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A new approach to the nonlinear vibration analysis of plates and shells of arbitrary shapes

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ABSTRACT : A new approach to the nonlinear analysis of vibration of plates and shells has been proposed in this paper. The mathematics of obtaining the governing differential equations for plate and shell problems in conjunction with the concept of 'Lines of Equal Deflections' has been developed for moderately large vibrations of structures and the process of derivation has been made simplified.

1 INTRODUCTION

Investigations related to nonlinear vibration of plates and shells are few in comparison with those for linear cases. This, probably, is due to the difficulties involved in solving the nonlinear partial differential equations of fourth order associated with such problems. Yet, when the middle plane of a plate exhibits large deflections of the order of the magnitude of the thickness, the nonlinear analysis becomes essential for the study of the frequency characteristics. As the deflection function and the stress function are in a coupled form in all types of analyses generally used, their solutions are very hard to be obtained. To get rid of such difficulties various computational methods are being developed and at the same time attempts are being made to find ways in restructuring the governing equations in a bit simpler form. The method of 'Lines of equal Deflections' or, 'Constant Deflection Contour' method (CDC) is one of such ways which may be applied the computational hazzards effectively in the study of nonlinear behaviour of structures subjected to moderately large vibrations.

The concept of CDC has earlier been introduced and developed by V. D. Kluish nikov [1] and J. Mazumdar [2-4]. The application of this method has so far been mainly restricted to linear cases only. It appears from a literature study that no researcher has ever applied this method for nonlinear vibration analysis. Very recently the present author has been able to use this method to study the large deflection static analysis of plates [5].

In this paper the CDC method has been introduced and a new approach has been proposed to obtain the basic equations governing the motion of plate and shell structures. It has been achieved that the dynamic behaviour of such structures does not necessarily depend on solving fourth order partial differential equations but it may be reduced to that of a set of third order ordinary differential equations.

2 CONSTANT DEFLECTION CONTOUR (CDC) METHOD

Let an elastic isotropic shallow shell be considered. The equation of its middle surface referred to a system of orthogonal coordinates (x,y,z) will be given by

$$z = (x^2/2R_1) + (xy/R_{12}) + (y^2/2R_2) \qquad (1)$$

For a shallow shell $r = (x^2 + y^2)^{\frac{1}{2}}$ is small compared to the radii of curvature R_1, R_2 and R_{12} (considered to be constants) everywhere in in the region.

When the shell vibrates in a normal mode, then at any instant 't', the intersection between the deflected surface and the parallels z = constant yield contours which after projection onto the z = 0 surface are the level curves called the "Lines of Equal Deflections". The family of curves may be represented by u(x,y) = const. [4]. If the boundary of the shell be subjected to any kind of support, then clearly the boundary, assumed to be a simple closed curve 'C', will belong to the family of lines of equal deflections and, without any loss of generality, one may consider u = 0 on the boundary, elsewhere, u = u(x,y) = constant. Jones et al [4] have treated the problem of transverse vibrations of shallow shells where the details of deduction of governing differential equations have been given. Jones et al [4] designated the curve $C = C_o$ as the boundary of the shell and $C = C_u$ for the rest of the family of curves.

Mazumdar and others [2-4] started from the application of D'Alembert's principle and summing the forces in the direction normal to the surface for derivation of the basic equation which involves too much of mathematical calculations. Here, the author of the present paper proposes a new approo approach which needs less labour and, perhaps, opens up a new avenue for nonlinear studies for structures of arbitrary shapes.

3 THE PROPOSED ANALYSIS

Instead of deriving the equations of motion as followed by Mazumdar et al [4] an attempt has been made to evaluate them directly from the wellknown and established equations intrducing the concept of lines of equal deflection. For brevity, the well established governing equations for shell vibrations may be written as

$$D\,\nabla^4 w = h\,S(w,\phi) - K_1\,h\,\phi_{yy} - K_2\,h\,\phi_{xx} - \rho h\,w_{tt} + q \qquad (2)$$

$$\nabla^4\phi = (E/2)\,S(w,w) - K_1 w_{yy} - K_2\,w_{xx} \qquad (3)$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, w is the deflection, h = the thickness, ϕ is the Airy stress function, K_1 and K_2 are the principal curvatures, is the density of the material, E is Young's modulus, q is the uniform load, if there be any, while $\nabla^2 = /x^2 + /y^2$ denotes the two-dimensional Laplacian operator and the operator 'S' stands for

$$S(m,n) = [\,m_{xx}\,n_{yy} - 2\,m_{xy}\,n_{xy} + m_{yy}\,n_{xx}]$$

and $w_x = \partial w/\partial x$, etc.

For corresponding plate problems K_1 and K_2 will assume zero values.

On assumption that there exists a function $u = u(x,y)$ with continuous partial derivatives and which defines the lines of equal deflection by the relation $u = u(x,y) =$ constant, one can make use of the following transformations

$$\partial w/\partial x = (dw/du)\, u_x , \quad \partial^2 w/\partial x^2 = (dw/du)\, u_{xx} + (d^2w/du^2) u_x^2$$

$$\partial^2 w/\partial x\, \partial y = (d^2w/du^2)\, u_x\, u_y + (dw/du)\, u_{xy} , \quad \text{etc.}$$

With the transformation after substituting the above values, the surface integrals are taken when equations (2) and (3) will reduce to

$$\begin{aligned} D \iint_\Omega \Big[(d^4w/du^4)\, A_1 + (d^3w/du^3)\, A_2 + d^2w/du^2\, A_3 + (dw/du)\, A_4\Big] d\Omega \\ = h \iint_\Omega \Big[(d/du)(dw/du).\, d\phi/du\,)\, A_5 - 2\,(dw/du\,.\, d\phi/du)\, A_6\Big] d\Omega \\ - K_1 h \iint_\Omega \Big[(d^2\phi/du^2)\, A_7\Big] d\Omega - K_2 h \iint_\Omega \Big[(d\phi/du)\, A_8\Big] d\Omega \\ - K_2 h \iint_\Omega \Big[(d^2\phi/du^2)\, A_9 + (d\phi/du)\, A_{10}\Big] d\Omega \\ - h \iint_\Omega w_{tt}\, d\Omega + \iint_\Omega q\, d\Omega \end{aligned} \tag{4}$$

$$\begin{aligned} \iint \Big[(d^4\phi/du^4)\, A_1 + (d^3\phi/du^3)\, A_2 + (d^2\phi/du^2)\, A_3 + (d\phi/du)\, A_4\Big] d\Omega \\ = (E/2) \iint_\Omega \Big[d/du\,.\,(dw/du)^2\, A_5 - 2\,(dw/du)^2\, A_6\Big] d\Omega \\ - K_1 \iint_\Omega \Big[(d^2w/du^2)\, A_7 + (dw/du)\, A_8\Big] d\Omega \\ - K_2 \iint_\Omega \Big[(d^2w/du^2)\, A_9 + (dw/du)\, A_{10}\Big] d\Omega \end{aligned} \tag{5}$$

Where, $A_1 = (u_x^2 + u_y^2)^2$

$$A_2 = 6\,(u_x^2\, u_{xx} + u_y^2\, u_{yy}) + 8\, u_x\, u_y\, u_{xy} + 2\,(u_x^2\, u_{yy} + u_y^2\, u_{xx})$$

$$A_3 = 3\,(u_{xx}^2 + u_{yy}^2) + 4\,(u_x\, u_{xxx} + u_y\, u_{yyy}) + 4\, u_{xy}^2 + 4\,(u_x\, u_{xyy} + u_y\, u_{xxy}) + 2\, u_{xx}\, u_{yy}$$

$$A_4 = u_{xxxx} + u_{yyyy} + 2\, u_{xxyy}$$

$$A_5 = u_x^2\, u_{yy} - 2\, u_x\, u_y\, u_{xy} + u_y^2\, u_{xx} \qquad A_6 = u_{xy}^2 - u_{xx}\, u_{yy}$$

$$A_7 = u_y^2 , \quad A_8 = u_{yy} , \quad A_9 = u_x^2 , \quad A_{10} = u_{xx} \tag{6}$$

While performing the above integrals one should bear in mind that the surface is bounded by the contour lines $u = 1$ and $u = u(x,y) =$ constant. The double integrals may be performed by the use of Green's theorem

wherever possible to transform them into line integrals or one may simply apply the rule

$$\iint_{\Omega} f(u)\, g(x,y)\, d\Omega = -\int_{1}^{u} f(u) \cdot \oint_{C_u} \frac{1}{p^{\frac{1}{2}}}\, ds \tag{7}$$

where $p = u_x^2 + u_y^2$ and 'ds' is an elementary arc of the closed contour. It has been verified that the results are same for both the cases.

Once the equation of the contour lines is known the integrals appearing in equations (4) and (5) can be easily evaluated to yield a set of third order ordinary differential equations. However, the deflection function and the stress function remain in a coupled form to retain the nonlinear characteristic of the analysis. Equations (4) and (5) then become the two basic equations to investigate the static as well as the dynamic behaviour of the structure. Free or forced vibrations may be considered by imposing the desired value on 'q'.

4 ILLUSTRATION

To testify the validity of the above equations a simple illustration may be considered. Consider the axisymmetric vibration of a shallow spherical shell of radius 'a' upon a circular base. When the shell vibrates in a normal mode the lines of equal deflections may be taken as for the corresponding plate problem [4] which will be a family of similar and similarly situated circles of the form

$$u = u\,(x,y) = 1 - (x^2 + y^2)/a^2 \tag{8}$$

for which equations (4) and (5) will be reduced to

$$\begin{aligned}(1-u)^2\,(d^3w/du^3) - 2\,(1-u)(d^2w/du^2) + (h/2)(1-u)(dw/du)(d\phi/du)\\ + (ha/4D)(1-u)(d\phi/du) + (ha^4/16D)\int_1^u w_{tt}\, du\\ = (qa^4/16D)(1-u)\,,\end{aligned} \tag{9}$$

$$(1-u)^2\,(d^3\phi/du^3) - 2(1-u)(d^2\phi/du^2) + (E/4)(1-u)(dw/du)^2 + (a/4)\frac{dw}{du} = 0 \tag{10}$$

While performing the double integrals it has been taken into account that the maximum deflection occurs at u = 1 whereas the deflection is zero at the support. Necessarily, two conditions on 'w' are

$$w\Big|_{u=0} = 0 \quad , \quad (dw/du)\Big|_{u=1} = 0 \tag{11}$$

irrespective of whether the base is rigidly clamped or simply supported.

5 METHOD OF SOLUTION

Since a closed form solution of equations (9) and (10) seems to be atmost

difficult and as one of the integrals in equation (9) remains to be performed, finding of approximate solutions becomes a necessity. It can be assumed that the oscillations primarily takes place in the transverse direction, perpendicular to the middle plane of the plate for which one may seek solutions for 'w' and 'Ø' in the following form :

$$w = f(u)\ F(t) \quad \text{and } \emptyset = f(u)\ \theta(t) \tag{12}$$

where $F(t)$ and $\theta(t)$ are unknown functions of time and $f(u)$, the spatial part, is to be chosen so as to satisfy the conditions on the boundary.

Substitution of (12) into equation (10) and then applying a Galerkin procedure one can obtain, after a laborious but trivial calculations, the following relationship

$$\theta(t) = A\ F^2(t) \tag{13}$$

and finally inserting (12) and (13) into equation (9) and using a Galerkin procedure the following Time Differential Equation may be obtained

$$B_1 \ddot{F} + B_2 F + B_3 F + B_4 F = Q^* \quad \left(\dot{F} = \frac{dF}{dt} \right) \tag{14}$$

where $B_i (i = 1,2,3,4)$ are all known quantities. Equation (14) is then sufficient for investigation into different aspects of large vibrations as well as for static analysis after setting $B_1 = 0$.

6 APPLICATION OF CONFORMAL MAPPING TECHNIQUE TO CDC-METHOD

For more complicated boundaries approximate techniques, such as, collocation, finite difference or finite element methods may be used. However, Laura et al [6] offered a very convenient tool to conformally transform a given domain onto a simpler one like a unit circle. The boundary conditions can then be satisfied identically. Since the governing partial differential equations are not invariant under transformation and become more complicated, a variational method may be used to solve them.

In this Section a conformal mapping technique will be extended to the problems described in the earlier Section with the application of constant deflection contour method.

Transforming the differential operators from cartesian to complex coordinates system by $z = x + iy$, $\bar{z} = x - iy$, equations (4) and (5) may be reduced to

$$16\ w_{zz\bar{z}\bar{z}} = -4h/D \Big[w_{zz}\ \emptyset_{\bar{z}\bar{z}} - 2\ w_{z\bar{z}}\ \emptyset_{z\bar{z}} + w_{\bar{z}\bar{z}}\ \emptyset_{zz} - \left[\rho h\ w_{tt} + q \right] / D \tag{15}$$

$$16\ \emptyset_{zz\bar{z}\bar{z}} = -4E \left[w_{\bar{z}\bar{z}}\ w_{zz} - w^2_{z\bar{z}} \right] \tag{16}$$

Let z = f( ) be an analytic function which maps the given shape in the z-plane onto a unit circle in the -plane. It may be mentioned here that the exact mapping function is not necessary, an approximate function may be used for practical purposes. On transformation from z-plane to the - plane, equations (15) and (16) will reduce to

$$16D \left[w_{\zeta\zeta\bar{\zeta}\bar{\zeta}} \; z' \, z' - w_{\zeta\zeta\bar{\zeta}} \; z' \, z'' - w_{\zeta\bar{\zeta}\bar{\zeta}} \; z'' \, z' + w_{\zeta\zeta} \; z'' \, z'' \right] + (4h) \; L(w,\phi) + \rho h w_{tt} - q = 0 \qquad (17)$$

$$16 \; (\; \ldots\ldots) + 2E \; L(w,w) = 0 \qquad (18)$$

where (.....) stands for the first term of equation (17) within the third bracket, z' = d/d , etc., etc.

The advantage of conformal technique approach is that the boundary of plate structure is now a circle in the -plane and necessarily the nature of the iso-deflection curves u = u( ) = constant are circles of radius 'r' (1). Therefore, it is easy to choose

$$u = u(\zeta,\bar{\zeta}) = 1 - \zeta\bar{\zeta} \, , \quad \zeta = r \, e^{i\theta} \qquad (19)$$

which are circles of radius $(1 - u)^{\frac{1}{2}}$ within the domain bounded by the closed curve u = 0.

As described earlier, equations (17) and (18) may be transformed to a set of ordinary differential equations by the transformation

$$w_{\zeta} = dw/du \; u_{\zeta} \, , \quad w_{\zeta\zeta} = (d^2w/du^2) \; u_{\zeta}^2 + (dw/du) \; u_{\zeta\zeta}$$

$$w_{\zeta\bar{\zeta}} = (d^2w/du^2) \; u_{\zeta} \; u_{\bar{\zeta}} + (dw/du) \; u_{\zeta\bar{\zeta}} \, , \text{ etc} \qquad (20)$$

The other calculations remaining same as described in Section-4.

7 CONCLUDING REMARK

The basic equations derived in the present analysis can be conviniently applied to wide range of problems appearing in engineering designs where the structures are prone to large vibrations. As a byproduct of the present analysis the static behaviour of structures under consideration due to a uniform load 'q' may also be studied by neglecting the inertial term. Free or forced vibrations may also be treated by setting suitable value to 'q'.

For comparison, equations (9) and (10) become identical with those obtained by Jones et al [4] when the base of the elliptical dome is taken to be a circle of radius 'a'.

Through a carefull observation the advantage of the present analysis may be considered to yielddfruitful results and may open new avenues in plate and shell problems. For, the fact is that structures of arbitray shapes cannot be analyzed, in general, by other methods. For example, by utilising the conformal mapping technique one may be able to study the nature of nonlinear vibration of structures having shapes as described under Table-1

TABLE-1 : Appropriate and Approximate Expressions for Mapping Function

$$z = f(\zeta) = \lambda(\zeta + \mu \zeta^{m+1})$$

Plate shapes	m	λ	μ
Equilateral triangular	3	1.1352	- 1.16667
Square	4	1.080	-0.1022
Pentagon	5	1.0526	-0.0669
Hexagon	6	1.0376	-0.0478
Heptagon	7	1.0279	-0.0351
Octagon	8	1.0219	-0.0276

Further investigations on this topic are being carried out and they are expected to be published soon.

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A P P E N D I X

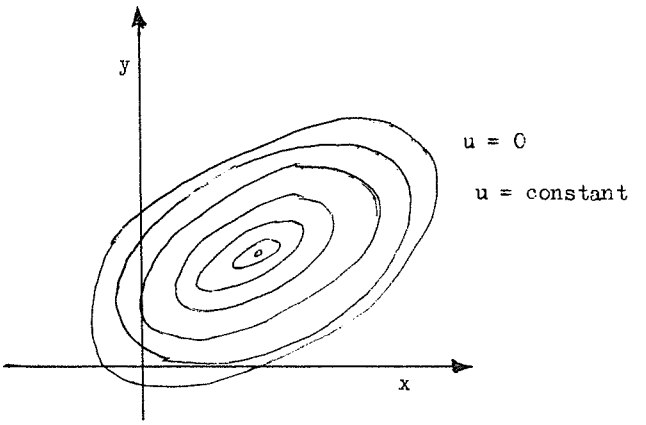

Fig. 1. Contour lines of equal Deflection

A-2 : Clamped edge : Ref. 4

$W = dW/du = 0$, for $u=0$.

For circumferential strain to be zero

$$(1 - u)\,(d^2\phi/du^2) - (1 - \quad)\,\phi/du \Big|_{u=0} = 0$$

A-3 : Simply supported edge (Plates)

On the boundary : $u = 0$, $W = 0$,

$$(1-u)\,(d^2W/du^2) - (1 - \nu)\,(dW/du) \Big|_{u=0} = 0$$

A-4 : Transformed boundary conditions on the -plane (Ref. 6)

For clamped edge:

$$W(\zeta, \bar{\zeta}) \Big|_{r=1} = W/\ r \Big|_{r=1} = 0$$

For simple support

$$W(\zeta, \bar{\zeta}) \Big|_{r=1} = M_n(\zeta, \bar{\zeta}) \Big|_{r=1} = 0$$