STRESS DISTRIBUTION IN A CUBIC SOLID CONTAINING A PENNY-SHAPED CRACK

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ABSTRACT
An axially symmetric stress distribution inside an infinite cubic elastic solid containing a penny-shaped crack is treated. It is assumed that the load is concentrated on one internal disc located at a finite distance from the crack. An analytical solution is presented for the displacement and stress distribution and for the stress intensity factors. Closed form solutions are given for the case of a point force as well as curves numerical results, showing the influence of this type of anisotropy.

Keywords: crack cubic material intensity factor.

1. INTRODUCTION
The stress distribution in an infinite isotropic solid containing a penny-shaped crack opened by pressure applied over its surface was first considered by Sack (1946) and Sneddon (1946) and subsequently by Payne (1953), Green & Zerna (1954) and Collins (1961). More recently, Dahan (1979) considered a transversely isotropic body. This paper presents in a closed form the complete solution of the problem concerning the opening of a crack embedded in a cubic elastic medium loaded by an asymmetrical system of normal stresses acting on one side of the crack. The general solution for the isotropic medium is then obtained through a limit operation.

Green & Zerna (1954) reduce the problem of determining the stresses in an infinite elastic solid containing a penny-shaped crack opened under normal pressure, the shearing stress being zero over the crack, to a problem in potential theory mathematically identical with the problem of determining the velocity potential for a flow past a rigid disk in a perfect fluid of infinite extent. By representing this potential as a complex variable integral, they reduce its determination to an application of the known solution of Abel’s integral equation. We extend their resolution method for determining the effect of a crack on the stress distribution due to a point force acting at an interior surface of an infinite elastic cubic solid. The problem is reduced to a system of simultaneous dual integral equations involving Bessel functions of zero and unit order and a closed form solution is obtained.

2. FORMULATION OF THE PROBLEM
In the problem that we consider here, we assume that there is symmetry about the z-axis (material axis and loading axis). The position of a typical point of the solid may be expressed in terms of cylindrical coordinates \((r, \theta, z)\). For a symmetrical loading of the solid situated in the plane \(z = h\), the displacement components are \((u_r, u_\theta, u_z)\) and the only non-vanishing components of the stress tensor will be \(\sigma_r, \sigma_\theta, \sigma_z, \sigma_\tau\).
We suppose that the radius of the crack is $r_0$. It is situated in the plane $z = 0$. Let the solid be divided into three domains: (1) the half-space $\Omega_1$ for $+\infty > z \geq h$; (2) the layer $\Omega_2$ for $h \geq z \geq 0$; and (3) the half-space $\Omega_3$ for $0 \geq z > -\infty$. The center of the crack is taken to be the origin.

![Diagram of crack and axisymmetrical loading]

If $p(r)\,$ is the loading on the plane $z = 0$, the presence of body forces implies the continuity of displacements $u, u_r$ and shear stress $\sigma_{rz}$, and the discontinuity of normal stress $\sigma_{zz}$ following the condition

$$\lim_{\varepsilon \to 0^+} [\sigma_{zz}(r, h - \varepsilon) - \sigma_{zz}(r, h + \varepsilon)] = p(r)$$

where $p(r)$ is an arbitrary function so defined for $r \geq 0$ that the Hankel transform of order zero $p^H(m)$ exists

$$p^H(m) = \int_0^\infty r J_0(mr) p(r) \, dm.$$  

Over the plane $z = 0$, the stresses and displacements are continuous in the exterior of the crack ($r_0 < r < +\infty$); the boundary of which is stress-free, \( i.e. \)

$$0 \leq r < r_0 \begin{cases} \sigma_{zz}(r, 0) = 0, \\ \sigma_{rz}(r, 0) = 0. \end{cases}$$

For the remaining conditions, it is assumed that the components of stress and displacements vanish as

$$\left(r^2 + z^2\right)^{1/2} \to +\infty.$$  

Using the results given in a previous paper (Ruimy, 2002), stress and displacement distributions can be expressed as
\[ \sigma_{(\theta,z)} = \int_{0}^{\infty} \left( -Ae^{-\gamma z} + C e^{-\gamma z} \right) \delta_1 + \left( -Be^{-\gamma z} + D e^{-\gamma z} \right) \delta_2 \right] m J_s (mr) \, dm \]
\[ \sigma_{(\theta,z)} = -\int_{0}^{\infty} \left[ (Ae^{-\gamma z} + C e^{-\gamma z}) \delta_1 + (Be^{-\gamma z} + D e^{-\gamma z}) \delta_2 \right] m J_s (mr) \, dm \]
\[ u_{(\theta,z)} = \int_{0}^{\infty} \left[ (Ae^{-\gamma z} - C e^{-\gamma z}) \delta_1 + (Be^{-\gamma z} - D e^{-\gamma z}) \delta_2 \right] m J_s (mr) \, dm \]
for \( i = 1, 2, 3 \), in each domain \( \Omega \).

3. SOLUTION

If we take into account the boundary conditions on the plane \( z = h \), the condition at infinity and the continuity of stresses \( \sigma_{(\theta,z)} \) and \( \sigma_{(\theta,z)} \) over the plane \( z = 0 \), we have the following relations between the twelve functions \( A_i, B_i, C_i, D_i \)

\[ A_i (m) = B_i (m) = C_i (m) = D_i (m) = 0 \]
\[ A_2 (m) = -p^{\mu} (m) e^{-\gamma z m} / 2 m^3 \delta_s (g_1 - g_2) \]
\[ B_2 (m) = -p^{\mu} (m) e^{-\gamma z m} / 2 m^3 \delta_s (g_2 - g_1) \]
\[ C_2 (m) = A_2 (m) e^{2 \gamma z m} + C_2 (m) \]
\[ D_2 (m) = B_2 (m) e^{2 \gamma z m} + D_2 (m) \]
\[ A_3 (m) = A_2 (m) + \left( C_2 (m) (p_1 s_2 g_2 + p_2 s_1 g_1) + D_2 (m) 2 p_2 s_2 g_2 \right) \left( p_1 s_2 g_2 - p_2 s_1 g_1 \right) \]
\[ B_3 (m) = B_2 (m) + \left( D_2 (m) (p_1 s_2 g_2 + p_1 s_2 g_2) + C_2 (m) 2 p_1 s_1 g_1 \right) \left( p_2 s_1 g_1 - p_1 s_2 g_2 \right) \]
with the notations

\[ s_1 = \sqrt{\frac{-x_3 + x_7 + \sqrt{(-x_3 + x_7)^2 - 4 x_9}}{2 x_9}} \]
\[ s_2 = \sqrt{\frac{-x_3 + x_7 - \sqrt{(-x_3 + x_7)^2 - 4 x_9}}{2 x_9}} \]
\[ x_3 = -a_{12} (h_2 - h_1) / (a_{12}^2 - a_i h_1) \]
\[ x_7 = -a_{41} h_1 + a_{12} (h_2 - h_1) / (a_{12}^2 - a_i h_1) \]
\[ x_9 = (h_2^2 - h_1^2) / (a_{12}^2 - a_i h_1) \]
where \( a_{11}, a_{22}, a_{44} \) are the three elastic coefficients of cubic materials. And
\[ p_k = 1 + x_3 s_k^2 \]
\[ q_k = x_3 s_k^2 \]
\[ g_k = x_3 - x_9 s_k^2 \]
\[ q_k = s_k^2 (a_{11} x_9 + 2 a_{11} x_3) - a_{44} \]
\[ h_1 = \frac{[6 a_{11} + 2 a_{12} + a_{44} + (2 a_{11} - 2 a_{12} - a_{44}) \cos (4 \theta) ]}{8} \]
\[ h_2 = \frac{[2 a_{11} + 6 a_{12} - a_{44} - (2 a_{11} - 2 a_{12} - a_{44}) \cos (4 \theta) ]}{8} \]
The application of the continuity of the displacements \( \alpha \) in the domain \( \sigma \). The classical method are not given here. However, we obtain

The mathematical solution of each pair of simultaneous equations is not evident. The different steps of the problem is mathematically solved. By using the expressions of the twelve functions \( A, B, C, D \), it is possible to write the stress and displacement distributions in the interior of each domain \( \Omega \) and to verify the respect of the limit conditions (continuity of stress and displacement on the planes \( z = 0 \) or \( h \)).

For example, when we consider the opening of the crack by a point force of magnitude \( P \) acting at an interior point \( (0,0,h) \), \( h > 0 \), the normal stress component \( \sigma_z \), in the domain \( \Omega^3 \) (i.e. \( 0 \geq z > -\infty \)), is given as

\[
\sigma_z(r,\theta,z) = \frac{P}{\pi r^2} \left\{ \frac{r^2}{2} \left[ \frac{s_1 g_1 (h-z)}{t^2 + s_1^2 (z-h)^2} - \frac{s_2 g_2 (h-z)}{t^2 + s_2^2 (z-h)^2} \right] \right\} - \frac{P}{2 \pi (g_1 - g_2)} \left\{ \int_0^\infty \left( \frac{g_1 t}{t^2 + s_1^2 h^2} - \frac{g_2 t}{t^2 + s_2^2 h^2} \right) \right\} \times \left\{ \int_0^\infty (p_2 s_1 g_1 e^{\nu mc} - p_1 s_2 g_2 e^{\nu mc}) \sin(m t) m J_0(m r) dm \right\} dt + g_1 g_2 \left( p_1 s_2 \arctg \frac{1}{s_1 h} - p_2 s_1 \arctg \frac{1}{s_2 h} \right) \left\{ \int_0^\infty (e^{\nu mc} - e^{\nu mc}) \sin(m t) m J_0(m r) dm \right\} - s_1 s_2 g_1 g_2 \left\{ \int_0^\infty \left( \frac{p_1}{t^2 + s_1^2 h^2} - \frac{p_2}{t^2 + s_2^2 h^2} \right) \left\{ \int_0^\infty (e^{\nu mc} - e^{\nu mc}) \cos(m t) m J_0(m r) dm \right\} dt \right\} .
\]
4. NORMAL STRESS ON THE PLANE  \( z = 0 \)

If we introduce the value \( z = 0 \) in the general expression of \( \sigma_{zz} \), we obtain

\[
\sigma(r, \theta, 0) = \frac{1}{2(g_1 - g_2)} \left\{ \int_0^\infty (g_1 e^{-s_1 \alpha h} - g_2 e^{-s_2 \alpha h}) \rho^H (m) m J_0 (mr) \, dm + \right.
\]
\[
+ \frac{2}{\pi} \left[ \int_0^\infty (g_1 e^{-s_1 \alpha h} - g_2 e^{-s_2 \alpha h}) \rho^H (\alpha) \sin(\alpha r_0) \, d\alpha \right] \times \left( \int_0^\infty \cos(\alpha r_0) J_0 (mr) \, dm \right) -
\]
\[
- \frac{2}{\pi} \int_0^\infty \left[ \int_0^\infty (g_1 e^{-s_1 \alpha h} - g_2 e^{-s_2 \alpha h}) \rho^H (\alpha) \alpha \cos(\alpha r_0) \, d\alpha \right] \times \left( \int_0^\infty \cos(\alpha r_0) J_0 (mr) \, dm \right) \right) \left\}
\]

It is interesting to note that

\[
\int_0^\infty \cos(mt) J_0 (mr) \, dm = \begin{cases} 
0 & \text{if } r < t \\
\frac{1}{(r^2 - t^2)^{1/2}} & \text{if } r > t
\end{cases}
\]

So, we verify that \( \sigma_{zz} = 0 \) for \( 0 \leq r < r_0 \) and for \( r_0 \leq r < +\infty \), we have

\[
\sigma(r, \theta, 0) = \frac{1}{2(g_1 - g_2)} \left\{ \int_0^\infty (g_1 e^{-s_1 \alpha h} - g_2 e^{-s_2 \alpha h}) \rho^H (m) m J_0 (mr) \, dm + \right.
\]
\[
+ \frac{2}{\pi} \left[ \int_0^\infty (g_1 e^{-s_1 \alpha h} - g_2 e^{-s_2 \alpha h}) \rho^H (\alpha) \sin(\alpha r_0) \, d\alpha \right] -
\]
\[
- \frac{2}{\pi} \int_0^\infty \left[ \int_0^\infty (g_1 e^{-s_1 \alpha h} - g_2 e^{-s_2 \alpha h}) \rho^H (\alpha) \alpha \cos(\alpha r_0) \, d\alpha \right] \left( r^2 - t^2 \right)^{1/2} \, dt \right\}
\]

5. STRESS INTENSITY FACTOR

The first stress intensity factor is defined by the limit

\[
k_1 = \lim_{r \to r_0} \left[ \frac{2 \pi (r - r_0)}{r_0} \right]^{1/2} \sigma_{zz}(r, \theta, 0)
\]

Using one of the definitions of Bessel function

\[
J_0 (mr_0) = \frac{2}{\pi} \int_0^\infty \cos(mt)(r_0^2 - t^2)^{-1/2} \, dt
\]

we obtain for the stress intensity factor

\[
k_1 = \frac{1}{(g_1 - g_2) \sqrt{r_0}} \int_0^\infty (g_1 e^{-s_1 \alpha h} - g_2 e^{-s_2 \alpha h}) \rho^H (\alpha) \sin(\alpha r_0) \, d\alpha 
\]

In the particular case of a point force \( P \), we obtain

\[
k_1 = \frac{P}{(g_1 - g_2) \sqrt{r_0}} \left( \frac{g_1 r_0}{r_0^2 + s_1^2 h^2} - \frac{g_2 r_0}{r_0^2 + s_2^2 h^2} \right)
\]
The profile of the curve is independent of the anisotropy. It is the same for an isotropic material. When the anisotropic elastic coefficients verify the condition of isotropy, these results agree with the solution given by Kassir & Sih (1975) for an isotropic material subjected to the same loading.

Fig.2 Variation of the stress intensity factor

REFERENCES