Vibration and wave localization within elastic constructions

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The phenomenon of vibration and wave localization within beams, plates and shells has been studied using linear and non-linear approach. Using linear approach, wave localization within FINITE constructions with inclusions (mass, frames, etc) can be explained by the phenomenon of trapped modes. The trapped modes have been studied by Ursell F. McEver, Evans and Kuznetchov. They have described the phenomenon of standing wave formation on the surface of ideal non-compressible liquid. Vorovich and Babeshko have studied vibration of a mass die on an elastic strip and shown the existence of the low frequency resonance and the possibility of high frequency resonance occurrence. The discrete frequency displacement to the continuous spectrum axis has been obtained approximately, and the conditions of the mixed spectrum formation are not associated with the parameters of the elastic strip. The trapped mode phenomenon within finite length constructions can be explained by the existence of a real discrete in parallel with a continuous spectrum. The existence and location of the real discrete spectrum depend on the type of a differential operator, inclusion parameters and their mutual arrangement.

Necessary and sufficient conditions for low frequency trapped modes are: 1. The continuous spectrum of a boundary value problem for an elastic construction without inclusion must be shifted from zero. 2. An inclusion which contacts with an elastic construction must possess a mass element.

Conditions of high frequency trapped modes initiation can not be stated as exactly as conditions of low-frequency trapped modes. It is reasonably safe to suggest that if the Green function of a waveguide (a construction) above the first cut-off frequency does not contain terms determining standing waves, the high frequency trapped modes are absent. Even if the construction has not got geometrical symmetry, the trapped
modes phenomenon can be obtained. In this case the vibration localization can be got by special set of inclusion parameters (mass, rigidity, distance between inclusions). The nonlinear approach has been used to explain the wave localization in the elastics construction.

Consider the formation of high-frequency trapping modes of oscillation in a number of infinite length systems. A prerequisite to the formation of such modes is the fact that a discrete spectrum of natural frequencies of oscillation goes out beyond the first boundary frequency on the continuous spectrum axis. For discrete values of frequencies to be found we set exacting boundary condition, not only radiation. Considering that the trapping modes of oscillation possess finite energy, a solution should be found in the class of functions \( f(x) \), which ensure the integrals \( \int_{-\infty}^{+\infty} f^2(x)dx \) and \( \int_{-\infty}^{+\infty} f^2(x)dx \) to be finite (for example in the case of a string). It can be shown that the last mentioned condition is tantamount to a search for a solution that does not contain waves at infinity. It is the requirement that is necessary for solving the succeeding problems.

**Axisymmetric oscillation of a cylindrical shell supported by two circular frames**

It has been shown that the BE-beams on an elastic foundation with two mass spring inclusions may possess a mixed spectrum. A discrete spectrum of the beam is located either below a boundary frequency or above it. It depends on the parameters \( c \) and \( M \). In the first case the expression

\[
c - M \omega^2 = \frac{Da^3}{2^{1/2} e^{-a^{21/2}} \sin(\pi/4 + a^{21/2}/2)} ; 1
\]

determines two natural frequencies \( \omega_n \) at given parameters \( c \) and \( m \). In the second case the corresponding spectrum of the parameters \( c_n, M_n, n = 1, 2, \ldots \) can be obtained at found trapping frequencies \( \omega_n \) and these frequencies become natural. This part of the article shows that expression (??) is a good approximation to mass and spring characteristics of the circular frames supporting the cylindrical shell. The free axisymmetric oscillation of the shell are analysed.

Trapping frequencies can be found whereby no transverse and longitudinal vibration waves take place, from a radiation condition (lack of
radiation at infinity). If the shell is subjected to a radial load harmonic in time, a resonance takes place at tuning.

Longitudinal vibration becomes substantial in this case.

The equations of steady-state oscillation of the shell with the frames have the form:

\[
\begin{align*}
\frac{\mu_0}{r} w_x + \frac{\omega^2}{c_0^2} u &= R_{1x} \delta(x) + R_{2x} \delta(x-l) \\
\frac{h^2}{12} w_{xxx} - \frac{\mu_0}{r} u_x + \left(\frac{1}{r^2} - \frac{\omega^2}{c_0^2}\right) w &= R_{1r} \delta(x) + R_{2r} \delta(x-l)
\end{align*}
\]

where \( r, h \) — radius and thickness of the shell; \( l \) — lengths between frames; \( R_{ix} \) — frame forces; \( \rho_f \) — frame density; \( F_f \) — area of the cross-section; \( E_f \) — Young’s modulus of frame;

\[
R_{1x} = \frac{\omega^2 \rho_f}{c_0^2 \rho_0} \frac{1}{1 - \mu^2} \frac{F_f}{h} u_0, \quad R_{2x} = -\frac{\omega^2 \rho_f}{c_0^2 \rho_0} \frac{1}{1 - \mu^2} \frac{F_f}{h} u_1,
\]

\[
R_{1r} = \left(\frac{\omega^2 \rho_f}{c_0^2 \rho_0} \frac{1}{1 - \mu^2} \frac{F_f}{h} - \frac{1}{r^2} \frac{F_f}{E_0}\right) w_0, \quad R_{2r} = \left(\frac{\omega^2 \rho_f}{c_0^2 \rho_0} \frac{1}{1 - \mu^2} \frac{F_f}{h} - \frac{1}{r^2} \frac{F_f}{E_0}\right) w_1.
\]

It is necessary to supplement the equation of motion by the known radiation condition at infinity. If the frames do not present, the starting problem has only a continuous spectrum with corresponding travelling longitudinal and transverse waves of vibration.

Dispersion dependence has the form:

\[
\Omega_{1,2}^2 = \frac{\omega^2}{c_0^2} = \frac{1}{2} \left[1 + (\alpha r)^2 + (\alpha r)^4 \frac{h^2}{12} \pm \sqrt{4(\alpha r)^2 \mu_0^2 + (1 - (\alpha r)^2 + (\alpha r)^4 \frac{h^2}{12})^2}\right]
\]

A longitudinal wave is defined by the dispersion dependence \( \Omega_{1}(\alpha) \). A bending wave of deformation in the shell is defined by \( \Omega_{2}(\alpha) \). The continuous spectrum of the shell occupies the whole axis \( 0 < \Omega < \infty \) as differentiated from the BE-beam. In this case occurrence of discrete natural frequencies would mean high-frequency trapping modes appearance. Using Fourier transform along coordinate \( x \) and solving the resulting algebraical equation, one has:

\[
\text{(2)}
\]

where

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\[ D(\alpha, \omega) = \left( \frac{\omega^2}{c_0^2 - \alpha^2} \right) \left( \frac{1}{r^2} - \frac{\omega^2}{c_0^2} + \frac{\alpha^4 h^2}{12} \right) + \frac{\alpha^2 \mu_0^2}{r^2} \]

\[ \kappa_1 = \frac{\rho_f F_f}{\rho_0 h}, \quad \kappa_2 = \frac{E_f F_f}{E_0 h} \]

We will find the discrete spectrum in the region of the frequencies \( 0 < \omega < c_0/r \), that is \( 0 < \Omega < 1 \).

Here, the dispersion equation is a denominator of (??) and has two real roots \( z_{12} = \pm \cdot \cdot \cdot (\omega) \) and four complex roots \( z_{3,4,5,6} = \pm a(\omega) \pm b(\omega) \), where \( a(\omega), b(\omega) \) are positive values.

A wave solution corresponding to the real roots must be equal to zero. When the discrete spectrum takes place, it means that the following equations are correct:

\[
\begin{align*}
\left( \frac{\omega^2}{c_0^2 - \alpha^2} \right) F_w + \frac{i \alpha \mu_0 P_w}{r} &= 0 \\
\frac{i \alpha \mu_0 P_w}{r} + \left( \frac{1}{r^2} - \frac{\omega^2}{c_0^2} + \frac{\alpha^4 h^2}{12} \right) P_u &= 0
\end{align*}
\]

here

\[
\begin{align*}
P_w &= \left( \frac{\alpha \omega^2}{c_0^2} - \frac{\alpha^2}{r^2} \right) \left( w_0 + w_l e^{-i\alpha l} \right) \\
P_u &= -\frac{\kappa_1 \omega^2}{c_0^2} \left( u_0 + u_l e^{-i\alpha l} \right)
\end{align*}
\]

It is obvious that the equation is valid when \( P_u = P_w = 0 \). We set

\[
\begin{align*}
w_0 + w_l e^{-i\alpha l} &= 0 \\
u_0 + u_l e^{-i\alpha l} &= 0
\end{align*}
\]

It means that the following relations are correct:

\[
\alpha_* = \frac{\pi n}{l}, \quad w_0 + w_l(-1)^n = 0, \quad u_0 + u_l(-1)^n = 0 \quad (3)
\]

The first relationship in (??) is the equation for determining the trapping frequencies. It has the form:

\[
\frac{\omega_n^2 r^2}{c_0^2} = \frac{1}{2} \left[ \frac{h^2 r^2 \pi^4 n^4}{12 l^2} + \frac{r^2}{l^2} \pi^2 n^2 + 1 - \left( \frac{h^2 r^2 \pi^4 n^4}{12 l^4} - \frac{r^2}{l^2} \pi^2 n^2 + 1 \right) + 4 \mu_0^2 \pi^2 n^2 \frac{r^2}{l^2} \right]
\]

\[ n = 1, 2, \ldots, n_b. \]
The value of $n_b$ is defined by the equation

$$
n_b^2 = \frac{1}{\pi^2} \left[ \left( 1 + \sqrt{1 + \frac{48 \mu_0^2 r^2}{\hbar^2}} \right) \frac{l^2}{2 r^2} \right]$$

The quadratic brackets are integer part of the term. The trapping frequencies defined by the equation are natural, if we can find parameters $\kappa_1$ and $\kappa_2$ such that the following simultaneous relationships are correct:

$$u(0) = u_0, \quad u(l) = u_l,$$

$$w(0) = w_0, \quad w(l) = w_l.$$

They have the form:

$$\begin{align*}
\begin{cases}
u_0 \left[ 1 - F_1(0) - (-1)^{n+1} F_1(l) \right] - w_0 \left[ F_2(0) + (-1)^{n+1} F_2(l) \right] = 0 \\
u_0 \left[ F_4(l) + (-1)^{n+1} F_4(0) \right] + w_0 \left[ (-1)^{n+1} (1 - F_3(0)) - F_3(l) \right] = 0
\end{cases}
\end{align*}$$

(4)

where

$$F_1(x) = -\frac{\kappa_1 \omega^2}{c_0^2} \frac{1}{2 \pi} \int_{-\infty}^{+\infty} \left( \frac{1}{r^2} - \frac{\omega^2}{c_0^2} + \alpha^4 \frac{\hbar^2}{12} \right) \frac{e^{i\alpha x}}{D(\alpha, \omega)} d\alpha$$

$$F_2(x) = \frac{1}{2 \pi^2} \mu_0 \frac{\kappa_1 \omega^2 - \kappa_2}{c_0^2 - r^2} \int_{-\infty}^{+\infty} \alpha \frac{e^{i\alpha x}}{D(\alpha, \omega)} d\alpha$$

$$F_3(x) = \frac{1}{2 \pi} \left( \frac{\kappa_1 \omega^2}{c_0^2} - \frac{\kappa_2}{r^2} \right) \int_{-\infty}^{+\infty} \left( \frac{\omega^2}{c_0^2} - \alpha^2 \right) \frac{e^{i\alpha x}}{D(\alpha, \omega)} d\alpha$$

$$F_4(x) = -\frac{1}{2 \pi^2} \mu_0 \kappa_1 \omega^2 \int_{-\infty}^{+\infty} \alpha \frac{e^{i\alpha x}}{D(\alpha, \omega)} d\alpha$$

$n = 1, 2, \ldots, n_b$

Expanding the determinant of the system (4), and retaining the main terms (in an effort to simplify we consider the case when $l/2 > 1$) one obtains:
\[
\left( \kappa_1 \frac{\omega_n^2}{c_0^2} - \kappa_2 \frac{1}{r^2} \right) \frac{3}{4} \frac{r^3}{h^2 a_0^3} = 1,
\]

\[
a_0 = \frac{\sqrt{2}}{2} \sqrt{\frac{1 - \Omega^2}{12(h/r)^2}}, \quad \Omega^2 < 1.
\]

By this means a spectrum of the parameters \(\kappa_{1n}\) and \(\kappa_{2n}\) is defined by the following expression:

\[
\kappa_{1n} \frac{\omega_n^2}{c_0^2} - \kappa_{2n} \frac{1}{r^2} = \frac{4}{3} \frac{h^2 a_0^3}{r^3}, \quad n = 1, 2, \ldots, n_b.
\]

It can be proved that the right part of the equation identically agrees with the expression \(\frac{1}{G_B(0)}\) where \(G_B(0)\) — the value of the Green function of the BE-beams.

The aim of this part of the paper is to describe some new physical effects which (as it has been shown in preliminary investigations) could appear in the interaction between elastic solid of a complicated form and flows of fluid. These basic effects can be described as follows:

a) a possibility of existence of localized in space time periodical modes (quasimodes) (in linear models);

b) a weak delocalization of these modes (as a result of resonances in nonlinear models);

c) a possibility of existence of coherent chaotic and quasiperiodic interaction of these modes;

In connection with c), we developed the theory of control of these coherent structures. By adjusting the elastic system parameters one can control the properties of localized modes and their time behaviour.

The existence of localized modes leads to sharp gradients of additional vibration fluid pressure which acts on the elastic solid (construction). In turn, it could leads to the destruction of the construction. The other aim is to know where maxima of this pressure are localized, in order to reinforce the construction.

**Mathematical model**

From abstract point of view, mathematical model has the form (we suppose that nonlinearities are small)

\[
Lu + \epsilon f(u) = 0,
\]

(6)
where \( u \) is a state of system, \( L \) is a linear operator of hyperbolic type (possibly, of very complicated form), \( f \) is a nonlinear operator. The main effects can be cleared even from simplest basic models for example the model of an elastic beam (with inclusions) contacting with the fluid

\[
Du_{xxxx} + Ku + mw u + V_1(x)u + V_2(x)u^3 = P, \quad P = \rho \phi_i(y = 0),
\]

\[
c_0^2 \Delta \phi = \phi_{tt}, \quad \frac{\partial \phi}{\partial y} = u_t, \quad y = 0.
\]

A mathematical analysis shows that there are possible effects \( a, b \) and \( c \) for potentials \( V_i \) of a singular form.

We are going to use different methods to resolve problems (1). First we will find localized modes and quasimodes for linear equation (\( \epsilon = 0 \)) following classical approaches in particular developed in Russia (V. P. Maslov, V. M. Babich et al.). The second step is an investigation of the dynamics in nonlinear models (6). It allows to consider the points \( b, c \) and it can be done by two ways: by traditional perturbation theory and by methods of infinite-dimensional KAM theory (namely, there exists a sequence of formal canonical transformations which transform the nonlinear term in (6) into terms of order \( \epsilon^2, \epsilon^3 \) etc. Thus asymptotically eq. (6) can be solved with the arbitrary prescribed accuracy).

As for point \( c \), for dissipative, authors developed mathematically rigorous analytic theory for dissipative systems where the existence of coherent chaotic structures consisting of localized modes has been proved and also it has shown that one can control the inertial form (and thus attractor of system). These results allow analytically to solve some important problems for dissipative systems (in particular, the Ruelle problem on existence of strange attractors). It is shown, for reaction diffusion systems, that by localized modes we can generate all structurally stable kinds of dissipative chaotic time behaviour and that we can control the attractor form by the system parameters.

Developed approaches can be applied also to conservative systems. It should solve problem \( c \) and create an algorithm control of coherent structures and their time behaviour in systems (6). We suppose that, for (6), also one can obtain all type of hamiltonian chaos and that one can control the chaos form adjusting the system parameters.

Although algorithm holds on analytic ideas however practical realization in actual situation is possible probably only by powerful computer methods (wavelets).