TWO-PARAMETER CHARACTERISATION OF CRACK TIP STRESS FIELD IN ELASTIC-PERFECTLY PLASTIC MATERIAL

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ABSTRACT

Effect of crack tip stresses on fracture behaviour of ductile materials is well established. Several detailed numerical and experimental studies have clearly revealed that a high hydrostatic stress (high constraint) near the crack tip significantly reduces the fracture toughness. Thus, characterisation of crack tip stresses plays an important role in fracture analysis of ductile materials. In this work asymptotic crack tip stress fields are proposed for a stationary plane strain crack in mode I loading under fully plastic condition. Detailed investigations have revealed that the general elastic-plastic crack tip field can be completely described by the 5-sector stress solution proposed in this article. A new constraint-indexing parameter T_{CS-2} is proposed which along with hydrostatic stress ahead of crack tip is capable of representing the entire elastic plastic crack tip stress fields over all angles around a crack tip. Excellent agreement is obtained between the proposed analytical solutions and the full-field finite element results for wide range of crack tip constraint.

INTRODUCTION

Characterisation of crack tip stresses has been an area of active research for many decades. Classical linear elastic fracture mechanics theory is based on the representation of crack tip stresses by a single parameter *K* (Stress intensity factor). However, it has been realized that the specimen geometry and loading conditions significantly affect the crack tip fields and, thus, the single parameter representation in terms of *K* or *J*-integral (HRR fields) have limited application to real cracked structures. For a Ramberg-Osgood material model, several alternative representation of crack tip stress fields have been suggested in the literature like *J-T* of Betegon and Hancock [1], *J-Q* by O'Dowd and Shih [2], and J-A_2 of Chao et al. [3]. However, it is important to understand that these two-parameter approaches can describe the state of stress only in the leading sector near the crack tip.

To construct the general elastic-plastic crack tip stress field Ibrgimov and Tarasyuk [4]) and Nemat-Nasser and Obata [5] first discussed the possibility of existence of elastic sector for plane strain mode-I crack in elastic-perfectly plastic material. Then, Li and Hancock [6] described the crack tip fields under small-scale yielding in terms of two plastic sectors and an elastic sector to account for the incomplete crack tip plasticity observed from detailed FE investigations. Their 3-sector solution was extended by Zhu and Chao [7] who, based on available FE results of Kim et al. [8], Zhu and Chao [9], proposed that the actual stress field of a stationary crack in elastic-perfectly plastic material under plane strain condition can be described by a 4-sector solution. Closed-form asymptotic solutions of crack tip fields were developed by them. Two undetermined parameters; T_p and T_π were proposed to characterise the state of stress near the crack tip. The proposed asymptotic solutions were compared with detailed FE results for various fracture specimens.

In this article asymptotic crack tip stress fields are developed for a stationary plane strain crack under mode-I loading. Incompressible, elastic-perfectly plastic material with Von-Mises yield criterion was assumed for the present study. Detailed investigations have revealed that the general elastic-plastic crack tip field can be completely described by the 5-sector stress solution proposed in this article. The 3-sector stress field proposed by Li and Hancock [6] and the 4-sector field proposed by Zhu and Chao [7] are subsets of the general elastic-plastic field proposed in this work. It is well known that loss of constraint at the crack tip leads to an elastic sector at the crack flank, thus, leading to incomplete crack tip plasticity. This study has revealed that cases arise where severe loss of crack tip constraint can lead to compressive yielding of crack flank. This particular situation leads to 5-sector stress field where the elastic sector is sandwiched between the two plastic sectors of uniform stress state. Such 5-sector stress field exists in an overmatched weld where the relatively higher strength of weld leads to shielding effect on the crack tip and, thus, leads to loss of crack tip constraint. Detailed 2-D elastic plastic finite element analyses were performed on mismatch welded middle tension, M(T) to examine the validity of the proposed 5-sector stress field.

Both under matched and over matched cases were analysed to simulate wide range of crack tip constraint. Excellent agreement is obtained between the proposed asymptotic crack tip stress field and finite element results. Detailed studies have revealed that, in the general case of elastic plastic crack tip field the T_π parameter proposed by Zhu and Chao [7] cannot be used as a constraint parameter to represent a unique state of stress at the crack tip. A new constraint-indexing parameter T_{CS-2} is proposed which along with T_p parameter, suggested by Zhu and Chao [7], is capable of representing the entire elastic plastic crack tip stress fields over all angles around a crack tip. Advantages of the proposed T_{CS-2} parameter over the T_π parameter are discussed. It is demonstrated that the proposed constraint parameters are adequate to represent the effects of specimen geometry, loading conditions, and the additional constraint that arises due to weld strength mismatch.

GOVERNING EQUATIONS

We consider here a stationary crack in an incompressible elastic-perfectly plastic material under plane strain condition. Only mode-I loading is considered here.

Equilibrium Equations

For elastic-perfectly plastic material equilibrium equations can be expressed as

$$\frac{d\sigma_{r\theta}}{d\theta}+\sigma_{rr}-\sigma_{\theta\theta}=0 \qquad \frac{d\sigma_{\theta\theta}}{d\theta}+2\sigma_{r\theta}=0 \tag{1}$$

Plane Strain Condition

If elastic response of the material is considered as incompressible then plane strain condition is same for both elastic and plastic sectors and can be expressed as

$$\sigma_{33}=\frac{1}{2}\left(\sigma_{11}+\sigma_{22}\right)=\frac{1}{2}\left(\sigma_{rr}+\sigma_{\theta\theta}\right) \tag{2}$$

Yield criterion

Von-Mises yield criterion for plane strain condition can be expressed as

$$\frac{1}{4}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)^2+\sigma^2{}_{r\theta}=k^2 \tag{3}$$

Here k is $\sigma_y/\sqrt{3}$, σ_y is the yield strength in tension. The in-plane stress components in the plastic sector can be expressed in terms of a stress function $\psi(\theta)$ [7]

$$\sigma_{rr}\left(\theta\right)=\sigma_m\left(\theta\right)-k\cos\psi\left(\theta\right) \qquad \sigma_{\theta\theta}\left(\theta\right)=\sigma_m\left(\theta\right)+k\cos\psi\left(\theta\right) \qquad \sigma_{r\theta}\left(\theta\right)=k\sin\psi\left(\theta\right) \tag{4}$$

Asymptotic Solution in Plastic Sector

When in-plane stress components satisfying the yield criterion, that is, eq. (4) is substituted in the equilibrium equations, eq. (1), the governing equations in plastic sector leads to two different stress solutions near the crack tip.

In constant stress sector (in which the mean stress σ_m is a constant quantity)

$$\psi\left(\theta\right)=2\theta+\psi_o \qquad \sigma_m\left(\theta\right)=C_1 \tag{5}$$

In fan field (where the mean stress σ_m is a linear function of the polar angle θ)

$$\psi\left(\theta\right)=\frac{\pi}{2}+n\pi \qquad \sigma_m\left(\theta\right)=-2k\theta\cos(n\pi)+C_2 \tag{6}$$

The constants ψ_o, C_1, C_2 and the integer n in eqs. (5) and (6) can be determined by boundary and continuity conditions.

Asymptotic Solution in Elastic Sector

In an elastic sector in addition to satisfying the asymptotic equilibrium equation, eq. (1), the deformation field has to be compatible. Stresses in such an elastic sector can be finally expressed as [10]

$$\sigma_{rr}(\theta) = -A\cos 2\theta - B\sin 2\theta + 2C\theta + D \tag{7a}$$

$$\sigma_{\theta\theta}(\theta) = A\cos 2\theta + B\sin 2\theta + 2C\theta + D \tag{7b}$$

$$\sigma_{r\theta}(\theta) = A\sin 2\theta - B\cos 2\theta - C \tag{7c}$$

The integration constants A, B, C and D can be determined from the boundary and continuity conditions.

Assembly of crack tip sectors

Asymptotic elastic and plastic sectors were assembled to maintain the continuity of tractions, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ across sector boundaries. However, radial stress σ_{rr} may be discontinuous across the sector boundaries. For a centred fan adjoining an elastic, or constant stress sector, $\sigma_{r\theta} = k$. There can be no stress jump in σ_{rr} and full continuity of all the stress components is required, that is,

$$\sigma_{\alpha\beta}\left(\theta^-_i\right) = \sigma_{\alpha\beta}\left(\theta^+_i\right) \tag{8}$$

Here θ^-_i and θ^+_i correspond to angles just before and after the border-delimitation angle θ_i, respectively.

ELASTIC-PLASTIC CRACK TIP STRESS FIELD UNDER MODE-I LOADING

For mode-I crack, the traction free conditions on the crack flank and symmetric deformation requires that

$$\sigma_{\theta\theta}(\pm\pi) = 0 \quad \sigma_{r\theta}(\pm\pi) = 0 \quad \sigma_{r\theta}(0) = 0 \tag{9}$$

The general elastic-plastic crack tip stress field, for an incompressible elastic perfectly-plastic material, actually comprises of 4 plastic sectors and one elastic sector as shown in Fig.1 (a). It is easy to visualise that when $\theta_4 = \pi$, the proposed 5-sector field degenerates to the 4-sector stress solution, Fig.1 (b). Since the first 3-sectors in proposed 5 sector field are identical to those of Zhu and Chao (2001) we omit the details and directly express the resulting equations as follows

$$\psi(\theta) = \begin{cases} 2\theta & 0 \le \theta \le \theta_1 = \dfrac{\pi}{4} \\ \dfrac{\pi}{2} & \theta_1 \le \theta \le \theta_2 \\ \dfrac{\pi}{2} + 2(\theta - \theta_2) & \theta_2 \le \theta \le \theta_3 \end{cases} \qquad \sigma_m(\theta) = \begin{cases} k(1+\pi) + T_p & 0 \le \theta \le \theta_1 = \dfrac{\pi}{4} \\ k\left(1 + \dfrac{3}{2}\pi - 2\theta\right) + T_p & \theta_1 \le \theta \le \theta_2 \\ k\left(1 + \dfrac{3}{2}\pi - 2\theta_2\right) + T_p & \theta_2 \le \theta \le \theta_3 \end{cases} \tag{10}$$

Here T_p is an undetermined constant in the asymptotic analysis. It has to be evaluated from full-field solution like SLF or finite element analysis. This T_p parameter was defined by Zhu and Chao [7] as

$$T_p = \sigma_{\theta\theta}^{app}(0) - \sigma_{\theta\theta}^{Prandtl}(0) \tag{11}$$

Thus, T_p essentially represents the hydrostatic stress ahead of the crack tip and carries the same physical meaning as that of Q proposed by O'Dowd and Shih [2]. The stress components in the second constant stress sector, using eq. (4), are as follows

$$\sigma_{rr}(\theta) = k\left(1+\frac{3}{2}\pi - 2\theta_2\right) + k\sin 2(\theta-\theta_2) + T_p \tag{12a}$$

$$\sigma_{\theta\theta}(\theta) = k\left(1+\frac{3}{2}\pi - 2\theta_2\right) - k\sin 2(\theta-\theta_2) + T_p \tag{12b}$$

$$\sigma_{r\theta}(\theta) = k\cos 2(\theta-\theta_2) \tag{12c}$$

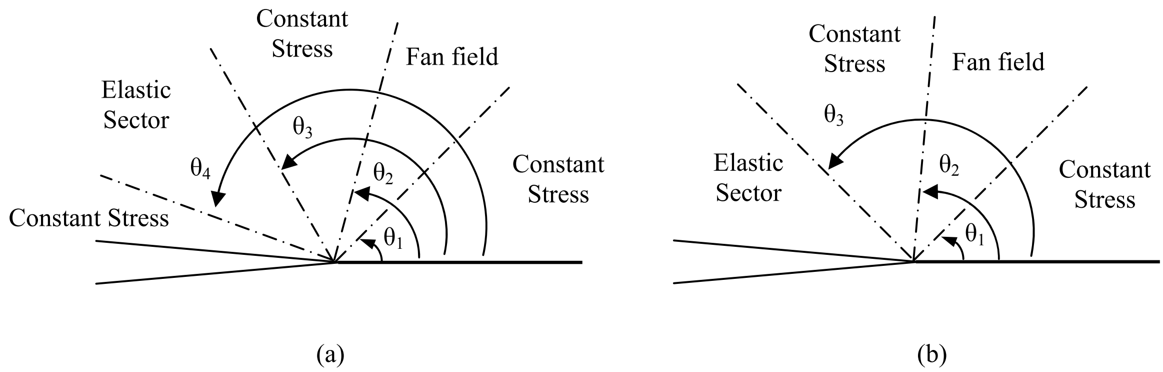


Fig.1: Description of elastic-plastic crack-tip stress fields under mode I by (a) Proposed 5-sector field and (b) 4-sector solution of Zhu and Chao [7].

In the third constant stress sector, adjacent to crack flank, the stress distribution can be obtained by using the traction free condition on the crack flank. The general solution can be expressed as

$$\psi(\theta) = 2\theta - \pi + n\pi \tag{13}$$

For compressive yielding of crack flank n=1 and the resulting stress function and stress components can be expressed as follows

$$\psi(\theta) = 2\theta \tag{14}$$

$$\sigma_{rr}(\theta) = -k(1+\cos 2\theta) \qquad \sigma_{\theta\theta}(\theta) = -k(1-\cos 2\theta) \qquad \sigma_{r\theta}(\theta) = k\sin 2\theta \tag{15}$$

For a fully continuous stress solution around the crack tip, the integration constants (A, B, C and D) that describes the stress distribution in the elastic sector can be expressed in terms of the border angles separating the two adjacent stress sectors by using the continuity conditions, eq. (8). Continuity conditions provide two additional equations, which are as follows

$$k\left(1+\frac{3}{2}\pi - 2\theta_2\right) + T_p = 2C\theta_3 + D \tag{16}$$

$$2C\theta_4 + D = -k \tag{17}$$

These two equations are insufficient to solve for three unknowns, that is, θ_2, θ_3 and θ_4. In other words, the T_p parameter alone can not characterise the crack tip stresses. It may be noted that the T_π parameter proposed by Zhu and Chao [7] that was used to describe the stress state in the elastic sector near the crack flank (in case of 4-sector field) losses its applicability in the general elastic plastic 5-sector stress solution. For various possible states of stress near the crack tip (for a 5-sector solution) the value of T_π parameter is fixed (T_π=-2k). In view of this limitation of T_π parameter we propose the hydrostatic/mean stress in the second constant stress sector as an independent and additional parameter which in conjunction with T_p parameter is used to describe the general state of stress near the crack tip in an elastic perfectly-plastic material.

We define the T_{CS-2} parameter as the difference of mean stress in the actual state of stress from the reference Prandtl field in the second constant stress sector.

$$T_{CS-2} = \sigma_m{}^{app} - k \tag{18}$$

In the second constant stress sector (θ_2<θ<θ_3) we have

$$\sigma_m = k\left(1 + \frac{3}{2}\pi - 2\theta_2\right) + T_p \tag{19}$$

The angular position of the border separating the fan field and the second constant stress sector can be expressed in terms of the two constraint indexing parameters as follows

$$\theta_2 = \frac{3}{4}\pi - \frac{\left(T_{CS-2} - T_p\right)}{2k} \tag{20}$$

For a given value of T_p and T_{CS-2} parameter, θ_2 can be obtained from Eq. (20). Eqs. (16) and (17) can then be used to evaluate the two unknown variables, that is, θ_3 and θ_4. Thus, a complete characterisation of crack tip stresses is possible by means of these two independent constraint indexing parameters. It may be easily demonstrated that a definite mathematical relation exists between the constraint parameters T_p and T_{CS-2} when transition occurs from the 5-sector field to the 4-sector stress field. This limiting condition corresponds to the situation when compressive yielding just initiates at the crack tip. Beyond this condition any further loss of crack tip constraint would spread compressive yielding in the region adjacent to crack flank and the resulting stress distribution cannot be described by the 4-sector field.

FINITE ELEMENT SIMULATION OF CRACK TIP STRESS FIELDS FOR DIFFERENT CONSTRAINTS

To verify the proposed asymptotic 5-sector field detailed full-field FE analyses were performed. All analyses were performed on mismatch welded middle tension, M(T), specimen, under large-scale plasticity (at limit state), to simulate a wide range of T_p and T_{CS-2} parameters representing different crack tip constraint levels. 2-D plane strain analyses were carried out on idealised weld without any heat affected zone (HAZ) (see Fig. 2). The two materials were assumed to have same elastic modulus and Poisson's ratio but mismatch in their yield strength. The elastic response of base and weld material was modeled as isotropic and almost incompressible (ν=0.49). The effects of geometry were modelled by changing the weld slenderness ratio (ψ) while the strength mismatch effects were incorporated by changing the mismatch ratio (M). These two parameters are defined as follows

$$M = \frac{\sigma_{YW}}{\sigma_{YB}} \qquad \psi = \frac{W-a}{H} = \frac{l}{H} \tag{21}$$

Material mismatch ratio M and weld slenderness ratio ψ were systematically varied to account all practical cases. Fig. 3 shows the FE discretisation scheme used in present investigation. To avoid problems associated with incompressibility eight-noded plane strain element with reduced integration were employed in all FE calculations. Due to symmetry only one-quarter of M(T) specimen was modelled. The numerical model employs the small-strain formulation. The material was model as isotropic, elastic-plastic obeying the Von-Mises yield criterion. Uniform rigid displacement was applied on the top edge of this FE model and the specimen was loaded to its limit state. For all the cases analysed, the values of constraint-indexing parameters, that is, T_p and T_{CS-2} parameters are provided in Table 1. Detailed comparison of analytical results with FE analyses is presented only for few representative cases (see Figs. 4 and 5).

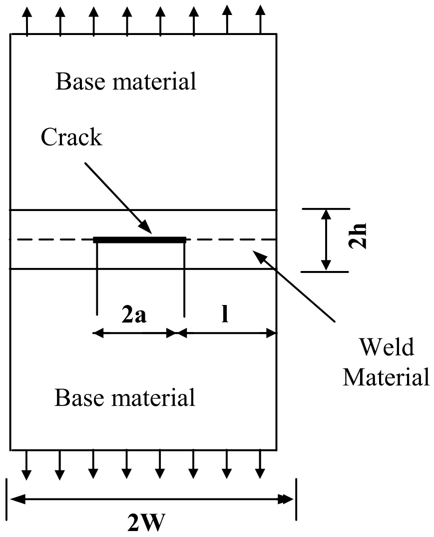


Fig.2: Idealised M(T) specimen with weld centre crack.

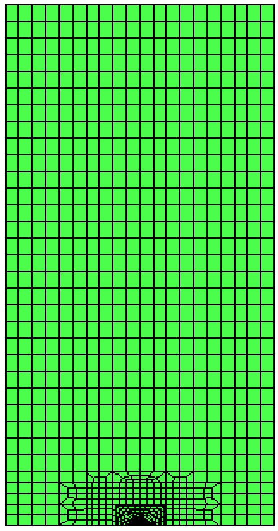


Fig.3: Finite element discretisation scheme used in present investigation.

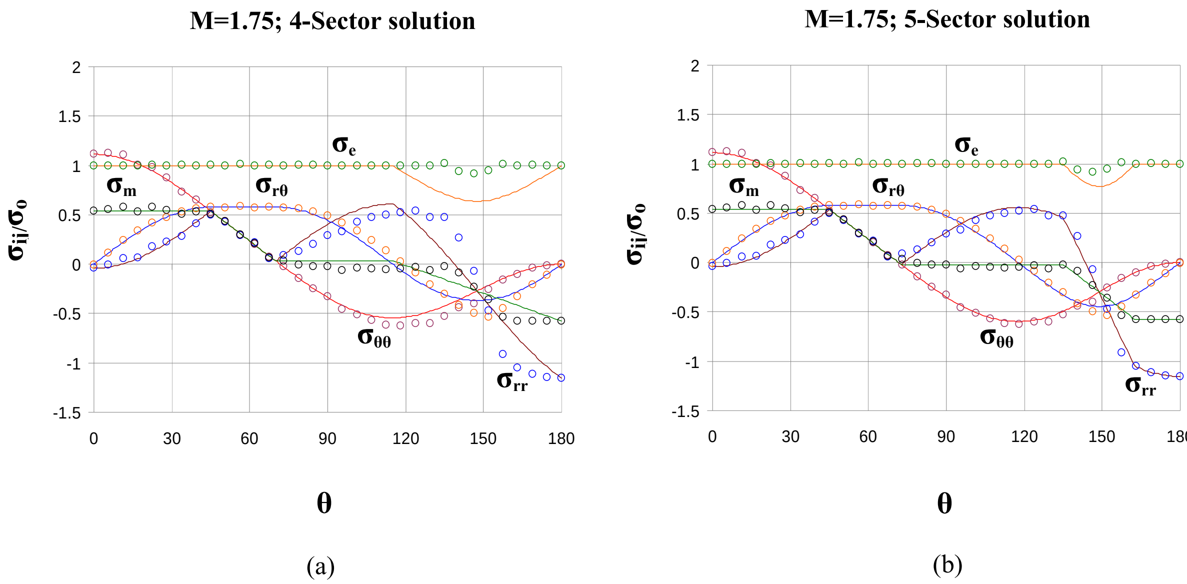
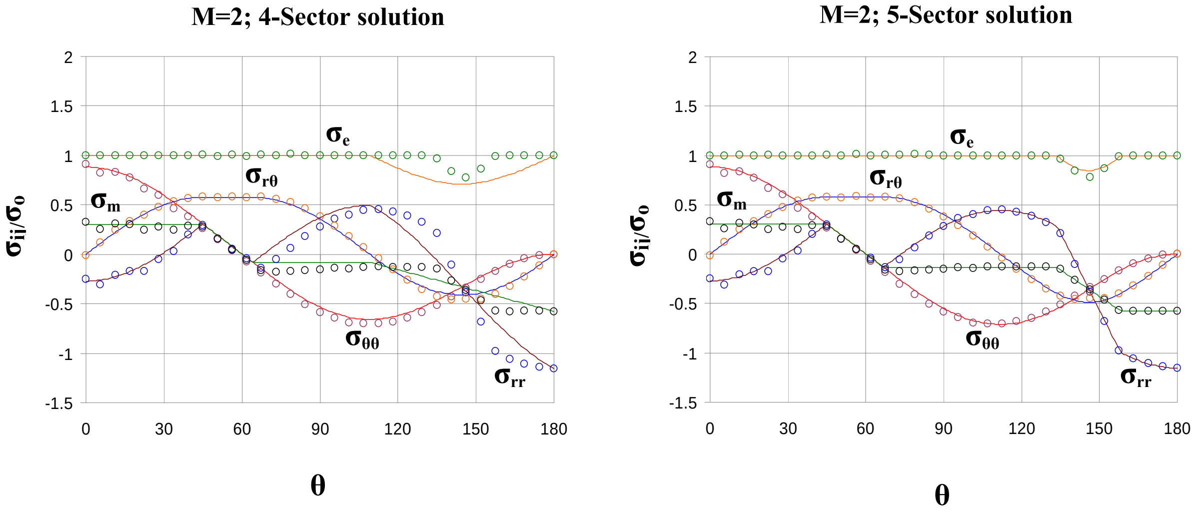


Fig.4: Comparison of crack-tip stresses for mismatch welded M(T) specimen with ψ=1.67 and M=1.75. (a) Asymptotic 4-sector stress solution proposed by Zhu and Chao [7]. (b) Proposed 5-sector stress solution. The open circles refer to results obtained from FEA.



(a) (b)

Fig.5: Comparison of crack-tip stresses for mismatch welded M(T) specimen with ψ=10 and M=2. (a) Asymptotic 4-sector stress solution proposed by Zhu and Chao [7]. (b) Proposed 5-sector stress solution. The open circles refer to results obtained from FEA.

Table 1: Numerical values of T_p, $T_{CS\text{-}2}$ and T_π parameters obtained from FE analyses.

Weld slenderness ratio (ψ)	Mismatch ratio (M)	T_p/k (FEA)	$T_{CS\text{-}2}/k$ (FEA)	Crack tip stress field classification
10	0.25	0	0	Prandtl field
	0.5	-0.371	0.0587	4-sector stress field
	0.75	-1.959	-0.0648	4-sector stress field
	0.9	-2.464	-0.3426	4-sector stress field
	1.25	-3.027	-0.791	4-sector stress field
	1.5	-3.243	-0.952	4-sector stress field
	1.75	-3.443	-1.133	5-sector stress field
	2	-3.611	-1.232	5-sector stress field
5	0.25	0	0	Prandtl field
	0.5	-0.376	0.0898	4-sector stress field
	0.75	-1.928	-0.0347	4-sector stress field
	0.9	-2.439	-0.312	4-sector stress field
	1.25	-3.076	-0.782	4-sector stress field
	1.5	-3.3	-0.995	4-sector stress field
	1.75	-3.498	-1.165	5-sector stress field
	2	-3.625	-1.247	5-sector stress field
3.33	0.9	-2.574	-0.389	4-sector stress field
	1.25	-3.011	-0.729	4-sector stress field
	1.5	-3.359	-1.04	4-sector stress field
	1.75	-3.475	-1.113	5-sector stress field
	2	-3.52	-1.18	5-sector stress field
	0.5	-0.401	0.0832	4-sector stress field
	0.75	-1.962	-0.0658	4-sector stress field
	0.9	-2.452	-0.323	4-sector stress field

2.5	1.5	-3.265	-0.979	4-sector stress field
	1.75	-3.32	-1.092	5-sector stress field
	2	-3.34	-1.109	5-sector stress field
1.667	0.25	-1.45	0.199	4-sector stress field
	0.5	-1.475	0.09	4-sector stress field
	0.75	1.987	-0.0378	4-sector stress field
	0.9	-2.559	-0.3893	4-sector stress field
	1.25	-3.072	-0.767	4-sector stress field
	1.5	-3.213	-0.948	4-sector stress field
	1.75	-3.212	-1.026	5-sector stress field
	2	-3.276	-1.058	5-sector stress field

CONCLUSIONS

Asymptotic crack tip stress fields are developed for a stationary plane strain crack in an incompressible elastic-perfectly plastic material under mode-I loading. Salient conclusions are as follows

1. The general elastic-plastic crack tip field can be fully described by the 5-sector stress solution proposed in this article. The 4-sector field proposed by Zhu and Chao [7] is a subset of this 5-sector field.
2. It is well known that the loss of constraint at the crack tip leads to an elastic sector at the crack flank. This study has revealed that cases arise where the severe loss of crack tip constraint can lead to compressive yielding of crack flank. This particular situation leads to 5-sector stress field.
3. A new constraint parameter T_{CS-2} is proposed which along with hydrostatic stress ahead of crack tip is capable of representing the general elastic plastic crack tip stress fields over all angles around a crack tip.
4. Detailed 2-D elastic plastic finite element analyses were performed to examine the validity of the proposed 5-sector stress field for a wide range of crack tip constraint. Excellent agreement is obtained between the proposed asymptotic crack tip stress field and the finite element results.
5. For elastic-perfectly plastic material at limit state, the proposed constraint parameters, that is, T_p and T_{CS-2} are adequate to represent the crack tip constraint arising due to specimen geometry and loading conditions as well as the additional constraint arising due to weld strength mismatch.

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