A Numerical Method For Stochastic Analysis of Nonclassically Damped Linear Systems

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INTRODUCTION

The stochastic analysis of non-classically damped systems is an imperative problem for the correct analysis of many industrial or civil buildings. As an example this analysis must be performed for the response of composite primary-secondary systems even when the two subsystems are individually classically damped. Indeed in a nuclear power plant system the supported equipement (the so-called secondary subsystem) must always remain operational, since its failure may compromise the safety of the whole system.
It is well known that a stochastic process can be idealized as a zero mean non-stationary Gaussian process and that the statistics of the respone can be obtained once the mean square response is evaluated (Nigam, 1983). Recently, various methods have been proposed to evaluate the non-stationary mean square response of structural systems subjected to a seismic input idealized as a Tajimi-Kanai-like filtered non-stationary process (see e.g. Di Paola et al, 1984; Langley, 1986). These methods require considerable computer time, especially for the evaluation of the statistics between output and input.
In this paper a new method is presented to evaluate the non-stationary mean square response of non-classically damped linear systems subjected to seismic input. This method, starting from a new definition of the impulse reponse function matrix (which avoids the traditionally used modal decomposition procedure), performs the analysis of non-classically damped systems by using similar quantities to those evaluated for classically damped ones. The only difference for the analysis of the two systems is in two different weighting vectors, which, for non-classically damped systems, take into account the complex eigenproperties of reduced equations of motion. Furthermore, the mean square response is obtained for both systems, by a common procedure. By means of this procedure, once the differential equations defining the evolution of the non-stationary cross-covariance of the output are written, by using Ito's rule, as a set of first order differential equations subjected to deterministic input, their solution can be obtained by means of step-by-step techniques widely used for deterministic analysis.

PRELIMINARY CONCEPTS AND DEFINITIONS

The equations of motion of an n-degree-of-freedom linear system subjected to a ground acceleration f(t) at its base can be written as follows:

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{x}} + \underset{\sim}{C}\,\dot{\underset{\sim}{x}} + \underset{\sim}{K}\,\underset{\sim}{x} = -\underset{\sim}{M}\,\underset{\sim}{r}\,f(t) \tag{1}$$

where $\underset{\sim}{r}$ is an influence vector; $\underset{\sim}{M}$, $\underset{\sim}{C}$ and $\underset{\sim}{K}$ are respectively the inertia, damping and stiffness matrices; $\underset{\sim}{x}$ is the vector of nodal displacements and the dot over a variable denotes its time derivative. If the matrix $\underset{\sim}{K}\,\underset{\sim}{M}^{-1}\,\underset{\sim}{C}$ does not commute with the matrix $\underset{\sim}{C}\,\underset{\sim}{M}^{-1}\,\underset{\sim}{K}$ (Caughey and O'Kelly, 1965), the system is usually called non-proportionally damped or non-classically damped. In this case the solution of equation (1) can be obtained by the 2n dimension state vector approach (Foss, 1958). For this purpose, this equation can be written in reduced form as follows:

$$\dot{\underset{\sim}{y}} = \underset{\sim}{D}\,\underset{\sim}{y} + \underset{\sim}{v}_0\,f(t) \tag{2}$$

where

$$\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{x} \\ \dot{\underset{\sim}{x}} \end{bmatrix} \;;\quad \underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{I} \\ -\underset{\sim}{M}^{-1}\,\underset{\sim}{K} & -\underset{\sim}{M}^{-1}\,\underset{\sim}{C} \end{bmatrix} \;;\quad \underset{\sim}{v}_0 = \begin{bmatrix} \underset{\sim}{0} \\ -\underset{\sim}{I} \end{bmatrix} \underset{\sim}{r} \tag{3}$$

The solution of equation (2), for zero start initial conditions, can be written as follows:

$$\underset{\sim}{y}(t) = \int_0^t \underset{\sim}{H}(t-\tau)\,\underset{\sim}{r}\,f(\tau)\,d\tau \tag{4}$$

where $\underset{\sim}{H}(t)$ is the so-called impulse response function matrix. If all the eigenvalues of $\underset{\sim}{D}$ are distinct, the matrix $\underset{\sim}{H}(t)$ can be written in the form (Muscolino, to appear)

$$\underset{\sim}{H}(t) = \{\sum_{k=1}^{n} [\underset{\sim}{L}_k\,h_k(t) + \underset{\sim}{S}_k\,\dot{h}_k(t)]\}\,\underset{\sim}{V}_0 \tag{5}$$

where $\underset{\sim}{L}_k$ and $\underset{\sim}{S}_k$ are two symmetric matrices of order 2n x 2n given respectively as:

$$\underset{\sim}{L}_k = -2[(\underset{\sim}{\psi}_k\,\underset{\sim}{\psi}_k^T - \underset{\sim}{\chi}_k\,\underset{\sim}{\chi}_k^T)\,\beta_k + (\underset{\sim}{\psi}_k\,\underset{\sim}{\chi}_k^T + \underset{\sim}{\chi}_k\,\underset{\sim}{\psi}_k^T)\gamma_k]$$
$$\underset{\sim}{S}_k = 2(\underset{\sim}{\psi}_k\,\underset{\sim}{\psi}_k^T - \underset{\sim}{\chi}_k\,\underset{\sim}{\chi}_k^T) \tag{6}$$

β_k and γ_k being respectively the real and imaginary parts of the k-th eigenvalue of matrix $\underset{\sim}{D}$, while $\underset{\sim}{\psi}_k$ and $\underset{\sim}{\chi}_k$ are respectively the real and imaginary parts of the k-th eigenvector normalized with respect to the matrix $\underset{\sim}{A}$; and $\underset{\sim}{V}_0$ is a matrix of order 2n x n given as:

$$\underset{\sim}{V}_0 = \begin{bmatrix} \underset{\sim}{I} \\ \underset{\sim}{0} \end{bmatrix} \;;\quad \underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{C} & \underset{\sim}{M} \\ \underset{\sim}{M} & \underset{\sim}{0} \end{bmatrix} \tag{7}$$

In equation (5) $h_k(t)$ is the following time-dependent function

$$h_k(t) = \frac{1}{\gamma_k}\,e^{\beta_k t}\,\sin(\gamma_k t),\quad t \geq 0;\quad h_k(t) = 0,\qquad t < 0 \tag{8}$$

and $\dot{h}_k$ is its time differentiation. By using equation (5), equation (3) can be written in alternative form as follows:

$$\underset{\sim}{y}(t) = \sum_{k=1}^{n} [\underset{\sim}{p}_k \int_0^t h_k(t-\tau)\,f(\tau)\,d\tau + \underset{\sim}{w}_k \int_0^t \dot{h}_k(t-\tau)\,f(\tau)\,d\tau] \tag{9}$$

where $\tilde{p}_k$ and $\tilde{w}_k$ are two vector defined as follows:

$$\tilde{p}_k = \tilde{L}_k \tilde{V}_0 \tilde{r}, \quad \tilde{w}_k = \tilde{S}_k \tilde{V}_0 \tilde{r} \tag{10}$$

Notice that the convolution integrals which appear in equation (9) are similar to those of a traditional classically damped system. Hence the only difference in the deterministic analysis of classically or non-classically damped systems is in the weighting vector $\tilde{p}_k$ and $\tilde{w}_k$.

It is to be emphasized that these convolution integrals can be seen as the Duhamel integral of a single oscillator having natural radian frequency $\omega_k^2 = (\beta_k^2 + \gamma_k^2)$ and damping ratio $\zeta_k = -\beta_k/\omega_k$ and forced by f(t). It follows that equation (9) can be rewritten in alternative form as follows:

$$\tilde{y}(t) = \sum_{k=1}^{n} \tilde{R}_k \tilde{z}_k(t); \quad \tilde{R}_k = [\tilde{p}_k, \tilde{w}_k] \tag{11),(12}$$

and

$$\tilde{z}_k^T = [\int_0^t h_k(t-\tau)\, f(\tau)\, d\tau \quad \int_0^t \dot{h}_k(t-\tau)\, f(\tau)\, d\tau] \tag{13}$$

is the vector solution, for zero start initial conditions, of the following first order differential equation:

$$\dot{\tilde{z}}_k = \tilde{D}_k \tilde{z}_k + \tilde{v}\, f(\tau) \tag{14}$$

where

$$\tilde{D}_k = \begin{bmatrix} 0 & 1 \\ -\omega_k^2 & -2\zeta_k\omega_k \end{bmatrix} ; \quad \tilde{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{15}$$

Since equation (14) is formally similar for classically and non-classically damped systems it follows that the deterministic analysis of both systems can be performed by a common formulation.

Notice that the formulation here described is also very useful for the stochastic analysis of non-classically damped systems, as will be shown in the next section.

NON-STATIONARY STRUCTURAL RESPONSE

It is well known that in order to evaluate the statistics of the response of a linear structure subjected to a non-stationary zero-mean Gaussian input process, the cross-covariances of the response has to be evaluated (Nigam, 1985). In this section a new method is presented to evaluate it.

In compact form the cross-covariances of the response can be evaluated by using the cross-covariance matrix. This matrix can be written in vectorialized form as follows:

$$E\{\tilde{y}^{[2]}\} = \sum_{k=1}^{n} \sum_{j=1}^{n} [(\tilde{R}_k \otimes \tilde{R}_s)\; E\{\tilde{z}_k \otimes \tilde{z}_s\}] \tag{16}$$

where $E\{\cdot\}$ means stochastic average, the symbol $\otimes$ means Kronecker or tensorial product (Fiedler, 1986) and the exponent in square brackets means tensor power, i.e.

$$E\{\tilde{y}^{[2]}\} = E\{\tilde{y} \otimes \tilde{y}\} \tag{17}$$

By using equation (16) it is evident that the nodal cross-covariance of the output can be easily obtained once the modal cross-covariances of the output are evaluated. In this section we are concerned to evaluate the latter quantities for seismic input. In this case the input process f(t) is usually defined as a filtered non-stationary stochastic process. By using a Tajimi-Kanai-like filter this process can be written as follows:

$$f(t) = \omega_g^2 \, u_g(t) + 2\zeta_g\omega_g \, \dot{u}_g(t) \tag{18}$$

where $u_g(t)$ and $\dot{u}_g(t)$ are the solution of a differential equation of a single oscillator subjected to a non-stationary white noise input, i.e.

$$\ddot{u}_g + 2\zeta_g\omega_g \, \dot{u}_g + \omega_g^2 \, u_g = \phi(t) \, \xi(t) \tag{19}$$

where ζ_g and ω_g are two parameters which define the characteristics of the filter, $\phi(t)$ is a deterministic shape function and $\xi(t)$ is a zero mean Gaussian white noise process.

Associating equation (19) to equation (14) these two equations can be written in compact form as follows:

$$\dot{\tilde{\underset{\sim}{z}}}_k = \tilde{\underset{\sim}{D}}_k \, \tilde{\underset{\sim}{z}} + \tilde{\underset{\sim}{v}} \, \xi(t) \, \phi(t) \tag{20}$$

where

$$\tilde{\underset{\sim}{z}}_k = \begin{bmatrix} \underset{\sim}{z}_k \\ u_g \\ \dot{u}_g \end{bmatrix} \quad \tilde{\underset{\sim}{D}}_k = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -\omega_k^2 & -2\zeta_k\omega_k & \omega_g^2 & 2\zeta_g\omega_g \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_g^2 & -2\zeta_g\omega_g \end{array}\right] ; \quad \tilde{\underset{\sim}{v}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{21}$$

Equation (20) represents a set of differential equations subjected to a Gaussian white noise vector process and to evaluate the statistics of the output it can be considered formally equivalent to an Itô's type stochastic differential equation:

$$d\tilde{\underset{\sim}{z}}_k = \underset{\sim}{g}_k(\tilde{\underset{\sim}{z}}_k, t) \, dt + \tilde{\underset{\sim}{v}} \, dw = \tilde{\underset{\sim}{D}}_k \, \tilde{\underset{\sim}{z}}_k \, dt + \tilde{\underset{\sim}{v}} \, dw \tag{22}$$

where w(t) is the so-called Wiener process such that

$$E[d\, w(t)] = 0: \quad E[d\, w(t) \otimes d\, w(t)] = 2\pi S_0 \, \phi^2(t) \tag{23}$$

In equation (22) $\underset{\sim}{g}_k(z,t)$ is the vector of drift coefficients and $2\pi S_0 \, \phi^2(t)$ is a deterministic function of the strength of the white process.

In the present analysis we will deal with the evaluation of the modal cross-covariances $E[\underset{\sim}{z}_k \otimes \underset{\sim}{z}_s]$ which are four elements of the vector $E[\tilde{\underset{\sim}{z}}_k \otimes \tilde{\underset{\sim}{z}}_s]$ (specifically the first two and the fifth and sixth). By applying Itô's rule (Itô ,1961 ; Jazwinski, 1970), this vector is obtained as the solution of the following differential equation:

$$\dot{E}[\tilde{\underset{\sim}{z}}_k \otimes \tilde{\underset{\sim}{z}}_s] = [\tilde{\underset{\sim}{D}}_k \otimes \underset{\sim}{I}_4 + \underset{\sim}{I}_4 \otimes \tilde{\underset{\sim}{D}}_s] \, E[\tilde{\underset{\sim}{z}}_k \otimes \tilde{\underset{\sim}{z}}_s] + \tilde{\underset{\sim}{v}}^{[2]} \, 2\pi S_0 \, \phi^2(t) \tag{24}$$

or in alternative form

$$\dot{\underset{\sim}{m}}_{ks} = \tilde{\underset{\sim}{D}}_{ks} \, \underset{\sim}{m}_{ks} + q(t) \, \tilde{\underset{\sim}{v}}^{[2]} \tag{25}$$

$$m_{ks} = E[\tilde{\underset{\sim}{z}}_k \otimes \tilde{\underset{\sim}{z}}_s]; \quad \tilde{\underset{\sim}{D}}_{ks} = [\tilde{\underset{\sim}{D}}_k \otimes \underset{\sim}{I}_4 + \underset{\sim}{I}_4 \otimes \tilde{\underset{\sim}{D}}_s] ; \quad q(t) = 2\pi S_0 \, \phi^2(t) \tag{26}$$

In these equations $\underset{\sim}{I}_r$ is an identity matrix or order r. In the next section a very effective computational method will be presented to solve equation (25) for non-stationary and stationary input processes.

COMPUTATIONAL ASPECTS IN EVALUATING THE CROSS-COVARIANCES

Equation (25) represents a set of first order linear differential equations with deterministic forcing function, whose solution can be easily numerically evaluated by a step-by-step technique. In particular, discretizing the time axis into small intervals of equal length Δt, and assuming that the forcing function is constant in each step, we can write (Borino and Muscolino, 1986)

$$\dot{\mathbf{m}}_{ks}(t_j,\Delta t) = \tilde{\Theta}_{ks}(\Delta t)\ \mathbf{m}_{ks}(t_j)+q(t_j)\ \tilde{\Gamma}_{ks}(\Delta t);\quad \tilde{\Gamma}_{ks}(\Delta t)=[\tilde{\Theta}_{ks}(\Delta t)-\mathbf{I}_{16}]\mathbf{D}_{ks}^{-1}\ \tilde{\mathbf{v}}^{[2]} \tag{27),(28}$$

Further simplifications in the analysis can be made by observing $\tilde{\Theta}_{ks}(\Delta t)$ is the fundamental matrix of equation (20) and can be written as:

$$\tilde{\Theta}_{ks}(\Delta t) = \tilde{\Theta}_k(\Delta t)\otimes\tilde{\Theta}_s(\Delta t) = (\tilde{\Phi}_k\otimes\tilde{\Phi}_s)\ \exp(\tilde{\Lambda}_{ks}\Delta t)\ (\tilde{\Phi}_k^{-1}\otimes\tilde{\Phi}_s^{-1}) \tag{29}$$

and

$$\tilde{\mathbf{D}}_{ks}^{-1} = (\tilde{\Phi}_k\otimes\tilde{\Phi}_s)\ \tilde{\Lambda}_{ks}^{-1}\ (\tilde{\Phi}_k^{-1}\otimes\tilde{\Phi}_s^{-1});\quad \tilde{\Lambda}_{ks} = (\tilde{\Lambda}_k\otimes\mathbf{I}_4 + \mathbf{I}_4\otimes\tilde{\Lambda}_s) \tag{30),(31}$$

In these equation $\tilde{\Lambda}_{ks}$ is a diagonal matrix, and $\tilde{\Lambda}_j$ and $\tilde{\Phi}_j$ are two matrices of order 4 x 4 listing the eigenvalues and eigenvectors of matrix $\tilde{\mathbf{D}}_j$ $(j = k,s)$ given in equation (21). This matrix possesses two pairs of complex and conjugate eigenvalues and eigenvectors. These quantities can be obtained in closed form solution. The four eigenvalues $\lambda_{i,j}$ of $\tilde{\mathbf{D}}_j$ are given respectively as (Muscolino,1986):

$$\lambda_{1,j} = -\zeta_j\omega_j + i\omega_{Dj};\ \lambda_{2,j} = -\zeta_g\omega_g + i\omega_{Dg};\ \lambda_{3,j} = \lambda_{1,j}^*,\ \lambda_{4,j} = \lambda_{2,j}^* \tag{32}$$

($i = \sqrt{-1}$ being the imaginary unit) while the corresponding eigenvectors $\tilde{\Phi}_{i,j}$ are vectors of order 4 given as:

$$\tilde{\Phi}_{1,j}^T = [\lambda_{1,j}^*/\omega_j^2\ \ 1\ \ 0\ \ 0\ ];\ \tilde{\Phi}_{2,j}^T = [r_{jg}\ \ q_{jg}\ \ \lambda_{2,j}^*/\omega_g^2\ \ 1];\ \tilde{\Phi}_{3,j} = \tilde{\Phi}_{1,j}^*;\ \tilde{\Phi}_{4,j} = \tilde{\Phi}_{2,j}^* \tag{33}$$

In these two equations the star means complex conjugate, $\omega_{Dj} = \omega_j\sqrt{1-\zeta_j^2}$ and q_{jg} and r_{jg} are complex quantities given respectively as:

$$q_{jg} = (\omega_g^2+\lambda_{2,j}\ 2\zeta_g\omega_g)/(\lambda_{2,j}^2+ 2\zeta_j\omega_j\ \lambda_{2,j}+\omega_j^2);\quad r_{jg} = q_{jg}/\lambda_{2,j} \tag{34}$$

From these equations it is evident that once the eigenproperties of the single oscillator given in equation (14) are known it is possible to evaluate all the eigenproperties of the matrix $\tilde{\mathbf{D}}_j$ which also takes into account the characteristics of the Tajimi-Kanai-like filter.

It is to be emphasized that in the stationary case, by using equation (25) it is possible, after very simple algebra, to obtain directly the stationary cross-covariances. Indeed in this case the cross-covariances are not time-dependent quantities ($\phi(t) = 1,\ \forall\ t$) and are given as:

$$E[\tilde{\mathbf{z}}_k\otimes\tilde{\mathbf{z}}_s] = 2\pi S_0\ \tilde{\mathbf{D}}_{ks}^{-1}\ \tilde{\mathbf{v}}^{[2]} = -2\pi S_0(\tilde{\Phi}_k\otimes\tilde{\Phi}_s)\ \tilde{\Lambda}_{ks}^{-1}(\tilde{\Phi}_k^{-1}\otimes\tilde{\Phi}_s^{-1})\ \tilde{\mathbf{v}}^{[2]} \tag{35}$$

The numerical solution proposed in this section is quite different from the other ones proposed in literature (Di Paola et al., 1984; Langley, 1986) to evaluate the stochastic response of systems subjected to filtered non-stationary input processes. The main difference is due to the fact that by using equation (27) the modal cross-covariances are evaluated at the same time as the cross-covariances between modal ouput and input and the variances of input. The numerical procedure proposed is very simple and unconditionally stabler and it is similar to that widely used in deterministic dynamic analysis of structural systems (see e.g. Borino and Muscolino, 1986).

Hence by using this technique the main computational difficulty in evaluating the stochastic response by the traditional methods is overcomed.Indeed no additional computer time is required to evaluated the convolution integral between the input and the output which makes the traditional techniques cumbersome. For good accuracy, the evaluation of these integrals requires a very small step interval and it may fail due to long time response because of overflow problems.

SUMMARY AND CONCLUSIONS

A new method is presented to evaluate the non-stationary mean-square response of classically and non-classically damped systems subjected to seismic input idealized as non-stationary Tajimi-Kanai-like filtered processes.
The proposed method requires the following steps: a) defining the input process as the solution of a second order differential equation forced by a non-stationary zero-mean Gaussian white noise process; b) writing by means of Ito's rule the differential equations which define the evolution of the non-stationary cross-covariances of the output as a set of first order differential equations subjected to deterministic input; c) solving these equations by using common techniques for deterministic analysis. This procedure overcomes the computational difficulties connected with the traditional ones. Indeed, no additional computational time is required to evaluate the operators which appear in the step-by-step solution with respect to the deterministic case and the stochastic averages between the input and output are evaluated at the same time as the evaluation of the mean square response.
Furthermore, starting from a new definition of the impulse response function matrix it has been possible to perform the stochastic analysis of non-classically damped systems by using similar quantities to those evaluated for classically damped ones. The only difference for the analysis of the two systems is in two different weighting vectors, which, for non-classically damped systems, take into account the complex eigenproperties of reduced equations of motion.

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