

# Cracked beam element formulation for structural analysis

Lucio Nobile<sup>(1)</sup>, Erasmo Viola<sup>(1)</sup>

1) DISTART-Department, Viale Risorgimento 2, 40136 Bologna, Italy

## ABSTRACT

In this paper, a finite element model for a cracked prismatic beam, under bending moment, axial and shear forces, is developed. The stiffness matrix is derived starting from an integration of stress intensity factors. Translational and rotational mass matrices are the same both for cracked and uncracked Timoshenko beam finite element. The stiffness matrix of the cracked element can be used in any finite element formulation to study static and dynamic behaviour of cracked structures. In the present study the transverse crack has been considered as open.

## INTRODUCTION

A crack on a structural member introduces a local flexibility which is a function of the depth of the crack. This flexibility changes the static and dynamic behaviour of the structures.

The fracture mechanics approach can yield the local compliances due to the cracked sections for which expressions for the Stress Intensity Factors (SIFs) can be developed. The local flexibility of the cracked region of the structural element was put into relation with the SIFs. A general method for extending fracture mechanics through the compliance concept to the analysis of a structure containing cracked members was considered in [1]. This paper also shows some examples of the application adopted for the deformation analysis of the cracked specimens. The formulation was also used to study the fatigue evolution of cracked frame-structures and the determination of the bending moment redistribution [2], [3]. At present, the stress intensity factors are theoretically obtained for various types of loading and specimen configurations. The formulas for the SIF as a function of the crack depth can be found in several handbooks [4]. Remarkably two simple methods for close approximation of stress intensity factors in cracked or notched beams were proposed [5], [6].

One was based on elementary beam theory estimation of strain energy release rate as the crack is widened into a fracture band, the other was based on elementary beam theory equilibrium condition for internal forces evaluated in the cross section passing through the crack tip. In this paper, the stiffness matrix for a two-node cracked Timoshenko beam element is derived. The equation of motion of the complete system in a fixed co-ordinate system includes translational and rotatory mass matrices. A parametric study of a transverse open crack has been carried out for various crack depths and crack locations. It also determines the changes in eigenfrequencies with two different beam models.

## STIFFNESS MATRIX OF CRACKED ELEMENT

For general loading, a local stiffness matrix relates forces to displacements. In this analysis, rotational and translational crack compliance are assumed in the local flexibility matrix. So bending, shear and axial effects will be included.

According to Okamura et al. [1], cracked member can be separated at the cracked section and both portions may be connected by a spring with spring constant  $D_b$ ,  $D_s$ ,  $D_a$ , corresponding to bending moment  $M$ , shear force  $T$  and axial force  $P$ , respectively. For general plane loading the strain energy of a beam element without a crack, is

$$W^{(0)} = \frac{1}{2EI} \left( M^2 L - MTL^2 + \frac{T^2 L^3}{3} \right) + \frac{1}{2} \frac{P^2 L}{EA} + \frac{1}{2} \frac{T^2 L}{GA} \quad (1)$$

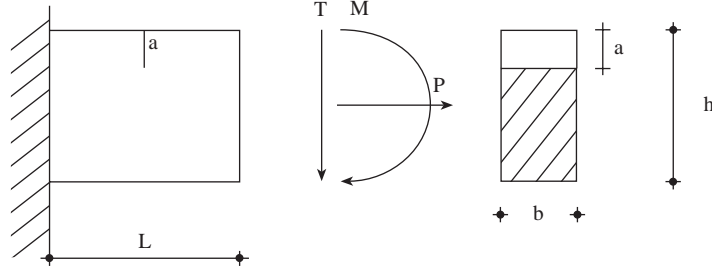
where  $E$  is the Young's modulus,  $G$  is the shear modulus,  $L$  the length of the beam element,  $A$  being the area of the cross-section and  $\Lambda = A/\chi$  a coefficient which depends on the shear coefficient  $\chi$ .

The additional strain energy due to the crack, as in fig. 1, is:

$$W^{(1)} = b \int_0^a \left[ \left( K_I^2 + K_{II}^2 \right) / E' + (1 + \nu) K_{III}^2 / E \right] da \quad (2)$$

where  $E' = E/(1 - \nu^2)$  for plane strain,  $E' = E$  for plane stress,  $\nu$  is the Poissons' ratio and  $K_I$ ,  $K_{II}$ ,  $K_{III}$  are the crack tip stress intensity factors for opening mode, sliding mode and tearing mode of crack surface displacements, respectively. It should be noted that the  $K$  values are linear factors containing the load linearly. The superposition of these three modes is sufficient to describe the most general 3-dimensional case of local crack-tip deformation and stress fields.

Since the cracked element is subject to three loading force systems applied simultaneously as shown in fig. 1, the total K is the algebraic sum of K values for each system applied separately. However, since different fields of stress occur for each mode, then the sums must be separated for each mode.



**Fig. 1. Cracked beam element.**

By considering the actions of bending moment M, shear force T and axial force P simultaneously, equation (2) becomes

$$W^{(1)} = b \int_0^a \left\{ \left[ (K_{IM} + K_{IT} + K_{IP})^2 + (K_{IIT})^2 \right] / E' \right\} da \quad (3)$$

where stress intensity factors are given by [4]

$$K_{IM} = \sigma_M \sqrt{\pi a} F_{IM}, \quad K_{IT} = T \sqrt{a} F_{IT}, \quad K_{IP} = \sigma_P \sqrt{\pi a} F_{IP}, \quad K_{IIT} = \sqrt{\pi a} F_{II} \quad (4)$$

with

$$F_{IM} = F_{IT} = F_I(\xi) = \sqrt{(2/\pi) \tan(\pi \xi/2)} \frac{0.923 + 0.199[1 - \sin(\pi \xi/2)]^4}{\xi \cos(\pi \xi/2)} \quad (5)$$

$$F_{IP}(\xi) = \sqrt{(2/\pi) \tan(\pi \xi/2)} \frac{0.752 + 2.02\xi + 0.37[1 - \sin(\pi \xi/2)]^3}{\xi \cos(\pi \xi/2)} \quad \xi \quad (6)$$

$$F_{II} = F_{II}(\xi) = (3\xi - 2\xi^2) \frac{1.122 - 0.561\xi + 0.085\xi^2 + 0.18\xi^3}{\sqrt{1 - \xi}} \quad (7)$$

where  $\xi = a/h$ .

Stresses  $\sigma_M$ ,  $\sigma_T$ ,  $\sigma_P$ ,  $\tau$  can be expressed as

$$\sigma_M = \frac{6M}{bh^2}, \quad \sigma_T = \frac{3TL}{bh^2}, \quad \sigma_P = \frac{P}{bh}, \quad \tau = \frac{T}{bh} \quad (8)$$

By substituting Eqs. (4)-(8) into Eq. (3) and assuming

$$R_I = \int_0^a a F_I^2 da, \quad R_{II} = \int_0^a a F_{II}^2 da, \quad Q = \int_0^a a F_{IP}^2 da, \quad Z = \int_0^a a F_I F_{IP} da \quad (9)$$

$$n = \frac{36\pi}{E'bh^4}, \quad m = \frac{\pi}{E'bh^2} \quad (10)$$

the strain energy due to the crack gets the form

$$W_c^{(1)} = n(M + TL/2)^2 R_I + mP^2 Q + \frac{6m}{h} (2MP + TPL) Z + mT^2 R_{II} \quad (11)$$

The flexibility matrices of cracked and uncracked element with constraints (fig. 1) may be calculated by means of Castigliano's theorem:

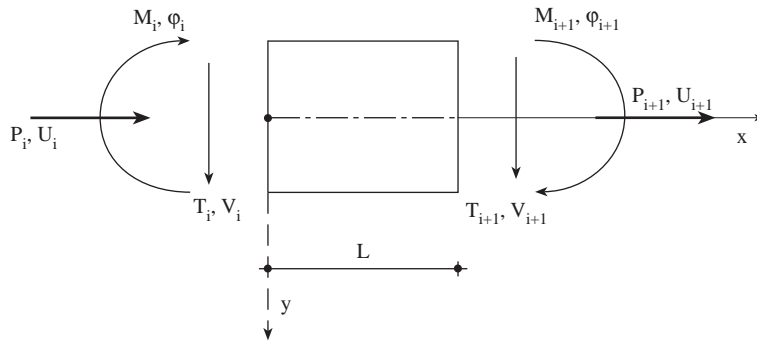
$$c_{ij} = \partial^2 W / \partial F_i \partial F_j \quad (i, j = 1, 2, 3) \quad (12)$$

where  $F_1 = P$ ,  $F_2 = T$ ,  $F_3 = M$  and  $W = W^{(0)}$  or  $W = W^{(1)}$ . For cracked element  $W = W^{(1)}$  and the flexibility matrix has the form

$$c^{(1)} = \begin{bmatrix} 2mQ & \frac{6mZL}{h} & \frac{12mZ}{h} \\ \frac{6mZL}{h} & \frac{nR_1 L^2}{2} + 2mR_{II} & nR_1 L \\ \frac{12mZ}{h} & nR_1 L & 2nR_1 \end{bmatrix} \quad (13)$$

In the case of uncracked element  $W = W^{(0)}$  and the  $3 \times 3$  flexibility matrix is

$$c^{(0)} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EJ} + \frac{L}{GA} & -\frac{L^2}{2EJ} \\ 0 & -\frac{L^2}{2EJ} & \frac{L}{EJ} \end{bmatrix} \quad (14)$$



**Fig. 2. Typical element with nodal displacements and nodal forces.**

Nodal forces on typical element must satisfy the equilibrium condition (see fig. 2), that can be written in the extended matrix form

$$\begin{bmatrix} P_i \\ T_i \\ M_i \\ P_{i+1} \\ T_{i+1} \\ M_{i+1} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -L & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_{i+1} \\ T_{i+1} \\ M_{i+1} \end{bmatrix} \quad (15)$$

Denoting by

$$\mathbf{A}^T = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -L & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \quad (16)$$

the stiffness matrix for the cracked element may be derived as:

$$\mathbf{k}^c = \mathbf{A} \left( \mathbf{c}^{(0)} + \mathbf{c}^{(1)} \right)^{-1} \mathbf{A}^T \quad (17)$$

in which the notation is obvious.

Introducing Eqs. (13), (14), (16) into Eq. (17) we obtain the cracked element stiffness matrix expressed in the extended form

$$\mathbf{k}^c = \begin{bmatrix} k_{11}^c & 0 & k_{13}^c & k_{14}^c & 0 & k_{16}^c \\ 0 & k_{22}^c & k_{23}^c & 0 & k_{25}^c & k_{26}^c \\ k_{31}^c & k_{32}^c & k_{33}^c & k_{34}^c & k_{35}^c & k_{36}^c \\ k_{41}^c & 0 & k_{43}^c & k_{44}^c & 0 & k_{46}^c \\ 0 & k_{52}^c & k_{53}^c & 0 & k_{55}^c & k_{56}^c \\ k_{61}^c & k_{62}^c & k_{63}^c & k_{64}^c & k_{65}^c & k_{66}^c \end{bmatrix} \quad (18)$$

which has the entries

$$k_{11}^c = -k_{14}^c = -k_{41}^c = k_{44}^c = \frac{EA(L + 6LR_1d)}{L^2(1 + \Omega)} \quad (19)$$

$$k_{13}^c = -k_{16}^c = k_{31}^c = -k_{34}^c = -k_{43}^c = k_{46}^c = -k_{61}^c = k_{64}^c = -\frac{EA\pi Z}{L^2(1 + \Omega)} \quad (20)$$

$$k_{22}^c = -k_{25}^c = -k_{52}^c = k_{55}^c = \frac{12EJ}{(1 + \Gamma)L^3 + 2\pi h R_{II}} \quad (21)$$

$$k_{23}^c = k_{26}^c = k_{32}^c = -k_{35}^c = -k_{53}^c = -k_{56}^c = k_{62}^c = -k_{65}^c = -\frac{6EJL}{(1 + \Gamma)L^3 + 2\pi h R_{II}} \quad (22)$$

$$k_{33}^c = k_{66}^c = \frac{EJL^2 \left[ 4 + \Gamma(1 + 2Qd) + 8Qd + 18R_1d + 36QR_1d^2 - 36d^2Z^2 + \frac{2\pi h R_{II}}{L^3} + \frac{4\pi^2 QR_{II}}{L^4} \right]}{\left[ (1 + \Gamma)L^3 + 2\pi h R_{II} \right] (1 + \Omega)} \quad (23)$$

$$k_{36}^c = k_{63}^c = \frac{EJL^2 \left[ 2 - \Gamma(1 + 2Qd) + 4Qd + 18R_1d + 36QR_1d^2 - 36d^2Z^2 + \frac{2\pi h R_{II}}{L^3} - \frac{4\pi^2 QR_{II}}{L^4} \right]}{\left[ (1 + \Gamma)L^3 + 2\pi h R_{II} \right] (1 + \Omega)} \quad (24)$$

where

$$d = \frac{\pi}{hL}, \quad \Gamma = \frac{12EJ}{GAL^2}, \quad \Omega = \left(2Qd + 6R_1d + 12QR_1d^2 - 12d^2Z^2\right) \quad (25)$$

It should be noted that the uncracked element stiffness matrix may be deduced from Eqs. (18)-(25). The stiffness matrix for Timoshenko beam element will be derived by neglecting the terms  $Z$ ,  $Q$ ,  $R_1$ ,  $R_{II}$ . Moreover, by setting  $\Gamma = 0$  in resultant equations, the above coefficients (18)-(24) reduce to elements of the Euler-Bernoulli beam model.

## MASS MATRIX

The consistent translational  $\mathbf{m}_u$  and  $\mathbf{m}_v$  and rotational  $\mathbf{m}_\vartheta$  mass matrices are assumed the same both for cracked and uncracked Timoshenko beam finite element. The above matrices can be expressed in the well known forms [7] and [8]. The total mass matrix of the two mode finite element is

$$\mathbf{m} = \mathbf{m}_u + \mathbf{m}_v + \mathbf{m}_\vartheta \quad (26)$$

The above mass matrices can be derived by using the Kinetic energy of the beam element of length  $L$

$$\tau = \frac{1}{2} \int_0^L \mu A \left( \frac{\partial u}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^L \mu A \left( \frac{\partial v}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^L \mu J \left( \frac{\partial \vartheta}{\partial t} \right)^2 dx \quad (27)$$

including the effects of both translational displacements  $u$ ,  $v$  and rotatory inertia, respectively. In (27)  $\mu$  is the mass density of the material. For a finite element formulation, the displacements  $u$ ,  $v$  and the rotation  $\vartheta$  can be expressed as

$$\begin{bmatrix} u \\ v \\ \vartheta \end{bmatrix} = \begin{bmatrix} N_{u1} & N_{u2} & N_{u3} & N_{u4} & N_{u5} & N_{u6} \\ N_{v1} & N_{v2} & N_{v3} & N_{v4} & N_{v5} & N_{v6} \\ N_{\vartheta1} & N_{\vartheta2} & N_{\vartheta3} & N_{\vartheta4} & N_{\vartheta5} & N_{\vartheta6} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \vartheta_1 \\ u_2 \\ v_2 \\ \vartheta_2 \end{bmatrix} \quad (28)$$

where  $N_{ui}$ ,  $N_{vi}$ ,  $N_{\vartheta i}$  ( $i = 1, 2, \dots, 6$ ) are the interpolation functions for axial displacement, transverse displacement and for the rotation of the cross-section about the positive  $z$  axis. So, the displacement field  $u = u(x)$ ,  $v = v(x)$ ,  $\vartheta = \vartheta(x)$  in the beam element is linearly interpolated by relation (28), where  $u_1$ ,  $v_1$ ,  $\vartheta_1$ ,  $u_2$ ,  $v_2$ ,  $\vartheta_2$  is the set of degrees of freedom for a two node beam element. It should be noted that in vibration displacement field functions  $u$ ,  $v$ ,  $\vartheta$  and nodal displacements  $u_1$ ,  $v_1$ ,  $\vartheta_1$ ,  $u_2$ ,  $v_2$ ,  $\vartheta_2$  are all time-dependent. However, for the purpose of deriving stiffness and mass matrices, they can all be regarded as constant in time. The resulting explicit form of shape functions for axial displacement  $u$  are:

$$N_{u1} = 1 - \frac{x}{L}, \quad N_{u4} = \frac{x}{L}, \quad N_{u2} = N_{u3} = N_{u5} = N_{u6} = 0 \quad (29)$$

for displacement  $v$  in the  $y$  direction, are

$$\begin{aligned} N_{v1} = N_{v4} = 0, \quad N_{v2} &= \frac{1}{1+\Gamma} \left[ 2 \left( \frac{x}{L} \right)^3 - 3 \left( \frac{x}{L} \right)^2 - \Gamma \left( \frac{x}{L} \right) + (1+\Gamma) \right] \\ N_{v3} &= \frac{-L}{(1+\Gamma)} \left[ \left( \frac{x}{L} \right)^3 - \left( 2 + \frac{\Gamma}{2} \right) \left( \frac{x}{L} \right)^2 + \left( 1 + \frac{\Gamma}{2} \right) \left( \frac{x}{L} \right) \right], \quad N_{v5} = \frac{1}{1+\Gamma} \left[ 2 \left( \frac{x}{L} \right)^3 - 3 \left( \frac{x}{L} \right)^2 - \Gamma \left( \frac{x}{L} \right) \right] \\ N_{v6} &= \frac{-L}{(1+\Gamma)} \left[ \left( \frac{x}{L} \right)^3 - \left( 1 - \frac{\Gamma}{2} \right) \left( \frac{x}{L} \right)^2 - \frac{\Gamma}{2} \left( \frac{x}{L} \right) \right] \end{aligned} \quad (30)$$

and for bending rotation may be written as:

$$\begin{aligned}
N_{\theta 1} = N_{\theta 4} = 0, \quad N_{\theta 2} &= \frac{-6}{(1+\Gamma)L} \left[ \left( \frac{x}{L} \right)^2 - \left( \frac{x}{L} \right) \right] \\
N_{\theta 3} &= \frac{1}{(1+\Gamma)} \left[ 3 \left( \frac{x}{L} \right)^2 - (4+\Gamma) \left( \frac{x}{L} \right) + (1+\Gamma) \right], \quad N_{\theta 5} = + \frac{6}{(1+\Gamma)L} \left[ \left( \frac{x}{L} \right)^2 - \left( \frac{x}{L} \right) \right] \\
N_{\theta 6} &= \frac{1}{(1+\Gamma)} \left[ 3 \left( \frac{x}{L} \right)^2 - (2-\Gamma) \left( \frac{x}{L} \right) \right]
\end{aligned} \tag{31}$$

The above shape functions are determined so that they exactly satisfy the homogeneous form of the static equations of equilibrium of a uniform Timoshenko beam axially loaded [7] [8]:

$$\frac{d}{dx} \left( EA \frac{du}{dx} \right) = 0, \quad \frac{d}{dx} \left( G\Lambda \left( \frac{dv}{dx} + \theta \right) \right) = 0, \quad - \frac{d}{dx} \left[ EJ \frac{d\theta}{dx} \right] + G\Lambda \left( \frac{dv}{dx} + \theta \right) = 0 \tag{32}$$

The explicit expressions for the elements of translational mass matrices  $\mathbf{m}_u$  and  $\mathbf{m}_v$  and for the rotational mass matrix  $\mathbf{m}_\theta$  are given as:

$$m_{ij}^u = \int_0^L N_{ui} \mu A N_{uj} dx, \quad m_{ij}^v = \int_0^L N_{vi} \mu A N_{vj} dx, \quad m_{ij}^\theta = \int_0^L N_{\theta i} \mu J N_{\theta j} dx \tag{33}$$

respectively, for  $i, j = 1, 2, \dots, 6$ . The mass matrices constructed in this way are said to be consistent in the sense that they are obtained for the same displacement approximation as the stiffness matrix. Letting

$$\mathbf{m}_t = \mathbf{m}_u + \mathbf{m}_v \tag{34}$$

where  $\mathbf{m}_t$  is the total translational mass matrix, Eq. (26) yields

$$\mathbf{m} = \mathbf{m}_t + \mathbf{m}_\theta \tag{35}$$

By substituting (29)-(31) into (33) and carrying out the integration over the beam-length  $L$ , the following mass matrices will be derived

$$\mathbf{m}_t = \frac{\mu A L}{(1+\Gamma)^2} \begin{bmatrix} \frac{(1+\Gamma)^2}{3} & 0 & 0 & \frac{(1+\Gamma)^2}{6} & 0 & 0 \\ \frac{13}{35} + \frac{7}{10}\Gamma + \frac{\Gamma^2}{3} & \left( \frac{11}{210} + \frac{11}{120}\Gamma + \frac{\Gamma^2}{24} \right) L & 0 & 0 & \frac{9}{70} + \frac{3}{10}\Gamma + \frac{\Gamma^2}{6} & - \left( \frac{13}{420} + \frac{3}{40}\Gamma + \frac{\Gamma^2}{24} \right) L \\ & \left( \frac{1}{105} + \frac{\Gamma}{60} + \frac{\Gamma^2}{120} \right) L^2 & 0 & \left( \frac{13}{210} + \frac{3}{40}\Gamma + \frac{\Gamma^2}{24} \right) & - \left( \frac{1}{140} + \frac{\Gamma}{60} + \frac{\Gamma^2}{120} \right) L^2 \\ & & \frac{(1+\Gamma)^2}{3} & 0 & 0 \\ & \text{SIMM.} & & \frac{13}{35} + \frac{7}{10}\Gamma + \frac{\Gamma^2}{3} & - \left( \frac{11}{210} + \frac{11}{120}\Gamma + \frac{\Gamma^2}{24} \right) L \\ & & & & \left( \frac{1}{105} + \frac{\Gamma}{60} + \frac{\Gamma^2}{120} \right) L^2 \end{bmatrix} \tag{36}$$

$$\mathbf{m}_\vartheta = \frac{\mu J}{L(1+\Gamma)^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5} & \left(\frac{1}{10} - \frac{\Gamma}{2}\right)L & 0 & -\frac{6}{5} & \left(\frac{1}{10} - \frac{\Gamma}{2}\right)L \\ 0 & \left(\frac{2}{15} + \frac{\Gamma}{6} + \frac{\Gamma^2}{3}\right)L^2 & 0 & -\left(\frac{1}{10} - \frac{\Gamma}{2}\right)L & -\left(\frac{1}{30} + \frac{\Gamma}{6} - \frac{\Gamma^2}{6}\right)L^2 & 0 \\ 0 & 0 & 0 & \frac{6}{5} & 0 & 0 \\ 0 & 0 & 0 & -\left(\frac{1}{10} - \frac{\Gamma}{2}\right)L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{2}{15} + \frac{\Gamma}{6} + \frac{\Gamma^2}{3}\right)L^2 \end{bmatrix} \quad (37)$$

SIMM.

The above mass matrices depend upon  $\Gamma$  and both these matrices reduce to the Bernoulli-Euler form by setting  $\Gamma = 0$ .

## NUMERICAL RESULTS

The equations of motion for the complete system in a fixed co-ordinate system may be derived by using the standard finite element method to the fixed-hinged plane frame represented in fig. 3

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{p} \quad (38)$$

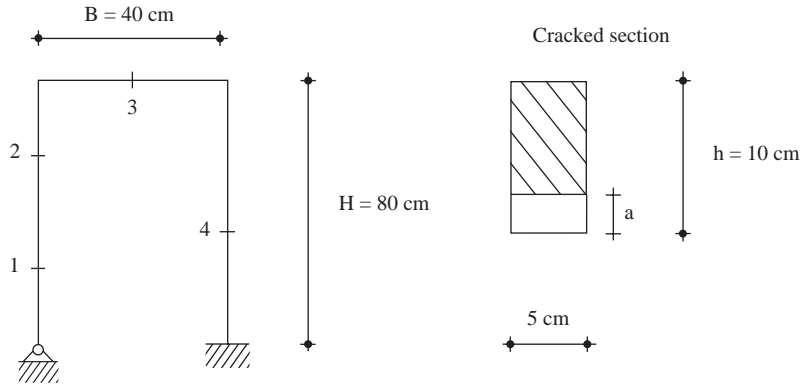
where  $\mathbf{M}$  is the mass matrix of the assembled system,  $\mathbf{K}$  the structural stiffness matrix and

$$\mathbf{x} = [u_1 v_1 \vartheta_1, \dots, u_i v_i \vartheta_i, \dots, u_n v_n \vartheta_n]^T \quad (39)$$

is the generalized displacement vector and

$$\mathbf{p} = [P_1 T_1 M_1, \dots, P_i T_i M_i, \dots, P_n T_n M_n]^T \quad (40)$$

is the external excitation vector. In numerical calculations, the crack is set only in one finite element of the discretized cracked structure.



**Fig. 3. Cracked fixed-hinged frame.**

The problem of determining the natural vibration frequencies and the associated mode shapes of a system always leads to solving the eigenproblem of the homogeneous linear system

$$\mathbf{K}\mathbf{x} = \omega^2 \mathbf{M}\mathbf{x} \quad (41)$$

where the mass and stiffness matrices are nearly always symmetric and positive definite. In the present work, four crack position are fixed, as shown in fig. 3 by 1, 2, 3, 4. For each position, three crack depths are considered. Frequencies  $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  for cracked and uncracked plane frame are shown in table 1, both Euler-Bernoulli theory and Timoshenko theory, for various crack depth and crack position, and assuming  $E = 2.06 \cdot 10^5 \text{ N/mm}^2$ ,  $\mu = 7750 \text{ kg/m}^3$ .

**Table 1. Frequencies for cracked and uncracked plane frame.**

Crack position	Crack depth a (cm)	Frequency (Hz)					
		Euler-Bernoulli theory			Timoshenko theory		
		$\omega_1$	$\omega_2$	$\omega_3$	$\omega_1$	$\omega_2$	$\omega_3$
1	2	102,182	517,103	1094,71	100,602	500,291	1082,97
	4	101,962	488,961	1068,15	100,363	474,309	1055,8
	6	101,456	427,864	956,227	99,7995	417,083	942,749
2	2	101,909	526,178	1090,82	100,307	508,534	1080,41
	4	101,035	525,349	1060,47	99,3464	507,573	1053,19
	6	99,7831	523,708	992,317	97,9317	505,8	988,677
3	2	101,165	522,092	1089,45	99,6217	504,956	1077,5
	4	97,9317	508,27	1066,22	96,4748	492,489	1053
	6	91,2621	479,352	1032,83	89,9573	466,105	1017,36
4	2	102,113	526,433	1098,51	100,544	508,846	1086,89
	4	101,648	526,239	1097,61	100,096	508,657	1085,63
	6	100,589	525,248	1093,91	99,0867	507,747	1080,06
Uncracked plane frame		102,256	526,457	1098,95	100,681	508,872	1087,46

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