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Computational methods in stochastic mechanics

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ABSTRACT

Of interest in this paper are mechanics problems where the loading, the geometry, or the properties of a continuum are stochastic in nature. The objective is to compute the statistical moments of the response (second-moment analysis), or probabilities associated with various states of a structure, including the states of damage and collapse (reliability analysis). Reviewed are several methods for solution of these problems that make use of conventional finite element and numerical integration procedures.

INTRODUCTION

Many problems in mechanics require consideration of the stochastic nature of materials and loads. At the microscopic level, most materials are composites with random distribution of anisotropic grains with random orientations. This randomness must be accounted for if one is interested in the micro-mechanical behavior of the material, as in micro-machines and micro-sensors. Furthermore, damage and fracture phenomena in structures are associated with the presence of defects, which are inherently random in size and distribution within the material. On the loading side, the stochastic nature of loads generated by such phenomena as earthquakes, wind, ocean waves, blast and turbine noise are well understood.

Computational stochastic mechanics involves the numerical solution of stochastic mechanics problems, typically by the finite element method and numerical integration. Beginning in early 1980's, this field has received increasing attention and it is presently an area of active research and development. There are two distinct aims in computational stochastic mechanics: second-moment analysis, which aims at computing the means and variances of response quantities, and reliability analysis, which aims at computing probabilities for realization of various structure states, including the states of damage and collapse. The main issues that arise include: (1) modeling and representation of material or loading random fields, (2) computation of the response and response gradients, (3) computation of second moments, and (4) formulation and computation of reliability. These are discussed in the following sections with emphasis on nonlinear structures, as this is the regime of interest in structural safety assessment.

DISCRETIZATION OF RANDOM FIELDS

The stochastic nature of a continuum or loading generally may be described in terms of a set of random fields distributed over the domain of the continuum. For both second-moment and reliability analysis, it is necessary to discretize such fields and represent them in terms of random variables. Existing methods for this purpose are briefly reviewed in this section.

Consider a scalar random field $z(\mathbf{x})$, $\mathbf{x} \in \Omega$, describing a material property constant (e.g., elastic modulus, yield stress, initial damage state) or loading over the domain Ω of a continuum. All random field discretization methods assume, either explicitly or implicitly, that $z(\mathbf{x})$ is Gaussian. In that case, the field is completely defined by its mean function $\mu(\mathbf{x})$, variance function $\sigma^2(\mathbf{x})$, and auto-correlation coefficient function $\rho(\mathbf{x},\mathbf{x}')$, the latter defining the correlation coefficient between the field values at locations $\mathbf{x}$ and $\mathbf{x}'$. Virtually all random field discretization methods represent the field in the form

$$\hat{z}(\mathbf{x}) = \mu(\mathbf{x}) + \mathbf{b}(\mathbf{x})^{\mathrm{T}}\mathbf{y} \tag{1}$$

where $\mathbf{b}(\mathbf{x}) = [b_1(\mathbf{x}) \cdots b_n(\mathbf{x})]^{\mathrm{T}}$ is a vector of deterministic functions and $\mathbf{y} = [y_1 \cdots y_n]^{\mathrm{T}}$ denotes a vector of standard normal variables. The difference between various methods lies in the selection of $\mathbf{b}(\mathbf{x})$. The number of random variables, n, determines the resolution of the representation, i.e., how closely $\hat{z}(\mathbf{x})$ approximates $z(\mathbf{x})$. For a given resolution, this number for each method is a measure of efficiency of the method. Naturally, for both second-moment and reliability analysis, it is desirable to have as few random variables as possible. We note that $\hat{z}(\mathbf{x})$ is Gaussian because of the linearity of the representation (1) in terms of $\mathbf{y}$.

In the most efficient representation of a Gaussian random field, the functions $\mathbf{b}(\mathbf{x})$ are obtained by solving an integral eigenvalue problem involving the auto-correlation coefficient function of the random field [5]. Unfortunately, exact solutions are seldom available. A closely related approach uses a set of orthogonal basis functions to approximately solve the integral eigenvalue problem [12]. In another approach, the functions $\mathbf{b}(\mathbf{x})$ are determined by minimizing the variance of the error between $\hat{z}(\mathbf{x})$ and $z(\mathbf{x})$ followed by discrete eigenvalue analysis [8]. This method provides the most efficient representation of a random field when the exact solution to the integral eigenvalue problem is not available. The latter reference includes comparisons of several discretization methods. A comprehensive mathematical treatment of this subject in the context of linear regression has recently been presented in [4].

The above discretization methods can be used for both material and load random fields. However, in many cases material properties cannot be modeled as Gaussian processes. Hence, there is need to define non-Gaussian random fields and represent them in a discrete form. One convenient way for this purpose is through a nonlinear transformation [6]. For example, $\tilde{z}(\mathbf{x}) = \exp[z(\mathbf{x})]$ defines a lognormal process that does not take on negative values. In particular, the transformation $\tilde{z}(\mathbf{x}) = F^{-1}\{\Phi[(z(\mathbf{x}) - \mu(\mathbf{x}))/\sigma(\mathbf{x})]\}$, where $\Phi[\cdot]$ denotes the standard cumulative probability function, produces a process $\tilde{z}(\mathbf{x})$, commonly known as a Nataf process, that has the marginal cumulative distribution function $F(\cdot)$ [8,9]. This transformation can be used to define a non-Gaussian random field with any desired marginal distribution. The discrete representation of $\tilde{z}(\mathbf{x})$ is obtained by simply using the discretized process $\hat{z}(\mathbf{x})$ in place of $z(\mathbf{x})$ in the above relations.

FINITE ELEMENT FORMULATION

Consider the motion of a body occupying the domain Ω in the three-dimensional space with coordinates $\mathbf{x} = (x_i, i = 1,2,3)$. Assume the body has mass density $\gamma(\mathbf{x})$ and is subjected to tractions $\tau_i(\mathbf{x},t)$ over its surface Γ and body forces $f_i(\mathbf{x},t)$ over its domain Ω, where t denotes time. Using a tensor notation, the equilibrium of the body is expressed in the variational form

$$\int_\Omega \delta u_i \gamma \ddot{u}_i d\Omega + \int_\Omega \delta u_{i,j} \sigma_{ij} d\Omega - \int_\Omega \delta u_i f_i d\Omega - \int_\Gamma \delta u_i \tau_i d\Gamma = 0 \tag{2}$$

where $u_i(\mathbf{x},t)$ and $\delta u_i(\mathbf{x},t)$, $i = 1,2,3$, are the components of displacement and variational displacement, respectively, $\ddot{u}_i(\mathbf{x},t) = \partial^2 u_i / \partial t^2$ are the components of acceleration, $u_{i,j}(\mathbf{x},t) = \partial u_i / \partial x_j$, $\sigma_{ij}(\mathbf{x},t) = D_{ijkl}(\varepsilon_{kl} - \varepsilon^p_{kl})$ are the components of stress, $\varepsilon_{ij}(\mathbf{x},t) = (u_{i,j} + u_{j,i})/2$ are the components of strain, $D_{ijkl}(\mathbf{x})$ are the elastic constants, and $\varepsilon^p_{kl}(\mathbf{x},t)$ are the components of plastic strain. In general, the material properties $\gamma(\mathbf{x})$, $D_{ijkl}(\mathbf{x})$ and those governing the evolution of the plastic strains in time are random fields. Similarly, the tractions $\tau_i(\mathbf{x},t)$ and body forces $f_i(\mathbf{x},t)$ are random fields in space and time. More generally, the geometry of the body may also be uncertain, in which case the domain Ω and surface Γ should be described as random fields. However, here we will assume that the geometry of the structure is known. See, e.g., [15] for an example involving random geometry.

We assume that the random fields describing the material properties and loads are all suitably discretized and represented in terms of a collected vector of standard normal variables $\mathbf{y}$ in the manner described in the preceding section. Note that any non-Gaussian random field needs to be represented in terms of a corresponding Gaussian field through a nonlinear transformation before discretization is applied. It should be clear that, after such discretization, the displacement, strain and stress responses of the body will also be functions of the random variables $\mathbf{y}$. We write these as $u_i(\mathbf{x},t,\mathbf{y})$, $\varepsilon_{ij}(\mathbf{x},t,\mathbf{y})$ and $\sigma_{ij}(\mathbf{x},t,\mathbf{y})$, respectively. The main objective in computational stochastic mechanics is in assessing the probabilistic nature of these response quantities. Specifically, in second-moment methods we are interested in the mean and variance/covariances of these responses, whereas in reliability analysis we are interested in the probabilities associated with the occurrence of these response quantities within specified domains.

Following the conventional finite element approach, the domain Ω is discretized into a grid of elements and the displacement field within each element is approximated in the form

$$\hat{u}_i(\mathbf{x},t,\mathbf{y}) = \sum_{k=1}^{n_e} N_k(\mathbf{x}) u_{ik}(t,\mathbf{y}) \tag{3}$$

where n_e is the number of finite element nodes, $N_k(\mathbf{x})$ is the shape function associated with node k of the element, and $u_{ik}(t,\mathbf{y})$ is the displacement at node k in direction x_i. The shape functions here are considered to be deterministic. However, if the geometry of the structure is random, the shape functions also need to be written as functions of $\mathbf{y}$ (see [15]).

Substituting (3) in (2) and following the usual finite element steps, we arrive at a discrete formulation of the problem, which we write in the matrix form

$$\mathbf{M}(\mathbf{y})\ddot{\mathbf{u}}(t,\mathbf{y}) + \mathbf{C}(\mathbf{y})\dot{\mathbf{u}}(t,\mathbf{y}) + \mathbf{R}(\mathbf{u}(t,\mathbf{y}),\mathbf{y}) = \mathbf{P}(t,\mathbf{y}) \tag{4}$$

in which $\mathbf{u}(t,\mathbf{y})$ is the vector of nodal displacements representing the collection of $u_{ik}(t,\mathbf{y})$ for all nodes and all directions, $\mathbf{M}(\mathbf{y})$ is the mass matrix, $\mathbf{C}(\mathbf{y})$ is the damping matrix (added to account for the effect of viscous damping when it is present), $\mathbf{R}(\mathbf{u}(t,\mathbf{y}),\mathbf{y})$ is the resisting force vector, and $\mathbf{P}(t,\mathbf{y})$ is the nodal load vector. For the sake of brevity, the specific variational forms of $\mathbf{M}(\mathbf{y})$ and $\mathbf{P}(t,\mathbf{y})$ are not given here. The reader can find them in standard finite elements texts, e.g., [16]. The resisting force vector is obtained by assembling the element vectors

$$\mathbf{R}^{(e)}(\mathbf{u}(t,\mathbf{y}),\mathbf{y}) = \int_{\Omega^{(e)}} \mathbf{B}(\mathbf{x})\sigma(\mathbf{x},t,\mathbf{y})d\Omega \tag{5}$$

where the superposed (e) denotes an element, $\mathbf{B}(\mathbf{x})$ is the matrix relating strains to the nodal displacements and involves the derivatives of the shape functions for the element, and $\sigma(\mathbf{x},t,\mathbf{y})$ is the stress vector. It is important to note that for an inelastic structure the resisting force vector is not only a function of the current displacements, but also of the entire response history. This is clear from (5) and the fact that for an inelastic structure stresses are functions of the net elastic strains, i.e., the difference between the total strains and the cumulative plastic strains. For a specific realization of random variables $\mathbf{y}$, the desired responses are computed by solving (4) and (5) by an iterative scheme for a sequence of discrete time steps [16].

As we will shortly see, certain second-moment and reliability methods require the gradient of the response with respect to selected realizations of the random variables $\mathbf{y}$. A brute-force, finite-difference approach would require many repeated solutions of the finite element problem. A more efficient approach is the direct differentiation method (DDM), where one derives the derivatives of the governing equations (4) and (5) and the relevant constitutive laws, and solves them along with the equations of motion [14]. The derivative of (4) with respect to an element y of $\mathbf{y}$ are written as

$$\mathbf{M}\frac{\partial\ddot{\mathbf{u}}}{\partial y} + \mathbf{C}\frac{\partial\dot{\mathbf{u}}}{\partial y} + \mathbf{K}\frac{\partial\mathbf{u}}{\partial y} = \frac{\partial\mathbf{P}}{\partial y} - \frac{\partial\mathbf{M}}{\partial y}\ddot{\mathbf{u}} - \frac{\partial\mathbf{C}}{\partial y}\dot{\mathbf{u}} - \frac{\partial\mathbf{R}}{\partial y}\Big|_{\mathbf{u}} \tag{6}$$

where $\mathbf{K} = \partial\mathbf{R}/\partial\mathbf{u}$ is the current tangent stiffness matrix and $\partial\mathbf{R}/\partial y|_{\mathbf{u}}$ denotes the derivative of the resisting force vector with respect to y with the nodal displacements fixed. The latter vector is computed by differentiating (5) and employing the derivatives of the constitutive law. Derivations for the case of J_2 plasticity under plane strain and plane stress conditions are given in [13,14]. The details are beyond the scope of this paper; however, the following important observation can be made here for the DDM approach: For a given value of the response $\mathbf{u}$, equation (6) for the derivative $\partial\mathbf{u}/\partial y$ is linear and involves the tangent stiffness matrix $\mathbf{K}$. It follows that, once the response at time t is computed by solving (4), the corresponding derivatives can be computed by solving the linear equation (6). Furthermore, if a Newton-type iteration is used to solve (4), then the tangent stiffness matrix is readily available in a factorized form, which greatly facilitates the solution of (6). This approach has been used in [13] for a variety of inelastic stochastic mechanics problems.

Unfortunately, most existing commercial finite element codes do not provide facilities for computing the response gradients. This hampers the implementation of stochastic methods in such codes. Although certain response gradients can be efficiently computed without modifying an existing finite element code, this is very difficult or impossible to do for the gradients with respect to material property constants for nonlinear structures, since these involve the gradients of the constitutive law. In [13], this was done by implementing gradient computation routines in the general-purpose finite element code FEAP [16], for which access to the source code was available. With increasing interest in computational stochastic analysis, it is becoming essential for general-purpose finite element codes to incorporate efficient routines for gradient computation. It is noted that these same gradients are needed in other applications as well, such as in optimal design.

SECOND MOMENT METHODS

As mentioned earlier, second-moment finite-element methods aim at computing the mean and variance/covariances of response quantities. A variety of methods have been developed for

this purpose [1,2,5,7,10,11] (see [13] for a review). Several of these are restricted to linear and/or static problems [2,5,11]. Although extension to nonlinear problems for some of these methods is possible, it is doubtful that they will lead to general-purpose procedures for solving nonlinear problems. A simple method that has more general applicability is the Taylor series expansion method [1], or its essential equivalent, the perturbation method [7,10]. Although these methods are limited to problems with small variabilities, they can be easily implemented in existing codes for both nonlinear and dynamic problems. Here, we briefly describe the Taylor series approach.

Consider the Taylor series expansion of the nodal displacement vector $\mathbf{u}(t,\mathbf{y})$ in terms of $\mathbf{y}$ around a central point $\mathbf{y}_0$:

$$\mathbf{u}(t,\mathbf{y}) = \mathbf{u}(t,\mathbf{y}_0) + (\mathbf{y}-\mathbf{y}_0)^{\mathrm{T}}\nabla\mathbf{u}(t,\mathbf{y}_0) + (\mathbf{y}-\mathbf{y}_0)^{\mathrm{T}}\nabla^2\mathbf{u}(t,\mathbf{y}_0)(\mathbf{y}-\mathbf{y}_0) + \cdots \tag{7}$$

Since $\mathbf{y}$ are standard normal, $\mathbf{y}_0 = \mathbf{0}$ may be used. The first-order term in $\mathbf{y}$ involves the gradient matrix $\nabla\mathbf{u}(t,\mathbf{y}_0)$, whose elements are the derivatives $\partial\mathbf{u}(t,\mathbf{y})/\partial y_i$ evaluated at $\mathbf{y}_0$. As we have seen, these derivatives are easily computed by DDM. For many problems, retaining only up to the first-order term is sufficient. The second-order term in $\mathbf{y}$ involves the Hessian matrix $\nabla\mathbf{u}(t,\mathbf{y}_0)$, whose elements are the second derivatives $\partial^2\mathbf{u}(t,\mathbf{y})/\partial y_i\partial y_j$ evaluated at $\mathbf{y}_0$. Computation of this matrix is much more difficult. The direct differentiation method unfortunately does not lead to easily implementable results. One possible alternative is to use finite differences on the DDM estimates of the first derivatives. In any case, the computations for the Hessian can be extremely time consuming if the number of random variables is large. With the coefficient matrices determined, the series in (7) is truncated at the first or second-order terms, leading to a polynomial expression of $\mathbf{u}(t,\mathbf{y})$ in $\mathbf{y}$. Expectation operations on the polynomial expression then yield the desired statistical moments. The advantage of this approach is that it can be used for nonlinear and/or dynamic problems. We note that the first-order approximation can be easily implemented in any existing finite element code that is capable of computing the response gradients.

RELIABILITY ANALYSIS

Structural reliability problems in general are formulated as probability integrals of the form

$$p = \int_{\mathcal{D}} \varphi(\mathbf{y})d\mathbf{y} \tag{8}$$

where $\varphi(\mathbf{y})$ is the standard normal density of the random variables $\mathbf{y}$, $\mathcal{D}$ is a domain in the outcome space of $\mathbf{y}$ that defines the event of interest, and p denotes the occurrence probability of the event. In the context of computational stochastic mechanics, two aspects are of importance. One is the formulation of the domain in terms of the standard normal variables $\mathbf{y}$ for various events of interest. The other is computation of the probability for a given event. We discuss these issues below. One should keep in mind that for typical structural reliability problems the dimension of $\mathbf{y}$ in (8) is large, especially when $\mathbf{y}$ represents random fields in a discretized form, and the magnitude of the event probability, p, is small.

Let $s(\mathbf{x},t,\mathbf{y})$ denote a response quantity of interest, e.g., a displacement, strain, or stress component, at location $\mathbf{x}$ and time t,. Suppose the event of interest is the occurrence of $s(\mathbf{x},t,\mathbf{y})$ equal to or greater than a given threshold s_0, i.e., the event $s_0 \le s_i(\mathbf{x},t,\mathbf{y})$. In that case the domain $\mathcal{D}$ can be defined as $\mathcal{D} = \{\mathbf{y}: g(\mathbf{y}) \le 0\}$, where $g(\mathbf{y}) = s_0 - s_i(\mathbf{x},t,\mathbf{y})$ is the limit-state function. Obviously the dependence of $s(\mathbf{x},t,\mathbf{y})$ and, therefore, $g(\mathbf{y})$ on $\mathbf{y}$ is algo-

rithmic in nature and requires numerical computation, i.e., finite element analysis and/or step-by-step time integration. More generally, $\mathcal{D}$ may be defined by a limit-state function involving several response quantities $s_i(\mathbf{x},t,\mathbf{y})$, $i = 1,2,\ldots$, e.g., the components of stress at a point in space and time. Most local failure criteria in mechanics can be formulated in this manner. In the field of structural reliability, problems that are defined by a single limit-state function are classified as "component" reliability problems, provided the limit-state function $g(\mathbf{y})$ is continuous and differentiable with respect to $\mathbf{y}$. For some nonlinear problems $s(\mathbf{x},t,\mathbf{y})$ and, therefore, $g(\mathbf{y})$, are smooth and continuous functions, but not differentiable with respect to $\mathbf{y}$. In such cases, it is possible to replace the actual system with a "smooth" approximation, for which the differentiability requirement holds. See [3] for an application involving a hysteretic oscillator.

A second class of structural reliability problems is defined by a set of limit-state functions $g_k(\mathbf{y})$, $k = 1,2,\ldots$, each describing a component problem as above, and a domain $\mathcal{D}$ that is defined in terms of unions and intersections of component events. These are known as "system" reliability problems and three distinct classes are recognized: "series system" problems, in which the domain is the union of component events, i.e., $\mathcal{D} = \{\mathbf{y}: \bigcup_k g_k(\mathbf{y}) \le 0\}$, "parallel system" problems, in which the domain is the intersection of component events, i.e., $\mathcal{D} = \{\mathbf{y}: \bigcap_k g_k(\mathbf{y}) \le 0\}$, and "general system" problems, in which the domain is defined as the union of intersections of subsets of the component events, i.e., $\mathcal{D} = \{\mathbf{y}: \bigcup_{C_l} \bigcap_{k \in C_l} g_k(\mathbf{y}) \le 0\}$. In the latter, the system event of interest occurs when all components within any "cut-set" C_l occur. As an example, a system reliability problem occurs when the event of interest concerns the responses $s(\mathbf{x}_k,t,\mathbf{y})$ at several different locations $\mathbf{x}_k$, $k = 1,2,\ldots$, relative to corresponding thresholds s_{0k}. Then, a series system problem occurs if the event of interest is the exceedance of a response above the corresponding threshold at <u>any</u> of the points, a parallel system problem occurs if the event of interest is the exceedance of the response above the corresponding threshold at <u>all</u> points, and a general system problem occurs when the event of interest is the exceedance of the response above the corresponding threshold at <u>all</u> points within <u>any</u> of several subsets of the points.

Before describing other classes of reliability problems, we briefly review the existing methods for computing the probability integral (8) for the component and system reliability problems. Given the large dimension of $\mathbf{y}$ and the small magnitude of p, numerical integration or conventional Monte Carlo simulation approaches are not appropriate. For this reason, approximate methods have been developed. The most effective among these are the first- and second-order reliability methods, FORM and SORM. These methods employ points in domain $\mathcal{D}$, known as "design points," that have locally maximum probability density in the standard normal space. Since the probability density in that space is rotationally symmetric and exponentially decaying in the radial direction, assuming the origin is not in $\mathcal{D}$, the design points are located on the boundary of $\mathcal{D}$ and locally have minimum distance from the origin. Due to the exponential decay of density, most of the contribution to the probability integral comes from the neighborhoods of these points. Based on these properties, in FORM and SORM, approximations to (8) are constructed by replacing the boundary of $\mathcal{D}$ at each design point with a first- or second-order approximating surface. Formulas for the probability contents of polyhedral or quadratic sets are then used to compute p. Hence, the main computational task in FORM and SORM is the finding of the design points. This is solved as a constrained optimization problem that minimizes the distance of a point from the origin, subject to the point being on the boundary of $\mathcal{D}$. Most efficient algorithms available for solution of

this problem require repeated computations of the limit-state functions $g_k(\mathbf{y})$ and their gradients with respect to $\mathbf{y}$. In the context of finite element reliability analysis, therefore, these approaches in essence require repeated computations of the response and response gradients for selected realizations of the random variables $\mathbf{y}$. The number of such repeated computations for each design point is typically of the order of 10 to 50, which is much smaller than that required by the Monte Carlo simulation technique for typical probability values of interest in reliability analysis.

A special class of problems in computational stochastic mechanics arises when one is interested in the extreme values of the response over intervals of space or time. Consider, for example, a maximum response of the body over its domain, $s_{\max}(t,\mathbf{y}) = \max_{\mathbf{x}\in\Omega} s(\mathbf{x},t,\mathbf{y})$, and let $\mathbf{x}_{\max}(t,\mathbf{y})$ denote the location of the maximum. One can show that in general $s_{\max}(t,\mathbf{y})$ is a continuous but not differentiable function of $\mathbf{y}$, and that $\mathbf{x}_{\max}(t,\mathbf{y})$ is a discontinuous function of $\mathbf{y}$, even when $s(\mathbf{x},t,\mathbf{y})$ is continuous and differentiable with respect to both $\mathbf{x}$ and $\mathbf{y}$. The reason is that the location of the maximum within the domain may change sharply with a small change in $\mathbf{y}$, whenever there are two or more maximums of equal magnitude within the domain. The same behavior can be shown for the maximum response in t at a given location $\mathbf{x}$, i.e., for $s_{\max}(\mathbf{x},\mathbf{y}) = \max_{t\in\mathcal{T}} s(\mathbf{x},t,\mathbf{y})$, where $\mathcal{T}$ denotes a time interval of interest. It follows that reliability problems dealing with the extreme values of a response over space or time domains, e.g., the probabilities of the events $\{s_0 \le \max_{\mathbf{x}\in\Omega} s(\mathbf{x},t,\mathbf{y})\}$ or $\{s_0 \le \max_{t\in\mathcal{T}} s(\mathbf{x},t,\mathbf{y})\}$, cannot be solved as component reliability problems.

One approach for solving reliability problems involving the extreme response is to convert them to series system problems. This is particularly effective in the finite element context for the extreme over a spatial domain, where the domain of interest is discretized into a finite number of elements. Let $s(\mathbf{x}_i,t,\mathbf{y})$ denote the representative value of $s(\mathbf{x},t,\mathbf{y})$ within the i-th element and $\mathcal{N}=\{1,2,..,N\}$ denote the set of element numbers. Then, $s_{\max}(t,\mathbf{y}) = \max_{\mathbf{x}\in\Omega} s(\mathbf{x},t,\mathbf{y}) \cong \max_{i\in\mathcal{N}} s(\mathbf{x}_i,t,\mathbf{y})$. It follows that $\Pr\{s_0 \le \max_{\mathbf{x}\in\Omega} s(\mathbf{x},t,\mathbf{y})\} \cong \Pr\{\bigcup_{i\in\mathcal{N}} s_0 - s(\mathbf{x}_i,t,\mathbf{y}) \le 0\}$, the latter defining a series system problem. This approach is effective when the response quantity of interest peaks at a few "hot spot" locations within the domain so that only a few of the component events $\{s_0 \le s(\mathbf{x}_i,t,\mathbf{y})\}$ significantly contribute to the probability p. An algorithm that automatically finds the elements with high contribution to the probability is described in [13]. The same approach can be used for the extreme in time, i.e., $\Pr\{s_0 \le \max_{t\in\mathcal{T}} s(\mathbf{x},t,\mathbf{y})\} \cong \Pr\{\bigcup_{i\in\mathcal{N}} s_0 - s(\mathbf{x},t_i,\mathbf{y}) \le 0\}$, where $\mathcal{N}$ now denotes a set of discrete points in time. However, this approach ceases to be effective when many elements (time points) contribute to the probability of interest. Examples include a body under homogeneous stress state or stationary stochastic response.

When the number of contributing elements (or time points) to the failure probability is many, an alternative approach is to compute the statistics of the number of out-crossings into domain $\mathcal{D}$ over the space or time interval of interest. It is well known that the mean of this number provides an upper bound to the probability of interest. For example, if $\upsilon(t)$ denotes the mean rate of up-crossing of $s(\mathbf{x},t,\mathbf{y})$ above the threshold s_0, then $\Pr\{s_0 \le \max_{t\in\mathcal{T}} s(\mathbf{x},t,\mathbf{y})\} \le \int_{\mathcal{T}} \upsilon(t)dt$. One can write $\upsilon(t) = \lim_{\delta t\to 0} \Pr\{s(\mathbf{x},t,\mathbf{y}) < s_0 \cap s_0 \le s(\mathbf{x},t+\delta t,\mathbf{y})\} / \delta t$. It follows that the mean up-crossing rate at a time t can be computed as a parallel system reliability problem with limit-state functions $g_1(\mathbf{y}) = s(\mathbf{x},t,\mathbf{y}) - s_0$ and $g_2(\mathbf{y}) = s_0 - s(\mathbf{x},t+\delta t,\mathbf{y})$. This approach has been used in [3] to determine the reliability of a hysteretic oscillator subjected to

stochastic excitation. The same approach can be used for the extreme over a spatial interval. However, very little experience exists for 2 or 3-dimensional domains.

Finally, one can consider reliability problems involving the maximum response in both space and time domains, i.e., problems involving $s_{\max}(\mathbf{y}) = \max_{\mathbf{x}\in\Omega, t\in T} s(\mathbf{x}, t, \mathbf{y})$. This class of problems can be solved by a combination of the methods described above, but to the author's knowledge no attempts have yet been made to solve them.

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