Nonlinear fluid-structure interactions in bounded domains:

a dual formulation

Valid R.

ENS Cachan / LMT, France

I-Abstract. A new nonlinear dual variational principle is presented for fluid-structure interactions in bounded domain. This principle takes account for geometrically large displacements and finite rotations for the shell tank and for the liquid as well, the latter being supposed weighty and viscid. All the stresses of both media are "dynamically admissible".

I-Introduction The problem of fluid-structure interactions is of primary importance in reactor technology and needs nonlinear computations ordinary solved through total Lagrange-Euler (TL) or better arbitrary Lagrange-Euler (ALE) formulations. However, in bounded domains even though the Lagrange formulation seem appropriate for the structure, the Euler formulation becomes not necessary for the fluid. It seems then opportune to extend in the nonlinear case some linear principles already presented and largely applied by means of the finite element method.

In that linear case, the structure was represented through a primal formulation, i.e. by its displacements, while the fluid was described in a dual scalar way by its pressure. The added mass of the fluid was then rigorously stated and calculated. The method needed supplementary unknowns to obtaining symmetric principles numerically convenient, as well known, (see ref. 1, refs. incl.).

The same ideas has been extended to the nonlinear case, but then the formulation revealed to be much more complicated, and it appeared that a dual-dual principle allowed to avoid the extra unknowns, while keeping the symmetry property.

Although assuming, besides a complete true dual modelling again with the liquid viscosity, some new difficulties arise owing to taking account of possible complete boundary conditions, in particular in the case of a shell structure tank, as well as the free surface of the liquid.

The theory will be decomposed into 4 parts: the nonlinear dual principle for a body in three-dimension (3-D), the same for a fluid, the same for a shell, and finally the liquid-shell interaction principle.

The whole presentation will be made in an intrinsic compact form

II-The nonlinear dual functional for 3-D body. This section is a recall of the nonlinear dual principle as a guide. Let a 3-D hyperelastic body of generic point \( M \), embedded in a 3-D Euclidean space \( E_3 \), \( U \) being the displacement, \( R \) the finite rotation mapping, the reference state being indicated by an index zero, \( \frac{\partial M}{\partial M_0} \), \( \alpha \) the primal density, and \( \rho \) the mass density.

The polar decomposition of \( \frac{\partial M}{\partial M_0} \) provides, the bar meaning the transposition mapping:

\[
\frac{\partial M}{\partial M_0} = \bar{R} [R_3 + h], \quad h = \bar{R}_3 \text{(local extension)}
\]

(2.1)

Starting from the primal principle written

\[
\int_{\Omega_0} \delta \alpha - \delta \mathcal{T} = 0, \forall \delta U, \quad \alpha = \begin{bmatrix} \frac{\partial M}{\partial M_0} & \frac{\partial M}{\partial M_0} \end{bmatrix} - 1_{E_3}
\]

(2.2)
where $T_0$ is the work of external forces. we bring the constraint (2.1) into (2.2) by of a Lagrange multiplier, which is proved to be $\tau$ precisely, it comes

$$
\begin{align*}
\delta \int_{\Omega_0} \left[ \delta \alpha + T_0 \left( \tau \left[ \frac{\partial M_0}{\partial \delta M_0} - R[1_E + h] \right] \right) \right] - \delta T_0 = 0, \forall V, R, \tau \ Adm.
\end{align*}
$$

$$
\varepsilon = \frac{1}{2} \left[ [1_E + h]^2 - 1_E \right] \Rightarrow \alpha = \alpha(h), \frac{\partial \alpha}{\partial h} = \tau
$$

(2.3)

$r$ is the Jaumann stress and $\tilde{r}$ is its transposed quantity in the associated linear space $(T_0(rh) = \tilde{r}h)$.

From the Legendre-Fenchel transform, the complementary energy density writes:

$$
\beta = \beta(r) = T_0(rh) - \alpha
$$

(2.4)

Now the elimination of $h$ gives:

$$
\begin{align*}
\int_{\Omega_0} \left[ \delta T_0 \left( \tau \frac{\partial U}{\partial \delta M_0} - \tau - r \right) - \beta - [\tilde{F} - \rho \tilde{U}] \delta U \right] - \int_{\partial \Omega_0} \tilde{F} \delta U = 0
\end{align*}
$$

(2.5)

An integration by parts gives:

$$
\begin{align*}
\int_{\Omega_0} \left[ \text{div} \tau + \tilde{F} \right] \delta U + \int_{\partial \Omega_0} [\tilde{n} \tilde{u} - \tilde{F}] \delta U = 0
\end{align*}
$$

(2.6)

and (2.6), the equilibrium equations

$$
\begin{align*}
\text{div} \tau + f = \rho \tilde{U} / \Omega_0, \quad \tau = \tilde{r}
\end{align*}
$$

(2.7)

Taking account of (2.7) and differentiating twice (2.6) with respect to the time makes possible the elimination of $U$. Hence the final dual principle:

$$
\begin{align*}
\int_{\Omega_0} \left[ -\text{div} \delta \tau \frac{\text{div} \tau}{\rho_0} + \frac{\partial}{\partial t} [\delta T_0(\tau - r - \beta)] \right] + \int_{\partial \Omega_0} \tilde{n} \text{div} \delta \tau \tilde{U} - \int_{\Omega_0} \frac{\partial \delta \tau}{\partial \rho_0} f = 0
\end{align*}
$$

(2.8)

with $\beta = \beta(r), 2r = \tau R + \tilde{r}R, \tau \in H(\text{div}, \Omega_0) + \text{I.C. (initial conditions)}$

Remarks: -) A virtual variation $\delta R$ restated the equilibrium equation $\tau = \tilde{r}$

-2) the Legendre-Fenchel transform made $\beta$ convex.

III-The nonlinear dual principle for a viscous fluid. Let us consider the case of a viscous fluid without gravity at first. The total stress $\sigma_T$ will be the sum of an elastic stress $\sigma$ (born from the pressure), and a viscous stress $\sigma'$. 

226
\[ \sigma_T = \sigma + \sigma' \]  
(3.1)

The energy volume \( \sigma_T \) = density \( \alpha \) is a function of the dilatation \( \text{det} \left( \frac{\partial M}{\partial M_0} \right)^{-1} \) or equivalently

\[ \frac{\rho_0}{\rho} = \sqrt{\text{det}(K)}, \text{with } K = \frac{\partial M}{\partial M_0} \frac{\partial M}{\partial M_0}, 2\varepsilon = K - 1 \]

Then it is found

\[ \sigma = \frac{\partial \alpha}{\partial \text{det}(K)} \frac{\text{Adj}(K)}{\sqrt{\text{det}(K)}} ; K = 2\varepsilon + 1, \varepsilon = [1_E + h]^2, \alpha = \alpha(h), r = \frac{\partial \alpha}{\partial h} \]  
(3.2)

Let us suppose a linear Cauchy viscous stress \( \sigma'_C \) s.t.

\[ \sigma'_C = k' \frac{\partial U}{\partial M}, (k' \in R) \]  
(3.3)

This assumption leads to a Piola-Kirchhoff viscous stress \( \sigma' \) and energy densities \( \alpha', \beta' \) primal and complementary respectively:

\[ \begin{aligned}
\sigma' &= k' \sqrt{\text{det}(K)} K^{-1} \varepsilon K^{-1} = \frac{\partial }{\partial \varepsilon} \frac{\partial }{\partial \varepsilon} (k' \varepsilon ) , \quad (k' \in R) \\
\alpha' &= k' \sqrt{\text{det}(K)} \frac{1}{2} T_r \left( \left( K^{-1} \varepsilon \right)^2 \right), K = \frac{\partial M}{\partial M_0} \frac{\partial M}{\partial M_0} , \varepsilon = \frac{1}{2} \left[ \frac{\partial M}{\partial M_0} \frac{\partial M}{\partial M_0} + \frac{\partial M}{\partial M_0} \frac{\partial M}{\partial M_0} \right] \\
\beta' &= \beta (r') = \tau B' r' , \quad B' = \tilde{B}' , 2r' = \tau' R + \tau R 
\end{aligned} \]  
(3.4)

The same method as in section 2 gives the following dual principle:

\[ \begin{aligned}
\int_{\Omega_0} \left[ - \text{div} \left( \frac{\partial \alpha'}{\partial \varepsilon} r + f \right) - \text{div} \tau \frac{\partial }{\partial \varepsilon} \frac{\partial \varepsilon}{\partial \varepsilon} + \frac{\partial \sigma}{\partial \varepsilon} \left[ \delta T_r (\tau - r - \delta \beta - \delta \beta') \right] \right] \\
+ \int_{\Omega_0} \frac{\partial \alpha' \partial}{\partial \varepsilon} \left[ \delta U_0 + \delta r' \tilde{U}_0 \right] = 0 \quad + I.C., \tau, r' \in H (\text{div}, \Omega_0), \tilde{R} R = R R = 1_E \\
\text{with } \delta \beta = \frac{\partial \sigma}{\partial \varepsilon} (\delta r' R + \tau' \delta R) 
\end{aligned} \]  
(3.5)

IV-The nonlinear principle for a shell . Let us recall here briefly the nonlinear dual principle for a shell structure modelled as a surface body. The middle surface \( \Sigma \) is supposed embedded in the 3-D linear Euclidean space \( E_3 \). Let \( m \) be the generic point of \( \Sigma \), \( E_2 \) the tangent plane of \( \Sigma \) at \( m \), and \( N \) the unit normal to \( \Sigma \) at \( m \). The middle surface is supposed endowed with a Riemannian connection by an orthogonal projector field \( \pi = \pi \).

In the case of a hyperelastic shell, the primal energy density \( \alpha \) is a function of the nonlinear deformations \( \Gamma, K, H_S \), the membrane, curvature and shear strain variations resp.

\[ \begin{aligned}
\Gamma &= \left[ \frac{\partial m}{\partial m_0} \frac{\partial m}{\partial m_0} - 1 \right] , K = \frac{\partial m}{\partial m_0} \frac{\partial N}{\partial m_0} \frac{\partial m}{\partial m_0} - \frac{\partial N}{\partial m_0} , H_S = \frac{\partial m}{\partial m_0} N - N_0 \\
\alpha &= \alpha(\Gamma, K, H_S) 
\end{aligned} \]  
(4.1)

227
From a surface polar decomposition \( \frac{\partial m}{\partial m_0} \), it comes:

\[
\frac{\partial m}{\partial m_0} = R\pi [1 - h_m] , \text{ with } h_m = \overline{h}_m \in L_2 \left( E_3, E_3 \right)
\]

\( R\overline{R} = \overline{R}R = 1_{E_3} \)

(4.2)

Following this line, we adopt the new nonlinear surface deformations

\[
\begin{align*}
\overline{R}\pi_0 \frac{\partial m}{\partial m_0} & \overset{\text{def}}{=} 1 + h_m \\
\overline{R}\pi_0 \frac{\partial N}{\partial m_0} & \overset{\text{def}}{=} h_f \\
\overline{R}\pi_0 N & \overset{\text{def}}{=} h_S \\
2\Gamma & = [1 + h_m]^2 - 1 \\
K & = [1 + h_m] h_f - \frac{g_{N_0}}{\partial m_0} \\
H_S & = [1 + h_m] h_{S} - N_0
\end{align*}
\]

(4.3)

s.t.

\[
\begin{align*}
\alpha & = \alpha_0 (\Gamma, K, H_S) = \alpha_0 (h) , \frac{\partial \alpha}{\partial h} = r = \left[ \begin{array}{c}
\tau_m \\
\tau_f \\
\tau_S
\end{array} \right] \\
\beta & = T_r (r h) - \alpha = \beta_0 (r) , \frac{\partial \beta}{\partial r} = h = \left[ \begin{array}{c}
h_m \\
h_f \\
h_S
\end{array} \right]
\end{align*}
\]

(4.4)

As before in the preceding sections, we introduce the constraints (4.3) into the primal principle with the Lagrange multipliers \( \tau_m, \tau_f, \tau_S \) resp. Elimination of the new deformations, by use of the complementary energy \( \beta \), allows to find the following mixed principle:

\[
\begin{align*}
& \left[ \delta \int_{\Sigma_0} T_r \left( \tau_m [1 - R\pi_0] + \tau_m \frac{\partial V}{\partial m_0} + \tau_f \frac{\partial N}{\partial m_0} + \tau_S N \right) - \beta \right] - \delta T_e = 0 \\
& \forall \left[ V, N, R, \tau_m, \tau_f, \tau_S \right] , Adm., \left( R \overline{R} = \overline{R}R = 1_{E_3} \right)
\end{align*}
\]

with \( \tau_m = \overline{\tau}_m = \tau_m R\pi_0 , \tau_f = \tau_f R\pi_0 , \tau_S = \tau_S R\pi_0 \)

(4.5)

Remarks - 1) The multipliers \( \tau_m, \tau_f, \tau_S \), are precisely the Piola-Lagrange stresses.

- 2) In the Kirchhoff-Love hypotheses, the deformation \( h_S = 0 \), which is a necessary and sufficient condition, may be incidentally regularized.

- 3) The main point of the theory was to adopt \( R \) and \( N \) as independent but coupled unknowns, the unknown \( N \) being ever normal to the deformed middle surface, due to the starting Euler formulation.

Owing to the same method applied in the preceding section, we find a dual variational principle for a nonlinear shell. By use of our compact following notations, it leads to the problem:

\[
\begin{align*}
\text{Find } \tau, R \in T \times \mathcal{R} \text{ s.t.}
& \int_{\Sigma_0} \left[ \delta G T^{-1} [G + f] + \int_{\Sigma_0} \frac{\partial^2}{\partial h^2} [\delta T_r (\tau_m - \tau_m) - \delta \beta] \right] + \int_{\partial \Sigma} \delta \nu_0 \left[ \tau_m \overline{V}_0 + \tau_f \overline{N}_0 \right] = 0, \forall \left[ \tau_m, \tau_f, \tau_S \right] \text{ Adm., } \tau_m = \overline{\tau}_m + I.C.
\end{align*}
\]

\( \nu_0 \text{ outward normal to } \partial \Sigma_0 \)

(4.6)

Remark: The equilibrium bidimensional equations which have been utilized are then for the...
shell:

\[
G + f = I \tilde{U}/\Sigma_0, \quad G = \left[ \frac{\overline{d\sigma_m}}{d\tau_f - \tau_S} \right], \quad U = \begin{bmatrix} V \\ N \end{bmatrix}, \quad \left\{ \begin{array}{l}
\overline{\nu_0} \tau_m - \overline{F}_m = 0/\partial \Sigma_0 F \\
\overline{\nu_0} \tau_f - \overline{F}_f = 0/\partial \Sigma_0 F
\end{array} \right.
\]

(4.7)

with the functional spaces:

\[
X = \tau \in \left\{ H(G, \Sigma_0), (\tau_1, \tau_2) = \int_{\Sigma_0} \left[ G_1 I^{-1} G_2 + \tau_1 B r_2 \right], \tau_1 B r_2 = \frac{\partial^2 \delta}{\partial \nu \partial \nu} (\tau) \right\} \\
T = \left\{ \tau \in X, G = 0, \nu_0 [\tau_m \cdot \tau_f] = 0 \quad \text{on} \quad \partial \Sigma_0 F \right\} \\
T_0 = T \setminus \left\{ \tau \in X, \nu_0 [\tau_m \cdot \tau_f] = \overline{F} \quad \text{on} \quad \partial \Sigma_0 F, \quad (\delta \tau, \tau) = 0 \right\} \\
\mathcal{R} = \left\{ R \in [L^2(\Sigma_0)]^g, \quad R(\overline{R}) = \overline{R} = 1_{\Sigma_0} \right\}
\]

(4.8)

V. The liquid-shell interaction principle. We consider now a tank modelled as a shell whose middle surface is \( \Sigma_0 \), which contains a liquid occupying a domain \( \Omega_0 \). The resulting variational principle is found by the association of the two preceding principles with appropriate linking conditions. But now the shell is decomposed into \( \Gamma_w \) the wet part and \( \Gamma_f \) the free surface, \( \Gamma_w, \Gamma_f \), both corresponding to the liquid boundary \( \partial \Sigma_0 \). Then

\[
\partial \Sigma_0 = \Gamma_w + \Gamma_f, \quad \Sigma_0 = \Sigma_0 U + \Gamma_w, \quad G = \left[ \frac{\overline{d\sigma_m}}{d\tau_f - \tau_S} \right], \quad U = \begin{bmatrix} V \\ N \end{bmatrix}
\]

(5.1)

We have called \( I \) the inertia matrix associated to the variable \( U \).

A new variable \( \eta \) is introduced for the vertical displacement of the liquid on the free surface \( \Gamma_f \). We call \( V_f \) a Lagrange multiplier whose aim is to equalizing the stresses on the common boundary \( \Gamma_w \). It is supposed also that a displacement set \( U_g \) could be possibly given on the dry part \( \Sigma_0 U - \Gamma_w \) of \( \Sigma_0 \).

Hence the liquid-shell interaction principle comes out, with the preceding notations, \( g \) being the gravity acceleration:

\[
\begin{align*}
\int_{\Sigma_0} \left[ - [G + \overline{F}] I^{-1} \delta G \right] + \int_{\Sigma_0} \left[ \frac{\partial^2 \delta}{\partial \nu \partial \nu} \left[ T_f (\tau_m - \tau_m) - \beta \right] \right] + \int_{\partial \Sigma_0} \delta \nu_0 \left[ \tau_m \dot{V}_g + \tau_f \dot{N}_g \right] &+ \int_{\Sigma_0 U - \Gamma_w} \delta \overline{G} \dot{U}_g \\
&+ \int_{\Omega_0} \left[ - \text{div}_\tau \frac{\text{div}r + \tau_f}{\rho_0} - \text{div}_\tau \frac{\text{div}r + \tau_m}{\rho_0} + \frac{\partial^2 \delta}{\partial \nu \partial \nu} \left[ T_f (\tau_f - \beta) - \beta_\beta \right] \right] \\
&- \int_{\Omega_0} \text{div}_\tau \underbrace{\frac{\text{div}r_{\rho_0}}{\rho_0} + \text{div}_\tau \tau_{\rho_0}}_{\text{I}} \\
&+ \delta \int_{\Gamma_f} \left[ \frac{\overline{\nu_0} \tau + \tau_f}{\overline{m}} \right] \dot{V}_f - \int_{\Gamma_f} \left[ \frac{\overline{\nu_0} \delta \tau r n_0}{\overline{m}} + \overline{\nu_0} \delta \tau r n_0 \right] \\
&- \int_{\Gamma_w} \left[ \frac{\overline{\nu_0} \tau + \tau_f}{\overline{m}} n_0 - \rho_0 \rho_0 \right] \delta \eta = 0, \\
\tau_{\nu} = \overline{m}, \\
\forall \left[ \tau_m, \tau_f, \tau_S, \tau, \tau_f, \tau = \overline{r}, \quad \tau' = \overline{r'} \right] &\quad \text{Adm.} + I.C.
\end{align*}
\]

(5.2)

Remarks.

1) The equilibrium equations in terms of Jaumann stresses \( \tau = \overline{r}, \quad \tau' = \overline{r'} \), \( \tau_m = \overline{m} \),
which are incidentally simply connected with the Piola-Lagrange stresses through the respective finite rotations, replace the constraint $\bar{R} = \bar{R} = 1_{\bar{R}}$ for the shell and the liquid as well. If these relations are not considered, the corresponding equivalent equations are taken into account by the principle from a virtual variation of $\delta \bar{R}$ in the different domains.

2) In (5.2), the different unknowns affecting the shell or the liquid are clearly specified by the integration domains.

VI-C: Conclusion. The preceding principle presented in an intrinsic compact form derives from a functional, then needs no supplementary unknowns for exhibiting symmetry. Moreover, it preserves the dynamic equilibrium, a fact which could be important for accurate stability conditions.

It is worth noting that the two unavoidable primal unknowns appeared in terms of $V_1$ and $\eta$.

The added mass, a very useful quantity in optimization problems, would be given by a double integration with respect to the time in terms of the liquid unknowns as

$$\mathcal{M} = \int_{V_0} \frac{1}{2} \rho_0 |\mathbf{U}|^2 = \int_{V_0} \frac{1}{2 \rho_0} |\text{div} \left[ \mathbf{\tau}^* + \mathbf{\tau}'^* \right]|^2 , \mathbf{\tau} = -\mathbf{\tau}^* , \mathbf{\tau}' = -\mathbf{\tau}'^* \}$$

(5.3)

(see ref. 1)

Let us remark also that, in the same way as in the linear case, a complete linearization of (5.2) would allow for a modal synthesis method, convenient for nonlinear transient computations in some frequency ranges. Independently of this possibility, many appropriate developments may be envisaged.

Finally, let us emphasize that all the preceding formulae can be easily developed in a full conventional writing as exposed in ref. 1, for numerical utilizations.

---

References
