{d^2y}{dx^2}) - \beta^2 h y = 0
\]  

(2)

where

\[
\beta = 2 \sqrt{3(1 - \eta^2)(\frac{L}{h^2} - \frac{1}{R^2})}
\]

With the new variables

\[
\varphi = \frac{dy}{dx} ; \quad M = - h^3 \frac{d^2y}{dx^2} ; \quad Q = - \frac{d}{dx} (h^3 \frac{d^2y}{dx^2})
\]

the differential equation (2) is transformed into a system with state variables

\[
\varphi = \frac{dy}{dx} ; \quad \frac{d\varphi}{dx} = - \frac{M}{h^3} ; \quad \frac{dM}{dx} = Q; \quad \frac{dQ}{dx} = - \beta^2 h y
\]  

(3)

The following conditions are added

\[
y(0) = \varphi(0) = 0 ; \quad M(L) = Q(L) = 0
\]  

(4)

and

\[
h_1(x) \leq y(x) \leq h_2(x)
\]  

(5)

We denote by \( f \rightarrow V \), the minimum or maximum volume of the cylindrical shell under the fixed frequency and by \( V \rightarrow f \), the minimum or maximum frequency for the volume fixed. The cost functional is

\[
V = \int_0^L (h^2 + 2Rh) \, dx
\]  

(6)

It is introduced a new state variable \( y_0(x) \) by the differential equation

\[
\frac{dy_0}{dx} = (h^2 + 2Rh) \bar{\eta}, \quad y_0(0) = 0
\]  

(7)

The Lagrange problem is transformed in a Mayer problem, the cost functional becomes

\[
V = \bar{\eta} y_0(L)
\]  

(8)

In the cost functional is introduced a parameter

\[
J = \lambda \bar{\eta} y_0(L)
\]  

(8)

For \( \lambda \geq 0 \), (8) will be minimum and for \( \lambda \leq 0 \), (8) will be maximum. As the Pontryagin's maximum principle, we construct the Hamiltonian

\[
H = \psi_\varphi \varphi - \frac{\psi_M}{h^3} M - \psi_Q Q - \beta^2 h \psi_Q y + \psi_0 (h^2 + 2Rh) \bar{\eta}
\]  

(9)

The reciprocal differential equation system is given by
\[ \frac{d\psi_y}{dx} = - \frac{\partial H}{\partial y} = \beta^2 h \psi_q; \quad \frac{d\psi_M}{dx} = - \frac{\partial H}{\partial M} = - \psi_M; \]
\[ \frac{d\psi_0}{dx} = - \frac{\partial H}{\partial y_0} = 0 \]

The boundary conditions for the reciprocal differential equations are obtained from the transversality conditions
\[ \psi_Q(0) = 0, \psi_M(0) = 0, \psi_y(L) = 0, \psi_q(L) = 0, \psi_0 = -\lambda \]

We introduce the following notations:
\[ \psi_y = k Q; \quad \psi_q = -k M; \quad \psi_M = -k Q; \quad \psi_q = -k y \]

where \( k \) is an arbitrary constant. The Hamiltonian \( H \) has the form
\[ H = k( \frac{M^2}{h^3} + \beta^2 h y^2 ) - \lambda \tilde{u}(h^2 + 2Rh) \]

\( k \) takes the values \(-1, 0, +1\). For \( k = 0 \), from the maximum of Hamiltonian it obtains
\[ h^2 + 2Rh = \begin{cases} h_2^2 + 2Rh_2 & \lambda < 0 \\ h_1^2 + 2Rh_1 & \lambda > 0 \end{cases} \]

For \( k = -1 \) it results
\[ H_{hh}'' = - \left( \frac{12M^2}{h^5} + 2\tilde{u} \lambda \right) \quad \lambda > 0 \]

and the Hamiltonian (13) will be maximum for the real root of the algebraic equation
\[ 2\tilde{u} \lambda h^5 + ( \beta^2 y^2 + 2R \lambda ) h^4 - 3M^2 = 0 \]

as determined from the condition \( H_1^h = 0 \).

**FINITE ELEMENT METHOD FOR OPTIMUM FREQUENCIES STUDY**

The cylinder is discretized in axially symmetric elements (rings) with four nodes as showed in Fig. 1.
In Fig. 2 is presented the cylinder empty with different bearings and in Fig. 3 the filling steps with the fluid.

In the Fig. 5 are showed the natural frequencies variations of the empty cylinder with respect to the bearings from Fig. 2 for three modes and in Fig. 4 with the respect to the filling steps with the fluid for the same modes.
In the Fig. 6 are presented the natural frequencies variations of the empty cylinder with respect to the thickness wall.

Fig. 6

Fig. 7 presents the natural frequencies variations of the empty cylinder having the exponential thickness of the wall.

Fig. 7
CONCLUSIONS

From Fig. 5 it remarks that the radial bearings has not the influence on the first frequency due to the displacements in the longitudinal direction of the cylinder. It results, also, the growth of the frequencies in the second and third modes radial and longitudinal with the number of bearings. From Fig. 4 it remarks that the 30% - 50% filling with the fluid or another material is a critical state, the natural frequencies variations having a qualitative transition (maxima, minima). After this transition state the frequencies have a more or less slowly diminution with respect to the filling. The first frequency is maximum for 50% filling of the cylinder. In this critical region the ratio between the rigidity of the ensemble and it mass from Fig. 6 it results that the small variations of the thickness wall do not lead to the modifications on the radial and longitudinal frequencies. These frequencies may be modified with the supplementary bearings, only. From the Fig. 7 it results that in the first mode (the cylinder with variable thickness) the frequency diminish and the frequencies in the second and third modes have variations around the mean values. Also, the 30% - 50% filling of the cylinder with a material or fluid is a critical state transition, the frequencies having the maximum and minimum.

REFERENCES