On the Application of G(θ) Method and its Comparison with De Lorenzi’s Approach

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INTRODUCTION

It has become common to characterize a material's static crack initiation under monotonically increasing deformation condition by using J-integral or the energy release rate G. A large number of methods have been proposed for calculating G or J, and the best of which include the virtual crack extension technique introduced by Hellen [1] and Parks [2]. The earlier formulations of this method were developed solely through a finite element approach and the resulting expressions were, therefore, based on finite element matrix formulations [1,2]. Recently, de LORENZI [3] has derived an analytical expression for the energy release rate by the application of the virtual crack extension principle to a continuum mechanics model, and a general 3-D expression including the effect of body force for the energy release rate has been deduced.

Because of some scatter in numerical analysis, either by the method of Hellen [1], Parks [2] or by that of de Lorenzi [3], one often has to use an average value for the energy release rate obtained through a sequence of different crack extensions. The computer cost would be, at this point, expensive with too long computer time consuming, especially when the material around the crack front in 3-D crack configuration problems undergoes a substantial amount of plastic flow [4]. To bypass this inconvenience we briefly present in this paper the principles of a new method: G(θ) method in which an infinitesimal crack length increment is assumed and G is formulated by means of a specific function, θ. This method was first developed by Destuynder et al. [5] and is used to calculate precisely the energy release rate in both linear and non-linear elastic fields, in 2-D as well as in 3-D media for both purely mechanical problems and thermo-mechanical problems.

In the final sections, we discuss the possibility of using G(θ) method to solve 3-D fracture problems with linear elastic materials or elasto-plastic materials following the deformation theory of plasticity.

PRINCIPLES OF G(θ) METHOD

Suppose that we have a planar linear elastic body of unit thickness Ω(l) . l indicates that the body contains a crack of length l. The cracked body Ω(l) is assumed to be subjected to three types of loads: surface tractions F and the body force f as well as a system of thermal loads, ΔT representing the prescribed temperature gradient between initial state Τ₀ and present state Τ. For the sake of simplicity, the crack faces are assumed to be free of tractions. Let σ be the stress tensor and U the displacement vector related to the applications of (F, f, ΔT). Then for a set of kinematically admissible displacement fields, V, we must have:

\[ \sigma = D \left( \frac{1}{2} (\nabla u + \nabla u^t) - \alpha \Delta T. Id \right) \]  

\[ \int_{\Omega(l)} \text{Tr} (\sigma : \nabla V) \, d\Omega = \int_{\partial \Omega(l)} F \nu d\Gamma + \int_{\Omega(l)} f V d\Omega \quad \forall V \text{ admissible fields} \]  

with D = elasticity tensor of Ω(l)
Id = a unit tensor
α = thermal expansion coefficient
\[ \frac{1}{2} (\nabla u + \nabla u^t) = \text{total strain tensor} = \frac{1}{2} (U_{ij} + U_{ji}, i) \text{ in cartesian coordinate system} \]
\( \text{Tr} = \text{Trace operator} \)

In equilibrium, we may express the potential energy \( W(I) \) of \( \Omega(I) \) as:

\[
W(I) = \frac{1}{2} \int_{\Omega(I)} \text{Tr} (\sigma : \nabla u) \, d\Omega - \frac{1}{2} \int_{\Omega(I)} \text{Tr} (\sigma) \, d\Omega - \int_{\Omega(I)} f \cdot u \, d\Omega - \int_{\Omega(I)} F \cdot u d\Gamma \quad (3)
\]

Now, let us consider an infinitesimal crack length extension \( \delta l \), so that the crack length becomes \( (1 + \delta l) \). \( \Omega (1 + \delta l) \) denotes the same solid containing a crack of length \( (1 + \delta l) \). In the changed state, the mechanical loads become \( F (1 + \delta l) \) and \( f (1 + \delta l) \), and the temperature gradient \( \Delta T (1 + \delta l) \). Let \( F(I) \) be a mapping function defined in \( \Omega(I) \), which maps all the mechanical arguments defined in \( \Omega (1 + \delta l) \) (forces, stress tensor etc.) into \( \Omega(I) \):

\[
F(I) = \text{Id} + \delta l \cdot \nabla \theta \quad (4)
\]

where \( \theta \) is a vector field defined in \( \Omega(I) \), equal to zero outside a neighbourhood \( (\Omega_0 + \Omega_1) \) of the crack tip \( P \), being assumed to be 1 and parallel to the crack plan within the domain \( \Omega_0 \), \( \theta \) varies therefore between 1 and 0 in the band \( \Omega_1 \) (fig. 1).

After extended, the stress tensor, displacement field and the potential energy are noted, in the configuration \( \Omega(I) \), respectively as \( \sigma(1 + \delta l), u(1 + \delta l) \) and \( W(1 + \delta l) \), being associated with the applications of \( F(I) = \text{Id} + \delta l \cdot \nabla \theta \), \( f(I) = \text{Id} + \delta l \cdot \nabla \theta \), \( \Delta T(1 + \delta l) \) by:

\[
\sigma(1 + \delta l) = \frac{1}{2} D \left[ (\nabla u)(1 + \delta l) + (\nabla u)^T (1 + \delta l) \right] - D \alpha \Delta T (1 + \delta l) \cdot \text{Id} \quad (5)
\]

\[
\int_{\Omega(I)} \text{Tr} [\sigma(1 + \delta l) \cdot (\nabla V)(1 + \delta l)] \, d\Omega = \int_{\Omega(I)} \frac{\partial \sigma(1 + \delta l)}{\partial \Omega} \cdot F(1 + \delta l) \, d\Omega + \int_{\Omega(I)} \frac{\partial \Omega}{\partial \Omega} \cdot f(1 + \delta l) \, d\Omega \quad (6)
\]

\[
\forall \in \text{admissible fields}
\]

and \( W(I + \delta l) = \frac{1}{2} \int_{\Omega(I + \delta l)} \text{Tr} [\sigma(1 + \delta l) \cdot (\nabla V)(1 + \delta l)] \, d\Omega - \frac{1}{2} \int_{\Omega(I + \delta l)} \text{Tr} [\sigma(1 + \delta l) \cdot \alpha \Delta T (1 + \delta l)] \, d\Omega - \int_{\Omega(I + \delta l)} f(1 + \delta l) \cdot U(1 + \delta l) \, d\Omega - \int_{\partial \Omega(I + \delta l)} F(1 + \delta l) \cdot U(1 + \delta l) \, d\Omega \quad (7)
\]

With the help of the mapping function \( F(I) \), we have for any function \( \phi \):

\[
(\nabla \phi) \cdot (1 + \delta l) = \nabla \phi : (\text{Id} + \delta l \cdot \nabla \theta)^{-1} = \nabla \phi : (\text{Id} - \delta l \cdot \nabla \theta) \quad (8)
\]

\[
\int_{\Omega(I + \delta l)} \phi \, d\Omega = \int_{\Omega(I)} \phi \, d\Omega + \int_{\Omega(I)} \phi \, d\Omega - \int_{\Omega(I + \delta l)} \phi \, d\Omega \quad (9)
\]

Since the energy release rate is defined as:

\[
G = \lim_{\delta l \to 0} \frac{W(I + \delta l) - W(I)}{\delta l} \quad (10)
\]

combining the previous equations, we obtain the derivative of potential energy with respect to the crack length in the \( \theta \) direction:

\[
- \frac{\partial G}{\partial \theta} = G = \int_{\Omega(I)} \text{Tr} (\sigma : \nabla u : \nabla \theta) \, d\Omega - \frac{1}{2} \int_{\Omega(I)} \text{Tr} (\sigma : \nabla u) \, d\Omega + \int_{\Omega(I)} \text{Tr} (\sigma) \cdot \nabla (\Delta T) \, d\Omega + \frac{1}{2} \int_{\Omega(I)} \text{Tr} (\sigma) \cdot \alpha \Delta T \, d\Omega + \int_{\Omega(I)} f \cdot u \, d\Omega + \int_{\partial \Omega(I)} \nabla f \cdot \theta \, d\Omega \quad (11)
\]

in which \( \sigma \) and \( u \) are solutions of system (1) - (2).

In the absence of thermal loads, \( \Delta T = \xi \), and if we neglect the effect of the body force, \( f = 0 \) then Eq. (11) reduces to:

\[
G(\theta) = \int_{\Omega(I)} \text{Tr} (\sigma : \nabla u : \nabla \theta) \, d\Omega - \int_{\Omega(I)} W \cdot d\Omega \quad (12)
\]

in such a case de Lorenzi's method [3] gives a similar formula:
\[ G(\Delta X_k) = \frac{1}{Ac} \int_{\Omega(l)} \left\{ \sigma_{ij} \frac{\partial u_j}{\partial X_k} - W \delta_{ik} \frac{\partial \Delta X_k}{\partial X_i} \right\} \, d\Omega \]  

where \( W \) is the strain energy density \( W = \frac{1}{2} \text{Tr} (\sigma : \nabla u) \), \( Ac \) the increase in cracked area generated by the virtual crack extension in a small amount \( \Delta l \), \( \Delta X_k \) a mapping function which maps the body containing the crack 1 into a body with a slightly increased crack length \( (l + \Delta l) \). \( \Delta X_k \) plays obviously a role like the mapping function \( \theta \) in Eq (12).

The resulting expression of \( G(\theta) \) method (12) is very similar to (13), but \( G(\theta) \) formulations are different to (13) essentially in the way that we consider an infinitesimal crack length increment \( \delta l \) rather than a finitely small quantity \( \Delta l \) for the derivation of the energy release rate, and a specific displacement vector \( \theta \) has been introduced to access the mechanical argument transformations between two different crack length configurations \( \Omega(l) \) and \( \Omega(l + \delta l) \).

EXTENSION OF \( G(\theta) \) METHOD TO 3D PROBLEMS

In the design of three-dimensional structures containing cracks with arbitrary profile, the knowledge of the energy release rate along the crack front is required. It is hoped that the presented \( G(\theta) \) method provides a more efficient tool of obtaining acceptably accurate energy release rate with relatively coarse meshes.

As an indication of how the \( G(\theta) \) method may be applied to any three-dimensional problems, consider a surface flaw in a finite thickness plate, schematically depicted in Fig. 2. After the stress-displacement analysis, for example, by finite element method, the stress tensor and displacement vector near the crack front are known. The local energy release rate \( G(\theta) \) at a typical node "i" is obtained assuming that the node "i" undergoes an infinitesimal local advance \( \delta l_i \) in the direction \( \theta_i \) normal to the crack front:

\[ G(\theta_i) = \int_{\Omega(l)} \left( \sigma : \nabla u \cdot \nabla \theta_i \right) \, d\Omega \ + \frac{1}{2} \int_{\Omega(l)} \left( \sigma : \nabla \Delta T \cdot \theta_i \right) \, d\Omega \ + \frac{1}{2} \int_{\Omega(l)} \left( \sigma : \nabla \Delta T \cdot \theta_i \right) \, d\Omega \ + \int_{\Omega(l)} f \cdot \nabla \theta_i \, d\Omega \ + \int_{\Omega(l)} \nabla f \cdot \theta_i \, d\Omega \]  

with the same notations adopted in the previous sections, and \( \theta_i \) being displacement vector equal to 1 at node "i". (Fig. 2).

For calculating the energy release rate along the crack front, \( G(S) \), at the point S, we write the equation:

\[ -\delta W = \int_{\text{crack front}} G(S) \cdot \delta l(S) \, dS \]  

where \( \delta l(S) \) is the increment of crack advance normal to the crack front at position S due to the advance of the node "i" by \( \delta l_i \), and \( dS \) is an element of arc length along the crack front.

To evaluate the line integral in (15) in order to derive estimates of \( G(S) \), we assume that \( G(S) \) and \( \delta l(S) \) can be approximated by a piece-wise linear function among the adjacent nodal points on the crack front:

\[ G(S) = \sum_{i=1}^{n} G_i \cdot M_i(S) \]

\[ \delta l(S) = \sum_{i=1}^{n} \delta l_i \cdot N_i(S) \]

According to experiences gained with the use of de Lorerzi's method, the best form functions of \( M_i \) and \( N_i \) in case of quadratic interpolating elements would be the following: in a segment of crack front of length \( l_i \) having 3 nodal points (see Fig. 3) when the local crack advance occurs at the corner points "i-1" or "i+1", we set:

\[ G(S) = G_{i+1} \cdot S + G_{i-1} \cdot (1 - S) \]

\[ \delta l(S) = 2 \cdot \delta l_i \cdot (S - \frac{1}{2}) (S - 1) \quad \forall S \in [0, 1] \]  

If instead, the local crack advance occurs at the middle point "i" of the segment \( l_i \), we set in such a case:

\[ 167 \]
\[ G(S) = G_{i-1} = G_1 = G_{i+1} \quad \forall S \in [0, 1] \] (17)

Substituting Eq. (16) or (17) into Eq. (15), a simple calculation leads to derive the energy release rate along the crack front:

\[ G_1 = \frac{2G(\theta_i)}{1_i} \quad \text{for the corner nodal points} \]

\[ G_i = \frac{3G(\theta_i)}{2_i} \quad \text{for the middle nodal points} \]

where \( G(\theta_i) \) is defined by Eq. (14), which has the following form for materials obeying the deformation theory of plasticity:

\[
G(\theta_i) = \int_{\Omega(1)} \text{Tr} \left( \sigma : \nabla \theta_i \right) d\Omega - \int_{\Omega(1)} W \cdot \text{div} \theta_i d\Omega + \int_{\Omega(1)} \text{Tr}(\sigma) \cdot \alpha \nabla T \cdot \theta_i d\Omega \\
+ \int_{\Omega(1)} f \cdot \text{div} \theta_i d\Omega + \int_{\Omega(1)} \nabla f \cdot \theta_i d\Omega
\] (18)

with \( W \) being the strain-energy density defined as:

\[ W = \int_0^{e} \sigma_{ij} d (\epsilon_{ij} - \alpha \Delta T) \] (19)

### NUMERICAL RESULT COMPARISONS

To benchmark the \( G(\theta) \) method described in preceding sections, the energy release rate was calculated and validated by a number of 2D as well as 3D crack configuration problems with both linear and non-linear elastic materials using the finite element GTETA procedure embedded in CASTEM 2000 calculation code, and the computed solutions were always compared against those obtained by alternate methods. A recent example aimed at the validation of GTETA procedure was carried out for a circular plate containing a center-surface-crack with an elliptical crack front as shown in Fig. 4. Because of the symmetry only a quarter of the plate was analyzed with 235 twenty-noded cubic elements and 50 fifteen-noded prismatic elements (Fig. 5). To this plate was applied a system of thermal loads. Shown in Fig. 6 are the prescribed temperature gradient fields between two different states. Obtained numerical results are summarized in the following table in contrast with those by de Lorenzi's formula.

### ENERGY RELEASE RATE ALONG THE CRACK FRONT

<table>
<thead>
<tr>
<th>POINTS</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G(\theta) ) method's results</td>
<td>9.40</td>
<td>10.54</td>
<td>11.52</td>
<td>12.78</td>
<td>13.00</td>
<td>11.73</td>
</tr>
<tr>
<td>de Lorenzi's results when ( \Delta a = 0.01 \text{ mm} )</td>
<td>9.36</td>
<td>10.48</td>
<td>11.50</td>
<td>12.72</td>
<td>13.05</td>
<td>11.72</td>
</tr>
</tbody>
</table>

### SUMMARY

From the consideration of an infinitesimal amount of crack advance, we have developed the virtual crack extension method, and a general expression for the energy release rate in an arbitrary 2-D or 3-D crack configuration has been derived from the continuum mechanics standpoint. Implementation of the proposed method in finite element procedure for solving any 3-D crack problem has been particularly discussed. For a 3-D elastic example containing a surface crack in a circular plate the numerical results of the energy release rate along the crack front using the proposed \( G(\theta) \) method are in excellent agreement with those obtained by de LORENZIS formula.
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REFERENCES


Fig. 3: $N_1, M_1$ selections when local crack increment takes place at the corner and the middle points.

Fig. 4: Studied circular plate containing one surface crack of elliptical profile.

Fig. 5: The mesh of one-quarter of the plate.

Fig. 6: Applied temperature gradient field for the energy release rate calculation along the crack front (unit °C).

A 3.23E+02
B 3.77E+02
C 3.50E+02
D 3.85E+02
E 3.77E+02
F 3.88E+02
G 4.03E+02