INTRODUCTION

A great deal of effort has been put, during the past few years, on the application of finite element methods to convective transport problems. Nevertheless, the degree of success has been less evident as compared to that in other disciplines such as structural mechanics. This is attributed to the fact that non-symmetric operators, such as convection terms, discretized by means of conventional Galerkin concept give rise to spurious oscillations between adjacent nodes. These oscillations can only be removed by severe mesh refinement which undermines the practical utility of the method. A similar situation has also been observed for central-difference type in the context of finite difference methods. To preclude such oscillations, alternate procedures have been proposed during the past years in finite element methods. One of these approaches (Morton and Parrott, 1980) was based on a Petrov-Galerkin weak formulation where special test functions are determined in order to ensure that the "unit C.F.L. property" is satisfied, first for scalar equations and then for linear first-order systems.

In a different approach (Brooks and Hughes, 1979 and 1982) exploited the interpretation of upwind differences as central differences plus artificial diffusion to construct upwind finite element schemes via the Galerkin weak formulation with added artificial diffusion. This upwind method can be implemented as a modification of the weighting function for the convective term, and application of the modified weighting function to all terms of the equation defines a consistent Petrov-Galerkin formulation. The ideas of the Lax-Wendroff scheme in finite differencing have motivated the development of the Taylor-Galerkin method (Donea, 1984). The fundamental idea at the basis of this method was to create high-order accurate time-stepping schemes to be coupled with the very accurate spatial discretization provided by the Galerkin weak formulation. The Taylor-Galerkin method has been successfully applied to the solution of advection and advection-diffusion linear problems in one and two dimensions (Donea, 1984; Donea et al., 1984).

Recently, a least-squares finite element method has been developed (Carey and Jiang, 1988) for first-order hyperbolic systems. This method is shown to lead to a discrete problem similar to that of Taylor-Galerkin approach and to produce good results for linear convective transport problems.

However, numerical experiments conducted on non-linear advection equations, such as the inviscid Burgers' equation, indicate that standard finite element schemes may become non-linearly unstable as steep solution gradients associated with shocks develop. This difficulty has led to the use of artificial dissipation techniques as a means to stabilize the calculation (see, e.g., Löhner et al., 1984; Hughes and Mallet, 1985; Selmin and Quartapelle, 1984 and Jiang and Carey, 1988).

In the past, we developed a space-time least-squares finite element method and applied it with some success to convection-dominated flow problems (Nguyen and Reynen, 1984-a) even in non-linear cases (Nguyen and Reynen, 1984-b) However, this method produced rather poor results for purely convective-transport equations.

By adding an artificial viscous dissipation term involving a time derivative we develop, in this paper, a class of space-time least-squares finite element schemes for linear and non-
linear convective-transport equations in order to improve the stability properties of numerical solutions. Thus, one obtains a class of finite element schemes depending on a free parameter $\varepsilon$ which is the amplitude of the artificial viscous dissipation term. It is interesting to mention here that all well-known finite element schemes for convective-transport equations can be derived from this formulation by varying the free parameter $\varepsilon$. The optimum least-squares space-time finite element scheme LSST for linear problems is obtained by ensuring that the scheme is stable, accurate and that the "unit C.F.L. property" is satisfied. For non-linear situations, a precise relation of the optimum value $\varepsilon_{opt}$ which is a function of $\Delta x$ and $\Delta t$ is proposed. The application of this method to the inviscid Burgers' equation demonstrates the accuracy and the stability of the least-squares space-time finite element schemes.

**LEAST-SQUARES WEAK FORMULATION IN TIME AND SPACE**

We consider the linear convection equation in one-dimension

$$u_t + a u_x = 0$$  \hspace{1cm} (1)

together with appropriate initial and boundary conditions. We will assume for simplicity that the convective velocity $a$ is positive constant and that the equation is to be valid over a finite interval $0 < x < L$.

The least-squares weak formulation in time and space is defined as

$$\delta \int \int (u_t + a u_x)^2 \, dx \, dt = 0$$  \hspace{1cm} (2)

In each element $e$ (Fig. 1), using the following transformations

$$dt = \Delta t \, d\zeta \ , \ dx = \Delta x \, d\eta$$

and taking variation with respect to $\delta u$, one obtains the contribution for an individual element ($\Delta x = \text{constant}$)

$$\int_0^1 \int_0^1 (u_{\zeta} + c u_{\eta}) \delta (u_{\zeta} + c u_{\eta}) \, d\zeta \, d\eta = 0$$  \hspace{1cm} (3)

where $c = \frac{a \Delta t}{\Delta x}$ is the Courant number for mesh size $\Delta x$.

Therefore, the least-squares weak formulation in time and space can also be regarded as a Petrov-Galerkin method in which a weighting function appears naturally and requires no "free" parameters.

$$w = u_{\zeta} + c u_{\eta}$$  \hspace{1cm} (4)

for all admissible test function $v = \delta u$. In each element ($\Delta x, \Delta t$), local approximations are taken in the form

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\[ u_e = P_j \left( U_j + \zeta \Delta U_j \right) \]  
(5)

where \( P_j(\eta) \) are the shape functions, \( U_j \) are the nodal values at the beginning of the time step and \( \Delta U_j \) are the increments of nodal values during a time-step \( \Delta t \). Then the weighting functions \( w_i \) take the form

\[ w_i = P_i + \zeta c \frac{dP_i}{d\eta} \]  
(6)

In the constant coefficient case where piecewise linear finite elements on uniform mesh \( \Delta x \) are used, (3) provides the fully discrete equation

\[ [1 + \left( \frac{1}{6} - \frac{1}{3} c^2 \right) \delta^2] \Delta U_j = - \frac{c}{2} (U_{j+1} - U_{j-1}) + \frac{1}{2} c^2 \delta^2 U_j \]  
(7)

where \( \delta^2 U_j = U_{j-1} - 2 U_j + U_{j+1} \)

This unconditionally stable scheme which has been used with success to solve convective problems with source term (Nguyen and Reynen, 1983) and two-phase flow problems (Nguyen, 1985) produced rather poor results for purely convective transport equations. In order to improve the performance of the least-squares space-time finite element scheme LSST for this kind of problem, the use of artificial dissipation techniques is proposed. Usually, the artificial dissipation is introduced by adding the term \( \epsilon u_{xx} \) or \( \epsilon u_{xt} \) into equation (1). In this paper, we introduce instead the term \( \epsilon u_{xxt} \) and maintain the weighting function defined in (6). Thus, we have the alternative weak formulation

\[ \int_0^1 \int_0^1 (u_{\zeta} + c u_\eta + \epsilon \frac{\Delta t}{\Delta x^2} u_\eta \eta \zeta) (v_{\zeta} + c v_\eta) \, d\zeta \, d\eta = 0 \]  
(8)

Using a basis of linear elements, the fully discrete equation takes the following form

\[ [1 + \left( \frac{1}{6} - \frac{1}{3} c^2 + \alpha c^2 \right) \delta^2] \Delta U_j = - \frac{c}{2} (U_{j+1} - U_{j-1}) + \frac{1}{2} c^2 \delta^2 U_j \]  
(9)

where \( \alpha = \frac{c}{a^2 \Delta t^2} \)

It is interesting to observe that all well-known finite element schemes can be derived from (9) by assuming different values of \( \alpha \). Then

(i) \( \alpha = \frac{1}{12} \) : The scheme (9) is identical to the classical least-squares finite element scheme based on Crank-Nicolson time-stepping (Carey and Jiang, 1988)
( ii) \( \alpha = \frac{1}{6} \) : The scheme (9) is identical to the third-order Taylor-Galerkin scheme TG3T (Selmin et al., 1985)

(iii) \( \alpha = \frac{1}{3} \) : The scheme is identical to the second-order Taylor-Galerkin scheme TG2S (Selmin et al., 1985)

(iv) \( \alpha = \frac{1}{3} - \frac{1}{6c} \) : The scheme (9) is identical to the Petrov-Galerkin scheme EPGII (Morton and Parrott, 1980)

(v) \( \alpha = \frac{1}{3} - \frac{1}{6c^2} \) : The scheme (9) is identical to the Lax-Wendroff finite difference scheme

However, the determination of the optimum value of \( \alpha \) is needed in order to ensure that the scheme is stable and accurate and that if possible the so-called "unit C.F.L. property" is satisfied.

**DETERMINATION OF THE OPTIMUM SCHEME**

**Stability Analysis**

A classical von Neumann stability analysis can now be applied to the recurrence relation (9) for our space-time least-squares formulation. Substituting a Fourier mode \( e^{ikx} \) (with \( j = \sqrt{-1} \)) into (9) and setting \( p=k\Delta x \) gives an amplification in one time step of

\[
G(p) = \frac{(2-6\alpha c^2 \cos p + j\alpha c^2 \sin p)}{(2-6\alpha c^2 + 2c^2) + (1+6\alpha c^2 - 2c^2) \cos p} = \frac{Z_1}{Z_2}
\]

which, in the asymptotic limit \( p \rightarrow 0 \), reduces to

\[
G(p) = 1 + j c p - \frac{1}{2} c^2 p^2 - j \left( \frac{1}{3} - \alpha \right) c^3 p^3 + O(p^4)
\]

to be compared with \( e^{rac{jcp}{1-p}} \) for the differential equation. It follows that the LSST scheme is stable if

\[
\left| \frac{Z_2}{Z_1} \right|^2 = 3 c^2 (1-\cos p)^2 \left[ (1 - 12\alpha) c^2 + 1 \right] > 0
\]

Hence:

(i) if \( \alpha \leq \frac{1}{12} \) the scheme (9) is second order accurate and unconditionally stable at all Courant numbers.

(ii) if \( \alpha > \frac{1}{12} \) the stability condition reads \( c^2 < \frac{1}{12\alpha - 1} \). The optimum LSST scheme is obtained with \( \alpha = \frac{1}{6} \) or with \( \varepsilon_{opt} = \frac{1}{6} a^2 \Delta t^2 \). Effectively in the case where \( \alpha = \frac{1}{6} \) the LSST scheme (9) is third order accurate. Since \( |G(p)| = 1 \) for \( c = 1 \) the LSST scheme possesses the so-called "unit C.F.L. property", i.e., signals are propagated without distortion when the characteristics pass through the nodes. Then the optimum LSST scheme for linear problems, is identical to the third-order Taylor-Galerkin scheme TG3T.
Non-linear extension

We now consider the extension of the previous space-time least-squares approach to the corresponding scalar non-linear equation

\[ u_t + a(u) u_x = 0 \]

Following the same procedure as in the linear case, we obtain again the recurrence relation (9). The nonlinearity of equation (9), due to the presence of the unknown \( U_i^{n+1} \) in the Courant number \( c \), requires a sufficient number of iterations in solution procedure. Iterations are performed until

\[ \sum_i \left| U_i^{n+1}_{k+1} - U_i^{n+1}_{k} \right| < \kappa \]

is obtained for a prescribed accuracy \( \kappa \). Here \( \kappa \) denotes the iteration number.

The optimum LSST scheme for non-linear cases can be based on numerical experiments and on the stability analysis. The equation (11) suggests the following correlation

\[ \alpha < \frac{c^2 + 1}{12c^2} \]

The rhs of the above relation is a decreasing function of \( c^2 \). Then it is reasonable to propose an optimum value of \( \alpha \) for non-linear problems

\[ \alpha_{opt} = \frac{Y^2 + 1}{12Y^2} \quad \text{with} \quad Y = \max[a(u)] \frac{\Delta t}{\Delta x} \quad (12) \]

The numerical results presented in the next section confirmed the validity of this choice.

NUMERICAL RESULTS

The performance of the optimum LSST scheme which is identical to the third-order Taylor-Galerkin scheme TG3T for linear cases is abundantly demonstrated (Donea, 1984) The object now is to test the optimum LSST scheme for representative non-linear problems and to compare results with those of non-optimum schemes. In particular, we will compare our optimum LSST results with those obtained by the scheme using \( \alpha = \frac{1}{4} \) for inviscid Burgers equations. All calculations employ a mesh of 50 linear elements on \([0,1]\). First we consider the inviscid Burgers' equation

\[ u_t + u u_x = 0 \]

with initial data

\[ u(x,0) = g(x) \]

corresponding to an initial slant step. The initial solution is shown in Figure 2.

The solutions at time \( t = 0.30 \) are plotted in Figure 3 (with \( \Delta t = 0.50 \Delta x \)) and in Figure 4 (with \( \Delta t = \Delta x \)) for the respective methods. The agreement between our solution and the exact solution is very good while the other solutions have both undershoot and overshoot oscillations.

A similar comparison is made in the case of the problem defined by the following initial condition

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\[ u(x,0) = \begin{cases} 
1 & \text{for } 0 < x < x^* \\
0 & \text{for } x^* < x < 1 
\end{cases} \]

The exact solution is a discontinuity which moves to the right with a speed equal to \( \frac{1}{2} \)

\[ u_{\text{exact}}(x,t) = \begin{cases} 
1 & \text{for } x < x^* + \frac{1}{2} \\
0 & \text{for } x > x^* + \frac{1}{2} 
\end{cases} \]

The different results at time \( t = 1 \) with \( x^* = 0.28 \) are shown in Figure 5 (\( \Delta t = 0.50 \Delta x \)) and in Figure 6 (\( \Delta t = \Delta x \)). The solutions obtained from the optimum LSST scheme are well behaved, but the other solutions oscillations are more pronounced.

Finally, we consider a test problem which has been chosen to assess the solution accuracy at the interior nodes in the presence of an outflow boundary. The initial and boundary conditions are

\[ u(0,t) = 0 \text{ and } u(x,0) = x \]

and the exact solution is the following

\[ u_{\text{exact}}(x,t) = \frac{x}{1+t} \]

The LSST numerical solutions and the exact solution at different times \( t = 1, 3, 5 \) with \( \Delta t = \Delta x \) are plotted in figure 7. An excellent agreement between numerical and exact solutions is observed. The following table gives the maximum values of the differences between numerical and exact solutions as a function of time and the ratio \( Y = \Delta t / \Delta x \)

<table>
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**CONCLUSIONS**

A class of space-time least-squares finite element schemes has been formulated for solving non-linear convective transport problems by adding an artificial viscous dissipation term \( \varepsilon u_{xx}t \) which appears to be effective in stabilizing non-linear situations. From numerical experiments, a precise relation of the optimum value \( \varepsilon_{\text{opt}} \) which is a function of \( \Delta x \) and \( \Delta t \) is proposed and an optimum least-squares finite element scheme can be derived and its numerical performance is illustrated by computing solutions to the inviscid Burgers' equation.

Much still remains to be done, however. Our studies have been largely confined to simple equations and one space dimension. We have begun follow up studies to further explore the advantages and limitations of the method and applications to higher dimensions and systems, which will be reported upon at a future date.
REFERENCES


Fig. 1 Nodal numbering and co-ordinate system for space-time elements

Fig. 2 Formation of a discontinuity. Initial condition
Fig. 3 Formation of a discontinuity. Solutions at $t=0.30$ with $Y = 0.50$

Fig. 4 Formation of a discontinuity. Solutions at $t = 0.30$ with $Y = 1$
Fig. 5 Propagation of shock. Solutions at $t=1$ with $Y=0.5$ and $x^*=0.28$

Fig. 6 Propagation of shock. Solutions at $t=1$ with $Y=1$ and $x^*=0.28$
Fig. 7 Influence of the outflow boundary conditions on the accuracy of the numerical solution