Matrix Decomposition Techniques and Bayesian Inference for Seismic Damage Detection in Structures

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ABSTRACT

In this paper we investigate some modalities of using matrix decomposition techniques and Bayesian models for damage detection in structures, in particular for the seismically induced damages. A stiffness matrix decomposition is employed for the damage detection in J.A. Escobar et al. in 2004 [1]. The modal shapes and vibration frequencies for the damaged state of the structure are used for building its lateral stiffness matrix from which the analytical model is adjusted by means of an iterative process allowing to detect the damaged structural elements. An alternative procedure was earlier proposed by Sohn and Law in 2000 [2] and it employs the Ritz vectors which offer certain advantages. The latter method is effectively based on the Bayes Theorem.

INTRODUCTION

The identification of the damages in a structure is a not very easy task. The mathematical tools needed in search of the damaged elements or substructures are naturally involving matrices and certain decomposition techniques of the matrix analysis. Both methods we survey and discuss in this paper present iterative algorithms for identifying damaged elements or substructures in a structure. The first of them, reported to 13 WCEE in 2004 [1], involves the singular value decomposition of matrices. We offer a concise presentation of the SVDs of matrices in the first section that follows. This factor decomposition of matrices is effectively involved in the procedure for damage detection in building structures proposed in [1]. We try to get a little deeper with the matrix formalism there involved, and the same approach regards an earlier devised method for the damage identification by use of the Ritz vectors and the Bayes Theorem [2].

SINGULAR VALUE DECOMPOSITION OF MATRICES

In this section of our paper we present the main properties and results connected with the SVDs (singular value decomposition) of matrices. Among the many possible decompositions of a given matrix in two, three (or more) factors, the (SVD) is one of the most productive. The SVD of a matrix naturally emerges from the problem of finding an optimal approximate solution to a system of linear equations which is inconsistent.

Approximate Solutions to Linear Systems and Projection Matrices

Starting from the basic matrix equation $Ax = b$ (with $A \in \mathbb{M}_{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$) the norm of the difference between the two sides of this equation may be regarded as an error. It comes to

$$E = \|Ax - b\|$$

where the norm that occurs in Eq. (1) is one of the possible matrix norms. This “error” is significant when the linear system $Ax = b$ is not consistent, what is equivalent to $b \notin \text{COLSP}_A$. An approximation statement of this problem consists in choosing a vector $\tilde{x}$ so as to minimize the error $E$, that equals the distance between the vectors (or points) $Ax, b$ in the column space of $A$. Thus searching for a “best” solution $\tilde{x}$ to minimize $E$ is the same as locating the point $p = A\tilde{x}$ so that it be closer to $b$ than any other point in the column space. In terms of geometry, $p$ should be the projection of $b$ onto the column space. This condition may be reformulated as the property of the error vector to be orthogonal on the column space of $A$, that is

$$b - A\tilde{x} \perp \text{COLSP}_A.$$  

Two ways for calculating the “approximate solution” $\tilde{x}$ and the projection $p = A\tilde{x}$ are considered in [3], namely:

(i) The vectors that are orthogonal on the $\text{COLSP}_A$ lie in the left nullspace of the matrix. Thus the error vector of Eq. (2) should be in the nullspace of $A^T$:
\[
A^T (b - A \overline{x}) = 0 \iff A^T A \overline{x} = A^T b. \tag{3}
\]

(ii) The error vector must be orthogonal to every column of \( A \):

\[
\begin{align*}
\{ a_1^T (b - A \overline{x}) = 0 \\
\vdots \\
\{ a_n^T (b - A \overline{x}) = 0 \\
\end{align*}
\iff \begin{bmatrix} a_1^T \\
\vdots \\
a_n^T 
\end{bmatrix} (b - A \overline{x}) = 0. \tag{4}
\]

Obviously, 0 in the equation of (3) is the column vector in the \( n \)-space and 0 in the 2\(^{nd} \) equation of (4) is the same, while the zeros in the system that occurs in (4) are the zero scalars; however, we do not use a specific notation like \( 0 \) for the zero vector. Another remark regards the notation for the columns of a matrix, namely \( a_j \ (j = 1, n) \Rightarrow A = [a_1 \ldots a_n] \).

The second equation in (4) is thus equivalent to the equation in (3). A third way to find the solution \( \overline{x} \) consists in taking (and using) the derivative of the square of the error in (1), that is

\[
E^2 = (Ax - b)^T (Ax - b) \tag{5}
\]

what gives

\[
\frac{d}{dx} E^2 = 0 \Rightarrow 2A^T Ax = 2A^T b. \tag{6}
\]

All these equivalent ways for minimizing the error in (1) involve a matrix with quadratic entries, namely \( A^T A \) : each entry in this matrix is the inner product of two columns of \( A \). If the columns of \( A \) are linearly independent then \( A^T A \) is invertible and

\[
\overline{x} = (A^T A)^{-1} A^T b. \tag{7}
\]

The projection of \( b \) on the column space is obtained from \( p = A \overline{x} \) and Eq. (7) as

\[
p = A(A^T A)^{-1} A^T b. \tag{8}
\]

The matrix that premultiplies the vector \( b \) in Eq. (8) is called a projection matrix :

\[
P = A(A^T A)^{-1} A^T. \tag{9}
\]

It follows from Eqs. (8) & (9) that \( p = Pb \) what means that \( p \) is the component of \( b \) in the column space, and the error \( b - Pb = (I - P)b \) is is the component in the orthogonal component. This provides a matrix formula for splitting a vector into two perpendicular components : \( p = Pb \) is in the column space of \( A \) (that is, \( \text{COLSP}_A \) in our notation or \( \mathbb{R}_A \) as denoted in [3]) while \( (I - P)b \in \mathbb{N}(A^T) \) = the left nullspace of \( A \) which is orthogonal to the column space.

The projection matrices have interesting properties. Two basic properties of \( P \) in Eq. (9) are its idempotence and its symmetry :

\[
P^2 = P \quad \text{and} \quad P^T = P. \tag{10}
\]

**Least Squares Fitting of Data**

If a series of experiments are carried out and the output \( b \) is expected to be a linear function of the input \( t \) than a linear expression of the form \( b = C + Dt \) corresponds to this assumption. If a number of \( m \) experimental results (or “readings”) are obtained for \( m \) values of the argument \( t_1, t_2, \ldots, t_m \) then a linear system can be obtained as consisting of the \( m \) equations \( C + Dt_i = b_i \ (i = 1, \ldots, m) \). This system can be rewritten as a matrix equation of the form \( Ax = b \) and the best solution \( [C \ D] \) is the one that minimizes the (square of the) error.
A simple example that models a variable load on a structure is given in [3] (page 159), with \( b = \) the reading from the strain gauge. A linear relationship of the form \( b = C + D t \) is accepted so far the response remains in the elastic domain. Data fitting by a straight line, given the measurements (or observation data) \( b_1, b_2, \ldots, b_m \) at the distinct points \( t_1, t_2, \ldots, t_m \), the straight line \( b = C + D t \) that minimizes the (square) error in (11) comes from the solution of the equations

\[
\sum \frac{b_i}{t_i} = \frac{C}{D} = \frac{\sum t_i}{\sum b_i}
\]

The Singular Value Decomposition

An \( m \times n \) matrix \( A \) admits an SVD decomposition if three matrices \( Q_1, Q_2, \Sigma \) exist such that

\[
A = Q_1 \Sigma Q_2^T\]

where \( Q_1, Q_2 \) are orthogonal matrices, that is \( Q_1^T Q_1 = I_m \) & \( Q_2^T Q_2 = I_n \), while \( \Sigma \) is a quasi-diagonal matrix. If \( A \) is an \( m \times n \) (hence rectangular) matrix then \( Q_1 \) is \( m \times m \) and its columns are eigenvectors of the matrix \( AA^T \) while \( Q_2 \) is \( n \times n \) and its columns are eigenvectors of the matrix \( A^T A \). If the rank of \( A \) is \( r \) then the \( r \) singular values on the diagonal of \( \Sigma \) are the square roots of the nonzero eigenvalues of both \( AA^T \) and \( A^T A \). The SVD allows to find the optimal solution \( \hat{x} \) to any problem in least squares, as presented in the previous section. The connections between the matrices \( AA^T, A^T A \) and the three factor matrices that occur in (13) is given by

\[
AA^T = (Q_1 \Sigma Q_2^T)(Q_2 \Sigma^T Q_1^T) = Q_1 \Sigma \Sigma^T Q_1^T \quad \text{and} \quad A^T A = \ldots = Q_2 \Sigma \Sigma^T Q_2^T.
\]

We investigate – in the next section – the possibilities to obtain the condensed stiffness matrix of a structure by means of a geometric transformation matrix. It will be seen that the condition of orthogonality of a certain vector on the column space of a matrix is a basic idea for identifying the damages in a structure.

**DAMAGE DETECTION IN STRUCTURES USING TRANSFORMATION MATRICES AND SVD**

In their contribution [1] to the 13 WCEE (Vancouver, August 2004), the three Mexican authors propose a method for detection and evaluation of damages based on the changes in the known modal shapes and vibration frequencies. The modal shapes and vibration frequencies for the damaged state of the structure are used to build its lateral stiffness matrix from which the analytical model is adjusted by means of an iterative process able to detect the damaged structural elements.

Let \( [K_d] \) denote the global stiffness matrix for the damaged state. If the structure consists of \( Ne \) elements, it can be expressed as

\[
E^2 = \| b - Ax \|^2 = \sum_{i=1}^{m} (b_i - C - D t_i)^2.
\]
where \( x_i \in [0,1] \) is a non-dimensional parameter that represents the contribution of the stiffness of the structural element \( i \) to the global stiffness matrix of the structure, and \( [K_e]_i \) is the stiffness matrix of the non-damaged element \( i \). Clearly, the coefficients \( x_i(i=1, \ldots, Ne) \) give an evaluation of the level of damage of the respective elements in terms of their stiffness. Since the number of degrees of freedom in the analytic pattern is usually (much) greater than the number of those that are possible to be measured, the damage detection method uses a matrix of geometric transformation to condense the degrees of freedom. If the damaged state of the structure is described (as in [1]) in terms of the rigid body displacements of the storey slabs, the condensed stiffness matrix related to these degrees of freedom is

\[
[K_d] = [T_d(x)]^T [K_e] [T_d(x)]
\]

where \( [T_d(x)] \) is the transformation matrix as depending of the damage parameter \( x \). It is not very clearly explained (in [1]) which is the nature of this damage parameter but \( x \) is later on considered as a (column) vector. This follows from an alternative expression of what could be called the altered (or damaged) stiffness matrix \( [\overline{K}_d] \), namely

\[
[\overline{K}_d] = \sum_{i=1}^{Ne} x_i [\overline{K}_e(x)]_i \quad \text{with} \quad [\overline{K}_e(x)]_i = [T_d(x)]^T [K_e(x)]_i [T_d(x)].
\]

The matrices \( [\overline{K}_d] \) and \( [\overline{K}_e(x)]_i \) that occur in (18) are of size \( Nm - N \) by \( Nm \) where \( Nm \) is the number of degrees of freedom measured in the structure. Since these matrices are symmetric, the number of relevant entries in each of them is \( Nm(Nm+1)/2 = nit \) (the number on independent terms, as they are called in [1]). The first equation in (18) may be rewritten by considering the “stiffness reducing coefficients” \( x_i(i=1, \ldots, Ne) \) as the components of a column vector \( \{x\} = [x_1 \ x_2 \ldots x_{Ne}]^T \). Eq. (4) in reference [1] for determining a current column (of “independent terms”) in the matrix that occurs in the first equation of (18) looks like

\[
\{\overline{k}_e\} = [S_k(x)] \{x\}
\]

where the left hand side in Eq. (19) is a column with \( nit \) components, hence a matrix of size \( nit \) by \( 1 \) while \( [S_k(x)] \) is the matrix of size \( nit \) by \( Ne \) that contains the independent terms of the matrices \( [\overline{K}_e(x)]_i \). It follows from these sizes of the matrices involved in expression (19) that the vector \( \{\overline{k}_e\} \) and the matrix \( [S_k(x)] \) are obtained by a process of linearizing \( [\overline{K}_e(x)]_i \). The upper triangle (including the main diagonal) of this symmetric matrix is turned into a vector with \( nit \) components by taking the first row with \( Nm \) entries, followed by the second row with \( Nm-1 \) entries (the first entry being omitted) and so on.

The matrix \( [\overline{K}_d] \) has to be adjusted with the fitness matrix obtained from the modal shapes and vibration frequencies measured in the structure. It may be assumed (as in Ref. [1]) that the transformation matrix \( [T_d(x)] \) could be the same for the damaged and undamaged state of the structure. In this way, an iterative procedure can be applied for detecting the damaged structural elements by successive approximations. Some details are given below.

The effect of the “noise” on the measured modal shapes and vibration frequencies can result in different matrices \( [\overline{K}_d] \) and \( [\overline{K}_m] = \text{the stiffness matrix of the structure resulting from the measurements. The error can be expressed in the usual way, as the “distance” between the two matrices, that is the norm of their difference, or by the square thereof:}

\[
E = \|[K_d] - [K_m]|| \quad \text{or} \quad E^2 = ([K_d] - [K_m])^T ([K_d] - [K_m])
\]

The expression of the squared error in (20) follows if the usual Euclidean norm is applied. If the earlier presented vectors of the “independent terms” are used, the previous error can be expressed as follows:
When the matrix \([\overline{K}_m]\) is affected by the noise in the measurements, it cannot be – in general – expressed as a linear combination of the matrices \([\overline{K}_k(x)]\). With our previous remark on the “linearization” of these matrices in terms of (column) vectors of size \(n_{it}\) that results in the first expression of the error in (21), this error will differ from zero. This shows that the linear system

\[
[S_k(x)\{x\} = \{\tilde{k}_m\}
\]

is inconsistent and we thus arrive to the problem presented in the previous section. According to Gilbert STRANG’s theory [3], the error \(E\) will be minimal if and only if the vector \(\{\tilde{k}_d\} - \{\tilde{k}_m\}\) is orthogonal on the column space of matrix \([S_k(x)]\):

\[
\{\tilde{k}_d\} - \{\tilde{k}_m\} \perp \text{COLSP}[S_k(x)].
\]

According to these arguments, the “best approximating” solution to the matrix equation (19) can be obtained as the solution to the optimization problem

\[
\begin{cases}
[S_k(x)\{x\} \leq \{\tilde{k}_m\}] , \\
\{0\} \leq \{x\} \leq \{1\} , \\
[S_k(x)^T[S_k(x)\{x\}] = [S_k(x)^T\{\tilde{k}_m\}] .
\end{cases}
\]

Remarks (1). Obviously, the vectors \(\{0\}\) & \(\{1\}\) that occur in the middle inequality of (24) are the (column) vectors with all their components equal to 0, respectively to 1. The orthogonality condition of (23) clearly comes from the condition (2) in the previous section, and this third equation in (24) is an instance of the second equation in (3). It is equivalent to an optimality condition. As regards the first inequality in (24), this could hold on the basis of engineering judgement.

In order to solve Eq. (22), the singular value decomposition is mentioned as possibly useful in Ref. [1]. It would be naturally applied to the matrix \([S_k(x)]\):

\[
[S_k(x)\{x\} = [U]\Sigma[V]^T
\]

where \([U]\) is an orthogonal matrix of size \(n_{it} – n\), \([V]\) is an orthogonal matrix of size \(N\) – by – \(N\) and \([\Sigma]\) is a quasidiagonal matrix of size \(n_{it} – 1\).

Remarks (2). The nature of the SVD in (25) depends on the size of the matrix \([S_k(x)]\): according to Theorem 7.3.5 and the subsequent corollaries in the basic monograph [5] by Roger Horn and Charles Johnson, the structure of the quasidiagonal matrix \([\Sigma]\) depends on whether \(m \geq n\) or \(m < n\) for \([S_k(x)] \in \mathcal{M}_{m,n}(r)\). A procedure for yielding a SDV for a matrix \(A\) is also presented in this monograph [4] at page 416. Let us recall (from the first section) that an approximate solution to an inconsistent linear system \(Ax = b\) is given by

\[
\bar{x} = (A^T A)^{-1} A^T b.
\]

Coming back to Eq. (22) which is – in fact – an underdetermined linear system, a way to obtain its solution is suggested in [1] by reducing the number of unknowns by associating the same factor \(x\) for the structural elements that have the same magnitude of damage. Such a reduction turns the size of matrix \([S_k(x)]\) from \(n_{it} – n\) to \(n_{it} – N_g\) with the number of elements with distinct damage magnitudes equal to \(N_g \leq N\). We do not recall the algorithm consisting of 10 steps in full detail since it is presented in [1], but we mention its iterative nature.

1 The matrices \([K_e]\), and \([T_d(x)]\) are obtained from the undamaged state of the structure.

2 The number of necessary iterations for the convergence of the algorithm is established.
The matrices \( \bar{K}_e(x) \) are calculated.

The matrix \( \bar{S}_k(x) \) is formed.

The factorization under the SVD in Eq. (25) is obtained.

The equation (linear system) \( \{ \bar{S}_k(x) \} = \{ \bar{K}_m \} \) of (22) is solved.

If \( \{ x \} = \{ x \}_n \) is the solution to the system in the previous step, the next vector is obtained as \( \{ x \}_{n+1} = \beta \{ x \}_n + (1 - \beta) \{ x \}_{n-1} \) where \( \beta \) is a convergence factor.

The matrices \( \{ K_{e,i} \} \) and \( \{ T_d(x) \} \) associated to the vector \( \{ x \}_{n+1} \) are calculated.

\( \{ \bar{K}_d \} \) is obtained and the error \( E = \| \{ \bar{K}_m \} - \{ \bar{K}_d \} \| \) is calculated.

Return to step 3 until the pre-established number of iterations is reached.

**Remarks (3).** The \( (n+1) \)–th vector given by its expression of Step 7 is – in fact – a (convex) linear combination of the vectors obtained in the previous two iterations. It is not very clearly explained how this factor is obtained. It is said (in [2]) to be a fraction of the sum of damage(s) obtained in iterations \( n-1 \) & \( n \). For the natural hypothesis that \( 0 \leq \beta \leq 1 \) implies that the \( (n+1) \)–th vector is actually a convex combination of the previous two ones. A more natural way to provide a convergence criterion for such an algorithm would consist in comparing the distance between two successive vectors with a given threshold.

**DAMAGE DETECTION BY USING LOAD-DEPENDENT RITZ VECTORS**

Another approach to the damage detection in a structure makes use of the load-dependent Ritz vectors in a Bayesian probabilistic procedure, as presented in H. Sohn and K.H. Law [2]. These Ritz vectors can be extracted experimentally from the traditional modal analysis using accelerometers. They were shown to be very efficient for the dynamic and earthquake analyses, eigenvalue problems and model reductions. The Ritz vectors simply serve as a basis to span the displacement space, what makes them to be easily employable in the techniques based on strain mode shape or mode shape curvature. In the just quoted reference, the authors incorporate the Ritz vectors into an previously devised Bayesian probabilistic framework for identifying the most probable damage event among different damage scenarios.

The Ritz vectors are generated by a recursive procedure and their independence is obtained by the Gram-Schmidt orthogonalization. It is shown in the article by H. Sohn and K.H. Law that the Ritz vectors exhibit a better sensitivity than the modal vectors. The damage detection is performed by changing damage locations and load patterns.

The theoretical formulation of the model used in [2] slightly differs but also has common points with the model by Escobar et al. of the previous section. As regards notations, the stiffness matrix of the system is denoted by \( K \) instead of \( \{ K_{e,i} \} \); let us recall that the latter notation stands for the global stiffness matrix of the system *in damaged state*. The authors of Ref. [2] consider a structure decomposition into \( N_{sub} \) substructures instead of \( Ne \) structural elements as in [1], what can be considered to be a generalization. The substructure stiffness matrices correspond to the matrices \( K_{e,i} \) in the previous section and they are denoted \( K_{s,i} \). The system stiffness matrix \( K \) is assembled from the substructure matrices in a similar way as in Eq. (16):

\[
K = \sum_{i=1}^{N_{sub}} \Theta_i K_{s,i}
\]  

where \( \Theta = \{ \Theta_i : i = 1, \ldots, N_{sub} \ & 0 \leq \Theta_i \leq 1 \} \). The elements in this set \( \Theta \) are non-dimensional parameters and each of them quantifies the contribution of the corresponding substructure to the system matrix. The analogy between (the roles of) \( \Theta \) and the vector \( \{ x \} \) in the previous section is quite obvious. Certainly, expression (27) of the stiffness matrix \( K \) could be equivalently written using a specific Schur product of matrices if \( \Theta \) is defined as a (column) vector instead of a simple set; but this is not an essential issue. It is important to emphasize the significance of these substructure-specific parameters: a substructure \( i \) is considered (in [2]) to be damaged when \( \Theta_i \) is less then a specified threshold. The advantages of the approach in [2] as compared to more “classical” approaches to damage detection consists in using Ritz vectors (RVs) instead of modal vectors (MVs) because: (i) RVs are more sensitive to damage than the MVs, and the
substructures of interest, 

(ii) substructures of interest can be made more observable using the RVs generated from particular load patterns, 

(iii) the computation of RVs is less expensive than that of MVs (eigenvectors), and 

(iv) the practical difficulties of modal testing encumbers the extraction of a large number of significant modes while a larger number of Ritz vectors can be extracted by imposing different load patterns on a structure.

The Ritz vectors appear as the components of a data set in a collection of $N_s$ data sets:

$$
\hat{\Psi}_{N_s} = \{ \hat{\psi}(n) : n = 1, \ldots, N_s \} \& \hat{\psi}(n) = [\hat{r}_1(n) \ldots \hat{r}_{N_s}(n)]^T.
$$

In this expression (28) $\hat{r}_i(n)$ denotes the $i$-th estimated Ritz vector in the $n$-th data set $\hat{\psi}(n)$. We have used a different notation for these Ritz vectors that occur in Eq. (3) of [2] as $\hat{\psi}$.

The identification of the most likely damaged substructures can be accomplished by means of an iterative procedure based upon the Bayes Theorem. If $H_j$ denotes a hypothesis on a damage event possibly involving any number of damaged substructures, the initial degree of belief in such a hypothesis is represented by the prior probability $P(H_j)$. According to Bayes’s Theorem, the posterior probability after observing the estimated data sets of Eq. (28) is given by

$$
P(H_j \mid \hat{\Psi}_{N_s}) = \frac{P(\hat{\Psi}_{N_s} \mid H_j)}{P(\hat{\Psi}_{N_s})}.
$$

The most likely damaged substructures are those involved in the hypothesis $H_{\text{max}}$ whose posterior probability is the highest over the range of the hypotheses on damage events:

$$
P(H_{\text{max}} \mid \hat{\Psi}_{N_s}) = \max_{\forall H_j} P(H_j \mid \hat{\Psi}_{N_s}).
$$

Since the explicit expression of the posterior probabilities involved in (29) and (30) are hard to be used, the use of an error function is recommended in [2], namely

$$
J(\hat{\Psi}_{N_s}, \Theta_{H_j}) = \frac{1}{2} \sum_{n=1}^{N_s} [\hat{\psi}(n) - \psi(\Theta_{H_j}) - e_M(\Theta_{H_j})]^T \psi [\hat{\psi}(n) - \psi(\Theta_{H_j}) - e_M(\Theta_{H_j})]
$$

where: $\psi$ is the covariance matrix of the vector in Eq. (28) and an analytical data set is defined in a similar way as in the second equation of (28):with the parameter vector $\Theta_{H_j}$ as the argument. The same argument occurs in the term $e_M(\Theta_{H_j})$ that occurs in Eq. (31) accounts for the output error implied by the deviations from the measured structural response and the response of the analytical model. Following a B&B (branch-and-bound) search scheme set up by the same two authors of Ref. [2]. It involves posterior probabilities and its basic steps are:

(i) Extend hypothesis $H_j$ to $H_j \cup D_i$ by adjoining the $i$-th substructure as damaged.

If $P(H_j \cup D_i \mid \hat{\Psi}_{N_s}) < P(H_j \mid \hat{\Psi}_{N_s}) \Rightarrow$ STOP extending $H_j \cup D_i$ ;

(ii) If $P(H_j \mid \hat{\Psi}_{N_s}) < P_{\text{max}} \Rightarrow$ STOP extending $H_j$.

If the damages are localized in a few substructures then the number of damage hypotheses that need to be examined by the B&B search is relatively small and the search becomes computationally feasible.

The generation of the load-dependent Ritz vectors starts from a factorization of the dynamic loading vector as a product of a spatial load vector by a time function:

$$
\mathbf{F}(s,t) = \mathbf{f}(s)u(t).
$$

The normalized Ritz vectors are recursively generated as follows:
\[
\tilde{r}_1 = \mathbf{K}^{-1} \mathbf{f}(s) \quad \text{and} \quad r_1 = \frac{\tilde{r}}{[\tilde{r}_1^T \mathbf{M} \tilde{r}_1]^{1/2}},
\]

\[
\mathbf{K} \tilde{r}_p = \mathbf{M} r_{p-1} \Rightarrow \tilde{r}_p = \mathbf{K}^{-1} r_{p-1}.
\]

Certainly, \( \mathbf{M} \) that occurs in Eqs. (35) & (36) is the mass matrix.

Let us close this survey on the damage detection by use of the Ritz vectors with the remark that an expression of the error term of (31) is also given in [2], involving a diagonal matrix whose relevant entries weight each error term taking into account the uncertainty of measurements and the sensitivities of the corresponding Ritz matrix entries conditional on the parameters of the damage state:

\[
S(r_{pk} \mid \Theta_{H_j}) = \frac{\Delta r_{pk}}{\Delta \Theta} = \sum \frac{\partial r_{pk}}{\partial \theta_i} \Delta \theta_i.
\]

Numerical examples related to an eight bay truss structure can be also found in [2].

**CONCLUDING REMARKS**

In this paper we have tried to go a little further with the matrix formalism involved in the research on the damage detection in structural systems due to the Mexican authors of the first reference below. Next we have examined some possibilities to adapt and extend another approach to damage detection in structures by a Bayesian methodology involving load-dependent Ritz vectors. Like for our other paper submitted to this SMiRT 19 Conference, this research has been mainly suggested by some interesting contributions to the 13 WCEE (Vancouver, August 2004). We have also approached the Bayesian inference for updating seismic fragilities and for estimating seismic vulnerability in our papers A. Vulpe & A. Carausu [6] and A. Vulpe, A. Carausu & G.E. Vulpe [7]. In the third section of our paper we have given a survey of a procedure for detecting damaged substructures in a structural system by use of the Ritz vectors and Bayesian inference. Some possibilities to extend the Bayesian methodology based on the Ritz vectors to the generic seismic fragility evaluation of components in nuclear power plants, in connection with the research reported in A. Yamaguchi, R.D. Campbell and M.K. Ravindra [8] to the SMiRT 11 Conference, have to be investigated. A synthesis approach to the Bayesian estimation of seismic fragilities is presented in Ref. [5] – a keynote to the SMiRT 16 Conference.

**REFERENCES**