π-θ Method: A New Method for the Stability Assessment of Crack Growth

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ABSTRACT

Based on the continuum mechanics model, thereinafter is performed a mathematical formulation for the second derivatives of potential energy with respect to crack length in the framework of linear elasticity, in order to analyse the stability of a cracked system. We employ for this perturbation domain technique with Lagrange coordinate; this leads to an explicit expression of the second derivatives which is particularly well amenable to the numerical computations. In the final section the second derivative is shown to be equal to a path independent integral termed D-integral.

INTRODUCTION

Consider a planar body, \( \Omega (l_1, l_2) \), containing two cracks of length \( l_1 \) and \( l_2 \). \( \Omega (l_1, l_2) \), having a boundary \( \partial \Omega (l_1, l_2) \), is supposed to behave as an isotropic elastic medium. Let \( \Gamma_f \) be a part of \( \partial \Omega (l_1, l_2) \) where traction \( f \) is prescribed. Suitable boundary conditions are assumed to exist for producing equilibrium, and let \( U, \sigma \) denote the corresponding displacement vector and stress tensor, respectively.

On the other hand, besides the existence of the real loading system, let us suppose a system of virtual mechanical load, \( F_\theta \), acting on \( \Omega (l_1, l_2) \) under the same boundary conditions. Naturally, to the application of \( F_\theta \) are connected a stress tensor \( \sigma_\theta \) and a displacement vector denoted as \( \theta \) and assumed to satisfy the following properties (in the axes of figure 1):

a) \( \theta \) is restricted in a set of disjointed neighbourhoods of crack tips, such as \( \theta = \sum_i \theta_i \) \((i = 1, 2)\) with \( \theta_1 \) and \( \theta_2 \) having an empty intersection;

b) \( \theta_i \) is a constant vector and parallel to the plane of \( i^{th} \) crack in a crack tip region, say \( \Omega_i \), within which the components of \( \theta_i \) are \( \theta_i = \left( \frac{\text{cos} \alpha_i}{\text{sin} \alpha_i} \right) \) with \( i = 1, 2 \);

c) Outside the neighbourhood \( (\Omega_1 + \Omega_2) \) of the crack tip, \( \theta_i \) is identically equal to zero. \( \theta_i \) varies, therefore, between \( (\theta_1) \) and \( (\frac{\text{cos} \alpha_1}{\text{sin} \alpha_1}) \) in the band \( \Omega_i \) \((i = 1, 2)\).

Let \( V \) be the space of all the kinematically admissible displacement fields on \( \Omega (l_1, l_2) \) and \( D \) the elasticity tensor. Then for the applications:

\[ f \sim (\sigma, u), \quad F_\theta \sim (\sigma_\theta, \theta) \]  \hspace{1cm} (1)

we have:

\[ \sigma = \frac{1}{2} D (\nabla u + (\nabla u)^T) \]  \hspace{1cm} (2)

\[ \int_{\Omega (l_1, l_2)} \text{Tr} (\sigma \cdot \nabla V) \, d\Omega = \int_{\Gamma_f} f \cdot V \, d\Gamma \quad \forall \, V \in \tilde{V} \]  \hspace{1cm} (3)

and:

\[ \sigma_\theta = \frac{1}{2} D (\nabla \theta + (\nabla \theta)^T) \]  \hspace{1cm} (4)

\[ \int_{\Omega (l_1, l_2)} \text{Tr} (\sigma_\theta \cdot \nabla V) = \int_{\Omega (l_1, l_2)} F_\theta \cdot V \, d\Omega \quad \forall \, V \in \tilde{V} \]  \hspace{1cm} (5)

where "T" denotes transpose, "Tr" trace operator of tensors.
REMARKS: 1. One deduces from Eq. (4) and (5) that the virtual loading $F_\theta$ and its stress tensor $\sigma_\theta$ are limited in the bands $\Omega_1^i$ ($i \in 1, 2$) surrounding the crack tips, i.e. all the material points not located in $\Omega_1^i$ have the fields $F_\theta$ and $\sigma_\theta$ equal to zero.

2. It is proved in [3, 6] that the energy release rate of the $i^{th}$ crack can be evaluated by:

$$G_i = \int_\Omega (l_1, l_2) Tr (\sigma \cdot \nabla u \cdot \nabla \theta) - \frac{1}{2} \int_\Omega (l_1, l_2) Tr (\sigma \cdot \nabla u) \text{div} \theta_i$$

with $\theta_i$ satisfying the aforementioned properties.

PERTURBATION DOMAIN TECHNIQUE FOR THE SECOND DERIVATIVE ASSESSMENT

In the absence of crack branching, a system of straight cracks as shown in Fig. 1, may interact and their growth regime may become unstable. A general description of such problems is given in [4, 7] in which the positivity of the second variations of energy has been shown to be a sufficient condition of non-bifurcation. In this section attention is focussed on the mathematical formulations of the second derivatives of potential energy with respect to crack lengths within the framework of Lagrange approach.

Let us assume that the $j^{th}$ crack length $l_j$ ($j \in 1, 2$) is virtually advanced a small quantity indexed by a parameter $\alpha$. Let, therefore, the cracked body after the crack increment be represented by $\Omega_1^j (l_1, l_2)$, and $F_\theta^j (x)$ a function defined in $\Omega_1 (l_1, l_2)$ that maps the configuration $\Omega (l_1, l_2)$ into $\Omega_1^j (l_1, l_2)$ or vice versa:

$$\Omega (l_1, l_2) \sim F_\theta^j \sim \Omega_1^j (l_1, l_2) \quad (j \in 1, 2)$$

The mapping function $F_\theta^j (x)$ in general is defined as follows:

$$F_\theta^j (x) = \text{Id} + \alpha \nabla \Pi_j$$

with $\text{Id}$ being a unit tensor and $\Pi_j$ defined in $\Omega_1 (l_1, l_2)$, being similar to $\theta_i$, i.e. $\Pi_j$ varies in a band surrounding the crack tip $j$; within/outside this band $\Pi_j$ is identically equal to constant/zero.

For sake of simplicity one assumes otherwise that the $j^{th}$ crack length increment takes place in such a way that the two cracked bodies are identical in outer shape:

$$\partial \Omega (l_1, l_2) = \partial \Omega_1^j (l_1, l_2)$$

so that the boundary loading $f$ on $\Gamma_f$ keeps unchanged during this small crack length advance. However, the virtual loading system $F_\theta$ acting according to the REMARK 1 on $\Omega_1^j$ (Fig. 1) may be in general altered. Let $F_\theta (j)$ represent this loading system after the $j^{th}$ crack increment. Then in the configuration $\Omega_1^j (l_1, l_2)$ the applications $f$ and $F_\theta (j)$ are connected to the displacement vectors and stress tensors, defined in $\Omega_1^j (l_1, l_2)$, by:

$$f \sim [\sigma (j), u (j)], \quad F_\theta (j) \sim [\sigma_\theta (j), \theta (j)] \quad \text{for } j \in 1, 2$$

being formulated in the way like the systems (2)-(3) and (4-5).

With the help of the mapping function (7) the solutions of Eq (9) may be transformed from the changed configuration $\Omega_1^j (l_1, l_2)$ into the initial configuration $\Omega (l_1, l_2)$. Let the solutions of Eq. (9) mapped into $\Omega (l_1, l_2)$ be represented by $(\sigma^j, u^j)$ and $(\sigma_\theta^j, \theta^j)$. Then we can for each material point set:

$$(\sigma^j, u^j) = F_\theta^j (x) \cdot (\sigma (j), u (j))$$

and

$$(\sigma_\theta^j, \theta^j) = F_\theta^j (x) \cdot (\sigma_\theta (j), \theta (j)) \quad \text{for } j \in 1, 2$$

Without proof we give the following proposition:

Proposition 1:

If $[\sigma (j), u (j)], [\sigma_\theta (j), \theta (j)]$ are solutions of Eq. (9) and $(\sigma^1, u^1), (\sigma^2, u^2)$ denote their transformations by Eq. (10), (11) into $\Omega (l_1, l_2)$, then $\exists \alpha > 0, \text{ sufficient small, for which we have:}$

$$(\sigma^1, u^1) = (\sigma, u) + \alpha (\sigma^\pi, u^\pi) + \alpha^2 \Omega$$

$$(\sigma_\theta^1, \theta^1) = (\sigma_\theta, \theta) + \alpha (\sigma_\theta^\pi, \theta^\pi) + \alpha^2 \Omega$$

$$(j \in 1, 2)$$
where \((\sigma, u)\) and \((\sigma_\theta, \theta)\) are solutions of Eq. (1), \((\sigma^\pi, u^\pi)\) and \((\sigma_\theta^\pi, \theta^\pi)\), representing the Lagrange variations of \((\sigma, u)\) and \((\sigma_\theta, \theta)\) when the \(j^{th}\) crack in \(\Omega (l_1, l_2)\) is advanced a small increment \(\alpha\), are solutions of the following equations:

\[
\sigma^\pi_j = \frac{1}{2} D \left[ \nabla u^\pi_j + t \nabla \theta^\pi_j - (\nabla u \cdot \nabla \theta^\pi_j + t \nabla \theta^\pi_j \cdot t \nabla u) \right]
\]

\[
\int_\Omega (l_1, l_2) \text{Tr} (\sigma^\pi_j \cdot \nabla V) \, d\Omega = \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla V \cdot \nabla \pi_j) \, d\Omega
\]

\[
- \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla V) \div \pi_j \, d\Omega \quad \forall \, V \in \tilde{V}
\]

(15)

and

\[
\sigma_\theta^\pi_j = \frac{1}{2} D \left[ \nabla \theta^\pi_j + t \nabla \theta^\pi_j - (\nabla \theta \cdot \nabla \theta^\pi_j + t \nabla \theta^\pi_j \cdot t \nabla \theta) \right]
\]

\[
\int_\Omega (l_1, l_2) \text{Tr} (\sigma_\theta^\pi_j \cdot \nabla V) \, d\Omega = \int_\Omega (l_1, l_2) \text{Tr} (\sigma_\theta \cdot \nabla V \cdot \nabla \pi_j) \, d\Omega - \int_\Omega (l_1, l_2) \text{Tr} (\sigma_\theta \cdot \nabla V) \div \pi_j \, d\Omega
\]

\[
+ \int_\Omega (l_1, l_2) F_\theta \div \pi_j \cdot V + \int_\Omega (l_1, l_2) \nabla F_\theta \cdot \pi_j \cdot V \, d\Omega \quad \forall \, V \in \tilde{V}
\]

(17)

SECOND VARIATION CALCULATIONS

Similar to Eq. (6), the energy release rate associated to the \(i^{th}\) crack in the configuration \(\Omega^\alpha (l_1, l_2)\) is expressed as:

\[
G_{i1}^j = \int_\Omega (l_1, l_2) \text{Tr} [\sigma (j) \cdot \nabla u (j) \cdot \nabla \theta_1 (j)] - \frac{1}{2} \int_\Omega (l_1, l_2) \text{Tr} [\sigma (j) \cdot \nabla u (j)] \div \theta_1 (j)
\]

(18)

with \(\Sigma \theta_1 (j) = \theta (j)\) and \([\sigma (j), u (j)], [\sigma_\theta (j), \theta (j)]\) being solutions of Eq. (9). Letting \(G_{i1}^j\) represent \(G_i^j\) mapped into the configuration \(\Omega (l_1, l_2)\):

\[
G_{i1}^j = F_{i1}^\Sigma \cdot G_i^j (j) \quad (i, j \in 1, 2)
\]

(19)

where \(G_{i1}^j\) can be easily carried out by substituting Eq. (12) (13) into Eq. (19), then, the variation of \(G_{i1}^j\) of Eq. (6) in the direction \(\pi_j\)

\[
\frac{\partial G_{i1}^j}{\partial \pi_j} = \lim_{\alpha \to 0} \frac{G_{i1}^j - G_{i1}^j}{\alpha} \quad (i, j \in 1, 2)
\]

(20)

may be calculated, and the following proposition holds:

Proposition 2:

With the same assumptions stated in the Proposition 1 on the relevant unknowns \((\sigma_j^\pi, u_j^\pi)\) and \((\sigma_\theta_j^\pi, \theta_j^\pi)\), the second variations of potential energy may be proved as:

\[
\frac{\partial G_{i1}^j}{\partial \pi_j} = - \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla u \cdot \nabla \pi_j \cdot \nabla \theta_1 \cdot \nabla \theta_1) \, d\Omega - \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla u \cdot \nabla \theta_1 \cdot \nabla \theta_1) \, d\Omega
\]

\[
+ \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla u \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j) \, d\Omega + \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla u \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j) \, d\Omega
\]

\[
+ \frac{1}{2} \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla u \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j) \, d\Omega
\]

\[
+ \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla u \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j) \, d\Omega
\]

\[
+ \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla u \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j) \, d\Omega
\]

\[
- \int_\Omega (l_1, l_2) \text{Tr} (\sigma \cdot \nabla u \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j \cdot \nabla \pi_j) \, d\Omega
\]

(21)

with \(i, j \in 1, 2\) and \((\sigma, u), (\sigma_\theta, \theta)\) being solutions of Eq. (1).
REMARKS: By substituting \((\sigma_j^\pi, u_j^\pi)\) and \((\sigma_j^\theta, u_j^\theta)\) defined respectively by (14) - (15) and (16) - (17) into (21), the right side of Eq. (21) may be shown to be symmetric with respect to \(\pi_j\) and \(\theta_i\), i.e.
\[
\frac{\partial G_{ij}}{\partial \theta_i} (\pi_j, \theta_i) = \frac{\partial G_{ij}}{\partial \pi_j} (\pi_j, \theta_i) \text{ for } i, j \in 1, 2
\]
This means that the matrix \(\frac{\partial G_{ij}}{\partial \theta_i}\) is symmetric.

SPECIAL CASE

Eq. (21) gives a general expression of the second variations of potential energy with respect to the crack length when an arbitrary mapping function \(\pi_j\) is taken into account. Because of its complexity, an excessive computation cost is required for accomplishing such a calculation. From the fact that the gradients of \(\pi_j\) and \(\theta_i\), are restricted in two bands around the crack tips, (21) will be simplified making these two bands disjointed (see Figure 2).
In such a case, all the terms having \((\partial \pi_j \times \partial \theta_i)\) will tend toward null and the following proposition gives the simplified expression of \(\frac{\partial G_{ij}}{\partial \theta_i}\):

Proposition 3:

If the bands supporting the gradient of \(\pi_j\) have an empty intersection with these of \(\theta_i\), the second variation of Eq. (21) reduces, in this case, to:
\[
\frac{\partial G_{ij}}{\partial \theta_i} = \int_{\Omega} (\sigma \cdot \nabla v \cdot \nabla \theta_i) \, d\Omega + \int_{\Omega} (\sigma \cdot \nabla u_j^\pi \cdot \nabla \theta_i) \, d\Omega - \int_{\Omega} (\sigma \cdot \nabla u_j^\pi \cdot \nabla \theta_i) \, d\Omega
\]
with \((\sigma, u)\) and \((\sigma_j^\pi, u_j^\pi)\) being defined by (2) - (3) and (14) - (15), and \(i, j \in 1, 2\).

PATH INDEPENDENT D INTEGRAL

Using Stokes' formula [5], one can obtain from Eq. (22) the expression of \(\frac{\partial G_{ij}}{\partial \theta_i}\) in term of a path integral, let us call it D integral, which in two-dimensional crack problems \(\partial \theta_i\) has the following form:
\[
D = \int_{C_i} \frac{1}{2} \, \text{Tr} (\sigma \cdot \nabla \nu + \sigma \cdot \nabla u_j^\pi) \, dX_2 - (\sigma \cdot \vec{n}^i + \sigma \cdot \vec{n}^j) \cdot \frac{\partial u_j^\pi}{\partial X_1} + \sigma \cdot \vec{n}^j \, dS, \quad (i, j \in 1, 2)
\]
where \(C_i\) is a path surrounding the crack tip \(i\) (see figure 3), \(\vec{n}^j\) being the outward normal unit vector to the curve \(C_i\). As a matter of fact, in two-dimensional planar bodies the contour integral D given by (23) can be shown to be path independent when evaluated along any path \(C_i\) beginning on the bottom of a crack face parallel to the \(X_1\)-axis, as in Fig. 3, and ending on the upper face.

SECOND VARIATIONS FOR THERMAL PROBLEMS

The measurement of the second variations by Eq. (22) is possible for any structure under any mechanical loading condition. Since thermal strains give rise to equivalent thermal loads over the total cracked domain, one has to add in Eq. (23) some surface integrals on the area enclosed by the curve \(C_i\):

Proposition 4:

Let \(T\) represent the prescribed temperature gradient between two considered states. In the aforesaid complex case with respect to the two vectors \(\pi_j\) and \(\theta_i\), the second derivatives of potential energy may be expressed for \(i, j \in 1, 2\) as:

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\[
\frac{\delta G_i}{\delta l_j} = \int_{\Omega} (1, l_2) \text{Tr} (\sigma_j^\pi \cdot \nabla \theta_i) \, d\Omega + \int_{\Gamma} (1, l_2) \text{Tr} (\sigma_i \cdot \nabla u_j^\pi \cdot \nabla \theta_i) \, d\Omega \\
- \frac{1}{2} \int_{\Omega} (1, l_2) \text{Tr} (\sigma_j^\pi \cdot \nabla u) \, \text{div} \theta_i \, d\Omega \\
- \frac{1}{2} \int_{\Omega} (1, l_2) \text{Tr} (\sigma_j^\pi \cdot \nabla u_j^\pi \cdot \nabla \theta_i) \, d\Omega + \frac{1}{2} \int_{\Omega} (1, l_2) \alpha \cdot \mathbf{T} \cdot \text{Tr} (\sigma_j^\pi) \, \text{div} \theta_i \, d\Omega \\
+ \frac{1}{2} \int_{\Omega} (1, l_2) \alpha \cdot \theta_i \cdot \nabla \mathbf{T} \cdot \text{Tr} (\sigma_j^\pi) \, d\Omega + \int_{\Omega} (1, l_2) \alpha \cdot \theta_i \cdot \nabla \mathbf{T} \cdot \text{Tr} (\sigma_j^\pi) \, d\Omega \\
+ \int_{\Omega} (1, l_2) \alpha \cdot \theta_i \cdot \nabla (\nabla \mathbf{T}) \cdot \pi_j \cdot \text{Tr} (\sigma_j^\pi) \, d\Omega
\]  

(24)

where \( \alpha \) represents the thermal expansion coefficient of used material, \((\sigma, u)\) and \((\sigma_j^\pi, u_j^\pi)\) are two solutions of the following equations:

\[
\sigma = D \left[ \frac{1}{2} \left( \nabla u + \nabla u^T \right) - \alpha \cdot \nabla \mathbf{T} \right] \text{Id}
\]  

(25)

\[
\int_{\Omega} (1, l_2) \text{Tr} (\sigma \cdot \nabla \mathbf{V}) \, d\Omega = \int_{\Gamma} \mathbf{f} \cdot \mathbf{V} d\Gamma \quad \forall \mathbf{V} \in \mathbf{V}
\]  

(26)

for solving \((\sigma, u)\)

and:

\[
\sigma_j^\pi = D \left[ \frac{1}{2} \left( \nabla u_j^\pi + \nabla u_j^\pi \right) - \alpha \cdot \nabla \mathbf{T} \right] \text{Id}
\]  

(27)

\[
\int_{\Omega} (1, l_2) \text{Tr} (\sigma_j^\pi \cdot \nabla \mathbf{V}) \, d\Omega = \int_{\Omega} (1, l_2) \text{Tr} (\sigma \cdot \nabla \mathbf{V} \cdot \nabla \mathbf{\pi}_j) \, d\Omega \\
- \int_{\Omega} (1, l_2) \text{Tr} (\sigma \cdot \nabla \mathbf{V}) \, \text{div} \pi_j \, d\Omega \quad \forall \mathbf{V} \in \mathbf{V}
\]  

(28)

for solving \((\sigma_j^\pi, u_j^\pi)\).

**REMARKS:**

1. Eq. (24), used to calculate the second derivatives of thermal potential energy, may also be proved to be connected to a path independent integral. It is the expression of the \( D \) integral when the thermal loads are considered in the formulations.

2. For testing the presented \( \pi-\theta \) method, several applications have been made for some simple-shaped structures under certain loading conditions for which we can deduce from the analytical solutions of energy release rate, the second derivatives of potential energy with respect to crack lengths. Obtained numerical results are in excellent agreement with these analytical solutions in both purely mechanical problems and thermo-mechanical problems.

**CONCLUSIONS**

Without any a priori assumption about the mode of crack propagation, an analytical expression of the second variations of potential energy has been derived by introducing a virtual loading system. Used to assess the growth pattern of a system of interacting cracks, the second variations have been rigorously proved, as an interesting consequence, to be related to a path independent integral - \( D \) integral. Finally, it must be noted that our derivation is equally valid for cracks growing off at an angle to the original crack plane and the principal results can also be extended in the context of elastic-plastic materials obeying the deformation theory of plasticity.

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REFERENCES


![Fig. 1: Virtual loading system $F_\theta$ and its associated $\theta$ displacement field for calculating the energy release rate](image1)

![Fig. 2: General case $(\pi \cap \theta \neq \phi)$ and simplified case $(\pi \cap \theta = \phi)$ for the mapping function $\pi_j$.](image2)

![Fig. 3: $\pi$ and $\theta$ for determination of the matrix of the second variations of a cracked body $\Omega(l_1,l_2)$](image3)