About Two New Efficient Nonlinear Shell Elements

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OBJECTIVE

The aim of the paper is to present the development of two shell elements for non-linear analysis. The first one is an axisymmetric curved shell element and it is developed for buckling analysis. The formulation is given, as well as some typical applications. The second one is an extension of the classical DKT (1) element to large strains taking into account all aspects of non-linearities. This element is used for the simulation of four point bending of cracked pipes. The whole experiment is simulated by the calculation taking into account very large strains at the crack tip and propagation of the crack.

THE AXISYMMETRIC CURVED SHELL ELEMENT

Basic equations

Definition of axis and displacement fields

Let us consider a curved axisymmetric shell as defined on Fig. 1.

The local reference axis are \( e_\theta, e_\phi, e_r \). The global axis are \( e_r, e_\theta, e_\phi \). The radius is \( r \).

The displacement field in the local axis are \( u, v, w \).

The displacement field in the global axis are \( U_r, U_\theta, U_\phi \).

Moreover we have \( v = U_\theta \).

We have the following formulas for the change of reference system.

\[
\begin{bmatrix}
U_r \\
U_\theta \\
U_\phi
\end{bmatrix} = \begin{bmatrix}
\sin \phi & \cos \phi & 0 \\
\cos \phi & \sin \phi & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
U \\
W
\end{bmatrix}
\]

Expression of the gradient of displacement field

With the notations, the gradient of displacement field can be expressed at any point across the thickness. We have then:

\[
\nabla u = \begin{bmatrix}
\frac{1}{1 + e^r} \left( \frac{\partial U}{\partial s} + \frac{U}{R} \right) - \frac{1}{r (1 + e \cos \phi)} \left( \frac{\partial U}{\partial \theta} + v \sin \phi \right) & \frac{\partial U}{\partial \theta} \\
\frac{1}{1 + e^r} \frac{\partial v}{\partial s} + \frac{v}{R} & \frac{\partial U}{\partial \theta} + v \cos \phi \\
\frac{1}{1 + e^r} \left( \frac{\partial w}{\partial s} - \frac{w}{R} \right) - \frac{1}{r (1 + e \cos \phi)} \left( \frac{\partial w}{\partial \theta} + v \cos \phi \right) & \frac{\partial w}{\partial \theta} \\
\end{bmatrix}
\]
R is the radius of curvature in a constant θ plane.
r is the radius of curvature in a constant z plane.
e is the distance between the mean line and the point M where the gradient is calculated.
\( \frac{\partial}{\partial n} \) is the derivative along the direction \( \text{e}_n \).

**Shell hypothesis**

We make the hypothesis that along the direction \( \text{e}_n \) the displacement field can be written as:

\[
\begin{align*}
\text{u}_e &= \text{u}_e = 0 + e \beta_1 \\
\text{v}_e &= \text{v}_e = 0 + e \beta_2 \\
\text{w}_e &= \text{w}_e = 0
\end{align*}
\]

**Expression of the deformation**

Linear deformation \( \epsilon \).
From Eq. (3) and (4), we obtain:

\[
\begin{bmatrix}
\frac{1}{1 + e/R} \frac{\partial w}{\partial s} + \frac{w}{R} \\
\frac{1}{1 + e/R} \frac{\partial v}{\partial s} \\
\frac{1}{1 + e/R} \frac{\partial w}{\partial s} - \frac{U}{R}
\end{bmatrix}
\begin{bmatrix}
\frac{-1}{r (1 + e \cos \phi)} \left( \frac{\partial U}{\partial \theta} + v \sin \phi \right) \\
\frac{-1}{r (1 + e \cos \phi)} \left( \frac{\partial v}{\partial \theta} - u \sin \phi - w \cos \phi \right) \\
\frac{-1}{r (1 + e \cos \phi)} \left( \frac{\partial w}{\partial \theta} + v \cos \phi \right)
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
0
\end{bmatrix}
\]

with the assumption:

\[
\text{4) } \epsilon = \frac{1}{2} (\nabla u + ^t \nabla u) \quad \text{and} \quad \gamma_{ij} = 2 \epsilon_{ij}
\]

We obtain with a first order Taylor expansion with respect to the variables \( \left( \frac{\epsilon}{R} \right) \)
\( \epsilon = \epsilon^m + e \epsilon^s \), where the membrane deformation \( \epsilon^m \) is given by:

\[
\begin{align*}
\epsilon^m_{\theta} &= \frac{\partial u}{\partial s} + \frac{w}{R} \\
\epsilon^m_{\phi} &= \frac{u}{r} \sin \phi - \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{w}{r} \cos \phi \\
\gamma^m_{\phi \theta} &= \frac{\partial v}{\partial s} - \frac{1}{r} \frac{\partial u}{\partial \phi} - \frac{\sin \phi}{r} v \\
\gamma^m_{s \phi} &= \beta_1 + \frac{\partial w}{\partial s} - \frac{u}{r} \\
\gamma^m_{s \theta} &= \beta_2 - \frac{1}{r} \frac{\partial w}{\partial \theta} + v \cos \phi
\end{align*}
\]

and the bending deformation \( \epsilon^b \) is given by:

\[
302
\]
\[
\epsilon_s = \frac{\partial \beta_1}{\partial s} - \frac{1}{R} \frac{\partial u}{\partial s} - \frac{w}{R^2}
\]
\[
\gamma^b_\theta = -\frac{1}{r} \left( \frac{\partial \beta_1}{\partial \theta} - \beta_1 \sin \phi \right) + \frac{\cos \phi}{r^2} \left( \frac{\partial v}{\partial \theta} - u \sin \phi - w \cos \phi \right)
\]
\[
\gamma^b_\phi = -\frac{1}{r} \frac{\partial v}{\partial \phi} + \frac{\cos \phi}{r^2} \left( \frac{\partial u}{\partial \phi} + v \sin \phi \right) - \frac{1}{r} \frac{\partial \beta_1}{\partial \phi} - \frac{1}{r} \beta_2 \sin \phi + \frac{\partial \beta_2}{\partial \phi}
\]

we have made the approximation that \( \gamma^b_\text{sn} \) and \( \gamma^b_\text{su} \) are zero.

Expression of the potential energy \( W^{\text{def}} \)

The potential energy \( W^{\text{def}} \) will be used for the derivation of the stiffness matrix and also for the whole formulation of the element shall be written.

(7) \( W^{\text{def}} = W^m + W^b + W^{\text{sh}} \)

let us note \( < > \), the integration over the mean surface of the shell, \( h \) its thickness, \( E \) the Young's modulus, \( \nu \) the Poisson's ratio and suppose the matricial is isotropic. Then \( G \) the shear modulus

(8) \( W^m = < h \sigma^m : \epsilon^m > \)

\[
\begin{pmatrix}
\sigma^m_s \\
\sigma^m_\theta \\
\sigma^m_\phi
\end{pmatrix}
= \frac{E}{1 - \nu^2}
\begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1 + \nu}{2}
\end{pmatrix}
\begin{pmatrix}
\epsilon^m_s \\
\epsilon^m_\theta \\
\epsilon^m_\phi
\end{pmatrix}
\]

\( \sigma^m_\phi \) is Young's modulus, \( \nu \) Poisson's ratio

\( W^b = < h \sigma^b : \epsilon^b > \) where:

\[
\begin{pmatrix}
\sigma^b_s \\
\sigma^b_\theta \\
\sigma^b_\phi
\end{pmatrix}
= \frac{E h^2}{12 (1 - \nu^2)}
\begin{pmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1 + \nu}{2}
\end{pmatrix}
\begin{pmatrix}
\epsilon^b_s \\
\epsilon^b_\theta \\
\epsilon^b_\phi
\end{pmatrix}
\]

(11) \( W^{\text{sh}} = G < \epsilon^m_\text{sn} h + \epsilon^m_\text{su} h > \)

Derivation of quadratic membrane deformation

To get the non linear deformation, we have to calculate:

(12) \( \epsilon = \frac{1}{2} [\nabla u : \nabla u] \)

This expression can be calculated directly from Eq. (3).

We shall limit the taylor expression only to zero order terms inwith respect to the variables \( \frac{\epsilon_s}{R} \) and \( \frac{\epsilon_\theta}{R} \).

From that we can easily calculate the quadratic deformation \( \epsilon^{Q} \).
\[
\begin{align*}
\epsilon_s &= \frac{Q_s}{R^3} + \left(\frac{w}{R}\right)^2 + \left(\frac{v}{R}\right)^2 + \left(\frac{w}{R} - u\right)^2 \\
\epsilon_\theta &= \frac{1}{r^2} \left( \frac{\partial u}{\partial \theta} + v \sin \phi \right)^2 + (u \sin \phi + w \cos \phi - \frac{\partial v}{\partial \theta})^2 + (\frac{\partial w}{\partial \theta} + v \cos \phi)^2 \\
\gamma_{s\theta} &= \frac{1}{r} \left( \frac{\partial u}{\partial \theta} + \frac{w}{R} \right) \left( \frac{\partial u}{\partial \theta} + v \sin \phi \right) + \frac{\partial v}{\partial \theta} \left( u \sin \phi + w \cos \phi - \frac{\partial v}{\partial \theta} \right) \\
&\quad + \left( \frac{\partial w}{\partial \theta} - \frac{u}{R} \right) (\frac{\partial w}{\partial \theta} + v \cos \phi).
\end{align*}
\]

This expression allows the torsional buckling prediction.

**Finite element**

A three node axisymmetric curved shell element is based on these equations:

![Shell Element Diagram](image)

The radius of curvature $R$ is constant on the element and calculated from the coordinates of the 3 modes. All degrees of freedom $u$, $v$, $w$, $\beta_1$, $\beta_2$ are quadratic over the element.

It is integrated with two Gauss points. It is a uniform reduced integration element. More details are given in [2].

**Example**

**Elastic buckling example**

The buckling of a long tube under axial compression is computed using the element implemented in INCA programs of CASTEM.

The tube has a radius $R$, a length $L$, a thickness $t$. The Young's modulus is $E$, the Poisson's ratio is $\nu$. It is clamped at its base, free at its top and submitted to a uniform axial load.

The critical axial load is:

\[(14) \quad P_{cr} = \frac{\pi^3 E R^3 t}{4 L^2}\]

The following numerical values are chosen:

$R = 10 \quad L = 300 \quad t = 1. \quad E = 20000 \quad \nu = 0.3$

The critical load is then: $P_{cr} = 1722.42$

The calculation with 30 elements gives a critical mode 1 axial load of: $P_{cal} = 1712.12$ which corresponds to an error of 0.6%.

**Torsional buckling**

The torsional buckling of a cylinder having the following properties is calculated:

$R = 1 \quad \begin{cases} L = 2 & t = 1 \quad E = 3 \quad E = 2.11 \quad \nu = 0.3 \\ L = 20 \end{cases}$

The critical values for simply supported edges are given by the following formula [3]:

\[(15) \quad \sigma_{cr} = 4.39 \frac{E}{1 - \nu^2} \left( \frac{L}{R} \right)^2 \sqrt{1 + 0.0257 (1 - \nu^2)^{3/4}} \left( \frac{L}{\sqrt{RT}} \right)^3\]
which gives in our case for \( L = 2 \) : \( \sigma_{cr} = 18.9 \times 10^6 \)

and for \( L = 20 \) \( \sigma_{cr} = 5.9 \times 10^6 \)

The result of the computation gives a coupled Fourier mode (symmetric and axisymmetric) 14 for the first case (see Fig. 2) and 5 for the long cylinder (see Fig. 3).

**Hemisphere under external pressure**

The plastic buckling of a hemisphere under external pressure is calculated. The problem is defined on Fig. 4.

The plastic bifurcation is calculated with INCA code of CASTEM SYSTEM and gives a plastic bifurcation for an external pressure of 0.5 MPa on axisymmetric mode which is given on Fig. 5. More details can be seen in [4].

**THE DKT LARGE STRAIN ELEMENT**

The DKT element is currently used in large displacement finite element method [1] [5] [6]. This element has been extended to perform large strain analysis taking into account the variation of thickness due to large strains.

It is used to perform a plastic failure mechanics analysis of a cracked 316 SS steel.

**Theoretical inputs**

The incremental analysis used is based on the updated Lagrangian incremental Jauman stress formulation (U.L.J.).

The chosen incremental stress strain relation is:

\[
\Delta S_{ij} = C_{ijkl} \Delta e_{ij}
\]

where \( \Delta S_{ij} \) is the increment of Kirchoff stress and \( \Delta e_{ij} \) is the linear part of the incremented Green strain tensor.

In equation (16) the constitutive matrix is given by:

\[
C_{ijkl}^* = C_{ijkl}^{EP} - \sigma_0 \delta_{ik} \delta_{jl} - \sigma_0 \delta_{ij} \delta_{kl} + \sigma_{ij} \delta_{kl}
\]

where \( \delta \) is the Kronecker symbol and \( C_{ijkl}^{EP} \) is the classical elastoplastic constitutive matrix.

**Variation of thickness**

The initial thickness is given at the beginning of the calculation the thickness of the shell is updated by the following procedure.

The normal plastic increment of deformation is evaluated at each integration point along the thickness by means of incompressibility constraint.

\[
\Delta e^{Pn} = - (\Delta e^{P1} + \Delta e^{P2})
\]

where \( \Delta e^{Pn} \) is the normal variation of plastic strain \( \Delta e^{P1}, \Delta e^{P2} \) are the two increments of tangential plastic strain. The mean variation of plastic strain \( \bar{\Delta e}^P \) across the thickness is then evaluated. From this the thickness \( h \) is updated with the following increment:

\[
\Delta h = \bar{\Delta e}^P h
\]

**Problem studied**

**Problem**

Fig. 6 shows the geometry studied. The cracked pipe is meshed with 630 elements as shown on Fig. 7. The pipe is submitted to a 4 point bending test. The Young's modulus is 2E11 MPa, the Poisson ratio being 0.3. The true stress, Logarithmic strain curve is given of Fig. 8.

**Crack propagation**

Two calculations are done and compared with experiment; one without crack propagation, another with crack propagation. The crack is supposed to propagate if the strain in the element in front of the crack tip is greater than 0.4; in this case, the node in front of the crack tip is released and the calculation is continued with this new boundary condition.
RESULTS AND CONCLUSIONS

The results of the two calculations are given of Fig. 9 and compared with experiment. This curve represents the applied moment, function of the crack opening displacement. One clearly sees that the two calculations give a similar curve until the point associated with a COD of 5.5 mm. At this point the crack propagation induces a decrease of the momentum, in the case of the calculation including the crack propagation. The calculation with the propagation is close to experimental curve until a certain propagation where we observe a stiffening, probably induced by a too coarse mesh. The comparison of the two calculations with experimental result show that the crack propagation occur slightly before the maximum momentum supported by the pipe. There are very high levels of strains at the crack tip, and we are there, clearly, in the frame of a ductile propagation mechanism.

REFERENCE


Figure 1: Definition of repairs
mode 14 - 14 $\sigma_a = 18.67 \times 10^6$
Fig. 2: Short cylinder under torsion

mode 5 - 5 $\sigma_a = 5.9 \times 10^6$
Fig. 3: Long cylinder under torsion

<p>| Locations of | z coordinate |</p>
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<tr>
<td>M</td>
<td>- 510</td>
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<tr>
<td>L</td>
<td>- 472</td>
</tr>
<tr>
<td>K</td>
<td>- 385.5</td>
</tr>
<tr>
<td>J</td>
<td>- 247.2</td>
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<tr>
<td>I</td>
<td>- 79.6</td>
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<tr>
<td>D</td>
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All numbers in $10^{-3}$ m

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<tr>
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<tr>
<td>M-N</td>
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Fig. 4: Geometry of the sphere

307
Fig. 5: Plastic buckling mode

Fig. 6: Definition of the geometry of the cracked pipe

<table>
<thead>
<tr>
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<th>$\delta$</th>
<th>$\epsilon_1$</th>
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<td>Notched</td>
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Fig. 7: Mesh used for the computation

Fig. 8: Stress strain curve

Fig. 9: Theoretical experimental comparison