

ANALYSIS OF CYLINDRICAL SHELLS BASED ON FOUR SHELL THEORIES BY SEGMENTATION METHOD

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ABSTRACT

A class of shell theories, which are derived from the three-dimensional elasticity equations by expanding the displacement vector in Taylor's series in the thickness coordinate, is first reviewed. A theory for general shell deformation is then developed based on a higher-order kinematic model for layered shells. The theory accounts for the effects of transverse shear deformation, transverse normal stress and transverse normal strain with an implicit nonlinear distribution of the tangential displacement component through the shell thickness. The theory is shown to result in a partial differential equation system of sixteenth order. The scope of the segmentation method is extended to the solution of composite layered cylindrical shells; cantilever and diaphragm supported boundary conditions. Governing first-order ordinary differential equations are derived for each of four theories, classical Love shell, Reissner-Nagdhi, and a higher-order shear deformation theory of 9 and 12 degrees of freedom. Reissner's mixed variational theorem is used to derive the equilibrium equations.

INTRODUCTION

Structural shell forms-a body bounded by two curved surfaces, are extremely important structural elements in applications such as nuclear reactors, pressure vessels, spacecrafts, missiles, etc. The development of the theoretical model (the theory) to describe its behaviour is an area of continuing interest. The so-called 'thin shell theory', originally formulated by Love in 1888, is firmly established by Love, [1, 2]. It is used extensively for analytical solutions and numerical analysis and further is continuously being applied to new problems to generate much needed design data. However, its use in many practical applications involving complex geometries and loadings, cut-outs, branches, intersections, contact problem involving shells, laminated shells, etc. is not at all effective because of the implicit simplistic assumptions in the theory. This calls for the development of more refined and higher-order shell theories.

Here, we are concerned with the derivation of a particular higher-order general shell theory. The behaviour of the shell, in any theory is governed by the behavior of an appropriate reference surface. This necessitates the transformation of the three dimensional (3D) elasticity equations into an approximate system of two dimensional (2D) shell equations. This transformation is an essential feature of any plate or shell theory. The situation is further complicated by the coupled nature of the membrane and bending behaviour especially in shells. These coupled deformations, in the form of stretching and curvature change of the reference surface, are required in predicting the strains that exist throughout the shell space.

In the present work, Governing equations are derived for a multilayered cylindrical shell subjected to symmetric loading using a higher order displacement Model. The manipulation of the governing equations into an equivalent system of first order ordinary differential equations is shown. Numerical results are presented for a cylindrical shell subjected to symmetric loading and different boundary conditions. Both single layered and multilayered cylindrical shells are analyzed. Numerical analysis of governing differential equations is done using segmentation method. Two computer programmes for segmentation method are developed in FORTRAN-77.

SHELL THEORY FORMULATION

Reissner [5] contended that the corrections to Love's first-approximation are desired, these should be obtained by simultaneously abandoning all of Love's assumptions. These developments had used displacements of the form:

$$\begin{aligned}
U_1 &= u_1 + z\theta_1 \\
U_2 &= u_2 + z\theta_2 \\
U_3 &= u_3
\end{aligned} \tag{4}$$

Where z is the co-ordinate normal to the reference surface, and u_1, u_2, θ_1 and θ_2 and u_3 have dependence upon the curvilinear co-ordinates α_1 and α_2 which describe the reference surface. Hildebrand, Reissner and Thomas [3] proposed a higher-order theory by dispensing with all the Love's assumptions. The term higher-order, as used herein, refers to the level of truncation of terms in the Taylor series expansion for displacements. They used displacement forms of the type:

$$U_i = u_i + z\theta_i + z^2 u_i^*, \quad (i=1, 2, \& 3) \tag{5}$$

They however, concluded, based on their simplistic analysis that the third terms in the tangential surface displacement components in relations (5) did not lead to improved results. It was this result, which perhaps prompted Naghdi [4] to develop a complete shell formulation based on the displacement forms:

$$\begin{aligned}
U_1 &= u_1 + z\theta_1 \\
U_2 &= u_2 + z\theta_2
\end{aligned} \tag{6}$$

$$U_3 = u_3 + z\theta_3 + z^2 u_3^*$$

Kant and Ramesh [6] presented a general orthotropic shell theory based on the displacement forms (5) and using a variational principle of displacements. Kant [7] extended this work for laminated orthotropic/composite shells where thermal strains are also considered.

In the present work, the displacement expansions for a general shell are written concisely as,

$$\begin{aligned}
U_1 &= u_1 + zu_1' + \frac{1}{2} z^2 u_1'' + \frac{1}{6} z^3 u_1''' \\
U_2 &= u_2 + zu_2' + \frac{1}{2} z^2 u_2'' + \frac{1}{6} z^3 u_2''' \\
U_3 &= u_3 + zu_3' + \frac{1}{2} z^2 u_3'' + \frac{1}{6} z^3 u_3'''
\end{aligned} \tag{7}$$

Fourth term in the displacement field is introduced for the improvement of bending behavior of the shell. The general shell equations, so derived, [8] are particularized for an axisymmetric shell, more specifically for a cylinder, for which the longitudinal (U_1) and the transverse (U_3) displacement components define the displacement field and Eq. (7) reduces to,

$$\begin{aligned}
U_1 &= u_1 + zu_1' + \frac{1}{2} z^2 u_1'' + \frac{1}{6} z^3 u_1''' \\
U_3 &= u_3 + zu_3' + \frac{1}{2} z^2 u_3'' + \frac{1}{6} z^3 u_3'''
\end{aligned} \tag{8}$$

The boundary-value problem is formulated in terms of 16 differential equations called the 'intrinsic equations' involving only 16 unknowns - called the 'intrinsic dependent variables'. The other variables are related to the intrinsic variables by algebraic relations. One of the two coordinates describing the problem is selected as a preferred coordinate in the basic method of analysis. In the present case, due to symmetric loads, the problem is described in terms of φ - the meridional coordinate only and therefore the choice of the preferred coordinate is a straight-forward one. Intrinsic equations are derived consisting of a system of 16 first-order ordinary differential equations with respect to φ , containing the 16 intrinsic variables which appear in the boundary conditions on the edge $\varphi = \text{constant}$. For the present problem, y , defining the vector of intrinsic variables consists of,

$$y = \left(u_1, u_1', u_1'', u_1''', u_3, u_3', u_3'', u_3''', N_{1\varphi}, M_{1\varphi}, P_{1\varphi}, M_{11}^*, Q_{1\varphi}, S_{1\varphi}, T_{1\varphi}, S_{11}^* \right)^t \tag{9}$$

The governing equations are arranged in such a way that the derivatives of the intrinsic variables with respect to φ (Axial coordinate) for a given value of φ are computed in a straight-forward manner, when the physical and geometrical parameters of the shell and the intrinsic variables themselves are known at that value of φ . Intrinsic

equations and the associated auxiliary relations are derived in a systematic manner for symmetrically loaded orthotropic/layered shells of revolution. These are numerically integrated by a specially developed procedure called the segmentation method [9].

EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS

For deriving the equilibrium equation, the principle of minimum potential energy is chosen due to its simplicity and also because its application gives simultaneously the natural boundary conditions that are to be used with the theory. The total potential, π for the shell element is defined as,

$$\pi = U - W_s - W_{e1} - W_{e2} \quad (10)$$

Where U is strain energy of the elastic shell,

W_s is the contribution to the 'load potential' and equals the external work done by body and surface tractions, and

W_{e1} and W_{e2} are another contribution to the 'load potential' and equal the external work done by edge forces.

For π to be stationary

$$\delta\pi = 0 \quad (11)$$

or

$$\delta(U - W_s - W_{e1} - W_{e2}) = 0 \quad (12)$$

The expression for the variation of the strain energy is given by (Reissner [5])

$$\delta U = \int \int \int_{\alpha_1 \alpha_2 z} (\sigma_1 \delta \varepsilon_1 + \sigma_2 \delta \varepsilon_2 + \sigma_3 \delta \varepsilon_3 + \tau_{12} \delta \gamma_{12} + \tau_{13} \delta \gamma_{13} + \tau_{23} \delta \gamma_{23}) A_1 A_2 (1 + k_1 z) (1 + k_2 z) d\alpha_1 d\alpha_2 dz \quad (13)$$

Term-by-term evaluation of the above integral is carried out, by making use of the relations given by Eqns. (13).

Under symmetric loading, $\tau_{12} = 0$ and $\tau_{23} = 0$, Further, all shell variables are independent of θ . Making these simplifications, and the following eight equations are obtained:

NUMERICAL EXAMPLES

Example 1

The first example is that of a circular cylindrical shell subjected to a unit transverse shear at one edge and clamped, at the other edge (Fig. 1a). This particular example is chosen principally due to the fact that all the discrete values of the stress-resultants and displacements due to Kant (HOST6)[6] are available. For comparative study the geometrical and material properties are taken to be, length $l = 25$ inches, radius $R = 10$ inches, thickness $h = 2.5$ inches, Young's modulus $E = 30.0 \times 10^6$ psi, and Poisson's ratio $\mu = 0.3$, $h/R = 1/4$. For the sake of one-to-one, correspondence with the results to be compared, the shell length is divided into 50 segments; the length of individual segment is 0.5 inch. Each segment is then divided into 5 subdivisions for numerical integration with the Runge-Kutta-Gill algorithm. Results using the theoretical formulation and the numerical technique for solution described earlier are compared explicitly with those of Kant [9] and Baluch [11] at discrete points along the length (Table 1). Important stress-resultants, such as M_{11}, Q_{11} , etc and the u_3 displacement component are further plotted and compared in Figs. 2-4 to see and verify the nature of variation for all four theories, Classical Love Shell[2], Reissner Naghdi [4], Kant(HOST6) [9], and Present (HOST8). The same example is also used for parametric study. For this study, shell length $l = 25$ inches, radius $R = 10$ inches, and Young's modulus $E = 30 \times 10^6$ psi are taken to be fixed. Thickness h is varied so as to obtain h/R ratio to be $1/50$. Number of segments and number of subdivisions within each segment that are found suitable for the parametric study corresponding to a particular combination of h/R ratio, without giving any numerical instability problem are summarized below for any future reference.

For $h/R = 1/5$, No. of segments = 50, No. of subdivision within each Segments = 5 and for $h/R = 1/50$, No. of segments = 400, No. of subdivision within each segments = 5, Length of each segments for $h/R=1/5$ is 0.5 and for $h/R=1/50$ is 0.0625.

(a) From the comparative study done it is seen that all the stress-resultants M_{11}, M_{22}, N_{22} and Q_{11} and the transverse displacement u_3 can be compared with the results of Classical Love Shell [2], Reissner Naghdi [4], Kant(HOST6)

[9] to establish the correctness of the theory. (b) From the parametric study made with reference to h/R , the following points, some of which are obvious, are observed: (i) Maximum values of M_{11} and M_{22} increase with increasing h due to increasing stiffness. (ii) Maximum value of N_{22} (N_{11} being zero because of one edge being free) increases with decreasing h due to increasing flexibility, making membrane action predominant. The same holds good for u_3 as well. (iii) The edge effect is seen to travel over a longer length with increasing h , is directly proportional to $h/2$. Following nondimensional parameters are used for this example.

$$\bar{u}_3 = \frac{Eh}{QR} u_3, \bar{M}_{11} = \frac{4}{QR} M_{11}, \bar{Q}_{11} = \frac{1}{Q} Q_{11} \tag{14}$$

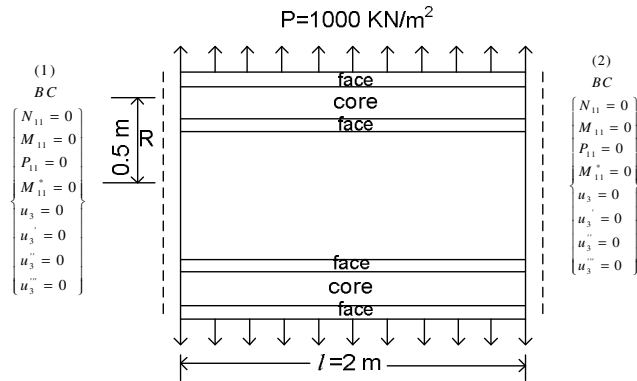
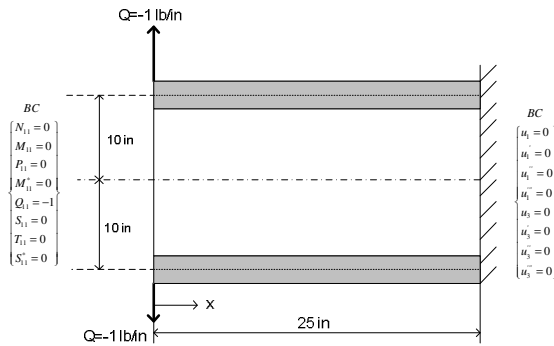


Fig. 1a: Cantilever circular cylindrical shell subjected to unit shear at the free edge

Fig. 1b: Diaphragm supported sandwich (0° /core/ 0°) cylindrical shell subjected to external pressure

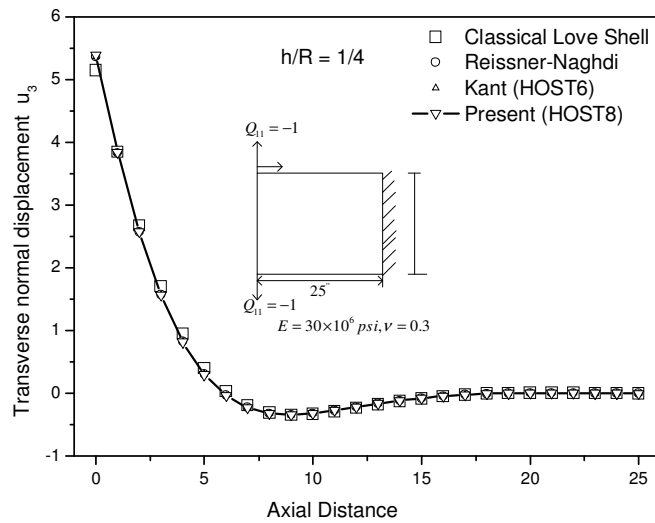


Fig. 2: Transverse normal displacement in thick shell – Example 1

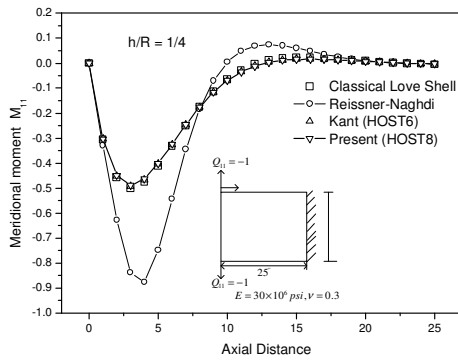


Fig. 3a Meridional moment in thick shell – Example 1

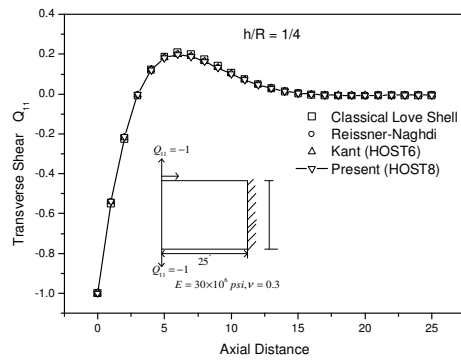


Fig. 3b Transverse shear in thick shell – Example 1

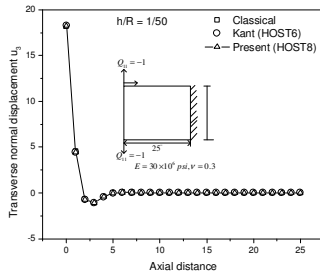


Fig. 4a Transverse normal displacement in thin shell – Example 1

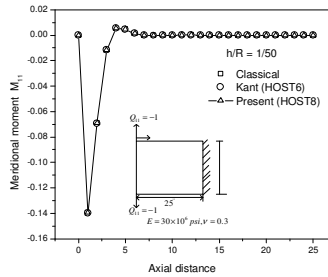


Fig. 4b Meridional moment in thin shell – Example 1

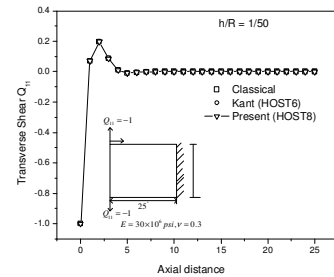


Fig. 4c Transverse shear in thin shell – Example 1

Table 1: Values of transverse normal displacement (u_3) along length of circular cylindrical shell (Example – 1)

| Inches | Classical Love Shell [2] | Reissner – Naghdi [4,5] | Kant HOST6 [9] | Present HOST8 | Baluch [11] |
|--------|--------------------------|-------------------------|----------------|---------------|-------------|
| 0 | 5.1413 | 5.3603 | 5.3640 | 5.3873 | 0.528375 |
| 1 | 3.8453 | 3.8295 | 3.8325 | 3.8438 | 0.380175 |
| 2 | 2.6768 | 2.5740 | 2.5763 | 2.5718 | 0.254550 |
| 3 | 1.7048 | 1.5750 | 1.5765 | 1.5653 | 0.155025 |
| 4 | 0.9495 | 0.8280 | 0.8295 | 0.8168 | 0.80850 |
| 5 | 0.4004 | 0.3057 | 0.3058 | 0.2961 | 0.29175 |
| 7 | -0.1927 | -0.2276 | -0.2285 | -0.2307 | -0.4050 |
| 8 | -0.3068 | -0.3191 | -0.3203 | -0.3195 | -0.31950 |
| 10 | -0.3305 | -0.3182 | -0.3197 | -0.3160 | -3.1500 |
| 12 | -0.2344 | -0.2186 | -0.2198 | -0.2162 | -0.21450 |
| 14 | -0.1260 | -0.1160 | -0.1168 | -0.1147 | -0.11325 |
| 16 | -0.0482 | -0.0448 | -0.0452 | -0.0445 | -0.04275 |
| 18 | -0.0068 | -0.0072 | -0.0073 | -0.0075 | -0.00675 |
| 20 | 0.0072 | 0.0062 | 0.0062 | 0.0058 | 0.00675 |
| 21 | 0.0078 | 0.0072 | 0.0073 | 0.0070 | 0.00750 |
| 22 | 0.0061 | 0.0062 | 0.0062 | 0.0061 | 0.00675 |
| 23 | 0.0035 | 0.0040 | 0.0041 | 0.0040 | 0.00450 |
| 24 | 0.0011 | 0.0017 | 0.0017 | 0.0017 | 0.00225 |
| 25 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

Example 2

A diaphragm supported ($0^\circ/\text{core}/0^\circ$) two layered cylinder (Fig. 1b) is analyzed and compared with elasticity theory. Subjected to external pressure of 1000 kg/cm^2 . The geometrical and material properties are $R = 0.5 \text{ m}$ and $l = 2 \text{ m}$. Again, In view of symmetry about mid length, only half shell is analyzed. Following material properties [10] are taken for the ($0^\circ/\text{core}/0^\circ$) sandwich cylinder:

Face Material properties are

$$E_r = 6.894 \times 10^6, E_\theta = 172.36 \times 10^6, E_z = 6.894 \times 10^6, G_{rz} = 1.378 \times 10^6, \\ \nu_{\theta r} = 0.25, \nu_{rz} = 0.25, \nu_{\theta z} = 0.25$$

Core Material properties are

$$E_r = 3.44 \times 10^6, E_\theta = 0.275 \times 10^6, E_z = 0.275 \times 10^6, G_{rz} = 0.413 \times 10^6, \quad (15) \\ \nu_{\theta r} = 0.0199, \nu_{rz} = 0.0199, \nu_{\theta z} = 0.25$$

Number of segments and number of subdivisions within each segment that are found suitable for the numerical study are summarized below for any future reference. For $h/R = 1/5$, number of segments are taken as 400, number of subdivision within each segments as 5 and length of each segments are taken as 0.0025. Non-dimensional parameters are used for this example are same as follows.

$$\bar{u}_3 = \frac{Eh}{p_o R^2} u_3, \bar{M}_{11} = \frac{4}{p_o Rh} M_{11}, \bar{Q}_{11} = \frac{4}{p_o \sqrt{Rh}} Q_{11} \quad (16)$$

Fig. 5-6a-b shows the distribution of transverse normal displacement, meridional moment, and transverse shear in thick sandwich shell. Following points are observed from the results. (i) Maximum values of M_{11} and Q_{11} are directly proportional to h , because stiffness and hence the bending action increase with increase in h . (ii) Maximum values of N_{11} increases with decrease in h due to increased flexibility. N_{22} attains the same maximum value away from the clamped edge, the distance being more in the case of thicker shell due to increased edge disturbance. The same holds good for u_3 too. (iii) The edge effect is seen to travel more in the case of larger h confirming a known fact.

Table 2 shows the comparison of numerical values obtained from the exact theory [12, 14] and from the present work for the isotropic case. Similarly, Table 3 shows the comparison of numerical values from the exact theory [13, 14]. From the comparison it is seen that the formulation derived in the present work is very accurate.

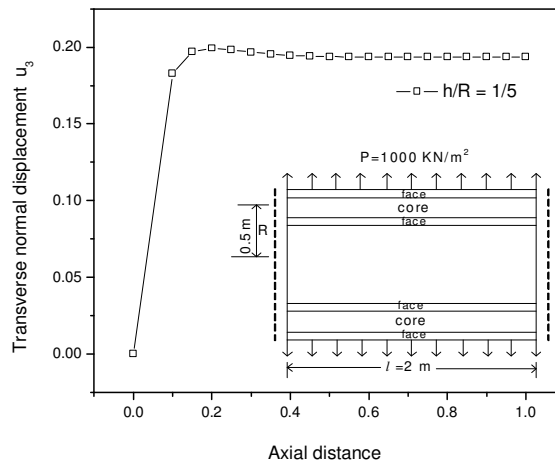


Fig. 5: Transverse normal displacement in thick sandwich shell – Example 2

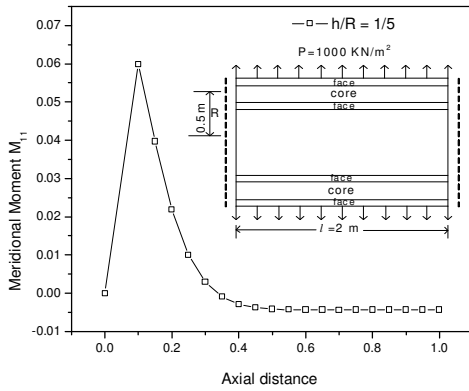


Fig. 6a: Meridional moment in thick sandwich shell – Example 2

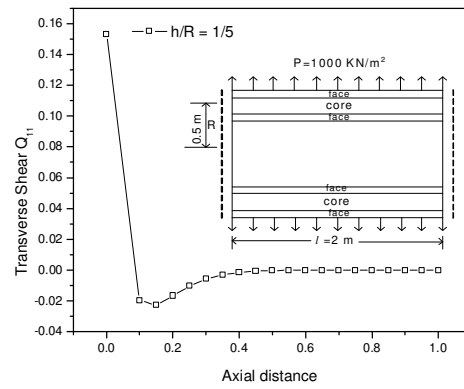


Fig. 6b: Transverse shear in thick sandwich shell – Example 2

Table 2: Comparison of radial displacements using various shell theories and elasticity theory for isotropic cylinder.

| | l/R=4 | | l/R=20 | |
|--------------------------------|--------------|--------------|--------------|--------------|
| h/R Ratios | 1/5 | 1/50 | 1/50 | 1/5 |
| No. of Segments | 100 segments | 400 segments | 400 segments | 400 segments |
| Classical [1,2] | 4.9880 | 50.0000 | 5.0000 | 5.0000 |
| Reissner – Naghdi (R- N) [4,5] | 4.9880 | 50.0000 | 5.0000 | 5.0000 |
| HOST 6 [9] | 4.9720 | 50.0000 | 4.9920 | 4.9920 |
| HOST 8 (Present) | 4.9800 | 50.0000 | 4.9920 | 4.9920 |
| Semi Analytical [14] | 5.2777 | 49.8475 | 5.3006 | 5.3006 |
| Timoshenko Plane Strain [12] | 5.3028 | 50.3452 | 5.3028 | 5.3028 |

Table 3: Comparison of radial displacements using various shell theories and elasticity theory for orthotropic cylinder

| | l/R=4 | | l/R=20 | |
|--------------------------------|--------------|-----------------------|--------------|--------------|
| h/R Ratios | 1/5 | 1/50 | 1/50 | 1/5 |
| No. of Segments | 100 segments | 400 segments | 400 segments | 400 segments |
| Classical [1,2] | 0.3240 | 3.2404 | 0.3240 | 0.3240 |
| Reissner – Naghdi (R- N) [4,5] | 0.3212 | 3.2597 | 0.3212 | 0.3212 |
| HOST 6 [9] | 0.3279 | 3.2597 | 0.3279 | 0.3279 |
| HOST 8 (Present) | 0.3275 | 3.2597*(800 Segments) | 0.3275 | 0.3275 |
| Semi Analytical [14] | 0.3435 | 3.2822 | 0.3438 | 0.3438 |
| Lekhnitskii Plane Strain [13] | 0.3405 | 3.2819 | 0.3405 | 0.3405 |

CONCLUSION

A higher-order general shell theory is presented for thick circular cylindrical shells of finite length cantilever and diaphragm supported boundary conditions. The complete derivation is new. The formulation incorporates the effects transverse shear deformation with warping of the cross-section, transverse normal strain and transverse normal stress. The displacement model is consistent and represents refined modes of both bending and membrane deformations. Besides thick shells, this theory should find immense application for laminated shells where the distribution of the tangential displacement components through the thickness is nonlinear.

Nomenclature

z = Normal coordinate along the thickness direction

\vec{R} = Position vector of a general point in shell space

\vec{r} = Position vector of a point on the reference surface

k_1, k_2 = Curvature of the lines of principal curvatures (along α_1 and α_2)

dS_1, dS_2 = Differential areas of the edge faces

\vec{U} = Displacement vector of any point in the shell space having components U_1, U_2, U_3 along α_1, α_2 and z directions

$u_i(u_1, u_2, u_3)$ = Linear displacements of a point of the reference surface along α_1, α_2 and z directions

u_1', u_2' = Rotations of the normal to the reference surface in α_1 and α_2 directions

u_1'', u_2'' = Curvature terms along α_1 and α_2 directions

u_3', u_3'' = Account for the transverse normal stress and strain

$\epsilon_i(\epsilon_1, \epsilon_2, \epsilon_3)$ = normal strains

γ_{ij} = Shearing strains

$\sigma_1, \sigma_2, \sigma_3$ = Normal stressess

$\tau_{12}, \tau_{13}, \tau_{23}$ = Shearing stressess

N_{11}, M_{11} , etc. = Stress-resultants in α_1 or meridional direction

N_{22}, M_{22} , etc. = Stress-resultants in α_2 or circumferential direction

Q_{11}, S_{11} , etc. = Transverse stress-resultants in z direction along α_1

Q_{22}, S_{22} , etc. = Transverse stress-resultants in z direction along α_2

$\bar{u}_i, \bar{u}_i', \bar{u}_i'', \bar{u}_i'''$ = Specified boundary displacement values

$\bar{N}_{11}, \bar{M}_{11}$ etc. = Specified boundary stress resultant values

h = Thickness of a homogeneous isotropic/orthotropic shell

α_1 or ϕ = Meridional coordinate of a shell of revolution

α_2 or θ = Circumferential coordinate of a shell of revolution

$\underline{p}^+, \underline{p}^-, \underline{m}^+, \underline{m}^-, \underline{n}^+, \underline{n}^-, \underline{o}^+, \underline{o}^-$ = Load vectors on the positive and negative external z surfaces and having components along α_1, α_2 and z directions

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