ABSTRACT

WHITE, ASHLEY WALLS. Conjugacy and Other Results in Leibniz Algebras. (Under the direction of Dr. Ernest Stitzinger.)

Leibniz algebras are a noncommutative generalization of Lie algebras. Many results in Lie algebras can be extended to Leibniz algebras. In this work we extend conjugacy results from Lie algebras to the Leibniz algebra cases and explore consequences of these results. In addition, we propose a method for extending Lie algebra results to Leibniz algebras via a construction we will call the companion Lie algebra to solvable Leibniz algebras, providing examples and consequences.

Conjugacy and Other Results in Leibniz Algebras

by
Ashley Walls White

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APPROVED BY:

Dr. Mohan Putcha	Dr. Kailash Misra
Dr. Radmila Sazdanovic	Dr. Ernest Stitzinger Chair of Advisory Committee

DEDICATION

For Bertie.

BIOGRAPHY

A North Carolina native, Ashley Walls White was born and grew up in Gastonia, NC. She holds a BS in Mathematics, a BS in Philosophy, and an MS in Mathematics, all from North Carolina State University. Ashley is passionate about teaching at the university level, as well as the K-12 level, and has served as an instructor at NCSU and the Duke TIP program for gifted K-12 students. Upon completion of her doctorate degree in 2017, Ashley plans to found an education and outreach foundation for science and mathematics in the Charlotte area, with goals to involve K-12 students and the community in active science research and inquiry based learning.

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TABLE OF CONTENTS

CHAPTER

1

INTRODUCTION

1.1 Leibniz Algebras

Leibniz algebras are a noncommutative generalization of Lie algebras, and were first introduced by Jean Louis Loday in [18]. Leibniz algebras have been studied at length in [4], [2], [15], and other works. One can consider a left Leibniz algebra, or a right Leibniz algebra with only slight differences in convention between the two, but with analogous results between both. In this work we will consider only left Leibniz algebras, as defined below.

Definition 1.1.1 *A (left)* **Leibniz algebra** *A is an $F-$vector space with a bilinear map,* $[,]: A \times A \to A$ *satisfying the Leibniz identity*

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

for $a, b, c \in A$.

Thus, in a left(right) Leibniz algebra, all left(right) multiplications are derivations. Notice that this multiplication is not necessarily antisymmetric. If this multiplication is, in fact, antisymmetric, then the Leibniz identity is precisely the Jacobi identity, and L is a Lie algebra. Cyclic algebras provide clear examples of algebras which are Leibniz but clearly not Lie:

Example 1.1.2 *Consider the cyclic algebra generated by a, $A = span\{a, a^2, a^3\}$ with the non-zero products $[a,a] = a^2$, $[a,a^2] = a^3$, $[a,a^3] = a^3$. A is a Leibniz algebra, but not Lie.*

It can easily be shown that A satisfies the Leibniz identity, but because $a^2 \neq 0$, A is not a Lie algebra. All Lie algebras are Leibniz, but any Leibniz algebra containing an element such that $a^2 \neq 0$ is not Lie.

1.2 Preliminaries

Many definitions from Lie algebras carry over directly to Leibniz algebras. Here we list several that occur often throughout this work.

Definition 1.2.1 *A Leibniz algebra is* **abelian** *if $A^2 = 0$.*

Definition 1.2.2 *For a Leibniz algebra A,* **Leib**(A) $= span\{[a,a] | a \in A\}$.

Example 1.2.3 *Consider the cyclic Leibniz algebra from above, $A = span\{a, a^2, a^3\}$. Here $A^2 = \{a^2, a^3\} = Leib(A)$.*

For any n-dimensional cyclic Leibniz algebra we have that $A^2 = \text{span}\{a^2, \dots, a^n\} = \text{Leib}(A)$, as shown in [15]. Consider a non-cyclic example that will be used throughout this chapter:

Example 1.2.4 *$A = span\{a, b, c\}$, with the non-zero products $[a,b] = c$, $[b,a] = c$. Here $Leib(A) = span\{c\}$ and $A^2 = span\{c\}$.*

Definition 1.2.5 *A subspace, S of A, is a* **subalgebra** *if $[S,S] \subseteq S$.*

In Leibniz algebras we must consider both left and right ideals, as well as left and right normalizers and centralizers.

Definition 1.2.6 *A subspace, S of A, is a left ideal if $[A,S] \subseteq S$, and a right ideal if $[S,A] \subseteq S$. S is an* **ideal** *of A if it is both a left and right ideal.*

Note that Leib(A) is an abelian ideal of A, and the minimal ideal for which the quotient of the algebra with the ideal is a Lie algebra [15], i.e. $A/\text{Leib}(A)$ is a Lie algebra.

Definition 1.2.7 *A Leibniz algebra A is* **simple** *if $A^a \neq Leib(A)$ and 0, $Leib(A)$, and A are the only ideals of A.*

Definition 1.2.8 *A Leibniz algebra A is semisimple if $Leib(A) = rad(A)$.*

The normalizer of a Leibniz algebra L is the intersection of the left and right normalizers. Similarly for the centralizer.

Definition 1.2.9 *The* **left centralizer** *of the subalgebra U is the subspace*

$$C_A(U) = \{x \in A | xU = 0\}$$

and the **centralizer** *of the subalgebra U in A is the subspace*

$$C_A(U) = \{x \in A | xU = Ux = 0\}$$

Definition 1.2.10 *The* **left normalizer** *of the subalgebra U is the subspace*

$$C_A(U) = \{x \in A | xu \in U, \forall u \in U\}$$

and the **normalizer** *of the subalgebra U in A is the subspace*

$$N_A(U) = \{x \in A | xu \in U, ux \in U, \forall u \in U\}$$

The majority of the the results in this work are for solvable Leibniz algebras, which we now define.

Definition 1.2.11 *For a Leibniz algebra A, the* **derived series** *of A is a series of ideals*

$$A \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \cdots$$

where $A^{(1)} = [A, A]$, $A^{(i+1)} = [A^{(i)}, A^{(i)}]$.

Definition 1.2.12 *A Leibniz algebra A is* **solvable** *if $A^{(m)} = 0$ for some m*

Definition 1.2.13 *For a Leibniz algebra A, the* **lower central series** *of A is a series of ideals*

$$A \supseteq A^{1} \supseteq A^{2} \supseteq \cdots$$

where $A^{i+1} = [A, A^{i}]$.

Definition 1.2.14 *A Leibniz algebra A is* **nilpotent** *if $A^n = 0$ for some n*

Example 1.2.15 *In the algebra $A = span\{a, a^2, a^3\}$, $A^{(1)} = [A, A] = span\{a^2, a^3\}$, and $A^{(2)} = [A^{(1)}, A^{(1)}] = 0$, thus the algebra is solvable. But $A^n = span\{a^3\}$, for any $n > 2$, so A is not nilpotent.*

Example 1.2.16 *In Example 1.2.4, $A^{(1)} = span\{z\}$, $A^{(1)} = 0$, and $A^2 = span\{z\}$, $A^3 = [A, A^2] = 0$, so A is both nilpotent and solvable.*

1.3 Overview of the Work

In Chapter 2 we examine properties of certain groups of maximal subalgebras, extending Towers' results in Lie algebras from [22] to their Leibniz algebra analogs. We find that, as in Lie algebras, maximal subalgebras which contain Engel subalgebras must necessarily be self normalizing.

In Chapter 3, we show that Cartan subalgebras of solvable Leibniz algebras are conjugate, and from this we are able to carry over several results from Lie algebras to the Leibniz algebra generalization. The conjugacy proofs in the Lie case often depend on anti-commutativity of the Lie bracket, thus it is necessary to find other paths in the Leibniz case.

Let $\mathfrak{X}$ denote a class of algebras. It is often of interest to study an algebra, L, that is minimally not in $\mathfrak{X}$, meaning every subalgebra of L is in $\mathfrak{X}$, but L is not itself in $\mathfrak{X}$. Such classes may be those of solvable, elementary, or triangulable algebras, among others. In Chapter 4 we will consider Leibniz algebras which are minimally non-$\mathfrak{X}$, for $\mathfrak{X}$ the classes of strongly solvable, supersolvable, and triangulable algebras. We present several results on two-generated Leibniz algebras, and provide a construction we will call the companion Lie algebra that can be used to show results on Leibniz algebras under certain circumstances.

CHAPTER

2

ENGEL AND CARTAN SUBALGEBRAS

2.1 Introduction

In this chapter we examine properties of certain sets of maximal subalgebras, extending Towers' results in Lie algebras from [22] to their Leibniz algebra analogs. We find that, as in Lie algebras, maximal subalgebras which contain Engel subalgebras must necessarily be self normalizing, and we provide other results on Engel and Cartan subalgebras.

2.2 Preliminaries

Let A be a Leibniz algebra. For $x \in A$, we denote the left multiplication operator as L_x, where

$$L_x(t) = [x, t] = xt \text{ for } x, t, \in A.$$

We will consider the Fitting decomposition of A relative to some operator, which is defined in the same manner as the Lie algebra case.

Definition 2.2.1 *For a Leibniz algebra A, and some linear transformation T, $A = A_0(T) + A_1(T)$ where $A_0(T)$ and $A_1(T)$ are the* **Fitting null** *and* **Fitting one** *components of T acting on A.*

$A_0(T) = \{a \in A : T^m(a) = 0$ *for some* $m > 0\}$, *and* $A_1(T) = \bigcap_{i=1}^{\infty} T^i A$.

An Engel subalgebra, $E_A(x)$ is the Fitting null component of A relative to the left multiplication operator.

Definition 2.2.2 $E_A(x) = \{t \in A : L_x^k(t) = 0$ *for some* $k\}$ *is the* **Engel subalgebra** *relative to* L_x.

Definition 2.2.3 *In a Leibniz Algebra A, a* **Cartan subalgebra** *is a nilpotent subalgebra which is self-normalizing.*

Barnes has shown in [4] that a subalgebra of a Leibniz algebra is Cartan if and only if it is a minimal Engel subalgebra. For cyclic Leibniz algebras, it is interesting to note the following proposition from [12].

Proposition 2.2.4 *Let A be a cyclic Leibniz algebra,* $A = span\{a, a^2, \ldots, a^n\}$. *Then* $E_A(a)$ *is the unique Cartan subalgebra of A.*

Example 2.2.5 *In the three dimensional cyclic Leibniz algebra,* $A = span\{a, a^2, a^3\}$, *with multiplications* $[a, a] = a^2$, $[a, a^2] = a^3$, $[a, a^3] = a^3$, $A_0 = E_A(a) = span\{a - a^3, a^2 - a^3\}$.

2.3 Maximal Subalgebras

For a subalgebra U of A define the core of U, U_A to be the largest ideal of A contained in U. The Frattini subalgebra of A, denoted $F(A)$, is the intersection of maximal subalgebras of A. The Frattini ideal, $\phi(A)$ is the largest ideal of A contained in the Frattini subalgebra. We will often consider Leibniz algebras where the Frattini ideal is zero, $\phi(A) = 0$. In this case we say that A is ϕ-free.

Following Towers [22], we will define the following sets of interest:

Definition 2.3.1 $\mathcal{G} = \{M : M$ *is a maximal subalgebra of A and* $E_A(x) \subseteq M$ *for some* $x \in A\}$

$G(A) = \bigcap_{M \in \mathcal{G}} M$ *if* $\mathcal{G}$ *is non-empty;* $G(A) = A$ *otherwise,*

$\gamma(A) = G(A)_A$

$\mathcal{G}$ is the set of maximal subalgebras which contain an Engel subalgebra, $G(A)$ is the intersection of such maximal subalgebras, or the whole algebra A if $\mathcal{G}$ is empty, and $\gamma(A)$ is the core of $G(A)$. We now state a few results on these sets.

Lemma 2.3.2 *Let A be a Leibniz algebra over a field F with at least dimA elements. Then $M \in \mathcal{G}$ if and only if M contains a Cartan subalgebra of L.*

Proof M contains an Engel subalgebra if and only if it contains a minimal Engel subalgebra, which are precisely the Cartan subalgebras, by [4].

Lemma 2.3.3 *Let A be a Leibniz algebra over a field F of characteristic zero, then $G(A) = \gamma(A)$.*

Proof Let A be a Leibniz algebra as above. In characteristic zero we have that $F(A) = \phi(A)$, by [8]. $G(A) \subseteq F(A)$, since $\mathcal{G}$ is contained in all maximal subalgerbas of A. Thus we have that $G(A) \subseteq \phi(A)$. If A is ϕ-free, then $G(A) = 0$, thus G(A) is an ideal and $G(A) = \gamma(A)$. Otherwise, note that for an ideal B of A, $F(A/B) = F(A)/B$, thus if $B = \phi(A) \neq 0$, $G(A) = \gamma(A)$.

Let M be an ideal of A, M is **nil** on A if each of its elements are nilpotent, i.e. if left multiplication on M by each $x \in A$ is nilpotent. Let U be a subalgebra of A. Recall the normalizer of U in A is $N_A(U) = \{x \in A | xu \in U, ux \in U, \forall u \in U\}$.

Proposition 2.3.4 *Let A be a Leibniz algebra and M be a maximal subalgebra of A. If $E_A(x) \subseteq M$ for some $x \in A$, then $N_A(M) = M$.*

Proof Let M be a maximal subalgebra of A that contains an Engel subalgebra. By [21] we know $M = N_A^r(M)$. Asssume $M \neq N_A^l(M) =$ A. By Barnes, we can assume that $x \in E_A(x)$. Then since M is a left ideal, the Fitting one component of L_x is contained in M. Thus $M = A$, a contradiction.

Lemma 2.3.5 *Let A be a Leibniz algebra, then*

(i) *$\gamma(A)$ is nil on A, and so nilpotent;*

(ii) *A is nilpotent if and only if $A = \gamma(A)$*

(iii) *A is nilpotent if and only if $\mathcal{G} = \emptyset$*

(iv) *A is nilpotent if and only if M is an ideal of A for all $M \in \mathcal{G}$;*

(v) *A is nilpotent if and only if A/B is nilpotent for some ideal B of A with $B \subseteq \gamma(A)$*

Proof (i) Suppose $t \in \gamma(A)$ and by way of contradiction that $E_A(t) \neq A$. Then there exists a subalgebra M such that $M \in \mathscr{G}$ and $E_A(t) \subseteq M$. $E_A(t)$ is the Fitting null component, so we have $A = E_A(t) + L_1(t)$. $L_1(t) \subseteq \gamma(A)$, so $A = M$, a contradiction. Thus $E_A(t) = A$ for all such t.

(ii) Let $A = \gamma(A)$, then $E_A(x) = A$ for all $x \in A$, thus every left multiplication operator in A is nilpotent, by Engel's theorem for Leibniz algebras [17], so A is nilpotent. The converse is clear.

(iii) A is nilpotent if and only if $A = \gamma(A)$, whence $G(A) = A$, and $\mathscr{G} = \emptyset$.

(iv) A is nilpotent if and only if every maximal subalgebra of A is an ideal, from [21]. Assume all maximal subalgebras of A in $\mathscr{G}$ are ideals of A and let $M \in \mathscr{G}$. Then $E_A(x) \subseteq M$ for some $x \in A$, so by Proposition 1.3, M it its own normalizer for all $M \in \mathscr{G}$. This contradicts that M is an ideal. The converse is clear.

(v) Assume A is nilpotent and B is an ideal of A such that $B \subseteq \gamma(A)$. $\gamma(A)$ is nilpotent, so B is nilpotent and A/B is nilpotent. Conversely assume A/B is nilpotent and by way of contradiction that $\mathscr{G} \neq \emptyset$. Then there is an $M \in \mathscr{G}$ and an $x \in A$ such that $E_A(x) \subseteq M$. Then $N_A(M) = M$, by Proposition 2.3.4. So we have $B \subseteq \gamma(A) \subseteq M$, and M/B is a maximal subalgebra of A/B, so M/B is an ideal of A/B, thus M is an ideal of A. But A is its own normalizer, contradiction, thus $\mathscr{G} = \emptyset$ and A is nilpotent.

Example Consider the 3 dimensional Leibniz algebra generated by a, with the multiplication $L_a(a^3) = 0$. A is nilpotent, and the only maximal subalgebra of A is A^2. Here $\mathscr{G} = \emptyset$ since
$E_L(a) = A \nsubseteq A^2$.

Lemma 2.3.6 *Let A be a Leibniz algebra over a field F and let B be an ideal A, then*

(i) $(E_A(x) + B)/B \subseteq E_{A/B}(x + B)$ *for all* $x \in A$

(ii) *if* $B \subseteq M$ *and* $M/B \in \mathscr{G}$ *then* $M \in \mathscr{G}$

(iii) $(\gamma(A) + B)/B \subseteq \gamma(A/B)$

(iv) *if F has at least dim A elements and* $M \in \mathscr{G}$ *then* $M/B \in \mathscr{G}$ *if* $B \subseteq M$

(v) *if F has at least dim A elements and* $B \subseteq \gamma(A)$ *then* $\gamma(A)/B = \gamma(A/B)$.

Proof (i) Let $y \in E_A(x)$. Then $L_x^k(y) = 0$, so $x^k y = 0$ for some k. Further, $(x+B)^k(y+B) \in B$. Thus $L_{x+B}^k(y+B) = 0$ in A/B, so $y + B \in E_{A/B}(x+B)$.

(ii) Let $B \subseteq M$ and $M/B \in \mathscr{G}$. Then $E_A(x) \subseteq M/B$ for some $x \in A$. Let $y \in E_A(x)$, then $L_x^k(y) = 0$, and $y \in M/B$, so $L_x^k(m+B) = 0$ in M/B for some $m \in M$, so $L_x^k(m) = 0$ in M. Thus $E_A(x) \subseteq M$, so $M \in \mathscr{G}$.

(iii) Straightforward, from (ii).

(iv) Assume $M \in \mathscr{G}$ and $B \subseteq M$. Then M contains an Engel subalgebra, and thus a Cartan subalgebra, C. Then M/B contains the Cartan subalgebra $(C+B)/B$ [4], so M/B contains an Engel subalgebra and $M/B \in \mathscr{G}$.

(v) Let $B \subseteq \gamma(A)$. Then $\gamma(A) + B = \gamma(A)$, so $\gamma(A)/B \subseteq \gamma(A/B)$ by (iii). Now let $y \in \gamma(A/B)$. By (iv) we have that $\gamma(A)/B \in \mathscr{G}$, so $\gamma(A)/B$ contains a Cartan subalgebra and $\gamma(A)/B$ is a maximal subalgebra of A.

In a solvale Leibniz algebra, a maximal subalgebra, M, contains an Engel subalgebra if and only if M is self-normalizing.

Proposition 2.3.7 *Let A be solvable, then $M \in \mathscr{G}$ if and only if $N_A(M) = M$.*

Proof If $M \in \mathscr{G}$, then M is its own normalizer by Proposition 2.3.4. Now assume A is a solvable Leibniz algebra and that M is a subalgebra of A such that $N_A(M) = M$. We want to show $M \in \mathscr{G}$, thus for some $k \in A$, $E_A(k) \subseteq M$. Let H be a minimal ideal of A, then $A = M + H$. (Note here that $E_A(x) = E_M(x) + E_H(x)$) Let K be a minimal ideal of M. If K is an ideal of A, then from the previous Lemma, if $K \subseteq M$, and $M/K \in \mathscr{G}$, then $M \in \mathscr{G}$. Suppose $M \notin \mathscr{G}$, then K is not an ideal of A and the core, M_A is zero.

For some $k \in K$, we have the Fitting decomposition of A with respect to the left multiplication operator, $A = E_A(k) + A_1(L_x(k))$. $A_1(L_x(k)) \subseteq H$, so $A = E_A(k) + H$. Since $M_A = 0$, there is some $k \in K$ and a maximal subalgebra B such that $E_A(k) \subset B$, and $A = H + B$. Then $E_H(k) = 0$, but $E_A(k) = E_M(k) + E_H(k)$, so $E_A(k) = E_M(k)$, thus $M \in \mathscr{G}$.

Proposition 2.3.8 *Let A be a solvable Leibniz algebra, then A is supersolvable if every maximal subalgebra had codimension 1 in A.*

Proof The proof is analogous to the Lie algebra case from [6].

Several of the following results involve the notion of supersolvable Leibniz algebras.

Definition 2.3.9 *A Leibniz algebra L is called supersolvable if there exists a chain of ideals* $0 = L_0 \subset L_1 \subset L_2 \subset \ldots L_{n-1} \subset L_n = L$, *where L_i is an i-dimensional ideal of L.*

Corollary 2.3.10 *Let A be a solvable Leibniz algebra. Then A is supersolvable if and only if M has codimension 1 for every $M \in \mathscr{G}$.*

Proof Suppose M has codimension 1 in A for every $M \in \mathscr{G}$, and let K be a maximal subalgebra not in $\mathscr{G}$. Since K is not in $\mathscr{G}$, K is not its own normalizer, thus K is an ideal in A, and as it is maximal, $\text{codim}(K) = 1$ in A. Then every maximal subalgebra has codimension 1, so by the above proposition, A is supersolvable. The converse follows directly from Proposition 2.3.7.

Corollary 2.3.11 *Let B be an ideal of A with $B \subseteq \gamma(A)$ such that A/B is supersolvable, then A is supersolvable.*

Proof Assume A/B is supersolvable. Then M/B has codimension one in A/B for all $M/B \in \mathscr{G}$. If $M/B \in \mathscr{G}$, then $M \in \mathscr{G}$, by Lemma 2.3.5, and M has codimension one in A, so A is supersolvable.

2.4 Further Results

We now define additional sets of interest for a Leibniz algebra A, in a familiar fashion:

Definition 2.4.1 $\mathscr{T} = \{M : M$ *is a maximal subalgebra of A and* $N_A(M) = M\}$

$T(A) = \bigcap_{M \in \mathscr{T}} M$ *if $\mathscr{T}$ is non-empty; $T(A) = A$ otherwise,*

$\tau(A) = T(A)_A$

$\mathscr{T}$ is the set of self-normalizing Maximal subalgebras in A, $T(A)$ is the intersection of such subalgebras, and $\tau(A)$ is the largest ideal of A contained in $T(A)$.

Corollary 2.4.2 *If A is solvable, then $\gamma(A) = \tau(A)$*

Proof If A is solvable $N_A(M) = M$ for all $M \in \mathscr{G}$, so $\mathscr{G} = \mathscr{T}$.

Let U be a subalgebra of a Leibniz algebra A. U is regular if we can choose a basis for U such that every vector of the basis is a root vector of A corresponding to a Cartan subalgebra C of A, or otherwise belongs to C. As with Lie algebras, say a Leibniz algebra, A, is semisimple if rad$(A) = 0$.

Proposition 2.4.3 *Let A be a semisimple Leibniz algebra over an algebraically closed field F of characteristic zero, then $M \in \mathscr{G}$ if and only if M is a regular maximal subalgebra of L.*

Corollary 2.4.4 *Let A be a semisimple Leibniz algebra over an algebraically closed field of characteristic zero, and suppose all maximal subalgebras of A belong to $\mathscr{G}$, then $A = sl(2, F)$*

Proof If A is a semisimple Leibniz algebra, then A is a Lie algebra, from [8] Corollary 2.7, so these follow immediately from [22] Proposition 2.9 and Corollary 2.10.

Define the abelian socle of A, Asoc(A), as the sum of the minimal abelian ideals of A.

Theorem 2.4.5 *Let A be a Leibniz algebra over an algebraically closed field of characteristic zero. Then the following are equivalent:*

(i) *All maximal subalgebras of A belong to $\mathscr{G}$*

(ii) $A/\phi(A) = Asoc(A/\phi(A)) \oplus S$ *where* $S \cong sl(2, F)$.

Proof We will consider the cases that $\phi(A) = 0$, and $\phi(A) \neq 0$. Assume (i) holds, and let $\phi(A) = 0$, then $A = \text{Asoc}(A) \oplus V$, where $V = S \oplus Z(V)$, with S a semisimple Lie algebra and $Z(V)$ is the center of V [8]. S satisfies the conditions of Corollary 2.4.4, thus $S = sl(2, F)$. We want to show $Z(V) = 0$, so assume otherwise. Let C be a Cartan subalgebra of A, and $A = C + A_1$ be the Fitting decomposition of A relative to C. Then $A_1 \subseteq A^{(1)} \subseteq \text{Asoc}(A) \oplus S$. If $Z(V) \neq 0$,then any maximal subalgebra of A containing Asoc$(A) \oplus S$ cannot contain a Cartan subalgebra, but all maximal subalgebras of A are in $\mathscr{G}$. Thus

$Z(V) = 0$, and $A = \mathrm{Asoc}(A) \oplus S$, thus $A/\phi(A) = \mathrm{Asoc}(A/\phi(A)) \oplus S$.

Now assume (ii) holds, still with $\phi(A) = 0$. Then $A = \mathrm{Asoc}(A) \oplus S$. Let $\mathrm{Asoc}(A) = A_1 + \cdots A_n$, then maximal subalgebras of A are of the form

$$\sum_{i=1, j \neq i}^{n} A_i \oplus S.$$

$S = sl(2, F)$ contains the Cartan subalgebra H, so each maximal subalgebra of A contains a Cartan subalgebra and is in $\mathscr{G}$.

If $\phi(A) \neq 0$, then (i) of Lemma 2.3.6 applies to $A/\phi(A)$, and the results follow.

CHAPTER

3

CONJUGACY OF CARTAN SUBALGEBRAS

3.1 Introduction

Conjugacy of Cartan subalgebras under certain condition was shown in sovlable and real Lie algebras by Barnes [6]. Omirov [19] treated conjugacy of Cartan subalgebras in complex Leibniz algebras. In this chapter we show the conjugacy of Cartan subalgebras in solvable Leibniz algebras, as well as real Leibniz algebras.

A self-centralizing minimal ideal in a solvable Lie algebra is complemented and all complements are conjugate [6]. Such an algebra is called primitive. If the minimal ideal is not self-centralizing, then there is a bijection which assigns complements to their intersection with the centralizer [20]. Such an intersection is the core of the complement. Towers extended this result to show that maximal subalgebras are conjugate exactly when their cores coincide, with the additional condition at characteristic p that L^2 has nilpotency class less than p. In this chapter, we will show that under this same condition, Cartan subalgebras are conjugate. Consequences of this result are shown and further results on Cartan subalgebras are mentioned.

3.2 Solvable Leibniz Algebras

Let A be an ideal in a solvable Leibniz algebra L. Also assume that at characteristic p, A has nilpotency class p, meaning $A^p = 0$, but $A^{p-1} \neq 0$. We will consider the group of automorphisms of L generated by the exponential operator acting on the the left multiplication operator, defined in the usual way.

Definition 3.2.1 *For $a \in A$, let $exp(L_a) = \sum_0^\infty \frac{1}{r!}(L_a)^r$.*

$I(L,A)$ is the group of automorphisms of L generated by $exp(L_a)$.

Definition 3.2.2 *Two subalgeabras U, V of L are* **conjugate** *when there exists $\beta \in I(L,A)$ such that $\beta(U) = V$.*

If A is abelian then $I(L,A)$ consists of all elements $I + L_a$. An algebra is primitive if it contains a self-centralizing minimal ideal, A. Barnes has shown in [4] that A is complemented and all complements are conjugate under $I(L,A)$, which extends his Lie algebra result. The socle, $\mathrm{Soc}(L)$, is the sum of all minimal ideals of L, and in the primitive case is equal to A. We record this as

Theorem 3.2.3 *([4], 5.13) Let P be a primitive Leibniz algebra with socle C. Then P splits over C and all complements to C in P are conjugate under $I(L,C)$.*

We need the following result, which is the Leibniz algebra version of Lie algebra results found in [5].

Lemma 3.2.4 *Let L be a solvable Leibniz algebra. If char$F = p$, also suppose that L^2 has nilpotency class less than p. Then,*

(i) *for all $x \in L^2$, exp_x exists,*

(ii) *if U is a subalgebra of L, then every $\alpha \in I(U,U^2)$ has an extension $\alpha^* \in I(L,U^2)$,*

(iii) *if A is an ideal of L, then every $\beta \in I(L/A,(L/A)^2)$ is induced by some $\beta^* \in I(L,L^2)$.*

Proof (i) For $x \in L^2$, $L_x(L) \subseteq L^2$. If char$F = 0$, then L^2 is nilpotent of class c for some c, so $L_x^c(L^2) = 0$, and so $L_x^{c+1}(L) = 0$. If char$F = p \neq 0$, then L^2 is nilpotent of class $c < p$, so $L_x^{p-1}(L^2) = 0$, thus $L_x^p(L) = 0$.

(ii) Let $\alpha \in I(U, U^2)$. Then $\alpha = \exp_{u_1} \exp_{u_2} \dots \exp_{u_n}$ for some $u_1 \dots u_n \in U^2$, where λ_{u_i} is the restriction of L_{u_i}, the left multiplication operator, to U. Then α is just the restriction of $\alpha^* = \exp_{u_1} \exp_{u_2} \dots \exp_{u_n} \in I(L, U^2)$.

(iii) Let $\beta \in I(L/A, (L/A)^2)$. Then $\beta = \exp_{u_1} \exp_{u_2} \dots \exp_{u_n}$, where λ_{u_i} is left multiplication in (L/A) with $u_i \in (L/A)^2$. Since $(L/A)^2 = L^2 + A/A$, $u_i = x_i + A$ for $x_i \in L^2$, and by (i), $\exp_{x_i}$ for $x_i \in L^2$ exists, so β is induced by $\beta^* = \exp_{x_1} \exp_{x_2} \dots \exp_{x_n} \in I(L, L^2)$.

We relax the condition that A is self-centralizing to obtain the Leibniz algebra extension of the result in [20]. We denote by $C_L(A)$ the centralizer of A in L. The Lie algebra instance of the following theorem was shown in [20], it leads immediately to the subsequent corollary, which we will use repeatedly.

Theorem 3.2.5 *Let A be a minimal ideal of a solvable Leibniz algebra, L. There exists a one to one correspondence between the distinct conjugate classes of complements to A under $I(L, A)$, and the complements to A in $C_L(A)$ that are ideals of L.*

Proof Let M and N be complements to A in L such that M and N are conjugate under $I(L, A)$. Define $M' = C_L(A) \cap M$ and $N' = C_L(A) \cap N$. Both M' and N' are complements to A in $C_L(A)$, and are ideals of L. We show that $M' = N'$. As M and N are conjugate, there exists a $\beta \in I(L, A)$ such that $N = \beta M$, and thus $N' = \beta M'$, but M' is an ideal of L, so for $a \in A$, $M' = \exp_a(M')$. Thus $M' = N'$. Conversely let M' be an ideal of L and a complement to A in $C_L(A)$. Then we have that $(A + M')/M'$ is its own centralizer in L/M', thus is complemented in L/M', and all complements are conjugate under $I(L/M', (A + M')/M')$. Let M and N be subalgebras of L that contain M' such that M/M' and N/M' are complement to $(A + M')/M'$ in L/M'. Then M/M' and N/M' are conjugate under $I(L/M', (A + M')/M')$, and by the correspondence theorem M and N are complements to A in L. From Lemma 3.2.4, any $\beta^* \in I(L/M', (A + M')/M')$ is induced by some $\beta \in I(L, A)$, so M and N are conjugate in $I(L, A)$.

Corollary 3.2.6 *Let L be a solvable Leibniz algebra and A be a minimal ideal of L with complements M and N. Then M and N are conjugate under $I(L, A)$ if and only if $M \cap C_L(A) = N \cap C_L(A)$.*

The corollary follows immediately from Theorem 3.2.5. Towers extends these results further in [24] for the Lie case, we will extend them now for Leibniz algebras. For a subalgebra U of L, its core, U_L is the largest ideal of L contained in U. U is core-free if $U_L = 0$. Say two subalgebras are conjugate in L if they are conjugate under $I(L, L) = I(L)$.

Lemma 3.2.7 *Let L be a solvable Leibniz algebra, and let M, K be two core free maximal subalgebras of L. Then M, K are conjugate under $exp(L_a) = 1 + L_a$ for some $a \in L$, thus they are conjugate in L.*

Proof Let A be a minimal abelian ideal of L. Then since M, K are maximal subalgebras $L = A + M = A + K$. Note that $C_L(A) = A + (M \cap C_L(A))$, and $M \cap C_L(A)$ is an ideal of L. So $M \cap C_L(A) \subseteq M_L$, but M is core-free, so $M \cap C_L(A) = 0$. Thus $C_L(A) = A$, and the result follows from Theorem 3.2.3.

Theorem 3.2.8 *Let L be a solvable Leibniz algebra over a field F. If char$F = p$, also suppose that L^2 has nilpotency class less than p. For maximal subalgebras M, K of L, M and K are conjugate under $I(L, L^2)$ if and only if $M_L = K_L$*

Proof Suppose M is conjugate to K under $I(L, L^2)$. Then $M = \beta K$ where $\beta \in I(L, L^2)$, $\beta = \exp_{x_1} \exp_{x_2} \cdots \exp_{x_n}$ for $x_i \in L^2$. Then $M_L = \beta K_L$, and further $\exp_{x_i} K_L = K_L$ when $x_i \in L$. So $\beta K_L = K_L$, thus $M_L = K_L$. Conversely let $M_L = K_L$. Then M/M_L and K/M_L are core-free maximal subalgebras of L/M_L, and are conjugate under $I(L/M_L, L/M_L)$ by Lemma 3.2.7, and thus also under $I(L/M_L, (L/M_L)^2)$. Then M and K are conjugate under $I(L, L^2)$ by Lemma 3.2.4.

We now define a Cartan subalgebra, just as in Lie algerbas, where $\mathcal{N}_L(C)$ is the normalizer of C in L.

Definition 3.2.9 *A* **Cartan subalgebra** *of a Leibniz algebra L is a nilpotent subalgebra C such that $C = \mathcal{N}_L(C)$*

We will show that Cartan subalgebras of solvable Leibniz algebras over a field F are conjugate in L, provided that when char$F = p$, L^2 has nilpotency class less than p. We will follow Barnes' Lie algebra approach from [4], with a notable exception. Barnes uses a cohomology argument from Lie algebras to establish a conjugacy relationship in his proof, instead we will make use of the following theorem for the Leibniz algebra case.

Theorem 3.2.10 *Let L be a solvable Leibniz algebra, and let M, K be complements of a minimal ideal A. If M and K are each Cartan subalgebras, then M and K are conjugate under $I(L, A)$.*

Proof M is a Cartan subalgebra and thus nilpotent, so consider the upper central series $Z_0(M) \subseteq Z_1(M) \subseteq \cdots \subseteq Z_j(M) = M$. Define $Z_j = Z_j(M) \cap C_L(M)$ to give the new series $0 \subseteq Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_j = M \cap C_L(A)$. We have that $[M, Z_j] \subseteq Z_{j-1}, [A, Z_j] = 0$, and $[L, Z_j] \subseteq Z_{j-1}$. Suppose for some j, $Z_{j-1} \subseteq K$ and by way of contradiction $Z_j \nsubseteq K$. Then $[L, Z_j] \subseteq Z_{j-1} \subseteq K$, and $[K, Z_j] \subseteq Z_{j-1} \subseteq K$, so $Z_j \subseteq \mathcal{N}_L(K) = K$, contradiction. Thus $Z_j \subseteq K$ for all j, so $M \cap C_L(A) = K \cap C_L(A)$, and the result follows from Corollary 3.2.6.

Theorem 3.2.11 *Let L be a solvable Leibniz algebra. If char$F = p \neq 0$, suppose further that L^2 is of nilpotency class less than p. Then the Cartan subalgebras of L are conjugate under $I(L, L^2)$.*

Proof If L is nilpotent the result holds. Suppose L is not nilpotent. Let H_1 and H_2 be Cartan subalgebras of L and let A be a minimal ideal of L. Then $(H_1 + A)/A$ and $(H_2 + A)/A$ are conjugate under $I(L/A, (L/A)^2)$, thus $H_1 + A$ and $H_2 + A$ are conjugate under $I(L, L^2)$ by Lemma 3.2.4, so suppose that $H_1 + A = H_2 + A$. If $H_1 + A$ is a proper subalgebra of L, then H_1 and H_2 are conjugate under $I(H_1 + A, (H_1 + A)^2)$, and thus under $I(L, L^2)$ by Lemma 3.2.4.

Now suppose that $H_1 + A = H_2 + A = L$. L is not nilpotent, so H_1 and H_2 are complements to A. By Theorem 3.2.10, H_1 and H_2 are conjugate under $I(L, A)$. Since A is a minimal ideal in L, and thus an irreducible L-module, we have $al = -la$, or $al = 0$ for all $a \in A, l \in L$ [4]. Then $(LA)L = -L(LA)$ or $(LA)L = 0$, so $(LA)L \subseteq LA$, and $L(LA) \subseteq LA$, so LA is an ideal in L. But since A is a minimal ideal, $LA = A$ or $LA = 0$. L is not nilpotent, so $LA \neq 0$. Thus $A = LA$, and as LA is a subalgebra of L^2, H_1 and H_2 are conjugate under $I(L, L^2)$.

3.3 Real Leibniz Algebras

Now consider a Leibniz algebra L over the field of real numbers, and the infinite series $\exp(Lx) = \sum_0^\infty \frac{1}{r!}(L_x)^r$. For $x \in L$, the series converges and $\exp(L_x)$ is an automorphism of L. Call $I^*(L)$ the group of automorphisms of L generated by $\exp(L_x)$ for $x \in L$. We find a criteria for two Cartan subalgebras of L to be conjugate under $I^*(L)$.

Theorem 3.3.1 *Let L be a real Leibniz algebra with radical R. The conjugacy classes under $I^*(L)$ of Cartan subalgerbas of L are in one to one correspondence with the conjugacy classes of $I^*(L/R)$ of Cartan subalgebras of L/R. Two Cartan subalgebras of L, H_1, H_2, are conjugate under $I^*(L) = I^*(L, L)$ if and only if $H_1 + R/R$ and $H_2 + R/R$ are conjugate under $I^*(L/R)$.*

Proof Assume H_1 and H_2 are conjugate under $I^*(L)$. Then $H_1 = \alpha H_2$ where $\alpha \in I^*(L)$. α induces an element, $\alpha^* \in I^*(L/R)$, so that $H_1 + R/R = \alpha^*(H_2 + R/R)$ and $H_1 + R/R$ and $H_2 + R/R$ are conjugate under $I^*(L/R)$.

Conversely assume $H_1 + R/R$ and $H_2 + R/R$ are conjugate under $I^*(L/R)$ then $H_1 + R/R = \alpha^*(H_2 + R/R)$ for $\alpha^* \in I^*(L/R)$. α^* is induced by some $\alpha \in I^*(L)$, so $H_1 + R = \alpha(H_2 + R)$, thus $H_1 + R$ and $H_2 + R$ are conjugate under $I^*(L)$. Then, say $H_1 + R = H_2 + R$. H_1 and H_2 are Cartan subalgebras of $H_1 + R$ and $H_2 + R$ respectively, so H_1, H_2 are Cartan subalgebras of $H_1 + R$ which is solvable, so H_1

and H_2 are conjugate under $I(H_1+R)$ by Theorem 3.2.11. For any subalgebra U of L, each element of $I^*(U)$ has an extension to an element in $I^*(L)$, so each element in $I(H_1+R)$ has an extension to an element in $I^*(L)$, thus H_1 and H_2 are conjugate under $I^*(L)$.

3.4 Consequences

The following conjugacy result is useful in extending results on Cartan subalgebras [21] to their Leibniz algebra counterparts.

Theorem 3.4.1 *Let L be a Leibniz algebra in which the intersection, J, of the terms in the lower central series is abelian. Let H and K be Cartan subalgebras of L. Then H and K are conjugate under exp (L_z) where $z \in J$.*

Proof If $J=0$, then L is nilpotent and $H=K$. If J is a minimal ideal of L, then H and K are complements to J in L and the result holds by Theorem 3.2.10. If B is an ideal of L and is maximal as an ideal in J, then $(H+B)/B$ and $(K+B)/B$ are Cartan subalgebras of L/B and are conjugate under $\exp(L_{(a+L/B)})$ in L/B by Theorem 3.2.10. Then $H+B$ is conjugate to $K+B$ under $\exp(L_a)$ in L where the image of H, H^*, is a Cartan subalgebra of $K+B$. By induction, the Cartan subalgebras H^* and K of $K+B$ are conjugate under $\exp(L_b)$, $b \in B$. Then H and K are conjugate under $\exp(L_{a+b})$ in L. Since $a+b \in J$ the result holds.

We now show the extension of several results from [21] to the Leibniz algebra case. Many of the following results are contingent upon the conjugacy of Cartan subalgebras in solvable Leibniz algebras, and we will consider the algebra L to be over a field of characteristic zero unless otherwise noted, so our conjugacy result applies. Consider the following set of algebras.

Definition 3.4.2 *Define the set $M(L)$ as the set of subalgebras, A, of L with the following properties:*

(1) A can be joined to a L by a chain of subalgebras of L each maximal and self-normalizing in the next

(2) No maximal subalgebra of A is self-normalizing in A

Notice that $M(L)$ is the set of subalgebras, A, of L such that A is nilpotent and A satisfies condition (1) above.

Denote by $N(L)$ the nilradical of L. Say a Leibniz algebra is nilpotent of length t if t is the smallest natural number such that there exists a chain of ideals of L, $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_t = L$ such that each factor is nilpotent. We show that when L is nilpotent of length 2, the Cartan subalgebras of L are precisely those subalgebras in $M(L)$, and when L is nilpotent of length 3 each element of $M(L)$ is contained in exactly one Cartan subalgebra of L. We will need the following two lemmas, the proof of Lemma 3.4.4 is completely analogous to the Lie algebra case in Barnes [5].

Lemma 3.4.3 *Let L be a Leibniz algebra and N an ideal of L, and D be a subalgebra in $M(L)$, then $D+N/N \in M(L/N)$.*

Proof D is nilpotent, so $D^m = 0$ for some m, and $(D+N)^m = D^m + N$, so $D^m + N/N = 0$ and $D+N/N$ is nilpotent. There exists a chain of ideals $D \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq L$, where $N_{K\,i+1}(K_i) = K_i$ and K_i is maximal in K_{i+1}. Consider the new chain $D+N/N \subseteq K_1/N \subseteq K_2/N \subseteq \cdots \subseteq K_n/N \subseteq L/N$. Clearly $N_{K\,i+1/N}(K_i/N) = K_i/N$ and K_i/N is maximal in K_{i+1}/N, so $D+N/N \in M(L/N)$.

Lemma 3.4.4 *Let L be solvable Leibniz algebra, A an ideal in L and U a subalgebra such that $A \subseteq U \subseteq L$, where U/A is a Cartan subalgebra of L/A. Then if H is a Cartan subalgebra of U, H is a Cartan subalgebra of L.*

Theorem 3.4.5 *Let L be a Leibniz algebra of nilpotent length 2 or less. Then the Cartan subalgebras of L are those subalgebras in $M(L)$.*

Proof If L is nilpotent the result holds, so assume L is nilpotent of length 2 and proceed by induction on the dimension of L. Let $D \in M(L)$, then there exists a chain of subalgebras $0 = D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots \subseteq D_t = L$, such that each D_i is maximal and self-normalizing in the next. If D is maximal in L, then $D = D_t$, thus D is a Cartan subalgebra of L and the result holds. Assume D is not maximal in L, then by induction D is a Cartan subalgebra of D_t, and as D_t is maximal and self-normalizing in L, $L = D_t + N(L)$. Notice $L/N(L) = (D_t + N(L))/N(L)) \cong D_t/(D_t \cap N(L))$. $L/N(L)$ is nilpotent, and $D_t \subseteq \mathcal{N}_L(D_t \cap N(L))$, so $D_t/(D_t \cap N(L))$ is a Cartan subalgebra of $L/(D_t \cap N(L))$. D is a Cartan subalgebra of D_t, so by Lemma 3.4.4, D is a Cartan subalgebra of L. Thus every element of $M(L)$ is a Cartan subalgebra of L.

Theorem 3.4.6 *Let L be nilpotent of length 3 or less, then each element of $M(L)$ is contained in exactly one Cartan subalgebra of L.*

Proof If L is nilpotent of length less than three, then the result holds by the previous theorem, so let L be nilpotent of length 3 and proceed by induction on the dimension of L. Let $D \in M(L)$, then

$D+N(L)/N(L)$ is a Cartan subalgebra of $L/N(L)$. Let E be a Cartan subalgebra of $D+N(L)$, then D is a Cartan subalgebra of L by Lemma 3.4.4. So $E+N(L)=D+N(L)$. Now let $S=E+N(L)$ and S_x be the Fitting null component of L_x acting on S. Define K to be the intersection of all S_x with $x\in D$. As D in nilpotent, $D\subseteq K$, we show K is a Cartan subalgebra of L. Assume $y\in\mathcal{N}_L(K)$, but $y\notin K$. Then $xy\in K$, but $y\notin S_x$, so $xy\notin S_x$, contradiction. So $y\in K$ thus K is its own normalizer in L. $K=K\cap(D+N(L))=D+(K\cap N(L))$, and since D is nilpotent by definition, and $(K\cap N(L))$ is a nilpotent ideal of K, K is nilpotent by [11].

Now assume D is contained in two Cartan subalgebras of L, E_1 and E_2. Then $E_1+N(L)=D+N(L)=E_2+N(L)$, so $E_1=E_1\cap(N(L)+D)=(E_1\cap N(L))+D$, and $E_2=(E_2\cap N(L))+D$. Suppose A is a minimal ideal of L. Then $E_1+A=E_2+A$, and E_1+A is such that the intersection of its lower central series, J, is abelian, so by Theorem 3.4.1 all Cartan subalgebras of E_1+A are conjugate under $\exp(L_z)=(I+L_z), z\in J$. E_1 and E_2 are both Cartan subalgebras of $A+E_1$, so are conjugate under $I+L_z, z\in J$. So we have $E_1\cap N(L)=(I+L_z)E_2\cap N(L)=(I+L_z)(E_2\cap N(L))=E_2\cap N(L)$, thus $E_1=(E_1\cap N(L))+D=(E_2\cap N(L))+D=E_2$.

The last theorem in this section which relies on the conjugacy of Cartan subalgebras in a solvable Leibniz algebra, L, shows that if the intersection of the lower central series of L is abelian but not zero, all complements to this intersection are conjugate by automorphisms of L of the form $I+\lambda_x$, as well as a generalization of this result.

Definition 3.4.7 *L^1_∞ is the intersection of the lower central series of L and for $i\geq 1$, L^n_∞ is the intersection of the lower central series of L^{n-1}_∞.*

Theorem 3.4.8 *Suppose that L^n_∞ is abelian but not 0 for some $n\geq 1$ and let H be a Cartan subalgebra of L^{n-1}_∞. Let H_0 be the Fitting null component of λ_H acting on L. Then H_0 is a complement of L^n_∞ in L. Furthermore, if K is a complement of L^n_∞ in L, then K is the Fitting null component of λ_J acting on L where J is a Cartan subalgebra of L^{n-1}_∞. All complements of L^n_∞ in L are conjugate by automorphisms of L of the form $I+\lambda_x, x\in L^n_\infty$.*

Proof We first show that H_0 is a complement of L^n_∞ in L by induction. So let $n=1$ and suppose $L^1_\infty=L^n\neq 0$ is abelian. Consider the Fitting decomposition of λ_H acting on L. $L=H_0+H_1$. H is a Cartan subalgebra, so $H=H_0$, and $L=H+H_1$. Consider L^k. $L^k=H^k+(\lambda_H)^{k-1}H_1=H^k+H_1$. So we have $L^n=H^n+H_1=H^n+(\lambda_H)^{n-1}H_1=H_1$, thus $H_1=L^1_\infty$. So $L=H_0+L^1_\infty$, and $H_0=H$ so all complements to L^1_∞ are conjugate by automorphisms of L of the form $I+\lambda_x, x\in L^n_\infty$.

Now let $n \geq 2$. Then $L_\infty^{n-1} = H + L_\infty^n$. $H \subseteq L_\infty^{n-1}$ so $H_1 \subseteq L_\infty^{n-1}$. And $L_\infty^n = H_1$, so $L = H_0 + L_\infty^n$. Now consider a complement K to L_∞^n in L. $L = K + L_\infty^n$. Then we have

$$L_\infty^{n-1} = L_\infty^{n-1} \cap L = L_\infty^{n-1} \cap (K + L_\infty^n) = (L_\infty^{n-1} \cap K) + L_\infty^n$$

And since $L_\infty^n \cap (L_\infty^{n-1} \cap K) = 0$, $L_\infty^{n-1} \cap K$ is a complement of L_∞^n in L_∞^{n-1}, thus is a Cartan subalgebra of L_∞^{n-1} and a nilpotent ideal in K. So K is the Fitting null component of $\lambda(L_\infty^{n-1} \cap K)$ acting on K. Hence $L = K + L_\infty^n$ where K is the Fitting null component of $\lambda(L_\infty^{n-1} \cap K)$ acting on L.

Finally let K_1 and K_2 be complements to L_∞^n in L, then each is the Fitting null component of λ_{H_1} and λ_{H_2}, Where H_1 and H_2 are Cartan subalgebras of L_∞^{n-1}. H_1 and H_2 are conjugate in L_∞^{n-1} under $(I + \lambda_x)$, $x \in L_\infty^n$. L is invariant under automorphisms of this form and thus K_1 and K_2 are conjugate by automorphisms of L of this form.

3.5 Further Results

We will end this chapter on several results relating to Cartan subalgebras. First we define an additional set of subalgebras, $\overline{M}(L)$.

Definition 3.5.1 *Define $\overline{M}(L)$ to the set of all subalgebras, A, of L with the following properties.*

(1) There exists a chain $A = A_0 \subset \cdots \subset A_n = L$ of subalgebras of L each maximal and self-normalizing in the next

(2) A_i/N is nilpotent, where N is the core of A_i in A_{i+1}

(3) There does not exist a subalgebra K of A such that K is maximal and self-normalizing in A and K/K_A is nilpotent

Lemma 3.5.2 *Let L be a nonnilpotent, solvable Leibniz algebra and let C be a Cartan subalgebra of L. Then L contains a self-normalizing maximal subalgebra M which contains C and K/K_L is nilpotent.*

Proof L is not nilpotent, so the lower central series does not terminate. Let K be the intersection of the members of the lower central series of L, $K = \bigcap_{j=1} L^j = L^k$. Let G be a maximal proper subalgebra of K which is a maximal ideal of L. From Barnes, [4], since C is a Cartan subalgebra of L, $(C+G)/G$ is a Cartan subalgebra of L/G. Further we have that there exists a maximal subalgebra M/G such that $(C+G)/G \subseteq M/G \subseteq L/G$. M/G is maximal and self-normalizing in L/G, and

$M/G + K/G = L/G$. $M/G \cap K/G$ is an ideal in M/G, and an ideal in K/G, so $M/G \cap K/G$ is an ideal in L/G. So $G \subseteq M \cap K \subset L$, but G is maximal so $G = M \cap K$ thus $M/G = M/(M \cap K) = (M+K)/K = L/K$. L/K is nilpotent, thus M/G is nilpotent. And since G is maximal, G is the largest ideal of L contained in M, thus $G = M_L$, so M/M_L is nilpotent, and $C \subseteq M$.

Theorem 3.5.3 *Let L be a solvable Leibniz algebra and let A be a subalgebra of L. Then A is a Cartan subalgebra of L is and only if $A \in \overline{M}(L)$*

Proof Let $A \in \overline{M}(L)$. Then there does not exist a subalgebra K of A such that K is maximal and self-normalizing in A and K/K_A is nilpotent, thus A must be nilpotent by Lemma 3.5.2, and A is a Cartan subalgebra of A_1. Let N be the core of A_i in A_{i+1}, then we have that A_i/N is nilpotent, and for all i up to n, A is nilpotent in A_i, thus A is nilpotent and self-normalizing in L.

Conversely, let A be a Cartan subalgebra of L. If L is not nilpotent, then by Lemma 3.5.2 there is a maximal subalgebra, H of L, which contains A and H/H_L is nilpotent. A is then a Cartan subalgebra of H, so if H is not nilpotent then there is a maximal subalgebra J of H which contains A and J/J_L is nilpotent, and we can continue this until we find a maximal subalgebra K containing A of which A is a Cartan subalgebra. In this case $K = A$ and the result holds.

Corollary 3.5.4 *Let H be a nilpotent subalgebra of L such that $L = H + N(L)$. Then there exists a unique Cartan subalgebra of C of L which contains H. C also contains the normalizer of H.*

The corollary follows from the previous theorem, and will be used in the next.

Theorem 3.5.5 *(1) Let K be a Cartan subalgebra of L and N be an ideal of L. Then $K + N/N$ is a Cartan subalgebra of L/N and if H is a subalgebra of L containing K and $H + N/N$ is nilpotent, then $H + N = K + N$.*

(2) Let $0 = N_0 \subset N_1 \subset \cdots \subset N_r = L$ be a chain of subalgebras of L such that each is an ideal in the next and N_{i+1}/N_i is nilpotent. Let $K \subseteq L$. If for each i, $K + N_i/N_i$ is nilpotent and $K + N_i/N_i$ is contained properly in no nilpotent subalgebra of L/N_i, then K is a Cartan subalgebra of L.

Proof (1) This follows from Corollary 6.3 of [4].
(2) Proceed by induction on r. If $r = 1$ the result holds, then $(K + N_1)/N_1$ is a Cartan subalgebra of L/Ni. $(K + N_0)/N_0$ is nilpotent by hypothesis. Note that $K + N_1 = K + N(K + N_1)$, so by Corollary 3.5.4, $K \in C$, where C is a Cartan subalgebra of $K + N_1$. Since $C = (C + N_0)/N_0$ is nilpotent, $C = K$. Then K is a Cartan subalgebra of $K + N_i$, and by Lemma 3.4.4, K is a Cartan subalgebra of L.

CHAPTER

4

STRUCTURE THEOREMS ON SOLVABLE LEIBNIZ ALGEBRAS

4.1 Introduction

Let $\mathfrak{X}$ denote a class of algebras. It is often of interest to study an algebra, L, that is minimally not in $\mathfrak{X}$, meaning every subalgebra of L is in $\mathfrak{X}$, but L is not itself in $\mathfrak{X}$. Such classes may be those of solvable, elementary, or triangulable algebras, among others. In particular, minimally non-$\mathfrak{X}$ Leibniz algebras which are ϕ-free have been studied in [9], and [24].

In this chapter we will consider Leibniz algebras which are minimally non-$\mathfrak{X}$, for $\mathfrak{X}$ the classes of strongly solvable, supersolvable, and triangulable algebras. We present several results on two-generated Leibniz algebras, and provide a construction we will call the companion Lie algebra that can be used to show results on Leibniz algebras under certain circumstances.

4.2 Companion Lie Algebras

Recall the Frattini ideal of a Leibniz algebra, denoted $\phi(L)$. We say that L is ϕ-free if $\phi(L) = 0$. In this section we construct what we will call the companion Lie algebra to a ϕ-free Leibniz algebra L, which is minimally not in a formation of solvable Leibniz algebras. Such correspondences have been considered previously in [3]. There the reverse construction is developed, obtaining a Leibniz algebra from a Lie algebra. We will use Barnes' definition of a formation from [7]. We will need the following notation for classes of algebras, where $\mathfrak{X}$ is a class of Leibniz algebras.

$$Q\mathfrak{X} = \{L/K | L \in \mathfrak{X}\}$$

$$R\mathfrak{X} = \{L | \ \exists \text{ ideals } K_i \text{ of } L \text{ with } (L/K_i) \in X \text{ and} \cap_i K_i = 0\}$$

Here $Q\mathfrak{X}$ is the class of quotients of Leibniz algebras, $R\mathfrak{X}$ is the class of subdirect sums. If $Q\mathfrak{X} = \mathfrak{X}$, $\mathfrak{X}$ is called a homomorph. Presenting this with other definitions we have:

Definition 4.2.1 *A non-empty class of solvable Leibniz algebras, $\mathfrak{X}$, which is Q-closed is called a homomorph, an R-closed homomorph is called a formation.*

Examples of such formations include the classes of nilpotent, supersolvable, and strongly solvable Leibniz algebras. Let $\mathfrak{F}$ be a subalgebra closed formation. If L is a solvable Leibniz algebra that is minimally not in $\mathfrak{F}$, and L is ϕ-free, we claim that L has a unique minimal ideal.

Proof Suppose A and B are both minimal ideals of L, then since L is solvable and $\phi(L) = 0$, they are each complemented by subalgeras H and K. As L is minimally not in $\mathfrak{F}$, H and K are both in $\mathfrak{F}$. So $L/(A \cap B) = H$, thus L is in $\mathfrak{F}$. Contradiction, so A is unique.

The socle of L, Soc(L), is the sum of the minimal ideals of L. Under the above conditions, the Soc(L) = A. Further we have that $A = N(L)$, where $N(L)$ denotes the nilradical, or the maximal nilpotent ideal of L. From [8], we have that since $\phi(L) = 0$, $N(L) = A = Z_L(A)$, which is contained in Leib(L) when L is not Lie. A is the kernel of $\mathfrak{L} = x \to L_x|_A$, a homomorphism from L to the derivation algebra of A, thus $H = L/A$ is Lie, and $A =$ Leib(L).

Thus we have that $L = A + H$, and if L is not a Lie algebra, then $A =$ Leib(L) is self-centralizing in L, and H is a Lie algebra. We can now state the definition of the companion Lie algebra.

Definition 4.2.2 *Let $\mathfrak{F}$ be a formation of solvable Leibniz algebras. Let L be minimally not in $\mathfrak{F}$ and suppose that $\phi(L) = 0$. The vector space direct sum constructed as $C = A + H$ with the product $[a+h, b+k] = [h,b]-[k,a]+[h,k]$, where the right hand products are the same as in L, is a Lie algebra, called the Companion Lie Algebra of L.*

We can show that a Leibniz algebra L has certain properties if and only if the companion Lie algebra C has these same properties. Thus we can construct the proofs of some results in Leibniz algebras via a companion Lie algebra argument. We will do just this in the next section.

4.3 Results via Companion Lie Algebras

Let $\mathrm{x} \in L$, and let x be the corresponding element in C. Denote left multiplication by x as L_{x}, and denote left multiplication by $x \in C$ by ad x. If $\mathrm{x} \in A$, then L_{x} and ad x are both nilpotent, though not necessarily equal. We begin with a definition, and make several claims.

Definition 4.3.1 *A Leibniz algebra, L, is strongly solvable if L^2 is nilpotent.*

Claim 4.3.2 *Suppose $\mathfrak{F}$ is a formation of strongly solvable Leibniz algebras and L, C, A, H are defined as above. If L is solvable, minimally non-strongly solvable, C is also non-strongly solvable.*

Proof If $\mathrm{x} \in H^2$, then L_{x} and ad x are either both nilpotent or both non-nilpotent. If all $\mathrm{x} \in H^2$ have ad x nilpotent, then left multiplication by each element in the Lie set $H^2 \cup A$ is nilpotent on both L^2 and C^2, thus L^2 is nilpotent if and only if C^2 is nilpotent. Thus if L is minimally non-strongly solvable, C is non-strongly solvable.

Claim 4.3.3 *With same conditions as the previous claim, if L is ϕ-free, C is also ϕ-free.*

Proof Suppose $\phi(C) \neq 0$. A is self-centralizing and the unique minimal ideal in both L and C, so H is a maximal subalgebra of both L and C, thus contains $\phi(C)$. Since A is the unique minimal ideal in C, $A \subseteq \phi(C)$, so $A \subseteq H$. Contradiction, thus $\phi(C) = 0$.

Claim 4.3.4 *With same conditions as the previous claims, if L is a solvable, minimally non-strongly solvable Leibniz algebra, C is also.*

Proof Suppose that $C = A + K$, then $K \cap A$ is equal to either A or 0 since A is a minimal ideal of C. If $K \cap A = A$, then $A \subseteq K$ and $K = C$, contradiction, so $K \cap A = 0$. In this case, K is isomorphic to H, which is strongly solvable in L, and thus in C, so K is strongly solvable.

If $C \neq A + K$, $A + K$ is strongly solvable, hence K is strongly solvable. Assume $A \subseteq K$. Then $K = K \cap (H + A) = (K \cap H) + A$, which is a subalgebra of L. Then $K^2 = (K \cap H)^2 + A$ in the Leibniz algebra L, and in the Lie algebra C. In L, both summands of K^2 are nilpotent Lie sets, and are both nilpotent in C. Thus K^2 is nilpotent in C, so K is strongly solvable.

Thus if L is a solvable, minimally non-strongly solvable, ϕ−free Leibniz algebra, the companion Lie algebra C also has these characteristics. Solvable, ϕ-free, minimally non-strongly solvable Lie algebras were described in Theorem 3.1 of [10]. We give the Leibniz algebra analog here.

Theorem 4.3.5 *Let L be a solvable ϕ-free minimally non-strongly solvable Leibniz algebra over field $\mathbb{F}$. Then $\mathbb{F}$ has characteristic $p > 0$, and $L = A + H$ is a semidirect sum, where*

1. *A is the unique minimal ideal of L,*
2. $\dim A \geq 2$,
3. $A^2 = 0$, *and*
4. *either $H = M \dotplus \mathbb{F}x$, where M is a minimal abelian ideal of H, or H is the three-dimensional Heisenberg algebra.*

Either L is Lie, or if not, then H is Lie, $A = Leib(L)$, and $AL = 0$.

Proof If L is a Lie algebra, then the result is Theorem 3.1 of [10]. If L is not Lie, then the companion Lie algebra, C, satisfies the conditions and conclusions of this theorem. Thus, L does also.

We now consider the class of supersolvable Leibniz algebras.

Definition 4.3.6 *A Leibniz algebra L is called supersolvable if there exists a chain of ideals* $0 = L_0 \subset L_1 \subset L_2 \subset \ldots L_{n-1} \subset L_n = L$, *where L_i is an i-dimensional ideal of L.*

Let L be a solvable, minimally non-supersolvable, ϕ-free Leibniz algebra. Supersolvable Leibniz algebras form a a formation, thus we can again use the companion Lie algebra construction to obtain results on L. Let $L = A + H$ as before, if L is Lie, the structure of L has been determined in [16], and [10].

Theorem 4.3.7 *Let L be a solvable, minimally non-supersolvable Lie algebra which is ϕ-free. Then the candidates for L are:*

1. *If L is strongly solvable, $L = A \dotplus \langle x \rangle$, where A is the unique minimal ideal of L and* $\dim A > 1$.
2. *If L is not strongly solvable, then F is of characteristic $p > 0$, L has unique minimal ideal A with basis $\{e_1, \ldots, e_p\}$, and one of the following holds:*
   (a) *$L = A + \langle x, y \rangle$ with antisymmetric multiplication $x e_i = e_{i+1}$, $y e_i = (\alpha + i)e_i$, with indices mod p, $yx = x$, and for all $a \in F$, $a = t^p - t$ for some $t \in F$, or*
   (b) *$L = A + \langle x, y, z \rangle$ with anti-symmetric multiplication $xe_i = e_{i+1}$, $ye_i = (i+1)\, e_{i-1}$, $ze_i = e_i$, with indices mod p, $yx = z$, $xz = yz = 0$, and F is perfect when $p = 2$.*

If L is not Lie, we find that the companion Lie algebra, C, satisfies the conditions of 4.3.7, thus L still has the structure laid out in the previous theorem with a few added conditions. We will construct the argument then state the theorem. Let $L = A + H$ as in the previous section, $A = \text{Leib}(L)$, H is a Lie algebra, A is the minimal ideal in C, and C is ϕ-free.

Claim 4.3.8 *Let L be a solvable, minimally non-supersolvable Lie algebra which is ϕ-free as above, then the companion Lie algebra C is not supersolvable.*

Proof $L = A + H$, and H is supersolvable by assumption. A is abelian by the assumption, however A is irreducible under the action of H acting on the left. Since L is not supersolvable, the dimension of A is greater than 1. The left action of H on A in C is the same as in L, and hence the dimension of A is greater than 1 in C, and C is not supersolvable.

Claim 4.3.9 *Let L be a solvable, minimally non-supersolvable Lie algebra which is ϕ-free, then all proper subalgebras of the companion Lie algebra C are supersolvable.*

Proof Let K be a proper subalgebra of C. if $C = A + K$, either $A \cap K$ is equal to 0 or A. In the first case K is isomorphic to H, and is thus supersolvable. In the second case, $K = C$, which is a contradiction. If $A + K$ is a proper subalgebra of C, then $A + K$ is supersolvable, so K is supersolvable. To show this assume $A \subset K$, then $K = (K \cap H) + A$. Note that $K \cap H$ is the same in L and in C. In L, the action of $K \cap H$ on A is simultaneously triangularizable. The left action of $K \cap H$ on A in C is the same as in L, hence K is supersolvable in C.

Thus C is solvable, minimally non-supersolvable, and ϕ-free, so we can now state the result.

Theorem 4.3.10 *Let L be a solvable, ϕ-free, minimally non-supersolvable Leibniz algebra. If L is a Lie algebra, then L is as in Theorem 4.3.7. If L is not Lie, then L is as in Theorem 4.3.7 with the added conditions that A = Leib(L) and AL = 0.*

Corollary 4.3.11 *If L is as in Theorem 4.3.10 and F is algebraically closed, then the solvable, minimally non-supersolvable Leibniz algebras are precisely the algebras at characteristic p such that either*

(a.) $L = A + B$ *where* $A = ((e_1, \ldots, e_p))$ *is the unique minimal ideal which is abelian,* $B = \langle x, y\rangle$ *with* $y e_i = -i e_i$ *and* $x e_i = -e_{i+1}$ *(indices mod p),* $y x = x$*, and either A = Leib(L) or L is Lie, or*

(b.) $L = A + B$ *where* $A = ((e_1, \ldots e_p))$ *is the unique minimal ideal which is abelian, B is the Heisenberg Lie algebra with basis* $\{x, y, z\}$ *and multiplication* $y e_i = (i+1)e_{i-1}$, $x e_i = -e_{i+1}$, $z e_i = e_i$, $yx = z$*, and either A = Leib(L) or L is Lie. Again, the index action is mod p.*

In both (a.) and (b.) we have $A = \text{Leib}(L)$, hence $AL = 0$, or L is Lie and multiplication of L on A is anti-symmetric.

4.4 Two-Generated Leibniz Algebras

An algebra, L, is two-generated if it is generated by precisely two elements, i.e. $L =< x, y >$ for some $x, y \in L$. We will consider this notion as it is related to strongly solvable and supersolvable Leibniz algebras as defined in the previous section. Bowman, Towers, and Varea prove the Lie algebra version of the next result from Theorem 3.1. of [10] , we obtain it from [13] instead.

Proposition 4.4.1 *Let L be a solvable Leibniz algebra that is minimally non-strongly solvable, then L is two-generated.*

Proof If L is not two-generated, then all subalgebras, particularly all two-generated subalgebras are strongly solvable. By Corollary 1 of [13], strong solvability is 2-recognizable, thus if all two-generated subalgebras of L are strongly solvable L must be also. Contradiction, so L must be two generated.

Proposition 4.4.2 *Let L be a solvable Leibniz algebra with each two-generated proper subalgebra strongly solvable. Then*

1. *L is either strongly solvable or two-generated, and*

2. *Every proper subalgebra of L is strongly solvable.*

Proof The proof of (1) follows the proof of the previous proposition. For (2), let S be a minimally non-strongly solvable subalgebra of L. If S is two-generated it is strongly solvable by hypothesis, thus it is not two-generated. But every two-generated subalgebra of S is strongly solvable, and as strong solvability is two-recognizable, S is also strongly solvable.

The following results are the similar to Theorem 3.5 of [10], for the Leibniz algebra case.

Theorem 4.4.3 *If L is solvable and minimally non-supersolvable, then L is two-generated.*

Proof If L is solvable and minimally non-supersolvable, then every two-generated proper subalgebra is supersolvable. If L is not two-generated then all two-generated subalgebras are supersolvable, but supersolvability is two-recognizable, [13], so L is supersolvable. Contradiction, thus L is two-generated.

Theorem 4.4.4 *If every two-generated proper subalgebra of L is supersolvable, L itself is not supersolvable, and L is ϕ-free, then L has the structure of one of the algebras in 4.3.10.*

Proof Let S be a proper subalgebra of L. If S is two-generated, then it is supersolvable by assumption. If S is non two-generated, then every two-generated subalgebra of S is supersolvable by assumption. Since supersolvability is two-recognizeable [13], S is also supersolvable. Thus, all proper subalgebras of L are supersolvable, and L is minimally non-supersolvable. Hence, Theorem 4.3.10 applies.

The next section describes results on triangulable Leibniz algebras and continues to provide results pertaining to two-generated Leibniz algebras.

4.5 Triangulable Leibniz Algebras

A Leibniz algebra L over a field $\mathbb{F}$ is triangulable on L-module M if when $\mathbb{F}$ is extended to K, the algebraic closure of $\mathbb{F}$, $K \otimes M$ admits a basis such that the representing matrices of $K \otimes L$ are upper triangular. L is said to be nil on M if left multiplication on M by each $x \in L$ is nilpotent. Then L acts nilpotently on M [11]. There is a maximal ideal of L that acts nilpotently on M. This ideal is denoted by nil(L). If L^2 is contained in nil(L), then the same holds in the algebra and module over K. By Theorem 1 of [14], in the algebraically closed case, L is triangulable on M if L^2 is contained in nil(L). Hence, in the general case, L is triangulable on M if L^2 is in nil(L). Note that these definitions still apply in the special case that L is a subalgebra of M. We state them in this context.

Proposition 4.5.1 *A subalgebra L of Leibniz algebra M is triangulable on M if and only if L^2 is contained in nil(L). If L is an ideal of M, then L is triangulable on M if and only if L^2 is nilpotent. M is triangulable on itself if and only if M is strongly solvable.*

The following two propositions follow exactly as in their Lie algebra versions, with left multiplication replacing ad in the proofs as shown in Lemma 4.1 and Lemma 4.5 of [10].

Proposition 4.5.2 *Let L be a Leibniz algebra, and let S and T be subalgebras of L that are nil on S such that S is contained in the normalizer of T. Then $S + T$ is nil on L.*

Proposition 4.5.3 *If S is a subalgebra of L such that $\phi(L)$ is contained in S, then nil $(S/\phi(L)) =$ nil$(S)/\phi(L)$.*

Proof Clearly $\phi(L)$ is contained in nil(L) since $\phi(L)$ is a nilpotent ideal in L. Let nil$(S/\phi(L)) = J/\phi(L)$. Clearly nil(S) is contained in J. Let $x \in J$. Now L_x acts nilpotently on $L/\phi(L)$, and $L = \phi(L) + L_0(x)$, the Fitting null component of L_x acting on L. Since L_0(x) is a subalgebra of L supplementing the Frattini ideal, L_0(x) is equal to L. This holds for all such x, and hence, J is contained in nil(S), and the result holds.

It is known that a solvable Leibniz algebra is triangulable on itself if all two-generated subalgebras of L are triangulable [14]. In [10], Lie algebras all of whose *proper* two-generated subalgebras are triangulable are similarly investigated. Our purpose is to find Leibniz algebra analogues to this and related ideas.

Theorem 4.5.4 *Let L be a solvable Leibniz algebra such that each two-generated proper subalgebra is triangulable on L. Then L is triangulable.*

Proof Each two-generated proper subalgebra of L is strongly solvable. Then by Proposition 4.4.2, every proper subalgebra of L is strongly solvable, as is each proper subalgebra of $L^* = L/\phi(L)$. If L is not triangulable, then L is not strongly solvable, and neither is L^*. Thus, Theorem 4.3.5 applies, $L^* = A + B$ as in the theorem, and $(L^*)^2 = A + B^2$. B is two-generated. Hence B is triangulable on L^*. Thus, B^2 acts nilpotently on L^* and hence, on A. Thus, $(L^*)^2$ is nilpotent and L^* is strongly solvable, a contradiction.

It is interesting to note that L is triangulable if and only if it is strongly solvable. If all two-generated subalgebras are triangulable on L, then L is triangulable (Theorem 4.5.4). However, if all two-generated subalgebras are strongly solvable, this does not guarantee that L is triangulable since a subalgebra can be strongly solvable without being triangulable on the algebra. The algebras in Theorem 4.3.5 are of this type.

Theorem 4.5.5 *If L is minimally non-triangulable, then L is two-generated and L/ϕ (L) is simple.*

Proof By Theorem 4.5.4, L is not solvable. If $L^* = L/\phi(L)$ is triangulable, then $(L^*)^2$ is nilpotent and L^* is solvable, a contradiction. Therefore, L^* is not triangulable, and hence not strongly solvable. Consider a proper subalgebra S of L and the corresponding proper subalgebra $S^* = S/\phi(L)$ of L^*. S is triangulable on L, hence S^2 is contained in nil(S) and $(S^*)^2$ is contained in nil(S^*). Hence, S^* is strongly solvable.

Note that the nilradical of L^* is not 0. Since $\phi(L^*) = 0$, nil(L^*) is complemented in L^* by a subalgebra T [8], which is strongly solvable. Hence, $L^* = \text{nil}(L^*) + T$ is solvable, a contradiction. Thus, L^* is semisimple. If L^* were to contain a proper ideal, that ideal would be strongly solvable and hence solvable, a contradiction. Thus, L^* contains no proper ideals, and L^* is simple.

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