

ANALYTICAL EVALUATION OF CRACK TIP CONSTRAINT FOR RIGID PLASTIC MATERIAL USING WORK PRINCIPLES

I.A. Khan¹, V. Bhasin¹, J. Chattopadhyay¹, K.K. Vaze¹ and A.K. Ghosh²

¹Reactor Safety Division, Bhabha Atomic Research Centre, Mumbai, INDIA-400085

²Health, Safety & Environment Group, Bhabha Atomic Research Centre, Mumbai, INDIA-400085

E-mail of corresponding author: imran@barc.gov.in

ABSTRACT

Many detailed studies have established that the existence of the J-annulus (J-controlled regime) is strongly dependent on the state of stress near the crack tip. To overcome this limitation of single parameter J-based characterisation, several approaches have been developed which rely on quantification of crack tip constraint. Thus, an accurate evaluation of crack tip stresses is a pre-requisite for detailed fracture analyses. In this work, analytical formulation of Modified Upper Bound (MUB) approach, recently proposed by Khan and Ghosh [1], is briefly discussed. Application of MUB approach for evaluation of crack tip constraint in homogeneous deeply cracked fracture specimens is described. Finally apart from accurate evaluation of limit load, detailed evaluation of crack tip constraint is performed, using MUB analysis, for a middle tension, M(T), specimen having strength mismatch weld.

INTRODUCTION

In elastic-plastic fracture mechanics J-integral has been widely used as a fracture parameter. As long as the HRR fields, that is, the J-dominated zone exists over size scale comparable to process zone (over which micro-separation process are active), a criterion for the onset of crack growth can be developed in terms of a critical value of J-integral [2]. However, several detailed studies have established that the existence of this J-annulus is strongly dependent on the state of stress near the crack tip. To overcome this limitation of single parameter J-based characterisation several approaches like $J-T$ of Betegon and Hancock [3], $J-Q$ by O'Dowd and Shih [2], and $J-A_2$ of Chao et al. [4] have been developed which rely on quantification of crack tip constraint. Thus, an accurate evaluation of crack tip stresses is a pre-requisite for a detailed fracture analyses.

For a rigid plastic material slip-line field (SLF) analysis can provide sufficiently accurate estimates of the crack tip constraint and the corresponding limit loads. Constructing complete SLF solutions involves discovering a field that satisfies (i) the Hencky equations for equilibrium and yield condition in the deformed region, (ii) the Geiringer equations for incompressibility and (iii) equilibrium and yield inequality in the rigid regions. As a result, constructing complete SLF is relatively difficult and theoretical solutions exist only for limited geometries and loading conditions.

Apart from SLF analysis, the extremum theorems of classical plasticity have also been extensively used for obtaining the limit load. The upper bound theorem, in particular, has been widely used in metal forming operations where no exact solutions for the load to cause unconstrained plastic deformation are available. The assumption which is often invoked to simplify the analysis is to consider the rigid mode of deformation, that is, the material is assumed to move in rigid blocks separated by lines of tangential displacement discontinuity [5]. This results in considerable simplification and useful upper bounds on limit loads can be easily obtained. Unfortunately, as a result of this simplifying assumption, particularly for the problems involving predominant bending loads, this upper bound approach provides (unacceptable) higher estimates of the limit loads. Thus, despite its efficiency the use of this upper bound analysis is quite restricted. Moreover the upper bound theorem do not provide any information about stress distribution.

Till now, these two methods of plastic analysis, that is, the upper bound theorem and SLF have remained more or less independent apart from the fact that both are upper bounds as they use kinematically admissible velocity fields. Recently, a new load bounding technique, Modified Upper Bound (MUB) approach, was proposed by Khan and Ghosh [1]. In this work, analytical formulation of MUB approach is briefly discussed. Application of MUB approach for evaluation of crack tip constraint in homogeneous deeply cracked fracture specimens is described. Finally apart from accurate evaluation of limit load, detailed evaluation of crack tip constraint is performed using MUB analysis for a middle tension, M(T), specimen having strength mismatch weld.

ANALYTICAL FORMULATION OF MODIFIED UPPER BOUND APPROACH

The basic expression of upper bound theorem can be expressed as [6]

$$\int_S T_i v_i^* dS \leq \int_V \sigma_{ij}^* d\xi_{ij}^* dV + \int_{S_p} k v^* dS_p \quad (1)$$

Here v^* denotes the tangential displacement increment discontinuity on a surface S_p for the kinematically admissible velocity field v_i^* , k the shear yield strength, T_i the traction acting on the surface S , $d\xi_{ij}^*$ the assumed rate of plastic strain increments as derivable from v_i^* in the usual way and σ_{ij}^* is a stress field, not necessarily statically admissible, derivable by way of the concept of the plastic potential from the strain increment field $d\xi_{ij}^*$. The equality sign is achieved only when the kinematically possible field v_i^* coincides with the actual field v_i .

The assumption which is often invoked to simplify the analysis is to consider the rigid mode of deformation, that is, the material is assumed to move in rigid blocks separated by lines of tangential displacement discontinuity. Under this condition $d\xi_{ij}^*$ becomes zero and Johnson and Mellor [5] sets the first integral of the right hand side of eq. (1) to zero. The velocity discontinuity v^* now refers to any fictitious mode of deformation assumed in the problem. This method requires no integration, in plane strain cases, as the tangential component of velocity discontinuity v^* is constant along any assumed deformation path. This results in considerable simplification and useful upper bounds can be easily obtained. Unfortunately, as a result of this simplifying assumption, particularly for the problems involving predominant bending loads, this upper bound approach provides (unacceptable) higher estimates of the limit loads. Thus, despite its efficiency the use of this upper bound analysis is quite restricted. To overcome this limitation Khan et al. [7] assumed that a rigid plastic body actually comprises of two distinct regions viz. Region I consisting of a set of volumes, say V_m ($m=1, 2, 3\dots$), comprising of rigid blocks of materials separated by surfaces of tangential displacement discontinuity (similar to the assumption made in classical upper bound approach) and Region II consisting of another set of volumes say V_n ($n=1, 2, 3\dots$), having deforming zone. They further assumed that the stress distribution in deforming zone is such that it satisfies differential equations of equilibrium at each and every point in this plastically deformed zone. However, neither information regarding the state of stress in region I was required nor it was necessary to prove that the stress distribution there satisfies the equilibrium equations. With these assumptions, Khan et al. [7] has proposed following expression of modified upper bound (MUB) approach

$$\int_S T_i v_i^* dS \leq \sum \int_{S_{Dm}} k v^* ds_m + \sum \int_{S_n} \sigma_{ij}^* v_i^* n_j dS_n \quad (2)$$

If the assumption of rigid mode of deformation is invoked over the whole body then second integral term of eq. (2) becomes zero and hence eq. (2) would reduce to the conventional form of upper bound theorem that is widely used in load bounding estimates.

APPLICATION OF MUB APPROACH FOR EVALUATION OF CRACK TIP CONSTRAINT OF HOMOGENEOUS DEEPLY CRACKED FRACTURE SPECIMENS

The cracked bend specimens are nowadays frequently used in fracture mechanics analysis. In order to ensure that a high crack tip constraint exist in these specimens the testing standards usually recommend deeply cracked geometries subjected to predominant bending load. This recommendation ensures that the fracture toughness so obtained would be a conservative estimate of the fracture toughness of the actual structure under investigation. In low strength metal specimens the cracked section is normally fully yielded before crack growth initiation. Under these conditions slip-line field (SLF) analysis, assuming that material is rigid-plastic, can provide sufficiently accurate estimates of stresses in plastic region and the corresponding limit load. SLF method requires integration of stress distribution of plastically deformed regions to arrive at global equilibrium equations. On the other hand the expression of MUB approach itself provides an equation of global equilibrium for force or bending moment. Minimum work principle may then be invoked to evaluate the unknown parameters of the plastic deformation field. This process of minimization automatically leads to other equations of global equilibrium which are identical to

those obtained from the SLF analysis. The equivalence of MUB approach with SLF analysis for a wide variety of problems was discussed in detail by Khan et al [7]. In order to describe the use of MUB approach for evaluation of crack tip constraint, a homogeneous deeply cracked pure bending specimen, SE(PB), is analysed here.

Single edge cracked specimen in pure bending (SE(PB)): For a deeply cracked ($a/W \geq 0.3$) SE(PB) specimen (assumed plastic field is shown in Fig. 1), the resulting expression for limit moment [1] can be expressed as

$$M_L = k \left[R^2 \left(\beta + \frac{\pi}{4} \right) + x(R + 0.5x) \right] \quad (3)$$

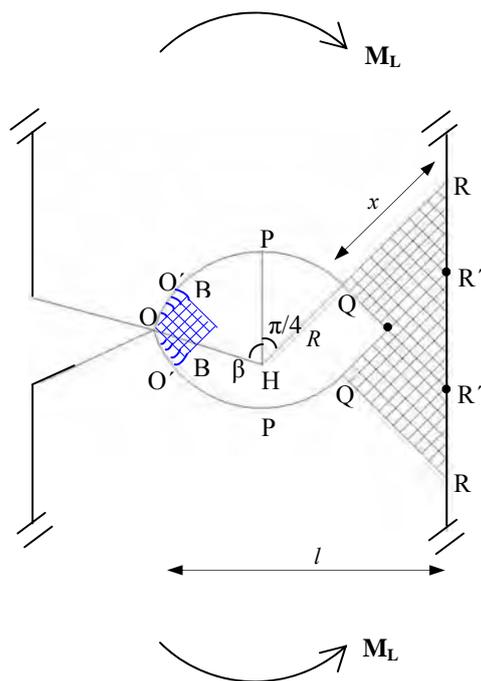


Fig.1: Assumed Plastic Deformation mechanism for SE(PB) specimen.

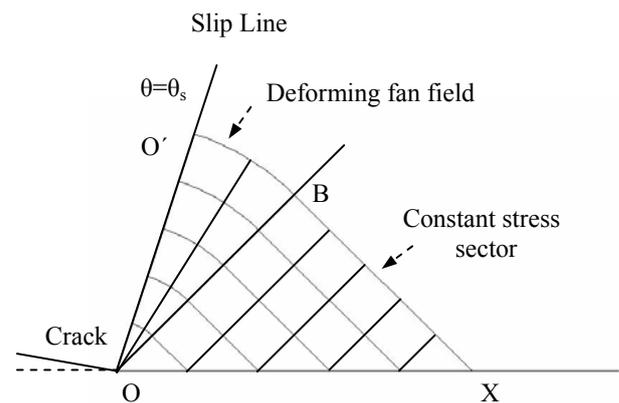


Fig. 2: Fully plastic crack-tip stress fields for deeply cracked SE(PB) specimens.

Eq. (3) represents the condition of global moment equilibrium about the hinge point. From geometry following relation can be easily obtained

$$R = \frac{l - \frac{x}{\sqrt{2}}}{\left(\sin \beta + \frac{1}{\sqrt{2}} \right)} \quad (4)$$

Here x and β are the two independent unknown parameters that would be evaluated using minimum work principle. Since the algebra involved is quite standard only important steps/equations are provided. Minimizing eq. (3) with respect to these two unknown parameters we have

$$\frac{\partial M_L}{\partial x} = k \left[\left(\beta + \frac{\pi}{4} \right) 2R \frac{\partial R}{\partial x} + R + x \frac{\partial R}{\partial x} + x \right] = 0 \quad (5)$$

$$\frac{\partial M_L}{\partial \beta} = k \left[R^2 + \left(\beta + \frac{\pi}{4} \right) 2R \frac{\partial R}{\partial \beta} + x \frac{\partial R}{\partial \beta} \right] = 0 \quad (6)$$

On further simplification, these two equations can be expressed as

$$(\sin \beta + \cos \beta) = 2 \cos \beta \left(\beta + \frac{\pi}{4} \right) \quad (7)$$

$$x = \frac{R}{\sqrt{2}} \left[\cos \beta - \sin \beta + 2 \sin \beta \left(\beta + \frac{\pi}{4} \right) - \sqrt{2} \right] \quad (8)$$

Eqs. (7) & (8) actually represent global equilibrium conditions which can also be obtained from detailed SLF analysis [8]. It is important to note that if the presence of compressive zone is neglected then eq. (3) would reduce to the classical upper bound solution proposed by Prager [9], that is,

$$M_L = 0.398 \sigma_y l^2 \quad (9)$$

which is about 10% higher than that obtained from detailed SLF/MUB solution.

Crack-Tip constraint for SE(PB) specimen: In the previous section, it was established that, for the assumed plastic deformation field, minimum work principle automatically satisfies the global equilibrium equations. It means that the state of stress in the regions having rigid mode of deformation and hence through out the body is identical with that obtained from detailed SLF analyses. Thus, the assumed plastic deformation field is in fact SLF and Hencky's theorem can now be used at any point in the plastically deformed regions to evaluate the state of stress. However, from the stress distribution we cannot determine the constraint parameter Q at the crack tip directly as the assumed plastic field can only give the stress components along the slip lines which radiate from the crack tip and which are inclined to the horizontal line with an angle larger than 45° [10]. From this kind of slip line fields the stress field surrounding the crack tip is not uniquely obtainable. Following Hao et al [10], possible crack-tip stress field is illustrated in Fig. 2. In this figure we assume, asymptotically, that a small segment of straight slip line OO' exists. It radiates from the crack tip and is connected to the arc OP in the global slip line field. Thus the stress components on these small lines are constant and equal to the components at the point O' on the arc $O'P$. The plastic deformation expand from the line OO' ahead of the crack tip, as shown in Fig. 2 and form a diamond-like plastic zone $OBXB$ like that in Prandtl field.

In Fig. 1, $OPQR'$ is a β slip line along which $\frac{\sigma}{2k} + \theta = \eta$. At point Q , $\sigma = -k$ and $\theta = \pi/4$. Thus, $\eta_Q = \eta_O = \frac{\pi}{4} - \frac{1}{2}$. At point O , $\theta = -\beta$ and on substituting $\psi = \pi/4 + \beta$, $\sigma_O = k(2\psi - 1)$. Also $O'BX$ is α -slip line along which $\frac{\sigma}{2k} - \theta = \xi$. Thus, $\xi_{O'} = \xi_B = \psi - \frac{1}{2} + \beta$. At point B , $\theta = -\pi/4$ and the normal pressure (hydrostatic stress) can be expressed in terms of slip angle β as follows.

$$\sigma_B = \sigma_x = k(2\psi - 1) + 2k \left(\beta - \frac{\pi}{4} \right) \quad (10)$$

In triangle OBX (that is actually a uniform stress zone) lying just below the crack tip, maximum tensile stress is given by the following expression

$$\sigma_{\theta\theta} = \sigma_B + k \quad (11)$$

Comparison of these local stresses near the crack tip and hence the constraint factor obtained using work principle and those from detailed SLF solution [11] is given in table 1.

Table 1: Comparison of results obtained from modified upper bound approach with SLF analysis [11] for SE(PB) specimen having $a/W=0.3 - 1$.

	R/l	ψ°	x/l	$2M_L/\sigma_Y l^2$	r_p	$(\sigma_m/\sigma_o)_{\theta=0}$	$(\sigma_{\theta\theta}/\sigma_o)_{\theta=0}$	Q
SLF	0.3886	117.04	0.502	1.2606	0.369	2.326	2.903	-0.064
MUB	0.3887	117.04	0.502	1.2606	0.369	2.326	2.903	-0.064

EVALUATION OF CRACK TIP CONSTRAINT FOR A MIDDLE TENSION, M(T), SPECIMEN HAVING STRENGTH MISMATCH WELD

As discussed earlier, SLF analysis has been widely used for evaluation of limit load and crack tip constraint for a rigid plastic material. However, it is well recognised that such classical methods are applicable to macroscopically homogeneous/single material. Welded structures have an abrupt material discontinuity at the base weld interface. Constructing SLF solutions for such problems is not straightforward as the stress connections conditions at the interface are not known. In past some investigators [10] have made an attempt to solve this problem analytically by assuming continuity of normal and shear stress along the slip line passing through the interface of base and weld material. However, this assumption is not well supported by the results obtained from detailed elastic-plastic finite element analysis. As a result, many subsequent investigations were made using detailed FE analysis to obtain accurate solutions of limit load and crack tip constraint for various fracture specimens having strength mismatch weld. In this work, MUB approach using a discontinuous stress solution is proposed to analyse M(T) specimen having a weld centre crack. Discontinuity is incorporated in the solution by assuming an unknown value of normal stress at the interface of two materials. MUB approach along with global equilibrium equations was utilized to obtain this unknown normal stress and hence the whole plastic field. In addition to limit load, detailed evaluation of crack tip stresses have been worked out. The results obtained were found to be in excellent agreement with the known FE solutions available in literature.

Consider the case of a M(T) specimen having a weld centre crack as shown in Fig. 3. Assumption of plane strain was made and analyses were carried out on idealised weld without any heat affected zone (HAZ). The two materials (base and weld) were considered as rigid plastic having mismatch in their yield strength. The effects of weld geometry were modelled by changing the weld slenderness ratio (ψ) while the strength mismatch effects were incorporated by changing the mismatch ratio (M). These two parameters are defined as follows

$$M = \frac{\sigma_{YW}}{\sigma_{YB}} \quad \psi = \frac{W-a}{H} = \frac{l}{H} \quad (12)$$

Material mismatch ratio M and weld slenderness ratio ψ were systematically varied to account all practical cases. $M>1$ correspond to an overmatch weld while $M<1$ refer to an undermatch weld. For an overmatch M(T) specimen the assumed plastic field is shown in Fig. 4. In this field it is assumed that a straight slip line emanate from the crack tip and crosses the base-weld interface. It is then merged into the fan field of angular extent γ whose centre lies at the base-weld interface. Near the free surface there is a region of constant stress (uniform tension) which merges with the fan field tangentially. It may be mentioned that this type of field was first suggested by Hao et al. [10], however, the authors did not provide any details of its analysis. The stress distribution in the constant stress region, CDE, can be expressed as

$$\sigma_{11} = 0, \quad \sigma_{22} = -2k \quad \text{and} \quad \sigma_{12} = 0 \quad (13)$$

From kinematics the relation between the rate of imposed displacement $\dot{\delta}$ and the tangential velocity v^* along the slip line AB can be easily established, that is,

$$v^* = \frac{\delta}{\sin\left(\frac{\pi}{4} + \gamma\right)} \tag{14}$$

Using stress distribution of the constant stress region *CDE* (eq. 13) in modified upper bound approach, eq. 2, the resulting expression for limit load can be expressed as

$$F_y = \frac{2\sigma_{yb}l}{\sqrt{3}} \left[\frac{(M-1)H/l}{\sin^2(\pi/4 + \gamma)} + \frac{\left(1 - \frac{x}{l} \cos(\pi/4 - \gamma)\right)}{0.5 \cos 2\gamma} + \frac{x}{l} \{(1 + 2\gamma) \sin(\pi/4 + \gamma) + \sin(\pi/4 - \gamma)\} \right] \tag{15}$$

Since γ is the only independent variable, as per MUB approach

$$\frac{dF_y}{d\gamma} = 0 \tag{16}$$

Thus, the limit load F_y and all the unknown parameters of assumed plastic field can be easily evaluated. When $\gamma=0$ the field reduces to that of homogeneous M(T) specimen.

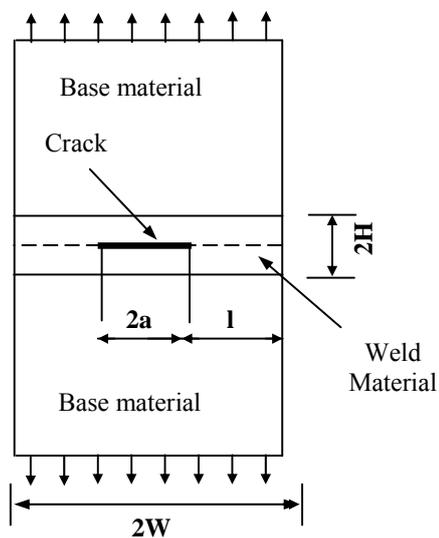


Fig.3: Idealised M(T) specimen with weld centre crack.

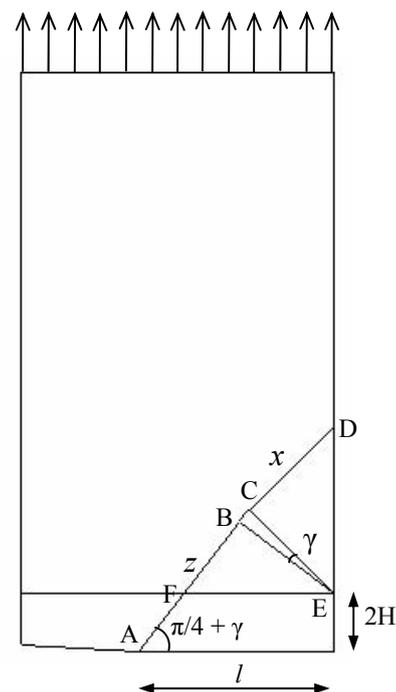


Fig.4: Assumed plastic field for a M(T) specimen with weld centre crack.

Once the parameter γ of assumed of global plastic field is known, the Hencky's equations may be used to construct stress field up to the base-weld interface. At this point it is assumed that there is a jump in the hydrostatic stress and the crack tip constraint can not be readily obtained.

The equations of global equilibrium can be expressed as follows

$$\frac{F_y}{2} = 2k_b x \cos(\pi/4 + \gamma) + \frac{k_b(1+2\gamma)x}{\sin(\pi/4 + \gamma)} + k_w H + \frac{\sigma_w^* H}{\tan(\pi/4 + \gamma)} \quad (17)$$

$$\frac{F_x}{2} = \frac{-k_b x \sin 2\gamma}{\sin(\pi/4 + \gamma)} + \frac{k_w H}{\tan(\pi/4 + \gamma)} - \sigma_w^* H = 0 \quad (18)$$

Here σ_w^* is the unknown value of hydrostatic stress that occurs at point F in the weld material. These two equations, that is, Eqs. (17) and (18) are insufficient to solve for three unknowns namely F_y , γ , and σ_w^* . For a wide variety of specimen geometry and loading conditions it has been established that the MUB approach provides results that are identical to those obtained from SLF analysis. Thus, if this equivalence of MUB approach and SLF analysis is assumed to hold good for the problems of strength mismatch weld also then the limit load F_y and γ obtained from MUB approach may be used to obtain σ_w^* from Eq. (17). As a cross check the value of σ_w^* so obtained was substituted in Eq. (18) to establish that the equilibrium equation in X-direction also get satisfied. This hypothesis was used to obtain crack tip constraint parameter h . For the case of M(T) specimen having weld centre crack results of limit load and crack tip constraint parameter h were obtained by Kim et al [12, 13] using detailed FE analysis. Detailed comparison of analytical results obtained from MUB approach with FE analyses [12, 13] is presented here (see Figs. 5 and 6). In Fig. 5 the normalised limit load represent the ratio of limit load of M(T) specimen having weld centre crack to that of homogeneous M(T) specimen. It may be observed from Fig. 5 that as the weld strength mismatch ratio M increases the difference between the limit load calculated on the assumption of continuity of stress at base-weld interface [10] and that obtained from detailed elastic-plastic FE analysis [12] becomes higher. This actually suggest that the assumption of continuous stress solution is not valid particularly for higher mismatch ratios. On the other hand the limit load obtained from MUB approach is in very good agreement with FE results [12] for all mismatch ratios. In Fig. 6 analytical solution of crack tip constraint obtained using MUB approach is compared with FE results of Kim et al. [13]. Reasonably good agreement was obtained between the two solutions.

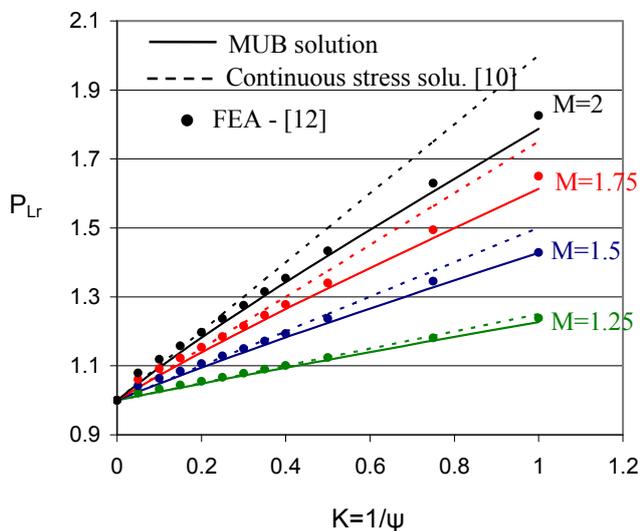


Fig. 5: Comparison of normalised limit load (MUB) of Overmatched M(T) specimen with FEA [12].

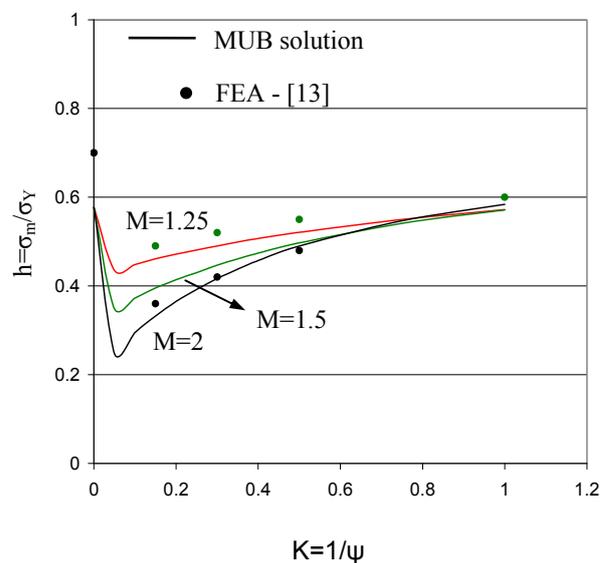


Fig. 6: Comparison of crack tip constraint parameter h of Overmatched M(T) specimen with FEA [13].

CONCLUSIONS

In this work analytical solutions of limit load and crack tip constraint for a rigid-plastic material under mode-I loading is described. For standard homogeneous fracture specimens MUB approach provide results that are in exact agreement with SLF solutions. Classical methods like SLF analysis are, however, applicable to macroscopically homogeneous/single material. Welded structures have an abrupt material discontinuity at the base weld interface. Constructing SLF solutions for such problems is not straightforward as the stress connections conditions at the interface are not known. Detailed FE analysis performed by Hao et al. [10] and Kim et al. [12] have revealed that SLF solutions based on continuity of stress at base-weld interface are not in good agreement with FE results. In this work MUB approach is successfully used to obtain analytical solution of limit load and crack tip constraint for a M(T) specimen having weld centre crack. Proposed analytical solutions are in good agreement with FE results [12,13].

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