Improved Prediction Method for Estimating Notch Elastic-Plastic Strains

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ABSTRACT

Notch stress-strain conversion (NSSC) rules are widely used to estimate non-linear and history dependent stress-strain behavior of the notch components or structures. This paper focuses on the estimation of stress and strain using the conventional NSSC rules, by considering the entire relaxation locus during the inelastic action. On the basis of local effects, net section collapse and the reference stress, a simple method for estimating stress-strain fields in the vicinity of stress concentrations is proposed. The accuracy of the method is compared with elastic-plastic finite element analysis for several notch configurations with different loading conditions and in plane stress, plane strain and axisymmetric configurations.

1. INTRODUCTION

Stress concentration configurations such as fillets, holes, welds, grooves and keyholes cannot be avoided when designing structures or mechanical components. Therefore, estimation of stresses and strains at notch is useful in the designing of a component or a structure with stress raisers against different types of failure modes, i.e. brittle fracture due to monotonic loading [1], low cycle fatigue (LCF) due to cyclic loading [2], creep at high temperature [3], and sometimes a combination of these failure modes [4]. The notch stress-strain conversion (NSSC) rules provide a relatively simple method for estimating local strains, especially for prediction of fatigue crack initiation (FCI) using either smooth specimen fatigue life data or other empirical relationships such as Manson-Coffin equation [5,6]. The most commonly used rules are Linear approximation [7], Neuber [8] and equivalent strain energy density (ESED) method [9]. Many experimental [10-12] and numerical [13,14] investigations have been carried out in order to verify these rules. Results show that Neuber rule predicts an upper bound on strains, Linear approximates a lower bound and the ESED approximation will between the two.

There are methods based on iterative linear elastic finite element analyses that make use of modification to the local elastic modulus of the component at each subsequent iteration in order to achieve inelastic-like strain and stress distributions. Seshadri [15] developed the elastic modulus adjustment procedure (EMAP) on a rigorous basis. By introducing the concept of “time-scaling” in conjunction with a uniaxial stress relaxation model, a local region constraint parameter characterizing multiaxiality and follow-up effects has been determined. This local region constraint parameter, which is a measure of the extent on interaction between local and remainder regions, has been shown to be useful in the assessment of multiaxial stress relaxation, creep damage, low-cycle fatigue, plastic collapse, and elastic-plastic fracture. The method for determining the constraint parameter, called the Generalized Local Stress and Strain (GLOSS) analysis, is based on two linear elastic finite element analysis (FEA) in which the first analysis is carried out with homogeneous material properties with mechanical and thermal loads applied to the structure. A second linear elastic FEA is performed by suitable reducing the elastic moduli of the elements that exceed the yield stress [15].

This paper focuses on the behavior in the small, and in the large, of relaxation process during redistribution from pseudoelastic state to inelastic state on stress-strain curve by considering reference stress [16] in the component. Based on the NSSC rules and the net-section stresses, a new approach is suggested for the estimation of the stress-strain at the notch root. The technique is demonstrated and the accuracy assessed, by comparing the results with inelastic finite element predictions for some practical notch configurations including plane stress, plane strain and axisymmetric geometries.

2. NOTCH STRESS-STRAIN CONVERSION RULES

For sake of discussion, consider a plate with a hole subjected to a tensile load. For a given load level, we examine the relationship between inelastic states (points $a'$ and $b'$) and the corresponding pseudoelastic points ($a$ and $b$) on the elastic curve. There is an implied relaxation trajectory that connects point $a$ to $a'$, and point $b$ to $b'$, as shown in Fig. 1. This trajectory is indicative of a progressive loss of local region constraint in the given component, and its slope is related to the constraint parameter. The assumption that the material remains isotropic, even after yielding locally, is reasonable given that small deformation theory is employed. The parameter $E_r$ can be estimated by applying NSSC rules; however, the accuracy of results would be mainly dependent on the choice of the NSSC rule.

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2.1 Linear rule

The linear rule is based on the assumption that the strains for pseudoelastic and inelastic states are the same, which leads to vertical line representing the relaxation locus. The linear rule can be written in terms of the strains as

\[ e_{eq}^{\text{ep}} = e_{eq}^{\text{el}} \]  

(1)

where \( e_{eq}^{\text{ep}} \) is the notch tip elastic-plastic equivalent strain and \( e_{eq}^{\text{el}} \) is that which would have been obtained in a linear elastic analysis. This rule gives a better estimation for plane strain compared to the plane stress condition [2]. Gowhari-Anaraki and Hardy [2] reported a strain range estimation method based on the linear rule that provides a lower bound estimation in comparison to the results obtained using inelastic FEA.

2.2 Neuber rule

Neuber published the theory of stress concentration [8] for a prismatic body subjected to pure shear loading, where a relationship between nominal and notch-tip stress and strain in terms of elastic stress and strain concentration factors was proposed. Neuber rule complemented the Manson-Coffin relationship for low cycle estimating fatigue life. However, the studies to verify Neuber rule by Leis et al. [10] and Conle et al. [11] revealed that it may overestimate the local inelastic strains and stresses. A generalization of Neuber rule proposed by Hoffmann and Seeger [17,18] addresses multiaxial proportional loading sequences. They extended Neuber rule by replacing the uniaxial stress and strain with equivalent stress and strain, i.e.,

\[ \sigma_{eq}^{\text{ep}} e_{eq}^{\text{ep}} = \sigma_{eq}^{\text{el}} e_{eq}^{\text{el}} \]  

(2)

where \( \sigma_{eq}^{\text{ep}} \) is the notch tip elastic-plastic equivalent stress, and \( \sigma_{eq}^{\text{el}} \) is that which would be obtained in an elastic analysis.

2.3 Equivalent strain energy density (ESED) rule

Molski and Glinka [9] proposed an alternate approximate techniques for use in place of the Neuber's rule, which is called as ESED method. This method is based on the assumption that the strain energy density at the notch root does not change significantly if the localized plasticity is surrounded by predominantly elastic material (by Hutchison [19] for cracks, and by Walker [20] for a deep sharp notch). In other words, the computation of the strain energy density at the notch root will yield identical results for either the elastic or the elastic-plastic material relationship. The ESED relation has the following form:

\[ \frac{1}{2} \sigma_{ij}^{\text{el}} e_{ij}^{\text{el}} = \int_0^{\epsilon_{ij}^{\text{ep}}} \sigma_{ij}^{\text{el}} d\epsilon_{ij}^{\text{ep}} \]  

(3)

It was demonstrated in some cases that the measured or FEA results for local strain lie between the predictions from the ESED rule and Neuber's rule. In these instances, it was difficult to determine which model best fit the experimental data. It has been suggested that estimations made from ESED rule and Neuber's rule will give lower and upper bounds on the local strain, respectively, which can be used to estimate the uncertainty with the life prediction (Sharpe et al. [21]).

The aforementioned NSSC rules can be expressed as a general relation between initial modulus of elasticity (elastic state) and effective modulus of elasticity (inelastic state) as
\[ E_s = \left( \frac{\sigma_y}{\sigma_{eq}} \right)^q E_0 \]  

(4)

where \( q \) is the elastic modulus adjustment parameter, which indicates the constraint in each part of a component or structure. The value of \( q \) calculated from different NSSC rules are presented in Table 1.

As illustrated in Fig. 2, in order to bring point \( a \) (which represents the equivalent stress and strain calculated from elastic solution) to the yield surface level, \( q \) would be dependent on the local constraint. Several regions are categorized by Adibi-Asl et al. [22] using the elastic modulus adjustment index, \( q \). Figure 2 is called the constraint diagram, and is useful to investigate the behavior of NSSC rules during relaxation.

Due to high localized plasticity region in the notch root, the peak stress is high compared to the primary stresses; therefore, a reasonable estimation of local inelastic stresses and strains are possible using the NSSC rules.

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### Table 1- NSSC Rules

<table>
<thead>
<tr>
<th>NSSC rule</th>
<th>Parameter ( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear [2]</td>
<td>( q = 1 )</td>
</tr>
<tr>
<td>Neuber [8]</td>
<td>( q = 2 )</td>
</tr>
<tr>
<td>ESED [9]</td>
<td>( q = \ln \left( \frac{2\sigma_y^2}{\sigma_{eq}^2 + \sigma_y^2} \right) / \ln \left( \frac{\sigma_y}{\sigma_{eq}} \right) )</td>
</tr>
</tbody>
</table>

### 3. REFERENCE STRESS

For the sake of discussion, consider a notched component subjected to a mechanical load. By increasing the load, the primary stress in the component increases and plastic zone becomes larger until net section yielding occurs. Figure 3-a is a plot of equivalent stress versus equivalent strain for an element in a given finite element discretization corresponding to the maximum stress location. The relaxation loci \( RL_1, RL_2 \) and \( RL_3 \) correspond to the redistribution of stress corresponding to points 1, 2 and 3, respectively. The trajectory of the relaxation locus, \( RL_2 \), is along 2-4-5-6. The initial portion of the locus (2-4) pertains to the redistribution of peak and secondary stresses. Referring to Fig. 3-a, the relaxation locus \( RL_1 \) corresponds to the plastic collapse process. If \( P_1, P_2 \) and \( P_3 \) are the external loads at points 1, 2 and 3 on linear elastic-line, the respective reference stress is related to the limit load by the relationship.

\[ \sigma_{ref} = \frac{P_k\sigma_y}{P_L} \quad (k = 1, 2, 3) \]

(5)

It follows that

\[ \frac{P_1}{\sigma_{ref1}} = \frac{P_2}{\sigma_{ref2}} = \frac{P_3}{\sigma_{ref3}} \]

(6)

where \( \sigma_{ref1}, \sigma_{ref2}, \) and \( \sigma_{ref3} \) are reference stresses (primary stresses) corresponding to \( P_1, P_2 \) and \( P_3 \), respectively, and \( P_L \) is the limit load.
4. GENERAL PROCEDURE FOR NOTCH STRESS STRAIN CONVERSION

In this section, a general method for the assessment of inelastic behavior based on the overall relaxation locus is introduced. Consider an elastic-perfectly plastic material behavior for which the stress is normalized as $\bar{\sigma} = \sigma / \sigma_y$, and strain is normalized as $\bar{\varepsilon} = \varepsilon / \varepsilon_y$ (where $\varepsilon_y = \sigma_y / E_y$). Equation (4) can be rewritten in terms of any arbitrary point ($\bar{\sigma}, \bar{\varepsilon}$) on the relaxation locus as

$$E_x = \left( \frac{\bar{\sigma}}{\bar{\sigma}_{eq}} \right)^q$$

where $E_x = E_y / E_0$ and $\sigma_{eq} = \sigma_{eq} / \sigma_y$.

The relationship between stress and strain for an arbitrary point using Eq. (7) can be obtained from Eq. (15), which describes the relaxation locus based on the traditional method, Neuber and ESED.

$$\bar{\varepsilon} = \bar{\sigma}_{eq} \bar{\sigma}^{(1-q)}$$

Also, the slope of relaxation locus at an arbitrary point can be determined by differentiating Eq. (8), i.e.,

$$\frac{dE_x}{d\bar{\sigma}} = (1-q) \left( \frac{\bar{\sigma}_{eq}}{\bar{\sigma}} \right)^q$$

With reference to Fig. 4, point A is the location of elastic solution, point X is an arbitrary point on the curve, point B is intersection of relaxation locus and yield surface, point C is location of reference stress, and point D is the location where relaxation locus is tangential to reference stress line, for which the corresponding strain is as yet undefined. The values of stress, strain and curve slope at the four points are reviewed in Table 2. In order to estimate location of point D, we assume that there is a point, say point X (unknown yet), where part of the relaxation locus between point X and D is a circle with the centre of ($\bar{\sigma}_0, \bar{\varepsilon}_0$), point O, and radius of R, and circle is tangential to the points X and D. Therefore, the line between points X and O can be expressed as

$$(\bar{\sigma}_O - \bar{\sigma}_X) = -E_x (\bar{\sigma}_O - \bar{\varepsilon}_X)$$

where $E_x = (1-q)(\bar{\sigma}_{eq} / \bar{\sigma}_X)^q$ (see Table 2).
Fig. 4- Graphical representation of general stress strain rule

Table 2- Values of relaxation locus parameters

<table>
<thead>
<tr>
<th>Point</th>
<th>$\bar{\sigma}$</th>
<th>$\bar{\varepsilon}$</th>
<th>$d\bar{\varepsilon} / d\bar{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$\sigma_{eq}$</td>
<td>$\varepsilon_{eq}$</td>
<td>$(1-q)$</td>
</tr>
<tr>
<td>$X$</td>
<td>$\sigma_X$</td>
<td>$\varepsilon_X$</td>
<td>$(1-q)(\sigma_{eq} / \sigma_X)^q$</td>
</tr>
<tr>
<td>$B$</td>
<td>1</td>
<td>$\sigma_{eq}^q$</td>
<td>$(1-q)\sigma_{eq}^q$</td>
</tr>
<tr>
<td>$C$</td>
<td>$\sigma_{ref}$</td>
<td>$\sigma_{eq}^q / \sigma_{ref}^{q-1}$</td>
<td>$(1-q)(\sigma_{eq} / \sigma_{ref})^q$</td>
</tr>
<tr>
<td>$D$</td>
<td>$\sigma_{ref}$</td>
<td>$\varepsilon_D$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

The lengths of $\overline{OX}$ and $\overline{OD}$ (Fig. 4), are as follow:

$$\overline{OX} = \sqrt{(\bar{\varepsilon}_O - \bar{\varepsilon}_X)^2 + (\bar{\sigma}_O - \bar{\sigma}_X)^2}$$

$$\overline{OD} = \sqrt{(\bar{\varepsilon}_O - \bar{\varepsilon}_D)^2 + (\bar{\sigma}_O - \bar{\sigma}_D)^2}$$

By equating $\overline{OX}$ and $\overline{OD}$, and making use of Eq. (10), the circle centre can be obtained as

$$\bar{\varepsilon}_O = \bar{\varepsilon}_D = \frac{-b + \sqrt{b^2 - 4c}}{2}$$

$$\sigma_O = (q-1)(\bar{\varepsilon}_O - \bar{\varepsilon}_X) \left(\frac{\sigma_{eq}}{\sigma_X}\right)^q + \sigma_X$$

where

$$b = 2\left(\frac{\sigma_{eq}}{\sigma_X}\right)^q [(q-1)(\sigma_{ref} - \sigma_X) - \sigma_X]$$

$$c = \sigma_X^2 + \sigma_{eq}^q \sigma_X^{2(q-1)} - \sigma_{ref}^q + 2(\sigma_{ref} - \sigma_X) \left[\sigma_X - (q-1)\sigma_X \left(\frac{\sigma_{eq}}{\sigma_X}\right)^q \right]$$

Considering the presence of primary stresses present, the relaxation locus goes through point $D$ rather than point $C$; therefore, the new expression for $q$ can be obtained by making use of Eq. (4) as

$$q' = \ln \left(\frac{\bar{\varepsilon}_D}{\sigma_{ref}}\right) / \ln \left(\frac{\sigma_{eq}}{\sigma_{ref}}\right) \text{ (where } \bar{\varepsilon}_D = \bar{\varepsilon}_O)$$  \hspace{1cm} (13)
Several points have been examined for the most appropriate location of point X; however, comparing the results with inelastic FEA, the best choice is point A. Therefore, Eq. (11) can be simplified as:

$$\bar{\sigma}_D = q\bar{\sigma}_{eq} - (q-1)\bar{\sigma}_{ref} + (\bar{\sigma}_{eq} - \bar{\sigma}_{ref})\sqrt{q^2 - 2q + 2}$$

(14)

Substituting Eq. (14) into Eq. (13), the modified value of q based on ESED [9] can be obtained as

$$q' = \frac{\ln(q\gamma - (q-1) + (\gamma - 1)\sqrt{q^2 - 2q + 2})}{\ln(\gamma)}$$

(15)

where $\gamma = \sigma_{eq}/\sigma_{ref}$. And q is calculated using ESED rule given in Table 2.

Finally, the modified predicted elastic–plastic strain at notch root using Eq. (8) is given by:

$$\varepsilon_{eq}^{notch} = \sigma_{eq}^{'} (or \varepsilon_{eq}^{notch} = (\sigma_{eq}/E)(\sigma_{eq}/\sigma_{y})^{\nu/2})$$

(16)

### 5. RESULTS AND VALIDATION

In this section, the method described in previous section is applied to several notch configurations, and the results are compared with the available NSSC rules and inelastic finite element analysis. Figure 5 depicts three geometries and dimensions that are analyzed. The material is assumed to be elastic-perfectly plastic in all cases. The various problems are modeled using the ANSYS software (university research version [23]). While modeling the notch components using FEA, it is necessary to use a fine mesh around the geometry discontinuity in order to simulate the high stress concentration at the notch. It should be mentioned that for cylinder the limit load is calculated using available analytical solutions; however, for the other two problems limit load is estimated using inelastic FEA.

#### 5.1. Thick-walled cylinder

To demonstrate the application of the proposed method, a closed ended thick-walled cylinder under internal pressure, Fig. 5-a, with inner radius of a and outer radius b is considered. Elastic-perfectly plastic material behavior model is assumed and material constants are $E=200$ GPa, $\nu=0.3$ and $\sigma_y=150$ MPa.

The limit load and internal pressure to cause plastic radius of c are given in [24] as

$$P_z = \frac{2}{\sqrt{3}}\sigma_y \ln \left( \frac{b}{a} \right)$$

$$P_i = \frac{2}{\sqrt{3}}\sigma_y \left[ 1 + 2\ln \left( \frac{c}{a} \right) - \left( \frac{c}{b} \right)^2 \right]$$

The equivalent stress and total elastic plastic strain in internal portion of cylinder are given as [25], respectively

$$\sigma_{eq} = \sqrt{3P_i \left( \frac{b}{a} \right)^2}$$

$$\varepsilon_{eq} = \frac{\sigma_y}{E} + \frac{2(2-\nu)}{3E} \left( \frac{c^2}{a^2} - 1 \right)$$
Assuming $b/a=2$ and $c/a=1.5$, the elastic-plastic strain using the aforementioned expressions is obtained as $\tilde{\varepsilon}_{\text{exact}}=2.416$. The predicted elastic-plastic strain by NSSC rules are shown in Table 3, and is compared with the value for strain obtained from exact solution. It is seen that the agreement between the present study and inelastic FEA is good (less than 1% error).

<table>
<thead>
<tr>
<th>NSSC rule</th>
<th>Parameter $q$</th>
<th>$\tilde{\varepsilon}$</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear</td>
<td>1.0</td>
<td>1.664</td>
<td>-31.12</td>
</tr>
<tr>
<td>ESED</td>
<td>1.244</td>
<td>1.884</td>
<td>-22.01</td>
</tr>
<tr>
<td>Neuber</td>
<td>2.0</td>
<td>2.768</td>
<td>14.56</td>
</tr>
<tr>
<td>Present study</td>
<td>1.749</td>
<td>2.437</td>
<td>0.86</td>
</tr>
</tbody>
</table>

5.2 Plate with a hole

A plate with a hole (Fig. 5-b) with a uniform tensile load $P$ is modeled in plane stress condition. The plate width is $2W=150$ mm, the length is $L=300$ mm, and the notch diameter is $2r=40$ mm. The modulus of elasticity is 200 GPa, and the yield stress is 150 MPa. The variation of equivalent stress versus equivalent elastic-plastic strain predicted by NSSC rules and inelastic FEA for plane stress and plane strain conditions are presented in Figs. 6-a and 6-b, respectively.

It can be seen from these figures that the elastic-plastic strains predicted by the present method are in agreement with nonlinear finite element method (NFEM) results for both plane stress and plane strain conditions. On the other hand, Neuber’s rule overestimates the inelastic strains, and the ESED and Linear rules underestimate the inelastic strain at notch root.

5.3 Round bar notch

For a round bar notch configuration (Fig. 5-c), $D=300$ mm, $r=25$ mm and overall length $L=500$ mm. The material is the same as for the plate with a hole. The variation of equivalent stress (elastically calculated) versus predicted elastic-plastic strain with different method including inelastic FEA is presented in Fig. 7.
CONCLUSION

Based on the entire relaxation locus, a new approach for the estimation of local behavior at notch root is presented. Notch root stresses and strains are calculated by employing the linear rule, Neuber's rule, Glinka's rule and the method suggested in this paper is compared with elastic–plastic finite element analysis predictions. The Neuber rule predicted conservative local strain amplitudes, especially when the local stress state is plane strain. The results from the ESED method are underestimates of the predictions compared to finite element analyses. The method proposed in this paper gives good estimation of strain and stress at the notch root.

REFERENCES