ABSTRACT

AN, MURAT. Clifford Algebraic Structure Study of the Interpolating Spinors and Electromagnetic Fields between the Instant Form Dynamics and the Light-Front Dynamics with an application to the Virtual Compton Scattering. (Under the direction of Chueng Ji.)

We study the common general Clifford algebraic structure shared by the Poincaré matrix and the electromagnetic field strength tensor and present the explicit interpolating spinor representations and electromagnetic fields between the instant form dynamics (IFD) and the light-front dynamics (LFD). We also apply our interpolating spinor representations to the analysis of virtual Compton scattering process in an attempt to support the analysis of the deeply virtual Compton scattering on $^4\text{He}$ target in Jefferson Lab.

We begin by constructing the generalized interpolating helicity spinors in chiral representation of $(J;0) \oplus (0,J)$ Lorentz group between the IFD and the LFD. Our analysis of interpolating scattering amplitudes shows that the broken symmetry under the longitudinal boost, $P^3 \leftrightarrow -P^3$, is restored only in the LFD.

We derive a direct connection between the polarization vectors in the $(1/2, 1/2)$ representation and the spin-1 spinors in the $(1,0) \oplus (0,1)$ representation by using the relationship between the gauge fields and the electromagnetic fields and express this connection as a $4 \times 6$ transformation matrix between the two representations. Interpolating polarization vectors lead to the interpolating electromagnetic fields in the $(1,0) \oplus (0,1)$ representation, which is used to derive the general solution of the Lorentz force equation. We investigate the kinematic and dynamic operator properties of the interpolating electromagnetic field strength tensor for the uniform constant fields and discuss how the number of fields corresponding to the dynamic operators changes between the IFD and the LFD.

As the Clifford algebra $(1,3)$ ($\text{Cl}_{1,3}$) has been a useful tool for the orthochronous Lorentz transformations, the complex Clifford algebra $\text{Cl}_{1,3}(\mathbb{C})$ is used for the derivation of spinors for the spin-1/2 and spin-1 particles. In addition to the well-known symmetric part of the tensor, composed of the photon polarization vector and its conjugate, we derive the anti-symmetric part of the tensor using the spinor definition of $\text{Cl}_{1,3}(\mathbb{C})$.

The dissertation is ended with an update on deriving the most general hadronic tensor structure of the virtual Compton scattering amplitude for a scalar target. Including the Bethe-Heitler process, we use the tensor representation derived from the Clifford algebra and confirm the previous results obtained by others on deeply virtual Compton scattering based on the dominance of the handbag diagrams. We also discuss the one-loop contribution with a scalar particle exchange in terms of Passarino-Veltman functions and its formal extraction of the Compton form factors for virtual Compton scattering.
Clifford Algebraic Structure Study of the Interpolating Spinors and Electromagnetic Fields between the Instant Form Dynamics and the Light-Front Dynamics with an application to the Virtual Compton Scattering

by
Murat An

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APPROVED BY:

__________________________  __________________________
Dean Lee                John Brown

__________________________  __________________________
Ronald Fulp              Danesha Seth Carley

__________________________
Chueng Ji
Chair of Advisory Committee
DEDICATION

This dissertation is dedicated to my loving and supportive wife Olga.
The author was born in a small city Muğla, Turkey. He went to Middle East Technical University to major physics and mathematics in 2001. In 2007 he was awarded by a scholarship from Turkey to study abroad. He came to Charlotte, NC for the language education in 2008 and was accepted by North Carolina State University in 2009.
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Chapter 1

Introduction

Dirac proposed three forms of relativistic Hamiltonian dynamics in 1949 [1], instant form dynamics (IFD) \((x^0 = 0)\), light-front dynamics (LFD) \((x^+ = (x^0 + x^3)/\sqrt{2} = 0)\) and point form \((x^2 = a^2 > 0, x^0 > 0)\). Although the point form’s exploration is recent [2], the instant form (the equal time \(t\)) and the light-front (the equal time \(\tau = t + z/c\)) have been the most popular choices.

There are several advantages of light-front dynamics over the instant form. First of all, the time we experience in real life is the light-front time \((x^+)\) rather than the instant time. We see a star’s light from years ago because of the speed of light is finite. The light from stars reaches us in different times, so every object’s time differs with distance. However, the light-front time fixes the time with distance so we experience the same light-front time regardless of the distance between them.
Secondly, the dispersion relation of the energy-momentum in instant form is given by

\[ k^0 = \sqrt{k^2 + m^2}, \]  

(1.1)

where \( k^0 \) is energy and \( \mathbf{k} \) is momentum vector.

In the light-front, the dispersion relation is given by

\[ k^+ k^- = |\mathbf{k}^\perp|^2 + m^2, \]

(1.2)

where \( k^- = (k^0 - k^3)/\sqrt{2} \) and \( k^+ = (k^0 + k^3)/\sqrt{2} \). Though Eq. 1.1 shows an irrational relation between energy and momentum, Eq. 1.2 yields a rational relation. It correlates the light-front energy \( k^- \) and \( k^+ \) and makes relations simpler. In the LFD when the light-front time is positive so is the momentum \( (k^+) \). This property of the LFD also leads to preventing the random creation of pair particles unless \( k^+ = 0 \).

There is no spontaneous creation in the light front vacuum. A constituent-type model in which all partons in a hadronic state are connected with hadrons instead of vacuum fluctuations can be obtained [3].

The interpolation between the IFD and the LFD provides a complete picture of the landscape between the two and clarifies the issue, if any, in linking them to each other. The same method of interpolating hypersurfaces has been used by Hornbostel [4] to analyze various aspects of field theories, including the issue of a nontrivial vacuum. The same kind of application to study the axial anomaly in the Schwinger model has also been presented [5], and other related works [6],[7], [8], [9],[10] can also be found in the literature.

LFD has the maximum number of kinematic generators compared to other forms of dynamics. There are seven kinematic generators which leave light-front time \( (x^+) \) invariant [11]. The study of the Poincaré algebra for any arbitrary interpolation angle started with [12] and recently provided the physical meaning of the kinematic vs. dynamic operators by introducing the interpolating time-ordered scattering amplitudes [13]. In particular, we demonstrated that the longitudinal boost invariance of each individual \( x^\perp \)-ordered scattering amplitudes is realized only at \( \delta = \pi/4 \) and the disappearance of the connected contributions to the current arising from the vacuum occurs independent of the reference frame only when the interpolation angle is taken to yield the LFD. This affirms the well-known saga of the longitudinal boost \( K^3 \) which maximizes the number of kinematic (i.e. interaction independent) generators in LFD as seven out of ten Poincaré generators. Dramatic character change of \( K^3 \) from “dynamic” for \( 0 \leq \delta \leq \pi/4 \) to “kinematic” in \( \delta = \pi/4 \) greatly benefits the use of LFD for the study of hadron physics.

The interpolation between IFD and LFD made it also clear that the disappearance of the connected contributions to the current arising from the vacuum in LFD doesn’t require the
infinite momentum frame (IMF). It thus resolves the confusion in the prevailing notion of equivalence between the LFD and the IMF. For the study of hadron physics in QCD, the built-in boost invariance together with the simpler vacuum property in LFD is certainly an appealing feature as it may save substantial computational efforts in getting QCD solutions that respect the full Poincaré symmetries.}

Although we want ultimately to obtain a general formulation for the QCD using the interpolation between the IFD and the LFD, we start from the simpler theory to discuss first the bare-bone structure that will persist even in the more complicated theories. Subsequent to our study of the simple scalar field theory [13] involving just the fundamental degrees of freedom such as the momenta of particles in scattering processes, we considered very recently interpolating the electromagnetic gauge degree of freedom between the IFD and the LFD and found that the light-front gauge in the LFD is naturally linked to the Coulomb gauge in the IFD through the interpolation angle [14]. We also extended our interpolation of the scattering amplitude presented in the simple scalar field theory [13] to the case of the electromagnetic gauge field theory but still with the scalar fermion fields known as the sQED theory [14] and analyzed the lowest scattering processes in sQED such as the analogues of the well-known QED process $e^+e^- \rightarrow \mu^+\mu^-$.}

1.1 Instant and Light-front Forms

The instant form quantization and the light-front quantization are the most popular choices among the three types of quantizations.

On light-front quantization, the equal time is defined as on the light-cone.

$$x^+ = \frac{1}{\sqrt{2}}(t + z),$$  \hspace{1cm} (1.3)

$$x^1 = x,$$ \hspace{1cm} (1.4)

$$x^2 = y,$$ \hspace{1cm} (1.5)

$$x^- = \frac{1}{\sqrt{2}}(t - z).$$ \hspace{1cm} (1.6)

The change of equal time brings a corresponding spatial coordinate change also $x^-$. The $x$ and $y$ coordinates remain unchanged. Here, the speed of light $c$ is taken as 1.

The light-front basis can be written with a transformation matrix $R$ from the instant form as
The metric tensor on light-front is

\[
\begin{pmatrix}
  x^+ \\
  x^1 \\
  x^2 \\
  x^-
\end{pmatrix} =
\begin{pmatrix}
  \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
  x^0 \\
  x^1 \\
  x^2 \\
  x^3
\end{pmatrix}.
\]

(1.7)

The metric tensor on light-front is

\[
g^{\mu\nu} = \begin{pmatrix}
  0 & 0 & 0 & 1 \\
  0 & -1 & 0 & 0 \\
  0 & 0 & -1 & 0 \\
  1 & 0 & 0 & 0
\end{pmatrix}.
\]

(1.8)

1.1.1 Lorentz Transformations

Lorentz transformation replaces the Galilean transformation with the additional dimension of Einstein’s special theory of relativity. A Lorentz transformation is a four dimensional transformation and is usually represented by $\Lambda^\mu_\nu$ as

\[x'^\mu = \Lambda^\mu_\nu x'^\nu,\]

(1.9)

However we use the inhomogeneous Lorentz transformation or Poincaré transformation which is described as

\[x'^\mu = \Lambda^\mu_\nu x'^\nu + a^\mu.\]

(1.10)

Here $a^\mu$ is a constant four vector. $\Lambda$ must satisfy

\[g_{\alpha\beta}\Lambda^\alpha_\mu \Lambda^\beta_\nu = g_{\mu\nu}.
\]

(1.11)

Weinberg [15] and some other authors refer to Poincaré transformations as Lorentz transformations in their work. Poincaré transformations (inhomogeneous Lorentz transformations) can be homogeneous Lorentz transformations when $a^\mu = 0$ in Eq. 1.10.

The Lorentz transformation can be written as an infinitesimal transformation up to the second order as

\[\Lambda^\mu_\nu = \delta^\mu_\nu + \delta\phi^\mu_\nu.\]

(1.12)

since it must satisfy the condition
\[ g_{\alpha\beta}(\delta_{\nu}^{\alpha} + \delta\phi_{\nu}^{\alpha})(\delta_{\rho}^{\beta} + \delta\phi_{\rho}^{\beta}) = g_{\mu\nu} \]
\[ g_{\mu\nu} + \delta\phi_{\mu\nu} + \delta\phi_{\nu\mu} + O(\phi^2) = g_{\mu\nu}. \] (1.13)

The Lorentz condition yields an anti-symmetric tensor

\[ \phi_{\mu\nu} = -\phi_{\nu\mu}. \] (1.14)

An infinitesimal Lorentz transformation with \( a^\mu = \epsilon^\mu \) in the Hilbert space can be written as

\[ U(1 + \phi, \epsilon) = 1 - \frac{1}{2} i \phi^{\mu\nu} M_{\mu\nu} + i \epsilon_{\mu} P_{\mu} + .... \] (1.15)

Here \( M_{\mu\nu} \) is an anti-symmetric Lorentz transformation which describes rotation and boost i.e. where

\[ M_{\mu\nu} = \begin{cases} 
M_{ij} & = \epsilon_{ijk} J_k \\
M_{i0} & = -K_i \end{cases} \quad (i, j, k = 1, 2, 3). \]

\( M_{\mu\nu} \) has a total of 6 components, and with \( P_{\mu} \), an inhomogenous Lorentz transformation has 10 generators. These generators construct a Lie algebra such as

\[ [M_{\mu\nu}, M_{\rho\delta}] = i(g_{\nu\rho} J_{\mu\delta} - g_{\mu\rho} J_{\nu\delta} + g_{\mu\delta} J_{\nu\rho} - g_{\nu\delta} J_{\mu\rho}), \] (1.16)
\[ [P_{\mu}, P_{\nu}] = 0, \] (1.17)
\[ [P_{\mu}, M_{\rho\delta}] = i(g_{\mu\rho} P_{\delta} - g_{\mu\delta} P_{\rho}). \] (1.18)

We can write \( M^{\mu\nu} \) as a matrix form and call it a Poincaré matrix

\[ M^{\mu\nu} = \begin{pmatrix}
0 & K^1 & K^2 & K^3 \\
-K^1 & 0 & J^3 & -J^2 \\
-K^2 & -J^3 & 0 & J^1 \\
-K^3 & J^2 & -J^1 & 0
\end{pmatrix}. \] (1.19)

There are ten Poincaré generators and these generators are separated as kinematic and dynamic generators as shown in Table 1.2, where kinematic generators do not alter the equal time, on the initial surface. Thus the Hamiltonian becomes invariant in such kinematic operators. As well, \([J^3, P^0] = 0\) and invariance is broken under dynamic generators \([K^3, P^0] = iP^3\).

The mapping from Lie algebra to Lie group may be expressed as

\[ \exp : so(3, 1) \to SO(3, 1), \] (1.20)
where the mapping is from the homogenous Lorentz algebra (Poincaré algebra) to the group of Lorentz transformation in $\mathbb{R}^{1,3}$.

By using a matrix exponential, the Lorentz transformation for boost and rotation groups can be written as

$$A(\phi, \theta) = e^{K \phi + J \theta}.$$  \hspace{1cm} (1.21)

Generators of the Lorentz transformations corresponds to important physical symmetries. The physical symmetry of rotation transformation $J$ corresponds to angular momentum and the physical symmetry of the boost transformation $K$ corresponds to motion in space-time. These transformation generators can be expressed in matrix form as

$$K^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (1.22)

and

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \hspace{1cm} (1.23)$$

The commutation relations of these matrices in general constructs three cyclic relations as

$$[J^i, J^j] = i\epsilon^{ijk} J_k; \hspace{1cm} (1.24)$$

$$[K^i, K^j] = -i\epsilon^{ijk} J_k; \hspace{1cm} (1.25)$$

$$[J^i, K^j] = i\epsilon^{ijk} K_k. \hspace{1cm} (1.26)$$

### 1.1.2 Light-front Lorentz Transformations

The four momentum in light-front form is the same as the coordinate changes: $P^+ = P_- = (P^0 + P^3)/\sqrt{2}$, $P^- = P_+ = (P^0 - P^3)/\sqrt{2}$, and $P_+ = P^-$ is the light-front energy.

The generators of the Lorentz group can be found by the light front T transformation in
Eq. 1.7 as

\[ M^{\mu\nu}_{LF} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & E^1 & E^2 & -K^3 \\ -E^1 & 0 & J^3 & -F^1 \\ -E^2 & -J^3 & 0 & -F^2 \\ K^3 & F^1 & F^2 & 0 \end{pmatrix}, \] (1.27)

where the light-front generators are expressed as

\[ E^1 = \frac{1}{\sqrt{2}} (K^1 + J^2), \] (1.28)

\[ E^2 = \frac{1}{\sqrt{2}} (K^2 - J^1), \] (1.29)

\[ F^1 = \frac{1}{\sqrt{2}} (K^1 - J^2), \] (1.30)

\[ F^2 = \frac{1}{\sqrt{2}} (K^2 + J^1). \] (1.31)

Among these ten Poincaré generators there are only three dynamic generators as shown in Table 1.2. Along with \( E^1 \) and \( E^2 \), \( K_3 \) also becomes a kinematic generator in LFD since \([P^+, K_3] = iP^+\). Therefore the initial surface \( x^+ = 0 \) becomes invariant under longitudinal boost.

### 1.2 Interpolating Form

The interpolating quantization where the equal time is defined between the instant time \( t \) and the light-front time \( \tau = (t+z)/\sqrt{2} \) by an interpolating angle \( \delta \) such that \( x^+ = t \cos \delta + z \sin \delta \). In general we set a space-time coordinate transformation between IFD and LFD by interpolating angle \( x^\mu = R^\mu_{\nu} x^\nu \), i.e.

\[ \begin{pmatrix} x^+ \\ x^1 \\ x^2 \\ x^- \end{pmatrix} = \begin{pmatrix} \cos \delta & 0 & 0 & \sin \delta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin \delta & 0 & 0 & -\cos \delta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \] (1.32)
where the interpolation angle is $0 \leq \delta \leq \pi/4$. The $\delta \to \pi/4$ limit gives us the light-front form and we recover the instant form at $\delta \to 0$ limit. The matrix form of $\mathcal{R}_{\mu}^\nu$ is

$$
\mathcal{R}_{\mu}^\nu = \begin{pmatrix}
\cos \delta & 0 & 0 & \sin \delta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sin \delta & 0 & 0 & -\cos \delta
\end{pmatrix},
$$

(1.33)

The metric can be written as

$$
g^{\hat{\mu}\hat{\nu}} = \mathcal{R}_{\alpha}^\mu g^{\alpha\beta} \mathcal{R}_{\beta}^\nu = \begin{pmatrix}
\mathbb{C} & 0 & 0 & \mathbb{S} \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
\mathbb{S} & 0 & 0 & -\mathbb{C}
\end{pmatrix},
$$

(1.34)

where $\mathbb{C}$ refers to $\cos(2\delta)$, $\mathbb{S}$ refers to $\sin(2\delta)$, and the inverse one is

$$
\mathcal{R}_{\mu\nu} = g_{\mu\alpha} \mathcal{R}_{\alpha}^\nu = \begin{pmatrix}
\cos \delta & 0 & 0 & -\sin \delta \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
\sin \delta & 0 & 0 & \cos \delta
\end{pmatrix}.
$$

(1.35)

The energy-momentum transformation will be the same as the coordinate transformation, as we have seen in the light-front case.

$$
P^+ = P^0 \cos \delta + P^3 \sin \delta, \quad (1.36)
$$

$$
P^1 = P^1, \quad (1.37)
$$

$$
P^2 = P^2, \quad (1.38)
$$

$$
P^- = P^0 \sin \delta - P^3 \cos \delta. \quad (1.39)
$$

**Interpolating Poincaré Matrix**

The Lorentz transformation generators given in the Poincaré matrix can be written in interpolating dynamics similar to Eq. 1.27 as

$$
M^{\hat{\mu}\hat{\nu}} = \mathcal{R}_{\alpha}^\mu M^{\alpha\beta} \mathcal{R}_{\beta}^\nu = \begin{pmatrix}
0 & E^1 & E^2 & -K^3 \\
-E^1 & 0 & J^3 & -F^1 \\
-E^2 & -J^3 & 0 & -F^2 \\
K^3 & F^1 & F^2 & 0
\end{pmatrix},
$$

(1.40)
and

\[ M_{\mu\nu} = g_{\mu\lambda} M^\lambda_{\nu\beta} g_{\nu\beta} = \begin{pmatrix} 0 & D^1 & D^2 & K^3 \\ -D^1 & 0 & J^3 & -K^1 \\ -D^2 & -J^3 & 0 & -K^2 \\ -K^3 & K^1 & K^2 & 0 \end{pmatrix}, \]  

(1.41)

where

\[ E^1 = K^1 \cos \delta + J^2 \sin \delta, \quad D^1 = -K^1 \cos \delta + J^2 \sin \delta, \]
\[ E^2 = K^2 \cos \delta - J^1 \sin \delta, \quad D^2 = -K^2 \cos \delta - J^1 \sin \delta, \]
\[ F^1 = K^1 \sin \delta - J^2 \cos \delta, \quad K^1 = -K^1 \sin \delta - J^2 \cos \delta, \]
\[ F^2 = K^2 \sin \delta + J^1 \cos \delta, \quad K^2 = -K^2 \sin \delta + J^1 \cos \delta. \]  

(1.42)

The Poincaré algebra is constructed by these generators and their commutation relations are given in Table 1.1. As we mentioned before there is one more kinematic generator in LFD. Since only dynamic generators alter the time component, \( K^3 \) is not a dynamic generator in LFD for \( x^+ \). Between \( 0 \leq \delta < \pi/4 \) \( K^3 \) is a dynamic generator for \( x^+ \) and at \( \delta = \pi/4 \), it is not a dynamic generator.

Table 1.1: Poincaré Algebra for any interpolation angle. The commutation relation reads [element in the first column, element in the first row] = element at the intersection of the corresponding row and column.

<table>
<thead>
<tr>
<th></th>
<th>( P_+ )</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_- )</th>
<th>( D^1 )</th>
<th>( D^2 )</th>
<th>( K^3 )</th>
<th>( K^1 )</th>
<th>( K^2 )</th>
<th>( J^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_+ )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-iCP^1</td>
<td>-iCP^2</td>
<td>iP^-</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( P_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>iP^+_1</td>
<td>0</td>
<td>0</td>
<td>iP^-1</td>
<td>0</td>
<td>-iP^2</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>iP^+_2</td>
<td>0</td>
<td>0</td>
<td>iP^-2</td>
<td>iP^1</td>
</tr>
<tr>
<td>( P_- )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>iP^+_3</td>
<td>0</td>
<td>iP^-3</td>
<td>iP^1</td>
</tr>
<tr>
<td>( D^1 )</td>
<td>iCP^1</td>
<td>-iP^+_1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>iK^3</td>
<td>iF^1</td>
<td>-iJ^3S</td>
<td>-iJ^3C</td>
<td>-iD^2</td>
</tr>
<tr>
<td>( D^2 )</td>
<td>iCP^2</td>
<td>0</td>
<td>-iP^-_1</td>
<td>0</td>
<td>iJ^3C</td>
<td>0</td>
<td>iF^2</td>
<td>iJ^3S</td>
<td>iK^3</td>
<td>iD^1</td>
</tr>
<tr>
<td>( K^3 )</td>
<td>iP^-_1</td>
<td>0</td>
<td>0</td>
<td>-iP^+_2</td>
<td>0</td>
<td>iF^1</td>
<td>-iE^2</td>
<td>0</td>
<td>iE^1</td>
<td>iE^1</td>
</tr>
<tr>
<td>( K^1 )</td>
<td>0</td>
<td>-iP^-_2</td>
<td>0</td>
<td>-iJ^3S</td>
<td>-iE^1</td>
<td>0</td>
<td>iJ^3C</td>
<td>-iP^-3</td>
<td>0</td>
<td>iK^1</td>
</tr>
<tr>
<td>( K^2 )</td>
<td>0</td>
<td>0</td>
<td>-iP^-_3</td>
<td>iJ^3S</td>
<td>-iK^3</td>
<td>iE^2</td>
<td>0</td>
<td>iJ^3C</td>
<td>0</td>
<td>-iK^1</td>
</tr>
<tr>
<td>( J^3 )</td>
<td>0</td>
<td>iP^2</td>
<td>-iP^1</td>
<td>0</td>
<td>iD^2</td>
<td>-iD^1</td>
<td>0</td>
<td>iK^2</td>
<td>-iK^3</td>
<td>0</td>
</tr>
</tbody>
</table>

Among the covariant Poincaré generators for any interpolation angle, \( K^1, K^2, J^3, P_1, P_2, P_- \) are always kinematic generators as they don't alter the \( x^+ = 0 \) plane. The set of kinematic and
Table 1.2: Kinematic and dynamic generators for different interpolation angles

<table>
<thead>
<tr>
<th>Angle</th>
<th>Kinematic</th>
<th>Dynamic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = 0$</td>
<td>$K^1 = -J^2, K^2 = J^1, J^3, P^1, P^2, P^3$</td>
<td>$D^1 = -K^1, D^2 = -K^2, K^3, P^0$</td>
</tr>
<tr>
<td>$0 \leq \delta &lt; \pi/4$</td>
<td>$K^1, K^2, J^3, P^1, P^2, P^-$</td>
<td>$D^1, D^2, K^3, P^+$</td>
</tr>
<tr>
<td>$\delta = \pi/4$</td>
<td>$K^1 = -E^1, K^2 = -E^2, J^3, K^3, P^1, P^2, P^+$</td>
<td>$D^1 = -F^1, D^2 = -F^2, P^-$</td>
</tr>
</tbody>
</table>

dynamic generators depending on the interpolation angle are presented in Table 1.2.

The subset of the Poincaré group that doesn’t alter the time evolution parameter $x^+$ is called the stability group. As we can see, in LFD we have the greatest number of kinematic operators, and therefore the largest stability group. Since kinematic transformations don’t alter $x^+$, individual time-ordered amplitudes must be invariant under kinematic transformations.

1.3 Outline of the Rest of the Dissertation

As we have introduced interpolating form dynamics between IFD and LFD and the Lorentz transformations in this chapter, Chapter 1, we continue with the spinor representation in interpolating form in Chapter 2. We discuss the interpolating transformation operator with the connection of the light-front boost to the Jacob-Wick transformation operator and derive the interpolating spinors in helicity spinor form in chiral representation using the interpolating transformation operator between IFD and LFD. We then find amplitudes of an annihilation and creation scattering amplitude analogous to the electron-muon annihilation in helicity spinors. We observe that at the light-front limit, the amplitudes become frame independent and discuss the reasons behind this phenomena.

In chapter 3, we investigate the interpolating form of the Lorentz force equation for a charged particle under a constant uniform electromagnetic field as continuation of gauge field interpolation work presented in [14]. We discuss the connection between the gauge fields and the electromagnetic fields, i.e. between the four-component polarization vector representation and the six-component spin-1 spinor representation. With the analogy between the electromagnetic field tensor for constant uniform fields and the Poincaré matrix, we discuss the kinematic and dynamic characteristics in electromagnetism. Similar to the Poincaré matrix, the number of dynamic generators gets reduced in the LFD.

Chapter 4 is about the Clifford algebra, especially Clifford algebra (1,3) also called geometric or space-time algebra, since we are using the Minkowski metric. Because the Clifford algebra can be used as the algebra of vectors and spinors at the same time, this algebra becomes handy in terms of calculation of spinors and Lorentz transformations. We observe that the Clifford algebra presents a more effortless method to get the spinor of interpolating form for spin-1/2 and
spin-1 spinors by using the same transformation. Additionally, the connection of 4 component polarization vectors and 6 component spin-1 spinors is obtained through this algebra and this helps us to the complete polarization tensor’s anti-symmetric part by the relation between these two different spin-1 spinor representations.

As an application of this development of spin-1 spinor representation, we investigate the general structure of hadronic tensor for deeply virtual Compton scattering in Chapter 5. We discuss the simple case where the target is scalar case and get the most general form. The amplitudes of deeply virtual Compton scattering with Bethe-Heitler process are studied as well. One loop calculation is discussed for a scalar particle exchange with the outgoing real photon in terms of Passarino-Veltman functions.

Our conclusion is summarized in Chapter 6.
Chapter 2

Interpolating Helicity Spinors and Scattering Amplitudes

This chapter is about the spinor of interpolating form between instant form and light-front form. We investigate fermion fields in terms of spinors and scattering amplitudes.

To study the properties of a spinor in general, we need to understand that a spinor for a particle is characterized by two pieces of information: the momentum of the particle, and the spin orientation. The helicity spinor defined in the IFD has the spin of the particle either aligned or anti-aligned with its momentum direction. As shown in Figure 2.1, there are two kinds of spinors in terms of spin orientation: Dirac spinor where the spin is in z-direction in the rest frame and helicity spinor where the spin direction is along with the momentum direction. There are two representations of spinor in terms of handedness of particles: Standard representation where the spinor is a combination of right-handed and left-handed spinors as shown in Eq. 2.2 and chiral or Weyl representation where right-handed and left-handed spinors are decoupled. We show first the construction of spinors in IFD and discuss the construction of LFD in chiral representation and compare it with the standard representation. Then we discuss the interpolating spinors. Furthermore, we show an example of amplitudes. We discuss amplitudes of scalar particle spinors and the scattering of fermions using interpolating spinors that we obtained. We discuss the effect of frame dependence on helicity. We investigate the instant front limit \( (\delta \to 0) \) and the light-front limit \( (\delta \to \pi/4) \) and how spin orientation changes between them.

2.1 Spinors

There are two main spinor representations: chiral and standard representation. Chiral (Weyl) and standard representation spinors are related by the transformation matrix \( S \), which is given
by

\[ S = S^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix}, \tag{2.1} \]

where \( I_2 \) is a \( 2 \times 2 \) identity matrix. The spinors are related as

\[ \psi_{SR} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_R + \phi_L \\ \phi_R - \phi_L \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} = S\psi_{CR}, \tag{2.2} \]

and the Dirac matrices transform as

\[ \gamma_{SR}^\mu = S\gamma_{CR}^\mu S^\dagger. \tag{2.3} \]

There are six generators of the Lorentz group: three boost \( K \) and three rotation \( J \) generators. We can decouple them in order to construct two \( SU(2) \) groups since \( SO(1,3) \sim \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.1.png}
\caption{An illustration of the relations between different conventions and names we use in this dissertation. The red letters C and S stand for chiral and standard representations respectively. The blue letters H and D stand for helicity spinor and Dirac spinor respectively. The solid black arrow points from the instant form to the light front form, and the interpolation angle \( \delta \) goes from 0 to \( \pi/4 \). Both the helicity and Dirac spinors can be generated for arbitrary interpolation angle. The dashed purple arrow indicates the Melosh transformation, which relates the instant Dirac spinor to the light-front helicity spinor.}
\end{figure}
\( SU(2) \otimes SU(2) \) as
\[
A = \frac{1}{2} (J + iK), \quad (2.4)
\]
\[
B = \frac{1}{2} (J - iK), \quad (2.5)
\]
and their commutation relations become
\[
[A_i, A_j] = i \epsilon_{ijk} A_k, \quad (2.6)
\]
\[
[B_i, B_j] = i \epsilon_{ijk} B_k, \quad (2.7)
\]
\[
[A_i, B_j] = 0, \quad (2.8)
\]
for \( i, j, k = 1, 2, 3 \).

We can describe boost and rotation generators in \( SU(2) \) as
\[
J_R = \frac{1}{2} \sigma, \quad K_R = i \frac{1}{2} \sigma, \quad (2.9)
\]
\[
J_L = \frac{1}{2} \sigma, \quad K_L = -i \frac{1}{2} \sigma, \quad (2.10)
\]
as right-handed and left-handed systems, it is also called type I \((0, 1/2)\) and type II \((1/2, 0)\) systems. We can combine them together in \( SU(2) \times SU(2) \) and write them down together as
\[
K = \frac{i}{2} \left( \begin{array}{cc} \sigma & 0 \\ 0 & -\sigma \end{array} \right), \quad (2.11)
\]
\[
J = \frac{1}{2} \left( \begin{array}{cc} \sigma & 0 \\ 0 & \sigma \end{array} \right). \quad (2.12)
\]
We observe that \( A \) and \( B \) components are formed in matrix representation and they are
\[
A = \frac{1}{2} \left( \begin{array}{cc} \sigma & 0 \\ 0 & 0 \end{array} \right), \quad B = \frac{1}{2} \left( \begin{array}{cc} 0 & 0 \\ 0 & \sigma \end{array} \right). \quad (2.13)
\]
We can represent \( A \) and \( B \) or right-handed or left-handed operators as \((J, 0) \oplus (0, J)\) in chiral representation of the Lorentz group.

We follow the spin description in Ryder [16] to construct spinors in general. The plane wave solutions of Dirac equation has four spinors with positive and negative energy and spin up and
down for each case at rest as

\[ \psi(x) = u(0)e^{imt}, \quad \text{positive energy}, \quad (2.14) \]
\[ \psi(x) = v(0)e^{-imt}, \quad \text{negative energy}. \quad (2.15) \]

With spin conditions the spinors at rest are

\[
\begin{align*}
  u^{(1)}(0) & = \begin{pmatrix} \sqrt{m} \\ 0 \\ \sqrt{m} \\ 0 \end{pmatrix}, &
  u^{(2)}(0) & = \begin{pmatrix} 0 \\ \sqrt{m} \\ 0 \\ -\sqrt{m} \end{pmatrix}, &
  v^{(1)}(0) & = \begin{pmatrix} \sqrt{m} \\ 0 \\ -\sqrt{m} \\ 0 \end{pmatrix}, &
  v^{(2)}(0) & = \begin{pmatrix} 0 \\ \sqrt{m} \\ 0 \\ -\sqrt{m} \end{pmatrix}. \quad (2.16)
\end{align*}
\]

In chiral representation that is spinor separated as right-handed and left-handed as \( \psi_{CR} = \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \).

Since the spinors are decoupled in chiral representation, we use the chiral representation for spinor constructions. In order to carry spinors into the moving frame in terms of Dirac spinors, we use Eq. 2.11 to describe an arbitrary boost to any direction by the rapidity \( \eta \)

\[
\begin{pmatrix} \phi'_R \\ \phi'_L \end{pmatrix} = \begin{pmatrix} e^{\frac{1}{2} \sigma \cdot \eta} & 0 \\ 0 & e^{-\frac{1}{2} \sigma \cdot \eta} \end{pmatrix} \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}. \quad (2.17)
\]

The momentum in the moving frame can be written in terms of the boost angles as

\[
\begin{align*}
  \cosh(\eta/2) & = \left( \frac{p^0 + m}{2m} \right)^{1/2}, \quad (2.18) \\
  \sinh(\eta/2) & = \left( \frac{p^0 - m}{2m} \right)^{1/2}, \quad (2.19) \\
  \tanh(\eta/2) & = \frac{|p|}{p^0 + m}, \quad (2.20)
\end{align*}
\]

where \( \eta = \sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2} \). We can write down each individual momentum as

\[
\begin{align*}
  p^0 & = \cosh(\eta), \quad (2.21) \\
  p^1 & = \frac{\eta_1}{\eta} \sinh(\eta), \quad (2.22) \\
  p^2 & = \frac{\eta_2}{\eta} \sinh(\eta), \quad (2.23) \\
  p^3 & = \frac{\eta_3}{\eta} \sinh(\eta), \quad (2.24)
\end{align*}
\]
then the spinors in the momentum frame are obtained from the rest spinors that are given by Eq. 2.16 by using $e^{\frac{i}{2} \sigma \eta} = \cosh(\eta/2) + \frac{\sigma \eta}{\eta} \sinh(\eta/2)$ as follows:

\[
\begin{align*}
\mathbf{u}^{(1)}(p) &= \frac{1}{2\sqrt{2}} \begin{pmatrix}
p^0 + m + p^3 \\
\sqrt{2}p^R \\
-\sqrt{2}p^R \\
\end{pmatrix}, & \mathbf{u}^{(2)}(p) &= \frac{1}{2\sqrt{2}} \begin{pmatrix}
p^0 + m - p^3 \\
\sqrt{2}p^L \\
-\sqrt{2}p^L \\
\end{pmatrix}, \\
\mathbf{v}^{(1)} &= \frac{1}{2\sqrt{2}} \begin{pmatrix}
p^0 + m + p^3 \\
\sqrt{2}p^R \\
-p^0 - m + p^3 \\
\sqrt{2}p^R \\
\end{pmatrix}, & \mathbf{v}^{(2)} &= \frac{1}{2\sqrt{2}} \begin{pmatrix}
p^0 + m - p^3 \\
\sqrt{2}p^L \\
\sqrt{2}p^L \\
\sqrt{2}p^R \\
\end{pmatrix}.
\end{align*}
\]  

(2.25)

(2.26)

where $p^R = (p^1 + ip^2)/\sqrt{2}$ and $p^L = (p^1 - ip^2)/\sqrt{2}$.

### 2.1.1 Light-front Boost

There are two types of transformation operators which bring the rest frame spinor to the moving frame. The first one is a pure boost in any direction

\[ T = e^{-i\beta \cdot K}. \]  

(2.27)

The second one is the transformation operator described by Jacob and Wick in [17] and given by

\[ T_{JW} = e^{-i\theta^+ \cdot \mathbf{J}^+} e^{-i\beta_3 K^3}. \]  

(2.28)

This transformation type describes a canonical boost in $z$-direction first, then rotations to the direction of the momentum. The rotation part enables the spin direction to align with the momentum direction. The initial particle spin direction is in the $z$-direction.

The light-front transformation operator is given by [18] as

\[ T_{LF} = e^{-i(\beta_1 E^1 + \beta_2 E^2)} e^{-i\beta_3 K^3}. \]  

(2.29)

The momentum transformation can be obtained by the light-front transformation operator is given by Eq. 2.29 and by using commutation relations in Appendix A.1.

\begin{align*}
T_{LF}^{-1} P^+ T_{LF} &= e^{\beta_3 P^+}, \\
T_{LF}^{-1} P^\perp T_{LF} &= P^\perp + \beta^\perp e^{\beta_3 P^+}, \\
T_{LF}^{-1} P^- T_{LF} &= e^{-\beta_3} (P^- + 2e^{\beta_3} \beta^\perp \cdot P^\perp + |\beta^\perp|^2 e^{2\beta_3} P^+).
\end{align*}

(2.30)

(2.31)

(2.32)
We write the momentum of moving particles from the rest frame $P_0^\mu = (m, 0, 0, 0)$ transformed as

\begin{align*}
P^+ &= e^{\beta_3}m, \\
P^\perp &= e^{\beta_3}m\beta^\perp, \\
P^- &= e^{-\beta_3}m + e^{\beta_3}m|\beta^\perp|^2.
\end{align*}

(2.33) \hspace{1cm} (2.34) \hspace{1cm} (2.35)

### 2.1.2 Light-front Spinors

In addition to the Dirac spinors, which can be in standard or chiral (Weyl) representation, we also have helicity spinors. The difference between the Dirac and helicity spinors is how to define momentum direction according to the spin direction. In Dirac spinor, the spin is in the $z$-direction (positive or negative) in the initial state and momentum is in an arbitrary direction. However, the helicity spinor momentum is defined along with the direction of the spin as we stated in Section 2.1.1 by applying the transformation, which is given by (2.28) by Jacob and Wick [17]. Ahluwalia and Sawicki [19] applied the light-front boost which is given by Eq. 2.29 for finding light-front spinors. In this definition, the spinors at rest are defined the same as the Dirac spinors in chiral representation as in Eq. 2.16. Then the transformation operator, which is given by Eq. 2.28, applies to these spinors to convert it to the moving frame and the light-front helicity operator is given by Eq. 2.29 and used as

\[ \psi_{LF} = T_{LF}\psi_{CR}(0). \]

(2.36)

First, we define $T_{LF}$ in $(1/2, 0) \oplus (0, 1/2)$ Lorentz group. We need to find right-handed and left-handed parts as

\[ T_{LF} = \begin{pmatrix} T_{LF}^R & 0 \\ 0 & T_{LF}^L \end{pmatrix}. \]

(2.37)

Then, we use the Eq. 2.9 and Eq. 2.10 notations to find $T_{LF}$ transformation

\begin{align*}
(\beta_1 E_1 + \beta_2 E_2)^R &= \frac{1}{\sqrt{2}}(\beta_1(i\sigma_1 + \sigma_2) + \beta_2(i\sigma_2 - \sigma_1)), \\
(\beta_3 K^3)^R &= i\beta_3 \sigma^3
\end{align*}

(2.38)

\begin{align*}
(\beta_1 E_1 + \beta_2 E_2)^L &= \frac{1}{\sqrt{2}}(\beta_1(-i\sigma_1 + \sigma_2) + \beta_2(-i\sigma_2 - \sigma_1)), \\
(\beta_3 K^3)^L &= -i\beta_3 \sigma^3.
\end{align*}

(2.39)
By using Eq. 2.33-Eq. 2.35, we get

\[
\begin{align*}
u^{(1)}_{CR}(p) &= \frac{1}{\sqrt{\sqrt{2}p^+}} \begin{pmatrix} \sqrt{2}p^+ + m \\ p^R \\ \sqrt{2}p^+ - m \\ p^R \end{pmatrix}, \quad
\begin{pmatrix} 0 \\ m \end{pmatrix}, \\
u^{(2)}_{CR}(p) &= \frac{1}{\sqrt{\sqrt{2}p^+}} \begin{pmatrix} -p^L \\ \sqrt{2}p^+ \\ p^L \\ -\sqrt{2}p^+ \end{pmatrix}.
\end{align*}
\] (2.40)

Converting them in standard representation by \(S\) transformation is given by Eq. 2.1, then the spinors are

\[
\begin{align*}
u^{(1)}_{SR}(p) &= \frac{1}{\sqrt{2\sqrt{2}p^+}} \begin{pmatrix} \sqrt{2}p^+ + m \\ p^R \\ \sqrt{2}p^+ - m \\ p^R \end{pmatrix}, \quad
\begin{pmatrix} -p^L \\ \sqrt{2}p^+ \\ p^L \\ -\sqrt{2}p^+ \end{pmatrix}, \\
u^{(2)}_{SR}(p) &= \frac{1}{\sqrt{2\sqrt{2}p^+}} \begin{pmatrix} \sqrt{2}p^+ + m \\ p^R \\ \sqrt{2}p^+ - m \\ p^R \end{pmatrix}.
\end{align*}
\] (2.41)

\[
\begin{align*}
u^{(1)}_{SR}(p) &= \frac{1}{\sqrt{2\sqrt{2}p^+}} \begin{pmatrix} \sqrt{2}p^+ - m \\ p^R \\ \sqrt{2}p^+ + m \\ p^R \end{pmatrix}, \quad
\begin{pmatrix} p^L \\ -\sqrt{2}p^+ + m \\ -p^L \end{pmatrix}, \\
u^{(2)}_{SR}(p) &= \frac{1}{\sqrt{2\sqrt{2}p^+}} \begin{pmatrix} \sqrt{2}p^+ - m \\ p^R \\ \sqrt{2}p^+ + m \\ p^R \end{pmatrix}.
\end{align*}
\] (2.42)

\[
\begin{align*}
u^{(1)}_{SR}(p) &= \frac{1}{\sqrt{2\sqrt{2}p^+}} \begin{pmatrix} \sqrt{2}p^+ - m \\ p^R \\ \sqrt{2}p^+ + m \\ p^R \end{pmatrix}, \quad
\begin{pmatrix} p^L \\ -\sqrt{2}p^+ + m \\ -p^L \end{pmatrix}.
\end{align*}
\] (2.43)

### 2.2 Interpolating Transformation

The interpolating transformation operator is defined to be consistent with the light front boost is given by Eq. 2.29 and the instant form transformation operator is given by Eq. 2.28 as

\[
T = e^{-i(\beta_1 \kappa^1 + \beta_2 \kappa^2)} e^{-i\kappa^3 \beta_3},
\] (2.44)

where we choose the sign of \(\beta_1\) and \(\beta_2\) as positive. This makes the transformation operator consistent with Soper’s notation in [20].

At instant form limit (\(\delta \to 0\)), the transformation is given by Eq. 2.44 goes to the instant form Jacob-Wick transformation is given by Eq. 2.28 and at light-front limit (\(\delta \to \pi/4\)), it becomes the light-front boost which is given by Eq. 2.29.

Similar to light-front form, we could write the momentum transformations in interpolating
form as

\[
T^{-1}P^1T = P^1 - \beta_1 P^+_\alpha \sin \alpha \cosh \beta_3 - \beta_1 P^-_\alpha \frac{\sin \alpha}{\alpha} (\cosh \beta_3 + \mathbb{S} \sinh \beta_3)
+ \mathbb{C} \beta_1 \left( \beta_1 P^1 + \beta_2 P^2 \right) \frac{\cos \alpha - 1}{\alpha^2},
\]  
(2.45)

\[
T^{-1}P^2T = P^2 - \beta_2 P^+_\alpha \sin \alpha \cosh \beta_3 - \beta_2 P^-_\alpha \frac{\sin \alpha}{\alpha} (\cosh \beta_3 + \mathbb{S} \sinh \beta_3)
+ \mathbb{C} \beta_2 \left( \beta_1 P^1 + \beta_2 P^2 \right) \frac{\cos \alpha - 1}{\alpha^2},
\]  
(2.46)

\[
T^{-1}P^-T = P^- \cos \alpha (\cosh \beta_3 + \mathbb{S} \sinh \beta_3) + P^+_\alpha \cos \alpha \cosh \beta_3 + \left( \beta_1 P^1 + \beta_2 P^2 \right) \frac{\sin \alpha}{\alpha},
\]  
(2.47)

\[
T^{-1}P^+_T = P^+_\alpha (\cosh \beta_3 - \mathbb{S} \cos \alpha \sinh \beta_3) + P^- \left[ (1 - \mathbb{S}^2 \cos \alpha) \sinh \beta_3 + \mathbb{S} (1 - \cos \alpha) \cosh \beta_3 \right]
\times \frac{(\beta_1^2 + \beta_2^2)}{\alpha^2},
\]  
(2.48)

where \( \alpha = \sqrt{\mathbb{C} (\beta_1^2 + \beta_2^2)} \). Substituting the rest frame has momentum eigenvalues is given by \( P_\mu = (\cos \delta M, 0, 0, \sin \delta M) \) \( P^1 = P^2 = 0, P^- = -MB, \) and \( P^+_\alpha = AM \).

\[
T^{-1}P^-T = M \cos \alpha (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3),
\]  
(2.49)

\[
T^{-1}P^1T = -\beta_1 M \frac{\sin \alpha}{\alpha} (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3),
\]  
(2.50)

\[
T^{-1}P^2T = -\beta_2 M \frac{\sin \alpha}{\alpha} (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3),
\]  
(2.51)

\[
T^{-1}P^+_T = \frac{M}{\mathbb{C}} \left[ \sin \delta \cosh \beta_3 \left( \frac{\cos \delta}{\sin \delta} - \mathbb{S} \cos \alpha \right) + \cos \delta \sinh \beta_3 \left( \frac{\sin \delta}{\cos \delta} - \mathbb{S} \cos \alpha \right) \right].
\]  
(2.52)

Also the expression \( P^+_\alpha = \mathbb{C} P^+_\alpha + \mathbb{S} P^- \) is very useful for our calculations

\[
T^{-1}P^+_T = M (\cos \delta \cosh \beta_3 + \sin \delta \sinh \beta_3).
\]  
(2.53)
These relations between Eq. 2.49-Eq. 2.53 bring us

\[
\cos \alpha = \frac{P_-}{M (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3)},
\]

(2.54)

\[
\frac{\beta_1 \sin \alpha}{\alpha} = \frac{-P^1}{M (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3)},
\]

(2.55)

\[
\frac{\beta_2 \sin \alpha}{\alpha} = \frac{-P^2}{M (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3)},
\]

(2.56)

and also

\[
\cosh \beta_3 = \frac{1}{M \sqrt{C}} \left( \cos \delta P^+ - \sin \delta \frac{P_-}{\cos \alpha} \right),
\]

(2.57)

\[
\sinh \beta_3 = \frac{1}{M \sqrt{C}} \left(- \sin \delta P^+ + \cos \delta \frac{P_-}{\cos \alpha} \right).
\]

(2.58)

The relation of \( P^+ \), \( P_\perp \), and \( P_- \) is given by

\[
(P^+)^2 - M^2 C^2 = P_-^2 + P_\perp^2 C.
\]

(2.59)

Since this quantity in Eq. 2.59 appears so often in our calculations, we now give it a special symbol to simplify our notation

\[
\mathbb{P} = \sqrt{(P^+)^2 - M^2 C} = \sqrt{P_-^2 + P_\perp^2 C}.
\]

(2.60)

Solving Eq. 2.54-Eq. 2.58, one gets the following useful relations between parameters \( \beta_1 \), \( \beta_2 \), \( \beta_3 \), \( \alpha \) and the momentum components:

\[
\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3 = \frac{\mathbb{P}}{M},
\]

(2.61)

\[
\cos \delta \cosh \beta_3 + \sin \delta \sinh \beta_3 = \frac{P^+}{M}.
\]

(2.62)
and
\[
\cos \alpha = \frac{P_1}{\mp}, \quad (2.63)
\]
\[
\sin \alpha = \sqrt{\frac{P^2 \mathbb{C}}{\mp}}, \quad (2.64)
\]
\[
e^{\beta_3} = \frac{P^+ + P}{M (\sin \delta + \cos \delta)}, \quad (2.65)
\]
\[
e^{-\beta_3} = \frac{P^+ - P}{M (\cos \delta - \sin \delta)}, \quad (2.66)
\]
\[
\frac{\beta_j}{\alpha} = \frac{P_j}{\sqrt{P^2 \mathbb{C}}}, \quad (j = 1, 2). \quad (2.67)
\]

### 2.2.1 Interpolating Transformation Operator

The helicity spinors are obtained by applying the transformation \( T \) is given by Eq. 2.44 to the initial state at rest that has a spin projection along the \( z \) direction. We may denote a generalized helicity spinor in a given interpolation angle as \( |j; p; j, m\rangle \) for a particle of spin \( j \) moving with momentum \( p \) and helicity \( m \). This state \( |j; p; j, m\rangle \) is obtained by the transformation \( T \) from the spin eigenstate \( |0; j, m\rangle \) at rest, which has a spin projection along the \( z \) direction satisfying \( J_3 |0; j, m\rangle = m |0; j, m\rangle \). Thus, we may specify \( |j; p; j, m\rangle = T |0; j, m\rangle \).

Following the procedure of Leutwyler and Stern [21], we may then define a new spin operator \( J_i \) for a moving particle as \( J_i = TJ_iT^{-1} \) to get
\[
J_3 |j; p; j, m\rangle = TJ_3T^{-1}T |0; j, m\rangle = m |j; p; j, m\rangle, \quad (2.68)
\]
where \( m \) is now not only the eigenvalue of the ordinary spin operator \( J_3 \) for the initial state at rest \( |0; j, m\rangle \) but also the eigenvalue of the operator \( J_3 \) for the generalized helicity spinor state \( |j; p; j, m\rangle \). It is straightforward to verify that \( J_i \) satisfies the SU(2) algebra as \( J_i \) does:
\[
[J_i, J_j] = TJ_iJ_jT^{-1} - TJ_jJ_iT^{-1} = T[J_i, J_j]T^{-1} = i\epsilon_{ijk}T J_k T^{-1} = i\epsilon_{ijk}J_k. \quad (2.69)
\]

As we have shown in the work [12], the new spin operator \( J_i \) commutes with the mass operator \( M \) defined by \( M^2 = P^\mu P_\mu = P^2 + P^2 \mathbb{C} - 2P_\mu P^\mu \mathbb{S} \mathbb{C} - P_\mu^2 \mathbb{S} \mathbb{C} \) for any generalized helicity spinor state, i.e. \( [J_i, M] |j; p; j, m\rangle = 0 \). As we discuss below, the operator \( J_3 \) intermediates between
the usual Jacob-Wick helicity operator in IFD and the light-front helicity operator in LFD and thus offers the role of general helicity operator in-between for any interpolation angle $\delta$.

The helicity operator with $T$ transformation is given by

$$J_3 = \frac{1}{M} (\sin \delta \cosh \beta_3 + \cos \delta \sinh \beta_3) \left( J_3 P_+ - K_1 P^2 + K_2 P^1 \right).$$

(2.70)

Using the Poincaré algebra for any arbitrary interpolation angle [12], we find that the new spin operator written in terms of the parameters $\beta_1$, $\beta_2$ and $\beta_3$ remains unchanged with or without including $T_3$ [12] as $[J^3, K^3] = 0$ and is given by

$$J_3 = J_3 \cos \alpha + (\beta_1 K_2 - \beta_2 K_1) \frac{\sin \alpha}{\alpha}.$$

(2.71)

We can then use the relations in Eq. 2.63 to rewrite it in terms of the particle’s momentum, and get

$$J_3 = \frac{1}{\mathbb{P}}(P_+ J_3 + P^1 K_2 - P^2 K_1),$$

(2.72)

where $\mathbb{P} \equiv \sqrt{(P^+)^2 - M^2 C} = \sqrt{P_+^2 + P_+^2 C}$. It is interesting to note that this operator $J_3$ can also be written in terms of the Pauli-Lubanski operator [21] $W^\mu = \frac{i}{2} \epsilon^{\mu \nu \alpha \beta} P_\nu M_{\alpha \beta}$ simply as $J_3 = W^+/\mathbb{P}$. In the instant form limit ($\delta \to 0$), $K_1 \to -J^2$, $K_2 \to J^1$, $P_+ \to P^3$ and $\mathbb{P} \to \sqrt{(P^0)^2 - M^2} = |\mathbb{P}|$, and thus the operator $J_3$ coincides with the familiar helicity operator $P \cdot J/|\mathbb{P}|$ in the IFD. In the light-front limit ($\delta \to \pi/4$), $K_1 \to -E_1$, $K_2 \to -E_2$, $P_+ \to P^+$ and $\mathbb{P} \to \sqrt{(P^+)^2} = P^+$, and thus the operator $J_3$ coincides with the light-front helicity operator $J_3 + \frac{1}{P^+} (P^2 E_1 - P^1 E_2)$ as discussed in [22]. Thus, $J_3$ intermediates between the usual Jacob-Wick helicity operator in IFD and the light-front helicity operator in LFD and it is reasonable to identify the operator $J_3$ as the general helicity operator for any interpolation angle $\delta$. The helicity eigenvalue for the state $|p, j, m\rangle$ is $m$ as previously given in Eq. 2.68. One should note, however, that the generalized helicity defined by the operator $J_3$ agrees with the ordinary notion of helicity defined usually by the spin parallel or anti-parallel to the particle momentum direction only in the IFD. For different interpolation angles, in general, there’s a relative angle between the spin orientation and the momentum direction according to the generalized helicity designated by $J_3$. In LFD, the transverse light-front boost operators $E_1$ and $E_2$ involve the rotations and they generate the angle between the spin orientation and the momentum direction.

If the particle is moving in $+z$ or $-z$ direction, so that $P^1 = P^2 = 0$, then the generalized
helicity operator is given by Eq. 2.72 becomes

\[ J_3 = \frac{P_- J_3}{P} = \frac{P_+}{|P_+|} J_3. \]  \hspace{1cm} (2.73)

Thus, for an arbitrary interpolation angle, the helicity sign of a particle moving in \( \pm z \) direction depends on the sign of \( P_\pm \). In the light-front limit, \( P_\pm \to P^+ \) which is always positive, and thus the light-front helicity of the particle is positive once the spin is parallel to the \( +z \) direction regardless of whether the particle is moving in the \( +z \) direction or the \( -z \) direction. This is dramatically different from the ordinary helicity defined in the IFD where \( P_\pm \to P^3 \). For a particle moving in the \( -z \) direction, the light-front helicity and the ordinary Jacob-Wick helicity is therefore opposite to each other. The swap of helicity amplitudes caused by such dramatic difference between the light-front helicity and the ordinary Jacob-Wick helicity has been noticed previously in the deeply virtual Compton scattering process [23].

### 2.2.2 Finding Interpolating Spinors

We use the interpolating boost Eq. 2.44, which is derived from the Jacob-Wick helicity operator Eq. 2.28 so the interpolating spinors are helicity spinors just like the LFD spinors and we use the same method to find the interpolating spinors. Here we take the spinors are in chiral representation is given by Eq. 2.16 and apply the interpolating boost is given by Eq. 2.44.

First, we discuss type I:\( (\frac{1}{2}, 0) \) \( J = \frac{\sigma}{2}, K = -i \frac{\sigma}{2} \) spinor representation for the left-handed part of the spinors.

The longitudinal part is

\[ e^{i \beta_3 K^3} = e^{-\beta_3 \frac{\sigma^3}{2}} = \begin{pmatrix} e^{-\frac{\beta_3}{2}} & 0 \\ 0 & e^{\frac{\beta_3}{2}} \end{pmatrix}. \]  \hspace{1cm} (2.74)

To find the transverse part, we define \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) from Eq. 1.42 and Eq. 1.42 as

\[ \mathcal{K}_1 = -A \frac{\sigma^2}{2} - iB \frac{\sigma^1}{2}, \]  \hspace{1cm} (2.75)

\[ \mathcal{K}_2 = A \frac{\sigma^1}{2} - iB \frac{\sigma^2}{2}, \]  \hspace{1cm} (2.76)

where we use the notation \( A = \cos \delta, B = -\sin \delta \) for this section and the next section since the equations are so long and this shortens them. The transverse part is

\[ e^{-i(\beta_1 \mathcal{K}_1 + \beta_2 \mathcal{K}_2)} = \cos \frac{\alpha}{2} + (\beta_1 (iA\sigma^2 - B\sigma^1) + \beta_2 (-iA\sigma^1 - B\sigma^2)) \frac{\sin \frac{\alpha}{2}}{\alpha} \]  \hspace{1cm} (2.77)
\[
\cos \frac{\alpha}{2} = \frac{(A-B)\beta_R}{\alpha} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \\
= \frac{1}{2 \cos \frac{\alpha}{2}} \left( \begin{array}{cc}
\cos \alpha + 1 & (A - B)\sin \frac{\alpha}{\alpha} \beta_L \\
-(A + B)\sin \frac{\alpha}{\alpha} \beta_R & \cos \alpha + 1
\end{array} \right).
\]

For the right-handed part, we use Type II: \((0, \frac{1}{2}) \ J \ = \frac{\vec{\sigma}}{2}, \ K = i \frac{\vec{\sigma}}{2}\) spinor representation.

The longitudinal part
\[
e^{\beta_3 \frac{\alpha}{2}} = \left( e^{\frac{\beta_3}{2}} \begin{array}{c}
0 \\
e^{-\frac{\beta_3}{2}}
\end{array} \right).
\]

\(K_1\) and \(K_2\) are now
\[
K_1 = -A \frac{\sigma^2}{2} + iB \frac{\sigma^1}{2},
\]
\[
K_2 = A \frac{\sigma^1}{2} + iB \frac{\sigma^2}{2},
\]

so the transverse part is
\[
e^{-i(\beta_1 K_1 + \beta_2 K_2)} = \frac{1}{2 \cos \frac{\alpha}{2}} \left( \begin{array}{cc}
\cos \alpha + 1 & (A + B)\sin \frac{\alpha}{\alpha} \beta_L \\
-(A + B)\sin \frac{\alpha}{\alpha} \beta_R & \cos \alpha + 1
\end{array} \right)
\]

Now our complete \(T\) is
\[
T = \left( \begin{array}{cc}
T_R & 0 \\
0 & T_L
\end{array} \right).
\]

But first we need to find corresponding values
\[
e^{\beta_3} = \frac{A + B}{MC} \left( P^+ + \frac{P_-}{\cos \alpha} \right),
\]
\[
e^{-\beta_3} = \frac{A - B}{MC} \left( P^+ - \frac{P_-}{\cos \alpha} \right),
\]
\[
\beta_R = -\frac{P_R}{M} \frac{\alpha}{\left( -B \cosh \beta_3 + A \sinh \beta_3 \right) \sin \alpha},
\]
\[
\beta_L = -\frac{P_L}{M} \frac{\alpha}{\left( -B \cosh \beta_3 + A \sinh \beta_3 \right) \sin \alpha}.
\]

Now our \(T_R\) and \(T_L\) are
\[
T_L = \frac{1}{2 \cos \frac{\alpha}{2}} \left( \begin{array}{cc}
(cos \alpha + 1)e^{-\frac{\beta_3}{2}} & (A - B)\frac{\sin \frac{\alpha}{\alpha} \beta_L e^{\frac{\beta_3}{2}}}{2} \\
-(A + B)\frac{\sin \frac{\alpha}{\alpha} \beta_R e^{-\frac{\beta_3}{2}}}{2} & (\cos \alpha + 1)e^{\frac{\beta_3}{2}}
\end{array} \right),
\]
\[ T_R = \frac{1}{2 \cos \frac{\alpha}{2}} \begin{pmatrix} (\cos \alpha + 1)e^{\frac{\beta}{2}} & (A + B)^{\sin \frac{\alpha}{2}} \beta e^{-\frac{\beta}{2}} \\ (-A + B)^{\sin \frac{\alpha}{2}} \beta e^{\frac{\beta}{2}} & (\cos \alpha + 1)e^{-\frac{\beta}{2}} \end{pmatrix}. \]  

Converting some part of these transformations into momentum space and leaving some part in terms of angles to make calculations easier

\[ T_L = N \begin{pmatrix} (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{-\frac{\beta_3}{2}} & -(A + B)P_L e^{\frac{\beta_3}{2}} \\ (A + B)P_R e^{-\frac{\beta_3}{2}} & (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{\frac{\beta_3}{2}} \end{pmatrix}, \]

\[ T_R = N \begin{pmatrix} (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{\frac{\beta_3}{2}} & -(A + B)P_L e^{\frac{\beta_3}{2}} \\ (A - B)P_R e^{\frac{\beta_3}{2}} & (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{\frac{\beta_3}{2}} \end{pmatrix}, \]

where \( N = (2 \cos \frac{\alpha}{2} M(-B \cosh \beta_3 + A \sinh \beta_3))^{-1} \).

We take the spin 1/2 spinors in chiral representation Eq. 2.16 and apply the interpolating boost \( T \) as \( u_{CR}(P) = T(P)u_{CR}(0) \)

\[ u_{CR}^{(1)}(P) = N\sqrt{M} \begin{pmatrix} (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{\frac{\beta_3}{2}} \\ (A - B)P_R e^{\frac{\beta_3}{2}} \\ (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{-\frac{\beta_3}{2}} \\ (A + B)P_R e^{-\frac{\beta_3}{2}} \end{pmatrix}, \]  

\[ u_{CR}^{(2)}(P) = N\sqrt{M} \begin{pmatrix} -(A + B)P_L e^{\frac{\beta_3}{2}} \\ (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{\frac{\beta_3}{2}} \\ -(A - B)P_R e^{\frac{\beta_3}{2}} \\ (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{-\frac{\beta_3}{2}} \end{pmatrix}, \]  

\[ u_{CR}^{(3)}(P) = N\sqrt{M} \begin{pmatrix} (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{\frac{\beta_3}{2}} \\ (A - B)P_R e^{\frac{\beta_3}{2}} \\ (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{-\frac{\beta_3}{2}} \\ -(A + B)P_R e^{-\frac{\beta_3}{2}} \end{pmatrix}, \]  

\[ u_{CR}^{(4)}(P) = N\sqrt{M} \begin{pmatrix} -(A + B)P_L e^{-\frac{\beta_3}{2}} \\ (P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{-\frac{\beta_3}{2}} \\ (A - B)P_L e^{\frac{\beta_3}{2}} \\ -(P_\perp + M(-B \cosh \beta_3 + A \sinh \beta_3)) e^{\frac{\beta_3}{2}} \end{pmatrix}. \]
From Eq. 2.57 and Eq. 2.58 we got

$$\cos^2 \alpha = \frac{P^2}{(P^+)^2 - M^2 C}. \quad (2.93)$$

We use $\mathbb{P}^2$ instead of $(P^+)^2 - M^2 C$ since we are boosting only in the positive $z$-direction $\cos \alpha$ only has a positive solution and $\mathbb{P} = \sqrt{(P^+)^2 - M^2 C}$ becomes always positive. Then

$$\cos \alpha = \frac{P_-}{\mathbb{P}}. \quad (2.94)$$

We can redefine the relation of Eq. 2.57 and Eq. 2.58 as

$$\cosh \beta_3 = \frac{1}{M C} (A P^+ + B \mathbb{P}), \quad (2.95)$$

$$\sinh \beta_3 = \frac{1}{M C} (B P^+ + A \mathbb{P}). \quad (2.96)$$

It would be easier to write down some common relations first

$$-B \cosh \beta_3 + A \sinh \beta_3 = \frac{\mathbb{P}}{M} \quad (2.97)$$

$$A \cosh \beta_3 - B \sinh \beta_3 = \frac{P^+}{M}. \quad (2.98)$$

We can now write down interpolating form of all spinors as
\[
\begin{align*}
\mathcal{B}^{(1/2)}(P) &= \begin{pmatrix}
\frac{\sqrt{P_{-} + P}}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} - P} \sqrt{A - B} \\
\frac{P_{-} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{-} + P} \sqrt{A + B} \\
\frac{P_{+} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A + B}
\end{pmatrix}, \\
\mathcal{B}^{(-1/2)}(P) &= \begin{pmatrix}
-P \frac{A + B}{2P} \sqrt{P_{+} - P} \sqrt{P_{+} + P} \sqrt{P_{-} + P} \sqrt{A - B} \\
-P \frac{P_{-} + P}{2P} \sqrt{P_{+} - P} \sqrt{P_{+} + P} \sqrt{A + B} \\
-P \frac{P_{+} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A + B}
\end{pmatrix}, \\
\mathcal{C}^{(1/2)}(P) &= \begin{pmatrix}
\frac{P_{-} + P}{2P} \sqrt{P_{+} - P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A - B} \\
-P \frac{A - B}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} - P} \sqrt{P_{-} + P} \sqrt{A + B} \\
-P \frac{P_{-} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A + B} \\
\frac{P_{+} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A + B}
\end{pmatrix}, \\
\mathcal{C}^{(-1/2)}(P) &= \begin{pmatrix}
P \frac{A - B}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} - P} \sqrt{P_{-} + P} \sqrt{A + B} \\
-P \frac{P_{-} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{-} - P} \sqrt{A + B} \\
-P \frac{P_{+} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A + B} \\
\frac{P_{+} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A + B}
\end{pmatrix}, \\
\mathcal{D}^{(1/2)}(P) &= \begin{pmatrix}
\frac{P_{-} + P}{2P} \sqrt{P_{+} - P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A - B} \\
-P \frac{A - B}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} - P} \sqrt{P_{-} + P} \sqrt{A + B} \\
-P \frac{P_{+} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A + B} \\
\frac{P_{+} + P}{2P} \sqrt{P_{+} + P} \sqrt{P_{+} + P} \sqrt{A + B}
\end{pmatrix}
\end{align*}
\]
The helicity operator Eq. 2.70 gives
\[ J_3 u_{CR}^{(1,2)} = \pm u_{CR}^{(1,2)}. \]  
(2.100)
since we have the interpolating helicity spinor now.

Spin-1 spinor representation is also presented in Appendix A.5 with the similar method of \((J,0) \oplus (0,J)\). Furthermore, in chapter 4, we examine the \((J,J)\) spinor case in spin 1 and compare with spin 1/2.

However, the generalized helicity spinor has its spin oriented at some angle away from the momentum direction in general because the interpolating kinematic operators \(K_1\) and \(K_2\) are given by Eq. 1.42 and Eq. 1.42. The \(T_{12}\) transformation involves the transverse rotations as well as the transverse boosts.

### 2.3 Time Ordered Scattering Amplitudes

First, we show the landscape of scalar particle scattering amplitudes before fermion amplitudes. In this scene, the lowest order tree-level Feynman diagram of a spinless particle’s amplitude is proportionate to the propagator of the intermediate particle from \([13]\)
\[ \Sigma = \frac{1}{s - m^2}, \]  
(2.101)
where \(s = (p_1 + p_2)^2\) is the Mandelstam variable.

For annihilation processes analogous to the \(e^+e^- \rightarrow \mu^+\mu^-\) annihilation process for the spinless case, we have two time ordered amplitudes as shown in Figure 2.2. These two amplitudes are
\[ \Sigma^a_{IFD} = \frac{1}{2q^0} \left( \frac{1}{P^0 - q^0} \right), \] (2.102)
\[ \Sigma^b_{IFD} = -\frac{1}{2q^0} \left( \frac{1}{P^0 + q^0} \right). \] (2.103)

where \( P = p_1 + p_2 \) for simplification and the covariant Feynman amplitude is the sum of them
\[ \Sigma_{IFD} = \Sigma^a_{IFD} + \Sigma^b_{IFD} = \frac{1}{s - m^2}. \] (2.104)

In the interpolating frame the corresponding amplitudes become
\[ \Sigma^a_\delta = \frac{1}{2q^+} \left( \frac{C}{P^+ - q^+} \right), \] (2.105)
\[ \Sigma^b_\delta = -\frac{1}{2q^+} \left( \frac{C}{P^+ + q^+} \right), \] (2.106)

They can be rewritten as
\[ \Sigma^a_\delta = \frac{C}{2w_q P^+ - 2w_q^2}, \] (2.107)
\[ \Sigma^b_\delta = \frac{C}{2w_q P^+ + 2w_q^2}, \] (2.108)

where \( w_q = \sqrt{P^2_\perp + C(P^2_\perp + m^2)}. \)

From the dispersion relations we obtain the expansion
\[ P^+ = P_z (\sin \delta + \cos \delta) + \frac{P^2_\perp + s}{2P_z} \cos \delta + O(1/P_z^3), \] (2.109)
\[ P^- = P_z (\sin \delta + \cos \delta) + \frac{P^2_\perp + s}{2P_z} \sin \delta + O(1/P_z^3), \] (2.110)
\[ w_q = P_z (\sin \delta + \cos \delta) + \frac{P^2_\perp + s}{2P_z} \sin \delta + \frac{P^2_\perp + m^2}{2P_z} (\cos \delta - \sin \delta) + O(1/P_z^3). \] (2.111)

The plots, in Figure 2.3 from [13] where \( \Sigma^a_\delta \) and \( \Sigma^b_\delta \) under \( K_3 \) boost for total momentum \( P_z = p_z + p_z' \) with no perpendicular momentum for convenience and \( P^2 = s \) is taken as \( 2(GeV)^2 \) and the mass of propagator particle \( m \) is taken as 1GeV, demonstrates that the second time ordered amplitude \( \Sigma^b_\delta \) vanishes at light-front limit \( \delta = \pi/4 \) and first time ordered amplitude becomes \( P_z \) independent because \( K_3 \) becomes kinematic at light-front.

When \( \Sigma^a_\delta \) and \( \Sigma^b_\delta \) are observed in IFD and LFD respectively. The second amplitude \( \Sigma^b \) vanishes in LFD as \( \Sigma^b_\delta \rightarrow \pi/4 \) 0 since the numerator goes zero while the denominator does not,
and the first amplitude $\Sigma^a$ becomes frame independent. Even though $\Sigma^b_\delta$ in IFD survives, this term can vanish by carrying it to the infinite momentum frame (IMF) by taking $P_z \to \infty$ which is defined by Weingberg in [24].

\begin{align}
\Sigma^a_{IFD} &= \frac{1}{2q^0} \left( \frac{1}{P^0 - q^0} \right) P_z = q \to \infty \frac{1}{s - m^2}, \\
\Sigma^b_{IFD} &= -\frac{1}{2q^0} \left( \frac{1}{P^0 + q^0} \right) P_z = q \to \infty 0.
\end{align}

The track of the minimum value of $\Sigma^b_{0 \leq \delta < \pi/4}$ is given by

\begin{equation}
P_z = -\sqrt{\frac{s(1 - C)}{C}},
\end{equation}

which exhibits a J-shaped curve.

Figure 2.3: Interpolating amplitudes.
2.4 Annihilation Amplitudes for Fermions

In this section, we want to investigate how different helicity amplitudes change from the instant form to the light front form. As an example of usage of interpolating dynamics, we calculate the helicity dependent amplitudes for a scattering process. We discuss this annihilation process at lowest order Feynman diagram as shown in Figure 2.4. We put the initial electron and positron on the z axis, and the $\theta$ angle is the angle between the momentum of the final muon and that of the initial electron. We dropped $(-ie)^2$ factor and $-1/q^2$ propagator because they are not connected with our investigation.

First, we look at this process at instant form then in interpolating form.

2.4.1 High Energy Annihilation Amplitudes

We begin with the simplest case were we take the high energy limit where the mass terms become very small as $E \gg m$. The amplitude of annihilation is given by

$$\mathcal{M} = [\bar{u}^{r_3}(3)\gamma^\mu v^{r_4}(4)] [\bar{v}^{r_2}(2)\gamma_\mu u^{r_1}(1)], \quad (2.115)$$

where $r = 1$ is spin up and $r = 2$ is spin down state.

From canonical solutions to Dirac equation $u^{(i)}(p)$ and $v^{(i)}(p)$ vectors are

$$u^1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_x + ip_y}{E + m} \\ \frac{p_x - ip_y}{E + m} \end{pmatrix}, \quad u^2 = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E + m} \\ \frac{-p_x}{E + m} \end{pmatrix}. \quad (2.116)$$
Similarly, the two antiparticles vectors are

\[ v^1 = N \begin{pmatrix} \frac{p_\perp - ip_y}{E + m} \\ \frac{-p_\perp}{E + m} \\ 0 \\ 1 \end{pmatrix}, \quad v^2 = -N \begin{pmatrix} \frac{p_\perp + ip_y}{E + m} \\ \frac{p_\perp}{E + m} \\ 1 \\ 0 \end{pmatrix}, \] (2.117)

where \( N = \sqrt{E + m} \). They can be derived from Eq. 2.25 and Eq. 2.26 with a transformation from chiral representation into standard representation with \( S \) matrix which is given by Eq. 2.1.

The square of the amplitude is

\[ |\mathcal{M}|^2 = [\bar{u}^{T_3}(3) \gamma^\mu v^{T_4}(4)][\bar{v}^{T_2}(2) \gamma_\mu u^{T_1}(1)][\bar{u}^{T_3}(3) \gamma^\mu v^{T_4}(4)]^* [\bar{v}^{T_2}(2) \gamma_\mu u^{T_1}(1)]^*. \]

After applying the Casimir trick and summing over finals spin with averaging the initial ones, the result becomes

\[ \langle |\mathcal{M}|^2 \rangle = \left[ (p_3 \cdot p_1)(p_4 \cdot p_2) + (p_3 \cdot p_2)(p_4 \cdot p_1) + (p_3 \cdot p_4)m_\mu^2 + (p_1 \cdot p_2)m_e^2 + 2(m_e m_\mu)^2 \right]. \] (2.118)

In order to make our calculations easier we assume that it is a center of mass interaction and very relativistic so that we can ignore the mass terms.

Electrons are colliding head to head so the momenta of electrons are given by

\[ p_1 = (\epsilon, 0, 0, \epsilon \hat{z}), \] (2.119)
\[ p_3 = (\epsilon, 0, 0, -\epsilon \hat{z}). \] (2.120)

We calculate with helicities of scattering particles left-handed (helicity -1) or right-handed (helicity +1).

\[
\begin{align*}
|\mathcal{M}(e^-e^+_L \rightarrow \mu^+_R \mu^+_L)|^2 &= (1 + \cos \theta)^2, \\
|\mathcal{M}(e^-e^+_L \rightarrow \mu^+_R \mu^-_L)|^2 &= (1 - \cos \theta)^2, \\
|\mathcal{M}(e^-e^+_R \rightarrow \mu^-_R \mu^-_L)|^2 &= (1 - \cos \theta)^2, \\
|\mathcal{M}(e^-e^+_R \rightarrow \mu^-_R \mu^+_L)|^2 &= (1 + \cos \theta)^2,
\end{align*}
\] (2.121) (2.122) (2.123) (2.124)

which agrees with Eq. (5.22)- Eq. (5.24) in [25]. The total amplitude is

\[ |\mathcal{M}|^2 = 16p^2(1 + \cos^2 \theta), \] (2.125)

which agrees with Eq. (5.14) in [25].
2.4.2 Amplitudes without Neglecting Mass Terms

The amplitude can be also written in terms of the current of description of interactions where the current is

\[ J^\mu_{nm} = \bar{u}^n \gamma^\mu u^m \]  

(2.126)

and the amplitude

\[ M = -i J^\mu_1 \bar{J}_2^\nu. \]  

(2.127)

In our case ("e + e → μ + μ" scattering analogy), the amplitude is

\[ \mathcal{M} = [\bar{u}(3)\gamma^\mu v(4)] [\bar{v}(2)\gamma^\mu u(1)]. \]  

(2.128)

We are using the center of mass frame, electron and positron are moving in z-direction opposite to each other as shown in Figure 2.4

\[ p_1 = (\epsilon, 0, 0, p_{\text{ini}}), \]  

(2.129a)

\[ p_2 = (\epsilon, 0, 0, -p_{\text{ini}}), \]  

(2.129b)

\[ p_3 = (\epsilon, p_{\text{final}} \sin \theta, 0, p_{\text{final}} \cos \theta), \]  

(2.129c)

\[ p_4 = (\epsilon, -p_{\text{final}} \sin \theta, 0, -p_{\text{final}} \cos \theta), \]  

(2.129d)

where \( \epsilon = \sqrt{m_{\text{ini}}^2 + p_{\text{ini}}^2} = \sqrt{m_{\text{final}}^2 + p_{\text{final}}^2}. \)

The amplitudes with this momenta become

\[ |\mathcal{M}_{++-}|^2 = |\mathcal{M}_{--+}|^2 = 2m_v^2 (\sqrt{s} + (2m_\mu - \sqrt{s}) \cos^2 \theta)^2, \]

\[ |\mathcal{M}_{+-+}|^2 = |\mathcal{M}_{-+-}|^2 = m_v^2 (2\sqrt{s} + 2(2m_\mu - \sqrt{s}) \cos^2 \theta)^2, \]

\[ |\mathcal{M}_{++-}|^2 = |\mathcal{M}_{--+}|^2 = s(2m_\mu + \sqrt{s} - (2m_\mu - \sqrt{s}) \cos^2 \theta), \]

\[ |\mathcal{M}_{+-+}|^2 = |\mathcal{M}_{-+-}|^2 = s(2m_\mu - \sqrt{s})^2 (1 - \cos^2 \theta)^2, \]

\[ |\mathcal{M}_{++-}|^2 = |\mathcal{M}_{--+}|^2 = |\mathcal{M}_{+-+}|^2 = |\mathcal{M}_{-+-}|^2 = 4m_v^2 (2m_\mu - \sqrt{s})^2 \cos^2 \theta (1 - \cos^2 \theta), \]

\[ |\mathcal{M}_{+--+}|^2 = |\mathcal{M}_{---+}|^2 = |\mathcal{M}_{+++--}|^2 = |\mathcal{M}_{---+}|^2 = s(2m_\mu - \sqrt{s})^2 \cos^2 \theta (1 - \cos^2 \theta). \]

where for convenience we represent right-handed helicities as + and left-handed helicities as – and for example + − + − means a right-handed electron collide with a left-handed positron and produce a right-handed muon and a left-handed anti-muon. s, is a Mandelstam variable, describing the center of mass energy as \( s = (p_1 + p_2)^2 = 4\epsilon^2. \).
Eventually, the total amplitude is

$$|\mathcal{M}|^2 = \left\{ 1 + \frac{4m_e^2}{s} + \frac{4m_\mu^2}{s} + \left( 1 - \frac{4m_e^2}{s} \right) \left( 1 - \frac{4m_\mu^2}{s} \right) \cos^2 \theta \right\}, \quad (2.130)$$

which agrees with Eq. (5.71) in [25]. When we take the massless case $m_e, m_\mu \to 0$ then the amplitude goes to Eq. 2.125.

### 2.4.3 Amplitudes in the Interpolating Form

Because we have helicity spinors, we can derive the amplitudes in terms of the initial and final helicity states in the interpolating form.

$$p_i^0 = \gamma p_i^0 + \gamma \beta p_i^z = \frac{E}{M} p_i^0 + \frac{P^z}{M} p_i^z, \quad (2.131a)$$

$$p_i^z = \gamma p_i^z + \gamma \beta p_i^0 = \frac{E}{M} p_i^z + \frac{P^z}{M} p_i^0, \quad (2.131b)$$

$$p_i^\perp = p_i^\perp, \quad (i = 1, 2, 3, 4) \quad (2.131c)$$

where $M = 2\epsilon$ is the center of mass energy and $E = \sqrt{(P^z)^2 + M^2}$ is the total energy in the boosted frame. We use these frame dependent four momenta in Eq. 2.99a and Eq. 2.99b to get frame dependent helicity spinors for an arbitrary interpolation angle $\delta$. The amplitudes and probabilities in Eq. 2.118 are then calculated using these generalized helicity spinors. We choose $\theta = \pi/3$ in the center of mass frame for the angle between incoming and outgoing particles as shown in Figure 2.4, and the center of mass energy $M = 2\epsilon = 4$ GeV, just a little above the sum of outgoing particle and antiparticle masses. We use $m_{\text{ini}} = 1$ GeV and $m_{\text{final}} = 1.5$ GeV for illustration which is analogous to the $e^+e^- \to \mu^+\mu^-$ annihilation process like proton and anti-proton to an excited proton and anti-proton pair.

The annihilation amplitudes and probabilities for all 16 different spin configurations are plotted in terms of both the interpolation angle $\delta$ and the total momentum $P^z$ in Figure 2.5 and Figure 2.6, where we denote the positive and negative helicities with “+” and “−”. For example, “++ −−” represents a process where a positive helicity incoming particle and a negative helicity incoming anti-particle annihilate to a positive helicity outgoing particle and a negative helicity outgoing anti-particle as described in the previous section.

From these plots you can see that the amplitudes between $0 \leq \delta < \pi/4$ depends on the reference frame. However, on the light-front limit, amplitudes are independent of the reference frame and each helicity dependent amplitude has a constant value.

Although each helicity amplitude changes with the reference frame and the interpolation
Figure 2.5: (Color online) fermion annihilation amplitudes (with the factor $-e^2 / q^2$ dropped) for 16 different spin configurations with the center of mass energy $M = 4$ GeV and the annihilation angle $\theta = \pi/3$. The two blue dashed lines are the boundary lines across which the values of helicity amplitudes suddenly change and the red solid line is the universal J-curve mentioned.
Figure 2.6: (Color online) fermion annihilation probabilities (with the factor $-e^2/q^2$ dropped) for 16 different spin configurations with the center of mass energy $M = 4$ GeV and the annihilation angle $\theta = \pi/3$. The two blue dashed lines are the boundary lines across which the values of helicity amplitudes suddenly change and the red solid line is the universal J-curve mentioned.
angle, the total probability, which is the sum of the probability of each diagram in Figure 2.6, is verified to be both frame independent and interpolation angle independent, as shown in Figure 2.7. The plots in Figures 2.5, 2.6, and 2.7 are also available in [26] and [27].

Figure 2.7: The total annihilation probability (with the factor \((e^2/q^2)^2\) dropped) as a sum of all spin contributions (helicity probabilities plotted in Figure 2.6) for the lowest tree diagram shown in Figure 2.4.

2.5 Summary and Conclusions

We generalize helicity spinors between the instant form and light-front form and find the generalized helicity operator. With these generalized helicity spinors, we calculated 16 helicity amplitudes and the squares of them that is probability of the annihilation process. From this picture, we can clearly see that the helicity probabilities for \(0 \leq \delta < \pi/4\) depend on the reference frame while these amplitudes are independent of the reference frame in the LFD.

Our work also suggests that the light-front spinor may be thought of as a generalized helicity spinor at the light-front limit, instead of the analog of the Dirac spinor.

The helicity amplitudes for the annihilation process analogous to the \(e^-e^+ \rightarrow \mu^-\mu^+\) annihilation are calculated at the lowest tree level, and plotted to show both the interpolation angle dependence and the frame dependence. The frame dependence originates from the fact that a certain helicity spinor corresponds to different spin configurations in different frames.
We observe that the same J curve that appeared in the time-ordered amplitudes in the $\phi^3$ theory [13] and the sQED theory [14], also shows up in every helicity amplitude we calculated. This J curve is independent of the specific kinematics, and has a universal shape that is only scaled by the center of mass energy. And finally, because the spin flips at large enough boosts in $+z$ and $-z$ direction, for interpolation angle $\delta < \pi/4$, there are two boundaries across which a certain helicity amplitude will shift its value sharply.

We investigate also that there is a critical point which causes a bifurcation in these helicity amplitudes. This phenomena is due to the change of sign for $P_-$ in Eq. 2.73 as we discussed in Section 2.2.1, and the critical interpolation angle $\delta_c$ is thus given by $P_- = 0$ and we have

$$\delta_c = \arctan \left( \frac{|P|}{E} \right).$$

(2.132)

where the initial two particles move only in the z-direction and yields the two boundaries.
Chapter 3

Interpolating Electromagnetic Field Strength Tensor

3.1 Introduction

One of the most beneficial advantages of the light-front dynamics (LFD) is having one less dynamic generator than the instant form dynamics (IFD). Since the kinematic operators leave the time invariant, their usage is beneficial in describing the characteristic of the motion with a simpler time variant expression. In this chapter, we use the comparison of the Poincaré matrix and the electromagnetic field strength tensor to examine how the Lorentz force equation varies with the fields which are analogous to the kinematic and dynamic operators in IFD. Then we examine the Lorentz force equation in interpolating form between instant form and light-front form and discuss how the $E_z$ field joins into a kinematic field from a dynamic field at the light-front limit.

We follow the footsteps of [14] where interpolating gauge and polarization vectors were studied and we connect the polarization vectors with the electromagnetic field strength tensor. In the next chapter, we will explain the connection between $(1/2, 1/2)$ Lorentz group and $(1, 0) \oplus (0, 1)$ Lorentz group spinors in more detail.

In the second section, we discuss the general solution found in [28] by using Pauli matrices as a vector basis. We discuss the similar conventions in the next chapter since Pauli matrices can be considered as three-dimensional Clifford algebra ($Cl_3$). We expand the solution and look at special conditions.

Then, we discuss the interpolating equation of motions. Then, by using interpolating dynamics in the previous chapter, we set an alternative method to solve equations of motions of a charged particle in a constant uniform electromagnetic field. We can remove the electromagnetic fields which corresponds to the dynamic generators by choosing the interpolating angle.
between fields in an appropriate way. In this method, we also decrease the degrees of freedom of the field strength tensor $F^{\mu\nu}$ by removing the fields which are analogous to dynamic operators. Ultimately, even though $E_z$ (analogous to the longitudinal boost operator $K^3$ in interpolating form) is still a dynamic operator, $E_z$ turns into a field analogous to a kinematic operator at the light front limit ($\delta \to \pi/4$) and the equations of motion change into a simpler form. In Appendix B, we showed some examples of the solution of the equation of motion problems in a constant uniform electromagnetic field and that the advantages of solving these equations of motion in fields which are analogous to the kinematic operators.

### 3.2 Electric and Magnetic Field Equation of Motion Solutions with Interpolating Quantization

There is a connection between Lorentz generators and electromagnetic fields, when we apply an electromagnetic field to a charge, electric and magnetic fields have an effect of boost and rotation transformation as Lorentz generators. We can compare the Poincaré matrix given by Eq. 1.19 with the electric and magnetic field tensor Eq. 3.1. They are given by

\[
M^{\mu\nu} = \begin{pmatrix}
0 & K^1 & K^2 & K^3 \\
-K^1 & 0 & J^3 & -J^2 \\
-K^2 & -J^3 & 0 & J^1 \\
-K^3 & J^2 & -J^1 & 0
\end{pmatrix},
\]

\[
F^{\mu\nu} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}.
\]

We can observe the correspondence of boost operators and electric fields $K \leftrightarrow E$ and rotation operators and magnetic fields $J \leftrightarrow B$ in this picture.

### 3.3 Polarization Vector and Electromagnetic Field Strength

We mentioned the difference between Dirac and helicity spinors in chapter 2. We encounter the same difference for gauge fields as a standard polarization vector that satisfies the Lorentz gauge condition ($\partial^\mu A_\mu = 0$). Helicity polarization vector satisfies the Coulomb gauge condition ($\nabla \cdot A = 0$) since the helicity transformation operator assures that the polarization vector direction stays perpendicular to the momentum direction $p \perp \epsilon$. The light-front gauge $A^+ = 0$, naturally connects to the Coulomb gauge $\nabla \cdot A = 0$ or $\partial^0 A_0 = 0$ through the interpolating
polarization vectors as shown in [13] and Eq. A.6.

In the previous chapter we discussed the difference between the helicity and Dirac spinors. Similarly, polarization vectors have these types of differences as standard and helicity polarization vectors, are given [29] also, corresponding to Lorentz gauge and Coulomb gauge fields. These polarization vectors can be connected to the electromagnetic field strength from $F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and can be written as $F^{\mu \nu} = -i \{ p^\mu A^\nu \}$ as which is discussed in [30]. By using this relation, we can write down a matrix which connects a 4 degrees of freedom polarization vectors in $(1/2, 1/2)$ Lorentz group and a 6 component spin-1 spinor $(1, 0) \oplus (0, 1)$ as

\[
\begin{pmatrix}
E_x \\
E_y \\
E_z \\
B_x \\
B_y \\
B_z
\end{pmatrix}
= -i
\begin{pmatrix}
p^1 & -p^0 & 0 & 0 \\
p^2 & 0 & -p^0 & 0 \\
p^3 & 0 & 0 & -p^0 \\
0 & 0 & -p^3 & p^2 \\
0 & p^3 & 0 & -p^1 \\
0 & -p^2 & p^1 & 0
\end{pmatrix}
\begin{pmatrix}
\phi \\
A_x \\
A_y \\
A_z
\end{pmatrix}, \quad (3.2)
\]

Similar to the spin-1/2 $A$ and $B$ notation in Eq. 2.4 and Eq. 2.5, $F^{\mu \nu}$ can be separated into right-handed and left handed as $E + iB \in (1, 0)$ and $E - iB \in (0, 1)$. Moreover, we can find the component of the $u$ spinor by expressing the electric and magnetic fields in terms of spherical harmonics as spin $+$, $-$, and 0; then we have six component spinors. Then, the $u$ spinor can be written as

\[
u(p, \lambda) = \frac{i}{\sqrt{2m}} \begin{pmatrix}
2(E^- + iB^-) \\
-E^3 - iB^3 \\
2(-E^+ - iB^+) \\
2(E^- - iB^-) \\
-E^3 + iB^3 \\
2(-E^+ + iB^+)
\end{pmatrix}, \quad (3.3)
\]

where $E^\pm = (E^1 \pm iE^2)/\sqrt{2}$ and $B^\pm = (B^1 \pm iB^2)/\sqrt{2}$.

We see that similar to the $A$ and $B$ convention in Eq. 2.4 and Eq. 2.5 to get spinors, $E + iB \in (1, 0)$ and $E - iB \in (0, 1)$ can be used and we find the general solution in Itzkson and Zuber’s book [28] by using Pauli matrices where $F^{\mu \nu}$ separates into $E + iB$ and $E - iB$.

3.3.1 Lorentz Force

A covariant Lorentz force is given by

\[
f^\mu = \frac{dp^\mu}{d\tau} = q F^{\mu \nu} u_\nu, \quad (3.4)
\]
where $\tau$ is proper time and defined as $\tau = \sqrt{t^2 - x^2 - y^2 - z^2}$, $p^\mu$ is four momentum, and $u_\nu$ is four velocity given by

$$u_\nu = \frac{dx_\nu}{d\tau} = \left( \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right)$$

$$= (u^0, -u^1, -u^2, -u^3) = \gamma(c, -v^1, -v^2, -v^3), \quad (3.5)$$

and the electromagnetic tensor is given in Eq. 3.1.

### 3.3.2 Lorentz Transformation and Electromagnetic Field Tensor

For simplicity, we work with the Lorentz transformation from 1.1.1 with two lower indices, i.e.

$$x_\mu = A_{\mu\nu}x^\nu, \quad (3.6)$$

where

$$A_{\mu\nu} = g_{\mu\rho}A_{\nu\rho}, \quad (3.7)$$

for infinitesimal form, it can be written as

$$A_{\mu\nu} = g_{\mu\nu} + \delta M_{\mu\nu}. \quad (3.8)$$

Since $g^{\mu\nu}A_{\mu\rho}A_{\nu\sigma} = g_{\rho\sigma}$ is conserved, $M_{\mu\nu}$ must be anti-symmetric such that $\delta M_{\sigma\rho} + \delta M_{\rho\sigma} = 0$. For homogeneous static electromagnetic fields the $F_{\mu\nu}$ generates Lorentz transformations, i.e., the trajectories (world lines) of particles are given by the Lorentz transformations of the initial momenta, because the equation of motion reads

$$\frac{du^\mu}{d\tau} = \frac{eF^{\mu\nu}}{m} u_\nu, \quad (3.9)$$

For a constant uniform $F_{\mu\nu}$, we indeed get Lorentz transformations.

### 3.4 General Solution of Motion in a Uniform Electromagnetic Field

In Itzkson and Zuber's book [28], the general solution of the motion of a charge under a constant uniform field is given, and we will follow their lead and expend their solution and compare it with our results. First, the four force in Eq. 3.4 is integrated and written in matrix form as

$$u^\mu(\tau) = \left( \frac{qF^{\mu\nu}}{e} \right)_\nu u_\nu(0). \quad (3.10)$$
Then the $2 \times 2$ Pauli matrix is introduced:

$$u = u_0 I + u \cdot \sigma.$$  \hspace{1cm} (3.11)

Using $\sigma \cdot a \cdot b = (a \cdot b)I + i \sigma \cdot (a \times b)$, one can write

$$\frac{du}{d\tau} = \frac{q}{m} \left( \frac{E + iB}{2} \cdot \sigma u + u \frac{E - iB}{2} \cdot \sigma \right),$$  \hspace{1cm} (3.12)

when we integrate, it becomes

$$u(\tau) = \exp \left( \frac{q}{2m} \frac{E + iB}{\sigma} \tau \right) u(0) \exp \left( \frac{q}{2m} \frac{E - iB}{\sigma} \tau \right).$$  \hspace{1cm} (3.13)

The three component complex vector $\mathbf{n}$ is defined as $E + iB$ and $a$ is defined as $(q/2m)\sqrt{\mathbf{n}^2}$ then

$$\exp \left( \frac{q}{2m} \frac{E + iB}{\sigma} \tau \right) = \cosh(a\tau)I + \frac{\mathbf{n} \cdot \sigma}{\sqrt{\mathbf{n}^2}} \sinh(a\tau).$$  \hspace{1cm} (3.14)

We consider the most general case so $u(0)$ is in any arbitrary direction, and use expansion of the exponential formula

$$u(\tau) = \left( \cosh(a\tau)I + \frac{\mathbf{n} \cdot \sigma}{\sqrt{\mathbf{n}^2}} \sinh(a\tau) \right) \left( u^0(0)I + u(0) \cdot \sigma \right) \left( \cosh(a^*\tau)I + \frac{\mathbf{n}^* \cdot \sigma}{\sqrt{\mathbf{n}^*}^2} \sinh(a^*\tau) \right),$$  \hspace{1cm} (3.15)

where $\mathbf{n}^*$ is the complex conjugate vector of $\mathbf{n}$. Using Pauli matrices multiplication again

$$u(\tau) = \left\{ \begin{array}{l}
\cosh(a\tau)u^0(0)I + \cosh(a\tau)u(0) \cdot \sigma + \frac{\mathbf{n} \cdot \sigma}{\sqrt{\mathbf{n}^2}} \sinh(a\tau)u^0(0) + \frac{\mathbf{n} \cdot u(0)}{\sqrt{\mathbf{n}^2}} \sinh(a\tau)I \\
+ \frac{i(\mathbf{n} \times u(0)) \cdot \sigma}{\sqrt{\mathbf{n}^2}} \sinh(a\tau) \cosh(a^*\tau) I + \frac{\mathbf{n} \cdot u(0)}{\sqrt{\mathbf{n}^2}} \sinh(a\tau) \sinh(a^*\tau) \\
+ \frac{\mathbf{n} \cdot \mathbf{n}^* u^0(0)}{\sqrt{\mathbf{n}^2 \mathbf{n}^*^2}} \sinh(a\tau) \sinh(a^*\tau) + \frac{i(\mathbf{n} \cdot u(0)) \cdot \mathbf{n}^*}{\sqrt{\mathbf{n}^2 \mathbf{n}^*^2}} \sinh(a\tau) \sinh(a^*\tau) \end{array} \right\} I$$

$$+ \frac{\mathbf{n}^* \cdot \sigma u^0(0)}{\sqrt{\mathbf{n}^*}^2} \sinh(a\tau) \cosh(a^*\tau) + \frac{u(0) \cdot \mathbf{n}^* \cdot \sigma}{\sqrt{\mathbf{n}^*}^2} \sinh(a\tau) \sinh(a^*\tau) + \frac{\mathbf{n} \cdot \sigma u(0)}{\sqrt{\mathbf{n}^2} \sqrt{\mathbf{n}^*}^2} \cosh(a\tau) \cosh(a^*\tau)$$

$$+ \frac{u(0) \cdot \sigma \cosh(a\tau) \cosh(a^*\tau) + \frac{\mathbf{n} \cdot \sigma u^0(0)}{\sqrt{\mathbf{n}^2}} \sinh(a\tau) \cosh(a^*\tau)}{\sqrt{\mathbf{n}^2} \sqrt{\mathbf{n}^*}^2} \sinh(a\tau) \sinh(a^*\tau)$$

$$+ \frac{i(\mathbf{n} \times u(0)) \cdot \sigma}{\sqrt{\mathbf{n}^2}} \sinh(a\tau) \cosh(a^*\tau) + \frac{i(\mathbf{n} \cdot u(0)) \cdot \mathbf{n}^* \cdot \sigma}{\sqrt{\mathbf{n}^2}} \cosh(a\tau) \sinh(a^*\tau)$$

$$+ \frac{i(\mathbf{n} \times \mathbf{n}^*) \cdot \sigma u^0(0)}{\sqrt{\mathbf{n}^2 \mathbf{n}^*^2}} \sinh(a\tau) \sinh(a^*\tau) - \frac{i(\mathbf{n} \times u(0)) \times \mathbf{n}^* \cdot \sigma}{\sqrt{\mathbf{n}^2 \mathbf{n}^*^2}} \sinh(a\tau) \sinh(a^*\tau).$$
Note that $\frac{n}{\sqrt{n^2}} = \left(\frac{n^*}{\sqrt{n^*}}\right)^*$ and $n \cdot n^* = E^2 - B^2$, and cross products can be expanded as

\[
\begin{align*}
\text{n} \times \text{n}^* &= (\text{E} + i\text{B}) \times (\text{E} - i\text{B}) = -2i\text{E} \times \text{B}, \\
\text{n} \times \text{u}(0) \times \text{n}^* &= -(\text{n}^* \cdot \text{n})\text{n} + (\text{n}^* \cdot \text{n})\text{u} \\
&= (\text{E} \times \text{u}(0) + i\text{B} \times \text{u}(0)) \times (\text{E} - i\text{B}) = 2i\text{u}(0) \times \text{E} \times \text{B}, \\
\text{n} \times \text{u}(0) &= \text{E} \times \text{u}(0) + i\text{B} \times \text{u}(0), \\
\text{u}(0) \times \text{n}^* &= -\text{E} \times \text{u}(0) + i\text{B} \times \text{u}(0), \\
(\text{n} \times \text{u}(0)) \cdot \text{n}^* &= (\text{n}^* \cdot \text{n})\text{u}(0) = 2i(\text{E} \times \text{B}) \cdot \text{u}(0).
\end{align*}
\]

The other terms:

\[
\begin{align*}
\text{n}^2 &= E^2 - B^2 + 2iE.B, \\
n^*^2 &= E^2 - B^2 - 2iE.B,
\end{align*}
\]

by using

\[
\sqrt{a + ib} = \left(\frac{\sqrt{a + \sqrt{a^2 + b^2}^2} + \text{sgn}(b)\sqrt{-a + \sqrt{a^2 + b^2}^2}}{\sqrt{2}}\right),
\]

for complex values inside the square root, it is derived by converting values into polar coordinates where $a = r \cos \theta$ and $b = r \sin \theta$ and $(a + ib)^{1/2} = r^{1/2}(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})$, therefore

\[
\begin{align*}
\sqrt{n^2} &= \sqrt{E^2 - B^2 + 2iE.B} = \left\{\frac{\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}}}{\sqrt{2}} + i\sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}}}{\sqrt{2}}\right\}, \\
\sqrt{n^*^2} &= \sqrt{E^2 - B^2 - 2iE.B} = \left\{\frac{\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}}}{\sqrt{2}} - i\sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}}}{\sqrt{2}}\right\},
\end{align*}
\]

then

\[
\sqrt{n^2n^*^2} = \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}.
\]
Then we can make some simplifications,

\[
\frac{n \cdot u(0)}{\sqrt{n^2}} \sinh(a \tau) \cosh(a^* \tau) + \frac{u(0) \cdot n^*}{\sqrt{n^*}^2} \cosh(a \tau) \sinh(a^* \tau) = 2Re \left( \frac{n \cdot u(0)}{\sqrt{n^2}} \sinh(a \tau) \cosh(a^* \tau) \right)
\]

\[
= \frac{1}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \left\{ (E + iB) \cdot u(0) \sqrt{E^2 - B^2 - 2iE \cdot B} \sinh(a \tau) \cosh(a^* \tau) \right.
\]

\[
+ (E - iB) \cdot u(0) \sqrt{E^2 - B^2 + 2iE \cdot B} \cosh(a \tau) \sinh(a^* \tau) \left\} \right.
\]

\[
= \frac{1}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \left\{ E \cdot u(0) \sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right.
\]

\[
+ B \cdot u(0) \sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right\} \sqrt{2}Re \left( \sinh(a \tau) \cosh(a^* \tau) \right)
\]

\[
+ \frac{1}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \left\{ E \cdot u(0) \sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right.
\]

\[
- B \cdot u(0) \sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right\} \sqrt{2}Im \left( \sinh(a \tau) \cosh(a^* \tau) \right). \quad (3.21)
\]

Similarly,

\[
\frac{n \cdot \sigma u^0(0)}{\sqrt{n^2}} \sinh(a \tau) \cosh(a^* \tau) + \frac{n^* \cdot \sigma u^0(0)}{\sqrt{n^*}^2} \cosh(a \tau) \sinh(a^* \tau)
\]

\[
= 2Re \left( \frac{n \cdot \sigma u^0(0)}{\sqrt{n^2}} \sinh(a \tau) \cosh(a^* \tau) \right)
\]

\[
= \frac{1}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \left\{ E \cdot \sigma u^0(0) \sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right.
\]

\[
+ B \cdot \sigma u^0(0) \sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right\} \sqrt{2}Re \left( \sinh(a \tau) \cosh(a^* \tau) \right)
\]

\[
+ \frac{1}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \left\{ E \cdot \sigma u^0(0) \sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right.
\]

\[
- B \cdot \sigma u^0(0) \sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right\} \sqrt{2}Im \left( \sinh(a \tau) \cosh(a^* \tau) \right). \quad (3.22)
\]
The cross vector part

\[
\frac{i(n \times u(0)) \cdot \sigma}{\sqrt{n^2}} \sinh(a\tau) \cosh(a^*\tau) + \frac{i(u(0) \times n^*) \cdot \sigma}{\sqrt{n^*^2}} \cosh(a\tau) \sinh(a^*\tau)
\]

\[
= 2iIm \left( \frac{(n \times u(0)) \cdot \sigma}{\sqrt{n^2}} \sinh(a\tau) \cosh(a^*\tau) \right)
\]

\[
= \frac{i}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \left\{ (E \times u(0) + iB \times u(0)) \cdot \sigma \sqrt{E^2 - B^2 - 2iE \cdot B \sinh(a\tau) \cosh(a^*\tau)}
\right.
\]

\[
+ \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \cdot \sqrt{2Im (\sinh(a\tau) \cosh(a^*\tau))}
\]

\[
+ \frac{1}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \left\{ (E \times u(0)) \cdot \sigma \sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \sqrt{2Re (\sinh(a\tau) \cosh(a^*\tau))} \right. \}
\]

\[
- \frac{1}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \left\{ (E \times u(0)) \cdot \sigma \sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \sqrt{2Re (\sinh(a\tau) \cosh(a^*\tau))} \right. \}
\]

\[
\text{(3.23)}
\]
In all the equations above, we have angle parts and we can expand angle relations as

\[
Re \left( \sinh(a) \cosh(a^*) \right) = \frac{1}{2} \left( \sinh(a + a^*) + \sinh(a - a^*) \right) = \frac{\sinh(2Re(a))}{2} \\
= \frac{1}{2} \sinh \left( q\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}}/(\sqrt{2m}) \right),
\]

\[
Im \left( \sinh(a) \cosh(a^*) \right) = \frac{1}{2} \left( \sinh(a + a^*) - \sinh(a - a^*) \right) = \frac{\sinh(2Im(a))}{2} \\
= \frac{1}{2} \sin \left( q\sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}}/(\sqrt{2m}) \right),
\]

\[
cosh(a) \cosh(a^*) = \frac{1}{2} \left( \cosh((a + a^*)\tau) + \cosh((a - a^*)\tau) \right) \\
= \frac{1}{2} \left( \cosh(2Re(a)\tau) + \cosh(2iIm(a)\tau) \right), \\
= \frac{1}{2} \left\{ \cosh \left( \frac{q\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}}}{\sqrt{2m}} \right) \right\},
\]

\[
\sinh(a) \sinh(a^*) = \frac{1}{2} \left( \cosh((a + a^*)\tau) - \cosh((a - a^*)\tau) \right) \\
= \frac{1}{2} \left( \cosh(2Re(a)\tau) - \cosh(2iIm(a)\tau) \right), \\
= \frac{1}{2} \left\{ \cosh \left( \frac{q\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}}}{\sqrt{2m}} \right) \right\},
\]

Now, we can write down each scalar and vector parts of the four velocity \( u^\mu(\tau) \) as
\[ u^0(\tau) = \frac{u^0(0)}{2} \left\{ \frac{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} + E^2 + B^2}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \cosh \left( \frac{q \tau E}{\sqrt{2m}} \right) \right. \]
\[ \left. \right. \]
\[ + \frac{\sqrt{(E - B)^2 + 4(E \cdot B)^2} - E^2 - B^2}{\sqrt{(E - B)^2 + 4(E \cdot B)^2}} \cos \left( \frac{q \tau E}{\sqrt{2m}} \right) \left\} \right. \]
\[ \left. \right. \]
\[ + \frac{E \cdot u(0)E + B \cdot u(0)B}{\sqrt{(E - B)^2 + 4(E \cdot B)^2}} \sinh \left( \frac{q \tau E}{\sqrt{2m}} \right) \]
\[ + \frac{E \cdot u(0)E - B \cdot u(0)B}{\sqrt{(E - B)^2 + 4(E \cdot B)^2}} \sin \left( \frac{q \tau E}{\sqrt{2m}} \right) \left\} \cosh \left( \frac{q \tau E}{\sqrt{2m}} \right) \cos \left( \frac{q \tau E}{\sqrt{2m}} \right) \right. \]
\[ \right. \]
\[ - \frac{1}{2\sqrt{(E - B)^2 + 4(E \cdot B)^2}} \left\{ (E \times u(0))E + (B \times u(0))B \right\} \sin \left( \frac{q \tau E}{\sqrt{2m}} \right) \]
\[ + \frac{1}{2\sqrt{(E - B)^2 + 4(E \cdot B)^2}} \left\{ (E \times u(0))B - (B \times u(0))E \right\} \sin \left( \frac{q \tau E}{\sqrt{2m}} \right) \]
\[ + \frac{(E \cdot u(0)E + (B \cdot u(0)B)}{\sqrt{(E - B)^2 + 4(E \cdot B)^2}} \left\} \cosh \left( \frac{q \tau E}{\sqrt{2m}} \right) \cos \left( \frac{q \tau E}{\sqrt{2m}} \right) \right. \]
\[ \right. \]
\[ \text{where} \]
\[ \mathbb{E} = \sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}}, \]
\[ \mathbb{B} = \sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}}. \]

The three cases cause some problems for numerical calculations because some terms in the denominator go to zero when they normally are supposed to be canceled with the terms.
in the numerator. \(E = 0\) from \(\cos\), \(\sin\) integration, \(B = 0\) from \(\cosh\), \(\sinh\), and \(E \cdot B = 0\) multiplication of these two from both hyperbolic and harmonic functions.

First, \(E = 0\) case, then the square root relations become

\[
\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \to \sqrt{B^4} = B^2, \tag{3.28}
\]
\[
\sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \to \sqrt{B^2 + \sqrt{E^4}} = \sqrt{2}|B|, \tag{3.29}
\]
\[
\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \to \sqrt{-B^2 + \sqrt{E^4}} = 0. \tag{3.30}
\]

The general solution changes into

\[
u^0(\tau) = u^0(0),
\]
\[
u(\tau) = u(0) \cos\left(\frac{q|B|\tau}{m}\right) + \frac{(B \cdot u(0))B}{B^2} \left(-1 + \cos\left(\frac{q|B|\tau}{m}\right)\right) - \frac{B \times u(0)}{|B|} \sin\left(\frac{q|B|\tau}{m}\right). \tag{3.32}
\]

When \(B = 0\) case, then square root relations change into

\[
\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \to \sqrt{E^4} = E^2, \tag{3.33}
\]
\[
\sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \to \sqrt{-E^2 + \sqrt{E^4}} = 0, \tag{3.34}
\]
\[
\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \to \sqrt{E^2 + \sqrt{E^4}} = \sqrt{2}|E|. \tag{3.35}
\]

The solution becomes

\[
u^0(\tau) = u^0(0) \cosh\left(\frac{q|E|\tau}{m}\right) + \frac{E \cdot u(0)}{|E|} \sinh\left(\frac{q|E|\tau}{m}\right), \tag{3.36}
\]
\[
u(\tau) = u(0) + \frac{(E \cdot u(0))E}{E^2} \left(-1 + \cosh\left(\frac{q|E|\tau}{m}\right)\right) + \frac{Eu^0(0)}{|E|} \sinh\left(\frac{q|E|\tau}{m}\right). \tag{3.37}
\]

Now, we are considering \(E \cdot B = 0\) case i.e. when they are perpendicular. The square root relations change as

\[
\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \to |E^2 - B^2|, \tag{3.38}
\]
\[
\sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \to \sqrt{-E^2 + B^2 + |E^2 - B^2|}, \tag{3.39}
\]
\[
\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \to \sqrt{E^2 - B^2 + |E^2 - B^2|}. \tag{3.40}
\]

There will be two cases, when \(E^2 - B^2 > 0\) and \(E^2 - B^2 < 0\).
When we take $E^2 - B^2 > 0$, then Eq. 3.38- Eq. 3.40 change as

$$\sqrt{(E^2 - B^2)^2 + 4(E.B)^2} \rightarrow E^2 - B^2,$$

$$\sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}} \rightarrow \sqrt{-E^2 + B^2 + |E^2 - B^2|} = 0,$$

$$\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}} \rightarrow \sqrt{E^2 - B^2 + |E^2 - B^2|} = \sqrt{2(E^2 - B^2)}.$$

The solution becomes

$$u^0(\tau) = -\frac{B^2 u^0(0)}{E^2 - B^2} + \frac{E^2 u^0(0)}{E^2 - B^2} \cosh \left(\frac{q\tau \sqrt{E^2 - B^2}}{m}\right) - \frac{(E \times B).u(0)}{E^2 - B^2},$$

$$\times \left(-1 + \cosh \left(\frac{q\tau \sqrt{E^2 - B^2}}{m}\right)\right) + \frac{E.u(0)}{\sqrt{E^2 - B^2}} \sinh \left(\frac{q\tau \sqrt{E^2 - B^2}}{m}\right),$$

$$u(\tau) = \frac{E^2 u(0)}{E^2 - B^2} - \frac{B^2 u(0)}{E^2 - B^2} \cosh \left(\frac{q\tau \sqrt{E^2 - B^2}}{m}\right)$$

$$+ \frac{E \times B u^0(0) + (E.u(0))E + (B.u(0))B}{E^2 - B^2} \left(-1 + \cosh \left(\frac{q\tau \sqrt{E^2 - B^2}}{m}\right)\right)$$

$$+ \frac{E u^0(0) - B \times u(0)}{\sqrt{E^2 - B^2}} \sinh \left(\frac{q\tau \sqrt{E^2 - B^2}}{m}\right).$$

When $E^2 - B^2 < 0$, this time Eq. 3.38-Eq. 3.40 becomes

$$\sqrt{(E^2 - B^2)^2 + 4(E.B)^2} \rightarrow B^2 - E^2,$$

$$\sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}} \rightarrow \sqrt{-E^2 + B^2 + |E^2 - B^2|} = \sqrt{B^2 - E^2},$$

$$\sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E.B)^2}} \rightarrow \sqrt{E^2 - B^2 + |E^2 - B^2|} = 0.$$

The solution becomes

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\[ u^0(\tau) = \frac{B^2 u^0(0)}{B^2 - E^2} - \frac{E^2 u^0(0)}{B^2 - E^2} \cos \left( \frac{q \tau \sqrt{B^2 - E^2}}{m} \right) + \frac{E \cdot u(0)}{\sqrt{B^2 - E^2}} \sin \left( \frac{q \tau \sqrt{B^2 - E^2}}{m} \right) \]
\[ - \frac{(E \times B) \cdot u(0)}{B^2 - E^2} \left( 1 - \cos \left( \frac{q \tau \sqrt{B^2 - E^2}}{m} \right) \right), \quad (3.43) \]

\[ u(\tau) = -\frac{E^2 u(0)}{B^2 - E^2} + \frac{B^2 u(0)}{B^2 - E^2} \cos \left( \frac{q \tau \sqrt{B^2 - E^2}}{m} \right) \]
\[ + \frac{-E \times B u^0(0) + (E \cdot u(0))E + (B \cdot u(0))B}{B^2 - E^2} \left( 1 + \cos \left( \frac{q \tau \sqrt{B^2 - E^2}}{m} \right) \right) \]
\[ + \frac{E u^0(0) - B \times u(0)}{\sqrt{B^2 - E^2}} \sin \left( \frac{q \tau \sqrt{B^2 - E^2}}{m} \right). \quad (3.44) \]

There is another special case when electric and magnetic fields have the same magnitudes as \( B^2 - E^2 = 0 \). The square root terms becomes

\[ \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2} \rightarrow 2E \cdot B, \quad (3.45) \]
\[ \sqrt{-E^2 + B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \rightarrow \sqrt{2}E \cdot B, \quad (3.46) \]
\[ \sqrt{E^2 - B^2 + \sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \rightarrow \sqrt{2}E \cdot B. \quad (3.47) \]
The solution of four velocity becomes

\[ u^0(\tau) = u^0(0) \left\{ \frac{(E + B)^2}{4E \cdot B} \cosh \left( \frac{q \sqrt{E \cdot B}}{m} \right) - \frac{(E - B)^2}{m^2} \cos \left( \frac{q \sqrt{E \cdot B}}{m} \right) \right\} \]

\[ + \frac{E \cdot u(0) + B \cdot u(0)}{2 \sqrt{E \cdot B}} \sinh \left( \frac{q \sqrt{E \cdot B}}{m} \right) + \frac{E \cdot u(0) - B \cdot u(0)}{2 \sqrt{E \cdot B}} \sin \left( \frac{q \sqrt{E \cdot B}}{m} \right) \]

\[ - \frac{(E \times B) \cdot u(0)}{2E \cdot B} \left\{ \cosh \left( \frac{q \sqrt{E \cdot B}}{m} \right) - \cos \left( \frac{q \sqrt{E \cdot B}}{m} \right) \right\}, \quad (3.48) \]

\[ u(\tau) = \frac{u(0)}{4E \cdot B} \left\{ - (E - B)^2 \cosh \left( \frac{q \sqrt{E \cdot B}}{m} \right) + (E + B)^2 \cos \left( \frac{q \sqrt{E \cdot B}}{m} \right) \right\} \]

\[ + \frac{E \times B u^0(0)}{2 \sqrt{E \cdot B}} \left\{ \cosh \left( \frac{q \sqrt{E \cdot B}}{m} \right) - \cos \left( \frac{q \sqrt{E \cdot B}}{m} \right) \right\} \]

\[ + \frac{(E + B) u^0(0)}{2 \sqrt{E \cdot B}} \sinh \left( \frac{q \sqrt{E \cdot B}}{m} \right) + \frac{(E - B) u^0(0)}{2 \sqrt{E \cdot B}} \sin \left( \frac{q \sqrt{E \cdot B}}{m} \right) \]

\[ + \frac{E \times u(0) - B \times u(0)}{2 \sqrt{E \cdot B}} \sinh \left( \frac{q \sqrt{E \cdot B}}{m} \right) - \frac{E \times u(0) + B \times u(0)}{2 \sqrt{E \cdot B}} \sin \left( \frac{q \sqrt{E \cdot B}}{m} \right) \]

\[ + \frac{(E \cdot u(0)) E + (B \cdot u(0)) B}{2E \cdot B} \left\{ \cosh \left( \frac{q \sqrt{E \cdot B}}{m} \right) - \cos \left( \frac{q \sqrt{E \cdot B}}{m} \right) \right\}. \quad (3.49) \]

Additionally to \( E^2 - B^2 = 0 \), we are also adding a \( E \cdot B = 0 \) condition. At first, because there is a \( E \cdot B \) term in the denominators, it looks like it is going to infinity, but it cancels with the term coming from hyperbolic and trigonometric functions. The four velocity for this condition becomes

\[ u^0(\tau) = u^0(0) \left( 1 + \frac{q^2(E^2 + B^2) \tau^2}{4m^2} \right) + \frac{E \cdot u(0) \tau}{m} - \frac{q^2(E \times B \cdot u(0) \tau^2)}{2m^2}, \quad (3.50) \]

\[ u(\tau) = u(0) \left( 1 - \frac{q^2(E^2 + B^2) \tau^2}{2m^2} \right) + \frac{q^2 E \times B u^0(0) \tau^2}{2m^2} + \frac{qE u^0(0) \tau}{m} - \frac{q B \times u(0) \tau}{m} \]

\[ + \frac{q^2((E \cdot u(0)) E + (B \cdot u(0)) B) \tau^2}{2m^2} \quad (3.51) \]
3.4.1 Interpolating Form

We can define the four force in interpolating form and the interpolating electromagnetic tensor as \( F^{\hat{\mu}\hat{\nu}} = \mathcal{R}_{\alpha}^\beta F^\alpha{}_{\beta} \mathcal{R}^\beta_{\gamma} \) by using Eq. 1.33. In it’s matrix form,

\[
F^{\hat{\mu}\hat{\nu}} = \begin{pmatrix}
0 & -E_x \cos \delta - B_y \sin \delta & -B_x \cos \delta + E_y \sin \delta & E_z \\
E_x \cos \delta + B_y \sin \delta & 0 & -B_x \cos \delta - E_y \sin \delta & B_y \\
E_y \cos \delta - B_x \sin \delta & B_z & 0 & -B_x \cos \delta - E_y \sin \delta \\
-E_z & -B_y \cos \delta + E_x \sin \delta & B_x \cos \delta + E_y \sin \delta & 0
\end{pmatrix}.
\]

One can observe that the interpolating Poincaré matrix (1.40) and field strength are similar. We can expand the interpolating Poincaré matrix as

\[
M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix}
0 & K^1 \cos \delta - J^2 \sin \delta & K^2 \cos \delta + J^1 \sin \delta & -K^3 \\
-K^1 \cos \delta + J^2 \sin \delta & 0 & -J^3 \cos \delta + K^1 \sin \delta & J^2 \cos \delta + K^1 \sin \delta \\
-K^2 \cos \delta + J^1 \sin \delta & J^3 & 0 & -J^1 \cos \delta + K^2 \sin \delta \\
K^3 & -J^2 \cos \delta - K^1 \sin \delta & J^1 \cos \delta - K^2 \sin \delta & 0
\end{pmatrix}.
\]

Interpolating four velocity can be driven from the instant four velocity by a transformation between interpolating space-time coordinates. Ordinary space-time coordinates are from Eq. 1.32, and the four velocity in interpolating form is given by

\[
u^i = \mathcal{R}_{\hat{\mu} i} u^\mu = (u^1, -u^2, -u^3, u^\tau).
\]

We can write down the equation of motion from the four force Eq. 3.4 in interpolating form as

\[
\frac{d u^\hat{i}}{d\tau} = -\frac{qF^{\hat{\mu}\hat{\nu}}}{m} u^\nu(\tau) - \frac{qF^{\hat{\mu}\hat{\nu}}}{m} u^\nu(\tau) + \frac{qF^{\hat{\mu}\hat{\nu}}}{m} u^\nu(\tau).
\]

One can see that the field analogous to the dynamic operator changes the \( u^\hat{i} \tau \) so \( F^{\hat{\mu}\hat{\nu}} \) and \( F^{\hat{\mu}\hat{\nu}} \) are dynamic operator analogous fields. However, we can eliminate them by choosing

\[
F^{\hat{\mu}\hat{\nu}} = 0, \quad \text{or} \quad F^{\hat{\mu}\hat{\nu}} = 0.
\]

We show that this kind of choice helps to simplify solving equations of motion as shown in Appendix B.1 since without dynamic operator analogous fields, we have reduced electromagnetic tensor fields and have a direct connection to \( t(\tau) \) and \( \tau \).

\( F^{\hat{\mu}\hat{\nu}} \) also becomes analogous to the dynamic generator for \( 0 \leq \delta < \pi/4 \) but at \( \delta = \pi/4 \), \( u^\tau \) becomes \( u^\tau \) and \( F^{\hat{\mu}\hat{\nu}} \) only becomes a scaling factor for the \( u^\tau(\tau) \) transformation as

\[
m \frac{d u^\tau(\tau)}{d\tau} = qE_z u^\tau(\tau).
\]
The integration of the expression above gives

\[ u^+(\tau) = e^\frac{g E_x}{m} \tau u^+(0). \]  

(3.58)

Figure 3.1: \( u^+(\tau) \) plot when \( E_x = B_y \tan \delta \) and \( E_y = B_x \tan \delta \) and \( u^+ \) becomes invariant at \( \delta = \{0, 46365, \pi/4\} \) respectively. The blue graph represents \( u^+(\tau) \) and the orange one is where it is invariant. Here \( B_x = 1T, B_y = -2T, B_z = 3T \), the initial velocity \( v_z(0) = 0.2c \), and \( m \) is taken as unit mass 1 for illustration.

Figure 3.2: \( \log(u^+(\tau)) \) plot when there is \( E_z \) and \( u^+ \) becomes invariant at \( \delta = \pi/4 \). The blue graph represents \( u^+(\tau) \) and the orange one is where it is invariant. Here \( B_x = 1T, B_y = -2T, B_z = 3T \), and \( m \) is taken as unit mass 1 for illustration.

3.5 Summary and Discussion

The connection between gauge fields and the electromagnetic field strength tensor is explained and established in interpolating form. The more detailed work on the connection between six
component spin 1 spinors and polarization vectors are shown in the next chapter.

We expanded the general solution in Itzykson and Zuber [28] which uses $\mathbf{E} + i\mathbf{B}$ and $\mathbf{E} - i\mathbf{B}$ transformation for the four velocity, which is similar to our $(J,0) \oplus (0,J)$ spinor expression in chiral representation. We write down these solutions for special cases. We put trajectories of one special case when $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{v}(0)$ are perpendicular and choose $\mathbf{v}(0)$ as the $z$-direction in Appendix B.3.

We discussed the relation between kinematic and dynamic operators analogous to electromagnetic fields in the Lorentz force equations. Kinematic operator motions are simpler than dynamic operator motions since there is no complication of time components.

The dynamic $E_z$ becomes a field which is analogous to the kinematic operator in LFD. We can observe this property from interpolating time transformation in interpolating form with a constant uniform electromagnetic field given by

$$ f^+ = m \frac{d\hat{u}^+(\tau)}{d\tau} = qF^+\hat{\varphi}u_\varphi = q(F^{+1}u_1 + F^{+2}u_2 + F^{+3}u_3), \quad (3.59) $$

where $F^{+1}$ and $F^{+2}$ are always analogous to the dynamic operators.
Chapter 4

Clifford Algebra and Spinors

4.1 Introduction

Since mostly our calculations are involving spinor algebra and vector algebra, Clifford algebra, which can combine these two, can be a useful tool for spinors and transformation as well. We defined Lorentz transformation under Clifford algebra and also spinor calculation. Clifford algebra and Clifford groups are the most convenient expression of spinors. We also show the relation between standard vector representation of spinors and Clifford algebra definition.

We begin with defining a spinor of spin-1 for complex Clifford algebra $(1,3) (Cl_{1,3}({\mathbb C}))$ by using $Cl_{1,3}({\mathbb C})$ for both $(1/2,1/2)$ and $(1,0) \oplus (0,1)$ orthochronous Lorentz group representations. The relationship between the two representations is found. As the electromagnetism can be described either by the gauge field $A^\mu$ or by the electromagnetic field $F^{\mu\nu}$, the two representations can be unified by one formula $u(p,\lambda) = \ell \wedge \bar{p}/m$.

We show that Clifford algebra presents a more convenient and easier path to find interpolating quantities and spinors. We introduce a Clifford basis for interpolating form and express the interpolating boost operator in terms of Clifford numbers. Thus, we have interpolating quantities of Lorentz transformations, and with the help of these transformations, we have interpolating spinors of spin-1/2 and spin-1 with the same boost operators and notation. Although Clifford quantities are basis independent, we can get interpolating quantities just taking interior products of them such as $F^{\mu\nu} = \gamma^{\mu\nu}.F$ and $F^{\mu\nu} = \gamma^{\mu\nu}.F$ with the same $F$. We derive electrodynamics in interpolating form just by taking the interpolating basis interior product of them for the field strength tensor, energy momentum tensor and so on.
4.2 Clifford Algebra

4.2.1 A Little History About the Clifford Algebra

The origin of the Clifford algebra is as old as the invention of imaginary numbers. Later on, they are begun to be used as 2D vector algebra, and it was the first step toward the usage of the natural numbers in vector algebra. The second biggest step was made by Sir William Rowan Hamilton in 1843. Representing 3D rotations was a central problem in the 19th century, and Hamilton invented quaternions as non-commutative expansions of the complex numbers to express a 3D rotation system. \{1, i, j, k\} are four elements of quaternions and three of them are specified as vectors. The fundamental formula for quaternions is \(i^2 = j^2 = k^2 = ijk = -1\).

One year after Hamilton’s invention, Herman Grassmann developed a non-commutative algebra that is called exterior algebra or later Grassmann algebra. He first mentioned it under the term theory of extension. In his publication, he described progressive and regressive products, and he gave a geometric meaning to exterior products. Two of the most important features of the exterior product are associativity \(a \wedge (b \wedge c) = (a \wedge b) \wedge c\) and non-commutativity \(a \wedge b = -b \wedge a\), where \(\wedge\) represents the exterior product. Grassmann algebra is now the fundamental foundation of supersymmetry and superstring theory.

In 1876, Clifford united Grassmann’s and Hamilton’s work by defining an interior product as an addition to exterior (progressive) products in Grassmann algebra. The bivectors in Clifford algebra are isomorphic to the quaternions. However, later this algebra was replaced by vector analysis which was developed independently by Josiah Willard Gibbs and Oliver Heaviside. At that time people didn’t like the idea of using 4 variables instead of 3 and changed it into a 3 vector system. The present day’s vector algebra was extracted out of the quaternion product. Clifford algebra has been forgotten for a long time after the vector analysis, however Clifford algebra resurfaced in the 1920’s after the need for algebra underlying quantum spins.

The first mathematical term, spinors, was discovered by Elie Cartan (1913) [31]. Later, Paul Ehrenfest utilized the term “spinors” in his work on quantum physics. It was Wolfgang Pauli who used spinors first in mathematical physics in 1927 with his Pauli matrices. In 1930 G. Juvet and Fritz found that they could use left ideals of a matrix algebra to represent spinors. In this algebra, the left column of matrices could be used as vectors and left minimal ideals for spinor algebra. The usage of the left minimal ideal of Clifford algebras began with Marcel Riesz in 1947. Recently, David Bohm and Basil Hiley, under the study of algebraic approaches to quantum theory constructed spinor algebra with left minimal ideals as of Riesz’s.
4.2.2 Complex Numbers

Complex numbers can be considered as the first application of Clifford numbers into vector space as they can represent 2 dimensional (2D) vectors as a real plane and a complex plane. Complex numbers can also be defined as part of Clifford algebra ($Cl_2$) on the real space $\mathbb{R}$.

A vector $(x, y)$ corresponds to $z = x + iy$. It is also possible to express it in polar coordinates

$$z = r(\cos \phi + i \sin \phi). \quad (4.1)$$

where $x = r \cos \phi$ and $y = r \sin \phi$.

The distance becomes the norm in complex space

$$|z| = \sqrt{zz} = \sqrt{x^2 + y^2}. \quad (4.2)$$

Since we can replace $z = x + iy$ in complex space by $(x, y)$ in real space ($\mathbb{R}^2$), we can write a corresponding real $2 \times 2$ matrix for imaginary number $i$

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.3)$$

Then complex numbers may be represented by $Mat(2, \mathbb{R})$ matrices are

$$a + ib \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \quad (4.4)$$

2 × 2 Real Matrix Representation of $\mathbb{C}$

Since $ii^* = 1$, $i^* \cong \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$z^* \to \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^T.$$

Then $zz^*$ is $|z|^2 \to \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix} = (a^2 + b^2) I$.

Moreover, if we have a polar angle $\theta$ then $R = \exp (i\theta)$ can be written as

$$R = \cos \theta + i \sin \theta \cong \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

This shows that the matrix in Eq. 4.4 satisfies all properties of a complex number $i$. 

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Rotations in 2D can be expressed as

\[ x' + iy' = R(x + iy), \tag{4.5} \]

and it shows that \( C \) is isomorphic to \( SO(2) \) group.

### 4.2.3 Quaternions

Algebra of quaternions or hypercomplex numbers, \( \mathbb{H} \), was invented by Hamilton as an expansion of complex numbers to describe the rotation group \( SO(3) \). He showed that such an algebra only can be derived by non-commutative multiplication.

Quaternions basically are four non-commutative numbers with a scalar or real part and can be written in form

\[ q = w + ix + jy + kz, \tag{4.6} \]

where \( i, j, k \) are imaginary units and satisfy the rules:

\[
\begin{align*}
    i^2 &= j^2 = k^2 = -1, \\
    ij &= ji = k, \quad jk &= -kj = i, \quad ki &= -ik = j.
\end{align*}
\]

However present day vector algebra was extracted by Gibbs and Heaviside and decreased from four component space to three components.

There have been many applications of quaternions in physics. Feza Gürsey described the Dirac equation in terms of quaternions [32] and Adler applied quaternion fields in quantum mechanics [33].

Rotations in \( \mathbb{R}^3 \) can be described as a spin-1/2 representations of rotation group. With \( R \) (a unit vector), a new vector is obtained by two-sided transformation as

\[ a' = RaR^*, \tag{4.7} \]

where \( R^* \) is the conjugate of hypercomplex numbers.

Quaternions construct an even subalgebra of Clifford algebra \((Cl_{0,3})\) like complex numbers are a subalgebra of Clifford algebra.

### Matrix Representations of Quaternions

The quaternions are extensions of complex numbers since \( i^2 = j^2 = k^2 = -1 \) and \( ij = k, \ jk = i, \ ki = j \) while \( ji = -k, \ kj = -i, \ ik = -j \). Also can be expressed as matrices as

\[ i \cong -i\sigma_1, \quad j \cong -i\sigma_2, \quad k \cong -i\sigma_3, \]

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where the matrices are in complex space with $i = \sqrt{-1}$.

We can do rotations in 3 dimensions with the quaternions as

$$R = \exp \left( \frac{ith_1 + jh_2 + kh_3}{2} \right),$$

where each $i, j, k$ is connected with the rotation about the $x, y, z$ axes, respectively.

The new coordinates would be

$$ix' + jy' + kz' = R\left( ix + jy + k z\right) R^{-1}$$

As an example, the rotation about the $z$-axis with $R = \exp \left( k \theta / 2 \right)$ and $q = ix + jy + k z$ can be written as

$$q' = RqR^{-1}$$

$$= \left( \cos \frac{\theta}{2} + k \sin \frac{\theta}{2} \right) (ix + jy + k z) \left( \cos \frac{\theta}{2} - k \sin \frac{\theta}{2} \right)$$

$$= i(x \cos \theta - y \sin \theta) + j(y \cos \theta + x \sin \theta) + k z.$$

One can see that $\mathbb{H}$ is isomorphic to $SO(3)$ group.

We can write quaternions by $2 \times 2$-matrices with entries in $\mathbb{C}$ as

$$i \cong \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad j \cong \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \cong \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

We can use the matrix representation of $i$ which is given in the previous section where $i \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$i \cong \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad j \cong \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad k \cong \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

As a conclusion, we see that complex numbers are convenient to use as a vector algebra to represent the rotation group $SO(2)$ and quaternions to represent the rotation group $SO(3)$. 
4.2.4 Clifford Algebra

This section is about the definition and some properties of Clifford algebra. The Clifford algebra \( Cl_{l,n} \) over \( \mathbb{R} \) is an associative algebra having \( l+n \) generators \( \gamma_1, \gamma_2, ..., \gamma_l, \gamma_{l+1}, ..., \gamma_{l+n} \) such that

\[
\gamma_1^2 = ... = \gamma_l^2 = 1, \quad \gamma_{l+1}^2 = ... = \gamma_{l+n}^2 = -1, \quad \gamma_i \gamma_j = -\gamma_j \gamma_i \quad (i \neq j)
\]

The quadratic form of a vector \( v = x_1 \gamma_1 + x_2 \gamma_2 + ... + x_l \gamma_l + ... + x_{l+n} \gamma_{l+n} \) of \( Cl_{l,n} \) is

\[
Q_{l,n} = v \times v^* = x_1^2 + ... + x_l^2 - x_{l+1}^2 - ... - x_{l+n}^2
\]

The interior and the exterior (wedge) products of \( a \) and \( b \) are

\[
a.b = \frac{ab + ba}{2}, \quad a \wedge b = \frac{ab - ba}{2}
\]

from adding and subtracting these two equations, we find that the Clifford product is

\[
ab = a.b + a \wedge b, \quad ba = a.b - a \wedge b
\]

The volume element for the \( \{\gamma_i\} \) frame is

\[
I_n = \gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n \quad (4.8)
\]

We also have a reciprocal \( \{\gamma^i\} \) is defined by

\[
\gamma^i \gamma_j = \delta^i_j, \quad \forall i, j = 1, ..., n. \quad (4.9)
\]

The reciprocal can also be constructed as

\[
\gamma^j = (-1)^{j-1} \gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_j \wedge ... \wedge \gamma_n I_n^{-1} \quad (4.10)
\]

where the \( \hat{\gamma}_j \) states that \( \gamma_j \) is missing from the expression.

4.2.5 Clifford Algebra \((1,3)\) \( Cl_{1,3} \)

Since we work in Minkowsky space, we use real and complex Clifford algebra \((1,3)\) \( Cl_{1,3} \) shortly. It is also refered as Geometric algebra [34] and space-time algebra [35]. The Clifford numbers of \( Cl_{1,3} \) are given by

\[
\gamma_0, \quad \gamma_1, \quad \gamma_2, \quad \gamma_3, \quad (4.11)
\]

where \( \gamma_0^2 = 1, \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -1 \) and \( \gamma_i \gamma_j = -\gamma_j \gamma_i \) for \( i \neq j \) (Non-commutative).
The metric is defined as $g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu$.

### 4.2.6 Grades

Clifford algebras are $\mathbb{Z}_2$ graded algebras and are separated as even and odd grades. Even multivector Clifford numbers are in even grade space and odd multivector Clifford number belongs to odd grade.

We have different vector spaces which map to real vector spaces as $\Lambda^k \mathbb{R}^4$ with $k = 0, 1, n$ when $n = p + q$ in $Cl_{p,q}$. The Clifford algebra $Cl_{1,3}$ is of 16 dimensions having a basis which consists of

$$
\begin{array}{ccc}
1 & \text{scalar} & \Lambda^0 \mathbb{R} \\
\gamma_0, \gamma_1, \gamma_2, \gamma_3 & \text{vector} & \Lambda^1 \mathbb{R}^4 \\
\gamma_{01}, \gamma_{02}, \gamma_{03}, \gamma_{03}, \gamma_{12}, \gamma_{13}, \gamma_{23} & \text{bivector} & \Lambda^2 \mathbb{R}^4 \\
\gamma_{0123}, \gamma_{023}, \gamma_{031}, \gamma_{012} & \text{trivector} & \Lambda^3 \mathbb{R}^4 \\
\gamma_{0123} & \text{volume element} & \Lambda^4 \mathbb{R}^4 \\
\end{array}
$$

The whole Clifford Algebra $(1,3)$ can be written in terms of a multivector structure as

$$
Cl_{1,3} = \mathbb{R} \oplus \mathbb{R}^4 \oplus \Lambda^2 \mathbb{R}^4 \oplus \Lambda^3 \mathbb{R}^4 \oplus \Lambda^4 \mathbb{R}^4.
$$

### 4.2.7 Lorentz Transformations

The restricted Lorentz transformation that preserves the time direction can be performed by rotor $R$. Rotor was first described by Clifford in showing that quaternion algebra is a special case of Grassmann algebra. Rotor is the exponentiation of a bivector $B$, as $R = e^{B/2}$, and they are a double cover of the Lorentz group

$$
v' = RvR^{-1}. \quad \text{(4.12)}
$$

When $B = \gamma_{ij}$ for $i, j = 1, 2, 3$, it is a rotational Lorentz transformation and when $B = \gamma_{0i}$ for $i = 1, 2, 3$ then it is a boost transformation.

### Matrix Representation of $Cl_{1,3}$

We use the most common example of matrix isomorphism between $Cl_{1,3}$ algebra and $Mat(4, \mathbb{C})$ group algebra, which involves Dirac gamma matrices. They are related by
4.3 Interpolation Quantization and Spinors with Clifford Algebra

The interpolation form is the form between the instant form and the light-front form with an interpolating angle \( \delta \) from chapter 1. The interpolating basis coordinates are

\[
\begin{align*}
 x^+ &= t \cos \delta + z \sin \delta, \\
 x^- &= t \sin \delta - z \cos \delta.
\end{align*}
\]  

(4.13)  

(4.14)

When we take the interpolating angle \( \delta = 0 \), it is the instant form and when \( \delta = \pi/4 \), it is the light-front form. We can derive how to change bases from \( \gamma_\mu x^\mu = \gamma_\bar{\mu} x^{\bar{\mu}} \) and the new basis in \( Cl_{1,3} \) is

\[
\begin{align*}
 \gamma_0 &= \gamma_0 \cos \delta + \gamma_3 \sin \delta, \\
 \gamma_1 &= \gamma_1, \\
 \gamma_2 &= \gamma_2, \\
 \gamma_3 &= \gamma_0 \sin \delta - \gamma_3 \cos \delta.
\end{align*}
\]  

(4.15)

The metric tensor is \( g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu \) and the interpolating metric becomes

\[
g_{\bar{\mu}\bar{\nu}} = \gamma_{\bar{\mu}} \cdot \gamma_{\bar{\nu}} = \begin{pmatrix}
 \gamma_0 \cdot \gamma_0 & \gamma_0 \cdot \gamma_1 & \gamma_0 \cdot \gamma_2 & \gamma_0 \cdot \gamma_3 \\
 \gamma_1 \cdot \gamma_0 & \gamma_1 \cdot \gamma_1 & \gamma_1 \cdot \gamma_2 & \gamma_1 \cdot \gamma_3 \\
 \gamma_2 \cdot \gamma_0 & \gamma_2 \cdot \gamma_1 & \gamma_2 \cdot \gamma_2 & \gamma_2 \cdot \gamma_3 \\
 \gamma_3 \cdot \gamma_0 & \gamma_3 \cdot \gamma_1 & \gamma_3 \cdot \gamma_2 & \gamma_3 \cdot \gamma_3
\end{pmatrix} = \begin{pmatrix}
 \mathbb{C} & 0 & 0 & S \\
 0 & -1 & 0 & 0 \\
 0 & 0 & -1 & 0 \\
 S & 0 & 0 & -\mathbb{C}
\end{pmatrix}.
\]  

(4.16)
According to this metric and our definition in Eq. 4.10, the interpolating reciprocal frame will be

\[
\begin{align*}
\gamma^0 &= \gamma_0 \cos \delta - \gamma_3 \sin \delta, \\
\gamma^3 &= -\gamma_0 \sin \delta - \gamma_3 \cos \delta,
\end{align*}
\] (4.17)

and of course \(\gamma^1 = -\gamma_1\) and \(\gamma^2 = -\gamma_2\). The Poincaré matrix with Clifford numbers can be presented as the wedge product of the basis with imaginary numbers in \(Cl_{1,3}\), and for the interpolating basis it will be

\[
M_{\mu\nu} = \frac{i}{2} \gamma^\mu \wedge \gamma^\nu = i \left( \begin{pmatrix} \gamma^0 \wedge \gamma^0 & \gamma^0 \wedge \gamma^1 & \gamma^0 \wedge \gamma^2 & \gamma^0 \wedge \gamma^3 \\ \\
\gamma^1 \wedge \gamma^0 & \gamma^1 \wedge \gamma^1 & \gamma^1 \wedge \gamma^2 & \gamma^1 \wedge \gamma^3 \\
\gamma^2 \wedge \gamma^0 & \gamma^2 \wedge \gamma^1 & \gamma^2 \wedge \gamma^2 & \gamma^2 \wedge \gamma^3 \\
\gamma^3 \wedge \gamma^0 & \gamma^3 \wedge \gamma^1 & \gamma^3 \wedge \gamma^2 & \gamma^3 \wedge \gamma^3 \end{pmatrix} \right). 
\] (4.19)

According to these, the new rotation and boost operators are

\[
\begin{align*}
J^1 &= \frac{i}{2} \gamma^2 \wedge \gamma^3 = \frac{i}{2} (-\gamma_{02} \sin \delta + \gamma_{23} \cos \delta) = -K^2 \sin \delta - J^1 \cos \delta, \\
J^2 &= \frac{i}{2} \gamma^3 \wedge \gamma^1 = \frac{i}{2} (\gamma_{01} \sin \delta + \gamma_{31} \cos \delta) = K^1 \sin \delta - J^2 \cos \delta, \\
J^3 &= \frac{i}{2} \gamma^1 \wedge \gamma^2 = \frac{i}{2} \gamma_{12} = J^3, \\
K^1 &= \frac{i}{2} \gamma^0 \wedge \gamma^1 = \frac{i}{2} (-\gamma_{01} \cos \delta + \gamma_{31} \sin \delta) = K^1 \cos \delta + J^2 \sin \delta, \\
K^2 &= \frac{i}{2} \gamma^0 \wedge \gamma^2 = \frac{i}{2} (-\gamma_{02} \cos \delta - \gamma_{23} \sin \delta) = K^2 \cos \delta - J^1 \sin \delta, \\
K^3 &= \frac{i}{2} \gamma^0 \wedge \gamma^3 = -\frac{i}{2} \gamma_{03} = -K^3.
\end{align*}
\] (4.20-4.25)

they are connected with the Poincare operators in 1 as \(J^1 = -F^2\), \(J^2 = F^1\), \(K^1 = E^1\), and \(K^2 = E^2\).

### 4.4 Spinors

We can get a complete picture of all spin representations of spin groups with Clifford algebra and also their relations. We can simply construct a spin representation without ad hoc constructions.

In this section, we correlate the vector potential and the field strength tensor and investigate interpolation in the field theory approach. First, we define the spinors with Clifford algebra for consistency spin-1/2 spinors and easiness of calculation with one frame for all spin-1 and spin-1/2 in our work. The Bargmann-Wigner equations give the connection between spin-1 and
spin-1/2 spinors and between \((1/2, 1/2)\) and \((1, 0) \oplus (0, 1)\) Lorentz group spinors. In Appendix A.3.1 and A.4, the polarization vectors and 6 component spin-1 spinor representations are given in standard or Dirac representation. Then, we compare them to Clifford spinors and show how they are connected.

There is an isomorphism between the algebra of \(Cl_{1,3}(\mathbb{C})\) and the algebra of \(Mat(2, \mathbb{C}) \oplus Mat(2, \mathbb{C})\) group matrices by the Artin-Wedderburn theorem in [36] and [37].

The relation between Clifford Spin group of \(Cl_{1,3}(\mathbb{C})\) and \(SO^+(1, 3)\) group from [38] can be shown as

\[ 1 \rightarrow \mathbb{Z}_2 \rightarrow Spin(1, 3) \rightarrow SO(1, 3), \quad (4.26) \]

where we define the rest frame spinors of \((1/2, 1/2)\) as vectors of \(Cl_{1,3}\) in Cartesian representation of their spins as it was discussed before in [15]. Then the \((1, 0) \oplus (0, 1)\) spinors may be derived from polarization vectors we got as \(u(\rho, \lambda) = \epsilon \wedge p/m\) by similar comparison of the vector potential \(A_\mu\) to the field strength tensor \(F_{\mu\nu}\). We compared our finding with standard methods of these Lorentz group spinors \(SO(1, 3)\) and \(SU(2)\) for spin-1 in Appendix A.3.1 and A.4. We find out that the Grassmann algebra basis is a better choice for expressing chiral (Weyl) spinors with circular polarizations.

**Spin-1/2 Spinors**

In this section, because of the connection with our spin-1 spinors, we consider definition of spin-1/2 spinors. According to Hestenes’ model, in order to construct spinors with Clifford algebra, we need to use matrices instead of column vectors because Clifford numbers are only isomorphic to matrices. We can use projection operators to express spinors in terms of the Clifford algebra. In [39], Hestenes uses \(U = \frac{1}{4}(1 + \gamma_0)(1 + \sigma_3)\) projection for spinors. In this equation the factors \(1/2(1 + \gamma_0)\) and \(1/2(1 + \sigma_3)\) are energy and spin projection operators. However, we will use \(i\gamma_{12}\) notation instead of \(\sigma_3\) for the spin operator, so our spin projection operator is a projective spin representation of \((1/2, 0) \oplus (0, 1/2)\)

\[ u = \frac{1}{2}(1 + \gamma_0)\frac{1}{2}(1 + i\gamma_{12}). \quad (4.27) \]

We know that \(u\) is a positive energy and positive helicity spinor since it has positive energy and spin projections. So, we need to apply some operators on our original spinor \(u\) which gives the first left column in the Dirac gamma matrix representation. We define two operators to achieve other spinors; charge and spin, raising and lowering operators with Clifford multiplications as

\[ Q^\pm = \frac{1}{2}(\mp \gamma_1 - i\gamma_2), \quad S^\pm = \frac{1}{2}(i\gamma_{23} \pm \gamma_{31}). \quad (4.28) \]
We can simplify them as

\[ Q = Q^+ + Q^-, \quad S = S^+ + S^- . \]  

(4.29)

Since we only have two states, particle or anti-particle or up or down spins, one of the \( \pm \) always eliminates the one state and changes the other state. By these operators, we can achieve all Dirac spinor representations from \( u \) as

\[ u^1 = u, \quad u^2 = Su, \quad v^1 = SQu, \quad v^2 = Qu. \]  

(4.30)

Dirac spinors in the moving frame are

\[ u^{(i)}(p) = Ru^{(i)}(0). \]  

(4.31)

where rotor \( R \) is an arbitrary boost to any direction \( R = e^{-\gamma_0 \phi/2} \) and it is in momentum space with \( \cosh(\frac{\phi}{2}) = \sqrt{\frac{E+m}{2m}} \) and \( \sinh(\frac{\phi}{2}) = \frac{p_0}{\sqrt{2m(E+m)}} \) transformation. This arbitrary boost can be written as

\[
R = \frac{1}{\sqrt{2m(p^0 + m)}} \begin{pmatrix}
p^0 + m & 0 & -p^3 & -p^1 + ip^2 \\
0 & p^0 + m & -p^1 - ip^2 & p^2 \\
-p^3 & -p^1 + ip^2 & p^0 + m & 0 \\
-p^1 - ip^2 & p^3 & 0 & p^0 + m
\end{pmatrix}.
\]  

(4.32)

There are different kinds of transformation choices like helicity Jacob-Wick transformation given by Eq. 2.28, light-front boost given by Eq. 2.29, and so on. One can simply arrange this choice by only defining our boost \( R \) with these spinors definitions.

If we use the standard representation of Dirac Gamma matrices as Clifford numbers, we can see that we have the same spinors in standard representation Eq. 2.116 as the first column of the \( 4 \times 4 \) matrices of \( u \) shown in Eq. 4.33. We can write \( u = R \) a square matrix spinor and the other spinors are a projection of it

\[
u \cong \begin{pmatrix}
u_1 & -\nu_2 & \nu_3 & \nu_4^* \\
u_2 & \nu_1^* & \nu_4 & -\nu_3^* \\
u_3 & \nu_4^* & \nu_1 & -\nu_2^* \\
u_4 & -\nu_3^* & \nu_2 & \nu_1^*
\end{pmatrix}.
\]  

(4.33)

Its conjugate with the reverse transformation operator is the bar notation of spinors as

\[ \bar{u}^{(1)} = (u^{(1)})^* R^{-1} . \]  

(4.34)

Antiparticle spinors are actually the charge conjugation of particle spinors. So instead of \( u^{(3)} \)
and \(u^{(4)}\), they are \(v^{(1)} = -i\gamma_2 u^{(1)*}\) and \(v^{(2)} = -i\gamma_2 u^{(2)*}\) in matrix representation or negative energy solutions (Feynman-Stueckelburg interpretation) \(v^{(1)}(p) = u^{(4)}(-p), v^{(2)}(p) = u^{(3)}(-p)\). In terms of Clifford numbers \(v^{(1)} = -iu^{(4)}\gamma_2\) and \(v^{(2)} = -iu^{(3)}\gamma_2\).

We can put Dirac spinors from Eq. 4.30 in terms of Clifford number as

\[
\begin{align*}
\mathbf{u}^{(1)}(p) &= R(1 + \gamma_0)(1 + i\gamma_{12})/4, \\
\mathbf{u}^{(2)}(p) &= R_i\gamma_3(1 + \gamma_0)(1 + i\gamma_{12})/4, \\
\mathbf{v}^{(1)}(p) &= R\gamma_3(1 + \gamma_0)(1 + i\gamma_{12})/4, \\
\mathbf{v}^{(2)}(p) &= R(-i\gamma_2)(1 + \gamma_0)(1 + i\gamma_{12})/4,
\end{align*}
\]

where \(R\) is given in Eq. 4.32.

**Spin-1 Spinors**

**Polarization Vectors**

By definition, spin-1 is a vector representation so we can use Clifford groups to define spin-1 representation along with the standard matrix-vector relation. It makes easier for the calculation of the relation of spinors.

We can begin with describing our vector as \(v = \gamma_\mu x^\mu\) instead of \(v = \{t, x, y, z\} = e_\mu x^\mu\) where \(e_\mu\) are column vectors for \(\mu = 0, 1, 2, 3\) in \(SO(1, 3)\) so we can use Clifford numbers instead of columns vectors to represent the polarization vector in the direction of \(i\) as \(\gamma_i\). Then we define the 0 helicity as the z direction vector and as \(\epsilon(0,0) = \gamma_3\) for \(\epsilon(p, \lambda)\) in \(Cl_{1,3}(C)\). Then, by using spin raising and lowering operators, \(S^+ = (i\gamma_{23} - \gamma_{31})/\sqrt{2}\) and \(S^- = (i\gamma_{23} + \gamma_{31})/\sqrt{2}\) which are given by Eq. 4.29 in the previous section, we can get spin +1 and −1 circular polarization vectors. The whole set of the polarization vectors are

\[
\begin{align*}
\epsilon(0, +) &= -(\gamma_1 + i\gamma_2)/\sqrt{2}, \\
\epsilon(0, 0) &= \gamma_3, \\
\epsilon(0, -) &= -(\gamma_1 + i\gamma_2)/\sqrt{2}.
\end{align*}
\]

We can make a direct connection between polarization vectors and spherical harmonics as

\(\epsilon(0, +) \sim Y^1_1, \quad \epsilon(0, 0) \sim Y^0_1, \quad \epsilon(0, -) \sim Y^{-1}_1\).

Polarization vectors in moving frame are

\[
\epsilon(p, \lambda) = R\epsilon(0, \lambda)R^{-1},
\]

where \(R\) is the same transformation given in Eq. 4.32.
(1, 0) ⊕ (0, 1) Lorentz Group Spinors Using the similar relation between electric and magnetic field with four potential as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = (\nabla \wedge A)^\mu$ where $\nabla = \gamma^\mu \partial_\mu$ and $A = \gamma^\nu A_\nu$, we can describe $u \sim \nabla \wedge A = -\gamma^j \partial_j / m$ since the vector potential can be written as $A = \gamma^\mu e^\mu(p) e^{ipx}$. Then $u(0, j) = -\gamma^j \wedge \gamma^0$ in the rest frame with normalization. In this situation our spinors on $(1, 0) \oplus (0, 1)$ representation will look like

$$
\begin{align*}
  u(0, +) &= (-\gamma_{01} - i\gamma_{02})/\sqrt{2}, & (4.43) \\
  u(0, -) &= (\gamma_{01} - i\gamma_{02})/\sqrt{2}, & (4.44) \\
  u(0, 0) &= \gamma_{03}. & (4.45)
\end{align*}
$$

Here we used the same notation for the separation of two $SU(2)$ subalgebras in $SU(2) \otimes SU(2)$ into $A = J + iK$ and $B = J - iK$ to project right-handed and left-handed spinor parts in chiral representation. One can see that in Clifford algebra, we could write down $A$ and $B$ such that $A = (1 + \gamma_5)J$ and $B = (1 - \gamma_5)J$ since $\gamma_5 J = iK$. Thus we would have the same operator as spin 1/2 spinor to project them into right-handed $(1, 0)$ or left-handed spinors $(0, 1)$. The electric and magnetic field can be expressed by the same way as $E + iB \in (1, 0)$ and $E - iB \in (0, 1)$ as these parts are related as $\gamma_{\mu\nu} F^{\mu\nu} = \nabla \wedge A$.

<table>
<thead>
<tr>
<th>$(1/2, 1/2)$</th>
<th>$(1, 0)$</th>
<th>$(0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>$\leftrightarrow (\gamma_{01} - i\gamma_{23})$</td>
<td>$(\gamma_{01} + i\gamma_{23})$</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>$\leftrightarrow (\gamma_{02} - i\gamma_{31})$</td>
<td>$(\gamma_{02} + i\gamma_{31})$</td>
</tr>
<tr>
<td>$\gamma_3$</td>
<td>$\leftrightarrow (\gamma_{03} - i\gamma_{12})$</td>
<td>$(\gamma_{03} + i\gamma_{12})$</td>
</tr>
</tbody>
</table>

Let’s make the same transformation for the spin $(1, 0) \oplus (0, 1)$ representation with similar ways of polarization vectors $u(p, \lambda) = Ru(0, \lambda)R^{-1}$ by using the spinors in Eq. 4.43, Eq. 4.44, and Eq. 4.45

$$
  \begin{align*}
    u(p, -) &= -\gamma_{01} \frac{(p^0 + m)^2 + (p^3)^2 - 2(p^L)^2}{2m(p^0 + m)} - i\gamma_{02} \frac{(p^0 + m)^2 + (p^3)^2 + 2(p^L)^2}{2m(p^0 + m)} \\
    &+ \gamma_{03} \frac{\sqrt{2}p^3 p^L}{(p^0 + m)^2} - i\gamma_{23} \frac{p^3}{m} + \gamma_{31} \frac{p^3}{m} + i\gamma_{12} \frac{\sqrt{2}p^L}{m} \\
    &= \frac{1}{2\sqrt{2}m(p^0 + m)} \left\{ - \frac{\gamma_{01} - i\gamma_{23}}{2}(1 - i\gamma_{12}) \left( \frac{p^0 + m + m - p^3}{2} \right)^2 - \frac{\gamma_{03} - i\gamma_{12}}{\sqrt{2}} \left( \frac{p^0 + m + m - p^3}{2} \right)^2 \\
    &- \frac{\gamma_{01} - i\gamma_{23}}{2}(1 + i\gamma_{12}) \left( \frac{p^0 + m + m - p^3}{2} \right)^2 + \frac{\gamma_{03} + i\gamma_{12}}{\sqrt{2}} \left( \frac{p^0 + m + m - p^3}{2} \right)^2 \right\}, & (4.46)
  \end{align*}
$$

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\[
\begin{align*}
\quad u(p, 0) &= -\gamma_{01} \frac{p^1 p^3}{m(p^0 + m)} - \gamma_{02} \frac{p^2 p^3}{m(p^0 + m)} \\
&\quad + \gamma_{03} \frac{p_0 (p^0 + m) - (p^3)^2}{m(p^0 + m)} + \gamma_{23} \frac{p^2}{m} - \gamma_{31} \frac{p^1}{m} \\
&= \frac{1}{\sqrt{2} m(p^0 + m)} \left\{ -\left(\frac{\gamma_{01} - i \gamma_{23}}{2}(1 - i \gamma_{12}) \right) \sqrt{2}(p^0 + m + p^3)p^L + \left(\frac{\gamma_{03} - i \gamma_{12}}{2}\right) \sqrt{2}(p^0(p^0 + m) - (p^3)^2) \\
&\quad + \left(\frac{\gamma_{01} - i \gamma_{23}}{2}(1 + i \gamma_{12}) \right) \sqrt{2}(p^0 + m - p^3)p^R \\
&\quad + \frac{\gamma_{03} + i \gamma_{12}}{2} \sqrt{2}(p^0(p^0 + m) - (p^3)^2) - \left(\frac{\gamma_{01} + i \gamma_{23}}{2}(1 + i \gamma_{12}) \right) \sqrt{2}(p^0 + m + p^3)p^R \right\}, \quad (4.47)
\end{align*}
\]

\[
\begin{align*}
\quad u(p, +) &= -\gamma_{01} \frac{(p^0 + m)^2 + (p^3)^2 - 2(p^R)^2}{2m(p^0 + m)} - \gamma_{02} \frac{(p^0 + m)^2 + (p^3)^2 + 2(p^R)^2}{2m(p^0 + m)} \\
&\quad + \gamma_{03} \frac{\sqrt{2}p^3 p^R}{m(p^0 + m)} + \gamma_{23} \frac{p^3}{m} - \gamma_{31} \frac{p^3}{m} - \gamma_{12} \frac{\sqrt{2}p^R}{m} \\
&= \frac{1}{\sqrt{2} m(p^0 + m)} \left\{ -\left(\frac{\gamma_{01} - i \gamma_{23}}{2}(1 - i \gamma_{12}) \right) \sqrt{2}(p^0 + m + p^3)^2 + \left(\frac{\gamma_{03} - i \gamma_{12}}{2}\right) \sqrt{2}(p^0 + m + p^3)p^R \\
&\quad + \left(\frac{\gamma_{01} - i \gamma_{23}}{2}(1 + i \gamma_{12}) \right) \sqrt{2}(p^0 + m + p^3)p^R \\
&\quad - \frac{\gamma_{03} + i \gamma_{12}}{2} \sqrt{2}\sqrt{2}(p^0 + m - p^3)p^R + \left(\frac{\gamma_{01} + i \gamma_{23}}{2}(1 + i \gamma_{12}) \right) \sqrt{2}(p^0 + m + p^3)p^R \right\}. \quad (4.48)
\end{align*}
\]

We can see one-to-one component correspondence between standard spinors in Eq. A.11, Eq. A.12, and Eq. A.13 and Clifford spinors from the expanded form Clifford spinors.

We can make connections with \(u(0, \lambda)\) vectors as

\[
\begin{align*}
\quad -(\gamma_{01} - i \gamma_{23})(1 - i \gamma_{12})/2 &\leftrightarrow (1, 0, 0, 0, 0, 0), \quad (4.49) \\
\quad (\gamma_{01} - i \gamma_{23})(1 + i \gamma_{12})/2 &\leftrightarrow (0, 0, 1, 0, 0, 0), \quad (4.50) \\
\quad (\gamma_{03} - i \gamma_{12})/\sqrt{2} &\leftrightarrow (0, 1, 0, 0, 0, 0), \quad (4.51) \\
\quad -(\gamma_{01} + i \gamma_{23})(1 - i \gamma_{12})/2 &\leftrightarrow (0, 0, 0, 1, 0, 0), \quad (4.52) \\
\quad (\gamma_{01} + i \gamma_{23})(1 + i \gamma_{12})/2 &\leftrightarrow (0, 0, 0, 0, 0, 1), \quad (4.53) \\
\quad (\gamma_{03} + i \gamma_{12})/\sqrt{2} &\leftrightarrow (0, 0, 0, 0, 1, 0). \quad (4.54)
\end{align*}
\]

In the expression above, the relation between Clifford number expressions (left-side) and the components of \(u\) spinor (right-side) seems complex. It is because the \(u\) spinor is defined by spin states and helicity while Clifford numbers are in Cartesian states \((x, y, z)\). We adopted J. Winnberg’s description of Grassmann algebra with a slight change in order to express spinors with helicity and spin states in the next section.
Grassmann Algebra and Clifford Spinors

By looking at the underlying vector space of a Clifford algebra, it is easily seen that it has the same dimension as Grassmann algebra on the same generators [40]. Chevalley used this isomorphism (of vector spaces) to construct Clifford algebras (over base fields of any characteristic) as a sub-algebra of the endomorphism algebra of that vector space. Hence, the Clifford algebra can be seen as a (non-trivial) deformation of the Grassmann algebra. Hence it is no surprise that spinor fields can be described by exterior form fields (differential forms), and this was already studied by W. Graf [41].

J. Winnberg showed that [42] there is an equivalence between the spin representation of a Clifford algebra and a Grassmann algebra which is the main algebra of supersymmetry and the relation to the orthogonal group.

We use a little different notation and redefine our Clifford basis as

\[
\begin{align*}
\theta_2 &= (-\gamma_1 - i\gamma_2)/\sqrt{2}, & \bar{\theta}_2 &= (\gamma_1 - i\gamma_2)/\sqrt{2}, \\
\theta_1 &= (\gamma_0 + \gamma_3)/\sqrt{2}, & \bar{\theta}_1 &= (\gamma_0 - \gamma_3)/\sqrt{2}.
\end{align*}
\]

(4.55)

(4.56)

We see that the expressions above satisfy the properties of Grassmann algebra.

\[(\theta_1)^2 = (\bar{\theta}_1)^2 = (\theta_2)^2 = (\bar{\theta}_2)^2 = 0 \quad \text{and} \quad \theta_i\theta_j = -\theta_j\theta_i \text{ for } i \neq j,\]

(4.57)

where the second expression includes with bar and without bar \(\theta_i\). So we can rewrite our polarization vectors with this new basis as well.

\[
\zeta(p, \lambda) = \gamma_{\mu}\epsilon^\mu = \theta_1\epsilon^+ - \theta_2\epsilon^L + \bar{\theta}_2\epsilon^R + \bar{\theta}_1\epsilon^-.
\]

(4.58)

where \(\epsilon^+ = (\epsilon^0 + \epsilon^3)/\sqrt{2}, \epsilon^- = (\epsilon^0 - \epsilon^3)/\sqrt{2},\)

\(\epsilon^R = (\epsilon^1 + i\epsilon^2)/\sqrt{2}, \epsilon^L = (\epsilon^1 - i\epsilon^2)/\sqrt{2}.\)

\[
u(p, \lambda) = \theta_1\theta_2u^1 + (\bar{\theta}_1\theta_1 - \theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2 - \bar{\theta}_2\theta_2)u^2 + \bar{\theta}_1\bar{\theta}_2u^3
\]

\[+ \bar{\theta}_1\theta_2u^4 + (\bar{\theta}_1\theta_1 - \theta_1\bar{\theta}_1 - \theta_2\bar{\theta}_2 + \bar{\theta}_2\theta_2)u^5 + \theta_1\theta_2u^6.\]

(4.59)

When we compare the Grassmann basis with the Clifford basis, we can see that the expression above looks much nicer than the one in the Clifford basis in Eq. 4.48 , Eq. 4.47, and Eq. 4.46. We also see how \(\theta_2\) and \(\bar{\theta}_2\) are related with \(\pm\) polarizations and \(\theta_1\) and \(\bar{\theta}_1\) are related with positive and negative helicity spinors components. Although \(\theta_1\) is not exactly \(R\) or \(L\) systems but a combination of them and each components of \(u(p, \lambda)\) in \(Cl_{1,3}\) Eq. 4.49, Eq. 4.51, Eq. 4.50, Eq. 4.52, Eq. 4.54, Eq. 4.53 can be the components in Eq. 4.59, respectively. Since helicity depends on the \(\theta_1\) and \(\bar{\theta}_1\), we can change chiral representation in Eq. 4.59.

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into standard representation by making \( \theta_1 \rightarrow (\theta_1 + \bar{\theta}_1) \) and \( \bar{\theta}_1 \rightarrow (\theta_1 - \bar{\theta}_1) \) transformations.

The relation between the polarization vector and the \( u \) spinor can be expressed by wedge product between them as

\[
u(p, \lambda) = \ell(p, \lambda) \wedge \bar{p}/m \tag{4.60}\]

It is also possible to express \( \bar{p} \) like \( \bar{p} = \theta_1 p^+ - \theta_2 p^L + \bar{\theta}_2 p^R + \bar{\theta}_1 p^- \)

So we are able to write down spinors as a function of each others like

\[
\begin{align*}
    u_1(p, \lambda) &= (p^L \epsilon^+ - p^+ \epsilon^L)/m, \\
    u_2(p, \lambda) &= (p^- \epsilon^+ - p^+ \epsilon^- - p^L \epsilon^R - p^R \epsilon^L)/\sqrt{2}m, \\
    u_3(p, \lambda) &= (-p^R \epsilon^- + p^- \epsilon^R)/m, \\
    u_4(p, \lambda) &= (p^L \epsilon^- - p^- \epsilon^L)/m, \\
    u_5(p, \lambda) &= -(p^+ \epsilon^- - p^- \epsilon^+ + p^L \epsilon^R - p^R \epsilon^L)/\sqrt{2}m, \\
    u_6(p, \lambda) &= (p^+ \epsilon^R - p^- \epsilon^L)/m. \\
\end{align*}
\tag{4.61}\]

The general form of polarization vector and \((1,0) \oplus (0,1)\) spinor can be written as

\[
\ell = \epsilon^\mu \gamma_\mu \text{ where } \epsilon^\mu = \{\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3\}, \tag{4.62}\]

and with respect to this, \( u = \ell \wedge \bar{p}/m \) becomes

\[
u = \{\epsilon^0 p^1 - \epsilon^1 p^0, \epsilon^0 p^2 - \epsilon^2 p^0, \epsilon^0 p^3 - \epsilon^3 p^0, \epsilon^2 p^3 - \epsilon^3 p^2, \epsilon^3 p^1 - \epsilon^1 p^3, \epsilon^1 p^2 - \epsilon^2 p^1\}. \tag{4.63}\]

If we write the relations of two spinors with a matrix, it would be

\[
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6
\end{pmatrix} = \frac{1}{\sqrt{2}m} \begin{pmatrix}
p^L & -p^+ & ip^+ & p^L \\
-(p^+ - p^-)/\sqrt{2} & -(p^R - p^-)/\sqrt{2} & i(p^R + p^L)/\sqrt{2} & (p^+ + p^-)/\sqrt{2} \\
-p^R & p^- & ip^- & p^R \\
p^L & -p^- & ip^- & -p^L \\
-(p^+ - p^-)/\sqrt{2} & (p^R - p^-)/\sqrt{2} & -i(p^R + p^L)/\sqrt{2} & (p^+ + p^-)/\sqrt{2} \\
-p^R & p^+ & ip^+ & -p^R
\end{pmatrix} \begin{pmatrix}
\epsilon^0 \\
\epsilon^1 \\
\epsilon^2 \\
\epsilon^3
\end{pmatrix}. \tag{4.64}\]

Here the \( u \) spinor is in chiral representation. This matrix is a universal transformation that changes polarization vectors into 6 component spin-1 spinors.

When we apply the matrix for connecting polarization vectors Eq. 4.64 with interpolating polarization vectors Eq. A.6, we get the same spinors as Eq. A.17 spin-1 spinors in \((1,0) \oplus (0,1)\) Lorentz groups.

We observe that electric and magnetic fields have similar behavior with rotation and boost
operators which are also mentioned in chapter 3 and we note that they are also defined by bivectors in Clifford algebra.

4.4.1 Anti-symmetry Polarization in Spin-1 Spinors

The conversion matrix we found on Eq. 4.64 helps us to understand many concepts on spinors. One of them is to make relations between $(J;0)$ Lorentz group type spinors with $(J,J)$ Lorentz group type spinors. As an example, you could look at how spin and handedness represented on each spinor type and make a connection between them.

The conversion matrix can be expressed also as $C = \sum u\epsilon_1$ since it converts polarization vector $\epsilon(p,\lambda)$ into 6 component spinors $u(p,\lambda)$. Here, $\epsilon_1$ can be defined as $\epsilon_1 = (\epsilon_e) \epsilon_1 = 1$. The opposite conversion is written as $C = \sum \epsilon u$ which converts $u(p,\lambda)$ spinor into polarization vector as $\epsilon(p,\lambda) = C u(p,\lambda)$. To derive to the symmetric part, we see that the multiplication of $C$ matrices already gives us the spin sums of spinors:

\[
CC' = \sum_\lambda u(p,\lambda)\epsilon^{-1}(p,\lambda)\epsilon(p,\lambda)\bar{u}(p,\lambda) = \sum_\lambda u(p,\lambda)\bar{u}(p,\lambda), \quad (4.65)
\]

\[
\bar{C}C = \sum_\lambda \epsilon(p,\lambda)\bar{u}(p,\lambda)u(p,\lambda)\epsilon^{-1}(p,\lambda) = -\sum_\lambda \epsilon_\mu(p,\lambda)(\epsilon^\nu)^*(p,\lambda). \quad (4.66)
\]

In spin-1/2 spinors, Eq. 4.65 can be written as $\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) = (\not{\gamma} + m)/2m$ because of the expression $(1 - \gamma_0)u(0,\lambda) = 0$ positive and negative energy projections. Similarly, we can write this expression for spin-1 as $\sum_\lambda u(0,\lambda)\bar{u}(0,\lambda) = (1 + \gamma_0)/2$, similarly $\gamma_0$ is written as $\gamma_0 = \begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}$. In Eq. 4.66, as it is seen, there is a superscript and subscript difference. We could define a different mapping between $\epsilon(p,\lambda)$ and $u(p,\lambda)$ in order to find an expression in terms of $\epsilon_\mu(p,\lambda)\epsilon^\nu(p,\lambda)$ both subscript indexes as $C' = \sum_\lambda u\epsilon^*$. In this case, Eq. 4.66 becomes

\[
\bar{C}C' = \sum_\lambda \epsilon(p,\lambda)\bar{u}(p,\lambda)u(p,\lambda)\epsilon^*(p,\lambda) = -\sum_\lambda \epsilon_\mu(p,\lambda)\epsilon^\nu(p,\lambda) = g_{\mu\nu} - \frac{P_{\mu}P_{\nu}}{m^2}. \quad (4.67)
\]

We can write the anti-symmetric part as a similar way to write down symmetric or spin average part as previously done by using the C matrix again

\[
C\gamma_5p^\#C' = A_{\mu\nu} = -ie_{\mu\nu\alpha\beta}p^\alpha s^\beta \frac{1}{m}, \quad (4.68)
\]

where $s$ spin vector is the same as longitudinal polarization $\epsilon(p,0)$ in Eq. A.4. $\gamma_5$ separates right-handed and left-handed spinors as $(1 \pm \gamma_5)/2u = u_{R,L}$ and, similar to $\gamma_0$, is written
as \( \gamma_5 = \begin{pmatrix} I_3 & 0_3 \\ 0_3 & -I_3 \end{pmatrix} \). This result is consistent with Passarino’s finding of the polarization tensor from spin-1/2 spinors in [43] which is connected with spin-1/2 anti-symmetric part with polarization vectors anti-symmetry.

We could also derive this formula by Clifford algebra as

\[
A = \epsilon(p,+)\epsilon^*(p,+) - \epsilon(p,-)\epsilon^*(p,-) = -R\gamma_{12}R^{-1} = -i\gamma_5 R\gamma_0 (R^{-1} R)\gamma_3 R^{-1} = -i\gamma_5 p^\prime \gamma^\prime /m,
\]

then

\[
A_{\mu\nu} = \gamma_{\mu\nu}.A = -i\epsilon_{\mu\nu\alpha\beta}p^\alpha s^\beta /m.
\]

since \( \gamma_5 = i\epsilon_{\sigma\delta\xi\eta}\gamma^\sigma\gamma^\delta\gamma^\xi\gamma^\eta \).

We can see that Eq. 4.70 gives us the exact result of \((\epsilon(p,+)\epsilon^*(p,+) - \epsilon(p,-)\epsilon^*(p,-))\) of using polarization vectors in Eq. A.4:

\[
A_{\mu\nu} = -i\epsilon_{\mu\nu\alpha\beta}p^\alpha s^\beta /m = \frac{-i}{m} \begin{pmatrix} 0 & p^2 s^3 - s^2 p^3 & p^3 s^1 - s^3 - p^1 & p^1 s^2 - s^1 p^2 \\ -p^2 s^3 + p^3 s^2 & 0 & p^0 s^3 - s^0 p^3 & -p^0 s^2 + s^0 s^2 \\ -p^3 s^1 + s^3 p^1 & -p^0 s^3 + s^0 p^3 & 0 & p^0 s^1 - s^0 p^1 \\ p^1 s^2 - s^1 p^2 & p^0 s^2 - s^0 p^2 & p^0 s^1 - s^0 p^1 & 0 \end{pmatrix}.
\]

(4.71)

Now, we want to express our anti-symmetric part as well as the symmetric part without mass terms in order to apply it to the real photon (massless) case. We took the notation in our previous work [14] for interpolation spinors. In this paper, the symmetric part of \pm averaged polarization vectors is already defined, and we add our anti symmetric part. First, we find the parts in terms of the interpolation basis then take the limit to the instant form.

First, we introduce vierbein \( p_\mu = n.pn_\mu + (p_\mu - n.pn_\mu) = n'_\mu + q_\mu \) as the separation of energy and vector parts of momentum.

Then we express the longitudinal polarization with \( n_\hat{\mu} \)

\[
s_\hat{\mu} = \epsilon(\hat{\mu},0) = \frac{p^\perp}{m^\perp}(p_\perp - \frac{m^2}{p^\perp},p_1,p_2,p_\perp)
\]

(4.72)

here \( \hat{\mu} = (+,1,2,-) \) is the interpolation basis. Then

\[
s_\hat{\mu} = \frac{p^\perp}{m^\perp}p_\hat{\mu} - \frac{m}{m^\perp}n_\hat{\mu}
\]

(4.73)

where \( n_\hat{\mu} = (1,0,0,0) \) converts \( \hat{\mu} \) into \( \mu \) and will be \( n^\mu g_{\mu\nu} = n_\mu = (\cos \delta,0,0,-\sin \delta) \). The anti-symmetric part is

\[
i\epsilon_{\mu\nu\alpha\beta}p^\alpha s^\beta /m = -i\epsilon_{\mu\nu\alpha\beta}p^\alpha n^\beta /m
\]

(4.74)
Since $P = \sqrt{q.n + q^2}$, the expression above is written in instant-form for the real photon case as

$$A_{\mu \nu} = -i \epsilon_{\mu \nu \alpha \beta} P^\alpha n^\beta / p.n \quad \text{with} \quad n^\mu = (1, 0, 0, 0) \quad (4.75)$$

The symmetric part in the interpolation basis is

$$S_{\hat{\mu} \hat{\nu}} = \sum_{\pm} \epsilon_{\hat{\mu}}(p, \lambda) \epsilon^*_{\hat{\nu}}(p, \lambda) = -g_{\hat{\mu} \hat{\nu}} + \frac{p.n(p_{\mu} n_{\nu} + n_{\mu} p_{\nu})}{2(P^2_C + P^2_\perp)} - \frac{C p_{\mu} p_{\nu}}{2(P^2_C + P^2_\perp)} + \frac{p^2 n_{\mu} n_{\nu}}{2(P^2_C + P^2_\perp)}. \quad (4.76)$$

where $C = \cos 2 \delta$ and $P_\perp = p^0 \sin \delta + p^3 \cos \delta$.

Then the general form of multiplication of polarization vectors for $\lambda = \pm$ is

$$\epsilon_{\hat{\mu}}(p, \lambda) \epsilon^*_{\hat{\nu}}(p, \lambda) = \frac{S_{\hat{\mu} \hat{\nu}}}{2} + \frac{A_{\hat{\mu} \hat{\nu}}}{2} = -g_{\hat{\mu} \hat{\nu}} + \frac{p.n(p_{\mu} n_{\nu} + n_{\mu} p_{\nu})}{2(P^2_C + P^2_\perp)} - \frac{C p_{\mu} p_{\nu}}{2(P^2_C + P^2_\perp)} + \frac{p^2 n_{\mu} n_{\nu}}{2(P^2_C + P^2_\perp)} - i \epsilon_{\hat{\mu} \hat{\nu} \hat{\alpha} \hat{\beta}} \frac{p^\alpha n^\beta}{2p.n}. \quad (4.77)$$

When we take the light-front limit

$$\epsilon_{\mu}(p, \lambda) \epsilon^*_{\nu}(p, \lambda) = \frac{-g_{\mu \nu}}{2} + \frac{p.n(p_{\mu} n_{\nu} + n_{\mu} p_{\nu})}{2(p.n)^2} - \frac{p^2 n_{\mu} n_{\nu}}{2(p.n)^2} - i \lambda \epsilon_{\mu \nu \alpha \beta} \frac{p^\alpha n^\beta}{2\sqrt{(n.p)^2}}. \quad (4.78)$$

and the general expression of multiplication of polarization in instant form is

$$\epsilon_{\mu}(p, \lambda) \epsilon^*_{\nu}(p, \lambda) = \frac{-g_{\mu \nu}}{2} + \frac{p.n(p_{\mu} n_{\nu} + n_{\mu} p_{\nu})}{2((p.n)^2 - p^2)} - \frac{p^2 n_{\mu} n_{\nu}}{2((p.n)^2 - p^2)} - i \lambda \epsilon_{\mu \nu \alpha \beta} \frac{p^\alpha n^\beta}{2\sqrt{(n.p)^2 - p^2}}. \quad (4.79)$$

and the general expression of multiplication of polarization vectors for a real photon ($p^2 = 0$) is

$$\epsilon_{\mu}(p, \lambda) \epsilon^*_{\nu}(p, \lambda) = \frac{-g_{\mu \nu}}{2} + \frac{p.n(p_{\mu} n_{\nu} + n_{\mu} p_{\nu})}{2p.n} - \frac{p^2 n_{\mu} n_{\nu}}{2p.n} - i \lambda \epsilon_{\mu \nu \alpha \beta} \frac{p^\alpha n^\beta}{2p.n}. \quad (4.80)$$

### 4.5 Interpolating Spinors

As we mentioned in Section 4.4, we could find the spinor’s expression for spin 1/2 and spin 1 spin groups in Clifford groups with a much easier and convenient way since we don’t need to separate right-handed and left-handed parts. We can use the same boost expression for spin 1/2 and spin 1 spinors together.

First, we can define the interpolating boost by bivectors. We take Eq. 2.44 and use Clifford
algebra corresponding for interpolating Lorentz transformations Eq. 4.20-Eq. 4.25, then we can write the interpolating boost as

\[ T = e^{\left\{ \frac{-\beta_1(1 - \sin \gamma_{01} + \cos \delta_{01}) + \beta_2(\cos \delta_{23} + \sin \gamma_{02})}{2} \right\}} e^{\left\{ -\frac{\beta_3\gamma_{03}}{2} \right\}}. \] (4.81)

We simply change the boost transformation into the interpolation transformation given by Eq. 2.44 and after the normalization, we get the same polarization vectors in our previous work, which is done with SO(1, 3) group in \cite{14}. The spinors at rest are the same as those given in Eq. 4.35-Eq. 4.38. Then, we apply the interpolating transformation operator given by Eq. 4.81. We can use the matrix representation of \( Cl_{1,3} \) as Dirac gamma matrices and take the first column of the outcome matrix as our spinors. We get the same spinors as given in Eq. 2.99a and Eq. 2.99b as

\[ u^{(1)}(P) = \frac{1}{\sqrt{P^\mu P^\nu}} \begin{pmatrix} (\cos \delta P^+ - \sin \delta P^\gamma)\sqrt{P^\pm + P^\parallel P^\perp} & 0 & 0 & 0 \\ -\sqrt{P^\pm + P^\parallel P^\perp} P^\gamma & 0 & 0 & 0 \\ -\sin \delta P^+ + \cos \delta P^\gamma \sqrt{P^\pm + P^\parallel P^\perp} & 0 & 0 & 0 \\ \sqrt{-P^\pm + P^\parallel P^\perp} P^\gamma & 0 & 0 & 0 \end{pmatrix}, \] (4.82)

where the first column is equivalent to Eq. 2.99a where \( \mathbb{P} = \sqrt{(P^\gamma)^2 - M^2 \mathbb{C}} \).

As for spin-1 spinors, different from the spin-1/2 we used in the vector representation of \( Cl_{1,3} \) and for interpolating spin-1 spinors we use the same rest frame spinor Eq. 4.39-Eq. 4.41. We apply the same interpolating transformation as spin-1/2, which is given by Eq. 4.81 with two-sides. Then, we can find each component of the polarization vectors as \( e^\mu(P, \lambda) = \gamma^\mu \cdot \mathcal{E}(P, \lambda) \). The spin-1 spinors in \((1, 0) \oplus (0, 1)\) can be derived from Eq. 4.43-Eq. 4.45, which is defined as \( u = \gamma \wedge \mathbb{P}/m \) by applying the same interpolating transformation. Each component of the spinor in chiral representation is related with Grassmann basis number since they give components in terms of spin states (\( \theta_2 \)) and helicity (\( \theta_1 \)). \( \theta_1 \) and \( \theta_2 \) are presented in Eq. 4.49-Eq. 4.53 in terms of Clifford numbers.

### 4.6 Electrodynamics with Clifford Algebra

Along with the Lorentz transformation and spinor, Clifford algebra can also help in the calculation of electrodynamics especially finding interpolating quantities like the stress field strength tensor, the energy-momentum tensor, and so on.

As in \cite{34} we can define the four derivative and electromagnetic vector potential as

\[ \nabla = \gamma^\mu \partial_\mu. \] (4.83)
where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $A$ is

$$A = \gamma^\mu A_\mu,$$

(4.84)

where $A^\mu = (\phi, A_x, A_y, A_z)$.

The field tensor becomes

$$F = \nabla \wedge A = \gamma_{\mu\nu} \partial^\mu A^\nu,$$

(4.85)

and we can see that $F^{\mu\nu}$ can be derived from this expression as $F^{\mu\nu} = F_\nu \gamma^{\mu\nu} = \gamma_{\alpha\beta} \gamma^{\mu\nu} A^\beta = \partial^\mu A^\nu - \partial^\nu A^\mu$ so we could write $F$ as

$$F = \gamma_{\mu\nu} F^{\mu\nu}.$$  

(4.86)

Basically, $F$ is frame independent, and we can use the same $F$ for interpolating form as well. Similarly, we can do the same thing for $\hat{A}$. We find $F^{\tilde{\mu}\tilde{\nu}}$ by using interpolating Clifford numbers Eq. 4.15 as

$$F^{\tilde{\mu}\tilde{\nu}} = \gamma^{\tilde{\mu}\tilde{\nu}} \cdot F.$$  

(4.87)

### 4.7 Summary

In this chapter, we showed that we can use Clifford algebra for interpolating form for both vector and spinor algebras. The same transformation operator can be used for vectors and Lorentz transformation and also spinors of spin 1/2 and spin 1 representation.

We defined spinors in terms of the Clifford spinor group in which Clifford algebra has similar a structure to the algebra of spin group generators. By using the matrix isomorphism of $Cl_{1,3}$ in Section 4.2.7, we are able to derive spinors in chapter 2 and Appendix A.3, A.4, and A.5.

Using these relation we showed Clifford spinor expressions for spin-1/2 and spin-1 spinor. Among the two spin-1 spinor representations, we were able to derive a connection between two Lorentz group spinors $(1,0) \oplus (0,1)$ and $(1/2,1/2)$ by a $6 \times 4$ matrix given by Eq. 4.64. This matrix enables us to write down the tensor relation of polarization vectors for symmetric and anti-symmetric cases by seperating spinors into two parts in $(J,0) \oplus (0,J)$ by the $(1 \pm \gamma^5)/2$ operator as shown in Appendix C.2. This process is also applicable to the interpolating spinors. Thus, we were able to complete the interpolating polarization tensor, where the sum is given in [14] by defining the interpolating polarization tensor without the sum. Therefore, we have both symmetric and anti-symmetric parts for the interpolating polarization vector as well, which can be useful for the calculation of a polarized photon, as we will see in the next chapter. The polarization tensor is also used for a polarized photon beam in the Bethe-Heitler process (lepton-antilepton) in Appendix D.3.
Chapter 5

Deeply Virtual Compton Scattering and Bethe Heitler Process Update

5.1 Introduction

Compton scattering is a significant process in particle physics. It has a special role for the beginning of QED in the sense that it proves that photons are quantized [44] and provides a unique tool for hadron physics studies.

Finding the general structure of virtual Compton scattering enables us to calculate the Compton scattering amplitudes of electrons and nucleon targets, especially for experiments in Jefferson lab. We are investigating the structure with different spin targets like spin-0 (such as Helium), spin-1/2, and spin-1.

Deeply virtual Compton scattering (DVCS) is one of the interactions between accelerated electrons and target nucleons in Jefferson lab (JLab) and other labs as well. It is key to understanding the interior structure of the nucleons through measuring the form factors by conducting experiments and comparing them with theory. With the 12GeV upgrade, we are expecting more precise data to compare with the hadronic structure tensor.

Perrottet [45] and Tarrach [46] have developed the most general tensor structure for low energy Compton scattering issues. X. Ji [47] constructed a hadronic tensor structure for high energy, but his assumptions of collinearity of the experimental setup of real photons and hadron detectors is not entirely accurate. There has to be an angle between two detectors in order to make measurements accurate.

Compton scattering plays an important role in extracting generalized parton distributions (GPDs) by Fourier transformation of quark nonlocal composite operators $Q_i$.

In this chapter, we attempt to derive a bare-bone structure of the hadronic tensor. We study the amplitudes of DVCS and Bethe-Heitler process (B HH). We investigate the hadronic tensor
structure as well. We compare the hadronic tensor structures in [23] and Belitsky [48].

Our main purpose is to obtain the most general structure of the hadronic tensor and find Compton from factors (CFFs) to analyze GPDs and the inner structure of hadrons.

Additionally, we added one loop calculation in terms of Passarino-Veltman functions [49] for scalar loops (biquark).

5.2 Generalized Hadronic Tensor Structure

To better fully understand the tensor structure, we first focus on a spin-0 hadron target example explicitly since it is the simplest case. We focus on the spin-1/2 target later, because this case is more difficult.

The hadronic tensor structure is Lorentz invariant, i.e. by multiplication of photon momentum $q_\nu$ or $q'_\mu$, the hadronic tensor vanishes as

$$q'_\mu T^{\mu\nu} = T^{\mu\nu} q_\nu = 0. \quad (5.1)$$

We have three independent momenta: $P = p + p', q, q'$ which expresses all four momenta.

For the first step, we find the tensor for sQED tree-level amplitude calculations shown in Figure 5.2.

The amplitude in terms of the tensor can be contracted with the photon-polarization vectors as

$$A(h, h') = (-ie)^2 \epsilon^*_\mu(q', h') T^{\mu\nu} \epsilon_\nu(q, h), \quad (5.2)$$

then the tensor for the tree-level diagram is found

$$T^{\mu\nu} = -2g^{\mu\nu} + \frac{(P + q)^\mu(P + q')^\nu}{s - M^2} + \frac{(P - q)^\mu(P - q')^\nu}{u - M^2}, \quad (5.3)$$
where $u$ and $s$ are Mandelstam variables and they hold the following relations

$$s - M^2 = q'(P + q) = (P + q').q,$$

$$u - M^2 = -q'(P - q) = -(P - q').q.$$ 

(5.4)

(5.5)

Multiplication by $q_{\nu}$ or $q'_{\mu}$ confirms the Eq. 5.1 condition.

After the tree-level case, we may generalize the tensor in such a way to hold the gauge invariance relation of Eq. 5.1 as

$$\tilde{T}^{\mu\nu} = t_0 g^{\mu\nu} + \sum_{i,j} t_{ij} k_i^{\mu} k_j^{\nu},$$

(5.6)

where $i, j = 1, 2, 3$ and $k_i$ are the three momenta of the system $P, q,$ and $q'.$ In order to make the tensor transverse, we use $\tilde{g}^{\mu\nu}$ as a projector to construct a kinematical singularities free tensor, given by

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{q^{\mu} q'^{\nu}}{q.q'},$$

(5.7)

where $\tilde{g}$ satisfies transverse relation

$$q_{\mu} \tilde{g}^{\mu\nu} = \tilde{g}^{\mu\nu} q_{\nu} = 0.$$ 

(5.8)

The tensor becomes

$$T^{\mu\nu} = \tilde{g}^{\mu\alpha} \tilde{T}_{\alpha\beta} \tilde{g}^{\beta\nu}.$$ 

(5.9)

We have

$$\tilde{P}_{L}^{\mu} \tilde{P}_{R}^{\mu}, \quad \tilde{P}_{L}^{\mu} \tilde{q}_{R}^{\nu}, \quad \tilde{q}'_{L} \tilde{P}_{R}^{\mu}, \quad \tilde{q}'_{L} \tilde{q}_{R}^{\nu}. $$

(5.10)
because of projector Eq. 5.7 where they are given by

\[ \hat{\kappa}_L = k_\alpha \tilde{g}^{\mu\alpha}, \quad \hat{\kappa}_R = k_3 \tilde{g}^{\beta\nu}, \quad (5.11) \]

in general.

Then it is possible to write down the most general scalar Compton scattering tensor as

\[ T^{\mu\nu} = \mathcal{H}_0 \tilde{g}^{\mu\nu} + \mathcal{H}_1 \tilde{P}_L \tilde{P}_R + \mathcal{H}_2 \tilde{P}_L \tilde{q}_R + \mathcal{H}_3 \tilde{q}_L \tilde{P}_R + \mathcal{H}_4 \tilde{q}_L \tilde{q}_R, \quad (5.12) \]

where \( \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \) and \( \mathcal{H}_4 \) are the five independent CFFs in VCS for a scalar hadron. In the tree-level tensor we have two form factors that can be written as

\[ \mathcal{H}_0 = -2, \quad \mathcal{H}_1 = \frac{1}{s - M^2} + \frac{1}{u - M^2}. \quad (5.13) \]

### 5.2.1 Comparison of two VCS Tensors and Form Factors

Since we compare the DVCS and BH process with [48], we compare the hadronic tensor of (5.12) to the hadronic tensor in [48], which is defined as

\[ T^{\mu\nu}(q, P, \Delta) = -\mathcal{P}^{\mu\nu} g_{\sigma\tau} \mathcal{P}^{\tau\nu} F_1 + \frac{\mathcal{P}^{\mu\sigma} P_\sigma P_\tau \mathcal{P}^{\tau\nu}}{2P.q_t} F_2 \]

\[ + \frac{\mathcal{P}^{\mu\sigma} (P_\sigma \Delta^\perp + \Delta^\parallel P_\tau) \mathcal{P}^{\tau\nu}}{2P.q_t} F_3 \]

\[ + \frac{\mathcal{P}^{\mu\sigma} (P_\sigma \Delta^\perp - \Delta^\parallel P_\tau) \mathcal{P}^{\tau\nu}}{2P.q_t} F_4 \]

\[ + \frac{\mathcal{P}^{\mu\sigma} \Delta^\perp \Delta^\parallel \mathcal{P}^{\tau\nu}}{M^2} F_5, \quad (5.14) \]

where \( \Delta^\perp = \Delta - \eta P_\mu \) and \( \eta = \Delta.q/P.q_t \) with \( q_t = (q + q')/2 \). \( \mathcal{P} \) and \( \tilde{g} \) are the same projection operators and we extract the terms as

\[ \tilde{g}^{\mu\sigma} g_{\sigma\tau} \tilde{g}^{\tau\nu} = g^{\mu\nu} - \frac{q^{\mu} q^{\nu}}{q.q'}, \quad (5.15) \]

\[ \tilde{g}^{\mu\sigma} P_\sigma P_\tau \tilde{g}^{\tau\nu} = \left( P^{\mu} - \frac{q^{\mu} P.q'}{q.q'} \right) \left( P^{\nu} - \frac{q^{\nu} P.q}{q.q'} \right), \quad (5.16) \]

\[ \tilde{g}^{\mu\sigma} P_\sigma \Delta^\parallel \tilde{g}^{\tau\nu} = \left( P^{\mu} - \frac{q^{\mu} P.q'}{q.q'} \right) \left( \Delta^{\nu} - \frac{q^{\nu} \Delta.q}{q.q'} \right) \quad (5.17) \]
We can relate their hadronic tensor with the tensor given in Eq. 5.12, and the relation between the CFFs of both tensors becomes

\[
\mathcal{H}_0 = -F_1, \quad \text{(5.18)}
\]

\[
\mathcal{H}_1 = \frac{\eta_2 F_5}{M^2} + \frac{F_2}{2P \cdot q_t} - \frac{\eta F_3}{P \cdot q_t}, \quad \text{(5.19)}
\]

\[
\mathcal{H}_2 = -\frac{F_5}{M^2} + \frac{F_3 + F_4}{2P \cdot q_t}, \quad \text{(5.20)}
\]

\[
\mathcal{H}_3 = \frac{F_5}{M^2} + \frac{F_3 + F_4}{2P \cdot q_t}, \quad \text{(5.21)}
\]

\[
\mathcal{H}_4 = -\frac{F_5}{M^2}. \quad \text{(5.22)}
\]

### 5.3 Cross Sections for Virtual Compton Scattering with Angular Dependence

In order to attempt to understand other structures of the hadronic tensor and also constructions of cross sections, we investigate virtual Compton scattering interactions. We have two parts: VCS and BH process shown in Figure. 5.3 as

\[
\mathcal{T} = \mathcal{T}_{VCS} + \mathcal{T}_{BH}, \quad \text{(5.23)}
\]

and the amplitude square is

\[
|\mathcal{T}|^2 = |\mathcal{T}_{VCS}|^2 + |\mathcal{T}_{BH}|^2 + \mathcal{T}_{VCS} \mathcal{T}_{BH}^* + \mathcal{T}_{BH} \mathcal{T}_{VCS}^*. \quad \text{(5.24)}
\]

Here, \(\mathcal{T}\) can be separated into two parts: hadronic part and leptonic part as

\[
\mathcal{T} = l^{\mu\nu} J_{\mu\nu}. \quad \text{(5.25)}
\]

The leptonic part is easy to calculate since it is just a point particle structure form with Feynman rules. However, since hadrons consist of smaller particles (quarks and gluons) we can not write them down as point particles because we don’t know the inner structure of the hadrons exactly. Instead, we can write a tensor structure in terms of form factors with incoming and outgoing momenta.

The lepton part of DVCS in Figure 5.3 is

\[
l_{DVCS}^{\mu\nu} = e^\nu(q') \sum_\lambda \bar{u}(k', \lambda') \gamma^\mu u(k, \lambda), \quad \text{(5.26)}
\]
Figure 5.3: Compton scattering between electron and Hadron have two parts: DVCS and Bethe-Heitler process

and the lepton part of BH process is

\begin{align}
\eta_{BH}^{\mu\nu} = \bar{u}(k') \left( e\gamma^\mu \frac{k' + \Delta}{(k' - \Delta^2)^2 - m^2} i e e^\nu(q') + i e e^\nu \frac{k - \Delta}{(k - \Delta)^2 - m^2} i e \gamma^\mu \right) u(k)
\end{align}

The hadronic tensor in [48] is given by

\begin{align}
T_{\mu\nu}(q, P, \Delta) = i \int dx e^{ix\cdot q} \langle p' | T j_\mu (x/2) j_\nu (-x/2) | p \rangle.
\end{align}

5.4 Hadronic Tensor with CFFs

The hadronic tensor in Eq. 5.47 changes when the interaction is defined as a handbag diagram in Figure 5.1, becoming

\begin{align}
T_{\mu\nu}(q, P, \Delta) = -\tilde{g}^{\mu\sigma} g_{\sigma\tau} \tilde{g}^{\tau\nu} \left(T_1 + \frac{(1 - \xi^2)(\Delta^2 - \Delta_{min}^2)}{4\xi M^2} \right) + \frac{\tilde{g}^{\mu\sigma} P_\sigma P_\tau \tilde{g}^{\tau\nu}}{2P_x q_t} \left(T_2 + \frac{\Delta^2}{4M^2} T_3 \right) + \frac{\tilde{g}^{\mu\sigma} P_\sigma \Delta_{\mu}^{\perp} \tilde{g}^{\tau\nu}}{2M^2} T_3,
\end{align}

where \( \tilde{g}^{\mu\nu} = g^{\mu\nu} - q^{\mu} q^{\nu}/q \cdot q', \Delta_{\mu}^{\perp} = \Delta_{\mu} - \eta P_\mu, \) and \( q_t = (q + q')/2. \)

Here, we like to note that we found an inconsistency between the hadronic tensor and DVCS and interference term amplitudes. We found out that simply by changing the coefficient of \( P^{\mu\sigma} P_\sigma P_\tau P^{\tau\nu} \) term as

\begin{align}
T_2 + \frac{\Delta^2}{2M^2} T_3,
\end{align}
instead of
\[ T_2 + \frac{\Delta^2}{4M^2} T_3 \] (5.31)
We get the right amplitudes. BH process isn’t affected by this because it is independent of the hadronic tensor.

5.5 Kinematics of VCS

The momentum conservation is given by \( k + p = k' + p' + q' \) and \( \Delta = p' - p - q - q' \). Here, \( q \) is the virtual photon in VCS (virtual Compton scattering) and \( \Delta \) is the virtual photon in the BH (Bethe-Heitler) process. In our calculations, we use the total momentum of hadrons \( P = p + p' \), and \( q_t = -(q + q')/2 \) the total momentum of photons. The definitions of some Lorentz invariant kinematic quantities are
\[
q^2 = -Q^2 \text{ virtuality of photon, } q'^2 = 0 \text{ real photon.}
\]
\[
x_B = \frac{q^2}{2p_{\gamma q}} = \frac{Q^2}{2p_{\gamma q}} \text{ the fraction of the nucleon’s momentum by the parton,}
\]
\[
y = \frac{p_{\gamma q}}{p_{\gamma k}} \text{ the fraction of the electron by the virtual photon.}
\]
\[
\xi = \frac{q^2}{P_{\gamma q}} \text{ and } \eta = \frac{\Delta q}{P_{\gamma q}}.
\]
Moreover, if the virtual photon is moving in the z-direction only with momentum \( q \) and the initial hadron is at rest \( p = (M,0,0,0) \), and the recoiled hadron’s momentum is given by \( p' = (E, |\vec{P}_2| \cos \theta_H, |\vec{P}_2| \sin \theta_H \sin \phi, |\vec{P}_2| \cos \theta_H) \), where \( \phi \) is the azimuthal angle between the lepton and hadron scattering planes, \( \theta_e \) and \( \theta_H \) are electron and hadron scattering angles between incoming and outgoing particles. The lepton momentum is \( k \).

Then the momenta in terms of angles are given by
\[
k = (E, E \sin \theta_e, 0, E \cos \theta_e) \quad (5.32)
\]
\[
q = (q^0, 0, 0, -|q^3|), \quad (5.33)
\]
\[
P = (M + E_2, |\vec{P}_2| \sin \theta_H \cos \phi, |\vec{P}_2| \sin \theta_H \sin \phi, |\vec{P}_2| \cos \theta_H), \quad (5.34)
\]
\[
q' = (q^0 + M - E_2, -|\vec{P}_2| \sin \theta_H \cos \phi, -|\vec{P}_2| \sin \theta_H \sin \phi,
- |\vec{P}_2| \cos \theta_H - |q^3|). \quad (5.35)
\]

5.6 Bethe-Heitler Process

First, we begin with a calculation of the Bethe-Heitler process for different conditions and hadronic tensors.

We take the calculation of a spin-zero target such as a Pion where the hadronic current is
\[
J_\mu = FP_\mu, \quad (5.36)
\]
which depends on momentum.

Then the amplitude becomes

\[ T_{BH} = \frac{e^3}{\Delta^2} F(\Delta^2)P_\mu \epsilon^\nu(q') \bar{u}(k', \lambda') [\gamma^\nu \frac{k - \Delta}{(k - \Delta)^2} \gamma^\mu + \gamma^\mu \frac{k' + \Delta}{(k' - q')^2} \gamma^\nu] u(k, \lambda). \] (5.37)

We separate the amplitude square into the leptonic and hadronic tensors from the current of Eq. 5.36 and the lepton tensor of Eq. 5.27.

We introduce

\[ \Delta^2 P_1 = \Delta^2 + 2k \Delta = 2k'q' \] and \[ \Delta^2 P_2 = \Delta^2 - 2k' \Delta = -2kq' \] then with the help of the expression \[ k \cdot k' = m^2 + \Delta^2/2 + k' \Delta - k \Delta, \] the leptonic tensor is

\[
L^{\mu \nu} = \frac{16}{P_1 P_2} \left\{ A g^{\mu \nu} + B \frac{k^{\mu} k^{\nu}}{\Delta^2} + C \frac{k'^{\mu} k'^{\nu}}{\Delta^2} + D \frac{k'^{\mu} \Delta^{\nu} + \Delta^{\mu} k'^{\nu}}{\Delta^2} + E \frac{k^{\mu} \Delta^{\nu} + \Delta^{\mu} k^{\nu}}{\Delta^2} \\
+ F \frac{k^{\mu} k^{\nu} + k'^{\mu} k'^{\nu}}{\Delta^2} + G \frac{\Delta^{\mu} \Delta^{\nu}}{\Delta^2} \right\}, \tag{5.38}
\]

where

\[
A = k'^2 + k^2 - 2\mu^2 \frac{1 + k'^2 - k^2}{P_1 P_2}, \quad B = 1 - 2\mu^2 \frac{P_2}{P_1}, \quad C = 1 - 2\mu^2 \frac{P_1}{P_2}, \quad D = -k' \Delta + 2\mu^2 \frac{k'^2 + k^2}{P_2}, \quad E = -k \Delta + 2\mu^2 \frac{k'^2 + k^2}{P_1}, \quad F = -G = -2\mu^2,
\]

with \( \mu^2 = m^2/\Delta^2, k_\Delta = k \Delta/\Delta^2, \) and \( k'_\Delta = k' \Delta/\Delta^2. \) We compare our lepton tensor \( L^{\mu \nu}/2 \) with the lepton tensor in [50]. We find that there are some differences in B and C in our definition of the lepton tensor and the lepton tensor in [50]. We checked our results with the real Compton scattering amplitude which is already given in Peskin in Section 5.6.

By using corrected leptonic and corrected hadronic tensors, we successfully derived the BH process amplitude square term which is given in [48] as Eq. (10).

**Real Photon Compton Scattering with BH Tensor**

![Compton scattering of real photons](Figure 5.4)

Figure 5.4: Compton scattering of real photons

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We can test our lepton tensor of the BH process of Eq. 5.38 with the real Compton scattering amplitude by just changing the out-going virtual photon $\Delta$ into an in-coming real photon $q$ as $\Delta \rightarrow -q$ with $q^2 = 0$. Then the amplitude of the BH with Eq. 5.38 changes as

$$|M|^2 = \epsilon_{\mu}(q)\epsilon^*_{\nu}(q) L_{BH}^{\mu\nu} |_{\Delta \rightarrow -q} = -g_{\mu\nu} L^{\mu\nu}$$

$$= \frac{4(-6m^8 + m^4(3s^2 + 14su + 3u^2) - m^2(s + u)(s^2 + 6su + u^2) + su(s^2 + u^2))}{(m^2 - s)^2(m^2 - u)^2}, \quad (5.39)$$

where $u$, $t$, and $s$ are Mandelstam variables and defined as

$$s = (k + q)^2 = 2k.q + m^2 = 2k'.q' + m^2,$$

$$t = (k' - k)^2 = -2k.k' + 2m^2 = -2q.q',$$

$$u = (q' - k)^2 = -2k.q' + m^2. \quad (5.40)$$

When we rearrange the amplitude

$$|M|^2 = 4 \left( \frac{m^2 - u}{s - m^2} + \frac{s - m^2}{m^2 - u} + 4m^2 \left( \frac{1}{s - m^2} - \frac{1}{m^2 - u} \right) + 4m^4 \left( \frac{1}{s - m^2} - \frac{1}{m^2 - u} \right)^2 \right), \quad (5.41)$$

it agrees with the solution in Peskin and Schroder [25] on page 162 because they have $m^2 - u = 2k.q'$ and $s - m^2 = 2k'.q'$. Then, the amplitude is

$$|M|^2 = \left( \frac{k.q'}{k.q} + \frac{k.q}{k.q'} + m^2 \left( \frac{1}{k.q} - \frac{1}{k.q'} \right) + m^4 \left( \frac{1}{k.q} - \frac{1}{k.q'} \right)^2 \right), \quad (5.42)$$

where $p$ is the lepton’s momentum which is $k$ in our notation, and $k$ is the photon’s momentum, which is $q$ in our notation.

### 5.7 Deeply Virtual Compton Scattering (DVCS)

The VCS amplitude with scalar hadronic tensors from Eq. 5.14 is

$$T_{vcs} = \left( \frac{\mp e^3}{Q^2} \right) T^{\mu\nu}\epsilon^*_{\mu}(q')\bar{u}(k')\gamma_\nu u(k), \quad (5.43)$$
where the sign ± refers to $e^\pm$ beam charge. The amplitude square can be written as

$$|T_{vcs}|^2 = T^{\mu\nu}\epsilon^*_\mu \bar{u}(k')\gamma_\nu u(k)[T_{\alpha\beta}\epsilon^*_\alpha \bar{u}(k')\gamma_\beta u(k)]^*$$

$$= T^{\mu\nu}(T^{\alpha\beta})^* \epsilon^*_\mu \epsilon_\alpha \bar{u}(k')\gamma_\nu u(k)\gamma_\beta u(k')$$

$$= -T^{\mu\nu}(T^{\alpha\beta})^* g_{\mu\alpha} Tr[\gamma_\nu (1 + \lambda \gamma^5)(k - m)\gamma_\beta (k' - m)]/2 \quad (5.44)$$

$$= H^{\nu\beta} L_{\nu\beta}. \quad (5.45)$$

We can separate into the hadronic and leptonic tensor similarly to the BH case as

$$L_{\nu\beta} = 2\{-g_{\nu\beta} k.k' + k_\nu k'_\beta + k'_\nu k_\beta + i\lambda \epsilon_{\nu\beta\delta\sigma} k^\delta k'^\sigma\}. \quad (5.46)$$

The hadronic term with the hadronic tensor is

$$H^{\nu\beta} = T^{\mu\nu}(T^{\alpha\beta})^* \epsilon^*_\mu (q')\epsilon^*_\alpha (q'), \quad (5.47)$$

where the polarization tensor from the previous chapter is given by Eq. 4.80 as

$$\epsilon_\mu (q', \lambda')\epsilon^*_\alpha (q', \lambda') = -g_{\mu\alpha}/2 + (q'\mu n_\alpha + n_\mu q'_\alpha)/2(q'.n) - q'\mu q'^\alpha/2(q'.n)^2 - i\lambda \epsilon_{\mu\alpha\delta\sigma} q^\delta n^\sigma/2(q'.n). \quad (5.48)$$

We notice that in this relation the second and third terms vanish because of Lorentz condition with photon momentum. We use the relation $P.q = P.q$ and $P.q_t = P.q$ since $P.\Delta = 0$.

We get rid of the momentum $k'$ terms by using momentum conservation $k' = k + p - p' - q' = k - \Delta - q' = k - q$.

Next, we expand in terms of $Q^2$ since we are taking kinematic quantities into the deeply virtual Compton scattering case where the energy of the virtual photon is very high $(Q^2 >> 1)$.

Then all the kinematic quantities for DVCS become

$$P.q = Q^2 x_B - Q^2/2 + \Delta^2/2, \quad P.q' = P.q, \quad P.k = Q^2 x_B y + k.\Delta, \quad (5.49)$$

$$q.q' = -Q^2/2 - \Delta^2/2, \quad k.q = -Q^2/2, \quad k.q' = -Q^2/2 - k.\Delta. \quad (5.50)$$

Now the amplitude becomes

$$|T_{DVCS}|^2 = e^2 Q^4 H^{\nu\beta} L_{\nu\beta} = 2e^6/ Q^4 T^{\mu\nu}(T^{\alpha\beta})^*(-g_{\nu\beta} k.k' + k_\nu k'_\beta + k'_\nu k_\beta)(g_{\mu\alpha} + i\lambda \epsilon_{\mu\alpha\delta\sigma} q^\delta n^\sigma/ q'.n), \quad (5.51)$$

where the hadronic tensor is a handbag dominant hadronic tensor given by (5.29).

The amplitude calculations have the same amplitude square as in Eq. (15) in [48] by using the hadronic tensor and leptonic tensor multiplication in Eq. 5.46 and Eq. 5.47.
5.8 DVCS-Bethe-Heitler Interference Term

Beside the square terms of the DVCS and BH process, we have the interference of the DVCS and BH process with the Pion interaction example by using Eq. 5.37, so the interference terms can be written as

\[
T_{BH} T^*_{DVCS} = \frac{e^6 F(\Delta^2)}{Q^2 \Delta^2} P_\mu \epsilon_\nu^* (q') \overline{u}(k', \lambda') [\gamma^\nu \frac{k - \Delta}{(k - \Delta)^2} \gamma^\mu + \gamma^\mu \frac{k' + \Delta}{(k - q')^2} \gamma^\nu] u(k, \lambda)
\]

\[\times (T^{\alpha \beta})^* \epsilon_\alpha (q') \overline{u}(k') \gamma^\beta u(k)] g_{\beta \beta'}, \tag{5.52}
\]

\[
T^*_{BH} T_{DVCS} = \frac{e^6}{Q^2 \Delta^2} T^{\alpha \beta} \epsilon_\alpha (q') [\overline{u}(k', \lambda') \gamma^\beta u(k, \lambda)] P_\mu F(\Delta^2)^* \epsilon_\nu(q')
\]

\[\times [\overline{u}(k', \lambda') [\gamma^\nu \frac{k - \Delta}{(k - \Delta)^2} \gamma^\mu + \gamma^\mu \frac{k' + \Delta}{(k - q')^2} \gamma^\nu] u(k, \lambda)]^*. \tag{5.53}
\]

We can separate this into the leptonic part and hadronic part as usual:

\[
T_{BH} T^*_{DVCS} = \frac{e^6}{Q^2 \Delta^2} H_{\mu \nu} L^\nu = \frac{e^6 F(\Delta^2)}{Q^2 \Delta^2} P_\mu (T^{\alpha \beta})^* \epsilon_\alpha (q') \epsilon^*_\nu(q')
\]

\[\times \left( \frac{Tr[\gamma^\nu (k - \Delta) \gamma^\mu \gamma^\beta \gamma^\nu]}{Q^2 P_2} + \frac{Tr[\gamma^\nu (k' + \Delta) \gamma^\mu \gamma^\beta \gamma^\nu]}{Q^2 P_1} \right), \tag{5.54}
\]

\[
T^*_{BH} T_{DVCS} = \frac{e^6}{Q^2 \Delta^2} H^\mu_{\nu \mu} L^\mu = \frac{e^6 F(\Delta^2)}{Q^2 \Delta^2} P_\mu (T^{\alpha \beta}) \epsilon_\alpha (q') \epsilon_\nu(q')
\]

\[\times \left( \frac{Tr[\gamma^\beta \gamma^\nu (k - \Delta) \gamma^\mu]}{Q^2 P_2} + \frac{Tr[\gamma^\beta \gamma^\gamma \gamma^\mu (k' + \Delta) \gamma^\nu]}{Q^2 P_1} \right), \tag{5.55}
\]

where the hadronic part is

\[
H^{\mu \nu, \beta} = F(\Delta^2) P_\mu (T^{\alpha \beta})^* \epsilon_\alpha (q') \epsilon^*_\nu(q') g^{\mu \nu'}, \tag{5.56}
\]

with \(T^{\mu \nu}\) from Eq. 5.29.

We can separate this into a symmetric and anti-symmetric part as

\[
H_S = -F(\Delta^2) P_\mu (T^{\nu \beta})^*/2, \tag{5.57}
\]

\[
H_A = -i \lambda' F(\Delta^2) P_\mu (T^{\nu \beta})^* \epsilon_{\alpha \nu p n} g^{\mu \nu'}/2 p n. \tag{5.58}
\]

The anti symmetry part of BH is

\[
\epsilon^{k P q q'} = -\epsilon^{k k' P \Delta} = \epsilon^{k k' P q} = -\epsilon^{k P q \Delta} = -\epsilon^{k' P q \Delta} = -\epsilon^{k' P q' \Delta}, \tag{5.59}
\]

\[
\epsilon^{k P q \Delta} = \epsilon^{k' P q \Delta} = \epsilon^{k q' \Delta} = \epsilon^{k q' q} = \epsilon^{k k' q} = \epsilon^{k q q'} = \epsilon^{P q q' \Delta} = \epsilon^{k k' P q} = 0,
\]

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where $\epsilon^{abcd} = \epsilon^{\mu
u\alpha\beta} a_{\mu} b_{\nu} c_{\alpha} d_{\beta}$ and only finding $\epsilon^{kPqq'}$ gives all other values so $\epsilon^{kPqq'}$ is given by

$$\epsilon^{kPqq'} = 2M |q^3| E \sin \theta_e |\vec{P}_2| \sin \theta_H \sin \phi = -\frac{Q^2 \sqrt{1 + \epsilon^2}}{x_B} \frac{Q^2 K}{y(1 + \epsilon^2)} \sin \phi = -\frac{Q^4}{x_B y \sqrt{1 + \epsilon^2}} K \sin \phi,$$

(5.60)

where $\epsilon = 2x_B M/Q$ and used quantities are defined in Appendix D.1.

Through this calculation, we get the same $|T_j|$ as in [48] by using the leptonic and hadronic structure of the interference of the DVCS and BH process.

### 5.9 Loop Calculations

#### 5.9.1 Feynman Rules

In order to get one loop correction of the hadronic tensor, we begin with the simple loop, scalar loop contribution, with biquark loop inside the hadron.

Recall the Feynman rules for scalar fields:

1. External lines with value 1
2. Internal lines with value $\frac{-i}{k^2 - m^2 + i\epsilon}$
3. Vertices joining 3 lines with value $i g$

Feynman rules for scalar Quantum Electrodynamics (sQED):

1. External lines with value 1
2. Internal lines of photon with value $\frac{-i\gamma^\mu}{k^2 + m^2}$
3. Vertices 2 scalar 1 photon of 3 lines with value $-2ie(2p - k)^\mu$

Short expressions:

$$\bar{P} = p + q, \quad R = p + q'.$$

(5.61)

The propagators are

$$d_0 = k^2 - M^2, \quad d_1 = (p - k)^2 - m^2, \quad d_2 = (p' - k)^2 - m^2, \quad d_3 = (\bar{P} - k)^2 - m^2$$
$$d_4 = \bar{P}^2 - m^2, \quad d_5 = (R - k)^2 - m^2, \quad d_6 = R^2 - m^2,$$

where $\mu$ is the mass of the remaining quarks, $M$ is the mass of the exchange biquark, and $m$ is the hadron mass.
5.9.2 Passarino-Veltman Integrals

The Passarino-Veltman integrals are given as

The generic integral

$$T_{n}^{\mu_1 \ldots \mu_p} = \frac{(2\pi \mu)^{4-d}}{i\pi^2} \int d^d k \frac{k^{\mu_1} \ldots k^{\mu_p}}{d_0 d_1 \ldots d_{n-1}},$$

(5.62)

with momenta of Figure 5.5

$$d_i = (k + r_i)^2 - m_i^2 + i\epsilon,$$

(5.63)

where $r_i$ is related with external momenta and is written as

$$r_i = \sum_{j=1}^{n} p_i, \quad j = 1, \ldots, n - 1,$$

$$r_0 = \sum_{i=1}^{n} p_i = 0.$$  

(5.64)
From Eq. 5.62, we can write down four scalar integrals, note that $4 - d = \epsilon$,

$$A_0(m_0^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^d k \frac{1}{k^2 - m_0^2}, \quad (5.65)$$

$$B_0(r_{10}^2, m_0^2, m_1^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^d k \prod_{i=0}^1 \frac{1}{[(k + r_i)^2 - m_i^2]}, \quad (5.66)$$

$$C_0(r_{10}^2, r_{12}^2, m_0^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^d k \prod_{i=0}^2 \frac{1}{[(k + r_i)^2 - m_i^2]}, \quad (5.67)$$

$$D_0(r_{10}^2, r_{12}^2, r_{23}^2, r_{30}^2, r_{13}^2, m_0^2, ..., m_3^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^d k \prod_{i=0}^3 \frac{1}{[(k + r_i)^2 - m_i^2]}, \quad (5.68)$$

where

$$r_{ij}^2 = (r_i - r_j)^2; \quad \forall i, j = (0, n - 1). \quad (5.69)$$

Keep in mind that the convention $r_0 = 0$ and $i\epsilon$ part in the denominator are suppressed for simplicity.

Here, we give the expansion of the $B_0$ and $A_0$ functions with dimensional regularization as an example.

$$\sum_{\text{loop}} (p) = A(p^2) + B(p^2)p, \quad (5.70)$$

$$B_0(p^2, m^2, M^2) = -e^2 \mu^\epsilon (2 - \epsilon) \frac{1}{16\pi^2} \int_0^1 dx (1 - x) \left\{ \Delta_\epsilon - \ln \left\{ -p^2 x(1 - x) + m^2 x + M^2 (1 - x) \right\} \right\}. \quad (5.71)$$

$$\mu^\epsilon (2 - \epsilon) = 2 \left\{ 1 + \epsilon (\ln \mu - \frac{1}{2}) + O(\epsilon^2) \right\} \quad (5.72)$$

$$\mu^\epsilon (2 - \epsilon) \Delta_\epsilon = 2 \left\{ \Delta_\epsilon + 2 (\ln \mu - \frac{1}{2}) + O(\epsilon) \right\} \quad (5.73)$$

$$B_0(p^2, m^2, M^2) = -\frac{2e^2}{16\pi^2} \int_0^1 dx (1 - x) \left\{ \Delta_\epsilon - 1 - \ln \left\{ -p^2 x(1 - x) + m^2 x + M^2 (1 - x) \right\} \right\}. \quad (5.74)$$

$$A_0(p^2, m^2, M^2) = e^2 \mu^\epsilon (4 - \epsilon) m \frac{1}{16\pi^2} \int_0^1 dx \left\{ \Delta_\epsilon - \ln \left\{ -p^2 x(1 - x) + m^2 x + M^2 (1 - x) \right\} \right\}. \quad (5.75)$$

$$\mu^\epsilon (4 - \epsilon) = 4 \left\{ 1 + \epsilon (\ln \mu - \frac{1}{4}) + O(\epsilon^2) \right\}, \quad (5.76)$$

$$\mu^\epsilon (4 - \epsilon) \Delta_\epsilon = 4 \left\{ \Delta_\epsilon + 2 (\ln \mu - \frac{1}{4}) + O(\epsilon^2) \right\}. \quad (5.77)$$
\[ A_0(p^2, m^2, M^2) = \frac{4e^2 m}{16\pi^2} \int_0^1 dx \left\{ \Delta_k - \frac{1}{2} - \ln \left\{ \frac{-p^2 x(1-x) + m^2 x + M^2}{\mu^2} \right\} \right\}. \quad (5.78) \]

### 5.9.3 Scalar Field with Scalar Loops

We begin the loop calculation of the DVCS at tree level with the seagull term. Here, we show how an integral expressed as Feynman parameters is related with Passarino-Veltman integrals.

![Figure 5.6: \( T_{sg}^{\mu\nu} \) diagram.](image)

\[ i\Gamma_{sg}^{\mu\nu} = \int \frac{d^4 k}{(2\pi)^4} \frac{i}{d_0 + i0}(-ig) \frac{i}{d_1 + i0} (2ie^2 g^{\mu\nu}) \frac{i}{d_2 + i0}(-ig) = e^2 g^2 \int \frac{d^4 k}{(2\pi)^4} \frac{-2g^{\mu\nu}}{d_0 d_1 d_2} \]
\[ = e^2 g^2 \int dxdydz \delta(x + y + z - 1) \int \frac{d^4 k}{(2\pi)^4} \frac{-2g^{\mu\nu}}{(xd_1 + yd_2 + zd_0)^4} \]
\[ = e^2 g^2 \int dxdydz \int \frac{d^4 k'}{(2\pi)^4} \frac{-2g^{\mu\nu}}{k^2 - M_{cov}^2}, \quad (5.79) \]

where \( k' = k - xp - yp' \) and \( M_{cov}^2 = (xp + yp')^2 - x(p^2 - m^2) - y(p'^2 - m^2) + (1 - x - y)M^2 \).

We can also carry the integral to Euclidean space (Wick rotation) as

\[ i\Gamma_{sg}^{\mu\nu} = e^2 g^2 \int dxdydz 2\pi^2 \int_0^\infty \frac{dk_E k_E^3}{(2\pi)^4} \frac{2g^{\mu\nu}}{(k_E^2 + M_{cov}^2)^3} = \frac{e^2 g^2 g^{\mu\nu}}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{1}{M_{cov}^2}, \quad (5.80) \]

or we can write it also as

\[ i\Gamma_{sg}^{\mu\nu} = -\frac{e^2 g^2 g^{\mu\nu}}{2\pi^2} C_0(m^2, m^2, s, m^2, M^2, m^2). \quad (5.81) \]
\[ i \Gamma_{Wst}^{\mu\nu} = \frac{(P + q)^\mu(P + q')^\nu}{(s - m^2)^2} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{d_0 d_1} \int d x \int d^4 k \frac{1}{(2\pi)^4 (xd_0 + (1-x)d_3)^2} \]

\[ = \frac{ie^2 g^2 (P + q)^\mu(P + q')^\nu}{4\pi^2 (s - m^2)^2} B_0(s, M^2, m^2). \quad (5.82) \]
\[ i \Gamma_{Ust}^{\mu \nu} = \frac{(P + q)^\mu}{s - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{(P + q - 2k)^\nu}{d_0 d_1 d_3} = \frac{(P + q)^\mu}{s - m^2} \int dx \, dy \]
\[ \times \int \frac{d^4k}{(2\pi)^4} \frac{(P + q - 2k)^\nu}{(xd_0 + yd_1 + (1 - x - y)d_3)^3} = -\frac{e^2 g^2 (P + q)^\mu (P + q')^\nu}{8\pi^2 (s - m^2) ((p \cdot q)^2 - m^2 q^2)} \]
\[ \times \left\{ \left[ (2m^2 - M^2 + p \cdot q)q^\nu (2p \cdot q + q^2) - q^2 (P^\nu + q''') \right] C_0 (m^2, q^2, s, M^2, m^2, m^2) 
\[ + B_0 (m^2, m^2, M^2) \{ p \cdot q (P^\nu + q''') - q'' (2m^2 + p \cdot q) \} + B_0 (s, m^2, M^2) \{ q'' (2m^2 + 3p \cdot q + q^2) 
\[ - (p \cdot q + q^2) (P^\nu + q''') \} \right\} + B_0 (q^2, m^2, m^2) \{ q^2 (P^\nu + q''') - q'' (2p \cdot q + q^2) \} \right\}. \quad (5.83) \]

---

**Figure 5.9:** $V_{st}^{\mu \nu}$ diagram.

---

\[ i \Gamma_{Vst}^{\mu \nu} = \frac{(P + q')^\nu}{s - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{(P + q - 2k)^\mu}{d_0 d_2 d_3} = \frac{(P + q')^\nu}{s - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{(P + q - 2k)^\mu}{xd_0 + yd_2 + (1 - x - y)d_3} \]
\[ \times \frac{e^2 g^2 (P + q')^\nu (P + q)^\mu}{8\pi^2 (s - m^2) m^2} \]
\[ \times \left\{ (2m^2 - M^2) C_0 (m^2, 0, m^2, m^2, M^2) + B_0 (m^2, m^2, M^2) - B_0 (0, m^2, m^2) \right\}. \quad (5.84) \]
There are also $u$ channel loops like

$$i\Gamma^{\mu
u}_{Tst} = \int \frac{d^4k}{(2\pi)^4} \frac{i}{d_0} \{-2ie(2p + q - 2k)\} \frac{i}{d_3} \{-2ie(P + q - 2k)\} \frac{i}{d_5} (-ig)$$

$$= e^2g^2 \int \frac{d^4k}{(2\pi)^4} \frac{4(P + q' - 2k)\nu(P + q - 2k)\mu}{d_0d_1d_2d_3}$$

$$= e^2g^2 \int dx dy dz dt \delta(x + y + z + t - 1) \int \frac{d^4k}{(2\pi)^4} \frac{4(P + q' - 2k)\nu(P + q - 2k)\mu}{(xd_0 + yd_1 + zd_2 + td_3)^4}. \quad (5.85)$$

They have similar loops to their $s$-channel partners.
5.9.4 CFFs with one Loop Calculation

We summarize the all loop contributions of s-channel and u-channel and we drop $e$ and $g$ coefficients since we don’t require them. We can write the CFFs of (5.12) with the outgoing real photon case so that we have $H_0$, $H_1$, and $H_2$ from (5.12) where the real photon part $q''_L$ vanishes because of the transverse property. Even though the loop tensors are with the real photon case so that we have $H$ coefficients since we don’t require them. We can write the CFFs of (5.12) with the outgoing photon momentum $q'$, we take loop contributions of CFFs without $q'$ momentum. First the tensor is in $T^{\mu
u}$. If $\tilde{g}^{\mu\nu}$ is applied, $T^{\mu\nu}$ changes to $\tilde{T}^{\mu\nu}$ and find CFFs

Seagull Diagrams:

\[ T_{kq}^{\mu\nu} = -\frac{\tilde{g}^{\mu\nu}}{2\pi^2} C_0(m^2, m^2, s, m^2, M^2, m^2). \tag{5.87} \]

The seagull only contributes to $H_0$ since the numerator does not depend on the momenta:

\[ H_0^{s_m} = \frac{1}{4\pi^2} C_0(m^2, m^2, s, m^2, M^2, m^2). \tag{5.88} \]

2 point $s$ and $u$ channel loops:

\[ \begin{array}{c|c}
W_{st}^{\mu\nu} & -\frac{(P+q')^\mu(P+q')^\nu}{4\pi^2(s-m^2)^2} B_0(s, m^2, M^2) \\
W_{ut}^{\mu\nu} & -\frac{(P-q')^\mu(P-q')^\nu}{4\pi^2(u-m^2)^2} B_0(u, m^2, M^2) 
\end{array} \]

The self energy diagrams in s channel and u channel contribute to $H_1$ only:

\[ H_1^{W_{st}} = -\frac{1}{4\pi^2(s-m^2)^2} B_0(s, m^2, M^2), \tag{5.89} \]

\[ H_1^{W_{ut}} = -\frac{1}{4\pi^2(u-m^2)^2} B_0(u, m^2, M^2). \tag{5.90} \]

3 point loops:

- $s$ channel

\[ U_{st}^{\mu\nu} = \frac{(P + q)^\mu}{8\pi^2(s-m^2)((p \cdot q)^2 - m^2 q^2)} \left\{ \{ (2m^2 - M^2 + p \cdot q)q^{\nu} (2p \cdot q + q^2) \\
- q^2 (P^{\nu} + q^{\nu}) \} C_0 (m^2, q^2, s, M^2, m^2, M^2) + B_0 (m^2, m^2, M^2) \{ p \cdot q (P^{\nu} + q^{\nu}) \\
- q^{\nu} (2m^2 + p \cdot q) \} + B_0 (s, m^2, M^2) \{ q^{\nu} (2m^2 + 3p \cdot q + q^2) \\
- (p \cdot q + q^2) (P^{\nu} + q^{\nu}) \} + B_0 (q^2, m^2, m^2) \{ q^2 (P^{\nu} + q^{\nu}) \\
- q^{\nu} (2p \cdot q + q^2) \} \right\}. \tag{5.91} \]

\[ V_{st}^{\mu\nu} = \frac{(P + q)^\mu (P + q')^\nu}{4\pi^2(s-m^2)m^2} \left\{ (\mu^2 + m^2 - M^2) C_0 (m^2, m^2, 0, m^2, M^2, m^2) \\
+ B_0 (m^2, M^2, mu^2) - B_0 (0, m^2, m^2) \right\}. \tag{5.92} \]
The vertex correction diagrams contributions to the CFFs are summarized as

\[ V_{\mu \nu}^{u} = \frac{(P - q')^{\mu}}{4\pi^{2}(u - m^{2})^{2}} \left\{ (m^{2} - u) \left\{ q''(3m^{2} - 2M^{2} + u) \right\} C_{0} \left( m^{2}, u, 0, m^{2}, M^{2}, m^{2} \right) \right. \]
\[ \left. - 2B_{0} \left( 0, m^{2}, m^{2} \right) + B_{0} \left( m^{2}, m^{2}, M^{2} \right) \left\{ (u - m^{2}) P^{\nu} + (m^{2} - u) q'' \right\} \right. \]
\[ \left. + (3m^{2} + u) q'' \right\} C_{0} \left( m^{2}, u, 0, m^{2}, M^{2}, m^{2} \right) + B_{0} \left( u, m^{2}, M^{2} \right) \left\{ (m^{2} - u) P^{\nu} + (u - m^{2}) q'' \right\} \]
\[ \left. - (m^{2} + 3u) q'' \right\}. \] (5.93)

\[ U_{\mu \nu}^{u} = \frac{(P - q')^{\nu}}{4\pi^{2}(u - m^{2})(q^{2} - m^{2} - u)^{2}} \left\{ ((m^{2} - u) q^{\mu} + q^{2} \left\{ q^{\nu} - P^{\nu} \right\} \right. \right. \]
\[ \times \left\{ (3m^{2} - 2M^{2} - q^{2} + u) C_{0} \left( m^{2}, u, q^{2}, M^{2}, M^{2} \right) - 2B_{0} \left( q^{2}, m^{2}, m^{2} \right) \right. \]
\[ + B_{0} \left( m^{2}, m^{2}, M^{2} \right) \left\{ - P^{\mu} (m^{2} + q^{2} - u) + q^{\mu} (m^{2} + q^{2} - u) \right\} \right. \]
\[ + q^{\mu} (3m^{2} - q^{2} + u) \right\} + B_{0} \left( u, m^{2}, M^{2} \right) \left\{ P^{\nu} \left( m^{2} - q^{2} - u \right) \right. \]
\[ \left. + q^{\nu} \left( -m^{2} + q^{2} + u \right) + q^{\mu} \left( -m^{2} + q^{2} - 3u \right) \right\}. \] (5.94)

The vertex correction diagrams contributions to the CFFs are summarized as

\[ H_{1}^{Vst} = \frac{1}{4\pi^{2}(q^{2} - m^{2} - s)^{2}} \left\{ q^{2}(3m^{2} + 2M^{2} + q^{2} - s)C_{0}(m^{2}, q^{2}, s, M^{2}, m^{2}, m^{2}) \right. \]
\[ + 2q^{2}B_{0}(q^{2}, m^{2}, m^{2}) + (m^{2} - q^{2} - s)B_{0}(s, m^{2}, M^{2}) \]
\[ - (m^{2} + q^{2} - s)B_{0}(m^{2}, m^{2}, M^{2}) \left\} , \right. \] (5.95)

\[ H_{1}^{Vut} = \frac{1}{8\pi^{2}m^{2}} \left\{ (2m^{2} - M^{2})C_{0}(m^{2}, 0, m^{2}, M^{2}, m^{2}, m^{2}) + B_{0}(m^{2}, m^{2}, M^{2}) \right. \]
\[ - B_{0}(0, m^{2}, m^{2}) \left\} , \right. \] (5.96)

\[ H_{1}^{Ust} = \frac{1}{4\pi^{2}(m^{2} - u)^{2} - 2q^{2}(m^{2} + u) + q^{4}} \left\{ \left( q^{2}(m^{2} - 2M^{2} - u) + (m^{2} - u)^{2} \right) \right. \]
\[ \times \left\{ C_{0}(m^{2}, u, q^{2}, m^{2}, M^{2}, m^{2}) + (q^{2} + u - m^{2})B_{0}(u, m^{2}, M^{2}) \right. \]
\[ + (q^{2} - u + m^{2})B_{0}(m^{2}, m^{2}, M^{2}) - 2q^{2}B_{0}(q^{2}, m^{2}, m^{2}) \right\} , \] (5.97)

\[ H_{1}^{Uut} = \frac{1}{4\pi^{2}(m^{2} - u)^{2}} \left( B_{0}(u, m^{2}, M^{2}) - B_{0}(m^{2}, m^{2}, M^{2}) \right) , \] (5.98)

\[ H_{2}^{Vst} = \frac{1}{4\pi^{2}(q^{2} - m^{2} + s)^{2}} \left\{ (s - m^{2})(3m^{2} - 2M^{2} - q^{2} + s)C_{0}(m^{2}, q^{2}, s, M^{2}, m^{2}, m^{2}) \right. \]
\[ - 2(s - m^{2})B_{0}(q^{2}, m^{2}, m^{2}) + (m^{2} - q^{2} + 3s)B_{0}(s, m^{2}, M^{2}) \]
\[ - (3m^{2} - q^{2} + s)B_{0}(m^{2}, m^{2}, M^{2}) \left\} , \right. \] (5.99)

\[ H_{2}^{Vut} = \frac{1}{4\pi^{2}(m^{2} - u)^{2}} \left( B_{0}(u, m^{2}, M^{2}) - B_{0}(m^{2}, m^{2}, M^{2}) \right) . \] (5.100)

\[ H_{2}^{Ust} = \frac{1}{4\pi^{2}(q^{2} - m^{2} + s)^{2}} \left\{ (s - m^{2})(3m^{2} - 2M^{2} - q^{2} + s)C_{0}(m^{2}, q^{2}, s, M^{2}, m^{2}, m^{2}) \right. \]
\[ - 2(s - m^{2})B_{0}(q^{2}, m^{2}, m^{2}) + (m^{2} - q^{2} + 3s)B_{0}(s, m^{2}, M^{2}) \]
\[ - (3m^{2} - q^{2} + s)B_{0}(m^{2}, m^{2}, M^{2}) \right\} , \] (5.101)

\[ H_{2}^{Uut} = \frac{1}{4\pi^{2}(m^{2} - u)^{2}} \left( B_{0}(u, m^{2}, M^{2}) - B_{0}(m^{2}, m^{2}, M^{2}) \right) . \] (5.102)
4 point loops:
s-channel

\[
T^{\mu\nu}_{st} = \frac{1}{4\pi^2} \left\{ (P + q)^\mu (P + q')^\nu D_0(m^2, t, m^2, s, m^2, q^2, M^2, m^2, m^2) \\
\quad + D_1 P^\mu (P - q'^\mu) + D_1 q'^\mu (P - q'^\mu) + D_2 P^\nu (P + q'^\mu) + D_2 q'^\nu (P + q'^\mu) \\
\quad + D_3 P^\nu (P - q'^\mu) + D_3 q'^\mu (P - q'^\mu) + P^\mu \{ D_2 (P + q'^\nu - q'^\mu) + (D_1 + D_3) (P - q'^\nu + q'^\mu) \\
\quad + 2 D_1 q'^\nu \} + 4 D_0 q'^\mu D_1 (P - q'^\mu) (P - q'^\mu + q'^\nu) + 2 D_1 q'^\nu (P - q'^\mu) + D_2 (P - q'^\mu) \times (P + q'^\nu - q'^\mu) + D_2 (P + q'^\nu - q'^\mu) \\
\quad + D_{12} (P + q'^\mu) (P - q'^\nu + q'^\mu) + D_{12} (P + q'^\nu - q'^\mu) \\
\quad + D_{23} (P + q'^\mu) (P - q'^\nu + q'^\nu) + D_{33} (P - q'^\mu) (P + q'^\nu + q'^\mu) \right\},
\]

(5.103)

and u channel

\[
T^{\mu\nu}_{ut} = \frac{1}{4\pi^2} \left\{ D_0 (m^2, t, q'^2, u, m^2, 0, M^2, m^2, m^2) (P - q')^\mu (P - q')^\nu \\
\quad + D_1 (P - q - q')^\mu P^\nu - D_1 (P - q + q')^\mu q'^\nu + D_2 P^\nu (P q - q')^\mu - D_2 q'^\nu (P + q - q')^\mu \\
\quad + D_3 P^\nu (P - q + q')^\mu - D_3 q'^\mu (P - q + q')^\mu + P^\mu \{ - q'^\nu D_1 + (P + q - q')^\nu D_2 + (D_1 + D_3) \times (P - q + q')^\nu \} + 4 D_0 q'^\mu - 2 D_{11} q'^\mu (P - q'^\nu + q'^\mu) + D_{11} (P - q'^\mu + q'^\mu) (P - q'^\nu + q'^\mu) \\
\quad + D_{12} (P - q'^\mu + q'^\mu) (P - q'^\nu + q'^\nu) - 2 D_{12} q'^\nu (P + q' - q'^\mu) + D_{12} (P + q' - q'^\mu) \times (P - q'^\nu + q'^\nu) - 2 D_{13} q'^\nu (P - q'^\mu + q'^\mu) + 2 D_{13} (P - q'^\mu + q'^\mu) (P - q'^\nu + q'^\nu) \\
\quad - 2 q'^\mu \{(P - q + q')^\nu D_1 + \frac{1}{2} (P + q - q')^\nu - 2 q'^\nu D_{11} + (P + q - q')^\nu D_{12} + \frac{1}{2} (P - q + q')^\nu (D_1 + D_3 \\
\quad + 2 (D_{11} + D_{13})) \} + D_{22} (P + q' - q'^\mu) (P - q'^\nu + q'^\mu) + D_{23} (P - q'^\mu + q'^\mu) (P - q'^\nu + q'^\mu) \\
\quad + D_{23} (P + q'^\mu - q'^\mu) (P - q'^\nu + q'^\nu) + D_{33} (P - q'^\mu + q'^\mu) (P - q'^\nu + q'^\mu) \right\},
\]

(5.104)

where \(D_i\) and \(D_{ij}\) can found from

\[
D^\mu = \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^d k k^\mu \prod_{i=0}^{3} \frac{1}{(k + r_i)^2 - m_i^2},
\]

(5.105)

\[
D^{\mu\nu} = \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^d k k^\mu k^\nu \prod_{i=0}^{3} \frac{1}{(k + r_i)^2 - m_i^2}
\]

(5.106)
with

\[ D^\mu = \sum_{i=1}^{3} r_i^\mu D_i, \]  
\[ D^{\mu \nu} = g^{\mu \nu} D_{00} + \sum_{i=1}^{3} r_i^\mu r_j^\nu D_{ij}. \]

We can further expand \( D_i \) and \( D_{ij} \) in terms of \( B_0 \) and \( C_0 \) functions but since they are long expressions we do not need to show them.

The four point box diagrams contribute each CFF as follows:

\[ \mathcal{H}_{0}^{Tst} = -\frac{D_{00}}{2\pi^2}, \]  
\[ \mathcal{H}_{1}^{Tst} = \frac{1}{4\pi^2} \left\{ D_0(m^2, t, m^2, s, m^2, q^2, M^2, m^2, m^2) + 2D_1 + 2D_2 + 2D_3 + D_{11} + 2D_{12} + D_{22} + 2D_{13} + 2D_{23} + D_{33} \right\}, \]  
\[ \mathcal{H}_{2}^{Tst} = \frac{1}{4\pi^2} \left\{ -D_0 - 2D_1 - 2D_3 - D_{11} + D_{22} - 2D_{13} - D_{33} \right\}. \]

\[ \mathcal{H}_{0}^{Tut} = -\frac{D_{00}}{2\pi^2}, \]  
\[ \mathcal{H}_{1}^{Tut} = \frac{1}{4\pi^2} \left\{ D_0(m^2, t, m^2, u, m^2, 0, M^2, m^2, m^2) + 2D_1 + 2D_2 + 2D_3 + D_{11} + 2D_{12} + D_{22} + 2D_{13} + 2D_{23} + D_{33} \right\}, \]  
\[ \mathcal{H}_{2}^{Tut} = \frac{1}{4\pi^2} \left\{ -D_0 - 2D_1 - 2D_3 - D_{11} + D_{22} - 2D_{13} - D_{33} \right\}. \]

5.10 Summary

In order to construct the most general tensor structure of hadrons in virtual Compton scattering, we investigated several different cases of the DVCS and Bethe Heitler process and some structures of hadronic tensors were given. We derived general leptonic tensor parts and tested them against the results of several papers ([50] and [48]). During our calculations we have found several miswritten equations in these papers. For example, in [50] the leptonic part minor sign and coefficient part are incorrect. To make sure our leptonic tensor for the BH process Eq. 5.38 was accurate, in Section 5.6 we calculated the real photon scattering amplitude which already exists in some sources and showed that it agrees with the amplitude in Peskin [25]. In [48], we found that either they used a different hadronic tensor than in the paper for the amplitude
calculation or the hardronic tensor is miswritten. We showed in our calculation that if we take the coefficient of $\mathcal{P}^{\mu\sigma} P_{\sigma} P_{\tau} \mathcal{P}^{\tau\nu}$ as

$$T_2 + \frac{\Delta^2}{2M^2} T_3,$$

instead of $T_2 + \frac{\Delta^2}{4M^2} T_3$, then we get the same amplitude squares as they do.

We wrote down first level loop calculations for scalar particle exchange, such as biquark model, then we can deduce the scalar integrals in Section 5.9 in terms of the Passarino-Veltman functions.
Chapter 6

Conclusion

In this dissertation, we investigated advantages of the light-front dynamics (LFD) over the Instant form dynamics (IFD). We studied how two forms differ by examining the interpolation between these two forms by interpolating form dynamics. One of the most important differences between IFD and LFD is the number of dynamic generators, which is less in LFD since $K_3$ joins to kinematic generators from the dynamic ones. We constructed the spinors in interpolating form for fermions and studied how amplitudes change in terms of frame dependence. We observed that since $K_3$ becomes a kinematic generator at the light-front limit, the amplitudes for fermions become frame independent in LFD.

With similar analogy to the Lorentz transformation, we studied the electrodynamics under a constant uniform field since the electromagnetic field strength tensor $F^{\mu\nu}$ and the Poincaré matrix $M^{\mu\nu}$ describe similar transformations. We expanded the general solution for the equations of motion which is found by Itzykson and Zuber [28]. We observed similarities with chiral properties of the fermion spinors transformation is given in chapter 2 in these solutions and examined them for special conditions. We also examined how interpolating fields, which are analogous dynamic generators, change at light front-limit by observing $u^\pm(\tau)$ since kinematic operators leave $u^\pm$ invariant. We examined some trajectories of these solutions with some critical points in Appendix B as well.

During the calculations, we used Clifford algebra as a unifying tool for Lorentz transformations and spinors as this algebra can combine vector and spinor algebra under one algebra. We can get spinors without separating into left-handed and right-handed notation as the standard derivation we used in chapter 2. The quantities in Clifford algebra are basis independent and we can take advantage of this property by getting the same quantities in instant form coordinates or in interpolating coordinates, just by taking the interior product of them as $f^\mu = \gamma^\mu.f$ or $f_\mu = \gamma_\mu.f$. We used Clifford algebra(1,3) in complex space $Cl_{1,3}(\mathbb{C})$ for spinor expression for spin-1/2 and spin-1 spinors under the same algebraic structure, then derived the spinors
in terms of Clifford numbers. We were able to find the interpolating photon propagator $\epsilon_\mu \epsilon^*_\nu$ with an anti-symmetric part, which is useful for deeply virtual Compton scattering (DVCS) and Bethe-Heitler (BH) process amplitude calculations.

As a continuation of finding the most general tensor structure of hadrons, we studied the scalar hadronic tensor structure for the DVCS and BH process with interference terms. This process is especially important in hadron physics in Jefferson Lab with 12GeV upgrade. We showed the equivalence between the hadronic tensor structure in [48] by relating the coefficients between these two hadronic tensors. We found some typos in the leptonic tensor in [50] and in the handbag dominant hadronic tensor [48] and checked the accuracy of our results. We applied the polarization vector tensor, which we derived in chapter 4 to DVCS, BH process, and interference amplitudes. One loop corrections of the real outgoing photon case are reduced to the Passarino-Veltman function terms in scalar integrals which is one step forward to finding exact results and finding of divergent terms.
REFERENCES


Appendix A

Appendix for Chapter 1 and Chapter 2

A.1 Table of Commutation Relation of LFD Poincaré Generators

Poincaré algebra on the light front can be written down as

Table A.1: Poincaré Algebra for LFD. The commutation relation reads [element in the first column, element in the first row] = element at the intersection of the corresponding row and column.

<table>
<thead>
<tr>
<th></th>
<th>$P^+$</th>
<th>$P^1$</th>
<th>$P^2$</th>
<th>$P^-$</th>
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<th>$E^2$</th>
<th>$J^3$</th>
<th>$F^1$</th>
<th>$F^2$</th>
<th>$K^3$</th>
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<td>0</td>
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<tr>
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<td>0</td>
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<td>0</td>
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<tr>
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<td>0</td>
<td>0</td>
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<td>0</td>
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<td>$iP^1$</td>
<td>0</td>
<td>$iP^-$</td>
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</tr>
<tr>
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<td>$iP^2$</td>
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</tr>
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A.2 The spin-1/2 Spinors in IFD and LFD

We can take the instant form and light-front limits of the interpolating spinors given by Eq. 2.99a and Eq. 2.99b. The light-front spinors are given in Table A.2 at light-front limit \((\delta \to \pi/4)\).

<table>
<thead>
<tr>
<th>Chiral Representation</th>
<th>Standard Representation</th>
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<td>(u^{(1)}_{CR}) = (\frac{1}{\sqrt{2\sqrt{2}P^+}}\begin{pmatrix} \sqrt{2}P^+ \ 0 \end{pmatrix})</td>
<td>(u^{(1)}_{SR} = \frac{M}{\sqrt{2\sqrt{2}P^+}}\begin{pmatrix} P_R \ \sqrt{2}P^+ - M \end{pmatrix})</td>
</tr>
<tr>
<td>(u^{(2)}_{CR}) = (\frac{1}{\sqrt{\sqrt{2}P^+}}\begin{pmatrix} M \ 0 \end{pmatrix})</td>
<td>(u^{(2)}_{SR} = \frac{1}{\sqrt{2\sqrt{2}P^+}}\begin{pmatrix} \sqrt{2}P^+ + M \ P_L \end{pmatrix})</td>
</tr>
<tr>
<td>(v^{(1)}_{CR}) = (\frac{1}{\sqrt{2\sqrt{2}P^+}}\begin{pmatrix} P_R \ -M \end{pmatrix})</td>
<td>(v^{(1)}_{SR} = \frac{1}{\sqrt{2\sqrt{2}P^+}}\begin{pmatrix} P_R \ \sqrt{2}P^+ + M \end{pmatrix})</td>
</tr>
<tr>
<td>(v^{(2)}_{CR}) = (\frac{1}{\sqrt{2\sqrt{2}P^+}}\begin{pmatrix} M \ P_L \end{pmatrix})</td>
<td>(v^{(2)}_{SR} = \frac{1}{\sqrt{2\sqrt{2}P^+}}\begin{pmatrix} P_L \ \sqrt{2}P^+ + M \end{pmatrix})</td>
</tr>
</tbody>
</table>

Table A.2: Spinors at light-front limit \((\delta \to \pi/4)\)

Similarly, we find the \(\beta_3\) functions from Eq. 2.57 become

\[
\cosh \beta_3 = \lim_{\delta \to 0} \frac{1}{MC} \left( AP^+ + \frac{BP_-}{\cos \alpha} \right) = \frac{P^0}{M}, \tag{A.1}
\]

\[
\sinh \beta_3 = \lim_{\delta \to 0} \frac{1}{MC} \left( BP^+ + \frac{AP_-}{\cos \alpha} \right) = \frac{P^3}{M \cos \alpha}. \tag{A.2}
\]

In the instant form limit, \(A \to 1\) and \(B \to 0\). Using the transformation matrix given by Eq. 2.1 as \(u_{SR} = Su_{CR}\). The spinors are given in Table A.3.

These spinors are equivalent to spinors given by Eq. 2.116.

A.3 Derivation of Polarization Vectors

Polarization vectors by definition are vector representations so we can use four vectors to describe them and Lorentz transformations in Eq. 1.22, Eq. 1.23 are transformations of these
Vectors. The boost operator defined as $B = e^{-iK\phi}$ by using these operators and the polarization vectors at rest are given by

$$\epsilon^\mu(0, +) = \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix}, \quad \epsilon^\mu(0, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \epsilon^\mu(0, -) = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \quad (A.3)$$

### A.3.1 Standard Polarization Vectors in IFD

Standard vectors when we boost the same rest spinors into any arbitrary direction while spin direction is in the z-direction initially.
Then the polarization vectors in the moving frame becomes

\[
\epsilon^\mu(p, +) = N \left( \begin{array}{c}
(p^0 + m)p^R \\
(p^0 + m)m/\sqrt{2} + p^1 p^R \\
i(p^0 + m)m/\sqrt{2} + p^2 p^R \\
p^3 p^R
\end{array} \right),
\]
\[
\epsilon^\mu(p, -) = N \left( \begin{array}{c}
(p^0 + m)p^L \\
(p^0 + m)m/\sqrt{2} + p^1 p^L \\
i(p^0 + m)m/\sqrt{2} + p^2 p^L \\
p^3 p^L
\end{array} \right),
\]
\[
\epsilon^\mu(p, 0) = N \left( \begin{array}{c}
(p^0 + m)p^3 \\
p^1 p^3 \\
p^2 p^3 \\
(p^0 + m)m + (p^3)^2
\end{array} \right). \tag{A.4}
\]

where \( N = 1/(m(p^0 + m)) \).

### A.3.2 Helicity Polarization Vectors in the Interpolating Dynamics

Since there is no right-handed and left-handed spinor difference in polarization vectors, we only change the transformation operator for interpolating polarization vector and apply Eq. 2.44 transformation into Eq. A.4. Then, the helicity polarization vectors become

\[
\epsilon_{\hat{\mu}}(P, +) = - \left( \begin{array}{c}
- \frac{B}{\sqrt{2}} \left( \frac{\beta_1 \sin \alpha}{\alpha} - \frac{i \beta_2 \sin \alpha}{\alpha} \right) \\
\frac{C}{\sqrt{2}} \left( \frac{\beta_1 \beta_2 (1 + \cos \alpha)}{\alpha^2} + \frac{i (\beta_1^2 + \beta_2^2 \cos \alpha)}{\alpha^2} \right) \\
- A \frac{\beta_1 \sin \alpha}{\alpha} + i \beta_2 \sin \alpha \alpha^2 \\
- \frac{B}{\sqrt{2}} \left( \frac{\beta_1 \sin \alpha}{\alpha} + i \beta_2 \sin \alpha \alpha^2 \right)
\end{array} \right),
\]
\[
\epsilon_{\hat{\mu}}(P, -) = \left( \begin{array}{c}
- \frac{B}{\sqrt{2}} \left( \frac{\beta_1 \sin \alpha}{\alpha} + i \beta_2 \sin \alpha \alpha^2 \right) \\
\frac{C}{\sqrt{2}} \left( \frac{\beta_1 \beta_2 (1 + \cos \alpha)}{\alpha^2} - \frac{i (\beta_1^2 + \beta_2^2 \cos \alpha)}{\alpha^2} \right) \\
- A \frac{\beta_1 \sin \alpha}{\alpha} - i \beta_2 \sin \alpha \alpha^2 \\
- \frac{B}{\sqrt{2}} \left( \frac{\beta_1 \sin \alpha}{\alpha} - \frac{i \beta_2 \sin \alpha}{\alpha} \right)
\end{array} \right),
\]
\[
\epsilon_{\hat{\mu}}(P, 0) = \left( \begin{array}{c}
A (-B \cosh \beta_3 + A \sinh \beta_3) + B \cos \alpha (A \cosh \beta_3 - B \sinh \beta_3) \beta_1 \sin \alpha (-B \cosh \beta_3 + A \sinh \beta_3) \beta_2 \sin \alpha (-B \cosh \beta_3 + A \sinh \beta_3) \\
A \cos \alpha (A \cosh \beta_3 - B \sinh \beta_3) + B (-B \cosh \beta_3 + A \sinh \beta_3) \beta_1 \sin \alpha (A \cosh \beta_3 - B \sinh \beta_3) \beta_2 \sin \alpha (A \cosh \beta_3 - B \sinh \beta_3) \beta_1 \sin \alpha (A \cosh \beta_3 - B \sinh \beta_3) \beta_2 \sin \alpha (A \cosh \beta_3 - B \sinh \beta_3)
\end{array} \right). \tag{A.5}
\]
Then the polarization vectors in terms of momenta are
\[ \epsilon_{\mu}(P, +) = \frac{1}{\sqrt{2}} \left( \|P\|, \frac{P_1 P_2 + i P_2 P_1}{\|P\|}, \frac{P_2 P_1 - i P_1 P_2}{\|P\|}, -\|P\| \right), \]
\[ \epsilon_{\mu}(P, -) = \frac{1}{\sqrt{2}} \left( \|P\|, \frac{P_1 P_2 - i P_2 P_1}{\|P\|}, \frac{P_2 P_1 + i P_1 P_2}{\|P\|}, -\|P\| \right), \]
\[ \epsilon_{\mu}(P, 0) = \frac{P^+}{\sqrt{M^2}} \left( P^+ - \frac{M^2}{P^+}, P_1, P_2, P_3 \right). \]  

(A.6)

A.3.3 Helicity Polarization Vectors in the IFD

We can take the light-front limit of the interpolating polarization vectors since they are naturally reduced into helicity spinors in IFD as \( \delta \to 0 \)
\[ \epsilon^{\mu}(P, +) = -\frac{1}{\sqrt{2}} \left( 0, \frac{P_1 P_2 + i P_2 P_1}{|P|}, \frac{P_2 P_1 - i P_1 P_2}{|P|}, -|P| \right), \]
\[ \epsilon^{\mu}(P, -) = -\frac{1}{\sqrt{2}} \left( 0, \frac{P_1 P_2 - i P_2 P_1}{|P|}, \frac{P_2 P_1 + i P_1 P_2}{|P|}, -|P| \right), \]
\[ \epsilon^{\mu}(P, 0) = \frac{1}{M|P|} \left( |P|, P^1 P^0, P^2 P^0, P^3 P^0 \right). \]  

(A.7)

A.4 \( (1, 0) \oplus (0, 1) \) Lorentz Group Spinors of Spin-1

Similar to spin 1/2 spinors, we can express \( (1, 0) \oplus (0, 1) \) in two separate \( SU(2) \) groups, right-handed and left-handed. Instead of Pauli matrices, we can use \( SU(2) \) matrices for spin-1, which are given by
\[ J_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{i}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]  

(A.8)

In this case, the rest frame spinors are given by
\[ u(0, +) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u(0, 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u(0, -) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \]  

(A.9)
The boost operators are: 

\[(K_1)_R = iJ_1, \quad (K_2)_R = iJ_2, \quad (K_3)_R = iJ_3; \quad (K_1)_L = -iJ_1, \quad (K_2)_L = -iJ_2, \quad (K_3)_L = -iJ_3\]

The complete boost operator with right-handed and left-handed representation combination is

\[K = \begin{pmatrix} K_R & 0 \\ 0 & K_L \end{pmatrix}. \quad (A.10)\]

With the boost transformation above, the spinors are transformed as 

\[u(p, \lambda) = e^{-iK_\lambda} u(0, \lambda)\]

and the spinors in the moving frame become

\[u(p, +) = \frac{1}{2\sqrt{2m(p^0 + m)}} \begin{pmatrix} (p^0 + m + p^3)^2 \\ 2(p^0 + m + p^3)p^R \\ 2(p^R)^2 \\ (p^0 + m - p^3)^2 \\ -2(p^0 + m - p^3)p^R \\ 2(p^R)^2 \end{pmatrix}, \quad (A.11)\]

\[u(p, 0) = \frac{1}{\sqrt{2m(p^0 + m)}} \begin{pmatrix} (p^0 + m + p^3)p^L \\ p^0(p^0 + m) - (p^3)^2 \\ (p^0 + m - p^3)p^R \\ -p^0(p^0 + m) - (p^3)^2 \\ -(p^0 + m + p^3)p^L \\ (p^0 + m + p^3)^2 \end{pmatrix}, \quad (A.12)\]

\[u(p, -) = \frac{1}{2\sqrt{2m(p^0 + m)}} \begin{pmatrix} 2(p^L)^2 \\ 2(p^0 + m - p^3)p^L \\ (p^0 + m - p^3)^2 \\ 2(p^L)^2 \\ -2(p^0 + m + p^3)p^L \\ (p^0 + m + p^3)^2 \end{pmatrix}. \quad (A.13)\]

The anti-particle spinors are

\[v(p, +) = \frac{1}{2\sqrt{2m(p^0 + m)}} \begin{pmatrix} (p^0 + m + p^3)^2 \\ 2(p^0 + m + p^3)p^R \\ 2(p^R)^2 \\ -(p^0 + m - p^3)^2 \\ 2(p^0 + m - p^3)p^R \\ -2(p^R)^2 \end{pmatrix}, \quad (A.14)\]
\[ v(p, 0) = \frac{1}{\sqrt{2m(p^0 + m)}} \begin{pmatrix} (p^0 + m + p^3)p^L \\ p^0(p^0 + m) - (p^3)^2 \\ p^0 + m - p^3 \\ p^0 + m + p^3 \end{pmatrix}, \quad (A.15) \]

\[ v(p, -) = \frac{1}{2\sqrt{2m(p^0 + m)}} \begin{pmatrix} 2(p^L)^2 \\ 2(p^0 + m - p^3)p^L \\ (p^0 + m - p^3)^2 \\ -2(p^L)^2 \\ 2(p^0 + m + p^3)p^L \\ -(p^0 + m + p^3)^2 \end{pmatrix}, \quad (A.16) \]

A.5 Generalized \((0, J) \oplus (J, 0)\) Helicity Spinors for Arbitrary Interpolation Angle for Spin 1

Following the notations in \[12\], we will use \(A = \cos \delta, B = -\sin \delta\) in the spinors listed below.

We also define \(\chi \equiv \frac{P^+ - \not{p}}{C} = \frac{P^+ - \sqrt{(P^+)^2 - M^2C}}{C}\) for convenience.

Spin-1 helicity spinors for any interpolation angle in chiral representation:
In Ahluwalia’s notation $P_R = P^R$ and $P_L = P^L$. One should also note that in our dissertation $P^+ = (P^0 + P^3)/\sqrt{2}$, and the spinors are all normalized so that $\bar{u}u = 2M$. 

\[
 u^{(+1)}_H = \frac{1}{2\sqrt{M^2 \bar{p}^2}} \left( \begin{array}{c} (P_+ + \bar{p})(P^+_\bar{p} + \bar{p}) \\ (A - B) \\ \sqrt{2}P^R(P^+_\bar{p} + \bar{p}) \\ (A - B)(P_+^R)^2(P^+_\bar{p} + \bar{p}) \\ (P_+^R + \bar{p}) \\ (A + B)(P_+^R)^2(P^+_\bar{p} - \bar{p}) \\ (P_+^R + \bar{p}) \end{array} \right),
\]

\[
 u^{(-1)}_H = \frac{1}{2\sqrt{M^2 \bar{p}^2}} \left( \begin{array}{c} (A + B)(P^L)^2(P^+_\bar{p} - \bar{p}) \\ (P_+^L + \bar{p}) \\ -\sqrt{2}P^L(P^+_\bar{p} - \bar{p}) \\ (A - B)(P^L)^2(P^+_\bar{p} + \bar{p}) \\ (P_+^L + \bar{p}) \\ -\sqrt{2}P^L(P^+_\bar{p} + \bar{p}) \\ (P_+^L + \bar{p})(P^+_\bar{p} + \bar{p}) \\ (A + B)(P^L)^2(P^+_\bar{p} - \bar{p}) \end{array} \right),
\]

\[
 u^{(0)}_H = \sqrt{\frac{M}{2\bar{p}^2}} \left( \begin{array}{c} -(A + B)P^L \\ \sqrt{2}P_+^L \\ (A - B)P^R \\ -(A + B)P^L \\ \sqrt{2}P_+^R \\ (A + B)P^R \end{array} \right). \quad (A.17)
\]
Appendix B

Appendix for Chapter 3

B.1 Interpolating Form Equation of Motion

$E_x$ and $B_y$ in the Interpolating Form Kinematic Solution

We take an example of a constant uniform electric field in the x-direction and magnetic field in the y-direction. Perpendicular to them, an initial velocity is in the z-direction. We can expand our expression for the interpolating form for kinematic solution along the interpolating time with existing electric and magnetic fields. First, the interpolating field tensor given by Eq. 3.52 becomes

\[
\mathbf{F}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix}
0 & -E_x \cos \delta - B_y \sin \delta & 0 & 0 \\
E_x \cos \delta + B_y \sin \delta & 0 & 0 & -B_y \cos \delta + E_x \sin \delta \\
0 & 0 & 0 & 0 \\
0 & B_y \cos \delta - E_x \sin \delta & 0 & 0
\end{pmatrix},
\]  
(B.1)

and the four velocity on interpolating form is $u^{\hat{\mu}} = (u^\hat{\tau}, u_x, 0, u^\hat{\zeta})$ where $u^\hat{\tau} = u_t \cos \delta + u_\zeta \sin \delta$ and $u^\hat{\zeta} = u_t \sin \delta - u_\zeta \cos \delta$ and $u^{\hat{\mu}} = (u^\hat{\tau}, -u_x, 0, u^\hat{\zeta})$.

We can arrange any ratio of $E_x/B_y$ with the interpolating angle $\delta$ and vanish the $E_x \cos \delta - B_y \sin \delta$ term in order to solve the equation of motion. The ratio between electric and magnetic fields are related with $\tan \delta$ and since the range of interpolating angle is $0 \leq \delta \leq \pi/4$, this ratio is between -1 and 1 so there is a $|B_y| \leq |E_x|$ limit.

We set the relation $E_x \cos \delta + B_y \sin \delta = 0$ in order to make it only kinematic operator analogous fields survive. The tensor given by Eq. B.1 becomes
The four force is \( \mathbf{u} = (u_x, -u_y, u_z) \) (we can ignore \( u_y \) since the motion is in the x-z plane), then the equations of motion are

\[
\begin{align*}
\frac{d^2 x^\tau}{d\tau^2} &= 0, \\
\frac{d^2 x(\tau)}{d\tau^2} &= \frac{qB_y}{m \cos \delta} \frac{dx^- (\tau)}{d\tau}, \\
\frac{d^2 y(\tau)}{d\tau^2} &= 0, \\
\frac{d^2 x^- (\tau)}{d\tau^2} &= \frac{qB_y}{m \cos \delta} \frac{dx(\tau)}{d\tau}.
\end{align*}
\]

First, we take advantage of being in kinematic region and find the interpolating time term, we just apply the initial conditions \( x^\tau (0) = 0 \) and \( \dot{x}^\tau (0) = u^\tau (0) \) and solve Eq. B.7 easily then

\[ x^\tau = u^\tau \tau. \]  

We take the integral of Eq. B.8 and Eq. B.10

\[
\begin{align*}
\frac{dx(\tau)}{d\tau} &= \frac{qB_y}{m \cos \delta} (\mathcal{S} x^\tau (\tau) - \mathcal{C} x^- (\tau)) + c_x, \\
\frac{dx^- (\tau)}{d\tau} &= \frac{qB_y}{m \cos \delta} x(\tau) + c_.
\end{align*}
\]
By initial conditions \(x(0) = 0, x^\cdot(0) = 0, \dot{x}(0) = 0, \) and \(\dot{x}^\cdot = u^\cdot(0)\), we find that \(c_x = 0\) and \(c_x^\cdot = u^\cdot(0)\) and

\[
\frac{d^2 x(\tau)}{d\tau^2} = \frac{qB_y}{m \cos \delta} \left( Su^\cdot(0) - \mathbb{C} \left( \frac{qB_y}{m \cos \delta} x(\tau) + u^\cdot(0) \right) \right). \tag{B.14}
\]

We use the change of variables as \(x(\tau) \rightarrow x'(\tau) - \frac{m \cos \delta}{qB_y} u^\cdot(0) + \frac{m \cos \delta}{\mathbb{C} qB_y} u^\cdot(0)\) and rewrite the Eq. B.8 as

\[
\frac{d^2 x'(\tau)}{d\tau^2} = - \frac{q^2 B_y^2 \mathbb{C}}{m^2 \cos^2 \delta} x'(\tau). \tag{B.15}
\]

The solution for \(x'(\tau)\) is

\[
x'(\tau) = c_1 \cos \left( \frac{qB_y \sqrt{\mathbb{C}} \tau}{m \cos \delta} \right) + c_2 \sin \left( \frac{qB_y \sqrt{\mathbb{C}} \tau}{m \cos \delta} \right). \tag{B.16}
\]

Change the variables to \(x(\tau)\) again with the initial conditions

\[
x(\tau) = \frac{m \cos \delta (\mathbb{C} u^\cdot(0) - Su^\cdot(0))}{qB_y \mathbb{C}} \left( -1 + \cos \left( \frac{qB_y \sqrt{\mathbb{C}} \tau}{m \cos \delta} \right) \right). \tag{B.17}
\]

The similar method for \(x^\cdot(\tau)\). Eq. B.10 changes into

\[
\frac{d^2 x^\cdot(\tau)}{d\tau^2} = \frac{q^2 B_y^2}{m^2 \cos^3 \delta} \left( Su^\cdot(0) - \mathbb{C} x^\cdot(\tau) \right), \tag{B.18}
\]

a small change \(x^\cdot(\tau) \rightarrow x^\cdot'(\tau) + \frac{\mathbb{C}}{q} u^\cdot(0)\), in order to make it tidy

\[
\frac{d^2 x^\cdot'(\tau)}{d\tau^2} = - \frac{q^2 B_y^2 \mathbb{C}}{m^2 \cos^2 \delta} x^\cdot', \tag{B.19}
\]

\(x^\cdot(\tau)\) has a hyperbolic solution

\[
x^\cdot'(\tau) = b_1 \cos \left( \frac{qB_y \sqrt{\mathbb{C}} \tau}{m \cos \delta} \right) + b_2 \sin \left( \frac{qB_y \sqrt{\mathbb{C}} \tau}{m \cos \delta} \right), \tag{B.20}
\]

and change \(x^\cdot'(\tau)\) into \(x^\cdot\) back with the initial conditions

\[
x^\cdot(\tau) = - \frac{Su^\cdot(0)\tau}{\mathbb{C}} + \frac{m \cos \delta u^\cdot(0)}{qB_y \sqrt{\mathbb{C}}} \sin \left( \frac{qB_y \sqrt{\mathbb{C}} \tau}{m \cos \delta} \right). \tag{B.21}
\]
The solution in general looks like

\[ x^+(\tau) = u^+(0)\tau, \] (B.22)

\[ x(\tau) = \frac{m \cos(\delta u^-(0) - Su^+(0))}{qB_yC} \left( -1 + \cos \left( \frac{qB_y\sqrt{C}\tau}{m \cos \delta} \right) \right), \] (B.23)

\[ y(\tau) = 0, \] (B.24)

\[ x^-(\tau) = \frac{-Su^+(0)\tau}{C} + \frac{m \cos \delta u^-(0)}{qB_y\sqrt{C}} \sin \left( \frac{qB_y\sqrt{C}\tau}{m \cos \delta} \right). \] (B.25)

We can express \( x \) and \( x^- \) in terms of the interpolation time \( x^+ \)

\[ x(x^+) = \frac{m \cos \delta(\delta u^+(0) - Cu^-(0))}{qB_yC} \left( -1 + \cos \left( \frac{qB_y\sqrt{C}x^+}{m \cos \delta} \right) \right), \] (B.26)

\[ y(x^+) = 0, \] (B.27)

\[ x^-(x^+) = \frac{-Sx^+}{C} + \frac{m \cos \delta u^-(0)}{qB_y\sqrt{C}} \sin \left( \frac{qB_y\sqrt{C}x^+}{m \cos \delta} \right). \] (B.28)

Since we assumed that \( E_x \cos \delta + B_y \sin \delta = 0 \), we can convert the \( \delta \) functions into field dependence as

\[ \tan \delta = -\frac{E_x}{B_y}, \] (B.29)

\[ \cos \delta = \sqrt{\frac{B_y}{E_x^2 + B_y^2}}, \] (B.30)

\[ \sin \delta = -\sqrt{\frac{E_x}{E_x^2 + B_y^2}}, \] (B.31)

\[ C = \frac{B_y^2 - E_x^2}{E_x^2 + B_y^2}, \] (B.32)

\[ S = -\frac{2E_xB_y}{E_x^2 + B_y^2}. \] (B.33)

Although we can describe the solution in terms of the interpolating time \( x^+ \), it is rather difficult to convert them into the instant form coordinates because it is a mixing of \( t \) and \( z \) variables and we cannot separate them from trigonometric functions. So we transform the solution Eq. B.22-Eq. B.25 into the instant form coordinates and change the interpolating angle...
expression into field expressions by using Eq. B.29-Eq. B.33 as

\[
t(\tau) = -\frac{1}{q(B_y^2 - E_z^2)^{3/2}} \left\{ qB_y \sqrt{B_y^2 - E_z^2} (E_x u_z(0) - B_y u_t(0)) \tau \\
+ mE_x (E_x u_t(0) - B_y u_z(0)) \sin \left( \frac{q \sqrt{B_y^2 - E_z^2}}{m} \tau \right) \right\},
\]  
(B.34)

\[
x(\tau) = \frac{m(u_z(0)B_y - u_t(0)E_x)}{q(B_y^2 - E_z^2)} \left\{ -1 + \cos \left( \frac{q \sqrt{B_y^2 - E_z^2}}{m} \tau \right) \right\},
\]  
(B.35)

\[
y(\tau) = 0,
\]  
(B.36)

\[
z(\tau) = -\frac{1}{q(B_y^2 - E_z^2)^{3/2}} \left\{ qE_x \sqrt{B_y^2 - E_z^2} (E_x u_z(0) - B_y u_t(0)) \tau \\
+ mB_y (E_x u_t(0) - B_y u_z(0)) \sin \left( \frac{q \sqrt{B_y^2 - E_z^2}}{m} \tau \right) \right\},
\]  
(B.37)

Similarly, we can find the solution for the \(|E_x| > |B_y|\) region and since we can express the interpolating angle in terms of electric and magnetic fields as \(\cos \delta = E_x/\sqrt{E_x^2 + B_y^2}\) and \(\sin \delta = B_y/\sqrt{E_x^2 + B_y^2}\), we write down expressions in terms of them:

\[
t(\tau) = \frac{1}{q(E_x^2 - B_y^2)^{3/2}} \left\{ qB_y \sqrt{E_x^2 - B_y^2} (E_x u_z(0) - B_y u_t(0)) \tau \\
+ mE_x (E_x u_t(0) - B_y u_z(0)) \sinh \left( \frac{q \sqrt{E_x^2 - B_y^2}}{m} \tau \right) \right\},
\]  
(B.38)

\[
x(\tau) = \frac{m(u_t(0)E_x - u_z(0)B_y)}{q(E_x^2 - B_y^2)} \left\{ -1 + \cosh \left( \frac{q \sqrt{E_x^2 - B_y^2}}{m} \tau \right) \right\},
\]  
(B.39)

\[
y(\tau) = 0,
\]  
(B.40)

\[
z(\tau) = \frac{1}{q(E_x^2 - B_y^2)^{3/2}} \left\{ qE_x \sqrt{E_x^2 - B_y^2} (E_x u_z(0) - B_y u_t(0)) \tau \\
+ mB_y (E_x u_t(0) - B_y u_z(0)) \sinh \left( \frac{q \sqrt{E_x^2 - B_y^2}}{m} \tau \right) \right\},
\]  
(B.41)

### B.2 Light-front Dynamics (LFD) Solutions

Even though we couldn’t solve all fields on any arbitrary directions case, we can take advantage of \(E_z\) being a field which is analogous to the kinematic operator in LFD and change the fields
into all kinematic analogous fields with a special case as we will mention and solve the equations of motion in existence of other fields in LFD.

**All Fields with Special case** \( E_x = -B_y \) and \( E_y = B_x \) in LFD

We can go further and define a special case where the electric and magnetic fields are equal and we can take advantage of the kinematics in light-front that are mixed states of the electric and magnetic fields as

\[
F_{LF}^{\mu \nu} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -E_x - B_y & -E_y + B_x & \sqrt{2}E_z \\
E_x + B_y & 0 & \sqrt{2}B_z & B_y - E_x \\
E_y - B_x & \sqrt{2}B_z & 0 & -B_x - E_y \\
-\sqrt{2}E_z & -B_y + E_x & B_x + E_y & 0
\end{pmatrix},
\]

(B.42)

Since the electric field is in the z-direction, the magnetic field is in the z-direction, and \( B_x - E_y \) are also analogous to the kinematic operators in the light-front, we can solve with these fields in the kinematic equation of motion. The electromagnetic field tensor is

\[
F_{LF}^{\mu \nu} = \begin{pmatrix}
0 & 0 & 0 & E_z \\
0 & 0 & -B_z & \sqrt{2}B_y \\
0 & B_z & 0 & -\sqrt{2}B_x \\
-E_z & -\sqrt{2}B_y & \sqrt{2}B_x & 0
\end{pmatrix},
\]

(B.43)

and the equations of the motion are

\[
\frac{d^2 x^+ (\tau)}{d\tau^2} = \frac{qE_z}{m} \frac{dx^+ (\tau)}{d\tau},
\]

(B.44)

\[
\frac{d^2 x (\tau)}{d\tau^2} = \frac{qB_z}{m} \frac{dy(\tau)}{d\tau} + \sqrt{2}qB_y \frac{dx^+ (\tau)}{d\tau},
\]

(B.45)

\[
\frac{d^2 y(\tau)}{d\tau^2} = \frac{qB_z}{m} \frac{dx(\tau)}{d\tau} - \sqrt{2}qB_x \frac{dx^+ (\tau)}{d\tau},
\]

(B.46)

\[
\frac{d^2 x^- (\tau)}{d\tau^2} = -\frac{qE_z}{m} \frac{dx^- (\tau)}{d\tau} + \sqrt{2}qB_y \frac{dx(\tau)}{d\tau} - \sqrt{2}qB_x \frac{dy(\tau)}{d\tau}.
\]

(B.47)

We take the integral first then rewrite the equation

\[
\frac{dx^+ (\tau)}{d\tau} = \frac{qE_z}{m} x^+ (\tau) + c_+,
\]

(B.48)

by the initial conditions \( c_+ = u^+(0) \) then

\[
\frac{d^2 x^+ (\tau)}{d\tau^2} = \frac{q^2E_z^2}{m^2} x^+ (\tau) + \frac{qE_z}{m} u^+(0),
\]

(B.49)
change $x^+$ into $x^+ \rightarrow x^+ - \frac{m}{qE_z} u^+(0)$ then we get solutions in hyperbolic functions, and with applying the initial conditions we get

$$x^+(\tau) = \frac{m u^+(0)}{qE_z} \left(-1 + e^{\frac{qE_z x}{m}}\right). \quad (B.50)$$

We take the integral of the rest

$$\frac{dx(\tau)}{d\tau} = \frac{qB_z}{m} y(\tau) + \frac{\sqrt{2}qB_y}{m} x^+(\tau) + c_x, \quad (B.51)$$
$$\frac{dy(\tau)}{d\tau} = -\frac{qB_z}{m} x(\tau) - \frac{\sqrt{2}qB_x}{m} x^+(\tau) + c_y, \quad (B.52)$$
$$\frac{dx^-(\tau)}{d\tau} = -\frac{qE_z}{m} x^-(\tau) + \frac{\sqrt{2}qB_y}{m} x(\tau) - \frac{\sqrt{2}qB_x}{m} y(\tau) + c_. \quad (B.53)$$

By the initial conditions, the constants are $c_x = 0$, $c_y = 0$ and $c_-=u^-(0)$.

By using the first degree differential expression we can write the equation of motion for $x(\tau)$ as

$$\frac{d^2x(\tau)}{d\tau^2} = \frac{qB_z}{m} \left(\frac{-qB_z}{m} x(\tau) - \frac{\sqrt{2}qB_x}{m} x^+(\tau)\right) + \frac{\sqrt{2}qB_y}{m} \left(\frac{qE_z}{m} x^+(\tau) + u^+(0)\right). \quad (B.54)$$

Similarly, the equation of motion of $y(\tau)$ is

$$\frac{d^2y(\tau)}{d\tau^2} = -\frac{qB_z}{m} \left(\frac{qB_z}{m} y(\tau) + \frac{\sqrt{2}qB_y}{m} x^+(\tau)\right) - \frac{\sqrt{2}qB_x}{m} \left(\frac{qE_z}{m} x^+(\tau) + u^+(0)\right). \quad (B.55)$$

Now, since we know the solution of $x^+(\tau)$ Eq. B.50, we can put the solution inside these equations

$$\frac{d^2x(\tau)}{d\tau^2} = -\frac{q^2 B_x^2}{m^2} x(\tau) + \frac{\sqrt{2}q(E_z B_y - B_x B_z)u^+(0)}{m E_z} e^{\frac{qE_z x}{m}} + \frac{\sqrt{2}qB_x B_z}{m E_z} u^+(0), \quad (B.56)$$
$$\frac{d^2y(\tau)}{d\tau^2} = -\frac{q^2 B_y^2}{m^2} y(\tau) + \frac{\sqrt{2}q(-E_z B_x + B_y B_z)u^+(0)}{m E_z} e^{\frac{qE_z y}{m}} - \frac{\sqrt{2}qB_y B_z}{m E_z} u^+(0). \quad (B.57)$$

Only additional terms as a function of $\tau$ are exponential terms. It is one of the advantages of $E_z$ being kinematic in the light-front and we can apply it also inside the change of variable along
with constants as

\[ x(\tau) \to x'(\tau) - \frac{\sqrt{2}m(E_z B_y - B_z B_y)u^+(0)}{q E_z(E_z^2 + B_z^2)} e^{\frac{qB_z\tau}{m}} - \frac{\sqrt{2}m B_z u^+(0)}{q B_z E_z} \]

\[ y(\tau) \to y'(\tau) + \frac{\sqrt{2}(E_z B_x - B_y B_z)u^+(0)}{m E_z(E_z^2 + B_z^2)} e^{\frac{qB_z\tau}{m}} + \frac{\sqrt{2}m B_y u^+(0)}{q B_z E_z} \]

by this change of variables, Eq. B.56 and Eq. B.57 becomes

\[ \frac{d^2x'(\tau)}{d\tau^2} = -\frac{q^2 B_x^2}{m^2} x'(\tau), \quad (B.60) \]

\[ \frac{d^2y'(\tau)}{d\tau^2} = -\frac{q^2 B_y^2}{m^2} y(\tau). \quad (B.61) \]

Then the solutions of changed variables are

\[ x'(\tau) = a_1 \cos \left( \frac{qB_z\tau}{m} \right) + a_2 \sin \left( \frac{qB_z\tau}{m} \right), \quad (B.62) \]

\[ y'(\tau) = b_1 \cos \left( \frac{qB_z\tau}{m} \right) + b_2 \sin \left( \frac{qB_z\tau}{m} \right), \quad (B.63) \]

and change back variables again into their original form and use the initial conditions

\[ x(\tau) = -\frac{\sqrt{2}m(B_x B_z - B_y E_z)u^+(0)}{q E_z(E_z^2 + B_z^2)} e^{\frac{qB_z\tau}{m}} + \frac{\sqrt{2}m(B_x B_z - B_y E_z)u^+(0)}{q B_z E_z} \sin \left( \frac{qB_z\tau}{m} \right) \]

\[ - \frac{\sqrt{2}m(B_x B_z + B_y B_z)u^+(0)}{q B_z E_z} \cos \left( \frac{qB_z\tau}{m} \right) + \frac{\sqrt{2}m B_y u^+(0)}{q B_z E_z} \]

\[ y(\tau) = -\frac{\sqrt{2}m(E_z B_x + B_y B_z)u^+(0)}{q E_z(E_z^2 + B_z^2)} e^{\frac{qB_z\tau}{m}} + \frac{\sqrt{2}m(E_z B_x + B_y B_z)u^+(0)}{q B_z E_z} \sin \left( \frac{qB_z\tau}{m} \right) \]

\[ + \frac{\sqrt{2}m (B_x B_z - B_y E_z) u^+(0)}{q E_z B_z (E_z^2 + B_z^2)} \cos \left( \frac{qB_z\tau}{m} \right) + \frac{\sqrt{2}m B_y u^+(0)}{q E_z B_z} \]

The last light-front coordinate’s equation of motion is

\[ \frac{d^2x^-(\tau)}{d\tau^2} = \frac{q^2 E_x^2}{m^2} x^-(\tau) - \frac{\sqrt{2}q^2(E_z B_y - B_z B_y)}{m^2} x(\tau) + \frac{\sqrt{2}q^2(E_z B_x + B_y B_z)}{m^2} y(\tau) \]

\[ + \frac{2q^2(B_x^2 + B_y^2)}{m^2} x^+(\tau), \quad (B.66) \]

after putting the solutions of \( x(\tau) \), \( y(\tau) \), and \( x^+(\tau) \) in this equation, it becomes

\[ \frac{d^2x^-(\tau)}{d\tau^2} = \frac{q^2 E_x^2}{m^2} x^-(\tau) + \frac{2q(B_x^2 + B_y^2) u^+(0)}{m B_z} \sin \left( \frac{qB_z\tau}{m} \right). \quad (B.67) \]
Since \( \sin \left( \frac{qB \tau}{m} \right) \) is also a repeating function as an exponential, we can apply the change of variables as

\[
x^-(\tau) \rightarrow x^-'(\tau) - \frac{2m(B_x^2 + B_y^2)}{qB_z(E_z^2 + B_z^2)} \sin \left( \frac{qB_z \tau}{m} \right), \quad (B.68)
\]
then the equation of motion turns into

\[
\frac{d^2x^-'}{d\tau^2} = \frac{q^2E_z^2}{m^2} x^-'(\tau). \quad (B.69)
\]

When we solve it with the initial condition then the solution becomes

\[
x^-(\tau) = \frac{m}{qE_zB_z(E_z^2 + B_z^2)} \left\{ B_z \left( (E_z^2 + B_z^2)u^-(0) + 2(B_x^2 + B_y^2)u^+(0) \right) \sin \left( \frac{qE_z \tau}{m} \right) - 2E_z(B_x^2 + B_y^2)u^+(0) \sin \left( \frac{qB_z \tau}{m} \right) \right\}. \quad (B.70)
\]

The whole solution of the equations of motion is

\[
x^+(\tau) = \frac{mu^+(0)}{qE_z} \left( -1 + e^{-\frac{qE_z \tau}{m}} \right), \quad (B.71)
\]

\[
x(\tau) = -\frac{\sqrt{2}m(B_xB_z - B_yE_z)u^+(0)}{qE_z(E_z^2 + B_z^2)} e^{-\frac{qE_z \tau}{m}} + \frac{\sqrt{2}m(B_xB_z - B_yE_z)u^+(0)}{qB_z(E_z^2 + B_z^2)} \sin \left( \frac{qB_z \tau}{m} \right) - \frac{\sqrt{2}m(B_xE_z + B_yB_z)u^+(0)}{qB_z(E_z^2 + B_z^2)} \cos \left( \frac{qB_z \tau}{m} \right) + \frac{\sqrt{2}mB_x}{qE_zB_z} u^+(0), \quad (B.72)
\]

\[
y(\tau) = -\frac{\sqrt{2}m(E_xB_x + B_yB_z)u^+(0)}{qE_z(E_x^2 + B_z^2)} e^{-\frac{qE_z \tau}{m}} + \frac{\sqrt{2}m(E_xB_x + B_yB_z)u^+(0)}{qB_z(E_x^2 + B_z^2)} \sin \left( \frac{qB_z \tau}{m} \right) + \frac{\sqrt{2}mB_y}{qE_zB_z} u^+(0), \quad (B.73)
\]

\[
x^-\tau) = \frac{m}{qE_zB_z(E_z^2 + B_z^2)} \left\{ B_z \left( (E_z^2 + B_z^2)u^-\left(0\right) + 2(B_x^2 + B_y^2)u^+(0) \right) \sin \left( \frac{qE_z \tau}{m} \right) - 2E_z(B_x^2 + B_y^2)u^+(0) \sin \left( \frac{qB_z \tau}{m} \right) \right\}. \quad (B.74)
\]

We can change to coordinates into instant coordinates as \( t = (x^+ + x^-)/\sqrt{2} \) and \( z = (x^+ - ... \)
\(x^-)/\sqrt{2}\)

\[
\begin{align*}
t(\tau) &= \frac{m}{2qE_zB_z(E_z^2 + B_z^2)} \left\{ -2E_z(B_x^2 + B_y^2)(u_t(0) + u_z(0)) \sin \left( \frac{qB_z\tau}{m} \right) \right. \\
&\quad + B_z \left( (E_z^2 + E_y^2)(u_t(0) - u_z(0)) + 2(B_x^2 + B_y^2)(u_t(0) + u_z(0)) \right) \sinh \left( \frac{qE_z\tau}{m} \right) \bigg\} \\
&\quad + \frac{m(u_t(0) + u_z(0))}{2qE_z} \left( -1 + e^{-\frac{qE_z\tau}{m}} \right), \quad \text{(B.75)}
\end{align*}
\]

\[
\begin{align*}
x(\tau) &= -\frac{m(B_xB_z - B_yE_z)(u_t(0) + u_z(0))}{qE_z(E_z^2 + B_z^2)} e^{\frac{qE_z\tau}{m}} + \frac{m(B_xB_z - B_yE_z)(u_t(0) + u_z(0))}{qB_z(E_z^2 + B_z^2)} \\
&\quad \times \sin \left( \frac{qB_z\tau}{m} \right) - \frac{m(B_xE_z + B_yB_z)(u_t(0) + u_z(0))}{qB_z(E_z^2 + B_z^2)} \cos \left( \frac{qB_z\tau}{m} \right) \\
&\quad + \frac{mB_z}{qE_zB_z} (u_t(0) + u_z(0)), \quad \text{(B.76)}
\end{align*}
\]

\[
\begin{align*}
y(\tau) &= -\frac{m(E_zB_x - B_yB_z)(u_t(0) + u_z(0))}{qE_z(E_z^2 + B_z^2)} e^{\frac{qE_z\tau}{m}} + \frac{m(E_zB_x - B_yB_z)(u_t(0) + u_z(0))}{qB_z(E_z^2 + B_z^2)} \\
&\quad \times \sin \left( \frac{qB_z\tau}{m} \right) + \frac{m(B_xB_z - B_yE_z)(u_t(0) + u_z(0))}{qE_zB_z(E_z^2 + B_z^2)} \cos \left( \frac{qB_z\tau}{m} \right) \\
&\quad + \frac{mB_y}{qE_zB_z} (u_t(0) + u_z(0)), \quad \text{(B.77)}
\end{align*}
\]

\[
\begin{align*}
z(\tau) &= \frac{m}{2qE_zB_z(E_z^2 + B_z^2)} \left\{ -2E_z(B_x^2 + B_y^2)(u_t(0) + u_z(0)) \sin \left( \frac{qB_z\tau}{m} \right) \right. \\
&\quad - B_z \left( (E_z^2 + E_y^2)(u_t(0) - u_z(0)) + 2(B_x^2 + B_y^2)(u_t(0) + u_z(0)) \right) \sinh \left( \frac{qE_z\tau}{m} \right) \bigg\} \\
&\quad + \frac{m(u_t(0) + u_z(0))}{2qE_z} \left( -1 + e^{-\frac{qE_z\tau}{m}} \right). \quad \text{(B.78)}
\end{align*}
\]

**Only \(E_x\) and \(B_y\) fields case**

The same situation but on the light-front form \((\mu = +, 1, 2, -)\) and the four velocity is \(u^\mu = (u^+, u_x, 0, u^-) = 1/\sqrt{2}(u_t + u_z, \sqrt{2}u_x, 0, u_t - u_z)\) and \(u_\mu = (u^-, -u_x, 0, u^+)\), then the field tensor becomes

\[
F^{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & -E_x - B_y & 0 & 0 \\
E_x + B_y & 0 & 0 & E_x - B_y \\
0 & 0 & 0 & 0 \\
0 & -E_x + B_y & 0 & 0
\end{pmatrix}, \quad \text{(B.79)}
\]

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Figure B.1: First plot is $|\mathbf{E}| > |\mathbf{B}|$ and second plot is $|\mathbf{E}| < |\mathbf{B}|$. $B_x$ and $B_y$ are pulling down in the -z direction opposite have force in the +z direction by $E_z$. Here $B_x = B_y = 0.1T$, $E_z/c = \{0.1, 0.05\}T$, $B_z = \{0.05, 0.1\}T$, and $v_0^z = 0.5c$.

and the four force is

$$f^\mu = qF^{\mu \nu}u_\nu = \frac{q}{\sqrt{2}}((E_x + B_y)u_x, (E_x + B_y)u^- + (E_x - B_y)u^+ + 0, (E_x - B_y)u_z). \quad (B.80)$$

The equation of motion from the four force is written as

$$\frac{d^2 x^+(\tau)}{d\tau^2} = \frac{q(E_x + B_y)}{\sqrt{2}m} \frac{dx(\tau)}{d\tau}, \quad (B.81)$$

$$\frac{d^2 x(\tau)}{d\tau^2} = \frac{q}{\sqrt{2}m} \left( (E_x + B_y) \frac{dx^- (\tau)}{d\tau} + (E_x - B_y) \frac{dx^+ (\tau)}{d\tau} \right), \quad (B.82)$$

$$\frac{d^2 y(\tau)}{d\tau^2} = 0, \quad (B.83)$$

$$\frac{d^2 x^-(\tau)}{d\tau^2} = \frac{q(E_x - B_y)}{\sqrt{2}m} \frac{dx(\tau)}{d\tau}. \quad (B.84)$$

In order to solve this problem, we can take advantage of the light-front variable to make the equation of motion look like the equation of motion of the simple electric field. First, we take the problem on the light-front with the condition that when electric and magnetic fields are equivalent in magnitude ($E_x = B_y$) then the fields, which are analogous kinematic generators,
vanishes.

\[ \frac{d^2 x^+}{d\tau^2} = \frac{q(\sqrt{2}E_x)}{m} \frac{dx}{d\tau}, \quad (B.85) \]
\[ \frac{d^2 x}{d\tau^2} = \frac{q}{m} (\sqrt{2}E_x) \frac{dx^-}{d\tau}, \quad (B.86) \]
\[ \frac{d^2 y}{d\tau^2} = 0, \quad (B.87) \]
\[ \frac{d^2 x^-}{d\tau^2} = 0. \quad (B.88) \]

First, we solve \( x^- \) with initial conditions \( x^-(0) = 0 \) and \( \dot{x}^-(0) = u^-(0) \). Then, it is

\[ x^-(\tau) = u^-(0)\tau. \quad (B.89) \]

Eq. B.86 becomes

\[ \frac{d^2 x}{d\tau^2} = \frac{q}{m} (\sqrt{2}E_x)u^-(0), \quad (B.90) \]

and with initial conditions \( x(0) = 0 \) and \( \dot{x}(0) = 0 \) the solution is

\[ x(\tau) = \frac{qE_x}{\sqrt{2}m}u^-(0)\tau^2, \quad (B.91) \]

and finally Eq. B.85 becomes

\[ \frac{d^2 x^+}{d\tau^2} = \frac{q^2 E_x^2}{m^2} u^-(0)\tau, \quad (B.92) \]

with initial conditions \( x^+(0) = 0 \) and \( \dot{x}^+(0) = u^+(0) \), the solution appears to be

\[ x^+(\tau) = \frac{q^2 E_x^2}{6m^2} u^-(0)\tau^3 + u^+(0)\tau. \quad (B.93) \]

When we solve the equation of motion for this situation (particle at the origin in the beginning), what we get is

\[ x^+(\tau) = \frac{q^2 E_x^2}{6m^2} u^-(0)\tau^3 + u^+(0)\tau, \quad (B.94) \]
\[ x(\tau) = \frac{qE_x}{\sqrt{2}m} u^-(0)\tau^2, \quad (B.95) \]
\[ y(\tau) = 0, \quad (B.96) \]
\[ x^-(\tau) = u^-(0)\tau. \quad (B.97) \]
Then we can convert light-front variables into instant form as

$$t(\tau) = u_t(0) \left( \frac{q^2 E_x^2}{6m^2} \tau^3 + \tau \right) - u_z(0) \frac{q^2 E_z^2}{6m^2} \tau^3, \quad (B.98)$$

$$x(\tau) = \frac{q E_x}{2m} (u_t(0) - u_z(0)) \tau^2, \quad (B.99)$$

$$y(\tau) = 0, \quad (B.100)$$

$$z(\tau) = u_t(0) \frac{q^2 E_x^2}{6m^2} \tau^3 + u_z(0) \left( -\frac{q^2 E_z^2}{6m^2} \tau^3 + \tau \right). \quad (B.101)$$

### B.3 2D Trajectory of the Charged Particle on the $E_x$ and $B_y$ Field

#### B.3.1 $E^2 > B^2$ Region Motion

In this section, we investigate the $E_x$ and $B_y$ field solution with the initial velocity $v_z^0$ which is solved by the interpolating method and the solutions are written in Eq. B.34-Eq. B.37.

We can see from these equations that the point where $E_x = B_y$ is the turning point of the motion. It changes from boost like motion to circular (rotation like) motion at this point since for $E_x \geq B_y$ is hyperbolic functions and for $E_x \leq B_y$ is circular functions.

However Eq. B.38-Eq. B.41 describe the trajectory of the charged particle in the region between $0 \leq B_y \leq E_x$ since the interpolating angle is defined between $0 \leq \delta \leq \pi/4$ and we solve the equation of motion for $E_x = B_y$ and $0 \leq E_x \leq B_y$ regions also.

When $E_x \geq B_y$, the particle just moves to the $x$-direction with acceleration in the $z$-direction constant speed but with relativistic effect Figure B.2.

When we apply a magnetic field it pulls a little to the -$x$ direction compare Figure B.3 and Figure B.4, then trajectory differs according to the initial velocity $v_z(0)$. First, when $v_z(0) \ll c$ at the beginning $v_z(0)$ be effective in the $z$-direction with a ratio $\sim \tau$ then the force from the magnetic field dominates the $z$-direction motion and the concave curve in Figure B.9. Secondly, when $v_z(0) \sim c$ it makes a convex curve since it has a higher $v_z(0)$ and it makes the changing of effects of force and initial velocity less visible, but still we observe the force on the -$x$ direction also due to the magnetic field (Figure B.5) and the same situation for Figure B.4 with a lower $E_x/B_y$ ratio.

Now, we will investigate the initial and later time motions separately.

Let’s take a look at $z(\tau)/x(\tau)$ ratio of Eq. B.39 and Eq. B.41 to find out where the charged
particle approaches as time goes.

\[
\frac{z(\tau)}{x(\tau)} = \frac{qE_x(v_x^0 E_x - B_y)\tau}{m(E_x - v_x^0 B_y)(-1 + \cosh \left( \frac{q\sqrt{E_x^2 - B_y^2} \tau}{m} \right))} + \frac{mB_y \sinh \left( \frac{q\sqrt{E_x^2 - B_y^2} \tau}{m} \right)}{m \sqrt{E_x^2 - B_y^2}(-1 + \cosh \left( \frac{q\sqrt{E_x^2 - B_y^2} \tau}{m} \right))}.
\] (B.102)

Expand the hyperbolic functions as

\[
cosh \left( \frac{q\sqrt{E_x^2 - B_y^2} \tau}{m} \right) = 1 + \frac{q^2(E_x^2 - B_y^2)\tau^2}{2m^2} + O(\tau^4) \tag{B.103}
\]

\[
sinh \left( \frac{q\sqrt{E_x^2 - B_y^2} \tau}{m} \right) = \frac{q\sqrt{E_x^2 - B_y^2} \tau}{m} + \frac{q^3(E_x^2 - B_y^2)^{3/2}\tau^3}{3!m^3} + O(\tau^5), \tag{B.104}
\]

and the ratio change into

\[
\frac{z(\tau)}{x(\tau)} = \frac{qE_x(v_x^0 E_x - B_y)\tau + mB_y(E_x - v_x^0 B_y) \left( \frac{q\sqrt{E_x^2 - B_y^2} \tau}{m} + \frac{q^3(E_x^2 - B_y^2)^{3/2}\tau^3}{3!m^3} + O(\tau^5) \right)}{m(E_x - v_x^0 B_y) \left( \frac{q^2(E_x^2 - B_y^2)\tau^2}{2m^2} + O(\tau^4) \right)}, \tag{B.105}
\]
Figure B.3: Trajectory of the charged particle when $E_x/B_y > c$ ($E_x/c = 0.01T$ and $B_y = 0.005T$) here is the blue trajectory is when $v_z(0) = 0.1c$, the red one is when $v_z(0) = 0.5c$, and the green one is when $v_z(0) = 0.9c$.

One thing we can see from the equation above is that at some specific speed $v_z(0)$, $\tau^1$ order vanishes in the numerator and it makes the $z(\tau)$ motion more dominant. The first order of $\tau$ in the numerator is $qE_x(v_z(0)E_x - B_y) + qB_y(E_x - v_z(0)B_y)$ and it vanishes when $v_z(0) = 0$, for the case $E_x \geq B_y$ the ratio turns into

$$
\frac{z(\tau)}{x(\tau)}|_{v_z=0} = \frac{qB_y\tau^3 + O(\tau^5)}{3m\tau^2 + O(\tau^4)}.
$$

(B.106)

It seems like the magnetic field kicks in at $\tau^3$ but the electric field is more dominant in higher order because of the $\sqrt{E_x^2 - B_y^2}$ factor and motion towards the $z$-direction. We observe from the trajectories with the existence of magnetic field motion are closer to the x-axis (Figure B.2, Figure B.3).

As time goes to infinity

$$
\left.\frac{z(\tau)}{x(\tau)}\right|_{\tau\to\infty} = \frac{qE_x(v_z^0E_x - B_y)\tau}{m(E_x - v_z^0B_y)(-1 + \cosh \left(\frac{q\sqrt{E_x^2 - B_y^2}}{m}\tau\right))} \bigg|_{\tau\to\infty} \\
+ \frac{B_y \sinh \left(\frac{q\sqrt{E_x^2 - B_y^2}}{m}\tau\right)}{\sqrt{E_x^2 - B_y^2}(-1 + \cosh \left(\frac{q\sqrt{E_x^2 - B_y^2}}{m}\tau\right))} \bigg|_{\tau\to\infty}.
$$

(B.107)

Here, hyperbolic functions can be approximately written as

$$
cosh \tau = \frac{e^\tau + e^{-\tau}}{2},
$$

(B.108)

$$
sinh \tau = \frac{e^\tau - e^{-\tau}}{2}.
$$

(B.109)
Only $e^{\tau}$ terms survive as $\tau \to \infty$ and the first term of Eq. B.107 vanishes because of $\cosh(const.\tau)$ then

\[
\frac{z(\tau)}{x(\tau)} \bigg|_{\tau \to \infty} = \frac{B_y}{\sqrt{E_x^2 - B_y^2}}. \tag{B.110}
\]

We see that as time goes to infinity, the $z/t$ ratio approaches the line of $B_y/(\sqrt{E_x^2 - B_y^2})$ and we can see that in the case of a non-magnetic field, there will be no $z$-direction motion. Additionally, we have a special case when $E_x = B_y/\sqrt{2}$. In this case, when the time goes to infinity, the $z/x$
ratio approaches 1 (Figure B.6 and Figure B.7).

Moreover, we can also write down axial motions of the trajectory separately rather than as the ratio as

\[
x(\tau)|_{\tau \to \infty} = \frac{m(E_xu_t(0) - B_yu_z(0))}{q(E_x^2 - B_y^2)} \left( -1 + e^{q\sqrt{E_x^2 - B_y^2}\tau/m} \right), \tag{B.111}
\]

\[
z(\tau)|_{\tau \to \infty} = \frac{E_x(u_z(0)E_x - u_t(0)B_y)\tau}{(E_x^2 - B_y^2)} + \frac{mB_y(u_z(0)E_x - B_yu_t(0))}{q(E_x^2 - B_y^2)^{3/2}} e^{q\sqrt{E_x^2 - B_y^2}\tau/m}. \tag{B.112}
\]

This helps to draw the motion of the long time approach (Figure B.7).

![Figure B.6: Trajectory of the charged particle when $E_x/B_y = \sqrt{2}c$ ($E_x/c = 0.141421356T$ and $B_y = 0.1T$) for 1/60th of second interval and the blue trajectory is when $v_z(0) = 0.1c$, the red one is when $v_z(0) = 0.5c$, and the green one is when $v_z(0) = 0.9c$](image)

For short time cases, we can use the approach $\tau \ll 1$ and express the Eq. B.105 with this limit as

\[
\frac{z(\tau)}{x(\tau)}|_{\tau \ll 1} = \frac{2mv_0^0}{q(E_x - u_0^0 B_y)\tau} + O(\tau). \tag{B.113}
\]

Rather than the ratio expression, we need to look limits for individual axis and for short time limit they become

\[
x(\tau) = \frac{q(u_t(0)E_x - u_z(0)B_y)^2}{2m(E_x^2 - B_y^2)\tau^2} + O(\tau^4), \tag{B.114}
\]

\[
z(\tau) = u_z(0)\tau + O(\tau^3). \tag{B.115}
\]

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Figure B.7: Trajectory of the charged particle when $E_x/B_y = \sqrt{2}c$ ($E_x/c = 0.0282842712T$ and $B_y = 0.02T$) for 1 hour interval and here the blue trajectory is when $v_z(0) = 0.1c$, the orange trajectory is when $v_z(0) = 0.5c$ and the green trajectory is when $v_z(0) = 0.9c$ and red dashed line is long time approach of these trajectories as $\tau \to \infty$ and it goes to $z/x = 1$ line as $E_x = \sqrt{2}B_y$. 
From the formula above, we can conclude that in the short time limit, trajectory more depends on $v_z(0)$ in Figure B.8, there is no z-direction motion when $v_z(0) = 0$, and with an increased magnetic field magnitude comparing to the electric field the motion will get closer to the z-axis as in Figure B.6. One can observe that with a higher initial velocity short time approach is getting closer to the exact trajectory because the short time approach is related with $v_z(0)$ and when it is higher then it is closer to the trajectory (Figure B.8).

Overall, from both approaches of long time and short time, we can conclude that initially the initial velocity dominates the motion and later in time forces of the electric and magnetic field determines the direction of the motion. There are three main regions of motions: initial time which is dominated by initial velocity mostly is shown in Figure B.8, mid region which is between the transition from the initial approached motion into the long time approached motion as shown in Figure B.6, and late time motion which is dominated by the force of fields and approaches to $z/x = B_y/\sqrt{E_x^2 - B_y^2}$ line independent of the initial velocity as you can see in Figure B.7.

\subsection*{B.3.2 $E^2 = B^2$ Region Motion}

At the light-front limit, we extend hyperbolic functions and take $E_x = B_y$ then the equations Eq. B.38-Eq. B.41 changes into Eq. B.98-Eq. B.101. At this limit, the trajectory of the charged particle changes as x component will change proportional to $\tau^2$ and z component first $\tau$ dominant then $\tau^3$ dominant as shown in Figure B.10.

\subsection*{B.3.3 $E^2 < B^2$ Region Motion}

We can see that when it exceeds the turning point ($E_x = B_y$) that it starts to make mixing motion with moving in the z-direction with waves in the x-region as in Figure B.11. From the figure, one could see that at $E_x/B_y = v_z(0)$ it changes from the positive x region to the negative x region and at this point only goes to the z-direction without any oscillations.

We can see that when we replace $E_x$ by $E_x = v_z(0)B_y$, Eq. B.39 and Eq. B.41 changes into

\begin{align}
  x(\tau) &= 0, \\
  z(\tau) &= u_z^0 \tau \quad \text{and} \quad u_z^0 \tau \sim \frac{v_z^0 t}{\sqrt{1 - (v_z^0)^2}}.
\end{align}

The trajectory is a relativistic speed motion in the positive z-direction only.

There is another important turning point in the motion. There are two oscillations on the charged particle: one in the x-direction and the other in the z-direction. However the z-direction oscillation is mixed with relativistic velocity but at some point oscillation exceeds the relativistic
speed and there is also movement in the negative z-direction which begins at a certain point and make a helical motion Figure B.12. We can learn the point when it changes into helical waves by looking into Eq. B.41. We need to calculate the change in z-direction $dz$ and find out when it turns negative. First, take the derivative of Eq. B.41

$$\frac{dz(\tau)}{d\tau} = u_z = \frac{E_x(B_yu_t(0) - E_xu_z(0)) + B_y(B_yu_z(0) - E_xu_t(0)) \cos \left( \frac{q\sqrt{B_y^2 - E_x^2}}{m} \right)}{B_y^2 - E_x^2}. \tag{B.118}$$

and rewrite the equation as

$$\frac{dz(\tau)}{d\tau} = u_z = \frac{u_t(0)E_xB_y(1 - \cos \left( \frac{q\sqrt{B_y^2 - E_x^2}}{m} \right)) + u_z(0)(B_y^2 \cos \left( \frac{q\sqrt{B_y^2 - E_x^2}}{m} \right) - E_x^2)}{B_y^2 - E_x^2}. \tag{B.119}$$

Now we are looking for negative $dz$ and since $B_y^2 - E_x^2$ is already taken as positive as Eq. B.119 becomes negative as

$$v_z(0)(B_y^2 \cos \left( \frac{q\sqrt{B_y^2 - E_x^2}}{m} \right) - E_x^2) + E_xB_y(1 - \cos \left( \frac{q\sqrt{B_y^2 - E_x^2}}{m} \right)) < 0, \tag{B.120}$$

and

$$v_z(0) < \frac{(\cos \left( \frac{q\sqrt{B_y^2 - E_x^2}}{m} \right) - 1)E_xB_y}{B_y^2 \cos \left( \frac{q\sqrt{B_y^2 - E_x^2}}{m} \right) - E_x^2}. \tag{B.121}$$

We take the minimum value of $\cos \left( \frac{q\sqrt{B_y^2 - E_x^2}}{m} \right) = -1$ since the negative $dz$ begins from this point, then

$$v_z(0) < \frac{2E_xB_y}{E_x^2 + B_y^2}. \tag{B.122}$$

$v_z(0) = 2E_xB_y/(E_x^2 + B_y^2)$ is the beginning of the helical motion and from this formula we can calculate the initial velocity for helical motion for certain electric and magnetic field or vice versa. Figure B.12 shows this transition of changing into a circular path for $E_x/c = 0.1T$ and $B_y = 0.373205T$, $v_z(0) = 0.5c$ as calculated from the formula.

The motion becomes more circular as the magnetic field ratio over electric field increase as shown in Figure B.12 and turns into helical waves. The closer to the $E_x/B_y = 1$ ratio, the more
frequency repeats.

As expected without any electric field, the charged particle follows a circular trajectory as shown in Figure B.13. The initial velocity and the magnitude of the magnetic field only effects the radius of the circle by the formula

\[ \frac{mv_z^2}{r_0} = qv_z B_y. \]  
(B.123)

where \( v_z = v_z^0 \) since the velocity will remain the same inside the circular trajectory then

\[ r_0 = \frac{mv_z^0}{qB_y}. \]  
(B.124)

For example, for a particle with 0.511 MeV/c mass, 0.5 T magnetic field, 0.9c velocity, and 1 eV positive charge, the radius would be

\[ r_0 = \frac{0.511 \text{MeV}/c^2 \cdot 0.9c \cdot 1 \text{eV}}{1\text{eV} \cdot 0.5 \text{Vs/m}} \approx 3\text{mm}. \]  
(B.125)

Like the \( E_x > B_y \) region we can also find the short time approximation for the \( B_y > E_x \) region. Similarly, we use circular functions extensions

\[
\cos \left( \frac{q \sqrt{B_y^2 - E_x^2} \tau}{m} \right) = 1 - \frac{q^2 (B_y^2 - E_x^2) \tau^2}{2m} + O(\tau^4), \]
(B.126)

\[
\sin \left( \frac{q \sqrt{B_y^2 - E_x^2} \tau}{m} \right) = \frac{q \sqrt{B_y^2 - E_x^2} \tau}{m} - \frac{q^3 (B_y^2 - E_x^2)^{3/2} \tau^3}{3!m^3} + O(\tau^5), \]
(B.127)

and Eq. B.39 and Eq. B.41 turn into

\[ x(\tau) = \frac{q(B_y u_z^0 - E_x u_x^0) \tau^2}{2m} + O(\tau^4), \]
(B.128)

\[ z(\tau) = u_z^0 \tau + O(\tau^3). \]
(B.129)

It is the same result as the \( E_x > B_y \) region as expected and short time trajectories for different initial velocity cases are shown in Figure B.14.

However, longtime approach is now different because hyperbolic functions replaced by the circular functions and the max limit of circular function is only 1. In this case, as \( \tau \to \infty \),
\[-1 \leq \cos \left( q \sqrt{\frac{B_y^2 - E_z^2}{m}} \right) \leq 1 \quad \text{and} \quad -1 \leq \sin \left( q \sqrt{\frac{B_y^2 - E_z^2}{m}} \right) \leq 1, \]

\[
\frac{m(B_y u_z(0) - E_x u_t(0))}{q(B_y^2 - E_z^2)} \leq x(\tau) \leq 0, \tag{B.130}
\]

\[
z(\tau) \to \infty. \tag{B.131}
\]

where \( z(\tau) \to \infty \) since \( \tau \) order in Eq. B.41.
Figure B.8: Initial time trajectories for less than 5 second motions with their short time approximations which dominated by initial velocity $v_z(0)$ in case of $E_x > B_y$ ($E_x/c = 0.005T$ and $B_y = 0.001T$). Here there are three cases: $v_z(0) = 0.1c$ case, real trajectory (blue color) and short time approximation (dashed cyan color), $v_z(0) = 0.5c$ case, real trajectory (red color) and short time approximation (dashed orange color), $v_z(0) = 0.9c$ case, real trajectory (green color) and short time approximation (dashed brown color).
Figure B.9: The trajectory change between the initial approached (orange dashed) and the long time approached (blue dashed) motions for $E_x/c = 0.005T$ and $B_y = 0.002T$ with $v_z(0) = 0.5c$.

Figure B.10: Trajectory of the charged particle when $E_x/c = B_y = 0.01T$ and the blue trajectory is when $v_z(0) = 0.1c$, the red trajectory is when $v_z(0) = 0.5c$, and the green trajectory is when $v_z(0) = 0.9c$. 
Figure B.11: Trajectory of the positive charge when $B_y > E_x$ ($E_x/c = 0.25T$ and $B_y = 0.5T$). The blue trajectory is when $v_z(0) = 0.1c$, the red one is when $v_z(0) = 0.5c$, and the green one is when $v_z(0) = 0.9c$.
Figure B.12: Trajectory of the charged particle when $E_x/B_y < 1$ ($E_x/c = 0.1T$ and $B_y = 0.373205T$) and here the red trajectory when $v^0_z = 0.5c$ is the turning point of helical waves, when $v^0_z = 0.9c$ the green trajectory has helical waves, and the blue trajectory has regular waves when $v^0_z = 0.1c$
Figure B.13: The trajectory of charged particle when $E_x = 0$ and $B_y = 0.5T > 0$, here respectively the blue trajectory is when $v_z(0) = 0.1c$, the red trajectory is when $v_z(0) = 0.5c$, and the green trajectory is when $v_z(0) = 0.9c$. 
Figure B.14: Initial time trajectories for less than 1 second motions with their short time approaches which dominated by initial velocity $v_z(0)$ in case of $B_y > E_x$ ($E_x/c = 0.001T$ and $B_y = 0.005T$). Three cases are: $v_z(0) = 0.1c$ case, real trajectory (blue color) and short time approximation (dashed cyan color), $v_z(0) = 0.5c$ case, real trajectory (red color) and short time approximation (dashed orange color), and $v_z(0) = 0.9c$ case, real trajectory (green color) and short time approximation (dashed brown color).
Appendix C

Appendix for Chapter 4

C.1 Lorentz Force Equation with Quaternions

Similar to the Itzykson and Zuber solution of the Lorentz force equation for a uniform constant field we discuss in Section 3.4, we can find a solution with quaternions also. In this approach, we use quaternions instead of Pauli matrices as

\[ \mathbf{u} = u^0 + \mathbf{u} \cdot \mathbf{J}, \]  

(C.1)

where \( \mathbf{J} = (\mathbf{i}, \mathbf{j}, \mathbf{k}) \). Therefore, Eq. 3.12 changes into

\[ \frac{d\mathbf{u}}{dt} = \frac{q}{m} \left( \frac{i\mathbf{E} - \mathbf{B}}{2} \cdot \mathbf{J} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{J} \right), \]  

(C.2)

when we integrate, it becomes

\[ \mathbf{u}(\tau) = \exp \left( \frac{q}{2m} \left( \frac{i\mathbf{E} - \mathbf{B}}{2} \cdot \mathbf{J} \right) \right) \mathbf{u}(0) \exp \left( \frac{q}{2m} \left( \frac{i\mathbf{E} + \mathbf{B}}{2} \cdot \mathbf{J} \right) \right). \]  

(C.3)

The three component complex vector \( \mathbf{n} \) is defined as \( i\mathbf{E} - \mathbf{B} \) and \( a \) is defined as \( (q/2m)\sqrt{\mathbf{n}^2} \) then

\[ \exp \left( \frac{q}{2m} \frac{i\mathbf{E} - \mathbf{B}}{2} \cdot \mathbf{J} \right) = \cosh(a\tau) + \frac{\mathbf{n} \cdot \mathbf{J}}{\sqrt{\mathbf{n}^2}} \sinh(a\tau). \]  

(C.4)

The rest is the same as the procedure we showed in Section 3.4. The main difference is Pauli matrices are part of \( Cl_3 \), i.e. \( (\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = 1 \) while quaternions are part of \( Cl_{0,3} \), i.e.
i^2 = j^2 = k^2 = -1. Similarly, the rotation and boost operators are related as

\begin{align}
-\imath \sigma^1 &\leftrightarrow \mathbf{i}, \quad \sigma^1 \leftrightarrow \imath \mathbf{i}, \\
-\imath \sigma^2 &\leftrightarrow \mathbf{j}, \quad \sigma^2 \leftrightarrow \imath \mathbf{j}, \\
-\imath \sigma^3 &\leftrightarrow \mathbf{k}, \quad \sigma^3 \leftrightarrow \imath \mathbf{k}.
\end{align}

\tag{C.5, C.6, C.7, C.8}

### C.2 4 × 6 Matrix and Spinors

In this section, we find the polarization vector tensor \( \epsilon_\mu \epsilon_\nu^* \) by using the 4 × 6 matrix which is given by (C.20) and can be expressed as

\[ \epsilon_\mu \epsilon_\nu^* = C_{4 \times 6} u_\mu \bar{u}_\nu C_{6 \times 4}^\dagger. \tag{C.9} \]

It can be also written as

\[ \epsilon_\mu \epsilon_\nu^* = C_{4 \times 6} u_\mu \bar{u}_\nu C_{6 \times 4}^\dagger = \frac{(\slashed{p} + m)(1 + \gamma_5 \slashed{\gamma})}{4m}. \tag{C.10} \]

By using Eq. A.11, Eq. A.12, and Eq. A.13 and taking the total sum, we get

\[ \sum \epsilon_\mu(p, \lambda) \epsilon_\nu^*(p, \lambda) = \frac{1}{m^2} \begin{pmatrix}
(p^0)^2 - m^2 & p^0 p^1 & p^0 p^2 & p^0 p^3 \\
p^0 p^1 & m^2 + (p^1)^2 & p^1 p^2 & p^1 p^3 \\
p^0 p^2 & p^1 p^2 & m^2 + (p^2)^2 & p^2 p^3 \\
p^0 p^3 & p^1 p^3 & p^2 p^3 & m^2 + (p^3)^2
\end{pmatrix}, \tag{C.11} \]

which is also can be written down as

\[ \sum \epsilon_\mu(p, \lambda) \epsilon_\nu^*(p, \lambda) = -g_{\mu\nu} + \frac{p^\mu p^\nu}{m^2}. \tag{C.12} \]

The antisymmetric part is

\[
\epsilon_\mu(+) \epsilon_\nu^*(+) - \epsilon_\mu(-) \epsilon_\nu^*(-) = \frac{1}{4m} \begin{pmatrix}
0 & -p^2 & p^1 & 0 \\
p^2 & 0 & \frac{(p^0)^2 + m p^0 - (p^3)^2}{m + p^3} & \frac{p^0 p^3}{m + p^3} \\
-p^1 & \frac{(p^0)^2 + m p^0 - (p^3)^2}{m + p^3} & 0 & \frac{p^1 p^3}{m + p^3} \\
0 & \frac{p^0 p^3}{m + p^3} & \frac{p^1 p^3}{m + p^3} & 0
\end{pmatrix}. \tag{C.13}
\]
The anti-symmetric part also can be written as

\[ A_{\mu\nu} = i\epsilon_{\mu\nu\alpha\beta} p^\alpha s^\beta / m, \]  

where \( p^\mu = (p^0, p^1, p^2, p^3) \) and \( s^\mu = ((p^0 + m)p^3, p^1 p^3, p^2 p^3, (p^0 + m)m + (p^3)^2) / (m(p^0 + m)) \). They are coming from the boosted expression of \( p^\mu_0 = (m, 0, 0, 0) \) and \( s^\mu_0 = (0, 0, 0, 1) \), momentum and spin at rest frame.

\[
\begin{align*}
A_{01} &= i\epsilon_{01\alpha\beta} p^\alpha s^\beta / m = i(p^2 s^3 - s^2 p^3) / m = -i p^2 / m, \\
A_{02} &= i\epsilon_{02\alpha\beta} p^\alpha s^\beta / m = i(p^3 s^2 - s^3 p^1) / m = -i p^1 / m, \\
A_{03} &= i\epsilon_{03\alpha\beta} p^\alpha s^\beta / m = i(p^1 s^2 - s^1 p^2) / m = 0, \\
A_{12} &= i\epsilon_{12\alpha\beta} p^\alpha s^\beta / m = i(p^0 s^3 - s^0 p^3) = i(p^0(p^0 + m) - (p^3)^2) / (m(p^0 + m)), \\
A_{13} &= i\epsilon_{13\alpha\beta} p^\alpha s^\beta / m = i(p^2 s^0 - s^2 p^0) / m = i p^2 p^3 / (m(p^0 + m)), \\
A_{23} &= i\epsilon_{23\alpha\beta} p^\alpha s^\beta / m = -i p^1 p^3 / (m(p^0 + m)).
\end{align*}
\]

The diagonal terms are all zero \( A_{\mu\mu} = 0 \) and other terms are anti-symmetric \( A_{\mu\nu} = -A_{\nu\mu} \).

For Interpolation form polarization vectors given in Eq. A.6, the anti-symmetric part is

\[
\epsilon_\mu(P,+)\epsilon_\nu^*(P,+) - \epsilon_\mu(P,-)\epsilon_\nu^*(P,-) = \frac{1}{iP} \begin{pmatrix}
0 & -P^2 \sin \delta & P^1 \sin \delta & 0 \\
-P^2 \sin \delta & 0 & P_- & -P^2 \cos \delta \\
P^1 \sin \delta & -P_- & 0 & P^1 \cos \delta \\
0 & P^2 \cos \delta & -P^1 \cos \delta & 0
\end{pmatrix}.
\]  

We can use the same expression we derived from the 4 × 6 matrix conversion in Eq. C.14, then
the anti-symmetric part of the interpolating polarization tensor becomes

\[ A_{01} = i \epsilon_{01\alpha\beta} p^\alpha s^\beta / m = i ((P^+ P_- \cos \delta - P^2 \sin \delta) P^2 / C - P^+ P^- P^3) / M^2 P \]

\[ = - P^2 \sin \delta (P_-^2 + M^2 C) / (i M C P) , \]

\[ A_{02} = i \epsilon_{02\alpha\beta} p^\alpha s^\beta / m = i (P^2 P^- P^+ - (P^+ P_- \cos \delta - P^2 \sin \delta) P^1 / C) / M^2 P \]

\[ = P^1 \sin \delta (P_-^2 + M^2 C) / (i M C P) , \]

\[ A_{03} = i \epsilon_{03\alpha\beta} p^\alpha s^\beta / m = i (P^1 P^- P^+ - P^- P^1 P^+) / M^2 / P = 0 , \]

\[ A_{12} = i \epsilon_{12\alpha\beta} p^\alpha s^\beta / m = i (P^0 (P^+ P_- \cos \delta - P^2 \sin \delta)) / C - P^3 (P^2 \cos \delta - P^+ P_- \sin \delta) P^2 / C)/ M P = - P_-(P_-^2 + M^2 C) / (i M C P) , \]

\[ A_{13} = i \epsilon_{13\alpha\beta} p^\alpha s^\beta / m = i (P^2 \cos \delta - P^+ P_- \sin \delta) / C - P^0 P^2 P^+ / M P \]

\[ = P^2 \sin \delta (P_-^2 + M^2 C) / (i M C P) , \]

\[ A_{23} = i \epsilon_{23\alpha\beta} p^\alpha s^\beta / m = i (P^0 P^1 P^+ - P^1 (P^2 \cos \delta - P^+ P_- \sin \delta)) / C / M P \]

\[ = - P^1 \cos \delta (P_-^2 + M^2 C) / (i M C P) , \]

with \( P^2 = P^+ P^+ - M^2 C , \) \( P^+ = P^0 \cos \delta + P^3 \sin \delta , \) and \( P_- = P^0 \sin \delta + P^3 \cos \delta . \) Then \( P^2 = (P^0)^2 \sin^2 \delta + (P^3)^2 \sin^2 \delta + 2 P^0 P^3 \cos \delta \sin \delta - M^2 \cos 2 \delta \) and \( P^+ P_- = ((P^0)^2 + (P^3)^2) \cos \delta \sin \delta + P^0 P^3 . \)

Using the formula

\[ A_{\mu\nu} = - i \epsilon_{\mu\nu\alpha\beta} p^\alpha n^\beta / P , \]  

(C.16)

with \( n^\mu = (\cos \delta, 0, 0, - \sin \delta) . \)

\[ A_{01} = - i \epsilon_{01\alpha\beta} p^\alpha n^\beta / P = - i (P^2 n^3 - n^2 P^3) / P = i P^2 \sin \delta / P , \]

\[ A_{02} = - i \epsilon_{02\alpha\beta} p^\alpha n^\beta / P = - i (P^3 n^1 - n^3 P^1) / P = i P^1 \sin \delta / P , \]

\[ A_{03} = - i \epsilon_{03\alpha\beta} p^\alpha n^\beta / P = - i (P^1 n^2 - n^1 P^2) / P = 0 , \]

\[ A_{12} = - i \epsilon_{12\alpha\beta} p^\alpha n^\beta / P = - i (P^0 n^3 - n^0 P^3) / P = i (P^+ \sin \delta + P^+ \cos \delta) / P = i P_- / P , \]

\[ A_{13} = - i \epsilon_{13\alpha\beta} p^\alpha n^\beta / P = - i (P^2 n^0 - n^2 P^0) / P , \]  

(C.17)

The \((1, 0) \oplus (0, 1)\) spinors, which are given by Eq. A.11, Eq. A.12, and Eq. A.13 and the polarization vectors which are given by Eq. A.4 are connected by the relation and in matrix
form

\[ C = \sum_{\lambda} u(p, \lambda) \epsilon^*(p, \lambda) \]

\[
\begin{pmatrix}
  p^L & -p^+ & ip^+ & p^L \\
  -(p^+ - p^-)/\sqrt{2} & -(p^R - p^L)/\sqrt{2} & i(p^R + p^L)/\sqrt{2} & (p^+ - p^-)/\sqrt{2} \\
  -p^R & p^- & ip^- & p^R \\
  p^L & -p^- & ip^- & -p^L \\
  -(p^+ - p^-)/\sqrt{2} & (p^R - p^L)/\sqrt{2} & -i(p^R + p^L)/\sqrt{2} & (p^+ - p^-)/\sqrt{2} \\
  -p^R & p^+ & -ip^+ & -2p^Lp^+ \\
\end{pmatrix}
\]

(C.18)

Its reverse matrix i.e.

\[ \epsilon = C^* u, \]

(C.19)

can be written as

\[
\begin{pmatrix}
  p^L & -p^+ & ip^+ & p^L \\
  -(p^+ - p^-)/\sqrt{2} & -(p^R - p^L)/\sqrt{2} & i(p^R + p^L)/\sqrt{2} & (p^+ - p^-)/\sqrt{2} \\
  -p^R & p^- & ip^- & p^R \\
  p^L & -p^- & ip^- & -p^L \\
  -(p^+ - p^-)/\sqrt{2} & (p^R - p^L)/\sqrt{2} & -i(p^R + p^L)/\sqrt{2} & (p^+ - p^-)/\sqrt{2} \\
  -p^R & p^+ & -ip^+ & -2p^Lp^+ \\
\end{pmatrix}
\]

(C.20)

The multiplications of these matrices give automatically the symmetric part of the spinor product as

\[
CC^* = \sum_{\lambda} u(p, \lambda) \bar{u}(p, \lambda)
\]

\[
\begin{pmatrix}
  m^2 & 0 & 0 & 2(p^+)^2 & 2\sqrt{2}p^Lp^+ & 2(pR)^2 \\
  0 & m^2 & 0 & 2p^Rp^+ & 2p^+p^- + 2p^Rp^L & 2p^Lp^- \\
  0 & 0 & m^2 & 2(p^R)^2 & 2\sqrt{2}p^Rp^- & 2(p^-)^2 \\
  2(p^-)^2 & -\sqrt{2}p^Lp^- & 2p(pR)^2 & m^2 & 0 & 0 \\
  -2\sqrt{2}p^Rp^- & 2p^+p^- + 2p^Rp^L & p^Lp^- & 0 & m^2 & 0 \\
  2(p^R)^2 & -2\sqrt{2}p^Rp^+ & 2(p^+)^2 & 2(p^-)^2 & 0 & m^2 \\
\end{pmatrix}
\]

(C.21)
\[ C^* C = \sum_{\lambda} \epsilon_{\mu}(p, \lambda)(-\epsilon^\nu)(p, \lambda) \]
\[ = \frac{1}{m^2} \begin{pmatrix}
    m^2 - (p^0)^2 & p^0 p^1 & p^0 p^2 & p^0 p^3 \\
    -p^0 p^1 & m^2 + (p^1)^2 & p^1 p^2 & p^1 p^3 \\
    -p^0 p^2 & p^1 p^2 & m^2 + (p^2)^2 & p^2 p^3 \\
    -p^0 p^3 & p^1 p^3 & p^2 p^3 & m^2 + (p^3)^2
\end{pmatrix}. \quad (C.22) \]

C.3 \((1, 0) \oplus (0, 1)\) Spinors and new Dirac Gamma Matrices

For spin-1 we can write a new whole set of Gamma matrices by replacing Pauli matrices with spin-1 \(SU(2)\) matrices \((S_1, S_2, S_3)\) as
\[ \gamma_0 = \begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0_3 & -2S_1 \\ 2S_1 & 0_3 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0_3 & -2S_2 \\ 2S_2 & 0_3 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0_3 & -2S_3 \\ 2S_3 & 0_3 \end{pmatrix}, \]
where \(I_3\) is identity \(3 \times 3\) matrix and \(O_3\) is \(3 \times 3\) zero matrix. These Gamma matrices are in chiral representation. In this definition, Gamma matrices are no more part of Clifford algebra since \(\gamma_\mu \gamma_\mu \neq \pm I_6\). Even though interior products are changed \(\gamma_\mu, \gamma_\nu\), the exterior products \([\gamma_\mu, \gamma_\nu]\) remain the same. Our definition of boost stays the same as spin 1/2 case as
\[ R = e^{-\gamma_0 \phi^i/2} = \begin{pmatrix} e^{S_i \phi^i} & 0_3 \\ 0_3 & e^{-S_i \phi^i} \end{pmatrix}. \quad (C.24) \]

We could define spinors in either with vectors or Gamma matrices as Eq. A.9 and \(u(0, \pm) = (I_6 + \gamma_0)(-\gamma_12, \gamma_12/2 \pm i\gamma_12)/8\) and \(u(0, 0) = (I_6 + \gamma_0)(-\gamma_12, \gamma_12/2 + i\gamma_12)/8\).

We can see that energy projection operator \(P_e\) defines particle and anti-particle spinors.

At rest frame it is \(P^e_{\pm} = (1 \pm \gamma_0)/2\), and we can see the \(P^e_+ u(0, \lambda) = u(0, \lambda), P^e_- v(0, \lambda) = 0, P^e_- u(0, \lambda) = 0, \) and \(P^e_- v(0, \lambda) = v(0, \lambda)\).

When we apply the boost, our spinors will be the same as Eq. A.12-Eq. A.13.

The energy projection operator will be
\[ P^e_+ = R(1 + \gamma_0)R^{-1}/2 = \frac{1}{2m^2} \]
\[ \times \begin{pmatrix}
    m^2 & 0 & 0 & 2(p^+)^2 & 2\sqrt{2}p^L p^+ & 2(p^L)^2 \\
    0 & m^2 & 0 & 2\sqrt{2}p^L p^R & 2p^R p^L + 2p^+ p^- & 2\sqrt{2}p^L p^- \\
    0 & 0 & m^2 & 2(p^R)^2 & 2\sqrt{2}p^R p^+ & 2(p^+)^2 \\
    2(p^-)^2 & -2\sqrt{2}p^L p^- & 2(p^L)^2 & m^2 & 0 & 0 \\
    -2\sqrt{2}p^R p^R & 2p^R p^L + 2p^+ p^- & -2\sqrt{2}p^L p^+ & 0 & m^2 & 0 \\
    2(p^R)^2 & -2\sqrt{2}p^L p^R & 2(p^+)^2 & 0 & 0 & m^2
\end{pmatrix}. \quad (C.25) \]

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\[ P_c^- = R(1 - \gamma_0)R^{-1}/2 = \frac{1}{2m^2} \]
\[
\begin{pmatrix}
  m^2 & 0 & 0 & -2(p^+_L)^2 & -2\sqrt{2}p^+_R p^+_L & -2(p^-)^2 \\
  0 & m^2 & 0 & -2\sqrt{2}p^+_R p^+_L & -2p^+_R p^+_L - 2p^+_R p^- & 2\sqrt{2}p^+_R p^- \\
  -2(p^-)^2 & 0 & m^2 & -2(p^+_L)^2 & -2\sqrt{2}p^- p^- & 2(p^-)^2 \\
  2\sqrt{2}p^- p_R & -2p^+_R p^+_L - 2p^+_R p^- & 2\sqrt{2}p^+_R p^- & 0 & m^2 & 0 \\
  -2(p^+_R)^2 & 2\sqrt{2}p^+_R p^+_L & 0 & 0 & m^2 & 0
\end{pmatrix}.
\]

(C.26)

### C.4 Electric and Magnetic Field Function in Momentum Space

We can take the expression in the previous section \( F^{\mu\nu} = (\nabla \times A) \gamma^{\mu\nu} = i(p^\times A) \gamma^{\mu\nu} \). When we combine with \( A^\mu = \epsilon^\mu(p, \lambda)e^{ip.k} \) and use polarization vectors in standard representation which are given by Eq. A.4, then we can write down electric and magnetic field functions as

\[ E(\vec{p}, x) = \frac{i}{2m} \begin{pmatrix} p^0 - (p^1)^2/(p^0 + m) \\ -p^1 p^3/(p^0 + m) \\ -p^2 p^3/(p^0 + m) \end{pmatrix}, \quad E(\vec{p}, y) = \frac{i}{2m} \begin{pmatrix} -p^1 p^2/(p^0 + m) \\ p^0 - (p^2)^2/(p^0 + m) \\ -p^3 p^2/(p^0 + m) \end{pmatrix}, \]

\[ E(\vec{p}, z) = \frac{i}{2m} \begin{pmatrix} -p^1 p^3/(p^0 + m) \\ -p^2 p^3/(p^0 + m) \\ p^0 - (p^3)^2/(p^0 + m) \end{pmatrix}, \]

(C.27)

\[ B(\vec{p}, x) = \frac{i}{2m} \begin{pmatrix} 0 & -p^3 \\ p^0 & 0 \\ p^2 & -p^1 \end{pmatrix}, \quad B(\vec{p}, y) = \frac{i}{2m} \begin{pmatrix} p^3 & 0 \\ 0 & -p^1 \end{pmatrix}, \quad B(\vec{p}, z) = \frac{i}{2m} \begin{pmatrix} -p^2 \\ p^1 \\ 0 \end{pmatrix}. \]

(C.28)

Keep in mind that \( \vec{E} = F^{0i} \) and \( \vec{B} = \epsilon^{ijk} F^{ij} \).
Appendix D

Appendix for Chapter 5

D.1 Kinematics of DVCS and BH Process with Angular Dependence

We can write down some energies and momenta in terms of these Lorentz invariant kinematics which we mentioned in Section 5.5 as

\[ P_1 : q = M q^0 = \frac{Q^2}{2 y_B} \]
\[ P_1 : k = M E = \frac{P_y q}{y} = \frac{Q^2}{2 y_B} \] then \[ q^0 = \frac{Q^2}{2 y_B M} \] and \[ q^3 = \sqrt{Q^2 + (q^0)^2} = \sqrt{Q^2 + \frac{Q^2}{4 y_B^2}}. \]

\[ P_1 : P_2 = E_2 M = M^2 - \Delta^2 / 2 \] then \[ E_2 = M - \Delta^2 / (2 M) \] and \[ |\vec{P}_2| = \sqrt{E_2^2 - M^2} = \sqrt{\Delta^4 / 4 M^2 - \Delta^2}. \]

There is a minimum value for \( \Delta^2 \) since there is an angle dependence between \( q \) and \( q' \). To find \( \Delta_{min} \), first we need to write down \( \Delta \) with \( q \) and \( q' \) as \( \Delta = q - q' \) then we can write \( q' \) as \( q' = -P_2 + P_1 + q = (q^0 + M - E_2, -|\vec{P}_2| \cos \theta_H, -|\vec{P}_2| \sin \theta_H \sin \phi, -|\vec{P}_2| \cos \theta_H + |q^3|) \) then

\[ \Delta^2 = q^2 - 2 q . q' = -Q^2 - 2 q^0 (q^0 + M - E_2) - 2 |q^3| (|\vec{P}_2| \cos \theta_H + |q^3|), \] (D.1)

where we can define \( \Delta = \Delta_{min} \) when \( \Delta^\perp = 0 \) that is \( \sin \theta_H = 0 \) that is also \( \cos \theta_H = 1 \) then

\[ \Delta_{min}^2 = -Q^2 - 2 q^0 (q^0 + M - E_2) - 2 |q^3| (|\vec{P}_2| + |q^3|). \] (D.2)

The calculation of \( k . \Delta \), from the definition of momentum of the interactions we could write

\[ k . \Delta = k . P_2 - k . P_1 = EE_2 - E |\vec{P}_2| \sin \theta_e \sin \theta_H \cos \phi - E |\vec{P}_2| \cos \theta_e \cos \theta_H - \frac{Q^2}{2 y_B} \] (D.3)

Here, we need to find the \( \cos \theta_e \) and \( \cos \theta_H \) expressions first. We will derive these angles from
In order to find
\[ q.k \text{ and } P_2.q \text{ expressions.} \]

\[ q.k = Eq^0 + E|q^3| \cos \theta_e \quad \Rightarrow \quad \cos \theta_e = \frac{q.k - Eq^0}{E \sqrt{(q^0)^2 + Q^2}} = -\frac{Q^2/2 - (q^0)^2/y}{E \sqrt{(q^0)^2 + Q^2}} \]
\[ = -\frac{Q(e^2 y + 2)}{2E\epsilon y \sqrt{1 + \epsilon^2}}, \quad (D.4) \]

\[ P_2.q = E_2q^0 + |\vec{P}_2||q^3| \cos \theta_H \quad \Rightarrow \quad \cos \theta_H = \frac{P_2.q - E_2q^0}{|\vec{P}_2| \sqrt{(q^0)^2 + Q^2}} = -\frac{Q^2 + \Delta^2 + \Delta^2 q^0 / M}{2|\vec{P}_2| \sqrt{(q^0)^2 + Q^2}} \]
\[ = -\frac{Q^2 + \Delta^2 + \Delta^2 Q/M}{2|\vec{P}_2| Q \sqrt{1 + \epsilon^2}}. \quad (D.5) \]

Then
\[ E|\vec{P}_2| \cos \theta_e \cosh \theta_H = -\frac{(y e^2/2 + 1)(-Q^2 + \Delta^2 + 2\Delta^2 x_B/\epsilon^2)}{2y(1 + \epsilon^2)}. \quad (D.6) \]

So the whole expression will look like

\[ k.\Delta = \frac{Q^2}{2y x B M} (M - \frac{\Delta^2}{2M}) - \frac{Q^2}{2y x B} + \frac{(y e^2/2 + 1)(-Q^2 + \Delta^2 + 2\Delta^2 x_B/\epsilon^2)}{2y(1 + \epsilon^2)} + K' \cos \phi \]
\[ = -\frac{2 x B \Delta^2}{Q^2 y e^2} + \frac{(y e^2/2 + 1)(-Q^2 + \Delta^2 + 2\Delta^2 x_B/\epsilon^2)}{2y(1 + \epsilon^2)} + K' \cos \phi \]
\[ = -\frac{Q^2}{2y(1 + \epsilon^2)} \left( 1 - \frac{\Delta^2}{Q^2} \left( 1 + \frac{y e^2}{2} + \frac{y e^2}{2} \right) \right) + K' \cos \phi. \]

In order to find $K'$, we need to find out $\sin \theta_e$ and $\sin \theta_H$ first,

\[ \sin \theta_e = \sqrt{1 - \cos^2 \theta_e} = \sqrt{1 - \frac{Q^2(1 + 2/y e^2)}{4E^2(1/\epsilon^2 + 1)}} = \sqrt{\frac{4E^2 y^2(1 + \epsilon^2) - Q^2(e^2 y^2 + 4y^2 + 4/\epsilon^2)}{4E^2 y^2(1 + \epsilon^2)}} \]
\[ = \frac{Q \sqrt{4 - 4y - \epsilon^2 y^2}}{2E y \sqrt{1 + \epsilon^2}}. \quad (D.7) \]

\[ \sin \theta_H = \sqrt{1 - \cos^2 \theta_H} = \sqrt{1 - \frac{(-Q^2 + \Delta^2 + 2\Delta^2 x_B/\epsilon^2)^2}{4|\vec{P}_2|^2 Q^2(1 + 1/\epsilon^2)}} \]
\[ = \sqrt{\frac{Q^2(\Delta^4/M^2 - 4\Delta^2)(1 + \epsilon^2) - (-Q^2\epsilon + \Delta^2 \epsilon + 2\Delta^2 x_B/\epsilon^2)^2}{4|\vec{P}_2|^2 Q^2(1 + \epsilon^2)}}, \quad (D.8) \]

and

\[ K' = E \sin \theta_e |\vec{P}_2| \sin \theta_H. \quad (D.9) \]
Figure D.1: Bethe Heitler process of incoming photon beam making Compton scattering with lepton and anti-lepton pair with virtual photon exchange with hadronic part.

D.2 Bethe-Heitler Lepton Tensor

\[ L^{\mu\nu} = \epsilon_{\alpha} \epsilon_{\beta} \epsilon^{\mu\alpha} \epsilon^{\nu\beta} = \frac{-e^4 g_{a\beta}}{(2k'.\Delta + \Delta^2)^2} [\bar{u}(k') \gamma^\alpha (k - \Delta + m) \gamma^\nu u(k)] [\bar{u}(k') \gamma^\beta (k' + \Delta + m) \gamma^\nu u(k)]^* \]

\[ - \frac{e^4 g_{a\beta}}{(-2k.\Delta + \Delta^2)^2} [\bar{u}(k') \gamma^\alpha (k - \Delta + m) \gamma^\nu u(k)] [\bar{u}(k') \gamma^\beta (k' + \Delta + m) \gamma^\nu u(k)]^* \]

\[ - \frac{e^4 g_{a\beta}}{(-2k.\Delta + \Delta^2)(2k'.\Delta + \Delta^2)} [\bar{u}(k') \gamma^\alpha (k' + \Delta + m) \gamma^\nu u(k)] [\bar{u}(k') \gamma^\beta (k - \Delta + m) \gamma^\nu u(k)]^* \]

\[ = \frac{e^2}{(2k'.\Delta + \Delta^2)^2} Tr[\gamma^\alpha (k' + \Delta + m) \gamma^\nu (k' + m) \gamma^\alpha (k' + \Delta + m)] \]

\[ - \frac{e^2}{(-2k.\Delta + \Delta^2)^2} Tr[\gamma^\alpha (k' - \Delta + m) \gamma^\nu (k' + m) \gamma^\alpha (k' + m)] \]

\[ - \frac{-e^2}{(2k'.\Delta + \Delta^2)(-2k.\Delta + \Delta^2)} Tr[\gamma^\nu (k' + \Delta + m) \gamma^\alpha (k' + m) \gamma^\nu (k + m) \gamma^\alpha (k + m)] \]

\[ - \frac{-e^2}{(2k'.\Delta + \Delta^2)(-2k.\Delta + \Delta^2)} Tr[\gamma^\alpha (k' - \Delta + m) \gamma^\nu (k' + m) \gamma^\nu (k' + m)]. \]

D.3 Bethe-Heitler Lepton-antilepton Case

In this section, we investigate another BH process where instead of lepton-lepton scattering, we have lepton-antilepton scattering as shown in Figure D.1.

The momentum conservation holds:

\[ p + q' \rightarrow k + k' + p', \quad q = p' - p = q' - k - k', \quad (D.10) \]

where \( q \) is the virtual photon, \( p \) is incoming hadron, \( k \) is lepton, and \( k' \) is anti-lepton. The
lepton current can be written as

\[ p^\mu = \bar{v}(k') \left( ie\gamma^\mu \frac{-\not{k} + \not{q}' - m}{(k - q')^2 - m^2} ie\gamma(q') - ie\gamma(q') \frac{-\not{k}' + \not{q} + m}{(k' - q)^2 - m^2} ie\gamma^\mu \right) u(k), \]

where there is a minus sign between the two propagators because of the charge difference between lepton and anti-lepton. The leptonic tensor is

\[ L^{\mu\nu} = \frac{e^4 e_\alpha^* e_\beta}{(2k.q')^2} [\bar{v}(k') \gamma^\mu (-\not{k} + \not{q}' - m) \gamma^\alpha u(k)]^* [\bar{v}(k') \gamma^\nu (-\not{k} + \not{q}' - m) \gamma^\beta u(k)] \\
+ \frac{e^4 e_\alpha^* e_\beta}{(2k.q')^2} [\bar{v}(k') \gamma^\alpha (-\not{k} + \not{q}' + m) \gamma^\mu u(k)]^* [\bar{v}(k') \gamma^\beta (-\not{k}' + \not{q}' + m) \gamma^\nu u(k)] \\
- \frac{e^4 e_\alpha^* e_\beta}{(2k.q')(2k'.q')} [\bar{v}(k') \gamma^\alpha (-\not{k} + \not{q}' + m) \gamma^\mu u(k)]^* [\bar{v}(k') \gamma^\beta (-\not{k}' + \not{q}' - m) \gamma^\nu u(k)] \\
- \frac{e^4 e_\alpha^* e_\beta}{(2k.q')(2k'.q')} [\bar{v}(k') \gamma^\alpha (-\not{k} + \not{q}' - m) \gamma^\mu u(k)]^* [\bar{v}(k') \gamma^\beta (-\not{k}' + \not{q}' - m) \gamma^\nu u(k)]. \]

We have symmetric and anti-symmetric polarization of the photon beam. The real photon only has two possible polarizations + or - (in circular polarization), there is no longitudinal polarization for the real photon. We can separate these polarizations into two parts as symmetric and anti-symmetric

\[ (\epsilon_\alpha^+ e_\beta^+ + \epsilon_\alpha^- e_\beta^-)/2 = -g_{\alpha\beta}/2, \]

\[ \epsilon_\alpha^+ e_\beta^+ - \epsilon_\alpha^- e_\beta^- = i\epsilon^{\alpha\beta\delta\gamma} \epsilon_\delta (q'.n). \]

where \( n \) is an arbitrary light-like vector which we derived from Eq. 4.80 and only \( g_{\alpha\beta} \) part of symmetric terms contributes because of the transverse property of the photon.

We use the Hadronic tensor that is given in [51] where the same process is investigated and
it is given by
\[ W_{\mu\nu} = \frac{1}{4\pi} \int d^4xe^{ipx} \langle p, s|J^\dagger_\mu(x), J_\nu(0)|p, s \rangle \]  
(D.14)
\[ = -w_1(q^2) \left( g_{\mu\nu} - \frac{p_\mu q_\nu}{q^2} \right) + w_2(q^2) \left( \frac{1}{m^2} \left( p_\mu - \frac{p.q}{q^2} q_\mu \right) \right) \left( p_\nu - \frac{p.q}{q^2} q_\nu \right) \]  
(D.15)
\[ + \frac{i\varepsilon_{\mu\nu\rho\sigma}q^\rho}{m^2} \left\{ S^\sigma \left( g_1(q^2) + \frac{p.q}{m^2} g_2(q^2) \right) - \frac{S.q}{m^2} p^\sigma g_2(q^2) \right\}. \]  
(D.16)

As one can see the anti-symmetric part of the hadronic tensor is \( W_{\mu\nu}^A(+) - W_{\mu\nu}^A(-) \) is
\[ W_{\mu\nu}^A = \frac{i\varepsilon_{\mu\nu\rho\sigma}q^\rho}{m^2} \left\{ S^\sigma \left( g_1(q^2) + \frac{p.q}{m^2} g_2(q^2) \right) - \frac{S.q}{m^2} p^\sigma g_2(q^2) \right\}, \]  
(D.17)
with \( S \) which is the same direction with the hadron momentum \( (p.S = +1) \).

The Kinematics of the photon decay Bethe-Heitler process:

The initial nucleon is at rest \( p^\mu = (m_i, 0, 0, 0) \) and the energy of the virtual photon is \( \nu = p.q/m_i = q^0 \). Moreover, the final nucleon mass is \( W^2 = m_f^2 = (p + q)^2 = m_i^2 + 2m_i\nu + q^2 \) so \( q^0 = \frac{m_f^2 - m_i^2}{2m_i} - \frac{q^2}{2m_i} \) we will call \( \Delta = \frac{m_f^2 - m_i^2}{2m_i} \).

The direction of the photon is in z-direction so \( q^\mu = (K, 0, 0, K) \) where \( K \) is the energy of the photon. Since the initial hadron is at rest, we could define the hadron spin as \( S^\mu = (0, 0, 0, 1) \) and the inner product with other momentums gives us \( S.q = K, S.k = k_z, S.k' = k'_z, \) and \( S.p = 0 \). By taking advantage of the relations of inner products with the real photon \( q' \), we could write \( k.q' = KE - Kk_z, k'.q' = KE' - Kk'_z \) and overall the spin inner product can be expressed as
\[ S.k = k_z = E - \frac{k.q'}{K}, \quad S.k' = k'_z = K - E - \Delta + \frac{q^2}{2m_i} - \frac{k'.q'}{K}, \quad S.q = \Delta - \frac{q^2}{2m_i} + \frac{k.q'}{K} + \frac{k'.q'}{K}. \]  
(D.18)

Then the amplitude becomes
\[ A = L_{A}\mu\nu W_{\mu\nu}^A, \]  
(D.19)
where only the anti-symmetric parts of the leptonic and hadronic tensors contribute since the amplitude of the polarized beam is
\[ A = \frac{d\sigma^\pm}{d\sigma^\mp} - \frac{d\sigma^\mp}{d\sigma^\pm}, \]  
(D.20)
where the arrow denotes the spin directions of the target and the beam. Then we get the same amplitude \( A \) given by Eq. D.19 with \( A \) in [51].