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Development of Pad Element for Detailed Core Deformation Analyses and its Verification

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ABSTRUCT

It is important to understand the deformation behaviors of FBR core in the aspects of safety and economy, and to realize more precise analyses of the core mechanics for sophistication of the core structure design. However, the interaction among subassemblies, which occurs as the result of contact between pad surfaces of adjacent subassemblies, makes the core mechanics complicated. Therefore, it is important to consider the characteristics of pads in the analysis in order to realize detailed analysis. The aim of this study is to develop a special 'PAD ELEMENT', which owes the role to treat the interaction between subassemblies at their pads precisely. In this paper, the detailed formulation of the 'PAD ELEMENT' and the result of its verification test are described.

1 INTRODUCTION

It is important to understand the deformation behaviors of FBR core in the aspects of safety and economy, i.e., the reactivity change in the transient period, the fuel life expectation, the removal of subassemblies, etc. Furthermore, the realization of precise analysis of the core deformation behaviors will contribute to sophistication of the core structure design.

The interaction among subassemblies, which occurs as the result of contact between pad surfaces of adjacent subassemblies, makes the core deformation complicated. Therefore, it is important to consider the characteristics of pads in the analysis in order to realize detailed analysis. The study of this subject is not new. Especially, IAEA conducted the international bench mark study by using special purpose codes for core mechanics from 1985 to1989 ([1]-[6]). However, these codes basically used a simple beam element to model a subassembly, and did not consider detailed characteristics of pad portion, i.e., the additional bending moment due to the friction at pad surface, the change of compression rigidity of pad cross-section due to contact pattern at pad, the deformation of pad cross-section due to thermal expansion, swelling and internal pressure etc.. Reinhall et al. pointed the importance of the effects of the additional bending moment due to the friction and the change of bending moment rigidity due to the interaction between duct and fuel pins, by comparing the computed results of the model of a few subassemblies by beam elements with those by shell elements[7]. If we could use shell elements to model each subassembly, it would be easy to consider these effects. In case we construct the core model by shell elements, we will face a difficulty in implementing numerical calculation. In order to avoid this difficulty we proposed the concept of modeling of the pad characteristics ([8],[9]).

The aim of this study is to develop a special 'PAD ELEMENT' which owes the role to treat the interaction between subassemblies at their pads precisely, and which is consistent in the modeling by beam elements in order to promise practical core deformation analyses. In this paper, the detailed formulation of the 'PAD ELEMENT' is described and the effects of some of the aforementioned characteristics at pad are discussed through comparative analysis results between the ordinary beam element model and the beam element model with 'PAD ELEMENT'.

2 FORMULATION OF 'PAD ELEMENT'

2.1 OUTLINE OF METHODOLOGY

This specific element is consisted of 6 ordinary beam elements which are arranged spoke-like(see Fig. 1) in order to express various characteristics of the pad of a subassembly, i.e., the flexibility of pad cross section including coupling effect, bending or torsional moment caused by friction forces between pads and uniform cross sectional deformation at pads such as thermal expansion, swelling, and so on.

This element has 7 nodes as shown in Fig. 1. In the figure the dashed line represents the shape of pad cross section. The satellite nodes from No.1 to No.6 are located at the center of six surfaces of the pad and the center node of No.7 is located at the center of the cross-section of the pad. A pad element is connected with six pad elements of adjacent subassemblies through contact elements at its satellite nodes and is also connected with beam elements, which correspond to the centerline of the subassembly at its center node. The bending moments due to friction at pad surfaces are transferred from satellite nodes to center node through spoke-like arms. The deformation of cross section of the pad due to thermal expansion etc. can be expressed by the extension of the arms. The coupling effect of the flexibility of pad cross section is considered by relating the compression force subjected to each pair of parallel surfaces of pad not only to the relative displacement of the same pair of parallel surfaces but also to those of two other pairs of parallel surfaces.

It is very easy to apply this pad element to conventional FEM codes, since this element can be simply connected to the node of beam element at the center node of the pad element.

2.2 FORMULATION BY BEAM ELEMENT

Taking the center as the origin of each beam coordinate, the beam axis as x_i-coordinate (i=1,6) and z-coordinate of the pad element as that of each beam as shown in Fig. 1. And assuming that 6 satellite nodes are free from moments, the equilibrium equation can be expressed by the following form,

$$\{f'_7\} = [k_0]\{u'_7\}. \qquad (2\text{-}1)$$

In the above equation the elements of the nodal force and nodal displacement vectors and the stiffness matrix are

$$\{f'_7\}^T = \lfloor f^i_{x7} f^i_{y7} f^i_{z7} m^i_{x7} m^i_{x7} m^i_{x7} \vdots f_{xi} f_{yi} f_{zi} \rfloor \quad (i = 1,6) \qquad (2\text{-}2a)$$

$$\{u'_7\}^T = \lfloor u^i_{x7} u^i_{y7} u^i_{z7} \theta^i_{x7} \theta^i_{x7} \theta^i_{x7} \vdots u_{xi} u_{yi} u_{zi} \rfloor \quad (i = 1,6) \qquad (2\text{-}2b)$$

$$[k_0] = \frac{EA}{l}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{3I_z}{Al^2} & 0 & 0 & 0 & \frac{3I_z}{Al} & 0 & -\frac{3I_z}{Al^2} & 0 \\ 0 & 0 & \frac{3I_y}{Al^2} & 0 & -\frac{3I_y}{Al} & 0 & 0 & 0 & -\frac{3I_y}{Al^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3I_y}{Al} & 0 & \frac{3I_y}{A} & 0 & 0 & 0 & \frac{3I_y}{Al} \\ 0 & \frac{3I_z}{Al} & 0 & 0 & 0 & \frac{3I_z}{A} & 0 & -\frac{3I_z}{Al} & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{3I_z}{Al^2} & 0 & 0 & 0 & -\frac{3I_z}{Al} & 0 & \frac{3I_z}{Al^2} & 0 \\ 0 & 0 & -\frac{3I_y}{Al^2} & 0 & \frac{3I_y}{Al} & 0 & 0 & 0 & \frac{3I_y}{Al^2} \end{bmatrix} \qquad (2\text{-}2c)$$

In the above matrix E, A, l, I_y, I_z are Young's modulus, cross section area, length and moments of inertia of area of the beam element, respectively.

2.3 CONSIDERATION OF CROSS-SECTION DEFORMATION

In order to consider the isotropic deformation of the hexagonal cross section due to thermal expansion, internal pressure, creep, swelling, etc. and the relative displacements between parallel surfaces of three pairs due to contact against adjacent subassemblies, pseudo-displacements are added to x direction displacement of node (i) as

$$u_{xi} \rightarrow u_{xi} - w_0 + w_{ij} \qquad (2\text{-}3)$$

By using this equation, Eqn.(2-1) is reformed as

$$\{f'_7\} = [k_0]\{u'_7\} - \frac{EA}{l}\{w_0\} + \frac{EA}{l}\{w_{ij}\} \quad (i = 1,6) \qquad (2\text{-}4)$$

where

$$\{w_0\}^T = \lfloor -w_0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \vdots w_0 \quad 0 \quad 0 \rfloor \qquad (2\text{-}5a)$$

$$\{w_{ij}\}^T = \lfloor -w_{ij} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \vdots w_{ij} \quad 0 \quad 0 \rfloor \qquad (2\text{-}5b)$$

$$(i,j) = (1,4),(2,5),(3,6), \qquad w_{ij} = w_{ji}$$

In Eqns.(2-5) w_0 and w_{ij} are the isotropic displacement for one beam and a half of the relative displacement between the pair nodes. w_0 might be approximately expressed as following.

$$w_0(p,\phi_t,T,t) = w_{01}(p) + w_{02}(p,\phi_t) + w_{03}(T,p,t) + w_{04}(T) \qquad (2\text{-}6)$$

In this equation p, ϕ_t, T and t denote internal pressure, neutron flux intensity, temperature and time, and the four terms of the right hand side are the displacements due to internal pressure, radiation creep and swelling, thermal creep and thermal expansion, respectively.

On the other hand, the coupling effect among compression stiffness of three pairs of surfaces is assumed by

$$\{w_{IJ}\} = [D]\{f_{IJ}\} \qquad (2\text{-}7)$$

where

$$\{w_{IJ}\}^T = \lfloor w_{14} w_{25} w_{36} \rfloor, \quad \{f_{IJ}\}^T = \lfloor f_{14} f_{25} f_{36} \rfloor, \quad [D] = a\begin{bmatrix} 1 & -\gamma & -\gamma \\ -\gamma & 1 & -\gamma \\ -\gamma & -\gamma & 1 \end{bmatrix} \qquad (2\text{-}8a,b,c)$$

Eqn.(2-8b) and Eqn.(2-8c) denote the relative displacement between the pair nodes and the corresponding compression forces, respectively. Eqn.(2-8c) denotes the compliance matrix including the coupling effect. The coupling effect can be considered by the parameter γ of non-diagonal terms.

The force vector of Eqn.(2-8b) is assumed to be the mean value of two x-direction forces, which are correspondent to the nodes of the pair surfaces. Then,

$$\{f_{IJ}\} = [H]\{f_x\} \qquad (2\text{-}9)$$

where

$$\{f_x\}^T = \lfloor f_{x1} f_{x2} f_{x3} f_{x4} f_{x5} f_{x6} \rfloor, \qquad [H] = -\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad (2\text{-}10a,b)$$

The following equation should be satisfied since the forces, which induce relative displacements between the pairs of parallel surfaces of the pad, are only compression.

$$f_{14}, f_{25}, f_{36} \geq 0 \qquad (2\text{-}11)$$

When the pad element is applied in the analysis, this condition is satisfied automatically because 'contact element' is applied to connect two satellite nodes of adjacent pad elements.

From Eqn.(2-7) and Eqn.(2-9),

$$\{w_{IJ}\} = [D][H]\{f_x\} \qquad (2\text{-}12)$$

2.4 TRANSLATION TO ELEMENT CO-ORDINATES

In order to construct 'PAD ELEMENT', the nodal vectors of each beam element are translated to those by pad element co-ordinates and are superposed at the center node.

Nodal displacement vectors and nodal force vectors by beam element co-ordinates and those by pad element co-ordinates are related by co-ordinate translation matrix.

$$\{u_i\} = [T_i]\{U_i\}, \quad \{f_i\} = [T_i]\{F_i\} \quad (i = 1,6) \qquad (2\text{-}13a,b)$$

where $\{u_i\},\{U_i\},\{f_i\},\{F_i\}$ are nodal displacement vectors and nodal force vectors by each beam element co-ordinates and pad element co-ordinates, respectively. $[T_i]$ is the co-ordinate translation matrix represented by

$$[T_i] = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \theta_1 = 0, & \theta_2 = \frac{\pi}{3}, & \theta_3 = \frac{2\pi}{3}, \\ \theta_4 = \pi, & \theta_5 = \frac{4\pi}{3}, & \theta_6 = \frac{5\pi}{3} \end{matrix} \qquad (2\text{-}14)$$

By applying Eqns.(2-13a,b) to Eqn.(2-4), the following equations are obtained.

$$\{\tilde{F}'_7\} = [\tilde{k}'_0]\{\tilde{U}'_7\} - \frac{EA}{l}\{\tilde{W}_0{}'\} + \frac{EA}{l}\{\tilde{W}_{ij}{}'\} \quad (i = 1,6) \qquad (2\text{-}15)$$

where

$$\{\tilde{F}^i_7\}^T = \lfloor F^i_{x7} \;\; F^i_{y7} \;\; F^i_{z7} \;\; M^i_{x7} \;\; M^i_{z7} \;\; M^i_{z7} \vdots F_{xi} \;\; F_{yi} \;\; F_{zi} \rfloor \qquad (2\text{-}16a)$$

$$\{\tilde{U}^i_7\}^T = \lfloor U^i_{x7} \;\; U^i_{y7} \;\; U^i_{z7} \;\; \Theta^i_{x7} \;\; \Theta^i_{z7} \;\; \Theta^i_{z7} \vdots U_{xi} \;\; U_{yi} \;\; U_{zi} \rfloor \qquad (2\text{-}16b)$$

$$[\tilde{k}^i_0] = [\tilde{T}_i]^T [k_0][\tilde{T}_i] \qquad (2\text{-}17)$$

where

$$[\tilde{T}_i] = \begin{bmatrix} [T_i] & [0] & [0] \\ [0] & [T_i] & [0] \\ [0] & [0] & [T_i] \end{bmatrix} \qquad (2\text{-}18)$$

$$\{\tilde{W}^i_0\}^T = w_0 \lfloor -\cos\theta_i \;\; -\sin\theta_i \;\; 0 \;\; 0 \;\; 0 \;\; 0 \vdots \cos\theta_i \;\; \sin\theta_i \;\; 0 \rfloor \qquad (2\text{-}19)$$

$$\{\tilde{W}^i_{ij}\}^T = w_{ij} \lfloor -\cos\theta_i \;\; -\sin\theta_i \;\; 0 \;\; 0 \;\; 0 \;\; 0 \vdots \cos\theta_i \;\; \sin\theta_i \;\; 0 \rfloor \qquad (2\text{-}20)$$

$$(i, j) = (1,4), (2,5), (3,6),$$

2.5 MATRIX EQUILIBRIUM EQUATION

By superposing Eqn.(2-15) of each beam element, the equilibrium equation of 'pad element' is obtained as follows.

$$\{F\} = [K_0]\{U\} - \frac{EA}{l}\{W_0\} + \frac{EA}{l}\{W_{IJ}\} \qquad (2\text{-}21)$$

where

$$\{F\}^T = \lfloor F_{x7}F_{y7}F_{z7}M_{x7}M_{y7}M_{z7} \vdots F_{x1}F_{y1}F_{z1} \vdots F_{x2}F_{y2}F_{z2} \vdots F_{x3}F_{y3}F_{z3} \vdots F_{x4}F_{y4}F_{z4} \vdots F_{x5}F_{y5}F_{z5} \vdots F_{x6}F_{y6}F_{z6} \rfloor \qquad (2\text{-}22a)$$

$$\{U\}^T = \lfloor U_{x7}U_{y7}U_{z7}\Theta_{x7}\Theta_{y7}\Theta_{z7} \vdots U_{x1}U_{y1}U_{z1} \vdots U_{x2}U_{y2}U_{z2} \vdots U_{x3}U_{y3}U_{z3} \vdots U_{x4}U_{y4}U_{z4} \vdots U_{x5}U_{y5}U_{z5} \vdots U_{x6}U_{y6}U_{z6} \rfloor \qquad (2\text{-}22b)$$

and

$$\{W_0\}^T = w_0 \left\lfloor 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \vdots 1 \;\; 0 \;\; 0 \vdots \frac{1}{2} \;\; \frac{\sqrt{3}}{2} \;\; 0 \vdots -\frac{1}{2} \;\; \frac{\sqrt{3}}{2} \;\; 0 \vdots -1 \;\; 0 \;\; 0 \vdots -\frac{1}{2} \;\; -\frac{\sqrt{3}}{2} \;\; 0 \vdots \frac{1}{2} \;\; -\frac{\sqrt{3}}{2} \;\; 0 \right\rfloor \qquad (2\text{-}23)$$

and also by assuming that $w_{ij}=w_{ji}$,

$$\{W_{IJ}\}^T = \left\lfloor 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \vdots w_{14} \;\; 0 \;\; 0 \vdots \frac{1}{2}w_{25} \;\; \frac{\sqrt{3}}{2}w_{25} \;\; 0 \vdots -\frac{1}{2}w_{36} \;\; \frac{\sqrt{3}}{2}w_{36} \;\; 0 \vdots -w_{14} \;\; 0 \;\; 0 \vdots -\frac{1}{2}w_{25} \;\; -\frac{\sqrt{3}}{2}w_{25} \;\; 0 \vdots \frac{1}{2}w_{36} \;\; -\frac{\sqrt{3}}{2}w_{36} \;\; 0 \right\rfloor \qquad (2\text{-}24)$$

Further, Eqn.(2-24) can be expressed by the nodal force vector by applying Eqn.(2-12) as follows.

$$\{W_{IJ}\} = [L][D][H][T_w]\{F\} \qquad (2\text{-}25)$$

where

$$[L]^T = \begin{bmatrix} \underset{3\times6}{[0]} & [L_1] & [L_2] \end{bmatrix}, \quad [T_W] = \begin{bmatrix} \underset{3\times6}{[0]} & [L_1] & \underset{3\times9}{[0]} \\ \underset{3\times6}{[0]} & \underset{3\times9}{[0]} & [L_2] \end{bmatrix} \qquad (2\text{-}26a,b)$$

$$[L_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{bmatrix}, \quad [L_2] = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{bmatrix} \qquad (2\text{-}27a,b)$$

Final matrix equilibrium equation of 'pad element' can be obtained by applying Eqn.(2-25) to Eqn.(2-21).

$$\{F\} + \{F_0\} = [K]\{U\} \qquad (2\text{-}28)$$

In the above equation $[K]$ is the stiffness matrix of 'pad element' and $\{F_0\}$ is the nodal pseudo-force vector due to isotropic deformation of the hexagonal pad cross section. This element with 24 d.o.f. can be easily equipped in the conventional FEM program. The details of these are described in Appendix.

3 VERIFICATION ANALYSES AND DISCUSSION

In order to verify the validity of the 'PAD ELEMENT', we performed parametric analyses by general purpose FEM code FINAS [10] in which 'PAD ELEMENT' is installed. Fig. 1 shows the structure of the analysis model, which consists of 19 subassemblies. The upper figure is the plane figure of the core model and the lower is the front view of the half centerline of fuel subassemblies, No.1, 2 and 8. The temperature distribution of the core is assumed to be axi-symmetric for simplicity and to vary in the axial direction. The gap between two pad surfaces of neighboring subassemblies is 1mm. Linear elastic analyses were implemented with considering friction between contact surfaces. Contacts occur between adjacent pads and between the pad of exterior subassemblies and restraint walls when the bowing of the subassemblies propagate due to thermal expansion. The reaction forces of six pad surfaces of subassembly No.2 due to contact is shown in Fig. 3. The difference of contact reaction forces between pads with and without taking the coupling effect of pad cross section flexibility into account were observed, even if the contact pattern at the pad is the same. Fig.3 shows the deflection of subassemblies of No.2 and No.8. In the case of taking the uniform cross section deformation, i.e. thermal expansion of beam elements, the deflection reduced because of decrease of initial gap clearance with temperature increase. As the result the number of contact surface of pad increased, and the values of contact reaction force also increased. Fig. 4 illustrates the moment distribution of subassemblies of No.2 and 8 whether 'PAD ELEMENT' is applied or not. It can be seen that there exist the discontinuous changes of the moment at the pad portions in case of applying 'PAD ELEMENT'. This is caused by propagation of the local bending moment realized by 'PAD ELEMENT'.

If the boundary conditions of support portion allow fuel subassemblies to move upward by external force, it is found that the lifting up of them occurred. The axial stress distribution is shown in Fig. 5, and Fig.6 shows the amount of lifting up at support portion. It is recognized that the axial stress of lifted subassembly is released below the lower pad. The amount of lift increased by taking the cross section deformation of pads due to thermal expansion, which was realized in 'PAD ELEMENT' into accounts.

4 CONCLUSION

In order to treat pad characteristics at contact, we develop 'PAD ELEMENT' which possesses the function of the flexibility of pad cross section including coupling effect, and can expresses bending and torsional moment induced by friction and uniform cross section deformation due to thermal expansion, swelling, etc. And the effects of some characteristics at pads are discussed through comparative analysis results between the ordinary beam element model and the beam element model with 'PAD ELEMENT'. Through this study it was proved that this element could be easily equipped in the conventional FEM program. And it was verified that the behavior i.e., change of initial gap clearance, coupling effect of pad cross section flexibility and local moment propagation etc., caused by pad characteristics were expressed by using 'PAD ELEMENT', while they could not be represented by conventional model.

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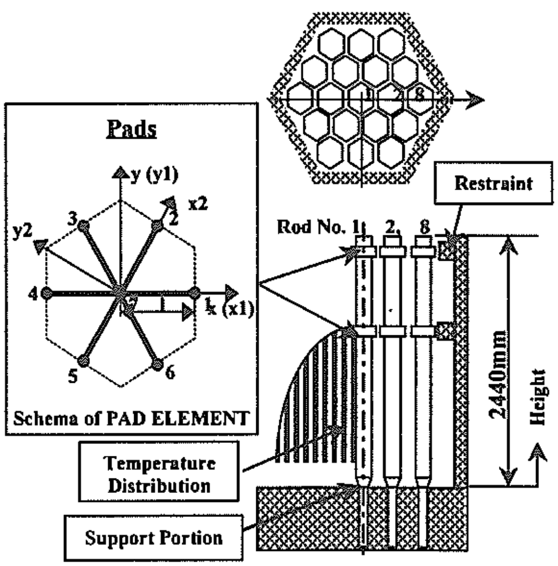

Fig.1 Concept of Analysis Model

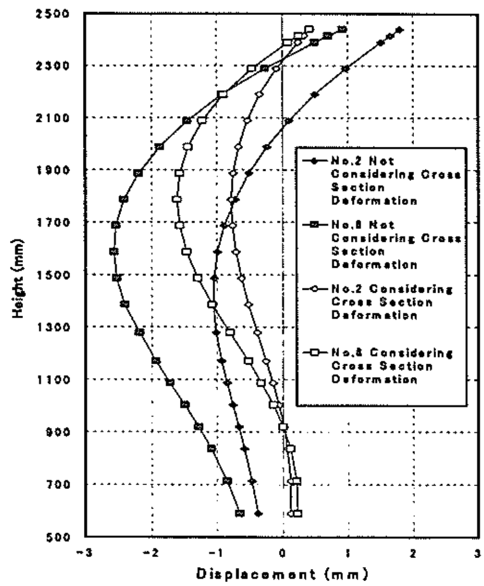

Fig. 3 Deflection of Subassemblies

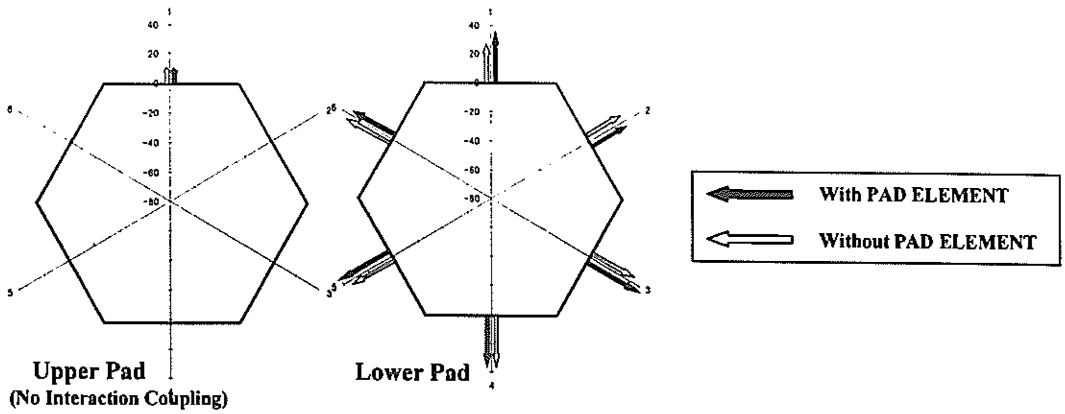

Fig. 2 Contact Reaction Force (Rod No.2)

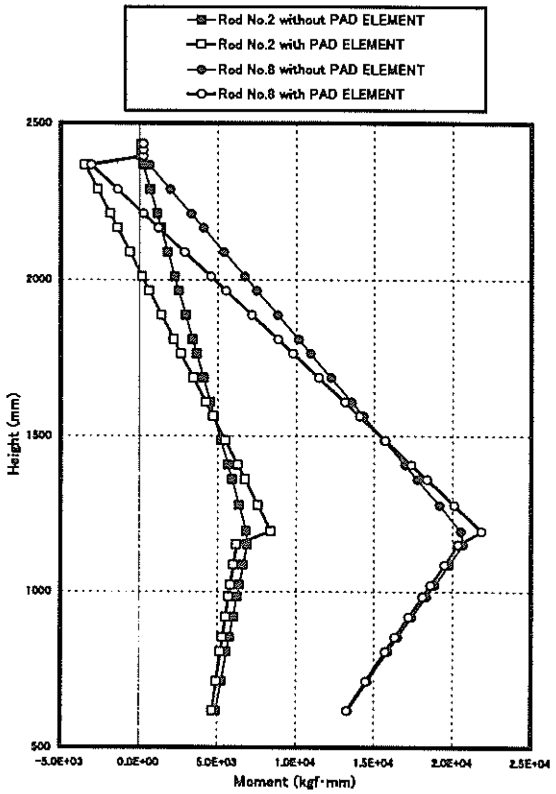

Fig. 4 Comparison of Moment Distribution

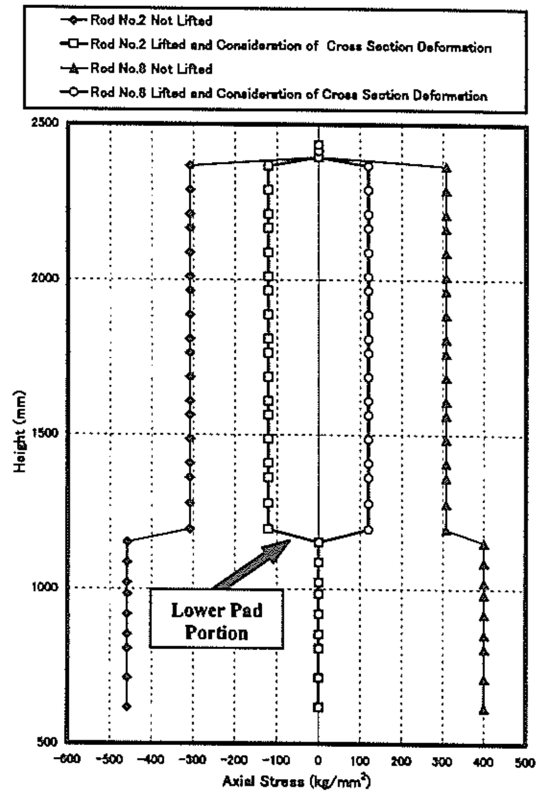

Fig. 5 Comparison of Axial Stress Distribution

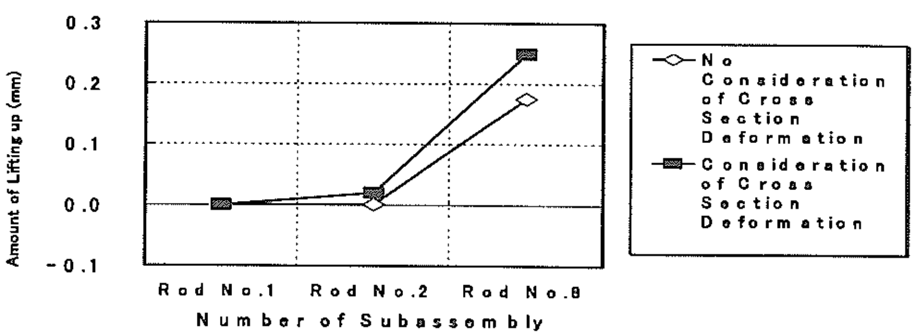

Fig. 6 Amount of Lifting up at Support Portion

Appendix

$$\{F_0\}^T = \frac{EA}{l} w_0 \lfloor \lfloor 0 \rfloor \ \lfloor 0 \rfloor \ \lfloor f_{01} \rfloor \ \lfloor f_{02} \rfloor \ \lfloor f_{03} \rfloor \ -\lfloor f_{01} \rfloor \ -\lfloor f_{02} \rfloor \ -\lfloor f_{03} \rfloor \rfloor \qquad \text{(A-1)}$$

where

$$\lfloor f_{01} \rfloor = \left\lfloor 1-\frac{8\beta}{\phi_1}(\phi_2-2\gamma) \quad 0 \quad 0 \right\rfloor, \ \lfloor f_{02} \rfloor = \left\lfloor \frac{1}{2}-\frac{4\beta}{\phi_1}(\phi_2-2\gamma) \quad \frac{\sqrt{3}}{2}-\frac{4\sqrt{3}\beta}{\phi_1}(\phi_2-2\gamma) \quad 0 \right\rfloor, \ \lfloor f_{03} \rfloor = \left\lfloor -\frac{1}{2}+\frac{4\beta}{\phi_1}(\phi_2-2\gamma) \quad \frac{\sqrt{3}}{2}-\frac{4\sqrt{3}\beta}{\phi_1}(\phi_2-2\gamma) \quad 0 \right\rfloor$$

(A-2a,b,c)

$$[K] = \begin{bmatrix} [K_0^{11}] & [0] & -[K_0^{33}] & -[K_0^{44}] & -[K_0^{55}] & -[K_0^{33}] & -[K_0^{44}] & -[K_0^{55}] \\ & [K_0^{22}] & [K_0^{23}] & [K_0^{24}] & [K_0^{25}] & -[K_0^{23}] & -[K_0^{24}] & -[K_0^{25}] \\ & & [K^{33}] & [K^{34}] & [K^{35}] & [K^{36}] & [K^{37}] & [K^{38}] \\ & & & [K^{44}] & [K^{45}] & [K^{46}] & [K^{47}] & [K^{48}] \\ & & & & [K^{55}] & [K^{56}] & [K^{57}] & [K^{58}] \\ & & & & & [K^{66}] & [K^{67}] & [K^{68}] \\ & SYM. & & & & & [K^{77}] & [K^{78}] \\ & & & & & & & [K^{88}] \end{bmatrix} \qquad \text{(A-3)}$$

where

$$[K^{33}] = \frac{EA}{l}\begin{bmatrix} 1-\frac{4\beta\phi_2}{\phi_1} & 0 & 0 \\ 0 & \frac{3I_z}{Al^2} & 0 \\ 0 & 0 & \frac{3I_y}{Al^2} \end{bmatrix}, \quad [K^{34}] = \frac{EA}{l}\begin{bmatrix} \frac{2\beta\gamma}{\phi_1} & \frac{2\sqrt{3}\beta\gamma}{\phi_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [K^{35}] = \frac{EA}{l}\begin{bmatrix} -\frac{2\beta\gamma}{\phi_1} & \frac{2\sqrt{3}\beta\gamma}{\phi_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(A-4a,b,c)

$$[K^{36}] = \frac{EA}{l}\begin{bmatrix} \frac{4\beta\phi_2}{\phi_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [K^{37}] = -[K^{34}], \ [K^{38}] = -[K^{35}], \quad [K^{44}] = \frac{EA}{l}\begin{bmatrix} \frac{1}{4}(1+\frac{9I_z}{Al^2})-\frac{\beta\phi_2}{\phi_1} & \frac{\sqrt{3}}{4}(1-\frac{3I_z}{Al^2})-\frac{\sqrt{3}\beta\phi_2}{\phi_1} & 0 \\ \frac{\sqrt{3}}{4}(1-\frac{3I_z}{Al^2})-\frac{\sqrt{3}\beta\phi_2}{\phi_1} & \frac{3}{4}(1+\frac{I_z}{Al^2})-\frac{3\beta\phi_2}{\phi_1} & 0 \\ 0 & 0 & \frac{3I_y}{Al^2} \end{bmatrix} \qquad \text{(A-4d,e,f,g)}$$

$$[K^{45}] = \frac{EA}{l}\begin{bmatrix} -\frac{\beta\gamma}{\phi_1} & \frac{\sqrt{3}\beta\gamma}{\phi_1} & 0 \\ -\frac{\sqrt{3}\beta\gamma}{\phi_1} & \frac{3\beta\gamma}{\phi_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [K^{46}] = \frac{EA}{l}\begin{bmatrix} -\frac{2\beta\gamma}{\phi_1} & 0 & 0 \\ -\frac{2\sqrt{3}\beta\gamma}{\phi_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [K^{47}] = \frac{EA}{l}\begin{bmatrix} \frac{\beta\phi_2}{\phi_1} & \frac{\sqrt{3}\beta\phi_2}{\phi_1} & 0 \\ \frac{\sqrt{3}\beta\phi_2}{\phi_1} & \frac{3\beta\phi_2}{\phi_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(A-4h,i,j)}$$

$$[K^{48}] = -[K^{45}], \quad [K^{55}] = \frac{EA}{l}\begin{bmatrix} \frac{1}{4}(1+\frac{9I_z}{Al^2})-\frac{\beta\phi_2}{\phi_1} & -\frac{\sqrt{3}}{4}(1-\frac{3I_z}{Al^2})+\frac{\sqrt{3}\beta\phi_2}{\phi_1} & 0 \\ -\frac{\sqrt{3}}{4}(1-\frac{3I_z}{Al^2})+\frac{\sqrt{3}\beta\phi_2}{\phi_1} & \frac{3}{4}(1+\frac{I_z}{Al^2})-\frac{3\beta\phi_2}{\phi_1} & 0 \\ 0 & 0 & \frac{3I_y}{Al^2} \end{bmatrix} \qquad \text{(A-4k,l)}$$

$$[K^{56}] = \frac{EA}{l}\begin{bmatrix} \frac{2\beta\gamma}{\phi_1} & 0 & 0 \\ -\frac{2\sqrt{3}\beta\gamma}{\phi_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [K^{57}] = \frac{EA}{l}\begin{bmatrix} \frac{\beta\gamma}{\phi_1} & \frac{\sqrt{3}\beta\gamma}{\phi_1} & 0 \\ -\frac{\sqrt{3}\beta\gamma}{\phi_1} & -\frac{3\beta\gamma}{\phi_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [K^{58}] = \frac{EA}{l}\begin{bmatrix} \frac{\beta\phi_2}{\phi_1} & -\frac{\sqrt{3}\beta\phi_2}{\phi_1} & 0 \\ -\frac{\sqrt{3}\beta\phi_2}{\phi_1} & \frac{3\beta\phi_2}{\phi_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{(A-4m,n,o)}$$

$$[K^{66}] = [K^{33}], \quad [K^{67}] = [K^{34}], \quad [K^{68}] = [K^{35}], \quad [K^{77}] = [K^{44}], \quad [K^{78}] = [K^{45}], \quad [K^{88}] = [K^{55}]$$

(A-4p,q,r,s,t,u)

and $\beta = \frac{aEA}{8l}$ (A-5) $\quad \phi_1 = \{1+8\beta(1-2\gamma)\}\cdot\{1+8\beta(1+\gamma)\}$, $\phi_2 = 1+8\beta(1-2\gamma)(1+\gamma)$ (A-6a,b)