Contributions to the Systems Reliability of Structures

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INTRODUCTION

There are many subjects left to be studied for evaluating the systems reliability of the structures. This paper discusses the recent researches on (1) selection of dominant failure paths, (2) structural mechanics model for generation of failure mode equations, (3) computation of component failure probability, and (4) evaluation of systems reliability.

SELECTION OF DOMINANT FAILURE PATHS

Consider a structural system with n elements. Elements are assumed to fail one by one up to some specific number p_q until structural failure results. The sequence of those elements to yield structural failure is symbolically denoted as $r_1, r_2, \ldots, r_p, \ldots,$ and r_{pq}, which is called a complete failure path. The set of the failed elememts to yield structural failure is called a failure mode. On the other hand, the sequence of the failed elements which do not yield structural failure, e.g., the failure path $r_1 \rightarrow r_2 \rightarrow \ldots \rightarrow r_p (p < p_q)$ is called a partial failure path. The probability $P^{(p)}_{fp(q)}$ of failure path $r_1 \rightarrow r_2 \rightarrow \ldots \rightarrow r_p$ is calculated as

$$P^{(p)}_{fp(q)} = P[\bigcap_{i=1}^{p} F^{(i)}_{ri(q)}] \tag{1}$$

where $F^{(i)}_{ri(q)}$ is the failure event that element r_i fails at the i-th order of sequence, i.e., $F^{(i)}_{ri(q)} = (Z^{(i)}_{ri(q)} \leqq 0)$. Superscript p denotes the length of the failure path and q is used to denote a particular failure path. When $p < p_q$, $P^{(p)}_{fp(q)}$ is the probability of a partial failure path while it is the probability of a complete failure path for $p = p_q$.

The probability $P^{(p)}_{fp(q)}$ is estimated by evaluating its lower and upper bounds, $P^{(p)}_{fp(q)(L)}$ and $P^{(p)}_{fp(q)(U)}$. For example, these bounds are given by the following formulas(Thoft-Christensen & Murotsu, 1986):

$$P^{(p)}_{fp(q)(L)} \leqq P^{(p)}_{fp(q)} = P[\bigcap_{i=1}^{p} F^{(i)}_{ri(q)}] \leqq P^{(p)}_{fp(q)(U)} \tag{2}$$

$$P^{(p)}_{fp(q)(U)} = \min_{j\varepsilon\{2,\dots,p\}} P[F^{(1)}_{r1(q)} \cap F^{(j)}_{rj(q)}] \quad (3)$$

$$P^{(p)}_{fp(q)(U)} = \min_{j\varepsilon\{1,2,\dots,p\}} P[F^{(j)}_{rj(q)}] \quad (4)$$

$$P^{(p)}_{fp(q)(L)} = \max \{0, 1-P[\bar{F}^{(1)}_{r1(q)}] - \sum_{i=2}^{p} \min_{j\varepsilon\{1,2,\dots,i-1\}} P[F^{(j)}_{rj(q)} \cap \bar{F}^{(i)}_{ri(q)}]\} \quad (5)$$

$$P^{(p)}_{fp(q)(L)} = \max \{0, P[F^{(1)}_{r1(q)}] - P[F^{(1)}_{r1(q)} \cap \bar{F}^{(2)}_{r2(q)}] - \sum_{j=3}^{p} \min(P^{(j-1)}_{fp(q)(U)}, P[F^{(1)}_{r1(q)} \cap \bar{F}^{(j)}_{rj(q)}])\} \quad (6)$$

Eq.(5) needs the safety margins at all the failure stages, while eq.(6) uses only the safety margins at the first and last stages and the upper bound of the preceding failure path probabilities.

There are too many failure paths in a redundant structure to generate all of them, which necessitates a procedure for selecting only the probabilistically significant failure paths. Efficient methods by using a branch-and-bound technique have been proposed(Murotsu et al, 1981; Thoft-Christensen & Murotsu, 1986), and the algorithmic procedure for the original version is given as follows:

- P_{fpM} : the maximum of the lower bounds for the probabilities of the selected complete failure paths
- X : the set of the failure paths to be selected for branching
- X_c : the set of the selected complete failure paths
- X_t : the set of the discarded failure paths
- x_s : a selected failure path
- x_c : a selected complete failure path
- γ : a bounding constant
- ϕ : a null set
- x_o : the artificial starting point of the failure path which can proceed to any one of the potential failure elements

Step 1 (initializing)
Set $P_{fpM}=0$, $X_c=\phi$, $X_t=\phi$, and $X=x_o$.
x_o is specified as a path for partitioning.

Step 2 (partitioning)
1. Proceed one failure stage by adding each of all the potential failure elements to the specified partial failure path. The resulting failure paths are added to the set X of the failure paths to be selected for branching.
2. Evaluate the upper bounds $P^{(p)}_{fp(q)(U)}$ for the probabilities of the new failure paths.

Step 3 (branching)
1. Select the failure path x_s with maximum upper bound probability among the newly partitioned failure paths.
2. Check the attainment of structural failure.
3. If structural failure is attained, go to step 4 for bounding by adding the selected failure path $x_c=x_s$ to the

set X_c of the selected complete failure paths. If not, go to step 2 for further partitioning by specifying the selected failure path as the failure path to be partitioned.

Step 4 (bounding)

1. Evaluate the lower bound $P^{(p_q)}_{fp(q)(L)}$ of the probability of occurrence for the selected complete failure path.
2. Update the maximum P_{fpM} of the lower-bound probabilities of the selected complete failure paths by setting

$$P_{fpM}=P^{(p_q)}_{fp(q)(L)} \text{ when } P_{fpM}< P^{(p_q)}_{fp(q)(L)}$$

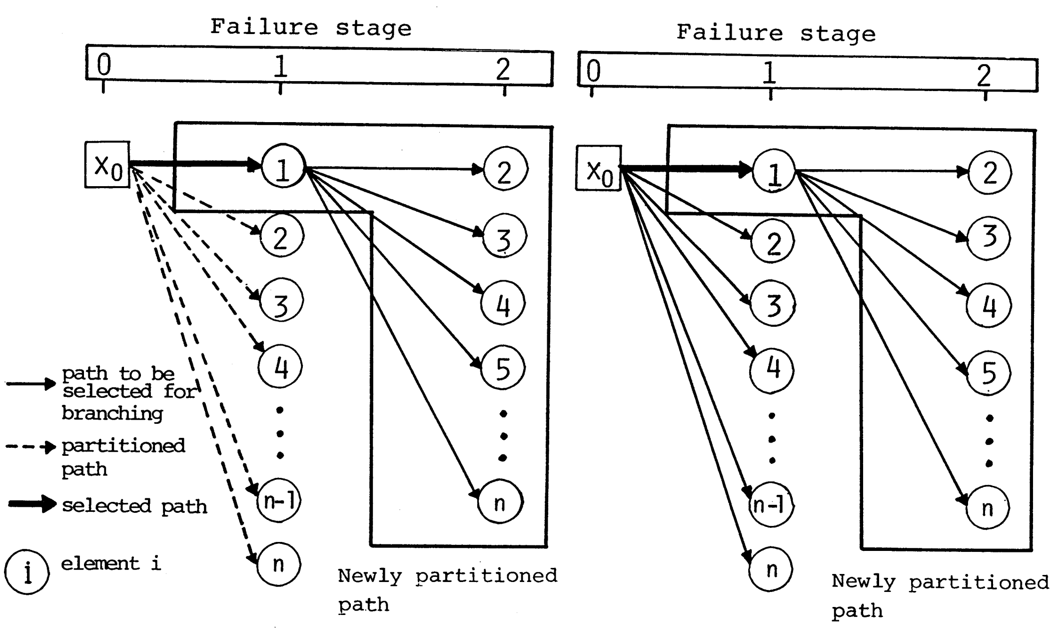
3. Discard the failure paths which have the probabilities of failure smaller than $10^{-\gamma}\cdot P_{fpM}$. Add the discarded failure paths to the set X_t of the discarded failure paths. Exclude the discarded failure paths and the selected complete failure path from the set X of the failure paths for branching. Consequently, the set X of the failure paths to be selected for branching is changed to

$$X \leftarrow X-X_t-X_c$$

Step 5 (terminating)

If $X=\phi$, i.e., there are no failure paths left for branching, the search is terminated. If not, go to step 3-2 by selecting the failure path x_s with the maximum upper-bound failure probability from the set of the failure paths with the largest path-length in the set X.

The selection of the branching path in step 3-1 and step 5 is restricted to the set of the newly partitioned failure paths and



(a) Depth-first branching rule. (b) Width-first branching rule.

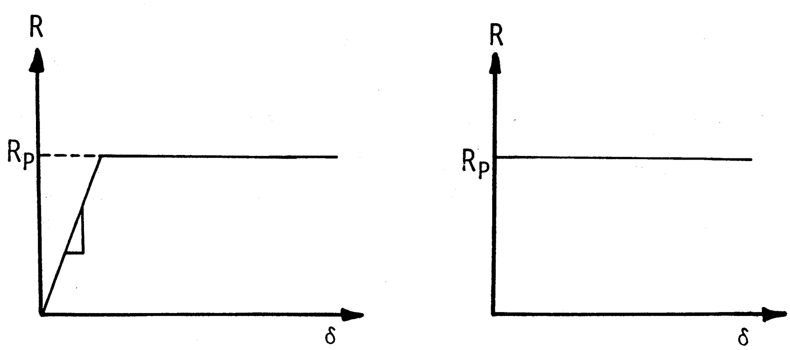
Fig. 1 Branching rules

that of the failure paths with the largest path-length, respectively. Thus, it is called a depth-first branching rule. A variation in the selection rule is to extend the set of the failure paths for selection to the set of all the potential failure paths for branching, i.e., the set X. This is called a width-first branching rule. Fig. 1 illustrates the two branching rules.

In order to calculate the upper bounds of the probability of occurrence for the failure paths, either eq.(3) or eq.(4) is applied while the lower bound is evaluated with either eq.(5) or eq.(6).

STRUCTURAL MECHANICS

An essential step in the computation of the probability of failure of structural systems is the generation of the failure mode equations. For large structures with high degree of redundancy, the incremental load method is often employed(Moses & Rashedi, 1983; Murotsu et al, 1980, 1981, 1985; Thoft-Christensen & Murotsu, 1986; Quek & Ang, 1986; Watwood, 1979). It is inherent in this approach that for ductile members, elasto-plastic force-displacement relationship is assumed rather than a rigid-plastic relationship, as shown in Fig. 2. The need to assume some relative stiffness values is essential to obtain the intermediate steps leading to the mechanisms although the final failure mode equation is independent of these values. This is consistent with the theory of plasticity. In most instances, however, some failure mode equations generated are not consistent with rigid-plastic analysis in the sense that some of the terms imply negative work. The treatment of such equations in the probability computations are often ambiguous and an attempt here is made to explain and reconcile such discrepancy.



(a) Elasto-plastic. (b) Rigid-plastic.

Fig. 2 Assumed member R-δ behaviour.

Consider for example the simple truss shown in Fig. 3, with all the failure paths and equations including the necessary constraints depicted in Fig. 4. The failure mode equation for path 2+ to 1-, namely $R_1^- - 0.2R_2^+ - 3.02S = 0$ is strictly incorrect from the theory of plastic analysis. From the elasto-plastic point of view, one can consider load increments as in the following: by using the force-displacement relationship of Fig. 2(a) with the appropriate capacities for the members such that member 2 yields first, member

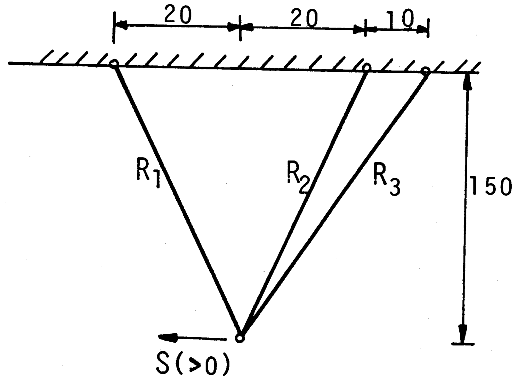
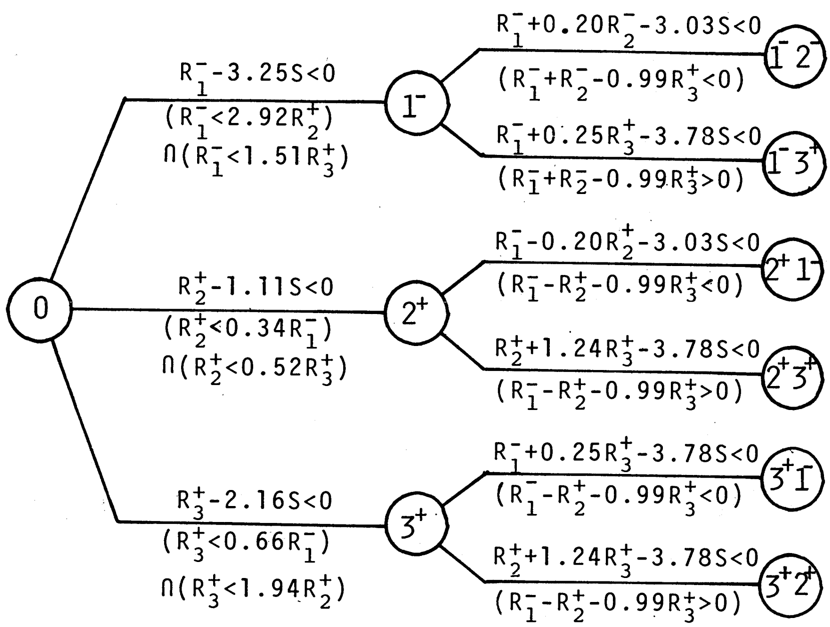


Fig. 3 Simple truss example.



Legend

N^s N: failed member number.

s: + for tensile, - for compressive.

( ) : inequality constraints to ensure that failure is in the sequence indicated.

Fig. 4 Failure paths and equations for simple truss.

2 will in fact continue to elongate whereas member 1 will be further shortened. There is no obvious violation of structural mechanics rule and such a failure mode may indeed be valid. In rigid-plastic theory, for the mechanism involving members 1 and 2, member 3 will not have any elongation since it is not loaded to its yield capacity although it is in tension. Hence, members 1 and 2 must fail in compression. In other words, member 2 should not extend in the first place. Hence, the equation obtained is invalid. This implies that if the assumption of rigid-plastic behaviour is desirable and the generation of the equations is made by the use of the load incremental method, only the consistent equations can be admitted in the probability computations. The intermediate branch equations including the constraints should not be used since rigid-plastic solution is strictly path-independent. This is true if all the complete machanisms are obtained in computing the system failure probability. For the example shown, the relevant set of equations needed to evaluate the system failure probability is

$$R_1^- + 0.20R_2^- - 3.02S = 0 \tag{7a}$$

$$R_1^- + 0.25R_3^+ - 3.78S = 0 \tag{7b}$$

$$R_2^+ + 1.23R_3^+ - 3.77S = 0 \tag{7c}$$

In conclusion, one must be clear at the start of the analysis whether the behaviour of each structural member is to be modelled as elasto-plastic or rigid-plastic. It is then obvious whether some of the generated modes are valid or should be neglected. The constraints, in theory, should be accounted in the computation of the failure probability if elasto-plastic behaviour is assumed although this is not done in practice due to computational difficulties.

COMPUTATION OF COMPONENT FAILURE PROBABILITY

In system reliability computations, the load and resistance variables are often assumed to be normally distributed. This simplifies the computation significantly especially if the limit state equation of each component is of the form g(X)=R-S, where R and S are the capacity and load effect of the component, respectively, and they are often assumed to be statistically independent. It is not unusual in structures that some components can resist load only either in tension or in compression. The strength of such components may appropriately be assumed to follow a lognormal or a truncated normal distribution function. In the case of the truncated normal as illustrated in Fig. 5, its density and distribution function is given by

$$f_R^t(r) = \frac{1}{\Phi\left[\frac{\mu_R}{\sigma_R}\right]} \frac{1}{\sigma_R} \phi\left[\frac{r-\mu_R}{\sigma_R}\right] \tag{8a}$$

$$F_R^t(r) = \frac{\Phi\left[\frac{r-\mu_R}{\sigma_R}\right] - 1}{\Phi\left[\frac{\mu_R}{\sigma_R}\right]} + 1 \tag{8b}$$

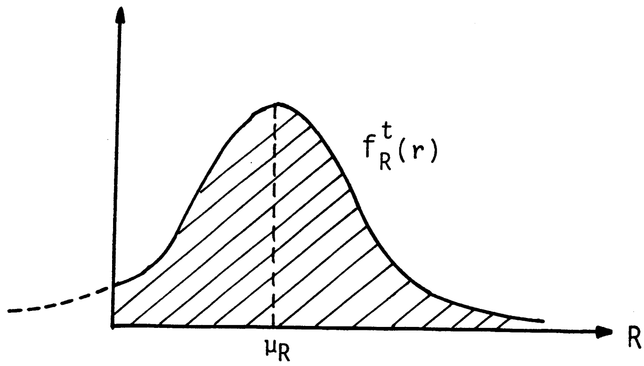


Fig. 5 Truncated normal density function.

Table 1. Failure probabilities computed by three methods with R being truncated normal [R(1.0, σ_R), S(0.05, 0.3)].

σ_R		Method (a)	Method (b)	Method (c)
0.1	P_f	1.3316×10^{-3}	1.3316×10^{-3}	1.3316×10^{-3}
	$\Delta P_f/P_f$	0	0	0
	Time (s)	0.16667	3.3333×10^{-2}	0
0.2	P_f	4.2088×10^{-3}	4.2090×10^{-3}	4.2090×10^{-3}
	$\Delta P_f/P_f$	0	4.7519×10^{-5}	4.7519×10^{-5}
	Time (s)	0.20000	5.0000×10^{-2}	0
0.3	P_f	1.2294×10^{-2}	1.2491×10^{-2}	1.2572×10^{-2}
	$\Delta P_f/P_f$	0	1.6024×10^{-2}	2.2613×10^{-2}
	Time (s)	0.28333	5.0000×10^{-2}	0
0.4	P_f	2.4469×10^{-2}	2.6419×10^{-2}	2.8717×10^{-2}
	$\Delta P_f/P_f$	0	7.9693×10^{-2}	1.7361×10^{-1}
	Time (s)	0.43333	0.10000	0
0.5	P_f	3.5324×10^{-2}	4.0250×10^{-2}	5.1631×10^{-2}
	$\Delta P_f/P_f$	0	1.3945×10^{-1}	4.6164×10^{-1}
	Time (s)	0.26667	0.10000	0

where μ_R and σ_R are the mean and standard deviation of the non-truncated distribution. The computation of the failure probability of each component may be approximated by one of the following:

a. $$p_f = \int_0^{\infty} [1-F_S(r)]\, f_R^t(r)\, dr \qquad (9a)$$

b. $$p_f \cong \Phi\left(- \frac{\mu_R^N - \mu_S}{\sqrt{(\sigma_R^N)^2 + \sigma_S^2}}\right) \qquad (9b)$$

c. $$p_f \cong \int_{-\infty}^{\infty} [1-F_S(r)]\, f_R(r)\, dr = \Phi\left(- \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right) \qquad (9c)$$

where μ_R^N and σ_R^N are the equivalent normal parameters of the truncated normal, using the principle of normal tail approximation. The results obtained using the three methods including their relative errors and computation times are tabulated in Table 1. It can be seen that numerical integration of the first equation using the Simpson's 3/8-formula with ten intervals produces accurate answers. For $\mu_R > 4\sigma_R$, it is considerably faster to use method (b) without loss of accuracy. Method (c) is superior if $\mu_R >> \sigma_R$. In systems reliability computations and in optimization problems with probability constraints, the computation time of the component failure probability contributes significantly to the overall computational time, which is often prohibitively long especially for practical structures. The trade-off between speed and accuracy is therefore of importance. The results of studies such as the one exemplified here will therefore serve as a guide to the range within which one should apply the appropriate method of computation to achieve an efficient solution.

SYSTEMS RELIABILITY

In estimating the failure probability of a structural system, an upper bound solution is obtained from the probability of initial damage to the system and a lower bound solution is obtained by taking the probability of the union of significant collapse modes(Quek & Ang, 1986; Thoft-Christensen & Murotsu, 1986) given by

$$P\left[\bigcup_{q \epsilon X_c} F_{r_{pq}}^{(pq)}(q)\right] \leqq P_f < P\left[\bigcup_{q \epsilon \{1,2,\ldots,n\}} (Z_q^{(1)} \leqq 0)\right] \qquad (10)$$

The second-order bound is used in estimating the union probabilities. For large structures, the significant collapse modes may be obtained using the branch-and-bound method(Thoft-Christensen & Murotsu, 1986). An alternate solution to the upper bound may therefore be obtained by considering all the incomplete failure paths when one applies this method together with the significant collapse modes. Using the same concept as above, the upper bound is given by

$$P_f \leqq P\left[\left(\bigcup_{q \epsilon X_c} F_{r_{s1}}^{(s1)}(q)\right) \bigcup \left(\bigcup_{q \epsilon X_t} F_{r_{s3}}^{(s3)}(q)\right)\right] \simeq P_{fU}^{(1)} \qquad (11)$$

where $F^{(s3)}_{r_{s3}(q)}$ is the event corresponding to $\min_{j=1,2,\ldots,p} P[Z^{(j)}_{r_j(q)} \leq 0]$.

Theoretically, one can further improve the solution by taking the intersection of all the branch events of each incomplete failure path instead of only using the initial event or the event leading to the lowest probability. This will lead to computational problems, first in the intersection and then in the union. The intersection of many events is computationally time-consuming and often inaccurate. The union of events will further be complicated unless one uses an approximate single equation to represent each set of intersection events in the final computation (Gollwitzer & Rackwitz, 1983). A natural consequence is to use two branch events for each incomplete path giving rise to the upper bound approximation

$$P_f \leq P[\bigcup_{q\varepsilon X_c} F^{(s1)}_{r_{s1}(q)}] + \sum_{q\varepsilon X_t} (F^{(s1)}_{r_{s1}(q)} \cap F^{(s2)}_{r_{s2}(q)}) \simeq P^{(2)}_{fU} \tag{12}$$

A further improvement to this is to use

$$P_f \leq P[\bigcup_{q\varepsilon X_c} F^{(s1)}_{r_{s1}(q)}] \cup [\bigcup_{q\varepsilon X_t} (F^{(s1)}_{r_{s1}(q)} \cap F^{(s2)}_{r_{s2}(q)})] \simeq P^{(3)}_{fU} \tag{13}$$

The difficulty with the above lies in the second-order union probability computation where the intersection of three events are needed. A conservative and simple estimate for such intersection

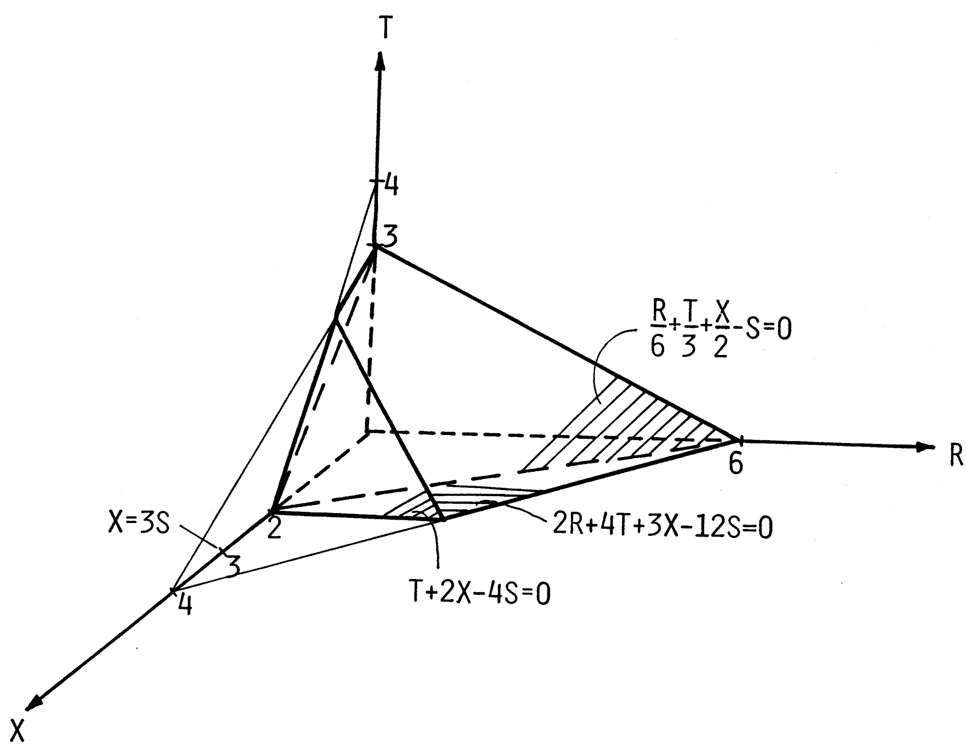


Fig. 6 Actual and approximated failure regions.

probability is to approximate the three hyperplanes for the three events by a hyperplane such that the region of integration bounded by the three hyperplanes encompassed that of the approximated hyperplane. As a simple example, consider the following event

$$D = (2R+4T+3X-12S< 0)\cap (X-3S< 0)\cap (T+2X-4S< 0)$$

being approximate by event

$$G = (\frac{R}{6} + \frac{T}{3} + \frac{X}{2} - S < 0)$$

and further exemplified in Fig. 6.

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