A Simplfied Inelastic Dynamic Analysis of the Maximum and Limit States of Structure

S. H. Lee, J. Zarka
Ecole Polytechnique, Palaiseau, France

ABSTRACT
A new direct method to evaluate the maximum and limit states of inelastic structures under dynamic loading, is presented. It is based on the full analytical solutions of a single-degree-of-freedom (SDOF) system which represents a kinematically hardening material and which is subjected to an instantaneous constant force (step loading). The introduction of the constitutive material nonlinearity as described by Zarka (1979), allows to propose rules to obtain the inelastic displacements, plastic strains, and residual stresses fields with only ***two or three static elastic analyses****, allowing a drastic saving in the calculation time and cost during dynamic inelastic analysis.*

INTRODUCTION

The inelastic dynamic analysis is now often performed for the safe structural design against the seismic events or explosions. During the last three decades, with the development of computers, many powerful methods of analysis such as direct integration methods (Houbolt, 1950, Newmark, 1959, Wilson *et al.*, 1973, Park, 1975, Hughes *et al.*, 1978) and modal superposition methods (Stricklin *et al.*, 1977, Liu and Lin, 1979, Shan *et al.*, 1979, Geschwinder, 1981, Polizzotto, 1986) were developed. Nevertheless, this type of analysis is very expensive and generally very complicate to use. The methods used in the domain of the dynamic plasticity adopt the step-by-step incremental procedure, which often leads to many difficult problems such as instability of algorithms, slow convergence of equilibrium iterations, inaccuracy of results in cases of severe plasticity and at last very long and expensive calculations. The inaccuracy of analysis, encountered in the limit state analysis, by the standard direct integration methods, is shown in details for example by Symonds *et al.* (1985, 1986, 1987) and implies the development of dynamic analysis algorithms.

On the other hand, in many practical cases, the informations on the maximum and limit states of the structure are very important for the protection from the local or global failures of the structure. Some methods such as the dynamic shakedown analyses (Ho, 1972, Corradi and Maier, 1974, Polizzotto, 1984) extended from the static shakedown theories and the displacement bound techniques (Martin, 1966, Maier and Corradi, 1974, Ponter, 1975, Capurso, 1975) give practical informations without performing the full dynamic analysis. But such analyses are based on convex analysis or mathematical programming and do not appear clearly in the mind of a structural engineer.

Therefore, it was necessary to develop a simplified method having the following characteristics :

1) inexpensive direct analysis instead of the step-by-step incremental analysis,
2) fast convergence, unconditional stability, practically acceptable accuracy, and easy usage.

In the spirit of above, Zarka *et al.* (1985) proposed a new approach to the dynamic analysis by expanding the simplified methods during quasi-static cyclic loadings (1979, 1980, 1989). This new method gives a good approximation of limit states in the case of very large loadings.

We have presented an extension of the direct method to evaluate the maximum and limit states, to more general cases of dynamic loadings (see for more details Lee, 1989).

In this special paper, we choose to focus on instantaneous constant forces as dynamic loading. In this case, complete analytical solutions can be found for a single-degree-of-freedom system. Using these analytical solutions, we can reduce our dynamic problem to a static elastoplastic problem which can be resolved by the standard simplified method.

ANALYTICAL DETERMINATION OF THE MAXIMUM AND LIMIT STATES OF A SDOF

Elastoplastic system during an instantaneous constant force

Equations of Motion

A simple elastoplastic system with kinematically hardening material is shown in Fig. 1. It is composed of one mass, one elastic spring, one inelastic internal spring, and one friction block. The governing equation of motion is written as

$$m\ddot{x} + k\,(x - \alpha) = f(t) = F \tag{1}$$

with the initial conditions :

$$x(0) = x_0,\ \dot{x}(0) = \dot{x}_0\ ,\ \alpha(0) = \alpha_0$$

where m denotes the mass, k the stiffness of elastic spring, $\ddot{x}(t)$, $\dot{x}(t)$, $x(t)$, and $\alpha(t)$ are, respectively, the acceleration, the velocity, the displacement, and the inelastic displacement at time t, f(t) is the time dependent force.

The equation of evolution is given in the symbolic form (Moreau, 1971) as follows :

$$\dot{\alpha} \in \partial\Psi_{C_0}(\sigma) \tag{2}$$

i.e.

$$\sigma = k\,(x - \alpha) - h\alpha \in C_0 = [\ -S\ ,\ S\ ] \tag{3}$$

and

$$\begin{cases} \dot{\alpha} \equiv 0 & \text{if } |\,\sigma\,| < S \\ \dot{\alpha} \geq 0 & \text{if } \sigma = S \\ \dot{\alpha} \leq 0 & \text{if } \sigma = -S \end{cases} \tag{4}$$

where C_0 is the fixed convex set (here a segment) which defines the elastic domain of the model, $\dot{\alpha}(t)$ is the velocity of inelastic displacement, S the threshold of plasticity, $\sigma(t)$ the stress on the friction block, and h the stiffness of the internal spring or the strain hardening coefficient.

First Integrals

Elastic response phases : In the phases of the elastic loadings or unloadings, the governing equation of motion is written as

$$m\ddot{x} + kx = F + k\alpha_0 \tag{5}$$

which has the first integral :

$$(\dot{x}/\omega)^2 + [x - (F/k + \alpha_0)]^2 = (\dot{x}_0/\omega)^2 + [x_0 - (F/k + \alpha_0)]^2 \tag{6}$$

where ω is the angular frequency of the elastic system defined by $\sqrt{k/m}$.

As shown by Newmark *et al.* (1971), the trajectories in the phase plane $(x, \dot{x}/\omega)$ are deduced from the first integral and are drawn in Fig. 2. They are half-circles centered in $(F/k + \alpha_0, 0)$ and with the radius :

$$r^{el} = \sqrt{(\dot{x}_0/\omega)^2 + [x_0 - (F/k + \alpha_0)]^2} \tag{7}$$

which is equal to the amplitude of the elastic vibration.

Elastoplastic response phases : During plastic loadings or plastic reloadings, the governing equation is given by :

$$m\,\ddot{x} + k_t\,x = \hat{F} \tag{8}$$

with

$$\begin{aligned} \hat{F} &= F - kS/(k + h) \quad \text{if } \dot{x} \geq 0 \\ \hat{F} &= F + kS/(k + h) \quad \text{if } \dot{x} \leq 0 \end{aligned} \tag{9}$$

where $k_t = kh/(k + h)$ is the tangential stiffness.

The first integral can be obtained in the same manner as in the elastic response :

$$(\dot{x}/\omega_t)^2 + (x - \hat{F}/k_t)^2 = [\dot{x}(t_y)/\omega_t]^2 + [x(t_y) - \hat{F}/k_t]^2 \tag{10}$$

where ω_t is the angular frequency of the tangential system given by $\sqrt{k_t/m}$, and $\dot{x}(t_y)$ and $x(t_y)$ are, respectively, the velocity and displacement at the time when the system begins to be plastified and are given by :

$$\dot{x}(t_y) = \sqrt{(k/m)\,[(\dot{x}_0/\omega)^2 + [x_0 - (F/k + \alpha_0)]^2 - (h\alpha_0/k - (F - \epsilon S)/k)^2]} \tag{11}$$

and

$$x(t_y) = \frac{k + h}{k}\,\alpha_0 + \epsilon S \tag{12}$$

with

$$\epsilon = \dot{x}_0/|\dot{x}_0| \text{ if } \dot{x}_0 \neq 0 \ , \ \epsilon = F/|F| \text{ if } \dot{x}_0 = 0 \text{ and } F \neq 0 \ , \ \epsilon = -\sigma_0/|\sigma_0| \text{ if } \dot{x}_0 = F = 0 \quad (13)$$

and

$$\sigma_0 = kx_0 - (k + h)\,\alpha_0 \,.$$

It is easy to see that the trajectories in the phase plane $(x, \dot{x}/\omega_t)$ represent also half-circles centered in $(\hat{F}/k_t, 0)$ and with the radius :

$$r = \sqrt{[\dot{x}(t_y)/\omega_t]^2 + [x(t_y) - \hat{F}/k_t]^2} \quad (14)$$

which is equal to the amplitude of the elastoplastic vibration.

The first radius of the elastoplastic system can be obtained as follows :

$$r_0 = \sqrt{\frac{k}{k_t}\left[\left(\frac{\dot{x}_0}{\omega}\right)^2 + \left(x_0 - \left[\frac{F}{k} + \alpha_0\right]\right)^2 - \left(\frac{h}{k}\alpha_0 - \frac{F - \epsilon S}{k}\right)^2\right] + \left(\frac{k}{k_t}\alpha_0 - \frac{F - \epsilon S}{k_t}\right)^2} \quad (15)$$

Determination of the Maximum State

The maximum state can be determined graphically or analytically according to the initial conditions and the applied force for the three cases :

1) $F\,\dot{x}_0 \geq 0$ ===> first extremum
2) $F\,\dot{x}_0 < 0$ and $F \geq S$ ===> second extremum
3) $F\,\dot{x}_0 < 0$ and $F < S$ ===> first or second extremum

where the extremum is defined as the response at the time when the kinetic energy is equal to zero. For simplicity, in this paper, we shall treat only the first case where the maximum total displacement and inelastic displacements are given by :

$$x_{max} = \frac{F}{k_t} - \frac{\epsilon S}{h} + \epsilon r_0 \quad (16)$$

$$\alpha_{max} = \frac{F - \epsilon S}{h} + \epsilon\,\frac{k}{k+h}\,r_0 \,. \quad (17)$$

It is worth noting that *for zero initial velocity, the maximum state is independent of the mass. The maximum displacements depend of the ratios of k/h and F/S. For all k/h, if the force is very large (F >> S), the inelastic displacement, shown in Fig. 3, is effectively equal to the double of the static reponse, i.e. 2(F - ϵS)/h, which is also the maximum displacement of the SDOF rigid-platic system* (Lee, 1989). Fig. 4 and 5 represent the maximum total displacement and the maximum stress in function of k/h and F/S. They are equal to the double of the static elastic response for the elastic case.

Determination of the Limit State

After some elastoplastic cycles, the system vibrates purely elastically, it reaches a limiting state where it shakesdown. The limit inelastic displacement can be considered as the last extreme displacement of the elastoplastic vibration. It is determined after calculating successively all extreme values of the elastoplastic response. In the phase plane, the radii of the first integrals reduce progressively each time the sign of the velocity changes. The system reaches its limiting state if the radius of the first integral of the elastoplastic system becomes inferior to a certain value, which is calculated from the criteria of plasticity, i.e.,

$$r_{n-1} \leq S/k_t \quad (18)$$

where n denotes nth extremum where the structure shakesdown.

The limiting total displacement and inelastic displacement are given in the quasi-analytical form

$$x_{lim} = \frac{F}{k_t} - (-1)^{n+1}\frac{\epsilon S}{h} + (-1)^{n+1}\epsilon r_{n-1} \quad (19)$$

$$\alpha_{lim} = \frac{F - (-1)^{n+1}\epsilon S}{h} + (-1)^{n+1}\epsilon\,\frac{k}{k+h}\,r_{n-1} \quad (20)$$

with

$$r_n = \sqrt{4(r_{n-1} - S/k_t)(S/k) + (r_{n-1} - 2S/k_t)^2} \ , \ n \geq 1 \quad (21)$$

The limit state is reached in an equilibrium domain dependent of the material and independent of the initial conditions; but the limiting values are functions of the initial conditions. The equilibrium domain grows in function of k/h from F/h to [(F-S)/h, (F+S)/h] which is the fixed equilibrium domain of the rigid-plastic system (Lee, 1989). Fig. 6 represents the inelastic displacement in function of k/h and F/S.

MAXIMUM AND LIMIT STATES OF THE MULTI-DEGREE-OF-FREEDOM (MDOF) SYSTEM

Maximum State of the Elastic System

Consider now a n-DOF elastic system. It is known that the mode shapes matrix and the natural frequencies matrix are given by :

$$\mathbf{\Phi} = [\Phi_1 \ \Phi_2 \ . \ . \ . \ \Phi_n] \ ; \ \mathbf{\Omega} = \mathrm{diag}[\omega_i], \ i = 1,2, ..., n \tag{22}$$

with orthogonality properties written as :

$$\mathbf{\Phi}^T \ \mathbf{M} \ \mathbf{\Phi} = \mathbf{I}, \quad \mathbf{\Phi} \ \mathbf{\Phi}^T = \mathbf{M}^{-1}$$

$$\mathbf{\Phi}^T \ \mathbf{K} \ \mathbf{\Phi} = \mathbf{\Omega}^2, \quad \mathbf{\Phi} \ \mathbf{\Omega}^{-2} \ \mathbf{\Phi}^T = \mathbf{K}^{-1} \tag{23}$$

where **M** is the mass matrix and **K** the stiffness matrix.
The equation of motion is :

$$\mathbf{M} \ \ddot{\mathbf{X}} + \mathbf{K} \ \mathbf{X} = \mathbf{F}(t) \ . \tag{24}$$

The substitution of the following transformation :

$$\mathbf{X}(t) = \mathbf{\Phi} \ \mathbf{Y}(t) \tag{25}$$

in the preceding equation (premultiplied by $\mathbf{\Phi}^T$) gives the equation of motion for the generalized modal displacements :

$$\mathbf{\Phi}^T\mathbf{M} \ \mathbf{\Phi} \ \ddot{\mathbf{Y}} + \mathbf{\Phi}^T\mathbf{K} \ \mathbf{\Phi} \ \mathbf{Y} = \mathbf{\Phi}^T\mathbf{F}(t) \ , \tag{26}$$

which, when considering the orthogonality, is written as :

$$\ddot{\mathbf{Y}} + \mathbf{\Omega}^2 \ \mathbf{Y} = \mathbf{\Phi}^T\mathbf{F}(t) \ . \tag{27}$$

This equation is composed of n individual equations and the solution of each equation can be found by the Duhamel integral. When the initial conditions are equal to zero, we have :

$$\mathbf{Y} = [y_1 \ y_2 \ . \ . \ . \ y_n]^T \tag{28}$$

$$y_i = \frac{1}{\omega_i} \int_0^t \Phi_i^T \ \mathbf{F}(\tau) \sin \omega_i(t - \tau) \ d\tau, \quad i = 1, 2, ..., n \ . \tag{29}$$

The vector of the generalized displacements can be rewritten as follows :

$$\mathbf{Y}(t) = \int_0^t \mathbf{H}(t - \tau) \ \mathbf{F}(\tau) \ d\tau \tag{30}$$

with

$$\mathbf{H}(\tau) = \mathbf{\Omega}^{-1}\mathbf{S}(\tau) \ \mathbf{\Phi}^T \ ; \ \mathbf{S}(\tau) = \mathrm{diag}[\sin \omega_i\tau] \ . \tag{31}$$

Integrating by parts, we find :

$$\mathbf{Y}(t) = \mathbf{G}(t - \tau) \ \mathbf{F}(\tau) \Big|_0^t - \int_0^t \mathbf{G}(t - \tau) \ \dot{\mathbf{F}}(\tau) \ d\tau \tag{32}$$

with

$$\mathbf{G}(\tau) = \mathbf{\Omega}^{-2}\mathbf{C}(\tau) \ \mathbf{\Phi}^T \ ; \ \mathbf{C}(\tau) = \mathrm{diag}[\cos \omega_i\tau] \ . \tag{33}$$

In the case of the constant force, we have $\dot{\mathbf{F}}(\tau) = \mathbf{0}$. The equation (32) becomes :

$$\mathbf{Y}(t) = [\mathbf{G}(0) - \mathbf{G}(t)] \ \mathbf{F} = [\mathbf{\Omega}^{-2}\mathbf{\Phi}^T - \mathbf{\Omega}^{-2}\mathbf{C}(t) \ \mathbf{\Phi}^T] \ \mathbf{F}. \tag{34}$$

Since all the degree-of-freedom are decoupled, it is evident that each degree-of-freedom is maximum when $\cos \omega_i t = -1$ and that the maximum state of the structure is reached when all the degree-of-freedom reach their maximum state at an instant t. This maximum state is possible mathematically if there exists no energy dissipation in the structure and a certain numerical error is permitted. So, we have :

$$\mathbf{Y}_{max} = 2 \ \mathbf{\Omega}^{-2}\mathbf{\Phi}^T \ \mathbf{F}. \tag{35}$$

The maximum displacement is found as follows :

$$\mathbf{X}_{max} = \mathbf{\Phi} \ \mathbf{Y}_{max} = 2 \ [\mathbf{\Phi} \ \mathbf{\Omega}^{-2}\mathbf{\Phi}^T] \ \mathbf{F} \tag{36}$$

which can be cast in the form :

$$\boxed{\mathbf{X}_{max} = 2 \ \mathbf{K}^{-1}\mathbf{F}} \tag{37}$$

This means that the maximum displacements of the elastic system subjected to a constant force are equal to the double of the static response to the same force applied statically and that at the time of the maximum state of the structure, the displacement of each degree-of-freedom is equal to the maximum value of the individual SDOF system whose characteristics are same as those of the DOF under considering. These results are also true for the stresses.

Hypothesis for the Maximum and Limit States of the Elastoplastic System

During the dynamic elastoplastic response, there exist temporal plastic energy dissipation and also spatial plastic energy distribution.

The result for the MDOF elastic system allows to make the hypothesis that *each degree-of-freedom of a MDOF elastoplastic system behaves as an independent SDOF elastoplastic system and that the maximum/limit state of the structure is attained when all the DOF reach their maximum/limit state.* This hypothesis enables to determine the local maximum/limit elastic stress field of a MDOF elastoplastic system from the known analytical solutions due to the temporal plastic energy dissipation for the SDOF system. So, we reach a static elastoplastic problem.

Using the above informations on the maximum and limit states, we impose as fictive loads the initial strains due to the dynamic elastic stresses on the structure and then we solve a static elastoplastic problem, from which we can find the spatial plastic energy distribution in the structure.

NUMERICAL METHODS

With our hypothesis, the analysis of the maximum and limit states is reduced to a static elastoplastic problem that we will solve with the quasi-static simplified method proposed by Zarka *et al.* (1979).

Review of the Evolution of Structures under Quasi-static Loadings

In this paper, we consider only the kinematically hardening materials.

The structure of a finite volume V with its boundary ∂V is subjected to body forces $\mathbf{X}^d(t)$ in V, surface forces $\mathbf{f}_i^d(t)$ on $\partial_{f_i}V$ of ∂V, displacements $\mathbf{u}_j^d(t)$ on $\partial_{u_j}V$, and initial strains $\epsilon_I^d(t)$ in V.

First we compute the **purely elastic response**. Assuming that the structure is made of a purely elastic material, we can obtain the displacement field $\mathbf{u}^{el}(t)$, the strain field $\epsilon^{el}(t)$ and the stress field $\sigma^{el}(t)$ using the routine ELAS :

$$\textbf{ELAS}\ (V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ \mathbf{X}^d,\ \mathbf{f}_i^d,\ \mathbf{u}_i^d,\ \epsilon_I^d,\ \mathbf{M}), \tag{38}$$

where the first three arguments represent the geometry of the structure, the subsequent four arguments are the loadings, and the last one denotes the material.

These fields are such that $\mathbf{u}^{el}(t)$ and $\epsilon^{el}(t)$ are kinematically admissible (K.A.) with $\mathbf{u}_j^d(t)$ on $\partial_{u_j}V$, and

$$\sigma^{el}(t) = \mathbf{L}(\epsilon^{el}(t) - \epsilon_I^d(t))\ (\text{or } \epsilon^{el}(t) = \mathbf{M}\sigma^{el}(t) + \epsilon_I^d(t)) \tag{39}$$

is statically admissible (S.A.) with $\mathbf{X}^d$ in V and $\mathbf{f}_i^d(t)$ on $\partial_{f_i}V$, where $\mathbf{M}$ and $\mathbf{L}=\mathbf{M}^{-1}$ are the symmetric positive definite elastic matrices.

Then, we consider the **real response**. In the real structure, we have the actual displacement and strain fields $\mathbf{u}(t)$ and $\epsilon(t)$ K.A. with $\mathbf{u}_j^d(t)$, and the actual stress field $\sigma(t)$ S.A. with $\mathbf{X}^d(t)$ and $\mathbf{f}_i^d(t)$. We can rewrite them as follows :

$$\mathbf{u}(t) = \mathbf{u}^{el}(t) + \mathbf{u}^{ine}(t) \tag{40}$$

$$\epsilon(t) = \epsilon^{el}(t) + \epsilon^{ine}(t) \tag{41}$$

$$\sigma(t) = \sigma^{el}(t) + \rho(t) \tag{42}$$

with

$$\epsilon^{ine}(t) = \mathbf{M}\rho(t) + \epsilon^p(t), \tag{43}$$

where $\mathbf{u}^{ine}$ and ϵ^{ine} are, respectively, the inelastic displacement and strain fields K.A. with 0 on $\partial_{u_j}V$, ρ denotes the residual stress field S.A. with $\mathbf{0}$ in V and 0 on ∂_{f_i}, and ϵ^p is the plastic strain field.

With our tool ELAS and the associated arguments :

$$\textbf{ELAS}\ \left(V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ 0^d,\ 0_i^d,\ 0_j^d,\ \boxed{\epsilon^p,\ \mathbf{M}}\right), \tag{44}$$

we can obtain :

$$\mathbf{u}^{ine} \Rightarrow \epsilon^{ine} \Rightarrow \rho = \mathbf{L}(\epsilon^{ine} - \epsilon^p). \tag{45}$$

We then introduce **the structural transformed parameters**. The yield criterion can be written as :

$$f(\sigma - y) = f(S - y) = f(S^{el} + \text{dev}\ \rho - y) = f(S^{el} - Y) \leq 0 \tag{46}$$

with

$$\mathbf{y} = \mathbf{C}\epsilon^p \Leftrightarrow \epsilon^p = \mathbf{C}^{-1}\mathbf{y}\ ;\ \mathbf{Y} = \mathbf{C}\epsilon^p - \text{dev}\ \rho, \tag{47}$$

where **S** and **S**^{el} denote, respectively, the deviatoric part of the actual and elastic stresses, **C**, the hardening modulus, and **y**, **Y**, the internal parameters and the structural transformed parameters, respectively.

In the **Y** space at any time, since $\sigma^{el}(t)$ is known, **Y** must belong to the known convex set $\hat{C}(\sigma^{el}(t)) = C_0 + S^{el}(t)$, which is locally built. The flow rules are written :

$$\dot{\epsilon}^p(t) \in - \partial\psi_{\hat{C}(\sigma^{el}(t))}(Y(t)) \,. \tag{48}$$

Now the inelastic strain field can be rewritten in the **Y** space as :

$$\epsilon^{ine} = M\rho + \epsilon^p = M\rho + C^{-1}(Y + \text{dev } \rho) = (M + C^{-1}\text{dev})\rho + C^{-1}Y = M'\rho + C^{-1}Y \tag{49}$$

where $M' = (M + C^{-1}\text{dev})$ is the modified elastic coefficient matrix.

When **Y** is supposed known, the ELAS routine is used to determine the inelastic fields :

$$\text{ELAS} \left\{ V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ 0^d,\ 0_i^d,\ 0_j^d,\ \boxed{C^{-1}Y,\ M'} \right\} \tag{50}$$

which gives :

$$u^{ine} \Rightarrow \epsilon^{ine} \Rightarrow \rho = L'(\epsilon^{ine} - C^{-1}Y) \Rightarrow y = Y + \text{dev}\rho \Rightarrow \epsilon^p = C^{-1}\, y, \text{ where } L' = M'^{-1}. \tag{51}$$

In many cases, there will exist at the same time an elastic part V_e and a plastic part V_p in the volume V. In this case, the field ϵ^p in V_e and the field **Y** on V_p are known and so we can calculate the inelastic fields using ELAS as follows :

$$\text{ELAS} \left\{ V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ 0^d,\ 0_i^d,\ 0_j^d,\ \boxed{\epsilon^p \text{ in } V_e,\ C^{-1}Y \text{ in } V_p,\ M \text{ in } V_e,\ M' \text{ in } V_p} \right\} \tag{52}$$

which gives the fields u^{ine}, from which the other associated unknown fields can be calculated using Eqs. (45) in V_e and Eqs. (51) in V_p .

Various applications of this formulation were made for the limit analysis of structures, the cyclic analysis (see Zarka *et al.*, 1989). Now, we shall show how we apply it during the dynamic analysis.

New Algorithm for the Dynamic Analysis

At first, we describe the algorithm for the uniaxial case such as for beam or bar structures.

- Use ELAS $(V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ X^d,\ f_i^d,\ u_j^d,\ \epsilon_I^d,\ E)$ (53)

to determine $u_s^{el} \Rightarrow \epsilon_s^{el} \Rightarrow \sigma_s^{el} = E(\epsilon_s^{el} - \epsilon_I^d)$ (54)

where E denotes Young's modulus and the subscript s represents the static response.

Maximum state

- Determine the dynamic maximum elastic stress field with a special routine which calculates the maximum and limit states of the SDOF elastoplastic system :

$$\text{SDOF}\ (E,\ h,\ \sigma_y,\ m,\ \epsilon_0,\ \epsilon_0^p,\ \dot{u}_0,\ \sigma_s^{el}) \Rightarrow \sigma^{el} \tag{55}$$

where ϵ_0, ϵ_0^p, and $\dot{u}_0$ are, respectively, the initial total strain, plastic strain, and velocity fields.

- Calculate the dynamic elastic displacement and strain fields :

$$\text{ELAS}\ (V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ 0^d,\ 0_i^d,\ 0_j^d,\ \sigma^{el}/E,\ E) \Rightarrow u^{el} \Rightarrow \epsilon^{el} \tag{56}$$

- Calculate the inelastic fields :

$$Y = \sigma^{el} - (\sigma^{el} \,/\, |\sigma^{el}|)\ \sigma_y \tag{57}$$

$$\text{ELAS}\ (V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ 0^d,\ 0_i^d,\ 0_j^d,\ 0 \text{ in } V_e,\ Y/h \text{ in } V_p,\ E \text{ in } V_e,\ E_t \text{ in } V_p) \tag{58}$$

$$\Rightarrow u^{ine} \Rightarrow \epsilon^{ine} \Rightarrow \rho = E_t\ (\epsilon^{ine} - Y/h) \text{ in } V_p,\ \rho = E\epsilon^{ine} \text{ in } V_e \tag{59}$$

where $E_t = Eh/(E + h)$ is the tangent modulus.

- Maximum response : $u = u^{el} + u^{ine}$; $\epsilon = \epsilon^{el} + \epsilon^{ine}$; $\sigma = \sigma^{el} + \rho$ (60)

Limit state

- Determine the dynamic limting elastic stress field :

$$\text{SDOF}\ (E,\ h,\ \sigma_y,\ m,\ \epsilon_0,\ \epsilon_0^p,\ \dot{u}_0,\ \sigma_s^{el}) \Rightarrow \sigma^{el} \tag{61}$$

- Calculate the limiting inelastic fields :

$$Y = \sigma^{el} - (\sigma^{el} \,/\, |\sigma^{el}|)\ \sigma_y \tag{62}$$

$$\text{ELAS}\ (V,\ \partial_{f_i}V,\ \partial_{u_j}V,\ 0^d,\ 0_i^d,\ 0_j^d,\ 0 \text{ in } V_e,\ Y/h \text{ in } V_p,\ E \text{ in } V_e,\ E_t \text{ in } V_p) \tag{63}$$

$$=> u^{ine} => \epsilon^{ine} => \rho = E_t (\epsilon^{ine} - Y/h) \text{ in } V_p, \rho = E\epsilon^{ine} \text{ in } V_e \quad (64)$$

- Calculate the dynamic maximum elastic stress field of the limiting elastic vibration :

$$SDOF (E, h, \sigma_y, m, \epsilon_0, \epsilon_0^p, \dot{u}_0, \sigma_s^{el}) => \sigma^{el} \quad (65)$$

- Determine the maximum elastic fields of the limting elastic vibration :

$$ELAS (V, \partial_{f_i} V, \partial_{u_j} V, 0^d, 0_i^d, 0_j^d, \sigma^{el}/E, E) => u^{el} => \epsilon^{el} \quad (66)$$

- Limit response : $u = u^{el} + u^{ine}$; $\epsilon = \epsilon^{el} + \epsilon^{ine}$; $\sigma = \sigma^{el} + \rho$ (67)

The general idea of our approach consists in :
Once the local elastic stress field is known after a static elastic analysis, we treat the SDOF elastoplastic mechanism at each point of the structure, and then we calculate, from the known analytical solutions, the local maximum or limit dynamic elastic stress field. In a second step, with the assumed transformed parameters Y in the plastic zone V_p and the zero plastic strains in the elastic zone V_e, the inelastic displacement field is determined using ELAS. According to the amount of the plasticity, we need some iteration procedures (usually a very few iterations).

EXAMPLES OF APPLICATIONS

Cantilever Beam

As a first example, we consider a cantilever beam subjected to a constant force (Fig 7). We model the beam using the 4x4 integration points per finite element and we use the Gauss quadrature. The element tangential stiffness matrix was constructed considering the yield criteria at each point of the element. The kinematically hardening elastoplastic behavior was assumed. The material data used in this problem were as follows ; elastic modulus E = 210 x 10^6 kN/m^2; yield stress σ_y = 250 x 10^3 kN/m^2; mass density ρ = 800 kg sec^2/m^4. Three different hardening moduli h = E, 0.1E, and 0.01E were tried. The geometry data were as follows; L = 2 m, b x d = 0.2 m x 0.3 m. The loadings consisted in F = 0.75 F_y, 1.0 F_y, 1.5 F_y, and 5.0 F_y, where F_y is the static yield force which causes the yielding in the extreme fibers, i.e. F_y = 15000 kN in the axial direction and F_y = 375 kN in the vertical direction.
For the appreciation of the results, we used the Newmark direct integration method with the pseudo force method and $\beta = 1/4$, $\gamma = 1/2$.
For h/E = 1 and F = 5F_y in axial direction, the beam was analysed. The results obtained for the new method were equal to the exact solutions (u_{max} = - 0.042989 m, u_{lim} = -0.02619 m) and in very good agreement with those for the step-by-step procedure, as shown in Fig. 8. The calculation times for the simplified and Newmark methods (Δt = 0.0288 T_1 = 0.0005 sec.) were 2.278 sec. and 84.372 sec., respectively, where T_1 is the first natural period of the structure.
The force F = 5F_y was also applied in the vertical direction with h/E = 1. Figure 9 shows the vertical displacement histories at the endpoint of the beam calculated by the direct integration method and the maximum and limit displacements by the new method. The maximum displacements obtained for the new simplified and Newmark methods were -0.1713 m and -0.1709 m. The calculation times for the new method and the Newmark method (Δt = 0.002 sec.) were 2.374 sec. and 185.637 sec., respectively. In the step-by-step calculation, it was impossible to obtain the complete limit state.
To verify our method further, we varied the loads and the hardening parameters. Table 1 gives the comparisons of the maximum vertical displacements at the endpoint of the beam obtained by the new simplified method (SM) and the step-by-step direct integration method (CM).

Table 1 Unit : x 10^{-2}m

h/E	F = 0.75Fy		F = 1.0Fy		F = 1.5Fy	
	SM	CM	SM	CM	SM	CM
1	-1.6101	-1.6111	-2.2117	-2.2181	-3.6084	-3.6423
.1	-1.6066	-1.6214	-2.2295	-2.3260	-4.4523	-4.6110
.01	-1.6033	-1.6237	-2.1555	-2.3553	-4.9498	-5.1519

One-bay Two-story Plane Frame

The frame shown in Fig. 10 was analysed. The material and section data were as follows :
E = 210 x 10^6 kN/m^2, h = E, 0.1E, and 0.01E, σ_y = 250 x 10^3 kN/m^2, ρ = 800 kg sec^2/m^4,
b x d = 0.6 m x 0.6 m for columns, b x d = 0.3 m x 0.6 m for beams.
The lateral forces F = 3600 kN, (approximately the static initial yielding load), were applied at the nodes 3 and 5. The time step sizes and duration of the calculation for the Newmark method were, respectively, Δt = 0.01 sec.= 0.014T_1 and 2 sec. in all the three analyses. The comparison of the results is shown in Table 2. The simplified method was averagely about 40 times faster than the Newmark method in computation times. Fig. 11 shows the displacement histories at the node 5 in the case

of h/E = 0.1 and the maximum and limiting values obtained with the simplified method.

Table 2 Unit : x 10^{-2}m

node	h/E = 1		h/E = 0.1		h/E = 0.01	
	SM	CM	SM	CM	SM	CM
3	2.7705	2.7894	2.7669	2.9419	2.7131	2.9833
5	6.0135	6.0846	6.0130	6.4115	5.9054	6.4971

10-element Plane Truss

The 10-element plane truss shown Fig. 12 was analysed too. The material and section data were as follows : E = 210 x 10^6 kN/m², h = E, 0.1E, and 0.01E, σ_y = 250 x 10^3 kN/m², ρ = 800 kg sec²/m⁴, L = 1 m, A = 0.01 m².

The vertical force F = 1700 kN, was applied at the node 5. The time step sizes and duration of the calculation for the Newmark method were, respectively, Δt = 0.001 sec.= $0.013T_1$ and 0.2 sec. for h=E, 0.1E and 0.3 sec. for h=0.01E. The comparison of the results is shown in Table 3. The computation times for the simplified and Newmark methods are 1.489 sec. and 28.078 sec. for h/E = 1, 1.539 sec. and 46.960 sec. for h/E = 0.1, and 1.483 sec. and 107.963 sec. for h/E = 0.01. Fig. 13 shows the displacement histories at the node 5 in the case of h/E = 0.1 and the maximum and limit values obtained with the simplified method.

Table 3 Unit : x 10^{-2}m

node	h/E = 1		h/E = 0.1		h/E = 0.01	
	SM	CM	SM	CM	SM	CM
5	-1.4620	-1.4519	-2.2214	-2.3123	-5.3872	-6.1650

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ILLUSTRATIONS

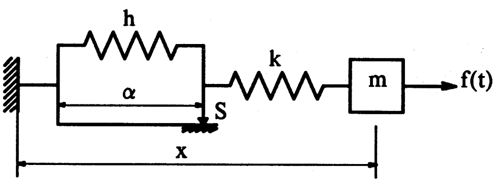

Fig. 1(a) : Single-degree-of-freedom elastoplastic model with kinematically hardening material

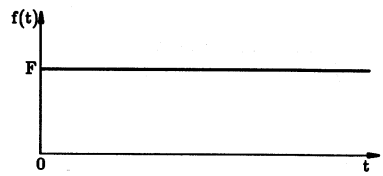

Fig. 1(b) : Instantaneous constant force

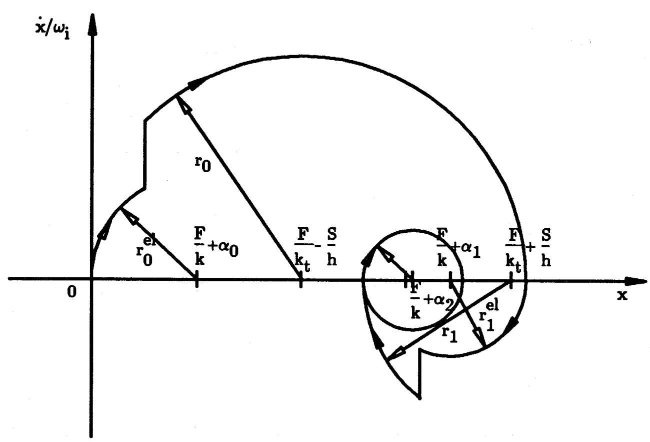

Fig. 2 : Trajectories in phase plane, $\omega_i = \omega$ for the elastic response, ω_t for the elastoplastic response

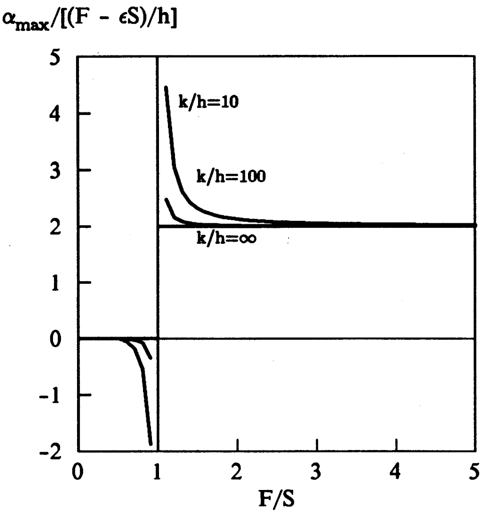

Fig. 3 : Maximum inelastic displacements under an instantaneous constant force

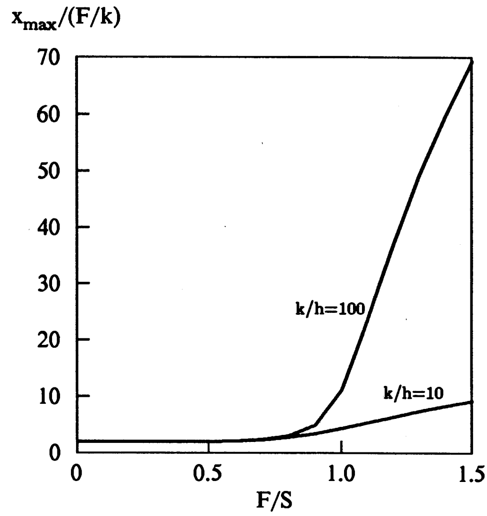

Fig. 4 : Maximum total displacements under an instantaneous constant force

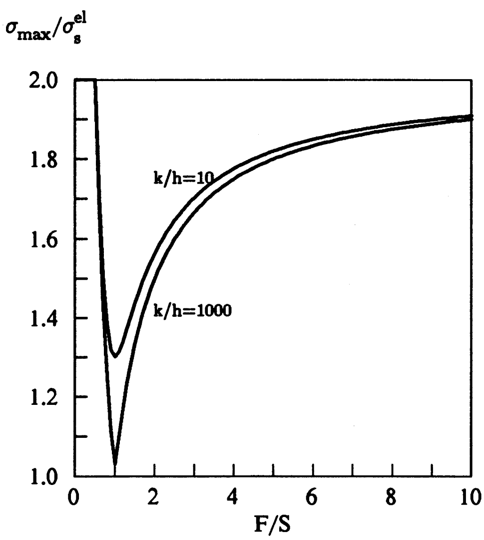

Fig. 5 : Maximum stresses in comparison with the static elastic stress σ_s^{el} under an instantaneous constant force

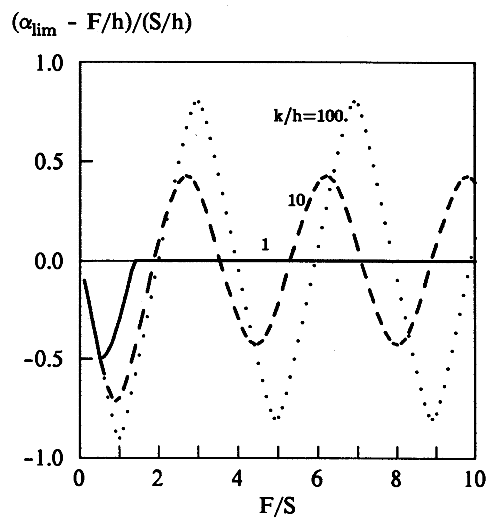

Fig. 6 : Limit inelastic displacements under an instantaneous constant force

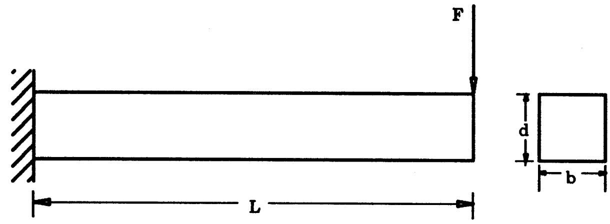

Fig. 7 : Cantilever beam

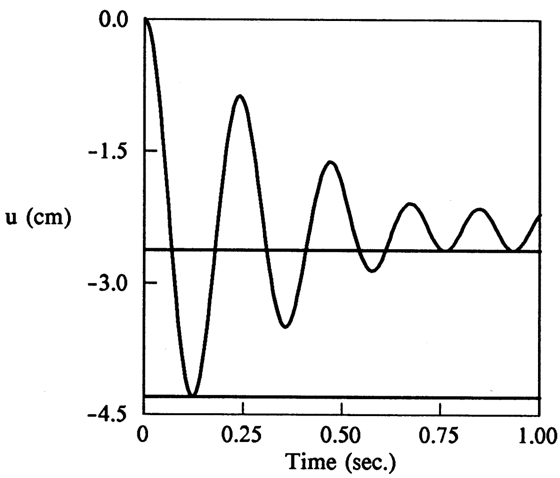

Fig. 8 : Axial diplacement-time history at the endpoint of the cantilever beam by the Newmark method and the maximum and limit displacements obtained for the simplified method ($F = 5F_y$, $h/E = 1$)

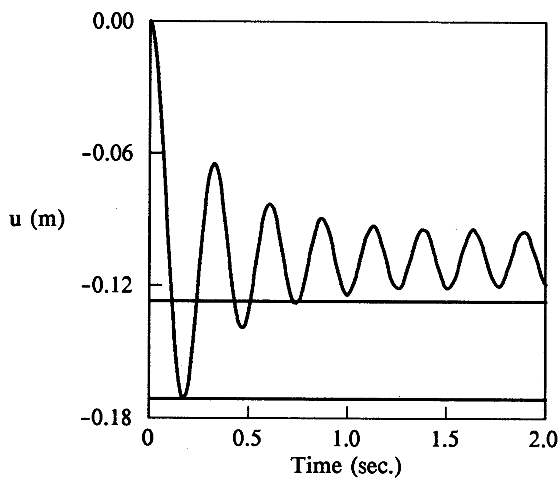

Fig. 9 : Vertical diplacement-time history at the endpoint of the cantilever beam by the Newmark method and the maximum and limit displacements obtained for the simplified method ($F = 5F_y$, $h/E = 1$)

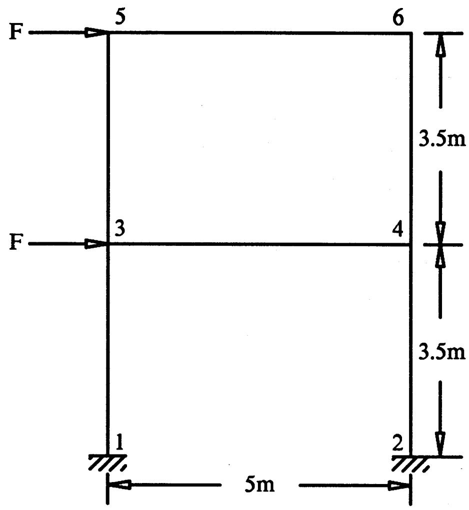

Fig. 10 : One-bay two-story plane frame

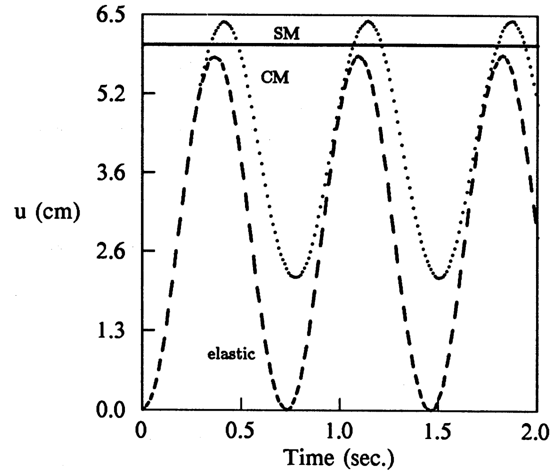

Fig. 11 : Elastic and elastoplastic lateral displacement-time histories at node 5 of the plane frame by the Newmark method and the maximum and limit displacements obtained for the simplified method (h/E = 0.1)

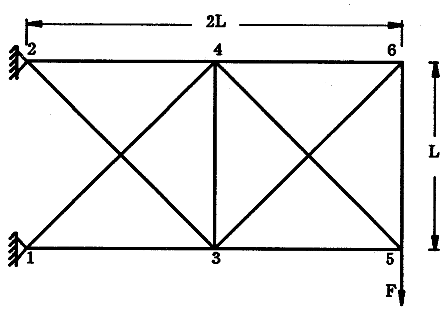

Fig. 12 : 10-element truss

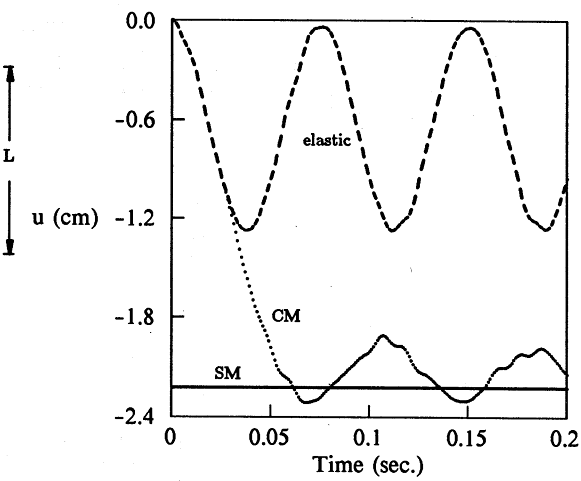

Fig. 13 : Elastic and elastoplastic vertical displacement-time histories at node 5 of the 10-element truss by the Newmark method and the maximum and limit displacements obtained for the simplified method (h/E = 0.1)