MESHFREE METHOD STUDY FOR FRACTURE MECHANICS PROBLEMS

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ABSTRACT

In this work, Element Free Galerkin method (EFGM) has been used to obtain the solution of edge crack and centre crack problems in LEFM under mixed mode loading as it provides a versatile technique to model static as well as moving crack problems without any requirement of re-meshing. Edge crack plate problem has been solved using different enrichment and smoothing technique and compare with exact solution. Numerical example solved shows the effectiveness of intrinsic enrichment over the other methods. The effect of crack interactions also has been studied for multiple cracks lying on the same face as well as on opposite faces under plane stress conditions. The SIFs obtained by EFGM using different weight functions & different domains are also compared as a part of parametric study with those obtained by exact method.

INTRODUCTION

Engineering design has existed since humans first started to construct their own structures. It is defined as the process of servicing a system, component or process to meet desired requirement. It is a decision-making process (often iterative), in which the basic sciences, mathematics and engineering sciences are applied to optimally convert resources to meet a stated objective. Unlike the other creative arts, engineering design requires rigorous scientific and mathematical justification for each and every detail. Adhoc procedures and abstract ideas have no place in engineering design. In spite of all these aims, engineering cannot claim perfection. Slight imperfection can become the source of a catastrophic failure of structures.

Belytschko and his co-workers [1] developed the EFGM to model arbitrary crack propagation without remeshing based on an idea of Lancaster and Salkauskas [2]. Belytschko *et al*. [3] first used EFGM for arbitrary shape elasticity problems along with a simple heat conduction problem. Organ *et al*. [4] developed continuous mesh-free approximation for domains with non-convex boundaries. Two techniques namely diffraction and transparency were compared to the original visibility criterion. Fleming *et al*. [5] proposed an enriched EFGM for crack tip fields by adding asymptotic fields to the trial function. A local mapping of enriched fields was also presented for curved cracks. Barry *et al.* [6] developed a small strain, three-dimensional elasto-plastic based EFGM formulation. Singular weight functions were used in moving least square for the determination of shape function. Rao and Rahman [7] developed an efficient meshfree method for the analysis of linear elastic structure subjected to single or mixed mode loading condition. Pannachet *et al.* [8] made some comments on the enforcement of constraint equations in EFGM. Both Lagrange's multiplier and Penalty approach were used to solve benchmark problems. Muravin [9] in his PhD thesis proposed a multiple crack weight method for the solution of interacting multiple cracks. Chen *et al.* [10] developed an EFGM formulation for determining the SIF for an interface crack in an orthotropic coating with a homogeneous substrate. Zhang *et al*. [11] developed an adaptive element free Gelarkin-finite element (EFGM-FEM) coupling model for thermal elasto-plastic coupling problem.

Aswani *et al.* [12] presented elasto-plastic analysis of an edge crack has been performed using element free Galerkin method. A model problem has been solved in plane stress condition under mode-I loading. Singh *et al.* [13] implemented and compared domain partitioning, lagrange multiplier and jump function approach to solve solid mechanics problems containing material discontinuities. Rajesh *et al.* [14] presented a coupling technique for integrating the element-free Galerkin method with the fractal finite element method (FFEM) for analyzing the homogeneous, isotropic, two-dimensional linear-elastic cracked structures subjected to mixed-mode loading conditions. Sharma and Tiwari [15] presented a detail study for the edge crack problem using EFGM.

REVIEW OF EFGM

In EFGM, the field variable u is approximated by moving least square approximation, $u^h(\mathbf{x})$ Fleming [16], which is given by

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) a_j(\mathbf{x}) \equiv \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{1}$$

where, $\mathbf{p(x)}$ is a vector of complete basis functions (usually polynomial), $\mathbf{a}(\mathbf{x})$ is a vector of unknown coefficients and m is the number of terms in the basis function. The unknown coefficients $\mathbf{a}(\mathbf{x})$ are obtained by minimizing a weighted least square sum of the difference between local approximation, $u^h(\mathbf{x})$ and field function nodal parameters u_I. The weighted least square sum denoted by $L(\mathbf{x})$ can be written in following quadratic form:

$$L(\mathbf{x}) = \sum_{I=1}^{n} w(\mathbf{x}-\mathbf{x}_I)[\mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) - u_I]^2 \tag{2}$$

where, u_I is the nodal parameter associated with node I at $\mathbf{x}=\mathbf{x}_I$ but these are not the nodal values of $u^h(\mathbf{x}=\mathbf{x}_I)$ because $u^h(\mathbf{x})$ as an approximant not an interpolant, $w(\mathbf{x}-\mathbf{x}_I)$ is the weight function having compact support associated with node I, and n is the number of nodes with domain of influence containing the point $\mathbf{x}$, i.e. $w(\mathbf{x}-\mathbf{x}_I) \neq 0$.

By setting $\frac{\partial L}{\partial \mathbf{a}} = 0$, a following set of linear equation is obtained

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\,\mathbf{u} \tag{3}$$

The approximation function can be obtained as

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \Phi_I(\mathbf{x}) u_I = \mathbf{\Phi}^T(\mathbf{x})\mathbf{u} \tag{4}$$

where n is the number of nodes in the domain of influence of an evaluation point,

Weight Function

Weight functions are defined to be smooth and monotonically decreasing at each node such that the whole domain is covered. Common shapes of weight functions in two dimensions are circles and rectangles. The choice of weight function can affect the MLS approximation of $u^h(x)$.The various weight functions used in the present analysis are as follows:

Cubicspline weight function: $w(r) = \begin{Bmatrix} 2/3 - 4r^2 + 4r^3 & 0 \le r \le 1/2 \\ 4/3 + 4r - 4r^2 + (4/3)r^3 & 1/2 \le r \le 1 \\ 0 & r > 0 \end{Bmatrix}$

Quarticspline weight function: $w(r) = \begin{Bmatrix} 1 - 6r^2 + 8r^3 - 3r^4 & 0 \le r \le 1 \\ 0 & r > 0 \end{Bmatrix}$

Quadratic weight function: $w(r) = \begin{Bmatrix} 1 - r^2 & 0 \le r \le 1 \\ 0 & r > 0 \end{Bmatrix}$

Cosine weight function: $w(r) = \begin{Bmatrix} \cos(\pi r/2) & 0 \le r \le 1 \\ 0 & r > 0 \end{Bmatrix}$

Rational weight function: $w(r) = \begin{Bmatrix} 1/(r^n + c) & 0 \le r \le 1 \\ 0 & r > 0 \end{Bmatrix}$

Gaussian weight function: $w(r)=\begin{cases} \dfrac{e^{-(rc^2)}-e^{-c^2}}{1-e^{-c^2}} & 0\le r\le 1 \\ 0 & r>0 \end{cases}$

PROBLEM FORMULATION

We consider the two-dimensional (2D) problem with small displacements on the domain Ω bounded by Γ. The equilibrium equation is

$$\nabla\cdot\sigma+\mathbf{b}=0 \text{ in } \Omega \tag{5}$$

with the following essential and natural boundary conditions:

$$\sigma.\mathbf{n}=\bar{t} \qquad \text{on} \quad \Gamma_t$$
$$\mathbf{u}=\bar{\mathbf{u}} \qquad \text{on} \quad \Gamma_u$$

where, σ is the stress tensor which is defined as $\sigma=D\varepsilon$, D is the linear elastic material property matrix, ε is the strain vector, $\mathbf{b}$ is a body force vector, u is the displacement vector, $\bar{t}$ is the traction force and $\bar{n}$ is the unit normal. Enforcing essential boundary conditions using Lagrange multiplier approach, the following discrete equations are obtained:

$$\begin{bmatrix}\mathbf{K} & \mathbf{G}\\ \mathbf{G}^{\mathbf{T}} & 0\end{bmatrix}\begin{Bmatrix}\mathbf{u}\\ \lambda\end{Bmatrix}=\begin{Bmatrix}\mathbf{f}\\ \mathbf{q}\end{Bmatrix} \tag{6}$$

$$\mathbf{K}_{IJ}=\int_{\Omega}\mathbf{B}^T\mathbf{D}\mathbf{B}_J\,d\Omega,\ \mathbf{G}_{IK}=-\int_{\Gamma_u}\Phi_I\mathbf{N}_k\,d\Gamma_u,\ \mathbf{f}_I=\int_{\Gamma_t}\Phi_I\bar{\mathbf{t}}\,d\Gamma_t+\int_{\Omega}\Phi_I\mathbf{b}\,d\Omega,\ \mathbf{q}_k=-\int_{\Gamma_u}\mathbf{N}_{\mathbf{k}}\bar{\mathbf{u}}\,d\Gamma_u \tag{7}$$

$$\mathbf{B}_I=\begin{bmatrix}\Phi_{I,x} & 0\\ 0 & \Phi_{I,y}\\ \Phi_{I,y} & \Phi_{I,x}\end{bmatrix},\ \mathbf{N}_k=\begin{bmatrix}N_k & 0\\ 0 & N_k\end{bmatrix} \tag{8}$$

$$\mathbf{D}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} \text{ for plane stress.}$$

where, E is modulus of elasticity and ν is the Poisson's ratio.

RESULTS AND DISCUSSIONS

Edge Crack with Different Weight Function

An edge crack plate under mode-1 loading i.e. under pure tension as shown in Fig. 1 has been solved by element free Galerkin method. The following data *D=2* units, *L=1* unit, *a=0.4* units, $\sigma_0=1$ unit, $E=207000$ units and $\nu=0.3$ was taken for the analysis. For all the cells, a 4×4 Gauss Quadrature was used except for the two around the crack tip, where 9×9 Gauss Quadrature was used to handle the crack tip singularity. A plane stress condition was assumed. K_I and K_{II} have been evaluated by interaction integral approach using uniform nodal spacing for 288 nodes. The SIFs obtained by EFGM using different weight functions and exact method are tabulated in Table 1. It has been noticed that cubic spline and rational weight functions perform better as compared to other used weight functions.

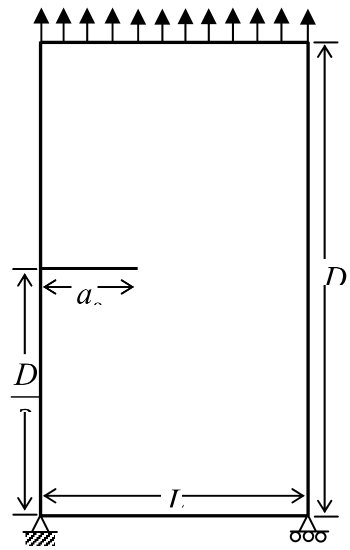

Fig. 1: Edge crack plate problem

Table 1 Stress Intensity Factor for the Edge Crack

Weight Function	Exact SIF (K_I)	Evaluated SIF (K_I)	% Error
Cubicspline	2.358	2.401	1.82
Quartispline		2.424	2.80
Quadratic		2.161	8.35
Cosine		2.286	3.05
R&R		2.416	2.46
Rational		2.398	1.70
Exponential		2.443	3.60
Gaussian		2.287	3.01

Edge Crack with Different Enrichment

An edge crack plate under mode-1 loading i.e. under pure tension as shown in Fig. 1 has been solved by element free Galerkin method. The dimensions and material property for plate is taken same above problem. The analysis is performed using cubic spline weight function. The essential boundary condition is imposed by Lagrange multiplier method. The stress intensity factor is computed using interaction integral approach.

For diffraction and transparency techniques, additional nodes are arranged along the crack line & at the crack tip to capture stress singularities. Such type of nodal arrangement is not required with extrinsic PU enrichment technique and intrinsic full basis enrichment technique.

Table 2: Comparison of mode-I stress intensity factor (K_I) for an edge crack

Parameters	Intrinsic	Extrinsic PU	Diffraction	Transparency	Exact
K_I	2.3988	2.3557	2.3396	2.4391	2.358
% error	1.73	0.10	0.89	3.44	-

With extrinsic PU enrichment technique heaviside & branch function are added to approximations to capture the stress singularities, whereas with full intrinsic basis enrichment, the entire near-tip asymptotic displacement field is included in the basis to captures the stress singularities. Thus, the arrangement of the nodes for enrichment techniques is much simpler than that of smoothing techniques. The SIFs obtained by EFGM using different enrichment and exact method are tabulated in Table 2. the stress intensity factors calculated by extrinsic PU enrichment technique are more accurate than those obtained by full intrinsic basis enrichment and diffraction techniques. Therefore, we can say that extrinsic PU enrichment technique is more accurate.

Edge Crack Interaction Problem

Edge crack interaction configuration shown in Fig. 3. This consider two right edge crack that are parallel to x-axis and one on the left edge is having inclination (α).The values of K_I and K_{II} have been evaluated at the tip of left edge crack only. K_I gives a symmetric variation with the change in inclination while K_{II} shows a decreasing-increasing trends as shown in **Fig.3.**

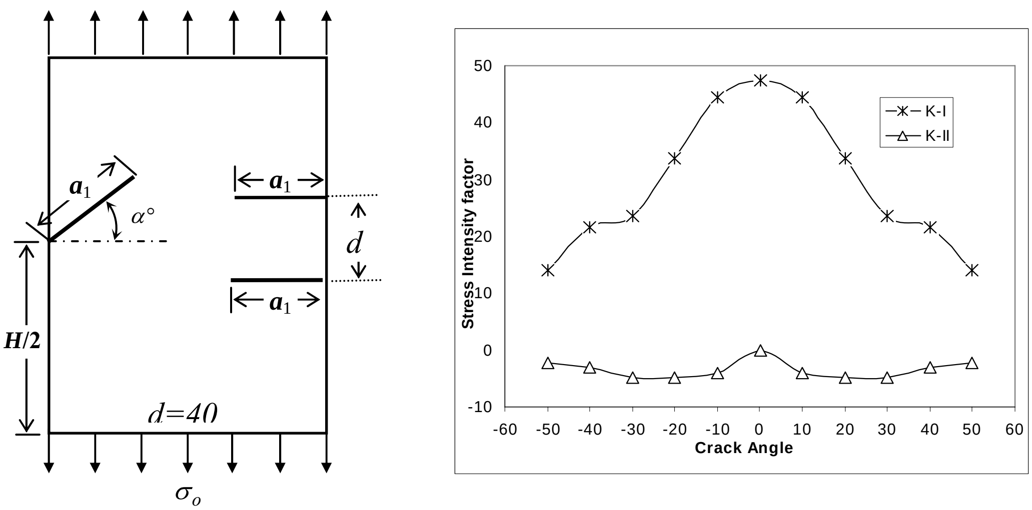

Fig. 3: Edge crack interaction problem

Edge Crack with Mixed Mode Loading

We consider a plate clamped on the bottom and subjected to shear traction $\tau = 1\, psi$ on the top and containing an edge crack of length $a = W/2 = 3.5\, in$ as Fig. 4. The material constants used are $E = 30 \times 10^6\ psi$ and $\nu = 0.25$, and plane strain conditions are assumed. The following values of stress intensity factors are taken from the reference.

Table 3 gives the results consisting of numerical values of stress intensity factors and error associated with them as function of domain size. For a given crack length (a = 3.5 in), the nodes in the entire domain (N=720) are kept constant. it is clear that stress intensity factors for mode-1 and mode-2 obtained by extrinsic PU enrichment technique remains nearly equal to reference values with small variations (maximum % error -2.61 and minimum % error 0.91 for mode-1 whereas maximum % error -6.08 and minimum % error 0.91 for mode-2) as the domain size changes.

Table 4 presents the results consisting of numerical values of stress intensity factors and error associated with them as the number of nodes in the domain varies. For a given crack length (a = 3.5), the size of domain (1 × 1) used for computing J integral is kept constant. It is clear that as the number of nodes increases the value of stress intensity factors obtained by extrinsic PU enrichment technique approaches to exact values.

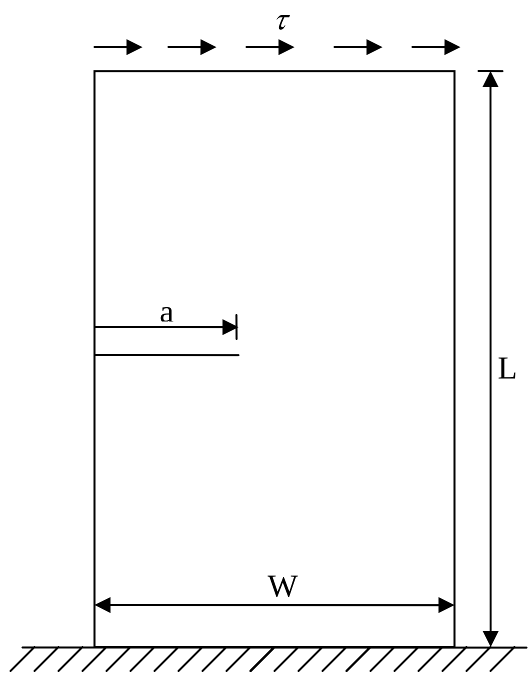

Fig. 4: Mixed Mode loading problem

Table 3: Numerical data for stress intensity factors as a function of domain size with constant number of nodes in the entire domain

Domain size	Modes	SIF ($psi\sqrt{in}$)	Ratio	% error	Exact SIF
1×1	Mode-1	34.0419	1.0012	-1.12	Mode-1= 34 $psi\sqrt{in}$ Mode-2= 4.55 $psi\sqrt{in}$
	Mode-2	4.6451	1.0209	-2.09	
2×2	Mode-1	34.7305	1.0214	-2.15	
	Mode-2	4.8268	1.0608	-6.08	
3×3	Mode-1	33.6922	0.9909	0.91	
	Mode-2	4.6911	1.0310	-3.10	
4×4	Mode-1	34.0373	1.0010	-1.11	
	Mode-2	4.6738	1.0272	-2.72	
5×5	Mode-1	34.8888	1.0261	-2.61	
	Mode-2	4.5913	1.0090	-0.91	

Table 4: Numerical data for stress intensity factors and error as a function of number of nodes in the entire domain with constant size of the domain used for computing J integral

Number of nodes	Modes	SIF ($psi\sqrt{in}$)	Ratio	% error	Exact SIF
528	Mode-1	36.2047	1.0648	-6.48	Mode-1= 34 $psi\sqrt{in}$ Mode-2= 4.55 $psi\sqrt{in}$
	Mode-2	4.8846	1.0735	-7.35	
576	Mode-1	35.1262	1.0331	-3.31	
	Mode-2	4.8349	1.0626	-6.26	
624	Mode-1	34.7751	1.0227	-2.28	
	Mode-2	4.7959	1.0540	-5.40	
672	Mode-1	34.4076	1.0119	-1.20	
	Mode-2	4.7168	1.0366	-3.66	
720	Mode-1	34.0419	1.0012	-0.12	
	Mode-2	4.6451	1.0209	-2.09	

CONCLUSION

In the present work, element free Galerkin method has been implemented for the analysis of edge and crack problem. It has been noticed that cubic spline and rational weight functions perform better as compared to other used weight functions. The stress intensity factors calculated by extrinsic PU enrichment technique are more accurate than those obtained by full intrinsic basis enrichment, transparency and diffraction techniques. On the basis of the above simulations, it can be predicted that crack inclination has a significant effect over the values of both mode-I and mode-II stress intensity factors. It has also noticed that domain size less significant in comparison number nodes in domain for accurate evaluation of stress intensity factor.

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