ABSTRACT

KUNDU, LOPAMUDRA. Information-Theoretic Limits on MIMO Antennas. (Under the direction of Dr. Brian L. Hughes.)

Multiple-input-multiple-output (MIMO) technique has become an essential element of the present generation wireless communication standards. Research on MIMO systems has shown that deployment of multiple antennas at both the transmitter and receiver can significantly increase the capacity of wireless systems in the presence of rich multipath, provided the antennas are spaced sufficiently far apart. Mobile transceivers, on the other hand, are constrained in size. Employing MIMO within mobile devices, therefore, requires close spacing between antennas. This leads to signal correlation between received signals and mutual coupling between the antenna elements that can profoundly degrade system performance. Prior works on compact MIMO systems have indicated that capacity loss due to coupling can be mitigated by using sophisticated multi-port matching at the receiver. However, these studies focused primarily on frequency-nonselective (FNS) matching that confers capacity benefit only over a small bandwidth. Since the coupled antennas are known to be highly frequency-selective (FS), it is unclear if the bandwidth limitations of FNS matching are inherent to compact MIMO receivers or whether FNS matching itself is a poor choice for coupled antennas over positive bandwidth.

While numerous studies exist regarding design of optimum matching network, little attention has been paid to the design of antenna array. For a given array, the existing literature suggests how matching network and communication algorithm can be jointly optimized to make the best use of that specific array. But from a more fundamental information-theoretic perspective, it is imperative to ask how to optimize the design of the array itself. In particular, arrays detect signal by observing current induced by incident
fields within an arbitrary conducting volume. Intuitively, this implies that the design of the array should be optimized so that it can capture all the useful information contained in the current within this conducting volume. In the design of wireless transceiver, array optimization therefore deserves equal attention as the design of optimum matching network.

In this dissertation, we address the aforementioned two important aspects of transceiver design, viz. matching network and antenna array. Firstly, we consider a general MIMO receiver front end with closely coupled antenna arrays over positive bandwidth. For this system, we investigate the impact of FS matching on the capacity of compact MIMO systems for specific array configurations. Next, we look into information-theoretic aspects of optimum antenna design for a narrowband system with optimum matching. We consider an arbitrary array formed from a conducting volume and derive bounds on capacity for the array and the underlying conductor. We introduce the notion of aperture capacity and associate it with a system of SNR eigenmodes existing within the aperture. We further show that the capacity of any M-port array formed from an arbitrary aperture is bounded above by that of M best SNR eigenmodes existing therein. Next, we use the theoretical tools developed in this work to study extensively different aspects of antenna design, e.g. array size and shape, various signal propagation environments, matching limitations etc. and the impact of each of these on the capacity of the overall system. Finally we summarize the main contributions of this dissertation and discuss scopes for future work.
Information-Theoretic Limits on MIMO Antennas

by
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To my parents and Sankha
BIOGRAPHY

Lopamudra Kundu received her Bachelor of Science degree with Honors in Physics from University of Calcutta, India in July 2005 and was placed in the First Class. Later, she joined the Department of Radio Physics & Electronics at University of Calcutta, India to pursue her second bachelor’s degree, Bachelor of Technology in Electrical Engineering, which she completed with distinction in July 2008. In August 2010, Lopamudra joined the graduate program in the Department of Electrical and Computer Engineering at North Carolina State University, Raleigh and began working towards her PhD under the advisement of Prof. Brian L. Hughes in Wireless Systems Engineering Lab since June 2011. She obtained her enroute master’s degree from NCSU in May 2012. Her research interest encompasses Information Theory, MIMO System, Broadband Matching, Antenna Design, Digital Signal Processing and Wireless Communication Algorithms. During the course of her PhD, she did summer internship at Microsoft Research Lab, Redmond under the mentorship of Dr. Ranveer Chandra from May 2015 to August 2015 and worked on efficient antenna design for small form factor mobile devices. Upon completion of her dissertation, Lopamudra will join the Platform Engineering Group at Intel, Santa Clara to begin her professional career as 5G/LTE Wireless Systems Engineer.
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“Sometimes our light goes out but is blown into flame by another human being. Each of us owes deepest thanks to those who have rekindled this light.”

-Albert Schweitzer
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Chapter 1

Introduction

With the proliferation of wireless technologies over the past two decades, the demand for broadband wireless data access has been rising exponentially [27]. This overwhelming growth of global mobile data traffic [39] is fueled not only by the advancement of mobile computing with widespread adoption of tablets and smartphones, but also by an increasing need to provide infrastructure for wireless machine-to-machine connections in various emerging applications like cloud computing [2], pervasive computing [115] and internet of things (IoT) [3]. But while the demand for faster data rate is increasing, the window of available spectrum is not. With limited power and bandwidth resources, design of communication systems therefore requires fundamentally new approach to cope with the rising demand for ubiquitous wireless connectivity around the globe.

1.1 Evolution of MIMO

In wireless communication, one of the most efficient ways to increase data rate without consuming additional bandwidth is through smart antenna techniques that use antenna
arrays to improve signal-to-noise ratio (SNR) of wireless links [105] and mitigate co-channel interference through beamforming [145]. In addition, arrays provide space diversity [1, 31, 125] to combat signal fading in multipath environment. During the past decade, multiple-input-multiple-output (MIMO) system, an evolution of smart antenna technique has shown that deployment of antenna arrays at both transmitter and receiver can dramatically improve wireless channel capacity in the presence of rich multipath [94]. Contrary to the traditional view of multipath propagation being an impairment to reliable communication, MIMO exhibits that multipath signaling, in lieu, can greatly improve achievable data rate if appropriate communication structure is employed [112, 146].

The advent of MIMO with the pioneering work of Foschini [44], Foschini and Gans [45], and Telatar [128] spurred an unprecedented surge in research endeavor across industry and academia alike in the late 90’s. Since then, a staggering volume of scholarly articles addressing numerous aspects of MIMO system has enriched the academic research [54, 69, 106, 113]. Alongside, the enormous potential of MIMO has opened up new avenues for industrial applications that could leverage this technique. For example, the high spectral efficiency and link robustness promised by MIMO through multiplexing and diversity schemes [87] have led to its commercial deployment in diverse wireless and cellular communication standards including Wi-Fi (IEEE 802.11n, 802.11ac), UMTS/HSPA+ (3G), WiMAX/LTE (4G), WLAN (IEEE 802.11n) and WMAN/WiMAX (IEEE 802.16e). These standards achieve vastly improved data rate compared to the previous standards by exploiting spatial degrees of freedom through MIMO signaling.

In MIMO system, the two main mechanisms employed to improve wireless system performance are diversity and multiplexing. While capacity gain can be obtained through multiplexing, beamforming combined with powerful coding techniques offer high diversity gain by optimizing received SNR. Space-time and space-frequency coding techniques can
alleviate signal fading by extracting temporal and spectral diversity. Space-time trellis
codes (STTC), for example, can extract remarkable diversity and coding gains [107, 125,
126], with the Achilles’ heel being decoding complexity that may increase exponentially
with transmission rate [100]. An alternative to STTC, namely space-time block codes
(STBC) emerged with Alamouti’s seminal work [1] that obtains full diversity order with
linear receiver complexity for a two-antenna transmit system and was later generalized
for arbitrary number of transmit antennas [127]. Though STBCs achieve full diversity,
they offer inferior performance to STTCs since the latter offer both diversity and coding
gains whereas STBCs do not provide coding gain. A comprehensive study on space-time
coding can be found in [50] and references therein. Spatial multiplexing schemes, pioneered
by Bell Laboratories Layered Space-Time Architecture (BLAST) [44, 148], on the other
hand, can significantly increase the data rate by transmitting independent data streams
per symbol period simultaneously using the dimensions of space and time and decoding
them at the receiver using sophisticated signal processing. The decoding complexity can
be greatly reduced using symbol interference cancellation as shown in [46]. In addition,
MIMO channel can be estimated prior to decoding as well, e.g. the use of training sequence
for channel estimation in Vertical-Blast (V-Blast) to facilitate equalization [90].

Practical signaling strategies for MIMO may not necessarily use either diversity or
multiplexing scheme purely. Instead space-time dimensions can be shared between these
two schemes. Diversity-multiplexing trade-off, therefore, is a fundamental design aspect of
MIMO system and has been extensively studied both from theoretical and practical design
perspectives in the literature [64, 125, 155]. An adaptive trade-off between these two gains
depending on channel conditions has been investigated in this context as well [65]. The
knowledge of channel condition at the link ends of the wireless channel is a key aspect for
deciding optimum transmission strategy for MIMO system. For example, when full channel
state information is available at the transmitter (CSIT), distribution of transmit power using \textit{water pouring} over space-time and/or space-frequency is optimum \cite{53,112,130} and leveraging jointly optimized linear precoder and decoder \cite{116} along with the optimum transmission strategy, capacity greater than or equal to those without CSIT can be achieved.

The channel model for a generic MIMO system with $N$ transmit and $M$ receive antennas is shown in Fig. 1.1 (cf., \cite{53,Ch. 10}, \cite{129, Ch. 7}), where $N$ transmitted symbols $x_1, \cdots, x_N$ are corrupted by fading path gains $h_{ij}$ of the wireless channel and additive noise $n_1, \cdots, n_M$ before reaching the receiver. Fading path gains may be \textit{frequency-flat} (i.e. $h_{ij}$ varies with time only) or \textit{frequency-selective} depending on whether the bandwidth of MIMO channel is small or large relative to the channel’s multipath delay spread \cite{53, Ch.4}. Frequency-selective MIMO channels offer diversity across three dimensions, viz. space, time and frequency, but at the same time suffers from intersymbol interference (ISI) \cite{53, Ch. 10}. In addition to conventional channel equalization approach (similar to SISO.
channels) to deal with ISI for MIMO channels [86], there have been extensive researches conducted on multicarrier modulation [7] and orthogonal frequency division multiplexing (OFDM) [9,67] that offer an alternative approach to complex MIMO channel equalization. MIMO-OFDM [121] now is the primary air interface for 4G and 5G broadband wireless standards.

Although the achievable data rate or channel capacity for MIMO system generally scales with the number of antenna elements, it is ultimately limited by the physical size of the array [73]. Mobile transceivers, for example, are constrained in space and therefore increasing the number of antennas requires close interelement spacing. In the following section, we address potential caveats of MIMO deployment in small, portable and handheld devices (e.g. laptop, cellphones, Wi-Fi access points and the like) that arise due to limited space constraint and briefly review prior works related to MIMO that take into account these issues.

1.2 Compact MIMO: Correlation and Coupling

Earlier studies on MIMO system primarily assumed rich scattering environment where the fading path gains could be modeled as independent random variables, i.e. the fading is spatially uncorrelated. Though this assumption holds true for arrays with elements separated by several wavelengths, such large spacing ceases to be practical in compact mobile devices. Therefore, more detailed MIMO channel models emerged later that incorporated (a) physical scattering through limited angle-of-arrival (AoA) and angle-of-departure (AoD) spread [73,117], and (b) propagation environment through separable correlation [119], rank-deficient channel [22] and virtual channel representation [117]. Extensive research on statistical models of fading path gains and the impact of fading
correlation on MIMO performance over the last two decades has constituted a rich collection of scholarly publications [19, 25, 26, 72, 74, 85, 104, 119, 131].

While these fading correlation studies modeled the propagation environment with intricate details befitting small form factor MIMO deployment [97], they do not encompass all the potential interactions that may occur due to proximity of elements in a compact array. For example, densely packed antenna array causes strong electromagnetic interaction between its elements as shown in Fig 1.2: current flowing through one element induces voltage in its neighbors, commonly known as mutual coupling [81]; the radiation pattern associated with each element becomes distorted due to scattering [120, 136]; antenna input impedance changes [5, Ch. 8], resulting in greater mismatch between the antennas and connected source and load impedances [75]; and the system noise becomes spatially correlated [32]. These effects, in turn, can profoundly impact MIMO performance in terms of received power, diversity gain and system capacity.

Many prior studies have looked into spatial signal correlation and how it affects the capacity of densely packed antenna array ([74] and references therein). Initial studies on the impact of mutual coupling, however, led to some contradictory results. While few
authors showed that mutual coupling, instead of being detrimental, may lead to pattern diversity [140] and in turn, can increase capacity by reducing correlation between channel coefficients [82,122], others either disagreed completely by claiming mutual coupling to be further correlating the spatial channels and therefore degrading capacity [43,68,71,104] or indicated that it depends on antenna spacing [28,84,139,141]. The discrepancies are largely due to difference in assumptions on system setup [33,51,79,81,88,98,143]. In particular, the impact of mutual coupling not only depends on the spacing between elements of the array, but also on the detailed aspects of the receiver front-end, such as antenna, matching network, amplifier properties and noise sources (internal and external) that were neglected in the initial studies.

Later works considered comprehensive circuit models for the receiver front end [98, 99,138,139] that incorporate not only the electric field correlation, but also the impact of antenna radiation pattern, matching network, amplifier characteristics [48,98] and physical model of system noise [32,33,47,48]. A striking conclusion of these works is that capacity loss due to coupling can be mitigated by the use of sophisticated impedance matching networks at the receiver [33,34,78,139]. In the next section, we discuss briefly about different kinds of matching networks and how bandwidth plays a crucial role in determining the optimality of their matching performance.

1.3 Impedance Matching and Bandwidth

A matching network is a special RF circuit that is often used in the receiver front end between the antennas and the amplifiers to alter the array impedance in order to maximize the power transfer [30] or minimize the noise figure [139,144]. Though mutual coupling may degrade the system capacity of densely packed array, appropriate use of antenna
load can undo this detrimental effect [81]. Since, antenna loads are usually fixed, an alternative approach is to use matching network in between antenna and the rest of the RF chain to undo the effect of coupling by diagonalizing the antenna impedance, as considered in [138,139]. Assuming spatially-white noise model, it was demonstrated that multi-port conjugate match can decouple the antenna array and maximize power transfer from antenna to the RF chain downstream. In addition to conjugate match, other matching techniques, e.g. 50 Ω match, self-match, input-impedance match etc. are also investigated in literature [41,78--80,102] and the general conclusion drawn from these works is that multi-port match always outperforms single-port match for narrowband system.

Several authors considered the impact of receiver noise modeling on the performance of compact transceivers [32,33,47,48]. Gans studied sky-noise dominant and amplifier noise dominant scenarios separately and showed that while matching network offers no increase in capacity in the former case [47], they offer significant improvement for the latter scenario [48]. A more general study on noise correlation and its impact on system performance with single and multi-port matching was presented in [32,33]. It was proved that for state-of-the-art receivers, capacity is maximized by multi-port minimum-noise-figure match [33]. Moreover, it was shown that multi-port matching, at least mathematically, can eliminate coupling at all antenna spacings and confer performance close to the ideal uncoupled system [32,33]. From circuit implementation perspective, however, conjugate matching networks often have forbiddingly complex structure due to multi-port interconnects. An alternative to multi-port matching is either to use suboptimal, uncoupled, single-port match [41,70] or to leverage symmetry of certain antenna structures (e.g. circular symmetry, centrosymmetry etc.) and decouple the array prior to matching [123]. Nonetheless, multi-port minimum-noise-figure match provides a
theoretical upper bound on the system performance which can be used as a benchmark to compare the optimality of different transceiver architectures [33].

Though impact of mutual coupling and its mitigation using optimal multi-port matching have been studied in depth, the existing literature primarily addressed the problem under *narrowband* assumption, i.e. the system parameters are assumed to be invariant over the system bandwidth. Strongly coupled antenna arrays, on the other hand, are inherently broadband [79,123] and it is imperative to look into the performance of matching network over the entire non-negligible bandwidth rather than at a single frequency. Lau et. al. observed that the multi-port conjugate match offers optimum performance over only a small bandwidth and collapses the system bandwidth at close antenna spacing [79]. Similar bandwidth behavior was observed in [123] as well. These results undermine the narrowband assumption for coupled MIMO system that most of the previous studies are based on.

Design of matching network for systems with relatively large bandwidth falls under the realm of broadband matching theory. The concept of broadband matching was first introduced by Bode [8] in the study of coupling networks for vacuum tubes, where he addressed the problem of designing a lumped, reciprocal two-port network to match an arbitrary passive load to a resistive generator. Later, Fano’s seminal work [40] extended Bode’s theory by replacing the frequency-dependent load with its Darlington’s equivalent [29] and derived the fundamental gain-bandwidth tradeoff for broadband matching. Youla [152] developed an alternative approach for solving broadband matching problem using complex-normalized scattering matrices [14,153], which is more general than Fano’s approach since it does not require Darlington’s equivalent representation of load. Later on, several authors contributed to the literature of broadband matching theory [18] and expanded its scope to frequency-dependent source [20,21,42], active load [17] and
multi-port networks [142].

While broadband matching theory for uncoupled loads has been studied in details, its extension to coupled loads has remained a relatively uncharted territory. One way to approach the problem is to decouple the loads and then apply single-port broadband matching theory [123, 124]. But this approach often requires detailed analytical model of antenna array impedance to derive matching bounds which may not be available. Alternatively, broadband decoupling and matching networks can be implemented in circuits and bandwidth limitation can be assessed through performance analysis of these circuits instead of deriving the theoretical broadband matching bounds [137, 154]. Nevertheless, these broadband matching networks are usually optimized for various performance metrics like envelope correlation coefficient, input reflection coefficient etc., which may not be capacity-optimal over the bandwidth. In this dissertation, we investigate broadband matching and its impact on capacity for coupled antenna arrays. The key idea is to design frequency-selective versions of matching techniques that are known to be capacity-optimum in the narrowband domain and look into the performance of these matching techniques over wide bandwidth for coupled MIMO systems.

1.4 Design of Antenna Array

Impact of mutual coupling on densely packed antenna elements depends on several aspects of transceiver design, array configuration being one of them. To date, virtually all studies on the capacity of compact MIMO systems have focused on relatively simple, ad-hoc arrays of co-located loops and dipoles or planar structures like patch and PIFA [55, 76, 132]. These conventional antenna arrays, however, may exhibit strong coupling while deployed in small terminals. An alternative approach to implement compact MIMO system is to
use single aperture multi-port antenna that circumvents the need to separate multiple antenna elements within limited space [13,15,83,111]. The common practices for MIMO antenna design with multiple ports are two-fold: either to exploit the inherent symmetry of the aperture geometry (like orthogonally-fed square patch) or to optimize the geometry itself using ad-hoc numerical optimization in order to increase isolation between the ports. None of these design strategies, however, offers physical insight into the underlying principles of antenna operation. Alongside, more systematic antenna design techniques like *theory of characteristic modes* exist in the literature that provide in-depth physical interpretation of radiation and scattering phenomena associated with antenna operation. Developed originally by Garbacz [49] and later extended by Harrington et al. [60--62], theory of characteristic modes is an useful design tool, especially in analysis, synthesis and optimization of antenna and scatterers [38]. Essentially, characteristic modes (or eigen modes) are current modes associated with arbitrarily shaped conducting bodies, having orthogonality properties both as currents on the conducting surface as well as radiation patterns in the far field [60]. As suggested by several authors, these properties of characteristic modes are crucial in improving MIMO antenna design, since by the virtue of orthogonality these modes achieve two most desired properties of a MIMO system, viz. maximum diversity and minimum coupling [16,37,89,96]. However, these eigen modes are not naturally impedance matched to common system impedances. Thus, when a single characteristic mode is purely excited, external matching network is required [16]. Moreover, while the eigenmode based antenna design offers an elegant solution for compact MIMO system, it is unclear that these current modes, when optimally matched to the system impedance, can maximize the overall system capacity. An information-theoretic aspect of multi-port antenna design is still lacking.

In this dissertation, we revisit the design of antenna array from a more fundamental
perspective. In principle, antennas detect transmitted information by indirectly observing the currents induced by incident fields within a conducting volume. We seek to quantify the information content in this induced current, study its variation with shape, size and material of the structure, and use this knowledge to design MIMO system that can capture all the useful information within the conducting volume.

1.5 Overview of the Dissertation

The organization of this dissertation is as follows. In Chapter 2, we revisit the capacity of a MIMO system of coupled receive antennas for a non-negligible bandwidth and explore the potential performance benefits of frequency-selective matching. We also determine the best possible MIMO capacity that can be achieved by physically-realizable, frequency-selective matching and gain insight into the design of matching networks that approach this optimal performance. Chapter 3 outlines an information-theoretic design approach of compact antenna array. We consider arbitrary receive array that observes the currents in a conducting volume $V$ and characterize the maximum narrowband MIMO capacity that can be achieved with this array. We introduce the concept of aperture capacity, which intuitively represents all of the useful information contained in the currents of $V$. We associate this information with a system of unit-power eigencurrents and prove that the capacity of any $M$-port array is always bounded above by the best $M$ eigencurrents. We further show that the aperture capacity of any substructure $\bar{V} \subseteq V$ is bounded by that of $V$, and suggest possible applications to antenna shape optimization. In Chapter 4, we study the performance limit of arbitrary antenna array with the variation of array configuration, direction of propagation and matching limitation. Finally, Chapter 5 summarizes the main contributions of this dissertation and suggests direction for future work.
Chapter 2

Frequency Selective Matching for Compact MIMO Systems

In the previous chapter, we have discussed how mutual coupling among receive antennas can profoundly reduce the capacity of MIMO systems and that the use of sophisticated antenna matching at the receiver can mitigate this capacity loss due to coupling. Further, capacity is maximized by a multi-port matching network that is designed to minimize the amplifier noise-figure [33] (MNF match). For a unilateral amplifier with no input noise [32], this reduces to the well-known multi-port conjugate match (MCM), as shown in Sec. A.1 of Appendix A. Most prior work on impedance matching has employed the narrowband assumption, where system behavior is modeled at a single frequency. While our goal is always to optimize the performance over an entire frequency band, it seems reasonable to employ narrowband models when (a) system parameters are fixed over the bandwidth of interest, and (b) system bandwidth is small enough so that broadband matching theory is not a significant factor. Broadband matching theory [40] establishes
constraints on our ability to realize arbitrary impedance matching networks over a positive bandwidth. Determining how small the bandwidth needs to be in order to effectively remove these constraints is far from clear, however, since there currently exists no general theory of broadband matching for multi-port impedances [79]. Even the single-port theory requires detailed analytical models of array impedances, which are not usually available.

By contrast, there have been few studies of the performance of these matching techniques over positive bandwidths, as noted in Sec. 1.3. In [78]-[80], the authors considered a frequency-nonselective (FNS) matching network for a $2 \times 2$ MIMO system that consists of the MCM match calculated at the center frequency and applied it over a 2.3% fractional bandwidth. This work showed that, for strong coupling, the behavior of the resulting system is highly frequency-selective over the band, and the FNS MCM network offered a significant capacity benefit only over very small bandwidths. Similar results are reported in [123]. These results tend to undermine the narrowband assumption. To date, virtually all studies on the capacity of compact MIMO systems have considered FNS matching. Since coupled antennas are known to be highly frequency-selective [78,123], it is unclear that FNS matching provides the best performance. In particular, it is natural to ask whether the bandwidth limitations of MCM observed in [78] are intrinsic to compact MIMO systems, or do they merely indicate that FNS matching is a poor choice for coupled antennas over larger bandwidths (as conjectured in [80]).

In this chapter, we revisit the capacity of a MIMO system with coupled receive antennas for a non-negligible bandwidth. We seek to evaluate capacity without assuming that the system is frequency-nonselective, and without neglecting the physical constraints imposed by broadband matching theory. In addition, we would like to explore the potential performance benefits of frequency-selective (FS) matching. Our goal is to determine the best possible MIMO capacity that can be achieved by physically-realizable, frequency-
Fig. 2.1: Block diagram of FS MIMO receiver front end

selective matching and also to gain insight into the design of matching networks that approach this optimal performance. The rest of the chapter is organized as follows. In Sec. 2.1, we describe the system model and, in Sec. 2.2, we deduce the formulas for ergodic capacity (with and without channel state information at the transmitter) and outage capacity for this system. Upper and lower bounds on the best possible MIMO capacity are derived in Sec. 2.3 that can be achieved by FS multi-port matching. In Sec. 2.4, we numerically evaluate these bounds for the case of $2 \times 2$ and $4 \times 4$ MIMO systems with coupled dipoles at the receiver. The conclusions are summarized in Sec. 2.5.

## 2.1 System Model

We consider a single user MIMO system [121] in which $n_T$ transmit antennas communicate with $n_R$ receive antennas using OFDM with $K$ frequency-flat sub-channels. The transmit antennas are assumed to be uncoupled and uncorrelated whereas the receive antennas are closely spaced that leads to mutual coupling and fading correlation between the received signals. A circuit model for the receiver is shown in Fig. 2.1. Each of the components is described below.
2.1.1 Antenna Array

An antenna necessarily converts incident electromagnetic field into voltage across its terminals. For closely spaced antennas in an array, the terminal voltage of one antenna depends not only on the incident electric field, but also on the currents flowing through the neighboring antenna elements. The relationship between antenna terminal voltages $v_A^k$ and currents $i_A^k$ is given by,

$$v_A^k = Z_A^k i_A^k + v_o^k$$  \hspace{1cm} (2.1)$$

where, $Z_A^k$ is $n_R \times n_R$ antenna impedance matrix and $v_o^k$ is the open circuit voltage at k-th frequency. The diagonal element $[Z_A^k]_{ii}$ is the self-impedance of i-th antenna in isolation and the off-diagonal term $[Z_A^k]_{ij}$ is the mutual impedance between i-th and j-th antennas in the array. For a frequency-selective MIMO-OFDM system, the open-circuit voltage induced by the transmitted signal $x^k$ can be modeled as

$$v_o^k = H^k x^k + n_o^k$$  \hspace{1cm} (2.2)$$

where $H^k$ is an $n_R \times n_T$ matrix of channel path gains at the k-th frequency and $n_o^k$ is the noise induced in the antenna array by its surrounding environment. We assume the columns of $H^k$ are i.i.d. zero-mean, circularly-symmetric complex random vectors with covariance $\Sigma_R^k$, denoted by $H^k \sim \mathcal{CN}(0, \Sigma_R^k)$. We further assume the noise is isotropic black-body radiation at the standard temperature $T_0$ (290 K), which can be modeled by Twiss’s formula (cf. [32]) $n_o^k \sim \mathcal{CN}(0, 4\bar{k}T_0 B R_A^k)$, where $R_A^k \triangleq \frac{1}{2}[Z_A^k + Z_A^{k\dagger}]$, $\bar{k} = 1.38 \times 10^{-23} \text{ J/K}$, $B$ is the subchannel bandwidth, and $[\cdot]^{\dagger}$ denotes conjugate-transpose.
2.1.2 Matching Network

A matching network is often used between the antennas and the amplifiers to alter the array impedance, usually to maximize the power transfer or minimize the noise figure. A FS matching network can be described by a $2 \times 2$ block impedance matrix at the $k$-th frequency:

$$Z^k_M = \begin{bmatrix}
Z^k_{M11} & Z^k_{M12} \\
Z^k_{M21} & Z^k_{M22}
\end{bmatrix}$$

(2.3)

where, each $Z^k_{Mij}$ is an $(n_R \times n_R)$ matrix. Ideally the network is designed with passive, reactive elements so it is lossless and reciprocal. Here the network is lossless if $Z^k_{M11} = -(Z^k_{M11})^\dagger$, $Z^k_{M22} = -(Z^k_{M22})^\dagger$, $Z^k_{M21} = -(Z^k_{M12})^\dagger$ and reciprocal if $Z^k_{M11} = (Z^k_{M11})^T$, $Z^k_{M22} = (Z^k_{M22})^T$, $Z^k_{M21} = (Z^k_{M12})^T$, where $[\cdot]^T$ denotes transpose of the operand. The cascade of the array and matching network is a linear network that can be represented by a Thevenin-equivalent open-circuit voltage $\bar{v}_o^k$ and impedance $\bar{Z}_A^k$, given by [63]

$$\bar{v}_o^k = M^k v_o^k, \quad Z_A^k = -M^k Z_{M12}^k + Z_{M22}^k$$

(2.4)

where $M^k = Z_{M21}^k (Z_{M11}^k + Z_A^k)^{-1}$. 

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2.1.3 Amplifier

The front-end amplifiers can be modeled by the well known Rothe-Dahlke [114] model, as shown in Fig. 2.1. The $i$-th amplifier consists of a noiseless impedance matrix

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$$

in series with internal noise modeled by a noise voltage source, $v_{ai} \sim \mathcal{CN}(0, 4kT_0Br_a)$, and independent noise current source, $i_{ai} \sim \mathcal{CN}(0, 4kT_0Bg_a)$, where $r_a$ is the equivalent noise resistance of the voltage source and $g_a$ is the equivalent noise conductance of the current source. The correlation impedance $z_{\text{cor}} = r_{\text{cor}} + jx_{\text{cor}}$ controls the correlation between the noise observed at the two ports of the amplifier. The noise statistics of an amplifier can be completely characterized by the parameters $\{r_a, g_a, z_{\text{cor}}\}$. When a source of impedance $z_s = r_s + jx_s$ is connected to the input of the amplifier, the noise factor $F_{\text{amp}}$ is defined as the ratio of the output noise power to the noise power contributed by the source alone. $F_{\text{amp}}$ takes on its minimum value $F_{\text{min}}$ at $z_s = z_{\text{opt}}$ where [32],

$$F_{\text{min}} = 1 + 2\left(g_ar_{\text{cor}} + \sqrt{g_a r_a + (g_a r_{\text{cor}})^2}\right)$$

$$z_{\text{opt}} = \frac{r_a}{g_a} + \frac{r_{\text{cor}}^2}{g_a} - jx_{\text{cor}}$$

2.1.4 Load

The receiver chains downstream from the front-end amplifiers are assumed to be electrically isolated and lumped together as the "load". The communication system observes the voltages across the load at each frequency, $\mathbf{v}_L^i$. The $i$-th load is modeled by an impedance $\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}$.
and any noise is modeled by a noise voltage \( v_{di} \sim \mathcal{CN}(0, 4kT_0Br_d) \), where \( r_d \) is the equivalent noise resistance.

## 2.2 Capacity

As in [32], we can use elementary circuit theory to show that the model of Sec. 2.1 leads to the observed voltage

\[
v^k_L = G^k[M^kH^kx^k + n^k]
\]  

(2.7)

where \( G^k \) is a non-singular matrix given by [32]

\[
G^k = \frac{z_{21}z_L}{z_L + z_{22}} \left( Z_A^k + \frac{z_{11}(z_L + z_{22}) - z_{12}z_{21}}{z_L + z_{22}} I \right)^{-1}
\]  

(2.8)

and \( n^k \sim \mathcal{CN}(0, 4kT_0B\bar{\Sigma}_n^k) \) is the combined noise due to all antenna, amplifier and downstream sources. Here

\[
\bar{\Sigma}_n^k \triangleq \left[ \bar{R}_A^k + r_a I + g_a(z_{22}Z_A^k + z_{\text{cor}} I)(Z_A^k + z_{\text{cor}} I)^\dagger + r_d \bar{K}^k \bar{K}^k \right]
\]  

(2.9)

where \( \bar{R}_A^k \triangleq \frac{1}{2}(Z_A^k + Z_A^k) \) and

\[
\bar{K}^k = \frac{1}{z_{22}z_L} \left[ z_{22}Z_A^k + (z_{11}z_{22} - z_{12}z_{21}) I \right]
\]  

(2.10)

### 2.2.1 Ergodic Capacity with CSIR: Equal Power Allocation

When the channel is perfectly known to the receiver (i.e. channel state information at the receiver or CSIR) and no channel state information (CSI) is available at the transmitter, equal power allocation scheme to all the space-frequency sub-channels is adopted [9].
The overall transmit power remains constant for every channel realization. Under these assumptions, we are interested in the capacity of (2.7), subject to the constraint that transmitted signal is spatially and temporally white, i.e. \( x^k \sim \mathcal{C}\mathcal{N}(0, (P/K_n)\mathbf{I}) \). From (2.2) and (2.4), note that we can write \( M^k \mathbf{H}^k = [\bar{\Sigma}^k]^{1/2} \mathbf{H}_w^k \) where \( \mathbf{H}_w^k \sim \mathcal{C}\mathcal{N}(\mathbf{0}, \mathbf{I}) \) and \( \bar{\Sigma}^k = M^k \Sigma^k_{R} M^k \dagger \).

Thus, the constrained ergodic capacity per unit bandwidth of (2.7) is given by

\[
C_e(\Sigma) = \frac{1}{K} \sum_{k=1}^{K} \mathcal{E} \left[ \log_2 \det(\mathbf{I}_n + \frac{1}{n_T} \mathbf{H}_w^k \Sigma^k \mathbf{H}_w^k) \right]. \tag{2.11}
\]

where \( \mathcal{E}[\cdot] \) denotes the expectation and \( \Sigma^k \) is the signal-to-noise ratio (SNR) matrix at the \( k \)-th frequency

\[
\Sigma^k \equiv \frac{P}{4kT_0KB} [\Sigma^k_{R}]^{1/2} [\Sigma^k_{n}]^{-1} [\Sigma^k_{R}]^{1/2}. \tag{2.12}
\]

### 2.2.2 Ergodic Capacity with Full CSI: Space-Frequency Water Filling

In the case when full CSI is available both at the transmitter and receiver for each channel realization, the power allocation scheme that leads to capacity is well known space-frequency mutual water-filling distribution [107], obtained by aligning the transmit covariance eigenvectors for each sub-carrier frequency \( k \) with those of \( \mathbf{H}_w^k \Sigma^k \mathbf{H}_w^k \). Denoting the ordered eigen values of a Hermitian matrix \( \mathbf{A} \) by \( \lambda_1(\mathbf{A}) \geq \lambda_2(\mathbf{A}) \geq \cdots \geq \lambda_N(\mathbf{A}) \), the full CSI ergodic capacity per unit bandwidth of (2.7) is given by

\[
C_f(\Sigma) = \frac{1}{K} \sum_{k=1}^{K} \sum_{m=1}^{M} \mathcal{E} \left[ \log_2 \{ 1 + P_{m,k} \lambda_{m,k}(\mathbf{H}_w^k \Sigma^k \mathbf{H}_w^k) \} \right]. \tag{2.13}
\]
where, \( M = \min\{n_T, n_R\} \), \( \lambda_{i,m,k} \) is the \( m \)-th ordered eigen mode on the \( k \)-th sub-channel for \( i \)-th channel realization and \( P_{m,k}^{i*} \) is the corresponding water-filling power allocation given by

\[
P_{m,k}^{i*} = \left[ \mu^i - \frac{1}{\lambda_{i,m,k}(H_k^\dagger\Sigma_k H_k)} \right]^+, \tag{2.14}
\]

where \([x]^+ = \max(0, x)\) and the power level \( \mu^i \) for the \( i \)-th channel realization satisfies

\[\sum_k \sum_m \mathbb{E}[P_{m,k}^{i*}] = 1.\]

### 2.2.3 Outage Capacity

For CSIR, since the transmitter cannot adapt the transmission strategy as per the CSI, poor channel states typically reduce ergodic capacity. An alternative notion of capacity with CSIR only is “outage capacity”. This is relevant for slowly fading channels where the instantaneous SNR \( \gamma \) is assumed to be invariant over a large number of symbols. Unlike ergodic capacity, the schemes designed to achieve outage capacity allow data to be lost for deep fades and henceforth offer higher data rate [23]. An important parameter \( P_{out} \), associated with outage capacity defines the probability that the system can be in outage, i.e. the transmitted symbols can not be reliably communicated through the channel. The outage probability is given by

\[P_{out}(\tau) = \mathbb{P}\{\gamma \leq \tau\},\]

where \( \tau \) is a non-negative threshold [32] and \( \mathbb{P}[\cdot] \) denotes probability. For received SNRs below \( \tau \), the system is in outage and the received symbols cannot be successfully decoded. Outage capacity [129] per unit bandwidth of (2.7) is given by

\[C_\tau(\Sigma) = \max\{R \geq 0 : \mathbb{P}[I(\Sigma) \leq R] \leq \tau\} \tag{2.15}\]

where, \( I(\Sigma) = \frac{1}{K} \sum_{k=1}^{K} \log_2 \det \left( I_{n_R} + \frac{1}{n_T} H_k^\dagger\Sigma_k H_k \right) \) [9]
2.3 Capacity Bounds for FS Matching

As noted in Sec. 1.3, most prior works on MIMO systems with coupling have focused exclusively on frequency-nonselective (FNS) matching. Since coupled antennas are known to be highly frequency-selective, it is unclear that FNS matching can provide the best performance. In this section, we explore the potential benefits of frequency-selective (FS) matching. To begin with, we consider FS extensions of the most popular FNS matching techniques considered in the literature [32], which are given by specifying the matching network at each OFDM frequency. However, there is a fundamental distinction between the FS and FNS techniques considered here. In principle, it is possible to build a matching network that approximates any lossless, reciprocal FNS matching network over a sufficiently small bandwidth. For larger bandwidths, however, broadband matching theory [123] imposes limits on the ability to realize arbitrary, frequency-selective matching networks. Thus, we can and will define desirable FS matching characteristics below; however, this does not mean that a physically-realizable matching network exists that can achieve these characteristics.

2.3.1 Upper Bound on Capacity

Various configurations of matching network exist in the literature, as summarized in Sec. 1.3. These matching techniques can be broadly classified into two categories, namely multi-port match and single-port match. Here, we consider two matching networks corresponding to each of these categories, viz. multi-port match for minimum-noise-figure and input-impedance match. Each of these matching techniques are described below.
Multi-port Matching for Minimum Noise Figure

For multi-port minimum-noise-figure (MNF) matching, the network is chosen so that $Z_k^A = z_{opt}I$, where $z_{opt} = r_{opt} + jx_{opt}$ is the source impedance that minimizes the amplifier noise factor (2.6). Impedance matrix of a lossless, reciprocal matching network that accomplishes MNF at a single frequency for FNS MIMO system is given by [33]. A FS version of the same impedance matrix would take the following form:

$$
Z_k^M = j\begin{bmatrix}
-X_k^A & \left(r_{opt}R_k^A\right)^{1/2} \\
\left(r_{opt}R_k^A\right)^{1/2} & x_{opt}I
\end{bmatrix} \tag{2.16}
$$

where, $X_k^A \triangleq \frac{1}{2}(Z_k^A - Z_k^{A\dagger})$ for $k$-th sub-carrier frequency. It was shown in [33] that this matching network maximizes the ergodic capacity of narrowband MIMO systems. Since (2.11) is a sum of the capacities of the narrowband subchannels, it follows that (2.16) maximizes each summand in (2.11) and so maximizes the capacity of our MIMO OFDM system as well. Similar argument holds for (2.13) and (2.15). However, since (2.16) is not necessarily physically realizable, the capacity associated with it must be considered only an upper bound on the potential performance of FS multi-port matching. For this reason, we call the capacity associated with (2.11), (2.13) and (2.15) with matching (2.16) the “MNF Upper Bound”.

Input Impedance Match

The input-impedance match (IIM) is a single-port matching technique that seeks to conjugate match each antenna individually. The optimum closed-form solution ($Z_I$) for a two-element antenna array with symmetric antennas was given in [78]. Following [78], a lossless, reciprocal FS network that achieves the IIM at each OFDM frequency for a
$2 \times 2$ MIMO system is given by

$$Z^k_M = j \begin{bmatrix} -x^k_I & \sqrt{(r_{opt}^k r_f^I)} I \\ \sqrt{(r_{opt}^k r_f^I)} I & x_{opt} I \end{bmatrix}$$

(2.17)

where $Z^k_I = r^k_I + j x^k_I$ and

$$Z^k_I = \sqrt{(R^k_{A11})^2 - (R^k_{A12})^2 + (X^k_{A12})^2 - \frac{(R^k_{A12})^2 (X^k_{A12})^2}{(R^k_{A11})^2} + j \left( \frac{R^k_{A12} X^k_{A12}}{R^k_{A11}} - X^k_{A11} \right)}$$

(2.18)

It was shown in [41] that IIM maximizes the ergodic capacity of a narrowband MIMO system over all single-port matching. Again, since (2.11) is a sum of the capacities of the narrowband subchannels, it follows that (2.17) maximizes each summand in (2.11) and so maximizes the capacity of MIMO OFDM system over all single-port FS matching networks. Once again, however, since (2.17) is not necessarily physically realizable, the capacity associated with it must be considered only an upper bound on the potential performance of FS single-port matching. For this reason, we call the capacity (2.11) with matching (2.17) the “IIM Upper Bound”.

### 2.3.2 Lower Bound on Capacity

As noted earlier, if the MNF network (2.16) is physically realizable, it maximizes the MIMO capacity as in (2.11), (2.13) and (2.15) over all multi-port matching networks. However, since there currently exists no general theory of multi-port broadband matching, there is no known test to determine whether these networks are realizable, and so the associated capacity is merely an upper bound.

We can develop lower bound on the capacity by applying any physically-realizable
Fig. 2.2: Multi-port matching as the cascade of decoupling network and single-port matching

matching network. For the IIM case, any single-port, FS matching network provides a lower bound. Here we consider the simple T-network illustrated in Fig. 2.3, where each component impedance \((A, B \text{ or } C)\) consists of a lumped inductor and capacitor. The impedance of the T-network connected to the \(i\)-th antenna then takes the form

\[
Z^k_{Ti} = j \begin{bmatrix}
X^k_{Ai} + X^k_{Bi} & X^k_{Bi} \\
X^k_{Bi} & X^k_{Ci} + X^k_{Bi}
\end{bmatrix}
\]

(2.19)

where the component reactances at frequency \(k\) are given by

\[
X^k_{ni} = 2\pi f_k L_{ni} - \frac{1}{2\pi f_k C_{ni}}, \quad n = A, B, C
\]

(2.20)

To develop a tight lower bound, we want to choose the parameters \(L_{ni}, C_{ni}\) so that \(Z^k_{Ti}\) is as close as possible to \(Z^k_{Mi}\), the upper bound in (2.17). To this end, we employ the genetic algorithm [147] to choose \(L_{ni}, C_{ni}\) so as to minimize the Frobenius norm summed over all
OFDM frequencies:

$$F(Z_T, Z_M) = \sum_{k=1}^{K} \text{Tr}\{(Z_T^k - Z_M^k)(Z_T^k - Z_M^k)^\dagger\}.$$  \hspace{1cm} (2.21)

where Tr[.] denotes the trace. We call the capacity associated with the optimal T-network the “IIM Lower Bound”.

Similarly, for the MNF case, any realizable FS multi-port network provides a lower bound. But for multi-port matching, the network design is very complicated and the level of difficulty scales with the dimension of MIMO system. However, for circulant symmetric antenna arrays, the design is analytically tractable owing to the symmetry of the array configuration. The impedance matrix $Z_A^k$ of such an array can be expressed as $Z_A^k = Q\Lambda_A^k Q^\dagger$, where Q is a frequency non-selective spatial unitary transformation that decouples the antenna array by diagonalizing its impedance matrix $Z_A^k$ to $\Lambda_A^k$. The diagonal entries of $\Lambda_A^k$ are called *eigen impedance* of the decoupled array.
\[ \Lambda^k_A = \begin{bmatrix}
\lambda^k_1 & 0 & \cdots & 0 \\
0 & \lambda^k_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda^k_{n_R}
\end{bmatrix} \quad (2.22) \]

The set of orthonormal eigenvectors for \( Z^k_A \) are given by the columns of \( Q \), which for an UCA with \( n_R \) antenna elements takes the following form:

\[ Q = \frac{1}{\sqrt{n_R}} \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & \alpha & \cdots & \alpha^{(n_R-1)} \\
1 & \alpha^2 & \cdots & \alpha^{2(n_R-1)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(n_R-1)} & \cdots & \alpha^{(n_R-1)(n_R-1)}
\end{bmatrix} \quad (2.23) \]

where \( \alpha = e^{-j2\pi/n_R} \). Physically, \( Q \) represents \( n_R \)-point DFT/IDFT in space [124]. One of the frequently used RF networks to implement spatial FFT/IFFT matrices in circuits is Butler matrix [12]. For UCA with \( n_R = 2 \) and impedance matrix of the form [124]

\[ Z^k_A = \begin{bmatrix}
z^k_{11} & z^k_{12} \\
z^k_{12} & z^k_{11}
\end{bmatrix} \quad (2.24) \]

the Butler matrix is given by,

\[ Q = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix} \quad (2.25) \]
with the associated eigen impedances as:

\[ \lambda_1^k = z_{11}^k + z_{12}^k, \quad \lambda_2^k = z_{11}^k - z_{12}^k \] (2.26)

Similarly, for a circular array with \( n_R = 4 \) and impedance matrix of the form [124]

\[ Z_A^k = \begin{bmatrix}
  z_{11}^k & z_{12}^k & z_{13}^k & z_{14}^k \\
  z_{14}^k & z_{11}^k & z_{12}^k & z_{13}^k \\
  z_{13}^k & z_{14}^k & z_{11}^k & z_{12}^k \\
  z_{12}^k & z_{13}^k & z_{14}^k & z_{11}^k 
\end{bmatrix} \] (2.27)

\[ Q \text{ takes the form:} \]

\[ Q = \frac{1}{2} \begin{bmatrix}
  1 & 1 & 1 & 1 \\
  1 & -j & -1 & j \\
  1 & 1 & 1 & -1 \\
  1 & j & -1 & -j 
\end{bmatrix} \] (2.28)

and the associated eigen impedances are given by

\[ \lambda_1^k = z_{11}^k + z_{12}^k + z_{13}^k + z_{14}^k, \quad \lambda_2^k = (z_{11}^k - z_{13}^k) - j(z_{12}^k - z_{14}^k) \]
\[ \lambda_3^k = z_{11}^k - z_{12}^k + z_{13}^k - z_{14}^k, \quad \lambda_4^k = (z_{11}^k - z_{13}^k) + j(z_{12}^k - z_{14}^k) \]

In particular for UCA, \( z_{12}^k \cong z_{14}^k \) which simplifies the eigen impedances to

\[ \lambda_1^k = z_{11}^k + 2z_{12}^k + z_{13}^k, \quad \lambda_2^k = z_{11}^k - z_{13}^k, \quad \lambda_3^k = z_{11}^k - 2z_{12}^k + z_{13}^k, \quad \lambda_4^k = \lambda_2^k \] (2.29)
Several authors have shown that Butler networks can be realized quite accurately over wide bandwidths ([93] and references therein). Assuming a perfect Butler network exists over the bandwidth of our system, $Z^k_A$ with circulant symmetry can thus always be simultaneously diagonalized at all frequencies and the multi-port matching network can be replaced by a bank of uncoupled single-port matching networks, i.e. $Z^k_M \triangleq \Lambda^k_M$ where,

$$
\begin{bmatrix}
\Lambda^k_{M11} & \Lambda^k_{M12} \\
\Lambda^k_{M21} & \Lambda^k_{M22}
\end{bmatrix}
= j
\begin{bmatrix}
-\Im \Lambda^k_A \\
(r_{opt} \Re \Lambda^k_A)^{1/2}
\end{bmatrix}
\begin{bmatrix}
\Im \Lambda^k_A & (r_{opt} \Re \Lambda^k_A)^{1/2} \\
1/2 & x_{opt}
\end{bmatrix}
$$

where $\Im \Lambda^k_A \triangleq \frac{1}{2j}(\Lambda^k_A - \Lambda^k_A^\dagger)$ and $\Re \Lambda^k_A \triangleq \frac{1}{2}(\Lambda^k_A + \Lambda^k_A^\dagger)$ respectively. Therefore, the input impedance $\bar{Z}^k_A$ seen looking into the matching network from its output port (as given in (2.4)) now becomes:

$$
\bar{Z}^k_A = \left( \Lambda^k_{M22} - \Lambda^k_{M21}(\Lambda^k_A + \Lambda^k_{M11})^{-1}\Lambda^k_{M12} \right)
$$

Fig. 2.2 shows how multi-port matching can be converted to a bank of uncoupled, single-port matching networks in cascade with a Butler network for antenna array with circulant symmetry. The two-port impedance matrix $Z^k_{Mi}$ of each of these $n_R$ single-port matching networks has the general form

$$
\begin{bmatrix}
Z^k_{M11i} & Z^k_{M12i} \\
Z^k_{M21i} & Z^k_{M22i}
\end{bmatrix}
= j
\begin{bmatrix}
-x^k_i \\
\sqrt{(r_{opt} r^k_i)^{1/2}}
\end{bmatrix}
\begin{bmatrix}
\sqrt{(r_{opt} r^k_i)^{1/2}} \\
x_{opt}
\end{bmatrix}
$$

where $\lambda^k_i = r^k_i + jx^k_i \quad \forall i = 1, \cdots, n_R$ is the $i$-th eigen impedance at $k$-th frequency, as defined in (2.22). The FS MNF match in equation (2.16) thus can be simplified to FS single-port match for circular array. We can now proceed as in the IIM case: we use
genetic algorithm to choose the parameters of a T-network to minimize (2.21), where \( Z_k^k \) is now obtained by replacing \( Z_k^k \) with \( \Lambda_k^k \) in (2.16), as shown in (2.30). We call the capacity associated with the optimal T-network the “MNF Lower Bound”.

2.4 Numerical Results

We now evaluate the capacity bounds for FS matching given in the previous section for the example of a MIMO system with \( n_T = n_R = \{2, 4\} \) antennas. For comparison, we also present results for the conventional FNS matching considered in prior literature. These FNS networks may not be realizable over wide bandwidths, but provide an indication of the relative merits of FS and FNS matching over smaller bands. The array configurations we consider here are uniform circular array (UCA) and uniform linear array (ULA) shown in Figs. 2.4a and 2.4b respectively.

2.4.1 System Parameters

We assume the transmit antennas are uncoupled, and the receive antennas are half-wavelength dipoles with inter-element spacing \( d \), where \( 0.1 \lambda \leq d \leq 1.0 \lambda \). Here \( \lambda \) is the wavelength corresponding to carrier frequency \( f_c = 900 \) MHz. The OFDM system consists
of $K = 64$ frequency-flat sub-carriers spread over a fractional bandwidth of 10% around $f_c$.

We consider a two-dimensional propagation model in which the received signal field consists of a large number of vertically-polarized plane waves that arrive uniformly from all directions in the plane orthogonal to the dipoles. We further assume fading is quasi-static, so the channel is constant over the duration of each OFDM symbol. Under these assumptions, the fading covariance in (2.2) is given by [32]

$$[\Sigma_k]_{ij} = \frac{1}{2\pi} \int_0^{2\pi} g_i^k(\phi) g_j^k(\phi)^* e^{j2\pi(i-j)\frac{d}{\lambda_k} \cos \phi} d\phi \quad (2.33)$$

where $\lambda_k$ is the wavelength corresponding to $f_k$ and $g_i^k(\phi)$ is the open-circuit voltage induced at frequency $f_k$ in the $i$-th antenna by a zero-phase, unit-amplitude plane wave arriving from azimuthal angle $\phi$.

The receive antenna impedance $Z_k^A$ and radiation patterns $g_i^k(\phi)$ were evaluated numerically using the Numerical Electromagnetics Code (NEC) [11], a well-known Method-of-Moments (MoM) program. Each wire dipole is $10^{-5}\lambda$ thick and divided into 25 computational segments. For each inter-element spacing $d$, the patterns $g_i^k(\phi)$ were evaluated in NEC for 32 equally-spaced angles and the results were used to numerically calculate the integral (2.33).

The amplifier used in this study is an economical SiGe, heterojunction bipolar transistor (HBT) low-noise amplifier (LNA) [92] designed for use in the cellular band. In high-gain mode with $f_c = 900$ MHz and $R_{\text{bias}} = 510$ $\Omega$, its impedance matrix and Rothe-Dahlke
noise parameters [114] are as follows

\[
\begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix} = \begin{bmatrix}
35.7 \angle -82.0^\circ & 2.74 \angle 91.8^\circ \\
325 \angle 119^\circ & 46.1 \angle -23.3^\circ
\end{bmatrix} \Omega
\]

and \( r_a = 9.45 \Omega, \ g_a = 3.24 \text{ mS}, \ z_{\text{cor}} = 35.3 \angle -114^\circ \). The amplifier is nearly unilateral \((z_{21} \gg z_{12})\) and \( F_{\text{min}} = 1.04 \text{ dB} \) is quite low, which makes it ideal for front-end amplifier. The downstream equivalent noise resistance is taken to be \( r_d \approx 240 \ \Omega \) [32] and the load resistance \( z_L \) is taken to be equal to \( r_d^* \) to ensure maximum power transfer to the load.

### 2.4.2 Multi-Port versus Single-Port Match

We would like to explore the potential performance benefits of employing FS matching. We begin with \( n_T = n_R = 2 \) and look into the effect of frequency-selective matching on ergodic capacity for both multi-port and single-port matching architectures. To this end, we plot in Fig. 2.5 the MNF and IIM Upper Bounds on capacity \( (C_e(\Sigma) \text{ in (2.11)}) \) versus inter-element spacing \( d/\lambda \) for an SNR of 20 dB. These bounds correspond to the FS
matching networks (2.16) and (2.17), respectively. For comparison, we also present the capacity obtained by using the conventional FNS versions of these matching techniques and, as a baseline, the MIMO capacity for uncoupled antennas \( (d/\lambda \gg 1) \). If frequency \( k_c \) corresponds to the carrier frequency, i.e. \( f_{k_c} = f_c \), then the FNS versions can be obtained by replacing \( k \) with \( k_c \) everywhere on the right-sides of (2.16) and (2.17).

For single-port matching, note that FS matching confers little benefit for \( d > 0.2\lambda \). For strong coupling \( (d < 0.1\lambda) \), however, the IIM Upper Bound exceeds the capacity of FNS IIM matching by \( 5 - 12\% \), which suggests that a modest increase in capacity is possible if (2.17) is realizable. For multi-port matching, the potential performance gains are even greater: For \( d < 0.2\lambda \), the MNF Upper Bound exceeds the capacity of FNS MNF matching by \( 5 - 40\% \), which suggests significant gains in performance are possible if (2.16) is realizable. In fact, the MNF Upper Bound even exceeds the capacity of an uncoupled MIMO system. These benefits vanish for \( d > 0.3\lambda \), where coupling is less significant.

These results suggest that FS matching has the potential to significantly boost MIMO capacity in the presence of strong coupling, provided the networks in (2.17) and (2.16)
are realizable. As noted earlier, however, we do not have a test to determine when this is possible. For this reason, we also derived capacity lower bounds that are obtained by attempting to fit a simple T-network, which is clearly realizable, to the networks in (2.17) and (2.16). In Fig. 2.6a, we plot the MNF and IIM Upper Bounds for an SNR of 20 dB together with the corresponding MNF and IIM Lower Bounds obtained by fitting the T-networks. For single-port matching, note that the IIM upper and lower bounds coincide everywhere to within the thickness of the plot lines; thus, the IIM Upper Bound can essentially be achieved for this example with a very simple FS matching network of the form given in Fig. 2.3. Similarly, for multi-port matching, the MNF Upper and Lower Bounds are also very close for all but the smallest $d$. Here again, most of the performance gains promised by the MNF Upper Bound can be achieved for this example with a Butler matrix and a bank of simple T-networks. Tables 2.1 and 2.2 show matching network parameters $L_n, C_n$ evaluated using genetic algorithm for MNF and IIM matches with $0.05 \leq d/\lambda \leq 0.55$.

Fig. 2.6b compares the FS MNF and IIM capacity lower bounds with the capacities of the FNS versions of these matching techniques as a function of SNR for $d/\lambda = 0.05$. For single-port matching, the capacity of FS IIM increases with SNR and is greater than the capacity of FNS IIM for both low and high SNRs. For multi-port matching, the impact of SNR is more striking, with the difference in capacities between FS and FNS matching increasing from 0.6 bps/Hz at 0 dB to 1.6 bps/Hz at 10 dB. These results suggest that FS matching can offer greater benefits at higher SNR. Interestingly, for SNR $\geq 5$ dB the frequency-selective single-port IIM match outperforms the frequency-nonselective multi-port MNF match. This contrasts with the narrowband case, where MNF matching optimizes capacity and thus outperforms all single-port matching techniques.
2.4.3 Uniform Circular Array (UCA)

So far, we have looked into FS MNF and FS IIM matching techniques for \( n_T = n_R = 2 \) antenna array and observed that FS MNF match outperforms FS IIM match for any SNR level or antenna inter-element spacing. Next, we explore the potential performance benefits of FS MNF for arrays with higher dimension. Since FS MNF match simplifies to FS single-port match for arrays with circulant impedance matrix and therefore becomes easily implementable in practice, we consider uniform circular array (UCA) in this section with \( n_T = n_R = \{2, 4\} \) and look into the effect of FS MNF match on both ergodic and outage capacities for \( 0.1 \leq d/\lambda \leq 1.0 \) and \( 0 \text{ dB} \leq \text{SNR} \leq 20 \text{ dB} \).

Ergodic Capacity

![Graph of Ergodic Capacity vs. d/λ (UCA)](image)

To this end, we plot in Figs. 2.7a and 2.7b the MNF Upper Bounds on ergodic capacity versus inter-element spacing \( d/\lambda \) for both CSIR \((C_e(\Sigma) \text{ in (2.11)})\) and full CSI
\( (C_f(\Sigma) \text{ in (2.13)}) \) with an SNR of 20 dB. These bounds correspond to the FS matching network (2.16). For comparison, we also present the ergodic capacity obtained by using the conventional FNS version of this matching technique and, as a baseline, the MIMO ergodic capacity for uncoupled antennas \((d/\lambda \gg 1)\), similar to the plots in the previous section. For both CSIR and full CSI, note that FS matching confers little benefit for \( d \geq 0.2\lambda \) with \( n_R = 2 \) and \( d \geq 0.3\lambda \) with \( n_R = 4 \) respectively, i.e. when the antennas are less coupled. However, for strong coupling \((d \leq 0.2\lambda)\), FS MNF upper bound exceeds the capacity of FNS MNF match by up to 16\% for CSIR and 15\% for full CSI with \( n_R = 2 \). For larger antenna array \((n_R = 4)\), these numbers increase to 35\% for CSIR and 28\% for full CSI. This suggests that significant gains in performance are possible to achieve if (2.16) is realizable. The benefit of using FS matching scales with the dimension of the antenna array as well since coupling between antenna elements is stronger for larger arrays, making the array more frequency-selective at small inter-element spacing. These benefits vanish for \( d > 0.3\lambda \), where coupling is less significant and system parameters are less frequency-selective.

In Figs. 2.7a and 2.7b, capacity lower bounds are plotted as well along with the upper bounds and capacity for FNS match. For \( n_R = 2 \), note that the upper bound can be essentially achieved for this example with very simple FS matching network of the form shown in Fig. 2.3 for both CSIR and full CSI scenarios. For larger antenna array \((n_R = 4)\), the MNF upper and lower bounds are also very close for all but the smallest \( d \) \((d = 0.1\lambda)\), where the capacity lower bound falls short of upper bound by about 6.6\% and 6.1\% for CSIR and full CSI respectively. Again, most of the performance gain promised by the MNF upper bound can be achieved for this example with a wideband Butler network and a bank of simple T-networks. It can be noted that the capacity performance for using FS match is close to identical for CSIR and full CSI scenarios, viz. 21.6 and 21.82 bps/Hz
respectively. However, since capacity for FNS match in full CSI scheme is more than that in CSIR, the relative merit of FS match is greater for CSIR compared to full CSI. For example, with $n_R = 4$, use of FS match increases capacity by 26% for CSIR, whereas for full CSI the increase in capacity is 20%.

Figs. 2.8a and 2.8b show the variation of ergodic capacity upper and lower bounds for FS MNF match along with FNS MNF match versus SNR at the closest antenna spacing ($d = 0.1\lambda$) with CSIR and full CSI respectively. For both CSIR and full CSI, ergodic capacity increases with SNR for $n_R = \{2, 4\}$. At low SNR, however, the relative benefit of FS match (lower bound) over FNS match is greater compared to high SNR. For CSIR, FS match (lower bound) outperforms FNS match by $14 - 32\%$ for $n_R = 2$ and $26 - 44\%$ for $n_R = 4$ for the SNR range between 0 dB and 20 dB, the relative merit being higher at low SNR. For full CSI, it is $14 - 23\%$ for $n_R = 2$ and $20 - 29\%$ for $n_R = 4$ respectively with the similar SNR range. The relative merit of FS match is greater for CSIR compared to full CSI for both high and low SNRs, though ergodic capacity (in bps/Hz) for full CSI
is always greater than that of CSIR for both FS and FNS match at all SNRs. While the relative performance gain for FS match increases with array size, it curbs with increasing SNR. Therefore, FS match confers greater relative benefit compared to FNS match at low SNR for both CSIR and full CSI scenarios. Table 2.3 enlists few matching network parameters $L_{ni}, C_{ni}$ evaluated using genetic algorithm for MNF match with $n_R = 4$ UCA.

**Outage Capacity**

![Outage capacity versus $d/\lambda$](image1)

![Outage capacity versus SNR](image2)

Fig. 2.9: Outage capacity for UCA

For CSIR, along with the study of ergodic capacity variation with inter-element spacing, outage capacity ($C_\tau(\Sigma)$ in (2.15)) versus $d/\lambda$ is also plotted in Fig. 2.9a for SNR = 20 dB and $P_{out} = 0.1$. Similar to ergodic capacity, FS match can offer significant performance benefit over FNS match in terms of outage capacity as well, the largest relative merit being 14% for $n_R = 2$ and 35% for $n_R = 4$ respectively at closest antenna
spacing, i.e. $d/\lambda = 0.1$. The outage capacity predicted by the upper bound can be achieved within the thickness of the plot by using a simple class of T-network for $n_R = 2$ and except for the smallest inter-element spacing ($d/\lambda = 0.1$), the upper and lower bounds on outage capacity closely coincide for larger antenna array ($n_R = 4$) as well.

Fig. 2.9b shows the variation of outage capacity upper and lower bounds for FS MNF match along with FNS MNF match versus SNR at the closest antenna spacing ($d = 0.1\lambda$) and for $P_{out} = 0.1$. Alike ergodic capacity, outage capacity scales with SNR for $n_R = \{2, 4\}$. At low SNR, however, the relative benefit of FS match (lower bound) over FNS match is greater compared to high SNR. FS match (lower bound) outperforms FNS match by $14 - 33\%$ for $n_R = 2$ and $26 - 45\%$ for $n_R = 4$ for the SNR range between 0 dB and 20 dB. The relative merit of FS match over FNS match is greater in terms of outage capacity compared to ergodic capacity with CSIR, for both high and low SNRs. While the relative performance gain for FS match increases with array size, it curbs with increasing SNR. Therefore, FS match confers greater relative benefit compared to FNS match at low SNR in terms of outage capacity, a trend similar to ergodic capacity as depicted before.
Fig. 2.11: Ergodic capacity for ULA

Fig. 2.10 plots outage capacity versus $P_{\text{out}}$ for the closest antenna inter-element spacing ($d/\lambda = 0.1$) and SNR = 20 dB. For $n_R = 2$, the relative merit between FS and FNS match decreases from 14.4% to 13.8% with increasing $P_{\text{out}}$, whereas for $n_R = 4$, it remains nearly the same at 26%. Therefore, under strong coupling, the relative gain in outage capacity by using FS match (lower bound) over that of FNS match is invariant across the range of outage probability $0.1 < P_{\text{out}} < 0.9$.

2.4.4 Uniform Linear Array (ULA)

Owing to the circulant symmetry of the impedance matrix, an UCA (or any circulant symmetric matrix in general) can be easily decoupled using frequency-independent spatial-unitary transformation similar to (2.23) and therefore, the analysis of broadband matching is considerably simplified. In case of a uniform linear array (ULA), however, this circular symmetry does not exist any more. Instead, $Z_A^k$ of a linear array is centrosymmetric in structure. Though an unitary-transformation for centrosymmetric impedance matrix exists
at each data-point over the frequency range, it is frequency-dependent in general [124]. This makes the matching problem much more complicated for general antenna array without circular symmetry. In particular, for ULA, the transformation (2.23) can only partially decouple the antenna array due to the centrosymmetry, unlike the perfect decoupling of UCA. In this case, we do not get diagonal impedance matrix like (2.22) using a frequency-invariant decoupling network like Butler matrix. But nonetheless, there remains fewer number of off-diagonal elements in $Z_A^k$ if Butler network is applied to linear array, indicating partial reduction in coupling, as shown in Sec. A.2 of Appendix A. Hence, it is worthwhile to explore the impact of this “partial-decoupling”, if any, on the broadband performance of linear array. In particular, we apply Butler network in (2.23) to a uniform linear array and consider only the diagonal elements of $Z_A^k$ post-decoupling, ignoring the non-zero off-diagonal terms. Next, we try to optimize a bank of FS single-port matching networks similar to (2.19) to match the eigen impedances, i.e. the diagonal terms in $Z_A^k$ post-decoupling. The optimized T-network parameters as formulated in (2.20)
are plotted in Figs. 2.12 and 2.13 for ULA with \( n_R = 2 \) and 4 respectively. Note that, for \( n_R = 2 \), an ULA is necessarily an UCA and hence is perfectly decoupled by Butler network whereas for \( n_R \geq 3 \) ULA ceases to be circulant symmetric.

To this end, we plot ergodic capacity \( (C_e(\Sigma) \text{ in (2.11)}) \) versus \( d/\lambda \) for ULA in Fig. 2.11a. Similar to the ergodic capacity plots for UCA in Sec. 2.4.3, both the upper and lower bounds on capacity using FS match along with the capacity for conventional
FNS MNF match are plotted in Fig. 2.11a for ULA, and as a baseline, the capacity under the condition of no coupling is shown as well. For \( n_R = 2 \), an antenna array is always circulant symmetric and hence, the capacity performance for both FS and FNS MNF is identical to that of UCA plotted in Fig. 2.7a. For larger ULA \( (n_R = 4) \), the gap between the plot-lines for FS MNF lower bound and FS MNF upper bound is worse compared to 4-element UCA. This is expected since ULA is not perfectly decoupled as UCA with \( n_R \geq 3 \). In particular, the FS MNF lower bound falls short of capacity upper bound by up to 33%, whereas for UCA, the difference between the bounds is maximum 7%, as can be seen in Fig. 2.7a. Nonetheless, it is interesting to note that even though ULA is not perfectly decoupled, single-port FS match based on only the diagonal elements of \( Z_k^A \) post decoupling confers significant capacity benefit over the conventional FNS match! Under strong coupling \( (d < 0.2\lambda) \), there is a considerable \( 20 - 24\% \) increase in capacity using FS match over FNS match. Therefore, single port FS MNF match can offer significant capacity enhancement over conventional FNS match for non-circular antenna arrays as well even though decoupling of antenna array over broad bandwidth is not feasible.

Fig. 2.11b shows the variation of ergodic capacity upper and lower bounds for FS MNF match along with FNS MNF match as a function of SNR at the closest antenna spacing \( (d = 0.1\lambda) \). Alike UCA, ergodic capacity for ULA scales with SNR for \( n_R = \{2, 4\} \) as well. At low SNR, the relative benefit of FS match (lower bound) over FNS match is greater compared to high SNR, a trend similar to UCA. FS match (lower bound) outperforms FNS match by \( 15 - 56\% \) for \( n_R = 4 \) ULA in the SNR range between 0 dB and 20 dB. While the relative performance gain for FS match increases with array size, it curbs with increasing SNR. Therefore, FS match confers greater benefit at low SNR similar to UCA.
2.5 Conclusions

Based on the results in previous section, we can offer several important conclusions: First, FS matching can significantly increase MIMO capacity in the presence of strong receiver coupling, for both UCA and ULA. Second, prior studies of MIMO capacity, which have exclusively assumed FNS matching, may therefore significantly underestimate capacity. For the particular examples of \((2 \times 2)\) and \((4 \times 4)\) systems of UCA, we found that FS MNF match can enhance the capacity achievable with the “decoupling and matching” strategy proposed in this chapter by up to 40%. Moreover, while the relative performance of FS match compared to FNS match increases with antenna array size, the difference between upper and lower bounds on capacity broadens under strong coupling as well for larger antenna array. This indicates that more complicated matching technique is possibly required to fill in the gap between the capacity bounds, since the antenna characteristics become exceedingly more frequency selective with increase in array dimension. Third, though the FS matching strategy presupposes perfect decoupling which is frequency-invariant only for circulant symmetric matrix, this strategy nonetheless may offer potential benefit for non-circular antenna arrays as well. In particular, for an example of \((4 \times 4)\) ULA, the ergodic capacity achievable with FS MNF match is up to 50% more than that by FNS match under strong coupling. Finally, these gains can be achieved by a very simple class of FS matching networks that are easily realized in hardware.
Table 2.1: Values of L and C for FS MNF match (lower bound)

<table>
<thead>
<tr>
<th>d/λ</th>
<th>Eigen impedance</th>
<th>(L_A) (nH)</th>
<th>(C_A) (pF)</th>
<th>(L_B) (nH)</th>
<th>(C_B) (pF)</th>
<th>(L_C) (nH)</th>
<th>(C_C) (pF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>(λ_1)</td>
<td>107.88</td>
<td>0.354</td>
<td>21.22</td>
<td>6.207</td>
<td>18.66</td>
<td>3.820</td>
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<tr>
<td></td>
<td>(λ_2)</td>
<td>61.89</td>
<td>0.495</td>
<td>2.66</td>
<td>28.153</td>
<td>0.006</td>
<td>7.533</td>
</tr>
<tr>
<td>0.1</td>
<td>(λ_1)</td>
<td>96.42</td>
<td>0.389</td>
<td>20.34</td>
<td>6.72</td>
<td>17.51</td>
<td>4.156</td>
</tr>
<tr>
<td></td>
<td>(λ_2)</td>
<td>79.41</td>
<td>0.407</td>
<td>5.45</td>
<td>13.493</td>
<td>2.65</td>
<td>5.99</td>
</tr>
<tr>
<td>0.15</td>
<td>(λ_1)</td>
<td>88.42</td>
<td>0.407</td>
<td>18.54</td>
<td>9.036</td>
<td>15.52</td>
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<tr>
<td></td>
<td>(λ_2)</td>
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<td>0.389</td>
<td>8.84</td>
<td>7.569</td>
<td>5.36</td>
<td>4.934</td>
</tr>
<tr>
<td>0.2</td>
<td>(λ_1)</td>
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<td>0.424</td>
<td>17.29</td>
<td>10.84</td>
<td>14.35</td>
<td>5.553</td>
</tr>
<tr>
<td></td>
<td>(λ_2)</td>
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<td>0.354</td>
<td>10.79</td>
<td>6.861</td>
<td>8.001</td>
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<td>79.58</td>
<td>0.442</td>
<td>14.87</td>
<td>26.225</td>
<td>12.38</td>
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<td></td>
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<td>5.234</td>
<td>10.965</td>
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<td>9.125</td>
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<tr>
<td></td>
<td>(λ_2)</td>
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<td>0.336</td>
<td>15.92</td>
<td>4.509</td>
<td>13.27</td>
<td>3.13</td>
</tr>
<tr>
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<td>(λ_1)</td>
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<td>(C_A) (pF)</td>
<td>(L_B) (nH)</td>
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Table 2.3: Values of L and C for FS MNF match (lower bound) with \( n_R = 4 \) UCA

<table>
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<tr>
<th>d/( \lambda )</th>
<th>Eigen impedance</th>
<th>( L_A ) (nH)</th>
<th>( C_A ) (pF)</th>
<th>( L_B ) (nH)</th>
<th>( C_B ) (pF)</th>
<th>( L_C ) (nH)</th>
<th>( C_C ) (pF)</th>
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<td>0.1</td>
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<td>22.11</td>
<td>13.93</td>
<td>1.27</td>
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<td>( \lambda_2, \lambda_4 )</td>
<td>93.38</td>
<td>0.371</td>
<td>11.85</td>
<td>5.80</td>
<td>0.82</td>
<td>11.06</td>
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<td>( \lambda_3 )</td>
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<td>6.74</td>
<td>11.51</td>
<td>1.83</td>
<td>11.62</td>
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<td>17.83</td>
<td>1760</td>
<td>0.79</td>
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<td>0.336</td>
<td>14.98</td>
<td>4.86</td>
<td>0.17</td>
<td>10.73</td>
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<td>( \lambda_3 )</td>
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<td>1.66</td>
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<td>6.12</td>
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<td>5.07</td>
<td>3.63</td>
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<td>88.42</td>
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In the previous chapter, we have shown that frequency-selective matching can improve the capacity of compact MIMO system significantly under strong coupling. We have demonstrated the proposed *decoupling and matching* technique for specific antenna array configurations, viz. circular and linear arrays. The results in chapter 2 provide a way of calculating capacity for a given receive array as a function of the signal propagation and noise environments and the array impedance. Therefore, if we are given a specific array, these results show how matching networks and communications algorithms can be jointly optimized to make the best use of that array. However, at a deeper level of abstraction, it is natural to ask whether information theory can provide any guidance *on the design of the array itself*. All antennas detect information by indirectly observing the currents
induced by the incident signal and noise fields in some volume of conducting material. How much information is contained in the currents within a given structure? How does this information vary with the shape and material of the structure, and with the number and placement of antenna ports? Most importantly, how can we use this knowledge to design MIMO antenna systems that efficiently capture all of the useful information within a given volume?

The aim of this chapter is to quantify the information about a transmitted signal that can be extracted from the currents induced in an arbitrary conducting volume $V$, and to use this information to determine the maximum capacity that can be achieved by any $M$-port array that observes the signal through $V$. Several authors have investigated the information contained in electromagnetic fields within a given volume, primarily to characterize the available degrees of freedom [10,109], or to decompose the information among incident spherical waves [56,57,59,95]. Perhaps the closest among these to our work is [57], which considered the maximum MIMO capacity that could be achieved in the presence of spherically-uniform signal multipath by a lossless antenna that fits within a given spherical volume at the receiver. The main differences between [57] and this work are: (1) we consider lossy antennas of arbitrary shape, more general signal propagation conditions, and models of physical noise sources such as sky noise and front-end noise; and (2) we focus here on narrowband MIMO capacity, whereas [57] considered the broadband characteristics of the array.

The rest of the chapter is organized as follows. In Sec. 3.1, we introduce the system model and present capacity metrics for arbitrary arrays and general signal propagation conditions. In Sec. 3.2, we consider an array that observes the conducting volume $V$ through a passive coupling network, and show how the capacity metrics are related to the properties of the underlying structure, $V$. In Sec. 3.3, we define the concept of aperture.
capacity, decompose the information in $V$ among a collection of unit-power eigencurrents, and derive bounds for the capacity of any $M$-port array that observes the signal through $V$. Finally, we present numerical examples in Sec. 3.4 and summarize our conclusions in Sec. 3.5.

3.1 System Model and Capacity Metrics

Consider a MIMO system in which $N$ transmit antennas send data to $M$ receive antennas. We assume the transmit antennas are identical, uncoupled and spaced far enough apart to ensure uncorrelated fading at the receiver. The receive antennas may be closely spaced, however, so the signals received by different antennas may exhibit mutual coupling and fading correlation. Before these signals are converted to digital form and processed by the communication algorithms, they first pass through an analog front-end which consists of a matching network followed by a collection of identical, uncoupled amplifiers, as illustrated in Fig. 3.1. In this chapter, we consider a general receiver in which both the array and front-end can be approximated as linear networks with internal noise sources, where all impedances and noise power spectral densities are frequency non-selective over the signal bandwidth. This model is largely drawn from [33], and this section extends the results of [33] to lossy antennas and more general signal propagation conditions.

3.1.1 Antenna Array

Let $\mathbf{v} \in \mathbb{C}^M$ denote the complex-baseband voltage across the terminals of the array and let $\mathbf{i} \in \mathbb{C}^M$ be the corresponding current flowing into the array. The relationship between the two can be described by

$$\mathbf{v} = \mathbf{Z}_A \mathbf{i} + \mathbf{v}_o,$$  \hfill (3.1)
where $\mathbf{Z}_A \in \mathbb{C}^{M \times M}$ is the impedance matrix and $\mathbf{v}_o$ is the open-circuit voltage induced by
the incident field. Here, diagonal elements of $\mathbf{Z}_A$ represent antenna self-impedances, while
off-diagonal elements represent mutual coupling between antennas. We can always write

$$\mathbf{Z}_A = \mathbf{R}_A + j\mathbf{X}_A , \quad (3.2)$$

where $\mathbf{R}_A \triangleq \frac{1}{2}(\mathbf{Z}_A + \mathbf{Z}_A^H)$, $\mathbf{X}_A \triangleq \frac{1}{2j}(\mathbf{Z}_A - \mathbf{Z}_A^H)$, and $(\cdot)^H$ denotes the conjugate-transpose.

If the antennas are reciprocal, then $\mathbf{Z}_A = \mathbf{Z}_A^T$ and $\mathbf{R}_A = \text{Re}[\mathbf{Z}_A]$ and $\mathbf{X}_A = \text{Im}[\mathbf{Z}_A]$ are
both real, symmetric matrices.

In [33], the capacity of MIMO systems with lossless antennas at the receiver was
considered. Here we explore the impact of finite conductivity on capacity. When internal
losses are small we assume, as in [62], that the antennas can be modeled by an ideal
lossless array with radiation resistance $\mathbf{R}_E$ in series with a loss resistance $\mathbf{R}_I$:

$$\mathbf{R}_A = \mathbf{R}_E + \mathbf{R}_I . \quad (3.3)$$

The radiation resistance $\mathbf{R}_E$ is closely related to the antenna patterns. Let the pattern
\( E_m(\theta, \phi) \) of antenna \( m \) be defined by the far-field approximation of the field produced at \((r, \theta, \phi)\) when the antenna is driven by the current \( i_m \),

\[
E(r, \theta, \phi) = \frac{1}{r} e^{-jkr} i_m E_m(\theta, \phi),
\]

where \( k = 2\pi/\lambda \) and \( \lambda \) is the wavelength. If all patterns are expressed relative to a common phase center, then for any lossless array in free space, we have [32, eq. 38]

\[
[R_E]_{mn} = \frac{1}{\eta} \iint_{S_1} E_m^T(\theta, \phi) E^*_n(\theta, \phi) \, dS
\]

where \( \eta \approx 120\pi \Omega \) is the impedance of free space, and \( \iint_{S_1} (\cdot) \, dS \) denotes the surface integral over the unit sphere \( S_1 \). Although (3.5) is proved in [32] for vertically-oriented dipoles, the proof clearly extends to arbitrary lossless antennas (Sec. B.1 of Appendix B).

Further note (3.5) is also a valid formula for the radiation resistance of the lossy array (3.3), since \( R_A \) and \( R_E \) have the same antenna patterns.

Assuming frequency-flat fading, the open-circuit voltage in (3.1) can be expressed as

\[
v_o = H_A x + n_o
\]

where \( x \in \mathbb{C}^N \) is the transmitted signal, \( H_A \in \mathbb{C}^{M \times N} \) is the channel matrix and \( n_o \in \mathbb{C}^M \) represents noise. We assume a Rayleigh fading environment with no correlation at the transmitter, so the columns of \( H_A \) can be modeled as independent and identically distributed (i.i.d.), zero-mean circularly-symmetric Gaussian random vectors, denoted by \( h_{A,i} \sim \mathcal{CN}(0, \Sigma_{h_A}) \), where \( 0 \in \mathbb{C}^M \) is the all-zero vector, \( \Sigma_{h_A} = \mathcal{E}[h_{A,i} h_{A,i}^T] \), and \( \mathcal{E}[\cdot] \) denotes the expectation. In general, \( \Sigma_{h_A} \) depends on the signal propagation environment and the radiation patterns of the receive antennas. In particular, we show in Sec. B.2 of
Appendix B that if the vertically and horizontally polarized components of the incident field are independent, identically-distributed and spatially white, then

\[ [\Sigma_{hA}]_{m,n} = \left( \frac{4\pi}{\eta k} \right)^2 \iint_{S_1} E_m^T(\theta, \phi) E_n^*(\theta, \phi) p(\theta, \phi) \, dS \, . \] (3.7)

Intuitively, we can interpret \( E|x_n|^2 p(\theta, \phi)/2\eta \) as the average incident signal power per steradian that arrives from direction \((\theta, \phi)\) from transmit antenna \(n\). In particular, when the incident signal power \(P_s\) arrives uniformly from all directions, then \(p(\theta, \phi) = \eta P_s/2\pi\) and hence (3.5) gives

\[ \Sigma_{hA} = \frac{2P_s\lambda^2}{\pi} R_E \, . \] (3.8)

Note we can always write \( H_A = \Sigma_{hA}^{1/2} H_w \) where \( H_w \) is an \( M \times N \) matrix with i.i.d. \( \mathcal{CN}(0,1) \) elements, and \( \Sigma_{hA}^{1/2} \) denotes the Hermitian positive-semidefinite square root of \( \Sigma_{hA} \) [66, pg. 406].

The term \( n_o \) in (3.6) represents antenna noise due to thermal noise from the loss resistance \( R_I \) as well as sky noise from the environment. The noise contributed by \( R_I \) can be modeled as Johnson-Nyquist noise. Twiss [133] has shown that a resistance \( R_I \) at temperature \( T_0 \) in Kelvin will produce an open-circuit noise voltage with distribution \( \mathcal{CN}(0, 4k_b T_0 B R_I) \), where \( k_b = 1.38 \times 10^{-23} \) J/K is Boltzmann’s constant and \( B \) is the system bandwidth in Hz. Twiss further observed that a lossless array with radiation resistance \( R_E \) surrounded by an isotropic black-body of temperature \( T_A \) [6, pg. 111] produces an open-circuit noise voltage with distribution \( \mathcal{CN}(0, 4k_b T_A B R_E) \). Thus, if we assume \( T_A \approx T_0 \), it follows from (3.3) that the combined antenna noise from both sources in (3.6) is distributed as \( n_o \sim \mathcal{CN}(0, 4k_b T_0 B R_A) \).
3.1.2 Front-End

In Fig. 3.1, we take the signals observed by the communication system to be the load voltages \( r_1, \ldots, r_M \). Prior to reaching the load, the signals pass through a front-end consisting of a matching network followed by a bank of amplifiers. The amplifiers are represented by Rothe-Dahlke [114] equivalent networks, where \( \mathbf{v}_a \sim \mathcal{CN}(0, 4kT_0B_r \mathbb{I}) \) and \( \mathbf{i}_a \sim \mathcal{CN}(0, 4kT_0B_g \mathbb{I}) \) are independent random vectors that represent amplifier noise. For simplicity, we call these circuits “amplifiers,” but any sequence of linear, noisy circuits can be represented this way [114]. Our only assumptions are that the “amplifiers” are linear, uncoupled and the impedances and noise power spectral densities are frequency non-selective over the signal bandwidth.

Throughout this section, we consider the special case of no matching network in Fig. 3.1. Using basic circuit analysis, we can show the load voltages in Fig. 3.1 are given by (e.g. [32, eq. 14])

\[
\mathbf{r} = \mathbf{G} \left[ \mathbf{v}_o - \mathbf{v}_a - (\mathbf{Z}_A + z_{\text{cor}} \mathbb{I}) \mathbf{i}_a \right],
\]

(3.9)

where \( \mathbf{G} = \frac{z_{21} z_L + z_{22}}{z_{21} + z_{22}} \left( \mathbf{Z}_A + \frac{z_{11}(z_L + z_{22}) - z_{12} z_{21}}{z_L + z_{22}} \mathbb{I} \right)^{-1} \). If the open-circuit voltage is observable, in the sense that every non-zero \( \mathbf{v}_o \) produces a non-zero contribution to \( \mathbf{r} \), then \( \mathbf{G} \) must be invertible. Since capacity is not affected by invertible transformations applied to the received signal, we can take the observed signal to be

\[
\mathbf{y} = \mathbf{G}^{-1} \mathbf{r} = \mathbf{v}_o - \mathbf{v}_a - (\mathbf{Z}_A + z_{\text{cor}} \mathbb{I}) \mathbf{i}_a = \mathbf{H}_A \mathbf{x} + \mathbf{n}_A,
\]

(3.10)

where \( \mathbf{n}_A \Deltaq \mathbf{n}_o - \mathbf{v}_a - (\mathbf{Z}_A + z_{\text{cor}} \mathbb{I}) \mathbf{i}_a \) is the overall system noise. Intuitively, (3.10) refers
the all voltages in (3.9) to the open-circuit array terminals.

### 3.1.3 Capacity Metrics

We would like to determine the capacity of (3.10) subject to a fundamental constraint on the transmitted power, say $\text{tr}(\Sigma_x) \leq P$ where $\Sigma_x \triangleq \mathcal{E}[xx^H]$.\footnote{The exact form of the power constraint depends on the physical interpretation of $x$. For example, if the transmit antennas are identical and lossless, and $x_i$ denotes the current driving transmit antenna $i$, then the average transmitted power is $(r_A/2) \sum_i \mathcal{E}[x_i]^2 = (r_A/2)\text{tr}(\Sigma_x)$, where $r_A$ is the antenna resistance.} The appropriate capacity metric for (3.10) depends on how rapidly the channel state matrix $H_A$ changes with time.

Under fast-fading conditions, we assume $H_A$ changes so rapidly with time, that the transmitter cannot estimate it accurately in real time; however, the receiver can estimate $H_A$ from pilot symbols embedded in the transmitted data. We call this situation channel state information at the receiver (CSIR). Here the appropriate metric is the ergodic capacity \[ C_e = \max_{\Sigma_x: \ 0 \leq \text{tr}(\Sigma_x) \leq P} \mathcal{E} \left[ \log \det (\mathbf{I} + \Sigma_x H_A^H \Sigma_{n_A}^{-1} H_A) \right] . \] (3.11)

The maximum in (3.11) is always attained by $\Sigma_x = (P/N)\mathbf{I}$. Writing $H_A = \Sigma_{hA}^{1/2} H_w$, where $H_w$ denotes a channel matrix with i.i.d. $\mathcal{CN}(0,1)$ entries, we can express (3.11) as [33, eq. 15]

\[ C_e = C_e(S_A) = \mathcal{E} \left[ \log \det \left( \mathbf{I} + \frac{1}{N} H_w^H S_A H_w \right) \right] . \] (3.12)

where $S_A$ is the SNR matrix introduced in [33, eq. 12]

\[ S_A \triangleq P \Sigma_{hA}^{1/2} \Sigma_{nA}^{-1} \Sigma_{hA}^{1/2} . \] (3.13)
Under slow-fading conditions, the channel changes sufficiently slowly that both the transmitter and receiver can estimate $H_A$. These conditions, which we call full CSI, enable the transmitter to perform space-time water-filling [129, pg. 346] along the eigenvectors of $H_w^H S_A H_w$. Denoting the ordered eigenvalues of a Hermitian matrix $A$ by $\lambda_1(A) \geq \cdots \geq \lambda_N(A)$, the full CSI capacity is
\[
C_f(S_A) = \min\{N,M\} \mathcal{E} \left[ \log \left( 1 + P_i^* \lambda_i(H_w^H S_A H_w) \right) \right],
\]
(3.14)
where the water-filling power allocations are given by
\[
P_i^* = \left[ \mu - \frac{1}{\lambda_i(H_w^H S_A H_w)} \right]^+ ,
\]
(3.15)
$[x]^+$ denotes the positive part of $x$, and the water level $\mu$ is chosen so that $\sum_i \mathcal{E}[P_i^*] = 1$.

For square matrices $A$ and $B$, recall $A \geq B$ means $A - B$ is non-negative definite.

Lemma 1: The capacity metrics depend on $S_A$ only through its eigenvalues, i.e.
\[
C_e(S_A) = C_e(A) , \quad C_f(S_A) = C_f(A) ,
\]
where $A = \text{diag}(\lambda_1(S_A), \cdots \lambda_M(S_A))$. Further, $S_A \leq S'_A$ implies
\[
C_e(S_A) \leq C_e(S'_A) , \quad C_f(S_A) \leq C_f(S'_A) .
\]
(3.17)

Proof of Lemma 1: $S_A$ is Hermitian, and so can be diagonalized by a unitary matrix [66, Thrm 4.1.5], say $S_A = U^H \Lambda U$ where $UU^H = U^H U = I$. Since $H_w$ and $UH_w$ have the same probability distribution, dropping $U$ in the capacity metrics does not change their

---

\[^{2}\text{From [128, pg. 4], the joint probability density function of the elements of } H_w \text{ is given by } p(H) = \pi^{-MN} \exp(-\text{tr}(H^H H)), \text{ from which it follows that } p(H) = p(UH).\]
values. Equation (3.17) is a direct consequence of Theorem 1 in [33].

3.1.4 Capacity-Optimal Matching

Thus far, we have assumed no matching is used between the antennas and front-end in Fig. 3.1. However, matching can significantly enhance both capacity metrics [98], [48], [33]. Front-ends similar to Fig. 3.1 were considered in [98] and [48], however, in the former antenna noise was neglected and in the latter additional constraints were imposed on the amplifiers. In these papers optimal matching schemes were suggested heuristically [98] or through numerical simulations [48]. More recently, a matching network that simultaneously maximizes both metrics was derived in [33]. This is a multiport generalization of the well-known minimum-noise figure match [114].

For $M = 1$, the noise contributed by the amplifier in (3.10) is often measured by the noise factor $F \triangleq \sigma_n^2 A / \sigma_n^2 o$. The noise factor takes its minimum value (e.g. [33, Appendix])

$$F_{\text{min}} = 1 + 2 \left(g_a r_{\text{cor}} + \sqrt{g_a r_a + (g_a r_{\text{cor}})^2}\right), \quad (3.18)$$

when the antenna impedance equals $z_{\text{opt}} = \sqrt{r_a / g_a + r_{\text{cor}}^2} - j x_{\text{cor}}$, where $g_a$, $r_a$ and $z_{\text{cor}} = r_{\text{cor}} + j x_{\text{cor}}$ are internal amplifier parameters.

For general $M$, the matching network in Fig. 3.1 is described by a block impedance matrix

$$Z_M \triangleq \begin{bmatrix} Z_{M11} & Z_{M12} \\ Z_{M21} & Z_{M22} \end{bmatrix}, \quad (3.19)$$

where each submatrix is in $\mathbb{C}^{M \times M}$. Ideally the network is designed with passive, reactive elements so it is lossless, reciprocal and noiseless. The network is lossless and noiseless if [63, pg. 13] $Z_M = -Z_M^H$ and reciprocal if $Z_M = Z_M^T$ where $[\cdot]^T$ denotes the transpose.
When matching is applied, the noise in the observation model (3.10) is changed to

\[ n_A = n_o - M^{-1} [v_a + (Z'_A + z_{cor} I) i_a], \]  

(3.20)

where \( M = Z_{M21} (Z_A + Z_{M11})^{-1} \) and \( Z'_A = -M Z_{M12} + Z_{M22} \). The covariance of this noise can be written as \( \Sigma_{n_A} = \Sigma_{n_o}^{1/2} F \Sigma_{n_o}^{1/2} \), where [33, eqs. 36-40]

\[ F = F_{min} I + g_a R_A^{-1/2} M^{-1} (Z'_A - z_{opt} I) (Z'_A - z_{opt} I)^H M^{-H} R_A^{-1/2}, \]  

(3.21)

is a matrix generalization of the noise factor. Since the second term in (3.21) is non-negative definite, it follows \( F \geq F_{min} I \) and hence

\[ \Sigma_{n_A} \geq F_{min} \Sigma_{n_o} = 4 k_b T_0 B F_{min} R_A, \]  

(3.22)

with equality if and only if \( Z'_A = z_{opt} I \). Thus, any network that satisfies \( Z'_A = z_{opt} I \) minimizes the noise in (3.10) in every dimension. There are many networks that satisfy this condition; in particular, one is given by [33, eq. 42]

\[ \begin{bmatrix} Z_{M11} & Z_{M12} \\ Z_{M21} & Z_{M22} \end{bmatrix} = j \begin{bmatrix} -X_A & (r_{opt} R_A)^{1/2} \\ (r_{opt} R_A)^{1/2} & x_{opt} I \end{bmatrix}. \]  

(3.23)

With optimal matching, the capacity metrics (3.12) and (3.14) are simultaneously maximized, and the SNR matrix (3.13) becomes

\[ S_A \triangleq \rho \Sigma_{h_A}^{1/2} R_A^{-1} \Sigma_{h_A}^{1/2}, \]  

(3.24)

where \( \rho = P/4 k_b T_0 B F_{min} \). These metrics depend on the amplifiers only through \( F_{min} \).
3.2 Arrays from Arbitrary Conducting Bodies

Arrays detect information by indirectly observing the currents induced by the incident signal and noise fields in some conducting body. In the last section, we calculated the capacity of a receive array in terms of the signal propagation environment and array impedance. In this section, we consider an array formed from a conductor of arbitrary shape, say $V$. We assume the array consists of a finite number of ports that observe and interact with $V$ through a passive coupling network. The goal of this section is to relate the information observed at the array ports to the properties of the underlying body $V$. In Sec. IV, we use this relationship to determine the maximum information that any $M$-port observation of $V$ can collect, as well as gain insights on how to construct arrays that achieve this maximum.

3.2.1 Fields and Currents in Conductors

We first briefly describe the relationship between the induced fields and currents in $V$; for further details, see [62, Sec. III]. Consider an arbitrary conducting body $V$ of conductivity $\sigma < +\infty$. The fields in $V$ can be represented as the sum of an incident field $E^i$, due to external sources, and a scattered field $E^s$ due to the induced volume current $J$. The relationship between the currents and fields is given by [62, eq. 50].

$$J = \sigma \left[ E^i + E^s \right]. \quad (3.25)$$

The scattered field can be further decomposed into a part that depends only on the shape of $V$

$$Z_{E}(J) = -E^s \quad (3.26)$$
and a part that depends only the conductivity

\[ Z_1(J) = \sigma^{-1}J \, . \]  

Consequently, (3.25) can be written as

\[ Z(J) \triangleq (Z_E + Z_1)(J) = E^i \, . \]  

For purposes of calculation, it is convenient to represent all the fields and currents in \( V \) with respect to a method of moments basis, say \( J = J(s) = J_1W_1(s) + \cdots + J_KW_K(s) \), for all \( s \in V \). Details on how to do this are discussed in [61], and the process is automated in the electromagnetic simulation software tool, FEKO [35]. For any \( K \)-dimensional basis, we may (with some abuse of notation) interpret \( J \) and \( E^i \) as vectors in \( \mathbb{C}^K \) and \( Z \in \mathbb{C}^{K \times K} \) as a generalized impedance matrix.

We generally observe the currents \( J \) through a collection of ports that couple to the conducting body, as illustrated in Fig. 3.2. The presence of these ports induces a corresponding field \( E \) in \( V \). We can therefore represent the total incident field \( E^i \) as the sum of \( E \) and the field due to the impressed signal and noise fields, denoted by \( -E_o \). From (3.28), the relationship between the fields and currents in \( V \) is thus

\[ E = ZJ + E_o \, , \]  

where \( Z = R + jX \). When the probes are absent, note \( E = 0 \) and (3.29) reduces to the usual boundary condition \( ZJ + E_o = 0 \).

Under flat-fading propagation conditions, \( E_o \) can be represented in a way similar to
\[ \mathbf{E}_o = \mathbf{H}_V \mathbf{x} + \mathbf{n}_V, \]  

where \( \mathbf{x} \) is the transmitted signal, and \( \mathbf{H}_V \in \mathbb{C}^{K \times N} \) is a matrix with columns that are i.i.d. \( \mathcal{CN}(0_K, \Sigma_h) \). Proceeding as in the Appendix, we can show that \( \Sigma_h \) is given by a formula similar to (3.7), where \( \mathbf{E}_m(\theta, \phi) \) now denotes the antenna pattern associated with the \( m \)-th basis function \( W_m(s) \), and \( \mathbf{n}_V \) represents antenna noise modeled by Twiss formula \( \mathbf{n}_V \sim \mathcal{CN}(0, 4k_bT_0BR) \).

### 3.2.2 Capacity Metrics for Conducting Bodies with Arbitrary Ports

Suppose we now construct a receive array that observes the currents on \( V \) via a collection of \( M \) probes, as illustrated in Fig. 3.2. The aim of this section is to relate the capacity metrics for this array to the properties of the underlying structure \( V \), as given in (3.30).
In general, if \( \textbf{v} \) and \( \textbf{i} \) denote the voltages across, and currents into, the ports, then any general coupling between the probes and \( V \) can be represented as

\[
\begin{align*}
\textbf{E} &= \textbf{Z}_{11}(-\textbf{J}) + \textbf{Z}_{12}\textbf{i}, \\
\textbf{v} &= \textbf{Z}_{21}(-\textbf{J}) + \textbf{Z}_{22}\textbf{i},
\end{align*}
\tag{3.31}
\]

where \( \textbf{Z}_{11} \in \mathbb{C}^{K \times K} \), \( \textbf{Z}_{12} \in \mathbb{C}^{K \times M} \), \( \textbf{Z}_{21} \in \mathbb{C}^{M \times K} \) and \( \textbf{Z}_{22} \in \mathbb{C}^{M \times M} \). Note the similarity between the coupling network of Fig. 3.2 and the matching network of Fig. 3.1. Here the coupling network is reciprocal, but it is not generally lossless or noiseless. Thus we have

\[
\textbf{Z}_{11} = \textbf{Z}_{11}^T, \textbf{Z}_{12} = \textbf{Z}_{12}^T, \textbf{Z}_{22} = \textbf{Z}_{22}^T.
\]

Let \( \textbf{Z}_A, \textbf{v}_o \) denote the driving-point impedance and open-circuit voltage seen looking into the probe ports. From elementary circuit theory, it is easy to show that

\[
\begin{align*}
\textbf{Z}_A &= -\textbf{M}\textbf{Z}_{12} + \textbf{Z}_{22}, \\
\textbf{v}_o &= \textbf{M}\textbf{E}_o,
\end{align*}
\tag{3.32}
\]

where \( \textbf{M} = \textbf{Z}_{21}(\textbf{Z} + \textbf{Z}_{11})^{-1} \). The only assumptions we make about the probes are: (1) they are shielded from the incident signal field, and so do not contribute additional information about the signal fields; (2) there are no redundant ports, so \( \textbf{M} \) has full rank; (3) the presence of the probes does not alter the relationship (3.29); and (4) the coupling network is passive, and so may absorb but not supply power. This last condition holds if and only if the resistance matrix of the coupling network is non-negative definite. Although it may not be apparent from (3.32), our model actually includes the thermal noise contributed by resistances in the coupling network. As long as the coupling network is at the same temperature as the antennas and the environment, the Twiss formula for the statistical distribution of noise in the antenna system depends only on \( \textbf{R}_A \), which is predicted by our model.
From (3.32), the covariance of the columns of $H_A$ in (3.6) is given by $\Sigma_{hA} = M\Sigma_{hv}M^H$, where $\Sigma_{hv}$ is the covariance of the columns of $H_V$ in (3.29). The resistance matrix of the array is given by $R_A = (1/2)(Z_A + Z_A^H)$, so the SNR matrix can be calculated as

$$S_A \triangleq \rho \Sigma_{hA}^{1/2}R_A^{-1}\Sigma_{hA}^{1/2} = \rho [M\Sigma_{hv}M^H]^{1/2}R_A^{-1}[M\Sigma_{hv}M^H]^{1/2}.$$ (3.33)

Thus, given the coupling network (3.31), we can in principle use (3.33) to calculate the corresponding capacity metrics (3.12) and (3.14).

### 3.3 Aperture Capacity

In this section, we characterize the maximum amount of information which can be extracted from an arbitrary conducting structure $V$. As in Sec. III, we consider a finite-element array consisting of ports that observe and interact with $V$ through a passive coupling network. We show the capacity of any such array is bounded above by a quantity called the *aperture capacity*, that represents all the information contained in the currents and fields induced in the conducting structure. This information can be partitioned among a set of information eigenmodes of the structure. We show that the capacity of any $M$-port array is bounded above by the information contained in the best $M$ information eigenmodes of the structure. We further show how ports, matching networks, and communications algorithms could be designed, at least in principle, in order to achieve these upper bounds.

As in Sec. II, let $Z \in \mathbb{C}^{K \times K}$ be the generalized impedance of the conducting body $V$, and let $\Sigma_{hv}$ denote the covariance of fading path gains on the surface of the body,
from (3.29). We define the \textit{aperture ergodic capacity} and \textit{aperture full CSI capacity} to be $C_e(S_V)$ and $C_f(S_V)$ calculated with the aperture SNR matrix:

$$S_V \triangleq \rho \Sigma_h^{1/2} R^{-1} \Sigma_h^{1/2}. \quad (3.34)$$

Note that $S_V$ depends on the number of transmit antennas, the propagation environment, and the shape and material of the conducting body. Intuitively, (3.34) represents the capacity of a "virtual array" in which there is one virtual antenna to observe each of the Method of Moments basis functions. In general, $C_e(S_V)$ and $C_f(S_V)$ are both increasing functions of the number of basis functions. In a properly-modeled system, however, these metrics should not depend on the basis chosen. Here we assume the basis is sufficiently rich that adding additional basis functions would not significantly increase the aperture capacity. Indeed, if it does, then the original basis was not sufficient to capture all of the useful information in the aperture.

\subsection*{3.3.1 SNR Eigencurrents}

The information in (3.34) can be associated with certain current eigenmodes on $V$. Consider the generalized eigenvalue problem of finding a scalar $\lambda$ and current $J$ such that

$$\lambda R J = \Sigma_h J. \quad (3.35)$$

When $R, \Sigma_h \in \mathbb{C}^{K \times K}$ are Hermitian and $R$ is positive definite, there exist $K$ generalized eigenvectors $J_k \in \mathbb{C}^K$ and eigenvalues $\lambda_k \in \mathbb{R}$ (cf. Lemma 3 in the Appendix) such that

$$J_k^H R J_k = \delta_{ik}, \quad J_k^H \Sigma_h J_k = \lambda_k \delta_{ik}, \quad 1 \leq k, i \leq K. \quad (3.36)$$

64
We call these SNR eigencurrents and, without loss of generality, assume they are ordered so that \( \lambda_1 \geq \cdots \geq \lambda_K \). When \( \mathbf{R} \) and \( \Sigma_{hv} \) are real, the eigencurrents \( \mathbf{J}_i \) can be chosen as real as well.

Let \( \mathbf{C} \) be a \( K \times K \) matrix whose \( i \)-th column is \( \mathbf{J}_i \) and let \( \mathbf{\Lambda} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_K) \). It follows that

\[
\mathbf{C}^H \mathbf{R} \mathbf{C} = \mathbf{I}_K, \quad \mathbf{C}^H \Sigma_{hv} \mathbf{C} = \mathbf{\Lambda}.
\] (3.37)

The first equality in (3.37) implies \( \mathbf{R}^{-1} = \mathbf{C} \mathbf{C}^H \). If we now define \( \mathbf{V} = \Sigma_{hv}^{1/2} \mathbf{C} \mathbf{\Lambda}^{-1/2} \), then \( \mathbf{V}^H \mathbf{V} = \mathbf{\Lambda}^{-1/2} \mathbf{C}^H \Sigma_{hv} \mathbf{C} \mathbf{\Lambda}^{-1/2} = \mathbf{I} \) and

\[
\mathbf{V}^H \mathbf{S}_V \mathbf{V} = \mathbf{\Lambda}^{-1/2} \mathbf{C}^H \Sigma_{hv} \mathbf{R}^{-1} \Sigma_{hv} \mathbf{C} \mathbf{\Lambda}^{-1/2} \\
= \mathbf{\Lambda}^{-1/2} \mathbf{C}^H \Sigma_{hv} \mathbf{C} \mathbf{\Lambda}^{-1/2} = \mathbf{\Lambda},
\] (3.38)

so \( \mathbf{V} \) is a unitary matrix that diagonalizes the SNR matrix \( \mathbf{S}_V \). It follows that \( \mathbf{\Lambda} \) can be recognized as the eigenvalues of the SNR matrix, i.e. \( \lambda_i = \lambda_i(\mathbf{S}_V), \ 1 \leq i \leq K \). We can therefore view \( \mathbf{J}_1, \ldots, \mathbf{J}_K \) as unit-power eigencurrents on \( V \) that partition the signal into \( K \) independent, parallel subchannels with i.i.d. noise, where the SNR of the \( i \)-th subchannel is \( \lambda_i(\mathbf{S}_V) \).

### 3.3.2 Bounds for Arbitrary Arrays

We now show that the capacity derived in (3.33) for an arbitrary \( M \)-port system on \( V \) is always bounded above by the capacity of the best \( M \) SNR eigencurrents. From Lemma 1, recall both capacity metrics depend only on the eigenvalues of the SNR matrix, and are increasing functions of these eigenvalues. For any \( 1 \leq M \leq K \), let
\( \Lambda_{V,M} \triangleq \text{diag}(\lambda_1(S_V), \ldots, \lambda_M(S_V)) \) consist of the \( M \) best eigenvalues of \( S_V \).

**Theorem 1:** For any passive coupling network, the eigenvalues of the \( M \)-port SNR matrix in (3.33) satisfies

\[
\lambda_i(S_A) \leq \lambda_i(S_V), \quad i = 1, \ldots, M. \quad (3.39)
\]

It follows that \( C_e(S_A) \) and \( C_f(S_A) \) are bounded above by the corresponding capacity of the best \( M \) SNR eigenmodes of the conducting body, i.e. \( C_e(\Lambda_{V,M}) \) and \( C_f(\Lambda_{V,M}) \).

**Proof:** We can express (3.33) as

\[
S_A = \rho \left[ M \Sigma_{hv} M^H \right]^{1/2} R_A^{-1} \left[ M \Sigma_{hv} M^H \right]^{1/2}
= \rho U^H \Sigma_{hv}^{1/2} M^H R_A^{-1} M \Sigma_{hv}^{1/2} U
\]

where \( U \triangleq \Sigma_{hv}^{1/2} M^H \left[ M \Sigma_{hv} M^H \right]^{-1/2} \) satisfies \( U^H U = I_M \). In the appendix, we prove for any passive coupling network that

\[
M^H R_A^{-1} M \leq R^{-1}. \quad (3.41)
\]

It follows from (3.34) that

\[
S_A \leq \rho U^H \Sigma_{hv}^{1/2} R^{-1} \Sigma_{hv}^{1/2} U = U^H S_V U. \quad (3.42)
\]

We now require the following lemma.

**Poincaré Separation Theorem** [66, Cor. 4.3.16]: For any \( 1 \leq M \leq K \), let \( S_V \in \mathbb{C}^{K \times K} \) be any Hermitian matrix and let \( U \in \mathbb{C}^{K \times M} \) be any matrix such that \( U^H U = I_M \). For
any $B$, let $\sigma_k(B)$ denote the eigenvalues of $B$ arranged in *ascending* order. Then

$$\sigma_k(S_V) \leq \sigma_k(U^H S_V U) \leq \sigma_{k+K-M}(S_V), \quad 1 \leq k \leq M. \quad \diamondsuit \quad (3.43)$$

The $k$-th eigenvalue of $S_V$ in ascending order equals the $(K - k + 1)$-th eigenvalue in descending order. Similarly, the $k$-th eigenvalue of $U^H S_V U$ in ascending order is the $(M - k + 1)$-th eigenvalue in descending order. Rewriting (3.43) is descending order, we have

$$\lambda_{k+K-M}(S_V) \leq \lambda_k(U^H S_V U) \leq \lambda_k(S_V), \quad 1 \leq k \leq M, \quad (3.44)$$

thereby proving (3.39) and the theorem. \diamondsuit

Theorem 1 provides an upper bound on the capacity of any $M$-port system that observes the received signal via the currents induced in $V$. What kinds of $M$-port systems achieve this upper bound?

For any $M$ let $C_M \in \mathbb{C}^{K \times M}$ be the matrix formed by extracting the first $M$ columns of $C$ in (3.37), so that

$$C_M^H R C_M = I_M, \quad C_M^H \Sigma_{hv} C_M = \Lambda_{V,M}. \quad (3.45)$$

Consider the $(K + M) \times (K + M)$ coupling network given by

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = j \begin{bmatrix} -X & R C_M V \\ V^H C_M^H R & X_{22} \end{bmatrix}. \quad (3.46)$$

where $V$ is any unitary matrix $V^H V = VV^H = I$ and $X_{22}$ is any real symmetric matrix.
From (3.33), it follows that $M = Z_{21} (Z + Z_{11})^{-1} = jV^H C_M^H$ and so

$$Z_A = -MZ_{12} + Z_{22} = I_M + jX_{22}. \quad (3.47)$$

Thus, the SNR matrix is given by

$$S_A \triangleq \rho \left[ M\Sigma_{h_v} M^H \right]^{1/2} R_A^{-1} \left[ M\Sigma_{h_v} M^H \right]^{1/2} = \rho M\Sigma_{h_v} M^H = \rho V^H C_M^H \Sigma_{h_v} C_M V = \rho V^H \Lambda_{V,M} V. \quad (3.48)$$

Thus, for any $V$, the eigenvalues of $S_A$ equal those of $\Lambda_{V,M}$ and the upper bound is attained. Note the coupling network (3.46) is reciprocal but not necessarily lossless. However, when $R$ and $\Sigma_{h_v}$ are real, then we can choose $C_M$ and $V$ to be real and (3.46) is lossless and reciprocal. It is not clear whether physically-realizable networks exist that achieve this coupling network, however, (3.46) provides insight into how optimal networks could be constructed.

For the particular case of $M = K$ and $V = I$, note that (3.46) yields a lossless network that partitions all of the currents in $V$ into the SNR eigencurrents. From (3.47), the resistance matrix of the partitioned system is $I$ and so the corresponding antenna noise is i.i.d., the fading path gains are independent, and the $i$-th port has SNR $\lambda_i(S_V)$.

### 3.3.3 Antenna Shape Optimization

Several authors have considered optimizing the shape of a multi-port antenna system in order to enhance directivity, or improve impedance matching and cross-polarization between ports [37, 150]. These methods often involve a computationally-intensive search among the substructures of some initial structure $V$. When the design objectives of
shape optimization include maximizing the capacity of an $M$-port substructure, the tools developed here provide a way to quantify the capacity loss associated with each substructure in a way that is independent of the placement of ports. This may lead to more efficient searches, since we can then focus attention on substructures that preserve most of the capacity of the initial structure $V$.

Suppose the initial structure $V$ is described by the generalized impedance $Z$ and current $J$, as in (3.29). Note that any smaller structure $\bar{V} \subseteq V$ can be described by a similar equation:

$$\bar{E} = \bar{Z}\bar{J} + \bar{E}_o,$$

where $\bar{E}, \bar{J} \in \mathbb{C}^K$ represent fields and currents in $\bar{V}$.

In general, we may choose to represent these structures with different basis functions. Let $W$ be a matrix\(^3\) that represents any current $\bar{J}$ relative to the basis of the larger structure, so $\bar{J}$ on $\bar{V}$ represents the same current as $J = W\bar{J}$ on $V$, and similarly $E = WE$. Since (complex) power calculations will produce the same results in both bases, we have

$$\bar{J}^H\bar{Z}\bar{J} = \bar{J}^H\bar{E} = J^HE = \bar{J}^HW^HZW\bar{J}$$

$$\bar{J}^H\bar{E}_o = J^HE_o = \bar{J}^HW^HE_o.$$

Since the equalities hold for all $\bar{J}$, it follows $\bar{Z} = W^HZW$, $\bar{E}_o = W^HE_o$, and so

$$\bar{R} = W^HRW, \quad \Sigma_{h\bar{V}} = W^H\Sigma_{hV}W.$$

\(^3\)If $W_i(s)$ and $\bar{W}_i(s)$ represent bases for $V$ and $\bar{V}$, respectively, it is not difficult to see that $W$ represents the change of basis $\bar{W}_i(s) = \sum_j [W]_{ij} W_i(s)$ for all $s \in \bar{V}$, as shown in appendix B.
The SNR matrix of the substructure $\bar{V}$ is therefore given by

$$S_{\bar{V}} \triangleq \rho \Sigma_{h_v}^{1/2} R^{-1} \Sigma_{h_v}^{1/2} = \rho [W \Sigma_{h_v} W^H]^{1/2} [W^H R W]^{-1} [W \Sigma_{h_v} W^H]^{1/2}.$$  \hfill (3.52)

**Theorem 2:** Let $S_V$ be the SNR matrix of $V$ and let $S_{\bar{V}} \in \mathbb{C}^{\bar{K} \times \bar{K}}$ be the SNR matrix for any substructure $\bar{V} \subseteq V$. Then

$$\lambda_i(S_{\bar{V}}) \leq \lambda_i(S_V), \ i = 1, \ldots, \bar{K}.$$  \hfill (3.53)

It follows the aperture capacities $C_e(S_{\bar{V}})$ and $C_f(S_{\bar{V}})$ of $\bar{V}$ are bounded above by the corresponding capacities of $V$, $C_e(S_V)$ and $C_f(S_V)$. Similarly, the capacities of the best $M$ eigencurrents of $\bar{V}$ are bounded above by the capacities of the best $M$ eigenmodes of $V$. \hfill \diamond

**Proof of Theorem 2:** For any $W$, observe (3.52) is a special case of (3.33) corresponding to

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = j \begin{bmatrix} -X & RW \\ W^H R & 0_{\bar{K} \times \bar{K}} \end{bmatrix}. \hfill (3.54)$$

This network is lossless and therefore passive, so we can apply Theorem 1. Note that (3.54) is not generally reciprocal, but the proof of Theorem 1 does not require reciprocity. \hfill \diamond

Theorem 2 states the aperture capacity of $V$ bounds the capacity of any substructure $\bar{V}$. In any shape optimization performed within $V$, the capacity will always be maximized by utilizing the entire structure. However, Theorem 2 could also be used to focus the search for substructures that exhibit other desirable traits. For example, suppose we
want to optimize the shape of a two-port antenna that fits within \( V \). The best achievable two-port capacity will be bounded by the first two eigenvalues of the SNR matrix of \( \bar{V} \). Thus, if we focus the search on substructures that satisfy \( \lambda_i(S_{\bar{V}}) \approx \lambda_i(S_V), \ i = 1, 2 \), we prioritize substructures that preserve most of the two-port capacity available in \( V \).

### 3.4 Examples

We now apply the results of Theorems 1 and 2 to the special case of antennas embedded in spherically-uniform multipath, including several numerical examples involving wire antennas. Additional applications and examples will be explored in Part II of this work.

#### 3.4.1 Lossless Antennas in Spherically-Uniform Multipath

Consider first an arbitrary array formed from an ideal lossless conductor, so \( R_A = R_E \) in (3.3). Suppose the incident signal field consists of a large number of plane waves that arrive uniformly from each direction with equal probability, so \( p(\theta, \phi) = \eta P_s / 2\pi \) in (3.7). From (B.8), the correlation of the fading path gains is then given by

\[
\Sigma_{h_A} = \frac{2P_s \lambda^2}{\pi} R_E .
\]  

(3.55)

Thus for any lossless array, the SNR matrix is given by

\[
S_A \triangleq \rho \Sigma_{h_A}^{1/2} R_A^{-1} \Sigma_{h_A}^{1/2} = \gamma I .
\]  

(3.56)
where $\gamma \triangleq 2\rho P_s \lambda^2 / \pi$. The ergodic capacity is therefore

$$C_\text{e}(S_A) = \mathcal{E} \left[ \log \det \left( I + \frac{\gamma}{N} H_w^H H_w \right) \right]. \quad (3.57)$$

This can be recognized as the classic capacity formula given by Telatar [128] for an ideal, uncoupled $N \times M$ MIMO system with i.i.d. fading between each pair of transmit and receive antennas. This is intuitively reasonable: when the signal covariance is proportional to the noise covariance, then optimal matching can, at least in principle, simultaneously decouple both into a sequence of independent channels with the same SNR $\gamma$. We may further note that $\gamma$ here represents the signal-to-noise at the output of the amplifier when a single lossless antenna is used with optimal matching. We therefore call $\gamma$ the ideal single-antenna SNR.

Note (3.57) depends on the receive array only through the number of ports. As noted in [128], since $\frac{1}{M} H_w^H H_w \rightarrow I_N$ as $M$ grows, we can approximate the capacity as $C_\text{e}(S_A) \approx N \log (1 + M \gamma / N)$ which grows without bound as $M$ increases. In particular, the aperture capacity of any structure represented by $K$ basis functions is given by (3.57) with $M$ replaced by $K$, so aperture capacity grows with the number of basis functions for any $V$.

This result is clearly not physically reasonable, and suggests the underlying physical model lacks one or more essential features of the real problem that constrain capacity. Several possibilities exist, including the following: (a) Any real array has internal losses; (b) the multipath model in (3.55) is infinitely rich; (c) we assume implicitly in (3.24) that we can match arbitrarily reactive impedances; and (d) the narrowband analysis presented here does not consider the bandwidth of the eigencurrents by, for example, examining the quality factors. In this chapter and the following chapters, we consider (a), (b) and (c).
The bandwidth of eigencurrents is left to future work.

### 3.4.2 Lossy Antennas in Spherically-Uniform Multipath

Now suppose the internal loss resistance $R_I$ in (3.3) is non-zero. For any body $V$, from (3.55) the SNR eigencurrents (3.37) satisfy

$$J_i^H [R_E + R_I] J_k = \delta_{ik} , \quad \gamma J_i^H R_E J_k = \lambda_k \delta_{ik} . \quad (3.58)$$

If we define $\eta_k \triangleq \lambda_k / \gamma$, it follows that $\eta_k$ is independent of $\gamma$ and

$$\eta_k = \frac{J_i^H R_E J_k}{J_k^H R_E J_k + J_k^H R_I J_k} . \quad (3.59)$$

If we were to drive $V$ with the eigencurrent $J_k$, note that $J_k^H R_E J_k$ would be the power radiated and $J_k^H R_I J_k$ would be the power dissipated in internal losses. Thus, $\eta_k$ represents the fraction of power that is radiated, which can be recognized as the radiation efficiency of eigencurrent $k$ [4, pg. 872]. Thus, Theorem 1 asserts that the capacity of any $M$-port antenna system that observes this structure is bounded by the capacity of the $M$ most efficient eigencurrents of $S_V$.

### 3.4.3 Wire Antennas in Spherically-Uniform Multipath

We now calculate numerical examples for a lossy wire antenna with spherically-uniform multipath, so the results of the last section apply. Consider a copper wire with conductivity $\sigma = 5.8 \times 10^7$ S/m, radius $r = 0.5$ mm and physical length $L = 0.5$ m. For this wire, we consider two electrical lengths $L = \lambda/6$ and $\lambda/2$, where $\lambda = c/f$ is the wavelength. The impedance and radiation resistance matrices, $Z$ and $R_E$, were obtained numerically
for $K = 127$ triangular basis functions using a method-of-moments code for linear wire antennas based on [103]. The efficiencies were then obtained by solving (3.58) in MATLAB.

In Fig. 3.3, we plot $\eta_k$ for wires of length $L = \lambda/6$ and $\lambda/2$. For $L = \lambda/6$, there are only two modes with $\eta_k \geq 0.1$; for $L = \lambda/2$, there are three modes with $\eta_k \geq 0.1$. In both cases, the efficiencies fall off rapidly for higher order modes with $k > 3$. Since a wire can be considered a substructure of any longer (or fatter) wire, from Theorem 2, the $\eta_k$ are increasing functions of $L$ and $r$. Thus, the $L = \lambda/2$ efficiencies are all greater than or equal to those of $L = \lambda/6$. As $L$ increases, the eigencurrents become more efficient and capacity increases monotonically.

Fig. 3.4 plots the aperture capacity $C_e(\Lambda_{V,M})$ versus the ideal single-antenna SNR $\gamma$ for $N = 2$ transmit antennas and several values of $M$. As shown in Theorem 1, $C_e(\Lambda_{V,M})$ is the information contained in the best $M$ eigencurrents and is an upper bound on the capacity of any $M$-port array that observes the wire. For $L = \lambda/6$, $\eta_1 \approx 1$ implies the first eigencurrent achieves a capacity equal to that of a single, lossless receive antenna.
The second eigencurrent increases capacity by 51%–70%. Each additional individual eigencurrent contributes almost nothing and collectively they offer capacity similar to that offered by first two modes. Similarly, for $L = \lambda/2$, the first two eigencurrents achieve the capacity of a lossless $M = 2$ array; the third, less efficient, eigencurrent increases capacity up to 20%. Each additional eigencurrent contributes literally nothing and collectively offer capacity almost the same as that offered by first three modes.

How close do conventional dipoles come to achieving the upper bounds in Figs. 3.4? Here we consider two examples: A conventional copper dipole with $M = 1$ feed in the center, and a less-conventional $M = 2$ dipole with two symmetrical feeds placed $\pm L/4$ from the center. The capacities of each antenna can be evaluated by the methods of [33], as extended in Sec. II. Fig. 3.5 shows the ergodic capacity of each dipole along with the corresponding aperture capacity $C_e(\Lambda V, M)$. For $M = 1, 2$ and both values of $L$, the dipole capacities coincide with the upper bounds to within the thickness of the plot lines. Thus, both antenna systems make optimal use of the observed currents on the wire.
3.5 Conclusions

In this chapter, we have derived information-theoretic bounds on the capacity that can be achieved by any MIMO receive array that observes the transmitted signal through the currents induced in a conducting structure $V$. These results enable us to investigate the impact on capacity of different structures $V$, port configurations, and signal propagation conditions. For the particular case of a copper wire in spherically-uniform multipath, the bounds suggest that most of the available information resides in the 2-3 most-efficient eigencurrents of (3.58), and we presented two simple port configurations that achieve these bounds. Other propagation conditions and array structures can be explored using the tools developed in this work.

The results of this chapter appear to extend easily to other wireless communication scenarios of interest. Non-uniform sky noise could be modeled by employing a covariance analogous to (3.7) for the open-circuit antenna noise in (3.6). This would change our results only by replacing $R$ in (3.37) with the open-circuit sky-noise covariance. The
assumption of flat-fading could easily be relaxed by replacing $p(\theta, \phi)$ in (3.7) with an appropriate power-delay profile (e.g., [129]), in which case the capacity would be expressed as an integral over frequency.

The capacity metrics derived in this chapter are based on the assumption that we can perfectly match arbitrary antenna impedances. However, some of the information counted in $C_e(\Lambda_{V,M})$ may be contained in very reactive eigencurrents, which may be difficult or impossible to match in practice. A more complete discussion of this issue and the dependence of aperture capacity on other parameters like directional propagation, shape, size etc. for planar structures are included in the next chapter.
Chapter 4

Aperture Capacity: Variation with Direction of Propagation, Array Configuration and Limitation on Q

In the previous chapter, we developed an information-theoretic framework for arbitrary MIMO antenna design. In this chapter, we use this framework to explore more versatile antenna configurations and signal propagation conditions. Firstly, we encompass directional fading to broaden the scope of wire antenna examples considered in Chapter 3 to more general scattering environment and alongside evaluate capacity limits for planar structures having various shapes and sizes. Secondly, we revisit the underlying assumptions based on which capacity metrics are derived in Chapter 3.

The rest of the chapter is organized as follows. In Sec. 4.1, we consider wire antennas embedded in scattering environment with directional multipath and show that the capacity metrics depends significantly on the angular spread of directional fading in the elevation
plane. Sec. 4.2 includes numerical examples of wire and patch antennas having different sizes and shapes to study the impact of antenna geometry on capacity. We show that capacity is almost shape-invariant for planar antennas with the small aperture area. A complete discussion on Q, the measure of reactivity of eigenmodes and its impact on aperture capacity is elaborated in Sec. 4.3. Finally we summarize the main contributions of this chapter in Sec. 4.4.

4.1 Directional Signal Propagation

We now explore the impact of directional propagation at the receiver on both aperture capacity and the number of ports needed to achieve this capacity. Consider the same example as in Sec. 3.4.3: a wire of length $\lambda/2$ aligned to the z-axis and centered at the origin of the spherical coordinate system in Fig. 4.1a. In Chapter 3, we calculated the aperture capacity of this structure for spherically-uniform multipath; here we consider non-uniform propagation.

The only aspect of aperture capacity that depends on the propagation environment is $\Sigma_{hV}$, the covariance of fading path gains on the surface of the body. As in Chapter 3, $\Sigma_{hV}$ is given by (3.30)

$$[\Sigma_{hV}]_{m,n} = \left(\frac{4\pi}{\eta k}\right)^2 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} E^T_m(\theta, \phi) E^*_n(\theta, \phi) p(\theta, \phi) \sin \theta \, d\phi \, d\theta,$$

(4.1)

where $E_m(\theta, \phi)$ is the antenna pattern of the $m$-th basis function, and the integral is expressed in spherical coordinates, so $dS = \sin \theta \, d\phi \, d\theta$. Here $(1/2\eta)p(\theta, \phi)$ represents the signal power per steradian incident from direction $\theta, \phi$ when the transmitted signal is normalized to unit power. Thus the total incident power is $P_s = (1/2\eta) \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} p(\theta, \phi) \sin \theta \, d\theta \, d\phi$. 

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In Fig. 3.4, we calculated the aperture capacity of this wire for spherically-uniform multipath, where $p(\theta, \phi) = \eta P_s / 2\pi$ for all $\theta, \phi$. We now consider multipath of the same power uniformly-distributed over all directions within $\theta_o > 0$ of the plane $\theta = 90^\circ$, as shown in Fig 4.1b. It follows that

$$p(\theta, \phi) = \begin{cases} \frac{\eta P_s}{2\pi \sin \theta_o}, & |\theta - \pi/2| \leq \theta_o \\ 0, & \text{else.} \end{cases}$$

Note that similar models have been used to represent the distribution of multipath around cellular base stations [135]. We also note that limiting the azimuth $\phi$ in the same way while holding power constant will not alter capacity, since the radiation patterns of the wire do not depend on $\phi$.

As shown in (3.37), the information captured by an arbitrary structure can be decomposed into a system of unit-power eigencurrents which partition the induced currents into independent, parallel subchannels with i.i.d noise, where the SNR of the $k$-th
subchannel is $\lambda_k$, the eigenvalue associated with the $k$-th eigencurrent. As in Chapter 3, it is convenient to express these eigenvalues as $\lambda_k = \gamma \eta_k$, where $\gamma$ is SNR of an ideal (lossless) single antenna and $\eta_k$ is the radiation efficiency of the $k$-th eigenmode.

For spherically-uniform multipath, we showed in Fig. 3.3 that virtually all of the capacity resides in the three most-efficient modes. Fig. 4.2 plots the radiation efficiency $\eta_k$ of these same modes versus elevation spread $\theta_o$ for the directional propagation model of Fig. 4.1b. Here $\theta_o \to 0$ corresponds to planar multipath, while $\theta_o = 90^\circ$ is the spherically-uniform multipath considered in Chapter 3. For planar multipath, note the efficiencies of all modes are significantly reduced, and virtually all information resides in one mode of efficiency 0.1. However, the efficiencies increase rapidly with elevation spread. Judging by where slopes are steepest, 75% of mode 1 efficiency comes from multipaths with elevations $< 20^\circ$ from the perpendicular. Similarly, the efficiency of mode 2 comes primarily from elevations between $20^\circ$ and $60^\circ$, and mode 3 from elevations between $30^\circ$ and $65^\circ$.

The relative impact of these three modes on aperture capacity depends not only on the efficiencies $\eta_k$ but also on the SNR $\gamma$. In Figs. 4.3a and 4.3b, we plot the $M-$
port aperture capacity $C_{e}(\Lambda_{V,M})$ versus elevation spread $\theta_o$ for SNRs of 0 dB and 20 dB, respectively. Recall that $C_{e}(\Lambda_{V,M})$ is an upper bound on the capacity that can be achieved when $M$ ports are coupled to the structure. Here we consider a system described by 99 basis functions, so $M = 99$ represents the full aperture capacity, i.e. all of the accessible information captured by the structure. At the lower SNR of 0 dB, we note that a conventional single-port dipole mounted in a propagation environment with elevation spread $< 10^\circ$, essentially achieves the full aperture capacity of the structure, and so is optimal. In environments with larger elevations spreads, however, capacity can be significantly increased by the addition of a second port, or possibly even a third. This situation is even more striking for SNR=20 dB: Here capacity can be significantly increased by the addition of a second port even when all multipath is within $5^\circ$ of the perpendicular. Thus, at high SNRs, the second mode in Fig. 4.2 apparently contributes significantly to capacity for small elevation spreads even though it is highly inefficient. This is perhaps not so surprising when one considers that capacity for high SNRs depends more on the signaling degrees-of-freedom than on received power (i.e., efficiency).

4.2 Antenna Size and Shape

The choice of the size and shape of an antenna are among the most important decisions that engineers make, yet few tools are available to understand in a systematic way how these decisions impact important metrics of wireless communication performance, such as capacity and signaling degrees-of-freedom. Of course, other desirable properties of antennas may influence the design process, such as directivity, gain and bandwidth (e.g., [37] and references therein). In addition, for a structure with multiple ports, we may wish to design the antenna so the ports are uncoupled and resonate at a desired frequency. In
this section, however, we set aside these other figures-of-merit and focus exclusively on using the results of Chapter 3 to investigate the impact of antenna size and shape on the available degrees-of-freedom and aperture capacity.

Consider a copper wire of length $L = 3$ m with conductivity $\sigma = 5.8 \times 10^7$ S/m and radius $r = 0.5$ mm operating at a frequency of $f = 300$ MHz, so $L = 3\lambda$ where $\lambda = c/f$ is the wavelength. For this structure, the impedance and radiation resistance matrices, $Z$ and $R_E$, were calculated numerically using $K = 255$ triangular basis functions, and the efficiencies were obtained by solving (3.37) in MATLAB. Since any shorter wire is a substructure of this wire, as explained in Sec. 3.3, we can evaluate the efficiencies of shorter wires by removing the basis functions corresponding to the missing conductor, which is equivalent to removing the corresponding rows and columns of $Z$ and $R_E$.

Fig. 4.4a plots the efficiencies of the first 7 eigenmodes of the wire versus the length in wavelengths, $L/\lambda$. Since each wire is a substructure of any longer wire, as proved in Sec. 3.3, the modal efficiencies are monotonically increasing functions of $L$. We further
note that the efficiency of each mode transitions from \( \approx 0 \) to \( \approx 1 \) within a relatively narrow interval of length that increases for higher order modes. Since at each length, all but one of the modal efficiencies are near zero or 1, it follows that the sum of the efficiencies

\[
D_F = \sum_{k=1}^{K} \eta_k
\]

is a measure of the number of (efficient) signaling degrees-of-freedom.

Fig. 4.4b plots \( D_F \) for the wire versus length in wavelengths, \( L/\lambda \). Over the range covered by the plot, we see that \( D_F \) grows slightly less than linearly with length; in fact, \( D_F \approx 3(L/\lambda)^{0.8} + 1 \) over this window, so the slope gradually decreases as suggested by the increasing lengths of the mode transitions in Fig. 4.4a. We note that the number of ports needed to capture the aperture capacity of the wire will grow with the number of efficient modes. We further observe that all of these calculations assume spherically-symmetric multipath; the presence of directional multipath can dramatically alter both the efficiencies and the degrees of freedom.
Note $D_F$ is also a good measure of aperture capacity in the low SNR regime. To see this, using (3.12) and the approximation $\ln \det(I + A) \approx \text{Tr}[A]$ for $\| A \|_2 \ll 1$, we find

$$C_e(S_V) \approx \mathcal{E} \left\{ \frac{1}{N} \text{Tr}[H_w^H S_V H_w] \right\} = \text{Tr}[S_V].$$

As noted above, the eigenvalues of $S_V$ are given by $\gamma \eta_k$, so it follows that

$$C_e(S_V) \approx \gamma \sum_{k=1}^{K} \eta_k = \gamma D_F, \ \gamma \ll 1.$$ 

As a second example, we consider an $L \times L$ copper patch positioned 5mm above a matching ground plane and operating at $f = 752.5$ MHz, so $\lambda = 39.87$ cm. We evaluate 10 values of $L$ ranging from $\lambda/3$ to $2\lambda$. For each $L$, the impedance and radiation resistance matrices, $Z$ and $R_E$, are calculated in FEKO using $K$ Rao-Wilton-Glisson basis functions, where $991 \leq K \leq 2130$, and the efficiencies are obtained by solving (3.37) in MATLAB. Because the currents are now two-dimensional, it is reasonable to expect more efficient modes for a given $L$, and that the number of significant modes will grow with area. These expectations are supported by Fig. 4.5a, which plots the efficiencies of the first 70 modes versus $(L/\lambda)^2$, the area in wavelengths. In this figure, we display one out of every five modes to make the plot more readable. Further note that only the circles represent data; the line segments that connect them are included to make the trends more readable and do not necessarily reflect the actual behavior between data points.

In Fig. 4.5a, once again we see that the efficiencies increase monotonically from 0 to 1 with size, and the change in area needed to effect this transition increases with the index of the mode. Fig. 4.5b plots the sum of the efficiencies $D_F$ versus area in wavelengths. For this range, we see that $D_F$ grows sublinearly with length; in fact, $D_F \approx 17.3(L/\lambda)^{2.7}$.
over this window, so the slope gradually decreases as suggested by the increasing length of the mode transitions in the figure. Note that there are a large number of efficient modes even for relatively small patches; for example, the $\lambda/3 \times \lambda/3$ patch has 6 modes with efficiencies greater than 0.1 (cf. Fig. 4.6a below). In principle, one would therefore need at least six ports to capture the aperture capacity of the structure for these propagation conditions.
All of the patches considered above are square. It is natural to ask whether other shapes might provide some advantage in terms of capacity or degrees-of-freedom. To this end, we compare three patches of equal area: a square, a circle and a rectangle.

For the square, consider again the $\lambda/3 \times \lambda/3$ patch above. Fig. 4.6a plots the radiation efficiencies of the first 20 eigencurrents of this patch. Note there are 6 modes with $\eta_k \geq 0.1$, significantly more than the $\lambda/2$ wire antenna considered in Chapter 3. The efficiencies first fall off slowly with $k$, and then more rapidly beyond the first few modes. Fig. 4.6b plots the corresponding aperture capacity $C_e(\Lambda_{V,M})$ versus the ideal single antenna SNR $\gamma$ for $N = 2$ transmit antennas and several values of $M$. Note the first two eigencurrents have $\eta_k \approx 0.98$, and hence achieve a capacity close to a lossless $M = 2$ array; the next four, less-efficient eigencurrents increase capacity up to 30% at high SNR. Beyond $k = 6$, the efficiency becomes too small to contribute significantly towards capacity, so virtually all of the capacity resides in the first 6 eigencurrents.

Next consider a circular patch of radius $r = \lambda/3\sqrt{\pi}$, which has the same area as the
square patch above and operates at the same frequency. Fig. 4.7 shows $\eta_k$ and $C_e(\Lambda_{V,M})$ versus $\gamma$ for this patch. The efficiency profiles in Figs. 4.6a and 4.7a are nearly the same, indicating that both shapes have eigenmodes with very similar radiation efficiencies. Consequently, the capacities available in the two patches are almost identical, as can be seen from Figs. 4.6b and 4.7b. This suggests that small square and circular patches yield roughly the same capacity at all SNRs of interest.

Lastly, consider a $(2/3)\lambda \times \lambda/6$ rectangle, which has the same area as the two patches above. We plot $\eta_k$ and $C_e(\Lambda_{V,M})$ versus $\gamma$ for this patch in Fig. 4.8. Unlike the circular patch, the efficiency profile of the rectangle is not quite the same as the square, as can be seen from Figs. 4.6a and 4.8a. The efficiencies fall off less rapidly in the rectangular shape, resembling a hyperbolic decay. However, it appears that the efficiencies are merely redistributed in the rectangular shape as compared to the square shape. For example, the first two modes have comparable efficiencies in both the rectangle and square; the third and fourth modes for rectangular patch have higher efficiencies than the square.
patch, whereas the fifth and sixth modes are lower in efficiencies for the rectangular patch by almost the same amounts. The sum of the first 12 eigenmode efficiencies is almost the same for both shapes, and their net contributions to aperture capacity are similar, as can be seen in Figs. 4.6b and 4.8b. Therefore, although the eigencurrent efficiencies are changed, the cumulative impact on aperture capacity is nearly the same in both the structures. Thus all three planar patches with the same aperture area have similar capacities at all SNRs of interest.

### 4.3 Quality Factor

The capacity metric $C_e(\Lambda_{V,M})$ was derived under the assumption that arbitrary port impedances can always be made resonant with an appropriate external matching network. However, it is natural to ask how much of this capacity could be achieved with naturally-resonant ports, which require no additional external matching. Further, some of the information contained in $C_e(\Lambda_{V,M})$ may reside in very reactive eigencurrents, which
may be difficult or impossible to match in practice. For these reasons, we would like to investigate how capacity changes when we impose a constraint on the reactivity of the ports, prior to matching.

For a single port with input impedance $Z = R + jX$, a simple way to limit the relative reactivity is to constrain the ratio

$$Q = \frac{X}{R}$$

For a simple reactive component such as an inductor, $Q$ is the well-known (unloaded) quality factor of the component, a measure of efficiency. However, an antenna port is not a simple reactive component, but rather a superposition of eigencurrents with possibly complex dynamical behaviors. Consequently, for ports this ratio does not necessarily measure efficiency. This ratio should also not be confused with the conventional antenna quality factor, which measures the ratio of power stored to power dissipated. We emphasize these distinctions by calling $Q$ the relative reactance.

In this section, we again consider the capacity achievable by a MIMO receiver that observes the transmitted signal through the currents induced in a conducting volume $V$. We first characterize the array impedances that are created when ports are introduced in $V$ and no external matching is used. We then use these results to bound the aperture capacity subject to a constraint on the average relative reactance of these ports. For the special case where relative reactance is constrained to zero, we obtain an upper bound the capacity that can be achieved by naturally-resonant ports.
4.3.1 Port without Matching

In (3.31), we considered a model of antenna ports that observe the currents on \( V \) through an arbitrary passive coupling network. This model is so general that it can encompass both port placement and external matching. Here we would like to separate port placement from matching, and thus we need a different model. In this section, we consider a port to be a device that can observe and interact with (i.e., load) a one-dimensional current at a selected location in \( V \). For simplicity, we consider here only ports that coincide with method-of-moments basis functions, although it will become apparent that the results also apply to any port that can be modeled as a linear combination of these basis functions.

Consider a volume described by the generalized impedance \( Z = R + jX \), where \( Z \in \mathbb{C}^{N \times N} \) is symmetric. Suppose we consider an \( m \)-port system formed from basis functions of this system. Without loss of generality, we take these basis functions to be \( i = 1, \ldots, m \), to simplify notation. Let \( Y = Z^{-1} \) be the admittance matrix of the structure, which is partitioned as

\[
Y = \begin{bmatrix}
Y_{11} & Y_{12} \\
Y_{T12} & Y_{22}
\end{bmatrix},
\]

where \( Y_{11} \in \mathbb{C}^{m \times m} \), \( Y_{12} \in \mathbb{C}^{m \times (N-m)} \) and \( Y_{22} \in \mathbb{C}^{(N-m) \times (N-m)} \). Then the impedance of the \( m \)-port is given by

\[
Z_A = Y_{11}^{-1}.
\]

We now express \( Z_A \) explicitly in terms of \( Z \). If \( Z \) is partitioned the same way as \( Y \),
\[
Z = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z^T_{12} & Z_{22}
\end{bmatrix}.
\] (4.4)

then from the matrix inversion lemma [66, pg. 406], it follows that

\[
Y_{11} = \left[Z_{11} - Z_{12}Z^{-1}_{22}Z^T_{12}\right]^{-1}
\] (4.5)

Subsequently, we can show

\[
Z_A = Z_{11} - Z_{12}Z^{-1}_{22}Z^T_{12} = M \begin{bmatrix}
Z_{11} & Z^T_{21} \\
Z_{21} & Z_{22}
\end{bmatrix}M^T \triangleq M \begin{bmatrix}
Z_{11} & Z^T_{21} \\
Z_{21} & Z_{22}
\end{bmatrix}M^H.
\] (4.6)

where \( M \in \mathbb{C}^{m \times N} \) is given by

\[
M = \begin{bmatrix}
I_m & -Z_{12}Z^{-1}_{22}
\end{bmatrix}.
\] (4.7)

This form is similar to the impedance produced by a general coupling network illustrated in Part I. We can also express the signal covariance for this \( m \)-port system in a similar way. Let us suppose the Thevenin equivalent for the entire structure be

\[
V = ZJ + V_0, \quad \Sigma_{hV} = E\left[V_0V_0^H\right]
\]

where, \( \Sigma_{hV} \) and \( Z \) are the fading covariance and generalized impedance of the entire
structure as defined earlier. The Norton Equivalent for the structure would be

\[
\mathbf{J} = \mathbf{YV} + \mathbf{J}_0, \quad \Sigma_{\mathbf{J}_0} = \mathbf{Y} \Sigma_{\mathbf{h}_V} \mathbf{Y}^H.
\]

where \( \mathbf{J}_0 = -\mathbf{YV}_0 \) is the short-circuit current. The short-circuit current of the first \( m \) basis functions can then be obtained as the principal \( m \times m \) submatrix of \( \Sigma_{\mathbf{J}_0} \), namely

\[
\begin{bmatrix}
\mathbf{Y}_{11} & \mathbf{Y}_{12}
\end{bmatrix} \Sigma_{\mathbf{h}_V} \begin{bmatrix}
\mathbf{Y}_{11} & \mathbf{Y}_{12}
\end{bmatrix}^H.
\]

Converting back to the impedance domain, we find the fading covariance of the \( m \)-port is

\[
\Sigma_{\mathbf{h}_{\mathbf{A}}} = \mathbf{Z}_{\mathbf{A}} \begin{bmatrix}
\mathbf{Y}_{11} & \mathbf{Y}_{12}
\end{bmatrix} \Sigma_{\mathbf{h}_V} \begin{bmatrix}
\mathbf{Y}_{11} & \mathbf{Y}_{12}
\end{bmatrix}^H \mathbf{Z}_{\mathbf{A}}^H = \begin{bmatrix}
\mathbf{I}_m & \mathbf{Y}_{11}^{-1} \mathbf{Y}_{12}
\end{bmatrix} \Sigma_{\mathbf{h}_V} \begin{bmatrix}
\mathbf{I}_m & \mathbf{Y}_{11}^{-1} \mathbf{Y}_{12}
\end{bmatrix}^H. \quad (4.8)
\]

Applying the matrix inversion lemma again, we have

\[
\mathbf{Y}_{12} = \mathbf{Z}_{11}^{-1} \mathbf{Z}_{12} \left[ \mathbf{Z}_{12}^T \mathbf{Z}_{11}^{-1} \mathbf{Z}_{12} - \mathbf{Z}_{22} \right]^{-1}.
\]

Thus, it follows from (4.5)

\[
\mathbf{Y}_{11}^{-1} \mathbf{Y}_{12} = \left[ \mathbf{Z}_{11} - \mathbf{Z}_{12} \mathbf{Z}_{22}^{-1} \mathbf{Z}_{12}^T \right] \mathbf{Z}_{11}^{-1} \mathbf{Z}_{12} \left[ \mathbf{Z}_{12}^T \mathbf{Z}_{11}^{-1} \mathbf{Z}_{12} - \mathbf{Z}_{22} \right]^{-1}
\]

\[
= \left[ \mathbf{Z}_{12} - \mathbf{Z}_{12} \mathbf{Z}_{22}^{-1} \mathbf{Z}_{12}^T \mathbf{Z}_{11}^{-1} \mathbf{Z}_{12} - \mathbf{Z}_{22} \right]^{-1}
\]

\[
= -\mathbf{Z}_{12} \mathbf{Z}_{22}^{-1} \left[ \mathbf{Z}_{12}^T \mathbf{Z}_{11}^{-1} \mathbf{Z}_{12} - \mathbf{Z}_{22} \right] \left[ \mathbf{Z}_{12}^T \mathbf{Z}_{11}^{-1} \mathbf{Z}_{12} - \mathbf{Z}_{22} \right]^{-1} = -\mathbf{Z}_{12} \mathbf{Z}_{22}^{-1}.
\]

From (4.8), we conclude that \( \Sigma_{\mathbf{h}_{\mathbf{A}}} = \mathbf{M} \Sigma_{\mathbf{h}_V} \mathbf{M}^H \).

Thus, choosing a subset of the basis functions as ports has the same mathematical
form as the general coupling networks considered in Part I. We note that this coupling network must be passive, so all the results proved in Part I apply here. In particular, the SNR matrix for the observation is

\[
S_A = \rho \left[ \mathbf{M} \Sigma_{hv} \mathbf{M}^H \right]^{1/2} \mathbf{R}_A^{-1} \left[ \mathbf{M} \Sigma_{hv} \mathbf{M}^H \right]^{1/2}
\]

\[
= \rho \mathbf{U}^H \Sigma_{hv}^{1/2} \mathbf{M}^H \mathbf{R}_A^{-1} \mathbf{M} \Sigma_{hv}^{1/2} \mathbf{U}
\]  

(4.9)

where \( \mathbf{U} \triangleq \Sigma_{hv}^{1/2} \mathbf{M}^H \left[ \mathbf{M} \Sigma_{hv} \mathbf{M}^H \right]^{-1/2} \) satisfies \( \mathbf{U}^H \mathbf{U} = \mathbf{I}_M \). Since the array is passive, we also have \( \mathbf{M}^H \mathbf{R}_A^{-1} \mathbf{M} \leq \mathbf{R}^{-1} \), so

\[
S_A \leq \rho \mathbf{U}^H S_V \mathbf{U} .
\]  

(4.10)

For a single impedance \( \mathbf{Z} = R + jX \), the quality factor is defined as \( Q = |X|/R \). But how should we define it for multiports? One possible approach would be to define it in terms of the quality factors of the single-ports that arise when the multiport is decoupled. Let us consider an \( m \times m \) impedance matrix \( \mathbf{Z} \). If we decouple the impedances via characteristic modes [60], we have

\[
\mathbf{J}^H \mathbf{R} \mathbf{J} = \mathbf{I}, \quad \mathbf{J}^H \mathbf{X} \mathbf{J} = \Lambda .
\]

The quality factor could be defined by mean squared eigenvalues of the decoupled port modes, namely

\[
Q = \sqrt{\frac{1}{m} \sum_{k=1}^{m} |\lambda_k|^2} .
\]

We now express these quality factors more directly in terms of \( \mathbf{Z} \). Note that the charac-
teristic values, $\mathbf{A} = \text{diag}(\lambda_1, \ldots, \lambda_m)$, are the eigenvalues of the real symmetric matrix

$$Q = R^{-1/2}XR^{-1/2}$$

It follows that the matrix $QQ^H$ has eigenvalues $|\lambda_1|^2, \ldots, |\lambda_m|^2$. Therefore, the quality factor metrics can be expressed as

$$Q = [(1/m)\text{tr}(QQ^H)]^{1/2} = \sqrt{(1/m)} \| Q \|_F$$

where $\| Q \|_F$ is the Frobenius norm of $Q$. Thus, the quality factor of $m$ ports is equal to well known matrix norm.

**4.3.2 $Q$ Constrained Capacity**

Suppose we wish to consider $m$-ports with the property that each port is uncoupled and has a quality factor of at most $Q$. What limits does this impose on capacity? How are these limits related to the underlying structure $V$?

From above, we note that the impedance of each $m$-port can be expressed as $Z_A = MZM^H$, where $M$ is given in (4.7). Then

$$R_A = MRM^H, \quad X_A = MXM^H$$

and the SNR and $Q$-matrices are

$$S_A = \rho R_A^{-1/2}M\Sigma_hvM^HR_A^{-1/2} \quad (4.11)$$

$$Q_A = R_A^{-1/2}MXM^HR_A^{-1/2} \quad (4.12)$$
Our goal is to determine how large $C_e(S_A)$ can be subject to the constraint $\| Q_A \|_F \leq Q$.

If we define $U = R^{1/2} M^H R_A^{-1/2}$, then we can express this problem as the maximization of

$$I(U) = C_e(U^H S_V U)$$

over all $N \times m$ matrices such that

$$U^H U = I_m, \| U^H Q_V U \|_F \leq Q$$

where $Q_V = R^{-1/2} X R^{-1/2}$ is the Q-matrix of the aperture.

In general, an optimization problem subject to $m$ inequality constraints and $k$ equality constraints can be expressed as

$$\min_{x \in \mathbb{R}^n} f(x), \text{ such that } g(x) \leq 0_m, \ h(x) = 0_k.$$ 

The Karush-Kuhn-Tucker (KKT) conditions give necessary conditions that must be satisfied by any optimum, and are most easily expressed in terms of the associated Langrangian, defined as

$$\mathcal{L}(x; \mu, \lambda) = f(x) + \lambda^T h(x) + \mu^T g(x),$$

where $\mu \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}^k$. If $x_o$ achieves a minimum (local or global), then there exist
\( \mu_o, \lambda_o \) such that

\[
\nabla_x L(x_o; \mu_o, \lambda_o) = 0_n \\
g(x_o) \leq 0_m, \quad h(x_o) = 0_k \\
\mu_o \geq 0_m, \quad \mu_{oj} g_j(x_o) = 0, \; j = 1, \ldots, m.
\]

When \( f, g, h \) are convex, these conditions are also sufficient for global optimality.

### 4.3.3 Special Case: One Port

For \( m = 1 \), maximizing \( C_e(U^H S_V U) \) in (4.13) is the same as minimizing the scalar

\[
f(U) = -U^H S_V U.
\]

Thus the Langrangian is given by

\[
L(U; \lambda, \mu) = -U^H S_V U + \lambda [U^H U - 1] + \mu [(U^H Q U)^2 - Q^2].
\]

In order for \( U_o \) to be the minimum, there must exist parameters \( \lambda_o \) and \( \mu_o \geq 0 \) such that

\[
\nabla_U L(U_o; \lambda_o, \mu_o) = -2S_V U_o + 2\lambda_o U_o + 4\mu_o (U_o^H Q U_o) Q U_o \\
= -2 [S_V - \lambda_o I - 2\mu_o (U_o^H Q U_o) Q] U_o = 0,
\]

where

\[
U_o^H U_o = 1, \quad (U_o^H Q U_o)^2 \leq Q^2, \quad \mu_o [(U_o^H Q U_o)^2 - Q^2] = 0.
\]

There are two types of solutions.
Type I: \( \mu_o = 0 \). In this case, the \( Q \) constraint is inactive and the necessary conditions become

\[
S_V U_o = \lambda_o U_o , \quad U_o^H U_o = 1 , \quad (U_o^H Q U_o)^2 \leq Q^2 .
\]

It follows that \( U_o \) is a unit-magnitude eigenvector of \( S_V \) with eigenvalue \( \lambda_o \), so the optimal value of \(-U^H S_V U\) is \(-\lambda_o\). Clearly, the global minimum is the unit-magnitude eigenvalue associated with the largest eigenvalue of \( S_V \). If this satisfies \(|U_o^H Q U_o| \leq Q\), then it is the global minimum.

Type II: \( \mu_o > 0, U_o^H Q U_o = \sigma Q, \sigma = \pm 1 \). In this case, the \( Q \) constraint is active and the necessary conditions become

\[
[S_V - 2\mu_o \sigma Q Q] U_o = \lambda_o U_o , \quad U_o^H U_o = 1 , \quad U_o^H Q U_o = \sigma Q .
\]

It follows that \( U_o \) is a unit-magnitude eigenvector of \( S_V - 2\mu_o Q Q \) with eigenvalue \( \lambda_o \), so the optimal value of the objective function is

\[
-U_o^H S_V U_o = -\lambda_o - 2\mu_o Q U_o^H Q V U_o ,
\]

so the minimum is obtained by the eigenvalue and unit eigenvector that give the largest value of \( \lambda_o + 2\mu_o \sigma Q U_o^H Q V U_o \). Here \( \mu_o \) and \( \sigma \) are chosen such that \( U_o^H Q U_o = \sigma Q \).

This suggests a way to plot the optimal value of \( f(U) \) versus \( Q \) parametrically in terms of \( r = 2\mu_o \sigma Q \). For \( r = 0 \), as noted above \( f(U) = -\lambda_1(S_V) \) and \( \sigma \) and \( Q(0) \) are defined as the sign and magnitude of \( U_o^H Q U_o \). For \( r \neq 0 \), we find the eigenvalues and eigenvectors of \( S_V - r Q \). Next, we choose \( \lambda_o \) and \( U_o \) to maximimze \( \lambda_o + r U_o^H Q V U_o \); we call this value \(-f(r)\). If we define \( \sigma_o \) and \( Q(r) \) as the sign and magnitude of \( U_o^H Q V U_o \)
and $\mu_o = r/(2\sigma_o Q(r))$, then all parameters satisfy the necessary conditions. We can subsequently plot $f(r)$ versus $Q(r)$ parametrically.

### 4.3.4 Two Ports: A Numerical Approach

We now consider a numerical approach to the general two-port problem, where $m = 2$, and $S_V$ and $Q_V$ are arbitrary, real symmetric $N \times N$ matrices. We would like to compute the maximum of the function

$$I(U) = C_e (U^H S_V U)$$

(4.15)

over all $N \times 2$ matrices $U$ such that

$$U^H U = I_m, \quad \|U^H Q_V U\|_F \leq \sqrt{2Q}.$$  

(4.16)

In this last constraint, $Q$ is a bound on the mean-squared reactance of the uncoupled ports.

**Small SNR:** For the special case of small signal-to-noise ratios, we can take the objective function to be

$$f(U) = \text{Tr}[U^H S_V U]$$

(4.17)

and the constraints can always be expressed as

$$\|U^H U - I_m\|_F = 0, \quad \text{Tr}[(U^H Q_V U)^2] \leq 2Q^2.$$  

(4.18)

One approach to solve this system is to perform unconstrained minimization of the
Langrangian

\[ L(U; \lambda, \mu) = -\text{Tr} \left[ U^H S_V U \right] + \lambda \| U^H U - I_m \|_F^2 + \mu \text{Tr} \left[ (U^H Q_V U)^2 \right] . \]  

(4.19)

Setting \( \lambda >> 1 \) will force the solution to satisfy \( U^H U = I_m \). For a given \( \mu \geq 0 \), if we find a minimum of this function, say \( U_o \) that satisfies \( U^H U = I_m \) then it will satisfy

\[-\text{Tr} \left[ U_o^H S_V U_o \right] + \mu \text{Tr} \left[ (U_o^H Q_V U_o)^2 \right] \leq -\text{Tr} \left[ U^H S_V U \right] + \mu \text{Tr} \left[ (U^H Q_V U)^2 \right] , \]  

(4.20)

for all \( U \) that satisfy \( U^H U = I_m \), or equivalently

\[-\text{Tr} \left[ U^H S_V U \right] \geq -\text{Tr} \left[ U_o^H S_V U_o \right] + \mu \left( \text{Tr} \left[ (U_o^H Q_V U_o)^2 \right] - \text{Tr} \left[ (U^H Q_V U)^2 \right] \right) , \]  

(4.21)

Thus \( U_o \) minimizes the function over all \( \text{Tr} \left[ (U^H Q_V U)^2 \right] \leq 2Q^2 \equiv \text{Tr} \left[ (U_o^H Q_V U_o)^2 \right] \). Choosing different values of \( \mu \) will yield different values of \( Q \).

This function can be minimized by gradient descent. For a function of a matrix, the chain rule yields

\[ L(U + \Delta U) = L(U) + \text{Tr} \left[ \left( \frac{\partial f}{\partial U} \right)^T \Delta U \right] + O \left( \| \Delta U \|_F^2 \right) \]  

(4.22)

Thus a simple gradient descent strategy would be to choose each step to be of the form

\[ \Delta U = -h \frac{\partial f}{\partial U} , \]

where \( h \) is a small positive number.

In order to calculate the gradient, we note that, for any matrices \( A \in \mathbb{R}^{N \times N} \) and
Let $B \in \mathbb{R}^{m \times m}$, we get from [108, pg. 13]

$$
\frac{\partial}{\partial U} \text{Tr} [U^H A U B] = A U B + A^H U B^H \quad (4.23)
$$

$$
$$

If $A$ is symmetric, these imply

$$
\frac{\partial}{\partial U} \text{Tr} [U^H A U] = 2 A U, \quad (4.24)
$$

$$
\frac{\partial}{\partial U} \text{Tr} [U^H U A] = 2 U A, \quad (4.25)
$$

$$
\frac{\partial}{\partial U} \text{Tr} [(U^H A U)^2] = 4 A U U^H A U. \quad (4.26)
$$

Also, it is noteworthy that

$$
\| U^H U - I \|_F^2 = \text{Tr} [(U^H U)^2] - 2 \text{Tr} [U^H U] + m.
$$

It follows from (4.25) and (4.26) that

$$
\frac{\partial}{\partial U} \| U^H U - I \|_F^2 = 4 U U^H U - 4 U = 4 [U U^H - I] U.
$$

Using (4.23) and (4.26), we conclude the gradient is given by

$$
\frac{\partial \mathcal{L}(U; \lambda, \mu)}{\partial U} = -2 S_V U + 4 \lambda [U U^H - I] U + 4 \mu Q_V U U^H Q_V U.
$$

We can now describe the complete gradient descent procedure: First let $U_0$ be the solution of the unconstrained problem, where the columns are the eigenvectors associated
with the two largest eigenvalues of $S_V$. Next choose a fixed $h > 0$, $\mu > 0$ and $\lambda \gg 1$ and define the recursion

$$U_{k+1} = U_k - h \frac{\partial L(U_k; \lambda, \mu)}{\partial U}, \quad k = 0, 1, 2, \ldots$$

In practice, we can stop the recursion when $\|U_{k+1} - U_k\|_F$ falls below some threshold $\tau h \ll h$, or equivalently

$$\left| \frac{\partial L(U_k; \lambda, \mu)}{\partial U} \right|_F < \tau.$$

Fig. 4.9 shows the variation of aperture capacity as a function of maximum allowable $Q$ for the example of a $\lambda/2$ dipole with two ports at SNR = 0 dB. When $\mu = 0$, the $Q$-constraint is inactive and the aperture capacity is maximum. With increasing $\mu$, the maximum allowable $Q$ decreases sharply and beyond $Q = 100$, capacity starts falling off with increasing $\mu$, as can be seen from Figs. 4.9a and 4.9b respectively. For a range of $\mu$ between 0 and 10, maximum allowable $Q$ decreases from 140 to nearly 0, which in turn,
reduces the available capacity by nearly 42% compared to the unconstrained aperture capacity. This shows that, for the example of a wire with two ports, the available capacity at low SNR may be significantly limited if matching restriction is imposed.

4.4 Conclusions

In this chapter, we have evaluated the maximum achievable narrowband MIMO capacity by any arbitrary antenna array using numerical examples of both one and two dimensional antenna structures and different signal propagation conditions. We have shown that for wire antennas with omnidirectional radiation pattern in azimuthal plane, aperture capacity changes significantly with the elevation angular spread of directional multipath. Wider elevation angular spread of DoA results in stronger excitation of existing eigenmodes in the aperture of the wire and the capacity associated with antennas embedded in spherically uniform multipath is an upper bound to the capacity conferred by directional multipath. For planar antennas embedded in spherically uniform multipath, aperture capacity decreases with reduction in aperture size, which is in accordance with Theorem 2 of Chapter 3. More interestingly, aperture capacity is nearly shape-invariant for patch antennas with similar size, but various shapes, e.g. square, circular and rectangular. Therefore, we have illustrated that the impact of aperture shape is insignificant on the available capacity for planar antenna structures. Finally, we express the problem of maximizing available capacity in any arbitrary antenna array as an optimization problem subject to constraints on maximum allowable $Q$ for eigenmodes whose contribution towards aperture capacity would be counted. We show that $Q$-constrained capacity can be up to 42% less than the unconstrained aperture capacity for the examples of wire antennas with two feeds.
The results of this chapter can be extended to other antenna structures and wireless communication scenarios. Antennas with non-omnidirectional radiation pattern in azimuth can easily be analyzed by considering $p(\theta, \phi)$ as a function of both $\theta$ and $\phi$ and replacing $p(\theta, \phi)$ with appropriate APS (angular power spectrum) function for non-uniform power density profile. In addition to planar antennas like patches, three dimensional antennas with similar volume and different shapes can be analyzed as well. High aspect ratio antenna structures may be of relevance for compact antenna design and hence, the impact of extremely asymmetric aperture shape on available capacity would be worthy to explore. Lastly, it would be of interest to study the impact of $Q$ on aperture capacity for systems with wide bandwidth, for which it will be important to characterize the broadband behavior of the eigencurrents.
Chapter 5

Conclusion and Future Work

In this chapter, we summarize the important contributions of this dissertation in Sec. 5.1 and conclude with suggestions for future work in Sec. 5.2.

5.1 Summary of Dissertation

In this dissertation, we have looked into the information-theoretic design aspects of two most important components in a compact MIMO transceiver, viz. antenna array and matching network.

In Chapter 1, we briefly reviewed the existing literature related to MIMO technology, the limitations of its deployment in small devices due to interactions between closely spaced antenna elements, viz. signal correlation and mutual coupling, design of impedance matching networks to mitigate capacity loss due to these interactions and the effect of positive system bandwidth on matching. We observed that the common objective of compact MIMO transceiver design in these prior works is to maximize capacity by using sophisticated narrowband (i.e. frequency non-selective or FNS) multiport impedance
matching, which is optimized over small bandwidth. This design approach, however, has the following drawbacks:

1. The narrowband multiport impedance matching technique maximizes capacity only over a small bandwidth and may not offer optimum performance when the system bandwidth is no more negligible.

2. Even for narrowband systems, multiport matching network is designed to maximize the capacity performance for a given antenna array. This design strategy does not guarantee overall system performance optimality of the transceiver, unless the antenna array is designed efficiently as well, prior to matching.

The lack of these design features has motivated the detailed study on design of frequency-selective (FS) matching networks as well as MIMO antenna arrays from information-theory perspective in the subsequent chapters.

In Chapter 2, we reviewed the capacity of compact MIMO systems over non-negligible bandwidth and showed that FS matching promises substantial capacity benefits over positive bandwidth. We proposed a *decoupling prior to matching* technique using which FS matching can be easily realized in hardware and hence, the predicted capacity gains can be closely achieved in practice. To demonstrate the efficacy of FS matching, we evaluated ergodic and outage capacity metrics of $2 \times 2$ and $4 \times 4$ UCAs and ULAs over 10% fractional bandwidth using both FS and FNS matching techniques. Simulation results showed that the benefits of FS match over FNS match scale with both array dimension and signal-to-noise ratio. This chapter establishes that prior studies on FNS matching over positive bandwidth have significantly underestimated capacity and that FS matching is crucial for optimizing capacity of closely coupled antenna arrays that are inherently frequency-selective.
Chapter 3 addressed information-theoretic design of arbitrary MIMO antenna array over narrow bandwidth. The main contributions of this chapter can be outlined as follows: First, we considered an arbitrary MIMO antenna array that detects the transmitted signal by observing current induced by the signal in a conducting volume $V$ and characterized the maximum narrowband MIMO capacity that can be achieved by this array using a front-end with optimum narrowband matching followed by a bank of noisy amplifiers. Second, we established that an upper bound to this capacity, namely aperture capacity, constitutes all the useful information contained in the currents of $V$ and this information, in turn, can be associated with a system of unit-power eigencurrents that can be interpreted physically as decomposing the induced current in $V$ into a set of scalar channels with independent signal and noise observations. Third, we bounded the capacity of any $M$-port array that observes the signal through $V$ by the best $M$ eigencurrents and also derived conditions under which the bound can be achieved. Finally, we showed that the capacity of any substructure $\bar{V} \subseteq V$ is bounded by that of $V$, which has potential application in antenna shape optimization.

Chapter 4 explored the information-theoretic MIMO antenna design guidance developed in Chapter 3 at a greater depth. We looked into directional propagation and its impact on aperture capacity for wire antennas, of which a special case, viz. spherically uniform multipath was studied in Chapter 3. We demonstrated that for a wire, the radiation efficiency of unit-power eigecurrents increases rapidly with elevation angle spread. Moreover, the relative impact of these modes on aperture capacity depends not only on the efficiency, but also on the SNR in that, the contribution of even a less efficient eigencurrent (i.e. when the elevation angle spread is small) towards capacity may be significant, when the SNR is high. We also illustrated how the number of efficient modes as well as their efficiency grow with the size of the aperture using the examples of wire
and patch. For small patch, we investigated three different shapes, viz. square, circular and rectangular, having the same area and found out aperture capacity to be nearly shape-invariant. Finally, we considered an $M$-port array with the property that each port is uncoupled with a quality factor of at most $Q$ and looked into the limits this may possibly impose on capacity. While the $Q$-constrained capacity for single-port array being not much different than unconstrained capacity, for $M = 2$, the available capacity may be up to 42% less than unconstrained capacity at low SNR if $Q$ is constrained.

5.2 Scopes for Future Work

The interdisciplinary work in this dissertation opens up several avenues for future research. In this section, we highlight some of these potential research directions that can be pursued to solve interesting open problems.

Frequency-selective adaptive matching

For compact MIMO systems with coupled antennas, frequency-selective matching promises capacity improvement over conventional frequency-nonselective matching over positive system bandwidth. In Chapter 3, the proposed *decoupling and matching* technique optimizes the matching network parameters based on the frequency-characteristics of the antenna array impedance matrix, which is assumed to be time-invariant. But in practice, the array impedance may change with the perturbations in the antenna near-field, e.g. movement of objects near the antenna, change in the signal propagation condition etc. These, in turn, may affect the array coupling, of which the accurate model is required for optimal frequency-selective matching design. Therefore, to retain the benefits of frequency-selective matching, the static matching network design proposed in this work...
needs to be extended to adaptive matching, where the frequency-selective matching is also reconfigurable in order to adapt to the changing array impedance characteristics over time. In addition, the decoupling technique used here is frequency-invariant, owing to the circulant symmetry in array geometry which can not decouple general, asymmetric array structures. Also, due to near-field loading-effects, even the symmetric array structures may cease to reflect the circular pattern in the array impedance. Hence, frequency-selective and adaptive decoupling network design remains another challenging open problem that needs to be addressed in future.

**Information-theoretic broadband antenna design**

The framework of information-theoretic MIMO antenna design developed in Chapter 3 is based on narrowband (i.e. single frequency) analysis of capacity and therefore, does not take into account the bandwidth of eigencurrents. To derive the performance limits of broadband MIMO antenna, it is imperative to characterize the broadband behavior of the eigencurrents. Due to the limitations imposed by broadband matching constraint, arbitrary antenna impedance can not be perfectly matched over a positive bandwidth and hence, the underlying assumption of perfect matching for the narrowband capacity metrics derived in Chapter 3 will not be a reasonable assumption anymore for broadband systems. Frequency variation of incident signal and noise fields will manifest into induced current in the antenna aperture, which in turn will impact the aperture capacity over a positive bandwidth. Thus, a natural extension of this work would be to extend the information-theoretic antenna design framework for broadband systems by modeling the frequency-selective behavior of eigenmodes and taking into consideration the broadband matching limitations in deriving the capacity metrics.
REFERENCES


APPENDICES
Appendix A

Details of Chapter 2

A.1 Equivalence between MNF and MCM Match

According to the standard amplifier noise model [6, pg. 15], it can be represented as a noisy two-port network with Thevenin equivalent impedance matrix as in equation (2.5) and open-circuit noise voltages \( \{n_1, n_2\} \) at the input and output ports respectively, where \( n_i \approx \mathcal{CN}(0, \sigma_i^2) \), with \( \rho_{12} \triangleq \frac{\mathbb{E}[n_1 n_2]}{\sigma_1 \sigma_2} \neq 0 \). In addition to the noise parameters \( \{\sigma_1, \sigma_2, \rho_{12}\} \), the Rothe-Dahlke noise model [114] is another alternative approach to characterize amplifier noise. The voltage and current sources at the input port as shown in Fig. 2.1 are related to the conventional noise parameters as

\[ v_a = n_1 + i_a (z_{11} - z_{\text{cor}}), \quad i_a = -\frac{n_2}{z_{21}} \]  

(A.1)

where, the correlation impedance \( z_{\text{cor}} \), defined so that \( \mathbb{E}[v_a i_a^*] = 0 \) is given by

\[ z_{\text{cor}} = r_{\text{cor}} + jx_{\text{cor}} = z_{11} - z_{12} \rho_{12} \frac{\sigma_1}{\sigma_2} \]  

(A.2)
The equivalent noise resistance of the voltage source and equivalent noise conductance of the current source are given by

\[ r_a = \frac{\mathcal{E}[|v_a|^2]}{4kT_0B}, \quad g_a = \frac{\mathcal{E}[|i_a|^2]}{4kT_0B} \quad (A.3) \]

In this model, the noise statistics of low-noise amplifiers (LNA) typically are characterized by parameters \( \{F_{\text{min}}, g_a, z_{\text{opt}}\} \), where \( F_{\text{min}} \) and \( z_{\text{opt}} \) are given by equation (2.6).

**Theorem 1:** For unilateral amplifier with zero input noise, multi-port MNF match converges to MCM.

**Proof:** For a unilateral amplifier, the gain from its output port to input port, i.e. reverse path gain is zero [110, pg. 524]. This implies \( z_{12} = 0 \) in Fig. 2.1 and \( z_{\text{cor}} = z_{11} \) in equation (A.2). When an impedance \( z_s \) is connected to the input port of an amplifier represented by Rothe-Dahlke model [114] with \( z_{12} = 0 \), the expression of noise current at the output port is [32]

\[ i_{\text{out}} = \frac{z_{21}}{(z_{11} + z_s)(z_{22} + z_L)} [v_a + (z_{\text{cor}} + z_s)i_a] \quad (A.4) \]

Given the above expression, the total noise current at the output of an unilateral amplifier is independent of source impedance \( (z_s) \) [48] only if \( v_a = i_a(z_{11} - z_{\text{cor}}) \). This implies noise-free input port, i.e. \( n_1 = 0 \) in equation (A.1) which makes \( r_a = 0 \) in equation (A.3) and in turn, \( z_{\text{opt}} = (r_{\text{cor}} - jx_{\text{cor}}) \) in equation (2.6). Therefore, for a unilateral amplifier with zero input noise, the noise parameters characterizing Rothe-Dahlke amplifier
noise model are summarized as

\[ z_{12} = 0, \quad z_{\text{cor}} = z_{11} \]  
\[ r_a = 0, \quad z_{\text{opt}} = z^*_{11} \]  
\[ n_1 = 0, \quad F_{\text{min}} = 1 + 4g_ar_{\text{cor}} \]

(A.5a)  
(A.5b)  
(A.5c)

Using these revised noise parameters, the FS MNF matching network in equation (2.16) takes the form

\[
Z^k_M = j \begin{bmatrix}
\Im(Z^k_{\Delta})^\dagger & (\Re z^*_{11}R^k_{\Delta})^{1/2} \\
(\Re z^*_{11}R^k_{\Delta})^{1/2} & \Im z^*_{11}I
\end{bmatrix}
\]

(A.6)

where the input and output port of the matching network are simultaneously conjugate matched with the receive antenna and front-end amplifier and is similar to MCM [138].

### A.2 Partial Decoupling of ULA

An \( n \times n \) centrosymmetric matrix \( Z \) is symmetric about its center and its \( \{i, j\} \)-th entry satisfies

\[
[Z]_{i,j} = [Z]_{n-i+1,n-j+1} \quad \forall 1 \leq i, j \leq n
\]

(A.7)

A \( 2 \times 2 \) centrosymmetric matrix has the same form as circulant matrix in (2.24). Uniform linear array (ULA) with \( n_R > 2 \) has impedance matrix that is primarily centrosymmetric in structure, but has some additional symmetries as well. For example, a \( 4 \times 4 \)
centrosymmetric impedance matrix $Z^k_A$ has the form

$$Z^k_A = \begin{bmatrix}
    z^k_{11} & z^k_{12} & z^k_{13} & z^k_{14} \\
    z^k_{21} & z^k_{22} & z^k_{23} & z^k_{24} \\
    z^k_{31} & z^k_{32} & z^k_{33} & z^k_{34} \\
    z^k_{41} & z^k_{42} & z^k_{43} & z^k_{44}
\end{bmatrix}$$  \hspace{1cm} (A.8)

For ULA, additionally $z^k_{12} = z^k_{21}$ and $z^k_{13} = z^k_{24}$. Therefore, the impedance matrix of a $4 \times 4$ ULA takes the form

$$Z^k_A = \begin{bmatrix}
    z^k_{11} & z^k_{12} & z^k_{13} & z^k_{14} \\
    z^k_{21} & z^k_{22} & z^k_{23} & z^k_{24} \\
    z^k_{31} & z^k_{32} & z^k_{33} & z^k_{34} \\
    z^k_{41} & z^k_{42} & z^k_{43} & z^k_{44}
\end{bmatrix}$$  \hspace{1cm} (A.9)

If the Butler matrix in (2.28) is applied to (A.9), then the resulting $\Lambda^k_A \triangleq Q^{-1} Z^k_A Q^{-\dagger}$ is no more diagonal as (2.22) for UCA, but $\lambda^k_{13} = \lambda^k_{31} = 0$, i.e. there are fewer non-zero off-diagonal terms in $\Lambda^k_A$ than in $Z^k_A$ of (A.9). The diagonal terms of $\Lambda^k_A$ or eigen impednaces are given by

$$\lambda^k_1 = \frac{(z^k_{11} + z^k_{22} + z^k_{14} + z^k_{23})}{2} + z^k_{12} + z^k_{13}, \quad \lambda^k_2 = \frac{z^k_{11} + z^k_{22}}{2} - z^k_{13},$$

$$\lambda^k_3 = \frac{(z^k_{11} + z^k_{22} - z^k_{14} - z^k_{23})}{2} - z^k_{12} + z^k_{13}, \quad \lambda^k_4 = \lambda^k_2$$
Appendix B

Details of Chapter 3

B.1 Radiation Resistance

Suppose that arbitrary currents $i = (i_1, \ldots, i_M)^T$ are applied to the array, so the power delivered is $(1/2)i^H \mathbf{R}_E i$. The electric field produced by these currents at $(r, \theta, \phi)$ in the far field is given by

$$ E(r, \theta, \phi) = \sum_{m=1}^{M} \frac{e^{jk r}}{r} i_m E_m(\theta, \phi) $$

where $k = 2\pi/\lambda$. Thus, the total power radiated is

$$ \frac{1}{2\eta} \iint_{S_1} |E(r, \theta, \phi)|^2 dS = \frac{1}{2\eta} \iint_{S_1} \left| \sum_{m=1}^{M} i_m E_m(\theta, \phi) \right|^2 dS. $$

For a lossless array, all the power delivered to the array is radiated, so

$$ (1/2)i^H \mathbf{R}_E i = \frac{1}{2\eta} \iint_{S_1} \left| \sum_{m=1}^{M} i_m E_m(\theta, \phi) \right|^2 dS. $$
Since this holds for all $i$, we can conclude that

$$[R_{E}]_{mn} = \frac{1}{\eta} \iint_{S_1} E_m^*(\theta, \phi)E_n^*(\theta, \phi) \, dS.$$  

Note the patterns are normalized, so $i_m = 1$ A produces the pattern $E_m(\theta, \phi)$ for each $m$.

### B.2 Fading Path Gain Correlation

We now derive the fading correlation formula given in (3.7). When antenna $m$ is driven by a current $i_m$, the resulting electric field is given in the far-field region by [4, eq. 2-92]

$$E(r, \theta, \phi) = -\frac{j\eta k e^{-jkr}}{4\pi r} i_m L_m(\theta, \phi), \quad (B.1)$$

where $L_m(\theta, \phi)$ is the vector effective length, $k = 2\pi/\lambda$, and $\eta$ is the impedance of free space. A plane wave $E_{inc}$ arriving from direction $\theta, \phi$ induces an open-circuit voltage in antenna $m$ given by [4, eq. 2-93]

$$v_{o,m} = L_T^m(\theta, \phi)E_{inc} \quad (B.2)$$

If the incident field is distributed over a solid angle, it can be described by an electric field density, say $e(\theta, \phi)$, which has units of $V/(m \cdot sr)$. The cumulative open-circuit voltage induced by the entire density would then be obtained by integrating over all directions:

$$v_{o,m} = \iint_{S_1} L_T^m(\theta, \phi)e(\theta, \phi) \, dS.$$

Now consider a single transmit antenna that sends data to $M$ coupled receive antennas.
In a flat-fading environment, the electric field density at the receiver takes the form

\[ e(\theta, \phi) = x h(\theta, \phi) , \]

where \( x \) is the transmitted symbol. The open-circuit voltage induced by the signal in antenna \( m \) is therefore \( x h_m \), where

\[ h_m \triangleq \iint_{S_1} \mathbf{L}_m^T(\theta, \phi) h(\theta, \phi) \, dS . \]

Note that \( h(\theta, \phi) \) can be expressed as the sum of two orthogonal polarized waves

\[ h(\theta, \phi) = h_\theta(\theta, \phi) \hat{\theta} + h_\phi(\theta, \phi) \hat{\phi} . \]

If we model \( h_\theta(\theta, \phi), h_\phi(\theta, \phi) \) as zero-mean, circularly-symmetric Gaussian random processes on \( S_1 \), then \( h_1, \ldots, h_M \) are zero-mean, circularly-symmetric Gaussian variables. The correlation between the components is then given by

\[ \mathcal{E}[h_m h_n^*] = \iint_{S_1} \iint_{S_1} \mathbf{L}_m^T(\theta, \phi) \mathcal{E}[h(\theta, \phi) h^H(\theta', \phi')] \mathbf{L}_n^*(\theta', \phi') \, dS \, dS' . \quad (B.3) \]

If we assume \( h_\theta(\theta, \phi), h_\phi(\theta, \phi) \) are independent, identically distributed, and spatially white, then \( \mathcal{E}[h_\theta(\theta, \phi) h_\phi^*(\theta', \phi')] = 0 \) and

\[ \mathcal{E}[h_\theta(\theta, \phi) h_\theta^*(\theta', \phi')] = \mathcal{E}[h_\phi(\theta, \phi) h_\phi^*(\theta', \phi')] = p(\theta, \phi) \delta(\theta - \theta') \delta(\phi - \phi') . \quad (B.4) \]
With this assumption (B.3) reduces to

$$E[h_m h_n^*] = \iint_{S_1} L^T_m(\theta, \phi) L_n(\theta, \phi) p(\theta, \phi) \, dS.$$  \hspace{1cm} (B.5)

Comparing (B.1) with the definition of the antenna patterns given in (3.4), we see $E_m(\theta, \phi) = -(j\eta k/4\pi)L_m(\theta, \phi)$ and so (B.5) can be rewritten as

$$E[h_m h_n^*] = \left(\frac{4\pi}{\eta k}\right)^2 \iint_{S_1} E^T_m(\theta, \phi) E_n^*(\theta, \phi) p(\theta, \phi) \, dS.$$  \hspace{1cm} (B.6)

thereby proving (3.7). Assuming $x$ and $h(\theta, \phi)$ are independent, we note

$$E[\|e(\theta, \phi)\|^2/2\eta] = E[\|x\|^2]E[\|h(\theta, \phi)\|^2/2\eta] = E[\|x\|^2]p(\theta, \phi)/2\eta$$

represents the average incident signal power per steradian that arrives from direction $\theta, \phi$. In particular, if the incident signal power $P_s E[\|x\|^2]$ is uniformly distributed on the sphere, then $p(\theta, \phi)$ is a constant such that $P_s = (1/2\eta) \iint_{S_1} p(\theta, \phi) dS$. It follows $p(\theta, \phi) = \eta P_s/2\pi$ and (3.7) reduces to

$$E[h_m h_n^*] = \frac{\eta P_s}{2\pi} \left(\frac{4\pi}{\eta k}\right)^2 \iint_{S_1} E^T_m(\theta, \phi) E_n^*(\theta, \phi) \, dS.$$  \hspace{1cm} (B.7)

Comparing with (3.5), we conclude the signal covariance for this case is given by

$$\Sigma = \frac{P_s}{2\pi} \left(\frac{4\pi}{k}\right)^2 R_E = \frac{2P_s\lambda^2}{\pi} R_E.$$  \hspace{1cm} (B.8)

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B.3 Generalized Eigenvectors

Lemma 3: Let $\mathbf{R}, \Sigma \in \mathbb{C}^{K \times K}$ be Hermitian matrices, with $\mathbf{R}$ positive definite.

(a) There exists a complex $M \times M$ matrix $\mathbf{C}$ and real diagonal $\Lambda$ such that

\[ \mathbf{C}^H \mathbf{R} \mathbf{C} = \mathbf{I}, \quad \mathbf{C}^H \Sigma \mathbf{C} = \Lambda. \quad (B.9) \]

(b) If $\mathbf{R}$ and $\Sigma$ are real, then $\mathbf{C}$ can be chosen to be real, so

\[ \mathbf{C}^T \mathbf{R} \mathbf{C} = \mathbf{I}, \quad \mathbf{C}^T \Sigma \mathbf{C} = \Lambda. \quad (B.10) \]

Proof: Since $\mathbf{R}$ is Hermitian and positive definite, there is a unitary matrix $\mathbf{U}$ such that $\mathbf{U}^H \mathbf{R} \mathbf{U} = \mathbf{D}$, where $\mathbf{D}$ is diagonal with positive elements on the diagonal. Note $\mathbf{A} = \mathbf{D}^{-1/2} \mathbf{U}^H \Sigma \mathbf{U} \mathbf{D}^{-1/2}$ is also Hermitian, so there is a second unitary matrix such that $\mathbf{V}^H \mathbf{A} \mathbf{V} = \Lambda$, where $\Lambda$ is diagonal and real. Setting $\mathbf{C} = \mathbf{U} \mathbf{D}^{-1/2} \mathbf{V}$, proves (B.9). For (B.10), observe when $\mathbf{R}$ is real then $\mathbf{U}$ can be chosen as real too. If $\mathbf{U}$ and $\Sigma$ are real, then $\mathbf{A}$ is real and so $\mathbf{V}$ can be chosen to be real as well. It follows that $\mathbf{C}$ is also real, thereby proving (B.10).

B.4 A Resistance Matrix Inequality

In this section, we prove inequality (3.41) for an arbitrary passive coupling network (3.31). Consider the network in Fig. 3.2 where $\mathbf{E}_o$ is now an arbitrary fixed complex vector. Let $\mathbf{Z}_A, \mathbf{v}_o$ be the impedance and open-circuit voltage seen looking into the ports, from (3.32). Suppose we attach a load $\mathbf{Z}_L$ to the network output, and let $\mathbf{v}, \mathbf{i}$ be the resulting
voltage and current into the network from the right. Since \( i = -(Z_A + Z_L)^{-1}v_o \), the power dissipated in the load is

\[
\text{Re}(i^H Z_L i) = v_o^H (Z_A^H + Z_L^H)^{-1} R_L (Z_A + Z_L)^{-1} v_o .
\]

From [30], the maximum power is dissipated when \( Z_L = Z_A^H \), which yields

\[
v_o^H R_A^{-1} v_o = E_o^H M^H R_A^{-1} M E_o .
\] (B.11)

Similarly, \( J = (Z + Z_1)^{-1} E_o \) where

\[
Z_1 = Z_{11} - Z_{12} (Z_L + Z_{22})^{-1} Z_{21} .
\] (B.12)

is the impedance seen looking from the aperture into the coupling network when \( Z_L = Z_A^H \). Thus, the power supplied by the source is

\[
\text{Re}(J^H Z J) = E_o^H (Z^H + Z_1^H)^{-1} R (Z + Z_1)^{-1} E_o
\]

\[
\leq E_o^H R^{-1} E_o ,
\]

where the second step follows by observing \( Z_1 = Z^H \) maximizes the power supplied. Since the power dissipated in \( Z_L \) is less than or equal to the maximum the source can supply, we have

\[
E_o^H M^H R_A^{-1} M E_o \leq E_o^H R^{-1} E_o .
\] (B.13)

This inequality holds for all \( E_o \), thereby proving (3.41).
B.5 Change of Basis in Method of Moments

Suppose for all continuous currents $J(s)$ on an arbitrary conducting volume $V$, there are two different MoM bases to represent the same current, say

$$J(s) = \sum_{i=1}^{K} J_i W_i(s) = \sum_{j=1}^{K} \bar{J}_j \bar{W}_j(s)$$

where, $J(s) = (J_1, \ldots, J_K)^T$ and $\bar{J}(s) = (\bar{J}_1, \ldots, \bar{J}_K)^T$ Further suppose that each basis function $\bar{W}_j(s)$ is a linear combination of $W_i(s)$’s, i.e.

$$\bar{W}_j(s) = \sum_{i=1}^{K} w_{ij} W_i(s)$$

It follows that

$$\sum_{i=1}^{K} = \sum_{j=1}^{K} \bar{J}_j \bar{W}_j(s) = \sum_{j=1}^{K} \bar{J}_j \left( \sum_{i=1}^{K} w_{ij} W_i(s) \right)$$

$$= \sum_{i=1}^{K} \left( \sum_{j=1}^{K} w_{ij} \bar{J}_j \right) W_i(s)$$

Hence, $J_i = \sum_{j=1}^{K} w_{ij} \bar{J}_j$ or equivalently $J = W \bar{J}$, where $W = \{w_{ij}\}$