BEHAVIOR OF A REINFORCED CONCRETE BEAM UNDER IMPACT LOADING: A SIMPLIFIED APPROACH

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Abstract

The motion of a reinforced concrete beam or one-way plate during and after dynamic loading is modeled by the classical partial differential equation and by the elastoplastic bilinear bending moment / curvature law for elastic and beyond elastic deformation. A resolution method is proposed by using the finite difference approximation. The progression of the beam deformation vs. time is stabilized by introducing some retroaction according to the implicit scheme. Beam spatial discretization concentrates the kinematics features (displacement, rotation, curvature, velocity and acceleration) and the mechanical properties (loading, bending moment and shear force) in nodes. The method consists of gathering each preceding beam characteristic or property in two vectors whose size is the total number of nodes. These vectors are node velocity and node bending moment. Computation of these vectors is made by iteration from preceding time step by using matrix products. Such a method of resolution can be easily programmed on spreadsheets. Comparison of the computed mid-span displacements with experimental results shows good accuracy and is used to choose the mechanical properties to be introduced for any predicting purpose.

Objective

Numerous codes are available to handle the calculations of reinforced concrete beams beyond their elastic limit under impact loading with high accuracy, good confidence and based on reliable tests. However, using such codes requires a certain degree of experience from the designer before obtaining reliable and rapid results. There is a need for a less sophisticated method giving quickly the influence of the necessary assumptions, for instance to get the response of a beam beyond its elastic limit when impacted. The aim of the article is to propose a simplified method allowing predicting the behavior of a beam or one-way plate, beyond elastic deformation, when subjected to a shock by direct integration of the equation of motion. The domain of beam modeling is beyond elastic domain of and before break, with displacements up to twice the beam section height.

In the proposed approach, computation is made on a spreadsheet software with finite difference method and iterative implicit method, with bilinear behavior law for reinforced concrete beams**.**

Principle of the simplified approach

The constitutive equations of motion for a slender beam undergoing dynamic loading are obtained on the one hand by the dynamic force equilibrium within each beam segment (the sum of inertia force, viscous damping force and beam reaction, when neglecting the deformation by shear strain, is opposed to the loading force) the well-known equation:

$$\rho S\left(\frac{\partial^2 w}{\partial t^2} + 2\xi\omega\frac{\partial w}{\partial t}\right) + \frac{\partial^2 M}{\partial x^2} - q = 0 \qquad \text{Eq. (1)}$$

with x, t : position of the section along the beam, time during the phenomenon
- w (x,t) : lateral deflection — M (x,t) : bending moment
- ξ : rate of critical damping — ω : pulsation of the 1st eigenmode of deformation
- ρ : volumic mass of material — S = B.H : area of current section
- q (x,t) : loading density — B : section width H : section height

and on the other hand by the bilinear elastic-perfectly plastic bending moment/curvature law:

$$\begin{cases} |M| \le M_{pl}\ ,\ \dfrac{\partial^2 w}{\partial x^2} = \dfrac{M}{E\,J} \\ |M| = M_{pl}\ ,\ \dfrac{\partial^2 w}{\partial x^2} = \dfrac{M_{pl}}{E\,J}\,sign(M) \end{cases} \qquad \text{Eq. (2)}$$

with E : Young's modulus of the beam $\frac{\partial^2 w}{\partial x^2}$: curvature of the beam

$J = \frac{B.H^3}{12}$: cross sectional moment of inertia of the current section

M_{pl} (> 0) : plastic bending moment of the current section

The resolution of the partial differential equation (1) with the behavior law (2) is made by using the finite difference method: the beam is split into N segments of unit length Δx giving N+1 nodes and time is discretized by a succession of

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time intervals Δt giving instant $t_n = n.\Delta t$. Two supplementary nodes are added, one at each extremity and outside the beam, in order to characterize the beam supports (see scheme in Figure 1).

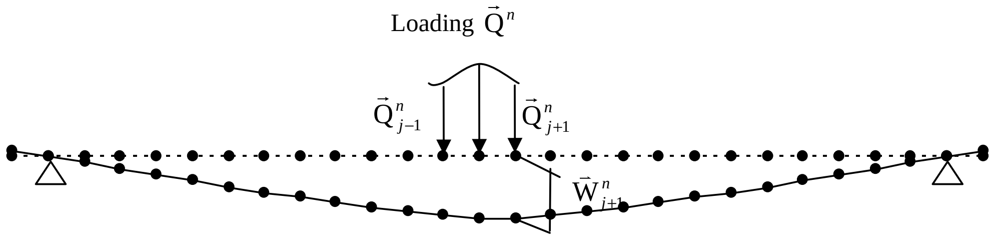

Figure 1 : Beam with N elements between supports and N+3 nodes

When noting W_j^n and M_j^n the deflection and bending moment at node j at the instant t_n, the second order derivative functions ∂²w/∂t² , ∂²w/∂x² and ∂²M/∂x² are respectively approximated by the second order derivatives at the central time or spatial station :

$$\frac{\partial^2 w}{\partial t^2} \approx \frac{W_j^{n+1} - 2W_j^n + W_j^{n-1}}{\Delta t^2} \text{ , } \frac{\partial^2 w}{\partial x^2} \approx \frac{W_{j+1}^n - 2W_j^n + W_{j-1}^n}{\Delta x^2} \text{ and } \frac{\partial^2 M}{\partial x^2} \approx \frac{M_{j+1}^n - 2M_j^n + M_{j-1}^n}{\Delta x^2} \text{ .}$$

In order to lower the order of the partial differential equation, the valuable unknown to be considered is the velocity $V_j^{n+1} = \left(W_j^{n+1} - W_j^n\right)/\Delta t$, the deflection W_j^{n+1} being then obtained by :

$W_j^{n+1} = W_j^n + \left(V_j^{n+1} + V_j^n\right)/2.\Delta t$.

A greater stability for the solution is obtained when considering the quantity ∂²M/∂x² for $t \in \left[t_n, t_{n+1}\right]$ as constant and equal to the half sum of its values at t_{n+1} and t_n : the solution is then of a closed type or according to an implicit scheme. It is convenient to consider the vectors $\vec{V}^n$ and $\vec{M}^n$ transporting the velocity and bending moment values of each of the N+3 nodes at the instant t_n and to consider the quasi tridiagonal derivative matrix [A] of rank N+3 (see details in appendix) in order to transform the derivation into a product [matrix]⊗vector according to the following notation:

$$\frac{\partial^2 \vec{M}^n}{\partial x^2} = \frac{1}{\Delta x^2}\left[A\right]\vec{M}^n \text{ and } \frac{\partial^2 \vec{V}^n}{\partial x^2} = \frac{1}{\Delta x^2}\left[A\right]\vec{V}^n \text{.}$$

The constitutive beam motion equations, after some simple rearrangements (see details in appendix), are transformed into a set of two equations with matrices and vectors:

$$\begin{cases} \vec{M}^{n+1} = [C_M]\,\vec{M}^n + \mu[D]\vec{V}^n - \lambda\,\Delta x^2\,[B^{-1}]\,\dfrac{\left(\vec{Q}^{n+1} + \vec{Q}^n\right)}{2} \\ \vec{V}^{n+1} = [C_V]\,\vec{V}^n - \lambda[D]\vec{M}^n - \dfrac{\lambda\mu}{2}\Delta x^2\,[D]\,\dfrac{\left(\vec{Q}^{n+1} + \vec{Q}^n\right)}{2} \end{cases}$$

The matrices $[C_M]$, $[C_V]$, [D] and $[B^{-1}]$ and the coefficients λ and μ depend on the derivative matrix [A], on the mechanical properties of the beam and on the spatial Δx and temporal Δt segmentations, the loading data being expressed within vectors $\vec{Q}$.

At each step of time, all the M_j^{n+1} absolute values are compared to the plastic limit M_{pl} and, when necessary, M_j^{n+1} is substituted by the plastic limit for the following step. Under such a form, the solution is naturally obtained by iteration from initial condition at t=0 and by respecting the limiting conditions (supports) with relationships between the 3 last nodes at each extremity. The spreadsheet software currently does accept to handle matrix with size up to 50 (25 to 30 nodes are generally sufficient to capture the useful deformation modes of the beam), the solution remains stable even after more than 10000 iterations, with a computation duration that remains quite reasonable (a few minutes).

Application to a one-way reinforced concrete plate hit by a high speed missile

Projectile tests on a reinforced concrete (RC) plate have been carried out by the Finnish company VTT in its own premises. Some results are presented at the Icone 2006 Conference [1]. Further results are presented in SMiRT 2007 Conference.

Reinforced concrete plate characteristics

The RC plate has the following characteristics: it is a one-way simply supported plate whose span length is 2,200 mm, with a thickness of 150 mm and a width of 2,000 mm.

The concrete characteristics are drawn from the European Code for concrete structures EC2 and correspond to a concrete with fck=50 MPa (cylinder) or fck=60 MPa (cube).

Reinforcement consists of a square mesh with 8 mm rods spaced by 50 mm on each face with a concrete cover of 20 mm.

The steel characteristics are as follows: elastic limit fy=537 MPa, first plastic plateau up to 5% deformation, then parabolic hardening up to an ultimate strength fu=637 MPa for 20% deformation.

These data allow the determination of the moment curvature law of RC sections that can be fissurated. The mechanical data to be introduced into the equations of resolution are the Young's modulus for elastic deformation and plastic plateau for beyond elastic deformation (see Figure 2). The Young's modulus of the fissurated sections is about 8,254 MPa to be compared to the value Ecm=37,000 MPa for non fissurated concrete: it means that the evolution of beam rigidity E.J due to fissuration is carried by the Young's modulus rather than by the cross sectional moment of inertia J.

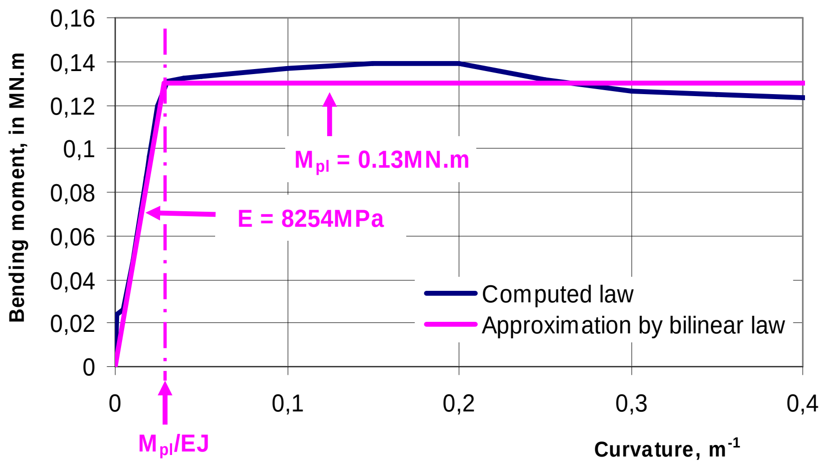

Figure 2 : Moment curvature law

Missile characteristics and loading

The missile, for the considered test, consists of an aluminium tube equipped with some additional steel parts that are necessary for its propulsion and guidance.

The missile is modeled by using the Riera approach: the geometric and mechanical characteristics of the aluminium tube are consistent with the assumption of soft impact. The necessary data are the impact velocity and the distributions of the mass density and of the buckling strength along the longitudinal missile axis.

The results of Riera model are the loading Q(t).

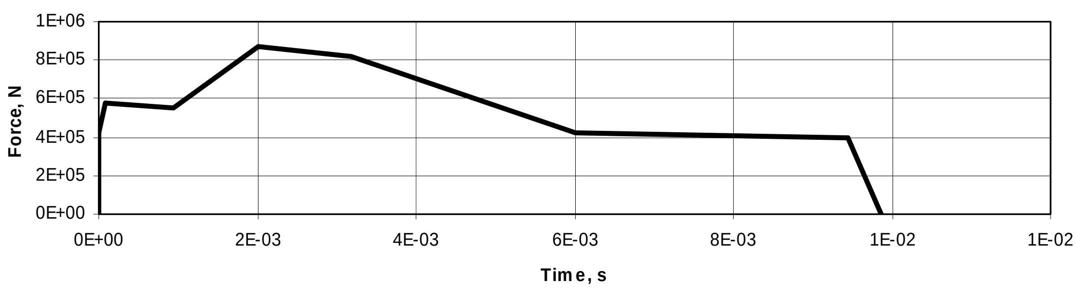

Figure 3: Loading diagram by Riera's

Computational characteristics

Some numerical constraints are to be respected: the spatial discretization shall be sufficiently refined to capture enough deformation modes and to allow the formation of plastified area.This leads to Δx ≈ H/2 and N ≈ 25 to 30 which is a good compromise for slender beam; the time step Δt shall be sufficiently small $\Delta t \approx \frac{1}{10}\frac{\Delta x}{V_s}$, with

$V_s = \sqrt{\frac{E.(1-\nu)}{(1-\nu).(1-2\nu).\rho}}$, V_s being the velocity of shear wave propagation within the beam and ν : Poisson's ratio.

With the latter condition, the classical Courant condition $\Delta t \leq \Delta x / V_s$ is respected and the discontinuity introduced by the non linearity of bilinear law does not affect the resolution stability.

Significant parameters for non-linear behavior

Among the numerous tests, we have considered those with permanent deformation and damages limited to cracks. Among the numerous experimental data acquired after each impact test, we have only considered the deflection at mid-span of the deformed one-way plate. The typical feature of the curve mid-span deflection vs. time is shown in Figure 4 below: the deflection increases at the beginning of the impact up to a maximum value and then oscillates with successive attenuations around a permanent deflection position. The interpretation of the phenomenon is simple: the impact loading causes a displacement that is elastic up to the yield limit of the most solicitated sections (first elastic phase), then plastic deformations occur up to the consumption of the kinetic energy of the missile when stopped (plastic phase); after its maximum position, the beam becomes elastic again (post plastic phase), the stored elastic deformation energy is released by oscillations with its progressive consumption by damping up to a final position that corresponds to the permanent plastic deformation.

The quantitative interpretation of the test consists in matching the experimental plate mid-span motion to the one given by dynamic equations. The sensitivity tests for the different parameters introduced to predict the beam motions show that the governing significant parameters are:

- the rigidity during the first elastic phase EJ (before any damaging by plasticity) and the damping rate ξ_e during this phase,
- the bending yield limit during the plastic phase M_{pl},
- the damaged rigidity EJ and the damping rate ξ_p during the post plastic phase.

It is worth noting that matching the experimental movement to the computed one is obtained accurately when giving the values obtained by the bilinear static bending curvature law, for the rigidity EJ of the first elastic phase and for the yield limit M_{pl} of the plastic phase. The damaged rigidity and the damping for post plastic phase are simply obtained by considering the frequency of the oscillations and by considering the decrement of the successive amplitudes of the oscillations. The ratio of the initial rigidity to the damaged rigidity is about 1.8. This can be graphically retrieved onto the following graph in Figure 4: the ratio is equal to the ratio of the elastic deflection before rest to the elastic deflection during the first elastic phase.

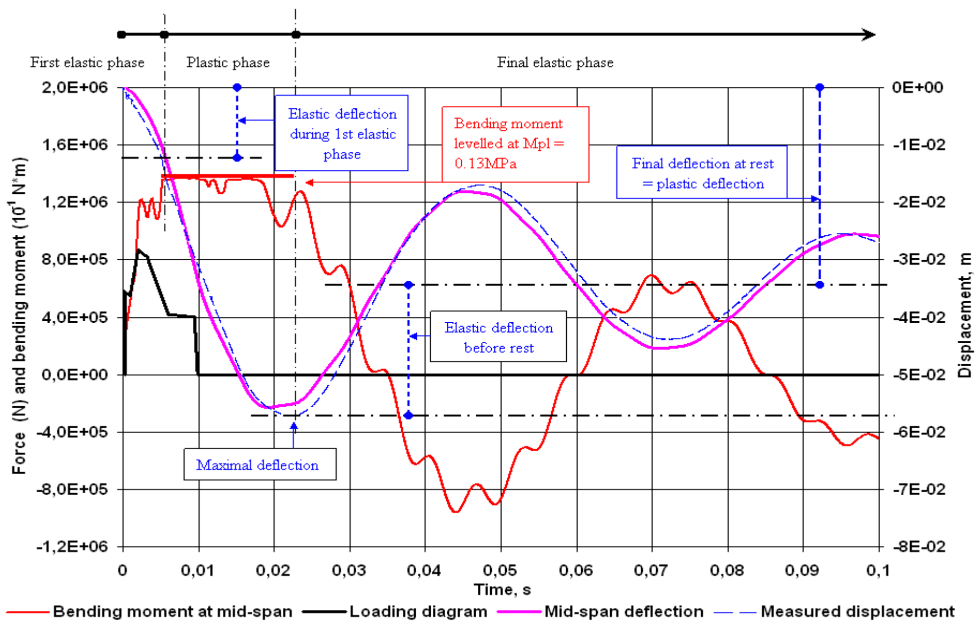

*Figure 4: Test 642 - Damping: 7% during 1st elastic phase and 10% beyond 1st elastic phase -Mpl: 0.135MN*m E given by moment-curvature law during first elastic phase E given by f= 19.5hz beyond first elastic phase.*

Conclusion

The resolution method of the equation of motion for a beam in the elastoplastic domain is proposed by using the finite difference approximation and with an implicit iterative method. The method allows programming on a spreadsheet. Comparison of simulation by the proposed method with experimental results (impact at high velocity of a deformable missile on a one-way RC plate) shows a good agreement and confirms both its efficiency and the use of static values for characteristics of bending moment / curvature law. Besides, such a resolution method allows numerous sensitivity studies, for dimensioning purpose or margin evaluation, regarding the shape effect of loading diagrams, the geometry of the beam and the mechanical characteristics of concrete and reinforcing steel.

References

[1] Saarenheimo A. et al.: "Numerical studies on impact loaded concrete structures" in Proceedings of Icone 14 (2006).

Appendix

Let us consider the classical equation of motion of a beam with only flexural deformation and undergoing a viscous damping for moderate deformations:

$$\rho.S.\left(\frac{\partial^2 w}{\partial t^2} + 2\xi\omega\frac{\partial w}{\partial t}\right) + \frac{\partial^2 M}{\partial x^2} - q = 0 \qquad \text{(A1)}$$

x, t : position of the section along the beam, time during the phenomenon

$w(x,t)$: lateral deflection $M(x,t)$: bending moment

ξ : rate of critical damping ω : pulsation of the 1st eigenmode of deformation

ρ : volumic mass of constitutive material (reinforced concrete)

S = B.H : area of current section B : section width H : section height

q (x,t) : loading density

The valuable unknown to consider is the velocity v defined by:

$$v = \frac{\partial w}{\partial t} \qquad \text{(A2)}$$

Replacing v in (A1) gives:

$$\frac{\partial v}{\partial t} + 2\xi\omega v + \frac{1}{\rho S}\frac{\partial^2 M}{\partial x^2} - \frac{q}{\rho S} = 0 \qquad \text{(A3)}$$

The resolution of the second order partial derivative equation (A3) is performed by the method of the finite difference method with the following approximations:

$$\frac{\partial v}{\partial t} \approx \frac{V_j^{n+1} - V_j^n}{\Delta t} \qquad \text{(A4)}$$

and

$$\frac{\partial^2 M}{\partial x^2} \approx \frac{M_{j+1}^n - 2.M_j^n + M_{j-1}^n}{\Delta x^2} \qquad \text{(A5)}$$

V_j^n and M_j^n being respectively the velocity and the bending moment with the classical notation for the subscripts: the lower subscript refers to the node position and the upper subscript refers to the instant. The beam is supposed to consist of N segments of unit length Δx and N+3 nodes considering that support modeling needs 2 nodes to characterize the freedom degrees in translation and rotation. In order to introduce some stability in the solution of eq. (A3), the velocity variation $\frac{\partial v}{\partial t}$ during the time between t_n and t_{n+1} is computed by considering, for the other terms of above equation, the mean value of their value at t_n and their value at t_{n+1}.

Consequently, equation (A3) becomes:

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + 2\xi\omega\left(\frac{V_j^{n+1} + V_j^n}{2}\right) + \left(\frac{1}{2\rho S}\right)\left(\frac{M_{j+1}^{n+1} - 2M_j^{n+1} + M_{j-1}^{n+1}}{\Delta x^2} + \frac{M_{j+1}^n - 2M_j^n + M_{j-1}^n}{\Delta x^2}\right) - $$

$$- \frac{Q_j^{n+1} + Q_j^n}{2\rho S} = 0 \qquad \text{(A6)}$$

After multiplying by Δt and introducing the parameter λ :

$$\lambda = \frac{\Delta t}{\rho S \Delta x^2} \qquad \text{(A7)}$$

and after rearranging, eq. (A6) becomes:

$$\frac{\lambda}{2}\left(M_{j+1}^{n+1}-2M_{j}^{n+1}+M_{j-1}^{n+1}\right)+\left(1+\xi\,\omega\Delta t\right)V_{j}^{n+1}=-\frac{\lambda}{2}\left(M_{j+1}^{n}-2M_{j}^{n}+M_{j-1}^{n}\right)+\left(1-\xi\,\omega\Delta t\right)V_{j}^{n}-$$
$$-\lambda\,\Delta x^{2}\left(\frac{Q_{j}^{n+1}+Q_{j}^{n}}{2}\right) \qquad \text{(A8)}$$

The above equation gives the first relation between the unknown V_j^{n+1} and M_j^{n+1} at the instant t_{n+1} from their value V_j^n and M_j^n at the instant t_n.

The second relation between V_j^{n+1} and M_j^{n+1} is given by the bilinear bending curvature law:

$$\begin{cases} |M|\le M_{pl}\ ,\ \dfrac{\partial^2 w}{\partial x^2}=\dfrac{M}{E\,J} \\ |M|=M_{pl}\ ,\ \dfrac{\partial^2 w}{\partial x^2}=\dfrac{M_{pl}}{E\,J}\,sign(M) \end{cases} \qquad \text{(A9)}$$

M_{pl} being the ultimate plastic bending moment of the section of the beam which is supposed with a uniform reinforcement.

By using the finite difference method the above equation is transformed into:

$$M_j^{n+1}=E\,J\,\frac{W_{j+1}^{n+1}-2W_j^{n+1}+W_{j-1}^{n+1}}{\Delta x^2} \qquad \text{(A10)}$$

The deflection W_j^{n+1} is deduced from the deflection at the precedent step W_j^n by considering the additional displacement resulting from a velocity equal to the mean value of the velocity V_j^{n+1} at the instant t_{n+1} and the velocity V_j^n at the instant t_n:

$$W_j^{n+1}=W_j^n+\frac{V_j^{n+1}+V_j^n}{2}\Delta t \qquad \text{(A11)}$$

After substituting W_j^{n+1} from eq. (A11) in the equation (A10), and after introducing the parameter μ :

$$\mu=\frac{E\,J\,\Delta t}{\Delta x^2} \qquad \text{(A12)}$$

the equation (A10) becomes:

$$M_j^{n+1}=M_j^n+\frac{\mu}{2}\left(V_{j+1}^{n+1}-2V_j^{n+1}+V_{j-1}^{n+1}\right)+\frac{\mu}{2}\left(V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}\right) \qquad \text{(A13)}$$

After rearranging, this is equivalent to:

$$M_j^{n+1}-\frac{\mu}{2}\left(V_{j+1}^{n+1}-2V_j^{n+1}+V_{j-1}^{n+1}\right)=M_j^n+\frac{\mu}{2}\left(V_{j+1}^{n}-2V_j^{n}+V_{j-1}^{n}\right) \qquad \text{(A14)}$$

In order to facilitate the handling of these scalars with 2 indices, let us consider the vectors that gather all these scalars, for each instant t_n, with the evident notation:

$$\vec{\mathrm{M}}^{n}={}^{t}\left(M_1^n,M_2^n,M_3^n,\cdots,M_j^n,\cdots,M_{N+3}^n\right) \quad \text{and} \quad \vec{\mathrm{V}}^{n}={}^{t}\left(V_1^n,V_2^n,V_3^n,\cdots,V_j^n,\cdots,V_{N+3}^n\right)$$

Let us introduce the matrix $[A]$ as the operator that allows the second derivation of a vector:

$$\begin{cases} \dfrac{\partial^2\vec{M}^n}{\partial x^2}=\dfrac{1}{\Delta x^2}[A]\,\vec{M}^n \\ \dfrac{\partial^2\vec{V}^n}{\partial x^2}=\dfrac{1}{\Delta x^2}[A]\,\vec{V}^n \end{cases} \qquad \text{(A15)}$$

The matrix $[A]$ is a quasi tridiagonal N+3 rank matrix with the following shape (the terms of the first and last line are shifted respectively to the right and to the left, which corresponds to the forward difference and to the backward difference of the approximation of the second order partial derivative:

$$
[A]=\begin{bmatrix}
1 & -2 & 1 & 0 & \cdots & & \cdots & & 0 & & & \cdots & & \cdots & 0\\
1 & -2 & 1 & 0 & & & & & & & & & & & \vdots\\
0 & 1 & -2 & 1 & 0 & & & & & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & 0 & & & \vdots\\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & & & \\
\vdots & & & & 0 & 1 & -2 & 1 & 0 & & & & & & \vdots\\
0 & & & & & 0 & 1 & -2 & 1 & 0 & & & & & 0\\
\vdots & & & & & & 0 & 1 & -2 & 1 & 0 & & & & \vdots\\
 & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
\vdots & & & 0 & & & & & & & & & & & \\
 & & & & & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
 & & & & & & & & & & 0 & 1 & -2 & 1 & 0\\
\vdots & & & & & & & & & & & 0 & 1 & -2 & 1\\
0 & \cdots & & \cdots & & 0 & & \cdots & & & \cdots & 0 & 1 & -2 & 1
\end{bmatrix} \tag{A16}
$$

With these notations, eqs (A8) and (A14) may be rewritten as:

$$
\begin{cases}
\dfrac{\lambda}{2}[A]\vec{M}^{\,n+1} + (1+\xi\,\omega\Delta t)\vec{V}^{\,n+1} = -\dfrac{\lambda}{2}[A]\vec{M}^{\,n} + (1-\xi\,\omega\Delta t)\vec{V}^{\,n} - \lambda\,\Delta x^2\,\dfrac{\left(\vec{Q}^{\,n+1}+\vec{Q}^{\,n}\right)}{2}\\
\vec{M}^{\,n+1} - \dfrac{\mu}{2}[A]\vec{V}^{\,n+1} = \vec{M}^{\,n} + \dfrac{\mu}{2}[A]\vec{V}^{\,n}
\end{cases} \tag{A17}
$$

This system of algebraic equations may be solved by linear combination: $\vec{V}^{\,n+1}$ is obtained by multiplying the second line by $\lambda[A]$ and by subtracting it to the first one, $\vec{M}^{\,n+1}$ is obtained by multiplying the first line by $\dfrac{\mu}{2}[A]$ and by adding the second one multiplied by $(1+\xi\,\omega\Delta t)$. Introducing the unit matrix $[I]$ of same rank as $[A]$ allows the following expressions:

$$
\begin{cases}
\left[(1+\xi\omega\Delta t)[I]+\dfrac{\lambda\mu}{4}[A]^2\right]\vec{V}^{n+1} = -\lambda[A]\vec{M}^{n} + \left[(1-\xi\omega\Delta t)[I]-\dfrac{\lambda\mu}{4}[A]^2\right]\vec{V}^{n} - \lambda\Delta x^2\,\dfrac{\left(\vec{Q}^{n+1}+\vec{Q}^{n}\right)}{2}\\
\left[(1+\xi\omega\Delta t)[I]+\dfrac{\lambda\mu}{4}[A]^2\right]\vec{M}^{n+1} = \left[(1+\xi\omega\Delta t)[I]-\dfrac{\lambda\mu}{4}[A]^2\right]\vec{M}^{n} + \mu[A]\vec{V}^{n} - \dfrac{\lambda\mu}{2}\Delta x^2[A]\dfrac{\left(\vec{Q}^{n+1}+\vec{Q}^{n}\right)}{2}
\end{cases} \tag{A18}
$$

Let us define the matrices

$$[B] = (1+\xi\,\omega\Delta t)[I] + \frac{\lambda\mu}{4}[A]^2 \tag{A19}$$

$$[C_V] = [B^{-1}]\left[(1-\xi\,\omega\Delta t)[I] - \frac{\lambda\mu}{4}[A]^2\right] \tag{A20}$$

$$[C_M] = [B^{-1}]\left[(1+\xi\,\omega\Delta t)[I] - \frac{\lambda\mu}{4}[A]^2\right] \tag{A21}$$

$$[D] = [B^{-1}][A] \tag{A22}$$

The matrix $[B]$ is symmetrical, definite and positive and therefore invertible and noted $[B^{-1}]$.

With the above definitions of matrix, the expression of eqs (A18) becomes:

$$\begin{cases} \vec{M}^{n+1} = [C_M]\,\vec{M}^{n} + \mu[D]\vec{V}^{n} - \lambda\,\Delta x^{2}\,[B^{-1}]\,\dfrac{\left(\vec{Q}^{n+1} + \vec{Q}^{n}\right)}{2} \\ \vec{V}^{n+1} = [C_V]\,\vec{V}^{n} - \lambda[D]\vec{M}^{n} - \dfrac{\lambda\mu}{2}\Delta x^{2}\,[D]\,\dfrac{\left(\vec{Q}^{n+1} + \vec{Q}^{n}\right)}{2} \end{cases} \quad \text{(A23)}$$

These equations give the value of the bending moment and the velocity at each node and at any time when knowing the value of the bending moment, the velocity and the deflection at each node at the beginning of the loading $\vec{Q}$, the deflection W_j^{n+1} being given by the equation (A11) which is: $W_j^{n+1} = W_j^{n} + \dfrac{V_j^{n+1} + V_j^{n}}{2}\Delta t$

The conditions at the supports are taken into account by the conditions applied to the supplementary nodes outside the supports:

- a simple beam-end support is marked by a free rotation i.e. the deflections of the 2 nodes around the support node are opposite;
- a clamped beam-end support is marked by blocked rotation i.e. the deflections of the 2 nodes around the support node are equal;
- a free end beam is marked by free rotation and free deflection i.e. the shear force

  $T_j^{n} = \dfrac{\partial M_j^{n}}{\partial x} = 0$, $\forall n$ and with $j = 2$ *or* $j = N + 2$