

ABSTRACT

MEEHAN, EMILY. Posets and Hopf Algebras of Rectangulations. (Under the direction of Nathan Reading.)

A rectangulation is a way of decomposing a square into rectangles. We consider two types of rectangulations, diagonal rectangulations and generic rectangulations.

Diagonal rectangulations with n rectangles are counted by the Baxter number $B(n)$. In this thesis, we first characterize a collection of posets on n , which we call Baxter posets, that are counted by $B(n)$. Given a diagonal rectangulation, the set of linear extensions of the corresponding Baxter poset is the fiber of a lattice homomorphism from the right weak order on permutations of $[n]$ to a lattice on diagonal rectangulations. Given a Baxter poset, we also describe how to obtain the unique Baxter permutation and the unique twisted Baxter permutation which are linear extensions of the poset.

Both diagonal rectangulations and generic rectangulations form bases for Hopf algebras which are isomorphic to sub Hopf algebras of the Malvenuto-Reutenauer Hopf algebra of permutations. The Hopf algebra of diagonal rectangulations is described in the work of Law and Reading. In this thesis, we describe a lattice on generic rectangulations which, like the lattice on diagonal rectangulations, is isomorphic to a quotient of the right weak order on permutations. Making use of this lattice, we describe the Hopf algebra of generic rectangulations. As is the case in the Hopf algebra of diagonal rectangulations, in the Hopf algebra of generic rectangulations, the product operation of two basis elements is a sum of elements in an interval in the lattice of generic rectangulations. Each element in this interval gives a way of combining the rectangulations. The coproduct operation is a sum of ways of splitting a rectangulation.

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Posets and Hopf Algebras of Rectangulations

by
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DEDICATION

To my family.

BIOGRAPHY

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Chapter 1

Introduction

A *rectangulation of size n* is an equivalence class of decompositions of a square S into n rectangles. Two decompositions R_1 and R_2 are members of the same equivalence class if and only if there exists a homeomorphism of the square, fixing its vertices, that takes R_1 to R_2 . We will consider two types of rectangulations, called diagonal rectangulations and generic rectangulations, exploring posets and natural algebraic operations related to these rectangulations.

In this chapter, we define diagonal and generic rectangulations and provide general background regarding Hopf algebras. After defining a combinatorial Hopf algebra, we provide two examples, the Malvenuto-Reutenauer Hopf algebra and the Hopf algebra of diagonal rectangulations. We then describe a method for constructing sub Hopf algebras of the Malvenuto-Reutenauer Hopf algebra and provide an example of this construction. Finally, we briefly outline the remaining chapters of the thesis.

1.1 Baxter Numbers and Diagonal Rectangulations

Definition 1.1.1. We say that a rectangulation is a *diagonal rectangulation* if, for some representative of the equivalence class, the top-left to bottom-right diagonal of S contains an interior point of each rectangle of the decomposition.

Diagonal rectangulations of size n are counted by the Baxter number

$$B(n) = \binom{n+1}{1}^{-1} \binom{n+1}{2}^{-1} \sum_{k=1}^n \binom{n+1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}.$$

The original combinatorial objects counted by the Baxter numbers, called Baxter permutations, first appeared in the work of Glen Baxter [4]. In investigating Eldon Dyer’s question about whether commuting continuous functions f and g from $[0, 1]$ into $[0, 1]$ have a common fixed point, Baxter used a subset of the permutations of $[2n - 1] := \{1, \dots, 2n - 1\}$ to encode maps between the fixed points of $g \circ f$ and the fixed points of $f \circ g$. In [7], Chung, Graham, Hoggatt, and Kleiman referred to these permutations as Baxter permutations. Noting that each Baxter permutation is completely determined by its odd entries, they mapped each Baxter permutation to a unique permutation of $[n]$, which they called a “reduced” Baxter permutation. We will follow current literature and refer to these “reduced” Baxter permutations as Baxter permutations.

Definition 1.1.2. A permutation $\sigma_1 \cdots \sigma_n$ is a *Baxter permutation* if and only if there exists no $i < j < k < l$ such that $\sigma_k + 1 < \sigma_i + 1 = \sigma_l < \sigma_j$ or $\sigma_j + 1 < \sigma_l + 1 < \sigma_i < \sigma_k$.

The Baxter numbers also count twisted Baxter permutations [14], certain triples of non-intersecting lattice paths [8], noncrossing arc diagrams consisting of only left and right arcs [18], certain Young tableaux [12], twin binary trees [12], diagonal rectangulations [2, 9, 14], and other families of combinatorial objects. Several of these Baxter objects, including diagonal rectangulations, can be obtained by pairing “twin” Catalan objects as described in Section 2.1. In Definition 2.1.6, we define a family of combinatorial objects, called adjacency posets, which arise naturally from diagonal rectangulations. Each adjacency poset captures the left-right and above-below adjacencies of rectangles in the corresponding rectangulation. In Definition 2.1.11, we define a collection of posets, which we call Baxter posets. By proving that a poset is a Baxter poset if and only if it is the adjacency poset of some diagonal rectangulation, we demonstrate that Baxter posets are also counted by the Baxter numbers. We then describe bijections between Baxter posets and related Baxter objects.

In our discussion of rectangulations, we will make use of the following terms.

Definition 1.1.3. Given a rectangulation R of a square S , we call a point in S a *vertex* of R if the point is the vertex of some rectangle of R . An *edge* of R is a line segment contained in the side of some rectangle of R such that the endpoints of the line segment are vertices and the segment has no vertices in its interior. A maximal union of edges forming a line segment is a *wall* of R .

$$\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes \text{id}} & A \otimes A \\
\text{id} \otimes m \downarrow & & \downarrow m \\
A \otimes A & \xrightarrow{m} & A
\end{array}
\qquad
\begin{array}{ccccc}
A \otimes \mathbb{K} & \longleftarrow & A & \longrightarrow & \mathbb{K} \otimes A \\
\text{id} \otimes \mu \downarrow & & \downarrow \text{id} & & \downarrow \mu \otimes \text{id} \\
A \otimes A & \xrightarrow{m} & A & \xleftarrow{m} & A \otimes A
\end{array}$$

Figure 1.1: Commutative diagrams for Definition 1.2.1.

The set of diagonal rectangulations is a subset of a second set of rectangulations, which we call generic rectangulations.

Definition 1.1.4. A rectangulation is a *generic rectangulation* if no four rectangles of the rectangulation share a vertex.

In some sense, even though combinatorial objects in bijection with diagonal rectangulations have been the focus of more mathematical research, as a collection of rectangulations, generic rectangulations are a more natural combinatorial object.

1.2 Hopf Algebras

The set of all diagonal rectangulations and the set of all generic rectangulations form bases for two combinatorial Hopf algebras, respectively called the Hopf algebra of diagonal rectangulations and the Hopf algebra of generic rectangulations. In Section 1.4, we provide results from [14] which describe the Hopf algebra of diagonal rectangulations. In Chapter 3, we describe the Hopf algebra of generic rectangulations.

To define a combinatorial Hopf algebra, we require the following definitions. These definitions agree with the definitions provided in [11], except here we take \mathbb{K} to be a field rather than a commutative ring.

Definition 1.2.1. A *unital associative algebra* (A, m, μ) is a vector space A over a field \mathbb{K} with linear maps $m : A \otimes A \rightarrow A$ and $\mu : \mathbb{K} \rightarrow A$ which satisfy the commutative diagrams in Figure 1.1. In the diagrams, the map id denotes the identity map on A , the map $A \rightarrow A \otimes \mathbb{K}$ sends $a \mapsto a \otimes 1_{\mathbb{K}}$, and the map $A \rightarrow \mathbb{K} \otimes A$ sends $a \mapsto 1_{\mathbb{K}} \otimes a$.

Definition 1.2.2. A *counital coassociative algebra* (A, Δ, ϵ) is a vector space A over a field \mathbb{K} with linear maps $\Delta : A \rightarrow A \otimes A$ and $\epsilon : A \rightarrow \mathbb{K}$ which satisfy the commutative

$$\begin{array}{ccc}
A \otimes A \otimes A & \xleftarrow{\Delta \otimes \text{id}} & A \otimes A \\
\uparrow \text{id} \otimes \Delta & & \uparrow \Delta \\
A \otimes A & \xleftarrow{\Delta} & A
\end{array}
\qquad
\begin{array}{ccccc}
A \otimes \mathbb{K} & \longrightarrow & A & \longleftarrow & \mathbb{K} \otimes A \\
\uparrow \text{id} \otimes \epsilon & & \uparrow \text{id} & & \uparrow \epsilon \otimes \text{id} \\
A \otimes A & \xleftarrow{\Delta} & A & \xrightarrow{\Delta} & A \otimes A
\end{array}$$

Figure 1.2: Commutative diagrams for Definition 1.2.2.

$$\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\Delta_A \downarrow & & \downarrow \Delta_B \\
A \otimes A & \xrightarrow{\phi \otimes \phi} & B \otimes B
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\epsilon_A \searrow & & \swarrow \epsilon_B \\
& \mathbb{K} &
\end{array}$$

Figure 1.3: Commutative diagrams for Definition 1.2.3.

diagrams in Figure 1.2. In these diagrams, the map $A \otimes \mathbb{K} \rightarrow A$ sends $a \otimes 1_{\mathbb{K}} \mapsto a$ and the map $\mathbb{K} \otimes A \rightarrow A$ sends $1_{\mathbb{K}} \otimes a \mapsto a$.

The left diagram of Figure 1.1 captures the associativity of m . The diagrams of Figure 1.2 are obtained by reversing the arrows in the diagrams in Figure 1.1 and replacing the multiplication map m by the comultiplication map Δ .

Definition 1.2.3. A *morphism of coalgebras* $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ is a linear map $\phi: A \rightarrow B$ such that the diagrams in Figure 1.3 commute.

The next definition makes use of coalgebra structures on \mathbb{K} and $A \otimes A$ where A is a coalgebra. Taking $\Delta_{\mathbb{K}}$ to be the canonical isomorphism from $\mathbb{K} \rightarrow \mathbb{K} \otimes \mathbb{K}$ and $\epsilon_{\mathbb{K}}$ to be $\text{id}: \mathbb{K} \rightarrow \mathbb{K}$ we have that $(\mathbb{K}, \Delta_{\mathbb{K}}, \epsilon_{\mathbb{K}})$ is a coalgebra. Given any two coalgebras $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$, we can construct a coalgebra $(A \otimes B, \Delta_{A \otimes B}, \epsilon_{A \otimes B})$ using the map $T: A \otimes B \rightarrow B \otimes A$ that sends $a \otimes b \mapsto b \otimes a$. Specifically, the comultiplication in $A \otimes B$ is obtained by first applying $\Delta_A \otimes \Delta_B$ to an element of $A \otimes B$ to obtain an element of $A \otimes A \otimes B \otimes B$, and then applying $\text{id} \otimes T \otimes \text{id}$ to the result to obtain an element of $A \otimes B \otimes A \otimes B$. The map $\epsilon_{A \otimes B}$ is obtained by first applying $\epsilon_A \otimes \epsilon_B$ to an element of $A \otimes B$ to obtain an element of $\mathbb{K} \otimes \mathbb{K}$ and then applying the canonical isomorphism from $\mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K}$ to obtain an element of \mathbb{K} .

Definition 1.2.4. A *bialgebra* $(A, m, \mu, \Delta, \epsilon)$ over a field \mathbb{K} is a vector space over \mathbb{K} such that:

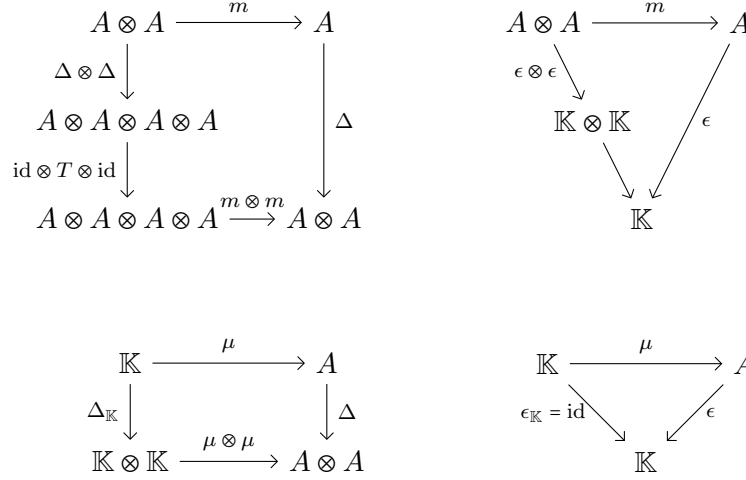


Figure 1.4: Commutative diagrams capturing the final item in Definition 1.2.4.

- (A, m, μ) is a unital associative algebra,
- (A, Δ, ϵ) is a counital coassociative algebra, and
- m and μ are homomorphisms of coalgebras.

Since $m : A \otimes A \rightarrow A$ and $\mu : \mathbb{K} \rightarrow A$, for these maps to be homomorphisms of coalgebras, we require the coalgebra structures on \mathbb{K} and $A \otimes A$ defined above. The condition that m and μ are coalgebra homomorphisms is equivalent to the commutativity of the four diagrams in Figure 1.4.

Definition 1.2.5. A *Hopf algebra* is a bialgebra $(A, m, \mu, \Delta, \epsilon)$ with an endomorphism $S : A \rightarrow A$, called the *antipode* such that the diagram in Figure 1.5 commutes.

Definition 1.2.6. A vector space A is a *graded vector space* if $A = \bigoplus_{n \geq 0} A_n$. A map $\phi : A \rightarrow B$ between graded vector spaces is a *graded map* if applying ϕ to any element of A_n results in an element of B_n for all n . A bialgebra $(A, m, \mu, \Delta, \epsilon)$ over a field \mathbb{K} is a *graded bialgebra* if A is a graded vector space and m, μ, Δ , and ϵ are graded maps.

The field \mathbb{K} admits the trivial grading $\mathbb{K} = \bigoplus_{n \geq 0} A_n$ in which $A_0 = \mathbb{K}$. Given a grading on a vector space $A = \bigoplus_{n \geq 0} A_n$, we obtain a natural grading on $A \otimes A$ by taking each element of $A_p \otimes A_q$ to be an element of the $p + q$ graded piece of $A \otimes A$.

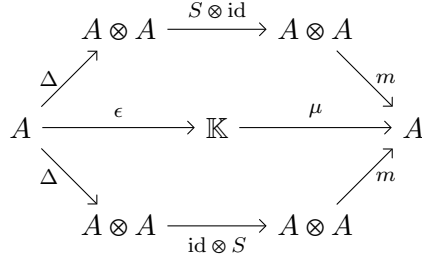


Figure 1.5: A commutative diagram for Definition 1.2.5.

Definition 1.2.7. Given a graded vector space $A = \bigoplus_{n \geq 0} A_n$ over a field \mathbb{K} , if $A_0 \cong \mathbb{K}$, then we say that A is *connected*.

The bialgebras we will consider are both graded and connected, so the following proposition allows us to obtain Hopf algebras without proving the existence of antipodes.

Proposition 1.2.8 ([11, Proposition 1.36]). *Every graded, connected bialgebra has an antipode.*

The Hopf algebra of diagonal rectangulations and the Hopf algebra of generic rectangulations are combinatorial Hopf algebras. A *combinatorial Hopf algebra* is defined by Aguiar, Bergeron, and Sottile [3] to be a graded, connected Hopf algebra A with a multiplicative linear functional $\zeta : A \rightarrow \mathbb{K}$. Informally, the term combinatorial Hopf algebra is used to refer to a graded Hopf algebra whose basis elements are indexed by some combinatorial object such that the product encodes some way of combining these combinatorial objects and the coproduct encodes some way of decomposing these combinatorial objects. We will make use of the informal definition of a combinatorial Hopf algebra.

1.3 The Malvenuto-Reutenauer Hopf Algebra

For us, the Malvenuto-Reutenauer Hopf algebra of permutations is an especially important example of a combinatorial Hopf algebra. We will consider two sub Hopf algebras of the Malvenuto-Reutenauer Hopf algebra, the Hopf algebra of twisted Baxter permutations and the Hopf algebra of 2-clumped permutations, which are respectively isomorphic to the Hopf algebra of diagonal rectangulations and the Hopf algebra of generic rectangulations.

Let S_n denote the set of permutations of $[n]$ and $\mathbb{K}[S_n]$ denote the vector space over \mathbb{K} whose basis elements are indexed by permutations of $[n]$. Let $\mathbb{K}[S_\infty] = \bigoplus_{n \geq 0} \mathbb{K}[S_n]$ denote the graded vector space over \mathbb{K} whose basis elements are indexed by permutations of $[n]$ for $n \geq 0$. The set of all permutations of the empty set contains a single element which we denote by \emptyset . To simplify notation, we refer to the basis element indexed by the permutation σ using the permutation σ itself (so elements of $\mathbb{K}[S_\infty]$ are denoted by linear combinations of permutations of any size). The vector space $\mathbb{K}[S_\infty]$ is connected since $\mathbb{K}[S_0] = \mathbb{K}[\emptyset] \cong \mathbb{K}$. Using the product and coproduct described below, we obtain a Hopf algebra, called the Malvenuto-Reutenauer Hopf algebra, from this graded, connected vector space [15].

To describe the product, we require the following definitions.

Definition 1.3.1. Given the one-line notation for a permutation $\psi = \psi_1 \cdots \psi_q \in S_q$ and some $p \in \mathbb{N}$, define the *shift of ψ by p* , denoted $\psi'_{[p]}$ to be $(\psi_1 + p) \cdots (\psi_q + p)$. We say that a permutation $v \in S_n$ is a *shifted shuffle* of the ordered pair (σ, ψ) where $\sigma \in S_p$ and $\psi \in S_q$, if $n = p + q$ and σ and $\psi'_{[p]}$ are subsequences of v .

Define the product $\sigma \bullet_{\text{MR}} \psi$ of two basis elements in the Malvenuto-Reutenauer Hopf algebra to be the sum of all shifted shuffles of the ordered pair (σ, ψ) .

Example 1.3.2. An example of this operation is

$$\begin{aligned} 213 \bullet_{\text{MR}} 21 &= 21354 + 21534 + 25134 + 52134 + 21543 + 25143 \\ &\quad + 52143 + 25413 + 52413 + 54213. \end{aligned}$$

Since $\bullet_{\text{MR}} : \mathbb{K}[S_p] \otimes \mathbb{K}[S_q] \rightarrow \mathbb{K}[S_{p+q}]$, this product is a graded map. The map $\mu : \mathbb{K} \rightarrow \mathbb{K}[S_\infty]$ is given by $1_{\mathbb{K}} \mapsto \emptyset$.

To define the coproduct, we will make use of the following map.

Definition 1.3.3. The map st takes a sequence s_1, \dots, s_p of distinct natural numbers and sends it to the unique permutation $\sigma_1 \cdots \sigma_p \in S_p$ such that for each $i \neq j$ in $[p]$, we have $s_i < s_j$ if and only if $\sigma_i < \sigma_j$. We call the resulting permutation the *standardization* of the sequence.

For example, $\text{st}(254) = 132 \in S_3$. The coproduct of a basis element in the Malvenuto-

Reutenauer Hopf algebra is given by

$$\Delta_{\text{MR}}(\sigma) = \sum_{i=0}^n \text{st}(\sigma_1, \dots, \sigma_i) \otimes \text{st}(\sigma_{i+1}, \dots, \sigma_n).$$

In this notation, $\text{st}(\sigma_1, \sigma_0)$ and $\text{st}(\sigma_{n+1}, \sigma_n)$ represent the empty permutation, denoted \emptyset .

Example 1.3.4. An example of this coproduct is

$$\begin{aligned} \Delta_{\text{MR}}(31254) = \emptyset \otimes 31254 + 1 \otimes 1243 + 21 \otimes 132 + 312 \otimes 21 \\ + 3124 \otimes 1 + 31254 \otimes \emptyset. \end{aligned}$$

Since $\Delta_{\text{MR}} : \mathbb{K}[S_n] \rightarrow \sum_{p+q=n} \mathbb{K}[S_p] \otimes \mathbb{K}[S_q]$, this coproduct is a graded map. The map $\epsilon : \mathbb{K}[S_\infty] \rightarrow \mathbb{K}$ is given by $\emptyset \mapsto 1_{\mathbb{K}}$ and $\sigma \mapsto 0_{\mathbb{K}}$ for all $\sigma \neq \emptyset$.

The product operation in the Malvenuto-Reutenauer Hopf algebra interacts nicely with a partial order on permutations called the right weak order.

Definition 1.3.5. Given $\sigma \in S_n$, define $\text{inv}(\sigma) = \{(\sigma_i, \sigma_j) \mid i < j \text{ and } \sigma_i > \sigma_j\}$. If $\sigma, \psi \in S_n$ then we say that $\sigma \leq \psi$ in the *right weak order* if and only if $\text{inv}(\sigma) \subseteq \text{inv}(\psi)$.

This definition implies that $\sigma \prec \psi$ in the right weak order if and only if σ can be obtained from ψ by transposing adjacent entries σ_i and σ_{i+1} of σ which satisfy $\sigma_i < \sigma_{i+1}$ in numerical order.

Definition 1.3.6. Let C denote a set of elements in a poset. If there exists a unique minimal element in $\{y \mid y \geq c \text{ for all } c \in C\}$, then that minimal element is called the *join* of C and is denoted $\vee C$. If there exists a unique maximal element in $\{y \mid y \leq c \text{ for all } c \in C\}$, then that maximal element is called the *meet* of C and is denoted $\wedge C$. If $\vee C$ and $\wedge C$ exist for every collection C in a poset, then we call the poset a *lattice*.

The right weak order on permutations is a lattice. For any collection of permutations, the join of these permutations in the right weak order is the unique permutation whose inversion set is found by taking the union and then the transitive closure of the inversion sets of the permutations in the collection. A non-inversion in a permutation σ is an ordered pair (σ_i, σ_j) such that $i < j$ and $\sigma_i < \sigma_j$. The meet of a collection of permutations in the right weak order is the unique permutation whose set of non-inversions is found by taking the union and then the transitive closure of the non-inversion sets of the permutations in the collection.

Given two permutations $\sigma \in S_p$ and $\psi \in S_q$, the product $\sigma \bullet_{\text{MR}} \psi$ is the sum of the elements of the interval $[\sigma\psi'_{[p]}, \psi'_{[p]}\sigma]$ in the right weak order on S_{p+q} . As we discuss in Section 1.5, many Hopf subalgebras of the Malvenuto-Reutenauer Hopf algebra can be described by considering certain lattice congruences on the right weak order.

1.4 The Hopf Algebra of Diagonal Rectangulations

Let dRec_n denote the set of diagonal rectangulations of size n and consider $\mathbb{K}[\text{dRec}_\infty] = \bigoplus_{n \geq 0} \mathbb{K}[\text{dRec}_n]$, the graded vector space over \mathbb{K} consisting of linear combinations of diagonal rectangulations of any finite size. Defining the product and coproduct as described below, we obtain the Hopf algebra of diagonal rectangulations [14, Theorem 6.8].

Definition 1.4.1. Let C denote a collection of line segments and points contained in a square. We say that C is a *partial diagonal rectangulation* if there exists a diagonal rectangulation D such that each line segment of D contains either a line segment or point of C . We call such a diagonal rectangulation D a *completion* of C .

In Figure 1.6, the rightmost diagram in the first row of the computation is a partial diagonal rectangulation. The rectangulations appearing in the second row of the figure are completions of this partial diagonal rectangulation. The sum of all completions of a partial diagonal rectangulation C is denoted by $\sum_{\text{compl}} C$.

Let $D_1 \in \text{dRec}_p$ and $D_2 \in \text{dRec}_q$. To find the product of D_1 and D_2 , denoted $D_1 \bullet_{\text{dR}} D_2$, we begin with a $p + q$ unit square with lower-left vertex at $(0,0)$. Place D_1 and D_2 in this square so that the upper-left corner of D_1 is at $(0,n)$, the lower-right corner of D_2 is at $(n,0)$, and the lower-right corner of D_1 and upper-left corner of D_2 are at (p,q) . Remove the bottom and right side of D_1 and the top and left side of D_2 from this diagram. Let C denote the union of the collection of remaining line segments, the boundary square, and $V = \{(p,q)\}$. Then $D_1 \bullet_{\text{dR}} D_2$ is the sum of all completions of C . Figure 1.6 shows the product of two diagonal rectangulations.

As is the case in the Malvenuto-Reutenauer Hopf algebra, the product in the Hopf algebra of diagonal rectangulations can be described as the sum of the elements of an interval in a lattice. Making use of a map ρ from permutations to diagonal rectangulations, this lattice is obtained from a lattice quotient of the right weak order. In Section 1.5, we describe the details of the construction of this lattice in greater generality.

$$\begin{aligned}
\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \bullet_{\text{dR}} \begin{array}{|c|} \hline \\ \hline \end{array} &= \sum_{\text{compl}} \begin{array}{|c|c|} \hline & \\ \hline & \cdot \\ \hline & \\ \hline \end{array} \\
&= \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}
\end{aligned}$$

Figure 1.6: The product of two diagonal rectangulations in the Hopf algebra of diagonal rectangulations.

Definition 1.4.2. We say that \mathcal{P} is a *path* in a diagonal rectangulation D if \mathcal{P} joins the upper-left vertex of the boundary square to the lower-right vertex of the boundary square and consists of down and right steps along edges of D .

For each such path, let $R_l(\mathcal{P})$ denote the union of the boundary of the square and edges of D below \mathcal{P} , and let $R_u(\mathcal{P})$ denote the union of the boundary of the square and the edges of D above \mathcal{P} . The coproduct in the Hopf algebra of diagonal rectangulations is

$$\Delta_{\text{dR}}(D) = \sum_{\mathcal{P}} \left(\sum_{\text{compl}} R_l(\mathcal{P}) \otimes \sum_{\text{compl}} R_u(\mathcal{P}) \right),$$

where the outer summation denotes the sum over all paths in D . Figure 1.7 shows the computation of the coproduct of the diagonal rectangulation shown in the left column of the figure. That column shows the six distinct paths in the diagonal rectangulation. The sum of the elements in the right column is the coproduct of this diagonal rectangulation.

1.5 \mathcal{H} -Families and Pattern Avoidance

In this section, we describe a method for constructing sub Hopf algebras of the Malvenuto-Reutenauer Hopf algebra using lattice homomorphisms.

Definition 1.5.1. A map f from a lattice L to a lattice M is a *lattice homomorphism* if $f(l_1 \wedge_L l_2) = f(l_1) \wedge_M f(l_2)$ and $f(l_1 \vee_L l_2) = f(l_1) \vee_M f(l_2)$ for all $l_1, l_2 \in L$.

In other words, a function $f : L \rightarrow M$ is a lattice homomorphism if f respects the meet and join operations of the lattices.

\mathcal{P}	$R_l(\mathcal{P})$	$R_u(\mathcal{P})$	$\sum_{\text{compl}} R_l(\mathcal{P}) \otimes \sum_{\text{compl}} R_u(\mathcal{P})$
	\emptyset		$\emptyset \otimes \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
			$\square \otimes \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array} + \square \otimes \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
			$\begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
			$\begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array} \otimes \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array}$
			$\begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array} \otimes \square + \begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array} \otimes \square$
		\emptyset	$\begin{array}{ c c } \hline & \\ \hline & \\ \hline \end{array} \otimes \emptyset$

Figure 1.7: The coproduct of a diagonal rectangulation in the Hopf algebra of diagonal rectangulations.

Definition 1.5.2. We say that an equivalence relation on a lattice is a *lattice congruence* if there exists a lattice homomorphism f such that each congruence class is a fiber of f .

Each fiber of a lattice homomorphism is an interval so we can choose to refer to each congruence class of a lattice congruence using the unique minimal element or the unique maximal element of that congruence class. Given a lattice congruence Θ on a lattice L and $x \in L$, let $\pi_{\downarrow}(x)$ and $\pi_{\uparrow}(x)$ respectively denote the minimal element and the maximal element of the equivalence class of x .

Conditions on lattice congruences that give rise to Hopf subalgebras of the Malvenuto-Reutenauer Hopf algebra are described in [16]. Since these conditions result in a Hopf algebra, such a family of congruences is called an \mathcal{H} -family of congruences. Let Θ denote an \mathcal{H} -family of congruences where Θ_n denotes the lattice congruence on S_n , and let Z_n^Θ denote the quotient S_n/Θ_n . For each n , we use Av_n^Θ to denote the collection of the minimal elements of the congruence classes. This notation is used because for each \mathcal{H} -family of congruences, these minimal elements can be described using pattern avoidance conditions.

Definition 1.5.3. Let $p = p_1 \cdots p_l \in S_l$ and \tilde{p} be obtained by inserting a single dash between some adjacent entries of p . We say that a permutation $\psi \in S_n$ *contains the pattern* \tilde{p} if there exists some subsequence $\psi_{i_1} \cdots \psi_{i_l}$ of ψ such that:

- For all $j, k \in [l]$, subsequence elements satisfy $\psi_{i_j} < \psi_{i_k}$ if and only if $p_j < p_k$.
- If p_j and p_{j+1} are not separated by a dash in \tilde{p} , then $i_j = i_{j+1} - 1$.

If ψ does not contain the pattern p , we say that ψ *avoids* p .

The first item in the above definition can be rephrased as the requirement that the relative order of the terms in the subsequence matches the relative order of the entries of p . The second item indicates that if p_j and p_{j+1} are not separated by a dash in \tilde{p} , then ψ_{i_j} and $\psi_{i_{j+1}}$ must be adjacent in ψ .

Example 1.5.4. Consider $\psi = 546312 \in S_6$. The subsequence 5612 is an occurrence of the pattern 3-4-1-2 in ψ , but is not an occurrence of the pattern 3-41-2 since the 6 and 1 are non-adjacent in ψ .

The pattern 3-41-2 is an example of a pattern which we will call an adjacent cliff pattern.

Definition 1.5.5. Let \tilde{p} be a pattern and $p = p_1 \cdots p_l \in S_l$ denote the associated permutation obtained by removing the dashes from \tilde{p} . Then \tilde{p} is an *adjacent cliff pattern* if $p_i = l$, $p_{i+1} = 1$, entries p_i and p_{i+1} are not separated by a dash in \tilde{p} , and every other pair of consecutive entries of \tilde{p} is separated by a dash. The pattern \tilde{p}' , with associated permutation $p' = p'_1 \cdots p'_l$, is a *scramble* of the adjacent cliff pattern \tilde{p} if \tilde{p}' is also an adjacent cliff pattern with $p'_i = l$, the first $i - 1$ entries of p' is a permutation of the first $i - 1$ entries of p , and $p'_{i+2} \cdots p'_l$ is a permutation of $\{p_{i+2}, \dots, p_l\}$. We say that permutations σ and ψ are related by an *adjacent cliff transposition of the pattern* p if one of these permutations, say σ , contains an occurrence $\sigma_{j_1} \cdots \sigma_{j_l}$ of the adjacent cliff pattern p such that $\sigma_{j_i} = l$ and transposing σ_{j_i} and $\sigma_{j_{i+1}}$ in σ results in the permutation ψ .

Example 1.5.6. The pattern 2-4-51-3 is a scramble of the pattern 4-2-51-3. The permutations 4316725 and 4316275 are related by an adjacent cliff transposition of the pattern 2-4-51-3.

A family of lattice congruences is an \mathcal{H} -family of congruences if and only if the family of congruences is defined by some collection \mathcal{C} of adjacent cliff patterns. Specifically, given a collection \mathcal{C} of adjacent cliff patterns, the elements of Av_n^Θ are the permutations that avoid all scrambles of these adjacent cliff patterns [16, Theorem 9.3]. The equivalence classes of a given \mathcal{H} -family of congruences are defined by declaring $[\sigma]_\Theta = [\psi]_\Theta$ if and only if σ and ψ are related by a sequence of adjacent cliff transpositions of patterns which are scrambles of elements of \mathcal{C} .

Consider the graded vector space $\mathbb{K}[Av_\infty^\Theta] = \bigoplus_{n \geq 0} \mathbb{K}[Av_n^\Theta]$. Define $r^\Theta : \mathbb{K}[S_\infty] \rightarrow \mathbb{K}[Av_\infty^\Theta]$ by $r^\Theta(\sigma) = \sigma$ if $\sigma \in Av_n^\Theta$ and $r^\Theta(\sigma) = 0$ otherwise. Let $c^\Theta : \mathbb{K}[Av_\infty^\Theta] \rightarrow \mathbb{K}[S_\infty]$ denote the map that takes each $\sigma \in Av_p^\Theta$ to the sum of the elements of the fiber $\Theta_p^{-1}(\sigma)$. If one wants c^Θ to embed $\mathbb{K}[Av_\infty^\Theta]$ as a sub Hopf algebra of the Malvenuto-Reutenauer Hopf algebra, then it must be the case that for all $\sigma \in Av_p^\Theta, \psi \in Av_q^\Theta$,

$$c^\Theta(\sigma \bullet_{Av} \psi) = c^\Theta(\sigma) \bullet_{MR} c^\Theta(\psi) \text{ and } (c^\Theta \otimes c^\Theta)(\Delta_{Av}(\sigma)) = \Delta_{MR}(c^\Theta(\sigma)).$$

The conditions on \mathcal{H} -families ensure that this is possible [16, Corollary 1.4]. Since r^Θ restricted to the image of c^Θ is the inverse of c^Θ , applying r^Θ and $r^\Theta \otimes r^\Theta$ respectively to these equalities, we obtain

$$\sigma \bullet_{Av} \psi = r^\Theta(c^\Theta(\sigma) \bullet_{MR} c^\Theta(\psi)) \text{ and } \Delta_{Av}(\sigma) = (r^\Theta \otimes r^\Theta)(\Delta_{MR}(c^\Theta(\sigma))).$$

From the conditions on \mathcal{H} -families, it follows that $\sigma \bullet_{Av} \psi = r^\Theta(\sigma \bullet_{MR} \psi)$. In other words, to find the product of σ and ψ in $\mathbb{K}[Av_\infty^\Theta]$, we find the sum of all shifted shuffles of the ordered pair (σ, ψ) and then eliminate permutations which are not elements of Av_{p+q}^Θ . This coincides with the sum of all elements of the interval $[\sigma\psi'_{[p]}, \pi_\downarrow(\psi'_{[p]}\sigma)]$ in the lattice Z_{p+q}^Θ , the quotient of the right weak order on S_{p+q} by the congruence Θ_{p+q} [13, Equation 6]. To find the coproduct of σ in $\mathbb{K}[Av_\infty^\Theta]$, we find the coproduct in the Malvenuto-Reutenauer Hopf algebra of the sum of all permutations that map to $[\sigma]_\Theta$ and then eliminate terms which are not elements of $\mathbb{K}[Av_\infty^\Theta] \otimes \mathbb{K}[Av_\infty^\Theta]$.

1.6 The Hopf Algebra **tBax**

The Hopf algebra described in this section is an example of a sub Hopf algebra of the Malvenuto-Reutenauer Hopf algebra that is constructed using the method from Sec-

tion 1.5.

The definition of a Baxter permutation provided in Section 1.1 (requiring that a Baxter permutation $\sigma = \sigma_1 \cdots \sigma_n$ contain no $i < j < k < l$ such that $\sigma_k + 1 < \sigma_i + 1 = \sigma_l < \sigma_j$ or $\sigma_j + 1 < \sigma_l + 1 < \sigma_i < \sigma_k$) can be rephrased using pattern avoidance conditions. Specifically, a Baxter permutation is a permutation that avoids the patterns 2-41-3 and 3-14-2. A second set of permutations counted by the Baxter numbers, called twisted Baxter permutations has a similar pattern avoidance description.

Definition 1.6.1. A *twisted Baxter permutation* is a permutation that avoids the patterns 2-41-3 and 3-41-2.

These pattern avoidance conditions are equivalent to the requirement that if $\sigma_i > \sigma_{i+1}$ then either all values numerically between σ_{i+1} and σ_i are left of σ_i in σ , or all of these values are right of σ_{i+1} in σ . For a proof that twisted Baxter permutations are counted by the Baxter numbers, see [14, Theorem 8.2].

Since the patterns 2-41-3 and 3-41-2 are adjacent cliff patterns, we can use them to construct an \mathcal{H} -family of congruences. Note that no additional patterns are obtained by considering the scrambles of these patterns. Defining $[\sigma]_{\Theta} = [\psi]_{\Theta}$ if σ and ψ are related by some sequence of adjacent cliff transpositions of the patterns 2-41-3 and 3-41-2 results in an \mathcal{H} -family of congruences in which the minimal element of each congruence class is a twisted Baxter permutation. In other words, making use of the results quoted in Section 1.5, the twisted Baxter permutations form a basis for a sub Hopf algebra of the Malvenuto-Reutenauer Hopf algebra. We use tBax to denote this Hopf algebra.

The product and coproduct operations in tBax are obtained as described in Section 1.5.

Example 1.6.2. Examples of the product and coproduct operations of tBax applied to specific basis elements are shown below.

$$\begin{aligned} 213 \bullet_{\text{tB}} 21 &= r^{\text{tB}}(21354 + 21534 + 25134 + 52134 + 21543 + 25143 + 52143 \\ &\quad + 25413 + 52413 + 54213) \\ &= 21354 + 21534 + 52134 + 21543 + 52143 + 54213 \end{aligned}$$

$$\begin{aligned}
\Delta_{\text{tB}}(2143) &= r^{\text{tB}} \otimes r^{\text{tB}}(\Delta_{\text{MR}}(2143 + 2413)) \\
&= r^{\text{tB}} \otimes r^{\text{tB}}(\emptyset \otimes 2143 + 1 \otimes 132 + 21 \otimes 21 + 213 \otimes 1 + 2143 \otimes \emptyset \\
&\quad + \emptyset \otimes 2413 + 1 \otimes 312 + 12 \otimes 12 + 231 \otimes 1 + 2413 \otimes \emptyset) \\
&= \emptyset \otimes 2143 + 1 \otimes 132 + 21 \otimes 21 + 213 \otimes 1 + 2143 \otimes \emptyset + 1 \otimes 312 + 12 \otimes 12 + 231 \otimes 1
\end{aligned}$$

Notice that the product $213 \bullet_{\text{tB}} 21$ is the sum of elements in the interval $[21354, 54213]$ in the lattice on twisted Baxter permutations obtained by taking the quotient of the right weak order by Θ .

The Hopf algebra of twisted Baxter permutations is isomorphic to the Hopf algebra of diagonal rectangulations described in Section 1.4 via a bijection ρ between these combinatorial families. The details of ρ are given in Section 2.2. Applying ρ to the product and coproduct computations shown in Example 1.6.2, we obtain the examples shown in Figure 1.6 and Figure 1.7.

1.7 Description of Remaining Chapters

In Chapter 2, we introduce a family of posets called Baxter posets that correspond to diagonal rectangulations. We prove that Baxter posets are counted by the Baxter numbers by showing that they are the adjacency posets of diagonal rectangulations. Given a diagonal rectangulation, we describe the cover relations in the associated Baxter poset. Given a Baxter poset, we describe a method for obtaining the associated Baxter permutation and the associated twisted Baxter permutation.

In Chapter 3, we explore the Hopf algebra of 2-clumped permutations, a sub Hopf algebra of the Malvenuto-Reutenauer Hopf algebra that, in turn, contains tBax as a sub Hopf algebra. As with the Hopf algebra tBax described in Section 1.6, the operations in the Hopf algebra of 2-clumped permutations can be described extrinsically in terms of the operations in the Malvenuto-Reutenauer Hopf algebra. Making use of a bijection between 2-clumped permutations and generic rectangulations, we can describe the Hopf algebra of 2-clumped permutations using generic rectangulations. We describe the cover relations in a lattice of generic rectangulations that is a lattice congruence of the right weak order on permutations. We then use this lattice to describe the product and coproduct operations

in the Hopf algebra of generic rectangulations. The descriptions we obtain are similar to the descriptions of the Hopf algebra of diagonal rectangulations provided in Section 1.4.

Chapter 2

Baxter Posets

In this chapter, we define Baxter posets and prove that they are also counted by the Baxter numbers. Baxter posets are closely related to Catalan combinatorics. Specifically, Baxter posets (and the closely related diagonal rectangulations) can be realized through “twin” Catalan objects. Additionally, the relationship between Baxter posets and diagonal rectangulations is analogous to the relationship between two Catalan objects, specifically sub-binary trees and triangulations of convex polygons. As a prelude to our discussion of Baxter posets, we describe a few Catalan objects and bijections between them.

2.1 Catalan and Baxter Objects

The Catalan number $C(n) = \frac{1}{n+1} \binom{2n}{n}$ counts the elements of S_n that avoid the pattern 2-31. The map τ_b , described below and illustrated in Figure 2.1, assigns a triangulation of a convex $(n+2)$ -gon to each element of S_n , and restricts to a bijection between permutations that avoid 2-31 and triangulations of polygons. Let $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ and let P be a convex $(n+2)$ -gon. For convenience, deform P so that P is inscribed in the upper half of a circle, and label each vertex of P , in numerical order from left to right, with an element of the sequence $0, 1, \dots, n+1$. For each $i \in \{0, \dots, n\}$, construct a path P_i from the vertex labeled 0 to the vertex labeled $n+1$ that visits the vertices labeled by elements of $\{\sigma_1, \dots, \sigma_i\}$ in numerical order. The union of these paths defines $\tau_b(\sigma)$, a triangulation of P .

Given a triangulation Δ of a convex $(n+2)$ -gon P , deform P (and Δ) as above.

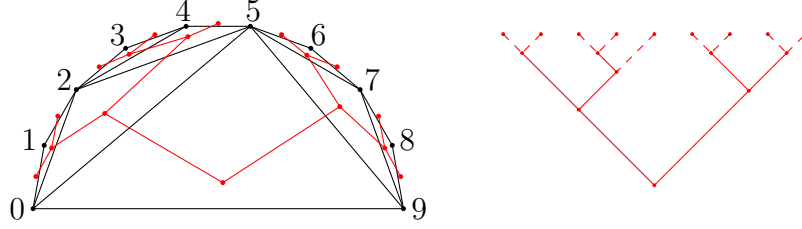


Figure 2.1: The Catalan objects obtained by applying τ_b and the dual graph construction to the 2-31 avoiding permutation 52143768.

Construct a graph with an edge crossing each edge of Δ except the horizontal diameter, as shown in red in the left diagram of Figure 2.1. (This is essentially the dual graph of Δ .) In what follows, we will call this the *dual graph construction*. Terminology for the resulting family of trees is mixed in the literature, with adjectives such as complete, planar, rooted, and binary appearing inconsistently. We will call the resulting tree a binary tree and provide a careful definition.

Definition 2.1.1. We say that a rooted tree is a *binary tree* if every non-leaf has exactly two children, with one child identified as the left child and the other as the right child.

The dual graph construction gives a bijection between triangulations of a convex $(n+2)$ -gon and binary trees with $2n+1$ vertices. The root of the binary tree corresponds to the bottom triangle of Δ and children are identified as left or right according to the embedding of Δ in the plane. For a reason that will become apparent later, we deform each binary tree resulting from this bijection as shown in the right diagram of Figure 2.1 so that the root is the lowermost vertex. Removing the leaves of a binary tree and retaining the left-right labeling of each child, we obtain a tree which we call a sub-binary tree.

Definition 2.1.2. A *sub-binary tree* is a rooted tree in which every vertex has 0, 1, or 2 children, and each child is labeled left or right, with at most one child of each vertex receiving each label.

The leaf-removal map is a bijection between binary trees with $2n+1$ vertices and sub-binary trees with n vertices. In the example shown in Figure 2.1, the edges removed by this map are shown as dashed segments.

We will make use of a second similar map from permutations to triangulations. The map τ_t described below restricts to a bijection between elements of S_n that avoid 31-2 and

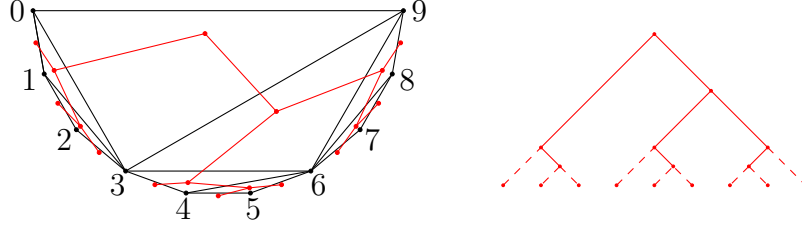


Figure 2.2: The Catalan objects obtained by applying τ_t and the dual graph construction to the 31-2 avoiding permutation 21547863.

triangulations of a convex $(n+2)$ -gon. Let $\sigma \in S_n$ and P a convex $(n+2)$ -gon. Deform P and label its vertices as shown in the example in Figure 2.2. For each $i \in \{0, 1, \dots, n\}$, construct the path P_i that begins at the vertex labeled 0, visits in numerical order each vertex labeled by an element of $[n] - \{\sigma_1, \dots, \sigma_i\}$, and ends at the vertex labeled $n+1$. The union of these paths is $\tau_t(\sigma)$. Performing the dual graph construction and then the leaf-removal map, we obtain corresponding binary and sub-binary trees. This time, we choose to deform the binary and sub-binary trees so that the root is the uppermost vertex, as illustrated in the right diagram of Figure 2.2.

Although a sub-binary tree is an unlabeled graph, for each sub-binary tree with n vertices, there exists a unique labeling of its vertices by the elements of $[n]$ such that every parent vertex has a label numerically larger than the labels of its left descendants and numerically smaller than the labels of its right descendants. An example of a sub-binary tree with such a labeling is show in Figure 2.3. Let T be a labeled sub-binary tree embedded in the plane as shown in Figure 2.3 and Δ_T the associated triangulation. View T as the Hasse diagram of a poset.

Definition 2.1.3. A total order L of the elements of T is a *linear extension* of T if $x <_T y$ implies that $x <_L y$.

The linear extensions of T , viewed as permutations in one-line notation, are exactly the permutations that map to Δ_T under τ_b . To see why, label each triangle of Δ_T according to the label of its middle (from left to right) vertex, as illustrated in Figure 2.3. The linear extensions of T are exactly the permutations that map to Δ_T because $x <_T y$ if and only if the triangle labeled y is “above” the triangle labeled x . Similarly, given a sub-binary tree T' , embedded in the plane as illustrated in Figure 2.2, and associated

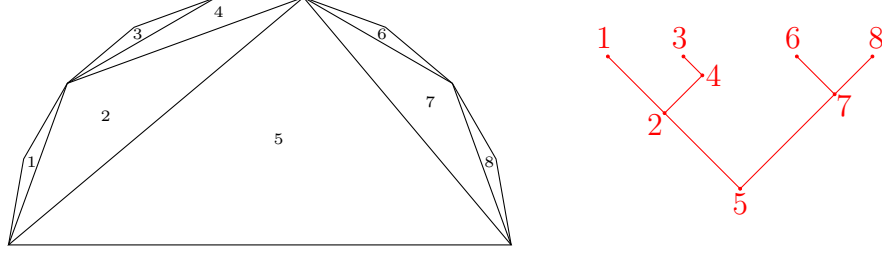


Figure 2.3: The labeling of a sub-binary tree obtained from the labeling of the triangles in the corresponding triangulation.

triangulation $\Delta_{T'}$, we obtain a labeling of T' such that the linear extensions of T' are exactly the permutations that map to $\Delta_{T'}$ under τ_t .

We now relate the Catalan objects described above to Baxter objects. Specifically, we will see that diagonal rectangulations are made by gluing together binary trees, and we will construct Baxter posets so that they play the same role for diagonal rectangulations that sub-binary trees play for triangulations.

Recall that a twisted Baxter permutation is a permutation that avoids the patterns 2-41-3 and 3-41-2 and that the twisted Baxter permutations in S_n are counted by the Baxter number $B(n)$. Twisted Baxter permutations are related to diagonal rectangulations by way of pairs of binary trees, called twin binary trees.

Definition 2.1.4. Given $\sigma \in S_n$, call the pair $(\tau_b(\sigma), \tau_t(\sigma))$ a pair of *twin binary trees*.

Gluing the twin binary trees associated with any permutation along their leaves, we obtain a decomposition of a square into n rectangles. We then rotate the resulting figure $\pi/4$ radians clockwise. Each decomposition resulting from this binary tree gluing map is a diagonal rectangulation.

Example 2.1.5. The result of applying the binary tree gluing map to the permutation 52147862 is shown in the left diagram of Figure 2.4. The binary trees which are glued together in this example are shown in Figures 2.1 and 2.2.

The binary tree gluing map restricts to a bijection between twisted Baxter permutations and diagonal rectangulations.

Given a diagonal rectangulation, label the rectangles of the decomposition according to the order in which they appear along the diagonal, labeling the upper-leftmost rect-

angle with 1 and the lower-rightmost rectangle with n . We refer to the rectangle with label i as “rectangle i .”

Definition 2.1.6. Given a labeled diagonal rectangulation D , construct a poset P by declaring $x <_P y$ if the interior of the bottom or left side of rectangle y intersects the interior of the top or right side of rectangle x , and then taking the reflexive and transitive closure of these relations. We call the resulting poset on $[n]$ the *adjacency poset* of D .

Adjacency posets are defined in [9, 14]. Remark 6.7 in [14] explains that, before taking the reflexive and transitive closure, these relations are acyclic. Thus the adjacency poset is a partial order on $[n]$. (A more general set of posets, corresponding to elements of the Baxter monoid, are defined in [10].) Each adjacency poset captures the “right of” and “above” relations of the diagonal rectangulation just as each sub-binary tree captures the “above” relations of the corresponding triangulation. Additionally, given an adjacency poset P and the corresponding diagonal rectangulation D , the set of linear extensions of P is the set of permutations that map to D under the binary tree gluing map [14, Remark 6.7]. We note that two permutations σ and ψ map to the same diagonal rectangulation if and only if $\tau_b(\sigma) = \tau_b(\psi)$ and $\tau_t(\sigma) = \tau_t(\psi)$. Thus, the set of linear extensions of the adjacency poset of a diagonal rectangulation is the intersection of the sets of linear extensions of the labeled sub-binary trees obtained from τ_b and τ_t .

As a diagonal rectangulation can be constructed from twin binary trees, the adjacency poset of a diagonal rectangulation can be constructed using the corresponding labeled sub-binary trees. Let D be a diagonal rectangulation, P the associated adjacency poset, and T_b and T_t respectively denote the corresponding labeled sub-binary trees obtained from τ_b and τ_t . By declaring $x <_P y$ if $x <_{T_b} y$ or $x <_{T_t} y$ and then taking the transitive closure, we obtain all of the relations of P . Although it is simple to use the relations of T_b and T_t to list the relations of P , it is not so straightforward to obtain a description of the Hasse diagram of P or to characterize the set of adjacency posets of diagonal rectangulations.

Definition 2.1.7. In any poset P , we say that y *covers* x , denoted $x <_P y$, if $x <_P y$ and there exists no z such that $x <_P z <_P y$.

In Theorem 2.3.2, the first main result of this chapter, we show that $x < y$ in the adjacency poset P if and only if, in the associated diagonal rectangulation, rectangles x and y form one of the configurations shown in Figure 2.7. This theorem allows us to

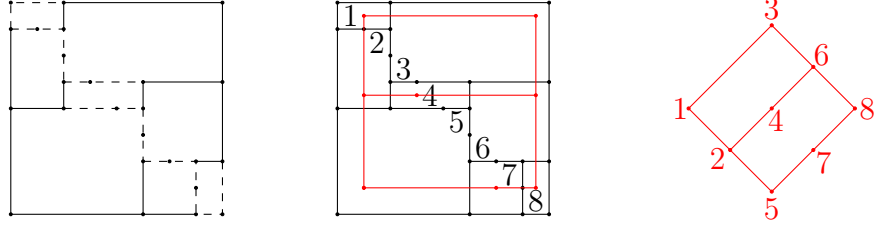


Figure 2.4: The rectangulation and adjacency poset obtained from the twisted Baxter permutation 52147863.

obtain a Hasse diagram for the adjacency poset from a diagonal rectangulation just as we easily obtain a sub-binary tree from a triangulation.

For our second result, which characterizes adjacency posets, we require the following definitions.

Definition 2.1.8. A poset P is *bounded* if it has an element that is greater than all other elements and an element that is less than all other elements.

Definition 2.1.9. Given a poset P on $[n]$, a *2-14-3 chain* is a chain $b <_P a <_P d <_P c$ of P such that $a < b < c < d$ in numerical order. We similarly define a 3-14-2 chain, a 2-41-3 chain, and a 3-41-2 chain.

Definition 2.1.11 will refer to a planar embedding of a poset, so we first provide a careful definition of a Hasse diagram and then a definition of a planar embedding of a poset.

Definition 2.1.10. Given a partially ordered set P , construct a graph G such that the vertices of G are labeled by the elements of P and there is an edge joining vertex x to vertex y if and only if $x <_P y$ or $y <_P x$. An embedding of G in \mathbb{R}^2 is a *Hasse diagram* for P if and only if for all $x <_P y$, vertex y is above vertex x in the plane and each edge of the embedding is a line segment. A *planar embedding* of a poset P is a Hasse diagram for P in which no two edges intersect.

Making use of these terms, we consider a subset of the posets on $[n]$.

Definition 2.1.11. A poset P on $[n]$ is a *Baxter poset* if and only if it satisfies the following conditions:

1. P is bounded.
2. If $x \in P$, then x is covered by at most two elements and covers at most two elements.
3. P contains no 2-14-3, no 3-14-2, no 2-41-3, and no 3-41-2 chains.
4. If $[x, y]$ is an interval of P such that the open interval (x, y) is disconnected, then $|x - y| = 1$.
5. There exists a planar embedding of P such that for every interval $[x, y]$ of P with (x, y) disconnected, if $w, z \in (x, y)$ and w is left of z , then $w < x < z$ in numerical order.

Condition 2 of Definition 2.1.11 implies that every open interval (x, y) of a Baxter poset consists of at most two connected components. Condition 5 implies that if (x, y) is disconnected, then the elements of one connected component are all smaller than x and y while the elements of the other connected component are larger than x and y . We call the embedding described in Condition 5 of Definition 2.1.11 a *natural embedding* of the Baxter poset.

We can now state the main result of Chapter 2.

Theorem 2.1.12. *A poset P is a Baxter poset if and only if it is the adjacency poset of a diagonal rectangulation.*

Remark 2.1.13. One might hope for an unlabeled version of the Baxter poset from which the labeled poset can be obtained, just as sub-binary trees have a canonical labeling. However, without “decorating” the poset with additional combinatorial information, this is not possible. This is quickly apparent since, when $n = 4$, of the 22 Baxter posets, 20 of these are chains. Decorating each poset to indicate the numerical order of each pair $x <_P y$ with (x, y) disconnected is insufficient. Additionally, decorating every edge of the Hasse diagram to indicate the numerical order of the elements of the cover relation does not allow us to determine a unique Baxter poset.

Recall that the original Baxter object, Baxter permutations, are elements of S_n that avoid the patterns 2-41-3 and 3-14-2. Given a diagonal rectangulation D , the set of permutations that map to D under the binary tree gluing map contains a unique twisted Baxter permutation and a unique Baxter permutation (see Theorem 2.2.1). Other authors (see [14, Proof of Lemma 8.4], [9, Proof of Lemma 6.6]) have described algorithms

for obtaining these permutations from a diagonal rectangulation. Our final results of this chapter describe how to obtain these pattern avoiding permutations directly from a Baxter poset. Here, we describe a method of obtaining the Baxter permutation.

Definition 2.1.14. Let P denote the planar embedding of a poset. The edges of the embedding separate the plane into maximal connected components. We call the closure of a bounded connected component a *region* of the embedding.

Let P be the natural embedding of a Baxter Poset. Assign an arrow to each region of the embedding as follows: If the maximal element of a region is greater (in numerical order) than the minimal element of that region, then that region is assigned a right-pointing arrow, and otherwise the region is assigned a left-pointing arrow. An example is shown in Figure 2.5.

Definition 2.1.15. If a region R_i of the natural embedding of a Baxter poset contains a right-pointing arrow and σ is a linear extension of P in which all labels of elements contained in the left side of R_i precede all labels of elements contained in the right side of R_i , then we say that σ *respects the arrow of R_i* . Similarly, we say that σ respects the arrow of a region R_i containing a left-pointing arrow if all labels contained in the right side of R_i precede all labels of elements contained in the left side of R_i . If σ respects the arrows of every region of P , then we say that σ *respects the arrows of P* .

The existence of a linear extension of P that respects the arrows of P should not be immediately obvious to the reader.

Theorem 2.1.16. *Given a Baxter poset P with its natural embedding, the unique Baxter permutation that is a linear extension of P is the unique linear extension that respects the arrows of the embedding.*

By adding a single relation for each region of the natural embedding of P , we obtain an alternate description of the map from an adjacency poset to its Baxter permutation. Specifically, for each region R with minimal element x and maximal element $x + 1$ we declare that the maximal element (with respect to the partial order P) of the left component of (x, y) is less than the minimal element of the right component. Similarly, for each region R with maximal element x and minimal element $x + 1$, we declare that the maximal element of the right component of (x, y) is less than the minimal element of the

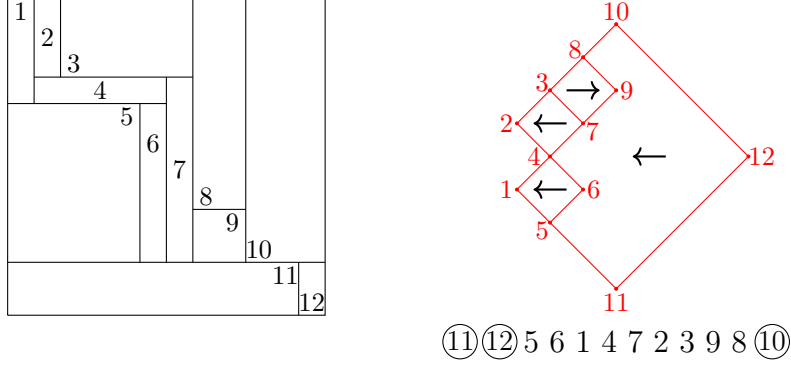


Figure 2.5: A diagonal rectangulation, the corresponding adjacency poset with an arrow assigned to each region, and the Baxter permutation obtained using Theorem 2.1.16

left component. By Theorem 2.1.16, the resulting partial order is a total order on $[n]$ and this total order is a Baxter permutation.

In Section 2.2, we describe the map ρ (mentioned in Sections 1.4 and 1.6) from permutations to diagonal rectangulations that coincides with the binary tree gluing map described in this section and provide some background related to diagonal rectangulations. We prove Theorem 2.3.2 (the characterization of the cover relations of the adjacency poset) in Section 2.3. Our main result regarding Baxter posets, Theorem 2.1.12, is proved in Section 2.4. Finally, in Section 2.5, we describe how to obtain a twisted Baxter permutation from a Baxter poset and then prove Theorem 2.1.16.

2.2 Diagonal Rectangulations

Recall that a rectangulation is a diagonal rectangulation if, for some representative of the equivalence class, the top-left to bottom-right diagonal of S contains an interior point of each rectangle of the decomposition. In our discussion of diagonal rectangulations, we often blur the distinction between an equivalence class and a representative of the equivalence class. We most often refer to a diagonal rectangulation using the distinguished representative with edges intersecting the diagonal in equally spaced points.

We now define a map ρ from S_n to the set of diagonal rectangulation of size n . Figure 2.6 shows the construction of $\rho(23154)$. The map ρ agrees with the map (described in Section 2.1) in which a diagonal rectangulation is constructed from a permutation by

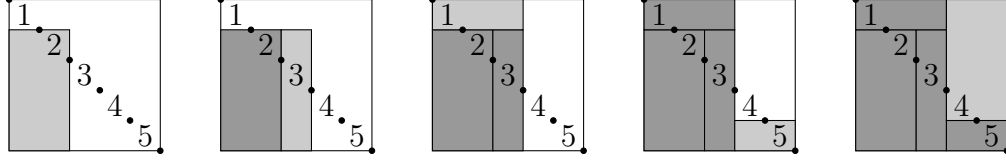


Figure 2.6: The map ρ applied to the permutation 23154.

gluing together twin binary trees and then rotating the result. Our description of ρ matches the description in [14, Section 6] and is essentially equivalent to maps described in [2, Section 3], [1, Section 4], and [9, Section 5].

Let $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ and S a square in \mathbb{R}^2 with bottom-left vertex at $(0, 0)$ and top-right vertex at (n, n) . Place $n + 1$ points at $(i, n - i)$ for $i \in \{0, \dots, n\}$. Label each of the n spaces between these points in order with an element of $[n]$, starting with 1 in the upper-leftmost space and finishing with n in the lower-rightmost space. We construct $\rho(\sigma)$ by considering the entries of σ sequentially from left to right. Let T_{i-1} denote the union of the left and lower boundaries of S and the rectangles of $\rho(\sigma)$ constructed using the first $i - 1$ entries of σ . In step i of the construction, we form a new rectangle that contains the diagonal label σ_i . We refer to this rectangle as *rectangle* σ_i . We construct rectangle σ_i as follows. If the point $u = (\sigma_i - 1, n - (\sigma_i - 1))$ is contained in T_{i-1} , then place the upper-left corner of rectangle σ_i so that it coincides with the uppermost point on the segment of T_{i-1} containing u . Otherwise, the upper-left corner of rectangle σ_i is the first point of T_{i-1} hit by the left-pointing horizontal ray with base point at u . Similarly, if the point $l = (\sigma_i, n - \sigma_i)$ is contained in T_{i-1} , then place the lower-right corner of rectangle σ_i so that it coincides with the rightmost point on the segment of T_{i-1} containing l . Otherwise, the lower-right corner of rectangle σ_i is the first point of T_{i-1} hit by the downward pointing vertical ray with base point at l . In the arguments that follow, we will use the observation that, by construction, the left side and bottom of rectangle σ_i are contained in T_{i-1} for all $i \in [n]$. The description of the construction of rectangle σ_i in $\rho(\sigma)$ can be rephrased follows: At step i , construct the largest possible rectangle such that the left side and bottom of this rectangle are contained in T_{i-1} and the rectangle contains only the diagonal label σ_i . We will also use the observation that, since the interior of each rectangle of a diagonal rectangulation D intersects the upper-left to bottom-right diagonal of S , no set of four rectangles of D share a vertex.

Theorem 2.2.1 ([14, Theorem 6.1, Corollary 8.7]). *The map ρ restricts to a bijection between twisted Baxter permutations and diagonal rectangulations. The map ρ also restricts to a bijection between Baxter permutations and diagonal rectangulations.*

Recall that we say that permutations σ and ψ are related by an adjacent cliff transposition of the pattern 2-41-3 or the pattern 3-41-2 if one of these permutations, say σ contains a subsequence $\sigma_i\sigma_j\sigma_{j+1}\sigma_k$ that is an occurrence of one of these two patterns and switching σ_j and σ_{j+1} in σ results in the permutation ψ .

Proposition 2.2.2 ([14, Proposition 6.3]). *Two permutations σ and ψ satisfy $\rho(\sigma) = \rho(\psi)$ if and only if they are related by a sequence of adjacent cliff transpositions of the patterns 2-41-3 and 3-41-2.*

Since $\sigma < \psi$ in the right weak order if and only if ψ can be obtained from σ by transposing adjacent entries σ_i and σ_{i+1} of σ which satisfy $\sigma_i < \sigma_{i+1}$ in numerical order, and the unique twisted Baxter permutation that maps to a fixed diagonal rectangulation D avoids the patterns 2-41-3 and 3-41-2, making use of Proposition 2.2.2, we see that the twisted Baxter permutation is the minimal element of the right weak order that maps to D under ρ .

Proposition 2.2.3 ([14, Proposition 4.5]). *Let D be a diagonal rectangulation and $\sigma \in S_n$ such that $\rho(\sigma) = D$. Then σ is a twisted Baxter permutation if and only if σ is the minimal element of the right weak order such that $\rho(\sigma) = D$.*

2.3 Adjacency Posets

In Section 2.1, we provided a definition of the adjacency poset of a diagonal rectangulation D . Specifically, we obtained the relations of the adjacency poset by declaring $x <_P y$ if rectangle x and rectangle y are adjacent with rectangle x left of or below rectangle y , and then taking the reflective and transitive closure of these relations. At times, we will make use of an equivalent definition of the adjacency poset.

Given a diagonal rectangulation D of size n in \mathbb{R}^2 with bottom-left corner at $(0,0)$ and top-right corner at (n,n) , define the partial order Q on $[n]$ as follows: if there exist a point p in the interior of rectangle x and a point q in the interior of rectangle y such that $q - p$ has positive coordinates declare $x \leq_Q y$, and then take the transitive closure of these relations.

Proposition 2.3.1. *Given a diagonal rectangulation D of size n , the adjacency poset P is the poset Q defined above.*

Proof. If $x \prec_P y$ then, by the definition of the adjacency poset, the interior of the bottom (or left side) of rectangle y intersects the interior of the top (or right side) of rectangle x . Thus there exist points $p \in \text{int}(\text{rectangle } x)$ and $q \in \text{int}(\text{rectangle } y)$ such that $q - p$ has positive coordinates. Therefore, by the definition of Q , we have that $x \leq_Q y$.

If $x \leq_Q y$, then there exist points $p \in \text{int}(\text{rectangle } x)$ and $q \in \text{int}(\text{rectangle } y)$ such that $q - p$ has positive coordinates. Consider the line segment joining p to q . If this segment passes through the vertex of some rectangle, since D contains only finitely many vertices, we may perturb p or q , obtaining points p' and q' , so that p' and q' are respectively in the interiors of rectangles x and y , the segment joining p' and q' contains no vertices of D , and $q' - p'$ has positive coordinates. Thus, we may assume that the segment joining p and q contains no vertices of D . The segment passes through the interiors of the sequence of rectangles $x = z_0, z_1, \dots, z_{m-1}, y = z_m$. For all $i \in [m]$, the segment exits rectangle z_{i-1} and enters rectangle z_i at a point in the interior of a side of both rectangles so $z_i \prec_P z_{i+1}$. Therefore $x \prec_P y$. \square

We note that the transitive closure in the definition of Q is required (since we have chosen to refer to each diagonal rectangulation using the representative with edges intersecting the diagonal in equally spaced points). Consider the rectangulation $\rho(312465)$ shown in Figure 2.10. Since the interior of the right side of rectangle 2 intersects the interior of the left side of rectangle 4, we have that $2 \prec_P 4$. Similarly, $4 \prec_P 6$, so by transitivity $2 \prec_P 6$. However, there do not exist $p \in \text{int}(\text{rectangle } 2)$ and $q \in \text{int}(\text{rectangle } 6)$ such that $q - p$ has positive coordinates.

We give a description of the Hasse diagram of the adjacency poset of a diagonal rectangulation by describing its cover relations.

Theorem 2.3.2. *Let D be a diagonal rectangulation and P the corresponding adjacency poset. Then $x \prec_P y$ if and only if rectangles x and y form one of the configurations shown in Figure 2.7.*

Proof. Let D be a diagonal rectangulation and P the adjacency poset of D . Assume that in D , rectangles x and y form one of the configurations shown in Figure 2.7. In each configuration, by definition, $x \prec_P y$. Assume that rectangles x and y form configuration (i)

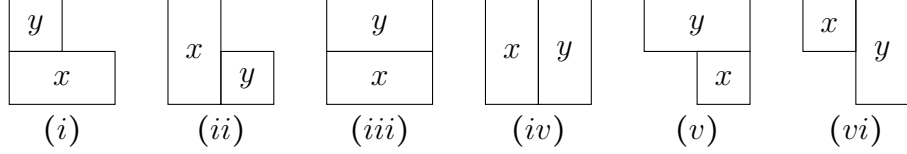


Figure 2.7: Configurations in a diagonal rectangulation that correspond to cover relations in the adjacency poset.

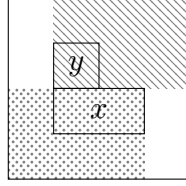


Figure 2.8: An illustration for the proof of Theorem 2.3.2.

and there exists some $z \in [n]$ such that $x <_P z <_P y$. Since $z <_P y$ and P is acyclic, $y \not<_P z$. Thus rectangle z contains no interior points in the lined region of Figure 2.8. Similarly, since $z \not<_P x$, rectangle z contains no interior points in the dotted region of Figure 2.8. Therefore, any rectangle z such that $x <_P z <_P y$ is completely contained in an unshaded region of Figure 2.8. However, by the definition of P , no label of a rectangle contained in the lower-right unshaded region of Figure 2.8 is covered by y . Similarly, in P no label of a rectangle contained in the upper-left unshaded region of Figure 2.8 covers x . Additionally, no label of a rectangle contained in the lower-right unshaded region is covered by the label of a rectangle contained in the upper-left unshaded region. Thus there exists no z such that $x <_P z <_P y$. Hence $x <_P y$. For the remaining configurations of Figure 2.7, similar considerations demonstrate that $x <_P y$.

To prove the other direction of the theorem, assume that $x <_P y$. Since the set of linear extensions of P is the fiber $\rho^{-1}(D)$ and $x <_P y$, there exists a linear extension $\sigma = \sigma_1 \cdots \sigma_n$ of P such that $x = \sigma_i$ and $y = \sigma_{i+1}$. Let T_{j-1} denote the union of the left and bottom boundaries of the square S and the partial diagonal rectangulation formed in the construction of $\rho(\sigma)$ after considering the first $j - 1$ entries of σ . Recall that the bottom and left edge of rectangle σ_j are contained in T_{j-1} for all $j \in [n]$. Using Definition 2.1.6 (the first definition of the adjacency poset), since $x <_P y$, we have that rectangles x and y are adjacent with rectangle x left of or below rectangle y . There

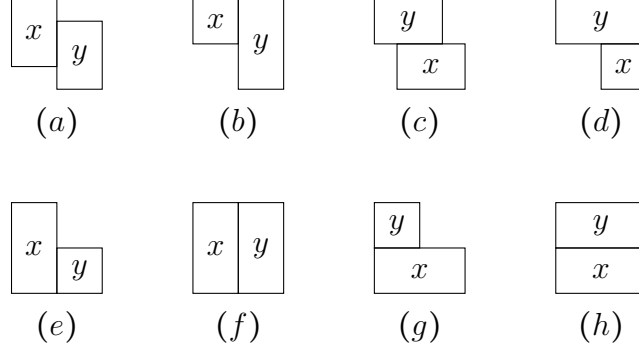


Figure 2.9: Relative locations of rectangles x and y used in the second half of the proof of Theorem 2.3.2.

are 18 possible configurations of adjacent rectangles in a rectangulation. Of these 18 configurations, only the 8 configurations shown in Figure 2.9 can possibly satisfy the condition that the bottom and left edge of rectangle y are contained in T_i . To complete the proof of the theorem, we observe that configurations (a) and (c) of Figure 2.9 cannot occur in any diagonal rectangulation. In a *diagonal* rectangulation, the upper-left to bottom-right diagonal of S passes through every rectangle of the rectangulation, but this is impossible in a rectangulation containing either of these configurations. Thus, if $x \prec_P y$, then rectangles x and y form one of the configurations shown in Figure 2.7. \square

Example 2.3.3. Figure 2.10 shows two diagonal rectangulations and their adjacency posets. The posets are constructed using the correspondence between cover relations of P and the rectangle configurations shown in Figure 2.7.

2.4 Characterization of Adjacency Posets

To prove Theorem 2.1.12, we require the following definitions and results. Recall that given a rectangulation R , a line segment that is not contained in the boundary of S and is a maximal (with respect to inclusion) union of edges of rectangles is called a wall of R . Given a planar embedding of a poset P , the embedding separates the plane into maximal connected components. Recall that we call the closure of each bounded connected component a region of the embedding.

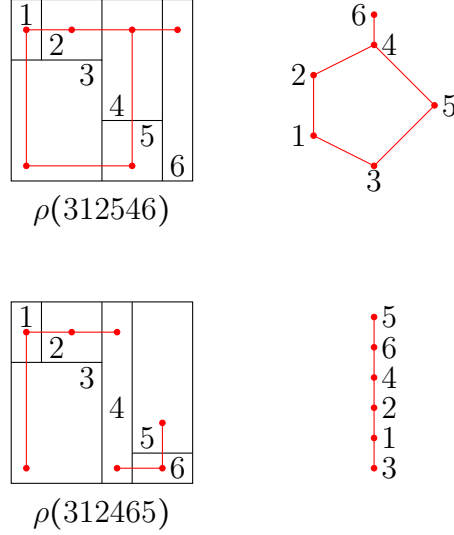


Figure 2.10: The adjacency posets corresponding to the diagonal rectangulations $\rho(312546)$ and $\rho(312465)$.

Definition 2.4.1. Given a planar embedding of a lattice P , for each $x \in P$, define $S(x)$ to be the union of the chains of P containing x and the horizontal line segments whose endpoints are contained in these chains. We say that x is *left of* y in the embedding if y is not contained in $S(x)$ and a left-pointing horizontal ray with vertex at y passes through $S(x)$. We similarly define *right of*.

In Figure 2.11, the gray region is $S(x)$. We note that since P is a lattice, x is left of y if and only if y is right of x . Furthermore, if x and y are incomparable in P , then either x is left of y or x is right of y .

Definition 2.4.2. Let $\mathcal{L} = \{L_1, \dots, L_l\}$ denote a collection of linear extensions of a poset P . We say that \mathcal{L} is a *realizer* of P if the intersection of these total orders is P . The *dimension* of a poset P is the size of the smallest realizer.

The following is a well-known result, which we will use to find a realizer of an adjacency poset. In the proposition and its proof, given $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, we declare $\sigma_i <_\sigma \sigma_j$ if and only if $i < j$. We will routinely pass between a permutation and its associated total order.

Proposition 2.4.3. *Let $[\sigma, \psi]$ be an interval in the right weak order on S_n . The elements of $[\sigma, \psi]$ are the linear extensions of the intersection of these total orders.*

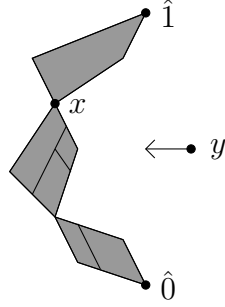


Figure 2.11: The shaded region shows $S(x)$. Since y is not contained in $S(x)$ and the left-pointing horizontal ray with base point at y intersects $S(x)$, we say that x is left of y .

Proof. Let $\sigma = \sigma_1 \cdots \sigma_n$ and $\psi = \psi_1 \cdots \psi_n$. Denote the intersection of the total orders σ and ψ by $\sigma \cap \psi$.

Let u be a linear extension of $\sigma \cap \psi$. If $(\sigma_i, \sigma_j) \in \text{inv}(\sigma)$ then, since $\sigma \leq \psi$ in the right weak order, $(\sigma_i, \sigma_j) \in \text{inv}(\psi)$. Thus $(\sigma_i, \sigma_j) \in \text{inv}(u)$. If $(u_i, u_j) \in \text{inv}(u)$, then either $u_j \not\prec_\sigma u_i$ or $u_j \not\prec_\psi u_i$. Since σ and ψ are total orders, we have that $(u_i, u_j) \in \text{inv}(\sigma)$ or $(u_i, u_j) \in \text{inv}(\psi)$. In the right weak order $\sigma \leq \psi$, so $(u_i, u_j) \in \text{inv}(\psi)$. We conclude that $u \in [\sigma, \psi]$.

Let $u = u_1 \cdots u_n \in [\sigma, \psi]$ and assume that u is not a linear extension of $\sigma \cap \psi$. Thus there exist $i, j \in [n]$ with $i < j$ such that $u_j <_\sigma u_i$ and $u_j <_\psi u_i$. If $u_j > u_i$ in numerical order, then $(u_j, u_i) \in \text{inv}(\sigma)$ and $(u_j, u_i) \notin \text{inv}(u)$, contradicting the assumption that $\sigma \leq u$ in the right weak order. If $u_j < u_i$ in numerical order, then $(u_i, u_j) \in \text{inv}(u)$ and $(u_i, u_j) \notin \text{inv}(\psi)$, contradicting the assumption that $u \leq \psi$ in the right weak order. Therefore, if $u \in [\sigma, \psi]$, then u is a linear extension of $\sigma \cap \psi$. \square

Since each congruence class of a lattice congruence on the right weak order is an interval [16, Section 2] and since each fiber of ρ is such a congruence class [14, Prop. 6.3], each fiber of ρ is an interval of the right weak order. Let D be a diagonal rectangulation and let L_1 and L_2 be respectively the minimum and maximum elements in the right weak order on S_n such that $\rho(L_1) = \rho(L_2) = D$. By Proposition 2.4.3, and since any poset is determined by its set of linear extensions, $\mathcal{L} = \{L_1, L_2\}$ is a realizer of the adjacency poset of D .

Given a linear extension $L = \sigma_1 \cdots \sigma_n$ of a poset P on $[n]$, let $\pi_L : [n] \rightarrow [n]$ be defined by $\pi_L(x) = i$ if and only if $x = \sigma_i$. The inverse of the permutation $\sigma_1 \cdots \sigma_n$ is $\pi_L(1) \cdots \pi_L(n)$.

If P has realizer $\mathcal{L} = \{L_1, L_2\}$, then the *projection of \mathcal{L}* denoted by $\pi_{\mathcal{L}}(P)$ is a map from $[n]$ to \mathbb{R}^2 given by $\pi_{\mathcal{L}}(x) = (\pi_{L_1}(x), \pi_{L_2}(x))$. This is an embedding of P into the componentwise order on \mathbb{R}^2 . To view this embedding of P as a Hasse diagram for P , we take “up” to be the direction of the vector $\langle 1, 1 \rangle$.

Theorem 2.4.4 ([20, p. 69]). *If P is a lattice with realizer $\mathcal{L} = \{L_1, L_2\}$, then the embedding of P into the componentwise order on \mathbb{R}^2 given by $\pi_{\mathcal{L}}(P)$ is a planar embedding of P .*

The following proposition is [5, p 32, Exercise 7(a)]. Since every Baxter poset is finite, bounded, and has a planar embedding, this proposition implies that every Baxter poset is a lattice.

Proposition 2.4.5. *A finite planar poset P is a lattice if and only if P is bounded.*

The following lemma is [6, Lemma 2.1]:

Lemma 2.4.6. *Let P be a bounded poset such that every chain of P is of finite length. If, for any x and y in P such that x and y both cover some element z , the join $x \vee y$ exists, then P is a lattice.*

We now have the necessary tools to prove the main result of Chapter 2, that adjacency posets and Baxter posets coincide.

(*Proof of Theorem 2.1.12*). Let D be a diagonal rectangulation of size n and P the associated adjacency poset. We first demonstrate that P satisfies the five conditions of Definition 2.1.11. The rectangle x of D whose lower-left corner coincides with the lower-left corner of S contains interior points below and left of interior points of all other rectangles of D . Thus for every $y \in [n] - \{x\}$, we have that $x <_P y$. Similarly, the label of the rectangle of D whose upper-right corner coincides with the upper-right corner of S is greater, in P , than every other element of P . Therefore, P is a bounded poset.

Observe that any rectangle x of D is the left rectangle of at most one of the configurations shown in Figure 2.7 and the bottom rectangle of at most one of the configurations shown in Figure 2.7. Thus, x is covered by at most two elements of P . Similarly, x covers at most two elements of P .

To show that P meets Condition 3 of Definition 2.1.11, for a contradiction assume that P contains a 2-14-3, a 3-14-2, a 2-41-3 or a 3-41-2 chain. This implies that some

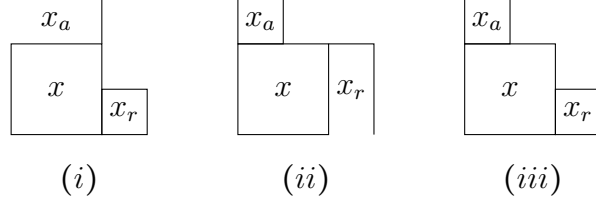


Figure 2.12: Given that $x \prec_P x_a$ and $x \prec_P x_r$ with $x_a \neq x_r$, in diagonal rectangulation D rectangles x, x_a and x_r form one of the three configurations shown.

linear extension σ of P contains this pattern with the “4” and “1” adjacent. By Proposition 2.2.2, transposing the “4” and “1” in this linear extension results in a permutation σ' such that $\rho(\sigma) = \rho(\sigma')$. Since the fiber $\rho^{-1}(D)$ is the set of linear extensions of P , the permutation σ' is also a linear extension of P . However, this contradicts the assumption that the “4” and the “1” are related in P .

Since the labeling of the rectangles of D comes from the map ρ from permutations to diagonal rectangulations, to demonstrate that P meets Condition 4 of Definition 2.1.11, we rely on observations about this map. Consider an interval $[x, y]$ of P such that (x, y) is disconnected. There exist $x_r \neq x_a$ such that $x \prec_P x_r$ and $x \prec_P x_a$. By Theorem 2.3.2, since no four rectangles of a diagonal rectangulation share a vertex, rectangles x, x_a , and x_r form one of the configurations shown in Figure 2.12. In Diagram (i), the left side of rectangle x_a is missing to indicate that the lower-left vertex of rectangle x_a coincides with or is left of the upper-left vertex of rectangle x . The bottom of rectangle x_r is missing in Diagram (ii) to similarly indicate that the lower-left vertex of rectangle x_r coincides with or is below the lower-right vertex of rectangle x .

First assume that rectangles x, x_a , and x_r are in the configuration shown in Diagram (i) of Figure 2.12 and let W be the vertical wall on the right side of rectangle x . The lower-right vertex of rectangle x and the lower-left vertex of rectangle x_r coincide, so rectangle x is the lowermost rectangle on the left side of W . By the definition of ρ , rectangle $x + 1$ is the uppermost rectangle adjacent to the right side of W and the lower-left corner of rectangle $x + 1$ is below the upper-right corner of rectangle x . Since the interiors of the right edge of rectangle x_a and the left edge of rectangle $x + 1$ intersect, we have that $x_a \prec_P x + 1$. Since the upper-right corner of rectangle $x + 1$ is strictly right of W and above rectangle x_r , we have that $x_r \prec_P x + 1$. We wish to show that $x + 1 = y$, i.e., there does not exist $z \prec_P x + 1$ such that $x_a \prec_P z$ and $x_r \prec_P z$. We will prove a stronger

statement: $x_a \vee x_r$ exists and $x_a \vee x_r = x + 1$. Since $x + 1$ is an upper bound for x_a and x_r , it suffices to demonstrate that any other upper bound z satisfies $x + 1 \leq_P z$. To obtain a contradiction, assume that $x + 1 \not\leq_P z$ for some upper bound z . We use an argument similar to the argument used in the proof of Theorem 2.3.2. Since $x <_P z$, we have that $z \not\leq_P x$. Thus rectangle z contains no interior points that are both left of the vertical line containing W and below the horizontal line containing the top of rectangle x . Since $x + 1 \not\leq_P z$, rectangle z contains no interior points that are both right of the vertical line containing W and above the horizontal line containing the bottom of rectangle $x + 1$. Thus z is contained in either the region left of the vertical line containing W and above the horizontal line containing the top of rectangle x or the region right of the vertical line containing W and below the horizontal line containing the bottom of rectangle $x + 1$. Note that these regions are disjoint, that rectangle x_a is contained in the first region, and that rectangle x_r is contained in the second region. In P , the label of a rectangle contained in the first region cannot cover the label of a rectangle contained in the second region and vice versa. Thus $x_a \not\leq_P z$ or $x_r \not\leq_P z$, a contradiction. Therefore $x_a \vee x_r = x + 1$.

When rectangles x, x_a and x_r form the configuration shown in Diagram (ii) of Figure 2.12, by considering the horizontal wall W above rectangle x and the rightmost rectangle above W , rectangle $x - 1$, we similarly show that $y = x - 1$ and that $x_a \vee x_r = x - 1$. In the case illustrated in Diagram (iii) of Figure 2.12, we first observe that since D is a diagonal rectangulation, the wall above or on the right side of rectangle x extends beyond the upper-right corner of rectangle x . In either case, using the previous arguments, we show that $y = x + 1$ or $y = x - 1$ and $y = x_a \vee x_r$.

To demonstrate that P meets Condition 5 of Definition 2.1.11, note that by Condition 1 of the definition, and since we verified that $y = x_a \vee x_r$ in each case of the proof of Condition 4, Lemma 2.4.6 implies that P is a lattice. Let L_1 and L_2 be respectively the minimum and maximum elements in the right weak order on S_n such that $\rho(L_1) = \rho(L_2) = D$. By Proposition 2.4.3, $\mathcal{L} = \{L_1, L_2\}$ is a realizer of P . By Theorem 2.4.4, the Hasse diagram obtained from $\pi_{\mathcal{L}}(P)$ is a planar embedding of P . Let $[x, y]$ be an interval of P such that (x, y) is disconnected. Let $x <_P x_l$ and $x <_P x_r$ where x_l is left of x_r in the planar Hasse diagram obtained from $\pi_{\mathcal{L}}(P)$. Let $\pi_{\mathcal{L}}(x_l) = (a, b)$ and $\pi_{\mathcal{L}}(x_r) = (c, d)$. Since x_l and x_r are incomparable with x_l left of x_r in the planar Hasse diagram, we have that $a < c$ and $b > d$ in numerical order. This implies that x_l precedes x_r in L_1 and x_l follows x_r in L_2 . Since $L_1 \leq L_2$ in the right weak order, $(x_l, x_r) \in \text{inv}(L_2)$.

Thus $x_l < x_r$ in numerical order.

Rectangles x, x_l , and x_r form one of the configurations shown in Figure 2.12 (with x_l replacing x_a). In every diagram of Figure 2.12, each rectangle x_i such that $x_l \leq_P x_i <_P y$ is contained in the region above the horizontal line containing the top of rectangle x and left of the vertical line containing the left side of rectangle y . Thus rectangle x_i intersects the diagonal of S in that region. This implies that $x_i < x$ in numerical order. Additionally, for each x_j such that $x_r \leq_P x_j <_P y$, since rectangle x_j intersects the diagonal of D in the region right of the vertical line containing the right side of rectangle x and below the horizontal line containing the bottom of rectangle y , we have that $x < x_j$ in numerical order. Thus one connected component of (x, y) contains elements numerically smaller than x and y while the other connected component contains elements numerically larger than x and y . Since $x_l < x_r$ in numerical order with x_l contained in the left component of (x, y) and x_r contained in the right component, given $w, z \in (x, y)$ such that w is left of z in this planar embedding of P , we have that $w < x < z$ in numerical order.

We have shown that the adjacency poset P satisfies each of the conditions in Definition 2.1.11, so P is a Baxter poset.

Now let P be a Baxter poset. To demonstrate that P is an adjacency poset, we first show that the set of linear extensions of P is a union of fibers of ρ . In what follows, we assume that P is embedded as described in Condition 5 of Definition 2.1.11. Let $\sigma = \sigma_1 \cdots \sigma_n$ be a linear extension of P and suppose $\psi = \sigma_1 \cdots \sigma_{j-1} \sigma_{j+1} \sigma_j \sigma_{j+2} \cdots \sigma_n$ such that $\rho(\sigma) = \rho(\psi)$. We will show that ψ is also a linear extension of P . Since $\rho(\sigma) = \rho(\psi)$ and $\sigma \lessdot \psi$ or $\psi \lessdot \sigma$ in the right weak order, by Proposition 2.2.2, the permutations σ and ψ are related by a single adjacent cliff transposition of the pattern 2-41-3 or the pattern 3-41-2. Let $a\sigma_j\sigma_{j+1}b$ be an occurrence of a 2-41-3, a 2-14-3, a 3-41-2, or a 3-14-2 pattern in σ . Since σ is a linear extension of P , the permutation ψ is also a linear extension of P if and only if σ_j and σ_{j+1} are incomparable in P . To proceed via contradiction, assume that σ_j and σ_{j+1} are comparable in P . Because σ_j precedes σ_{j+1} in σ and σ is a linear extension of P , we have that $\sigma_j <_P \sigma_{j+1}$. This implies that $\sigma_j \lessdot_P \sigma_{j+1}$ since any σ_k such that $\sigma_j <_P \sigma_k <_P \sigma_{j+1}$ would be between σ_j and σ_{j+1} in every linear extension of P (and in particular in σ). By Condition 3 of Definition 2.1.11, at least one of $\{a, b\}$ is incomparable with at least one of $\{\sigma_j, \sigma_{j+1}\}$. We assume that a is incomparable with σ_j or σ_{j+1} and note that if b is instead incomparable with σ_j or σ_{j+1} , then the argument is analogous. Since a precedes σ_j in σ , our assumption implies

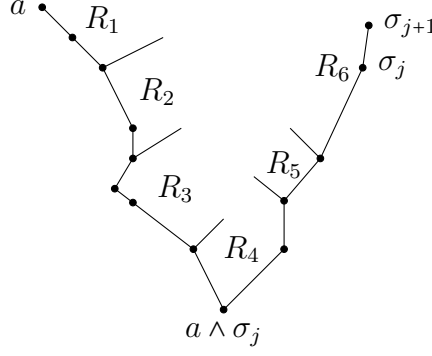


Figure 2.13: An illustration for the proof of Theorem 2.1.12.

that either $a <_P \sigma_{j+1}$ and a and σ_j are incomparable, or a is incomparable with both σ_j and σ_{j+1} . In either case, a and σ_j are incomparable.

By Proposition 2.4.5, P is a lattice so we may consider $S(a)$ and $S(\sigma_j)$. First assume that a is left of σ_j and consider the maximal chain C_1 of P from a to the minimal element of P , denoted $\hat{0}$, that follows the right boundary of $S(a)$. Let C_2 denote the maximal chain of P from σ_j to $\hat{0}$ that follows the left boundary of $S(\sigma_j)$. Note that C_1 and C_2 intersect at $a \wedge \sigma_j$ and let C'_1 and C'_2 denote the chains from a and σ_j to $a \wedge \sigma_j$ obtained by truncating C_1 and C_2 respectively. Figure 2.13 shows an example of the chains C'_1 and C'_2 . Each edge of C'_1 and C'_2 is the edge of a region of P that lies right of C'_1 and left of C'_2 . Starting at a , traveling down C'_1 to $a \wedge \sigma_j$, label the sequence of regions right of and adjacent to C'_1 with R_1, \dots, R_l . Starting at $a \wedge \sigma_j$, and traveling up C'_2 to σ_j , continue by labeling the sequence of regions left of and adjacent to C'_2 with R_l, R_{l+1}, \dots, R_m . In Figure 2.13, $l = 4$ and $m = 6$. For each $i \in [m-1]$, by Condition 2 of Definition 2.1.11, the region R_i shares an edge with the region R_{i+1} . (Otherwise C_1 is not the right boundary of $S(a)$ or C_2 is not the left boundary of $S(\sigma_j)$.) Since P is a lattice, for $i \in [m]$, each region R_i has a minimal element, denoted r_i contained in the boundary of R_i . Each r_i is a vertex of $C'_1 \cup C'_2$. (If the edge of some region is contained in $C'_1 \cup C'_2$ and that region's minimal element is not on $C'_1 \cup C'_2$, then again either C_1 is not the right boundary of $S(a)$ or C_2 is not the left boundary of $S(\sigma_j)$.) For each $i \in [l]$, the minimal element r_i is contained in the left side of the region R_{i+1} . Thus, by Condition 5 of Definition 2.1.11, we have that $a < r_1 < \dots < r_l = a \wedge \sigma_j$ in numerical order. For each $i \in \{l+1, \dots, m\}$, the minimal element r_i is contained in the right side of region R_{i-1} . Thus $a \wedge \sigma_j = r_l < r_{l+1} < \dots < r_m < \sigma_j$

in numerical order. Combining these strings of inequalities, we conclude that $a < \sigma_j$ in numerical order.

In a similar way, construct a sequence of regions S_1, \dots, S_p using the section of the right boundary of $S(a)$ from a to $a \vee \sigma_{j+1}$ and the section of the left boundary of $S(\sigma_{j+1})$ from σ_{j+1} to $a \vee \sigma_{j+1}$. If $a <_P \sigma_{j+1}$, then $a \vee \sigma_{j+1} = \sigma_{j+1}$. Whether $a <_P \sigma_{j+1}$ or a and σ_{j+1} are incomparable in P , using the sequence of maximal elements of these regions together with Condition 5 of Definition 2.1.11, we obtain a chain of inequalities and conclude that $a < \sigma_{j+1}$ in numerical order. However, combining the conclusions that $a < \sigma_j$ and $a < \sigma_{j+1}$ contradicts to the assumption that $a\sigma_j\sigma_{j+1}b$ is an occurrence of a 2-41-3, a 2-14-3, a 3-41-2 or a 3-14-2 pattern.

If σ_j is left of a in P , then to construct sequence of regions R_1, \dots, R_m , let C_1 be the right boundary of $S(\sigma_j)$ and C_2 be the left boundary of $S(a)$. To construct the sequence of regions S_1, \dots, S_p , use the right boundary of $S(\sigma_{j+1})$ and the left boundary of $S(a)$. Using these sequences and the corresponding chains of inequalities, we conclude that in numerical order $\sigma_j < a$ and $\sigma_{j+1} < a$. This conclusion again contradicts the assumption that $a\sigma_j\sigma_{j+1}b$ is an occurrence of a 2-41-3, a 2-14-3, a 3-41-2, or a 3-14-2 pattern. In both cases, we see that σ_j and σ_{j+1} are incomparable in P . Therefore the set of linear extensions of P is a union of fibers of ρ .

Any two linear extensions of a poset are related by a sequence of adjacent transpositions. Consider two linear extensions σ and ψ of P that differ by an adjacent transposition. To complete the proof that P is an adjacency poset, we will show that $\rho(\sigma) = \rho(\psi)$. Specifically, we demonstrate that σ and ψ are related by an adjacent cliff transposition of the pattern 2-41-3 or the pattern 3-41-2. Suppose that $\sigma = \sigma_1 \cdots \sigma_j \sigma_{j+1} \cdots \sigma_n$ and $\psi = \sigma_1 \cdots \sigma_{j-1} \sigma_{j+1} \sigma_j \sigma_{j+2} \cdots \sigma_n$. Since σ_j precedes σ_{j+1} in σ but σ_{j+1} precedes σ_j in ψ , we have that σ_j and σ_{j+1} are incomparable in P . This implies that $\sigma_j \wedge \sigma_{j+1} \notin \{\sigma_j, \sigma_{j+1}\}$ and $\sigma_j \vee \sigma_{j+1} \notin \{\sigma_j, \sigma_{j+1}\}$. Without loss of generality, up to swapping σ and ψ , we can assume that σ_j is left of σ_{j+1} in P . Consider sequences of regions R_1, \dots, R_m and S_1, \dots, S_p , defined as in the previous paragraph, replacing a with σ_j . Using these sequences of adjacent regions and the resulting inequalities, we obtain $\sigma_j < \sigma_j \wedge \sigma_{j+1} < \sigma_{j+1}$ and $\sigma_j < \sigma_j \vee \sigma_{j+1} < \sigma_{j+1}$ in numerical order. By definition, $\sigma_j \wedge \sigma_{j+1} <_P \sigma_j$ and $\sigma_j \wedge \sigma_{j+1} <_P \sigma_{j+1}$, so $\sigma_j \wedge \sigma_{j+1}$ precedes σ_j and σ_{j+1} in σ and ψ . Similarly, σ_j and σ_{j+1} precede $\sigma_j \vee \sigma_{j+1}$ in σ and ψ . Thus the sequence $(\sigma_j \wedge \sigma_{j+1})\sigma_j\sigma_{j+1}(\sigma_j \vee \sigma_{j+1})$ is an occurrence of a 2-41-3, a 2-14-3, a 3-41-2, or a 3-14-2 pattern in σ . \square

2.5 Twisted Baxter and Baxter Permutations from Baxter posets

Definition 2.5.1. Let P be a poset. We say that a subset I of the elements of P is an *order ideal* of P if and only if for every $a \in I$, if $b <_P a$, then $b \in I$. We say that an ordering $a_1 \cdots a_i$ of a subset of the elements of P is a *partial linear extension* of P if $\{a_1, \dots, a_j\}$ is an order ideal of P for all $j \in [i]$.

Given a poset P on $[n]$, the permutation σ is a linear extension of P if and only if σ satisfies the definition of a partial linear extension. Given a partial linear extension $\sigma_1 \cdots \sigma_{i-1}$ of P , we define $A_i \subseteq [n]$ by $u \in A_i$ if and only if $\sigma_1 \cdots \sigma_{i-1} u$ is a partial linear extension of P . We label this set A_i because it forms an antichain (a set of pairwise incomparable elements) of P .

Theorem 2.5.2. *Given a Baxter poset P , the unique twisted Baxter permutation $\sigma = \sigma_1 \cdots \sigma_n$ that is a linear extension of P is obtained by choosing $\sigma_i = \min(A_i)$ for each $i \in [n]$.*

Note that $\min(A_i)$ denotes the smallest, in numerical order, element of A_i . If a Baxter poset P is given a natural embedding, then this selection is equivalent to choosing the leftmost (in the embedding) element of A_i for each $i \in [n]$.

Proof. Let P be a Baxter poset and D the associated diagonal rectangulation. By Theorem 2.1.12, the total order σ is a linear extension of P if and only if $\rho(\sigma) = D$. Since ρ restricts to a bijection between diagonal rectangulations and twisted Baxter permutations (Theorem 2.2.1), there is a unique linear extension $\sigma = \sigma_1 \cdots \sigma_n$ of P that is a twisted-Baxter permutation. To construct σ one entry at a time, we must describe a method for choosing σ_i from A_i . By Proposition 2.2.3, the permutation σ is the minimal element of the right weak order such that $\rho(\sigma) = D$. That is, σ is the linear extension of P that contains the fewest inversions. Therefore, $\sigma_i = \min(A_i)$ for all $i \in [n]$. \square

The following results will be used in the proof of Theorem 2.1.16. The next lemma is equivalent to Corollary 4.2 in [14] which states that σ is a Baxter permutation if and only if σ^{-1} is a Baxter permutation. The description of Baxter permutations provided in this lemma is a rephrasing of Definition 1.1.2.

Lemma 2.5.3. *The permutation σ is a Baxter permutation if and only if σ contains no subsequence $\sigma_i\sigma_j\sigma_k\sigma_l$ such that $|\sigma_l - \sigma_i| = 1$ and the subsequence is an occurrence of the pattern 2-4-1-3 or the pattern 3-1-4-2.*

By Theorem 2.2.1, there exists a unique linear extension of P that is a Baxter permutation.

Lemma 2.5.4. *Let P be a Baxter poset and σ be the unique Baxter permutation that is a linear extension of P . Then σ respects the arrows of P .*

Proof. Let P be a Baxter poset with a natural embedding. Let σ denote a linear extension that does not respect the arrow of some region R of P . Let \min_R and \max_R respectively denote the minimal and maximal elements of R . By Condition 4 of Definition 2.1.11, we have that \min_R and \max_R differ in value by one. Since σ does not respect the arrow of R , there exists a subsequence $\min_R\sigma_i\sigma_j\max_R$ of σ such that σ_i and σ_j are contained in the boundary of R , one of these contained in the left component of (\min_R, \max_R) and the other contained in the right component of (\min_R, \max_R) , and this subsequence is an occurrence of a 2-4-1-3 or a 3-1-4-2 pattern. Thus, by Lemma 2.5.3, the permutation σ is not a Baxter permutation. \square

We make several useful observations about the map ρ . Justifications of some of these observations can be found in the proof of Theorem 2.1.12. Given a diagonal rectangulation D , if W is a horizontal wall of D and rectangle a is the leftmost rectangle below and adjacent to W , then rectangle $a - 1$ is the rightmost rectangle above and adjacent to W and a precedes $a - 1$ in every permutation σ such that $\rho(\sigma) = D$. Each rectangle below and adjacent to W has a label larger than a and each rectangle above and adjacent to W has a label smaller than $a - 1$. Similarly, if W is a vertical wall of D and rectangle a is the lowermost rectangle left of and adjacent to W , then rectangle $a + 1$ is the uppermost rectangle right of and adjacent to W and a precedes $a + 1$ in every permutation σ such that $\rho(\sigma) = D$. Additionally, every rectangle left of and adjacent to W has label smaller than a and every rectangle right of and adjacent to W has label larger than $a + 1$.

The lemma below follows from the definition of a Baxter permutation, the above observations, and Lemma 2.5.3.

Lemma 2.5.5. *Let D be a diagonal rectangulation and $\sigma = \sigma_1\cdots\sigma_n \in S_n$ such that $\rho(\sigma) = D$. If σ is a Baxter permutation, then σ satisfies the following properties:*

- If rectangles σ_i and σ_j are adjacent to a horizontal wall W with rectangle σ_i below W and rectangle σ_j above W , then σ_i precedes σ_j in σ and
- If rectangles σ_i and σ_j are adjacent to a vertical wall W with rectangle σ_i left of W and rectangle σ_j right of W , then σ_i precedes σ_j in σ .

To complete the proof of Theorem 2.1.16, we will refer to generic rectangulations. In this section, we need generic rectangulations exclusively to prove Lemma 2.5.9, a lemma about diagonal rectangulations, so we only provide the required background related to generic rectangulations from [17]. Additional background concerning generic rectangulations can be found in Chapter 3. Recall that a rectangulation R is a generic rectangulation if and only if there exists no set of four rectangles of R that share a vertex.

As with diagonal rectangulations, there is a map γ that takes a permutation on $[n]$ to a generic rectangulation of size n (see Section 3.3) and restricts to a bijection between a subset of S_n and generic rectangulations containing n rectangles. We will not need a complete description of γ in this section, so we instead quote the required results.

Theorem 2.5.6 ([17, Theorem 4.1]). *The map γ restricts to a bijection between generic rectangulations containing n rectangles and permutations of $[n]$ that avoid scrambles of the patterns 2-4-51-3 and 3-51-2-4.*

The next proposition relates adjacent cliff transpositions involving these patterns to the map γ and is analogous to Proposition 2.2.2.

Proposition 2.5.7 ([17, Proposition 4.3]). *Two permutations σ and ψ satisfy $\gamma(\sigma) = \gamma(\psi)$ if and only if they are related by a sequence of adjacent cliff transpositions of scrambles of the patterns 2-4-51-3 and 3-51-2-4.*

The map γ labels each rectangle of the constructed generic rectangulation with an element of $[n]$. Given a generic rectangulation R , this labeling of rectangles is unique i.e., if $x, y \in S_n$ such that $\gamma(x) = \gamma(y)$, then the labeling of the rectangles obtained from $\gamma(x)$ agrees with the labeling of the rectangles obtained from $\gamma(y)$. Thus we can refer to the rectangle of R with label i as *rectangle i* .

Given a generic rectangulation R and a wall W of R , we record the order in which the rectangles adjacent to W appear along W .

Definition 2.5.8. Let W be a horizontal wall of R . Temporarily label each vertex contained in W as follows. If the vertex is the upper-left vertex of some rectangle x , then label

the vertex with x . Otherwise, the vertex is the lower-right vertex of some rectangle y , and we label it with y . The left-to-right ordering of the vertices along W provides an ordering of these vertex labels, and we call this ordering the *wall shuffle of W* denoted σ_W . Similarly, if W' is a vertical wall of R , we temporarily label the vertices contained in W' . We label a vertex with x if it is the lower-right vertex of rectangle x . Otherwise, the vertex is the upper-left vertex of some rectangle y and we label the vertex with y . The bottom-to-top order of these labels along W gives us $\sigma_{W'}$, the wall shuffle of W' .

The map γ constructs a generic rectangulation R from a permutation in two steps. Given $\sigma \in S_n$, we first construct $\rho(\sigma)$. Then, for each wall of $\rho(\sigma)$, the vertices are labeled as described above. Finally, the vertices (and the attached edges) are reordered along each wall so that the wall shuffle of each wall is a subsequence of σ . In this section, the key point is that, to specify a generic rectangulation, it suffices to identify the associated diagonal rectangulation and an order of the vertices along each wall (i.e. a wall shuffle for each wall).

Given a Baxter permutation σ , the conditions given in Lemma 2.5.5 specify the wall shuffles of the generic rectangulation $\gamma(\sigma)$. As a result, we can make use of generic rectangulations to prove the following lemma.

Lemma 2.5.9. *Let D be a diagonal rectangulation. Then there is a unique permutation σ such that $\rho(\sigma) = D$ and such that σ satisfies the properties given in Lemma 2.5.5. This permutation σ is the Baxter permutation associated with D .*

Proof. Let D be a diagonal rectangulation and σ the unique Baxter permutation such that $\rho(\sigma) = D$. The permutation σ satisfies the properties given in Lemma 2.5.5. Assume that there exists a second permutation ψ such that $\rho(\psi) = D$ and ψ satisfies the properties given in Lemma 2.5.5. Since $\rho(\sigma) = \rho(\psi)$ and the wall shuffles of $\gamma(\sigma)$ agree with the wall shuffles of $\gamma(\psi)$, we have that $\gamma(\sigma) = \gamma(\psi)$. Thus, by Proposition 2.5.7, the permutations σ and ψ are related by a sequence of adjacent cliff transpositions of scrambles of the patterns 2-4-51-3 and 3-51-2-4. This implies that some subsequence of σ is an occurrence of a scramble of the pattern 2-4-15-3, the pattern 2-4-51-3, the pattern 3-15-2-4, or the pattern 3-51-2-4. First, assume that $\sigma_i \sigma_j \sigma_k \sigma_{k+1} \sigma_l$ is an occurrence of the pattern 2-4-15-3 in σ . This means that $\sigma_k < \sigma_i < \sigma_l < \sigma_j < \sigma_{k+1}$ in numerical order. However, this implies that the subsequence $\sigma_j \sigma_k \sigma_{k+1} \sigma_l$ is an occurrence of the pattern 3-14-2 in σ , contradicting our assumption that σ is a Baxter permutation. If σ contains

an occurrence of one of the other seven patterns, then we similarly show that σ is not a Baxter permutation. We conclude that the unique permutation mapping to D under ρ and satisfying the properties of Lemma 2.5.5 is the Baxter permutation σ . \square

Lemma 2.5.10. *Let D be a diagonal rectangulation with Baxter poset P naturally embedded in the plane. If a linear extension σ of P respects the arrows of P then σ satisfies the properties of Lemma 2.5.5.*

Proof. To show that σ satisfies the properties of Lemma 2.5.5, we will show that σ satisfies these properties for each possible configuration of rectangles adjacent to the wall.

First assume that on at least one side of the wall W there is only one adjacent rectangle. Let W be a horizontal wall with a single rectangle, rectangle r_1 , below W and sequence of rectangles r_2, \dots, r_l above W . For all $i \in \{1, \dots, l-1\}$, an interior point of rectangle i is strictly below and left of an interior point of rectangle $i+1$. Thus, by the definition of the adjacency poset and Theorem 2.1.12, we have that $r_1 <_P r_2 <_P \dots <_P r_l$. If W is horizontal with a single rectangle, rectangle r_l , above W and sequence of rectangles r_1, \dots, r_{l-1} below W , then we reach the same conclusion. In either case, in P , the labels of the rectangles adjacent to W form a chain and, in this chain, all labels of rectangles below W precede all labels of rectangles above W . When W is a vertical wall with a single rectangle either left of or right of W , the argument is the same. In these cases, we conclude that the labels of rectangles adjacent to W form a chain in P and the labels of rectangles left of W precede the labels of rectangles right of W in this chain. Thus every linear extension of P satisfies the properties of Lemma 2.5.5 for walls that are adjacent to exactly one rectangle on at least one side.

Now assume that on both sides of the wall W there are at least two adjacent rectangles. We will prove the claim that if W is a horizontal wall, then the labels of rectangles adjacent to W form a subset of the labels adjacent to some region of P . Let W be horizontal and, as illustrated in the left diagram of Figure 2.14, label from left to right the rectangles adjacent to and below W with the sequence b_1, \dots, b_i . Label the rectangles adjacent to and above W , again from left to right, a_1, \dots, a_j . Since D is diagonal and rectangles b_1 and a_1 are the leftmost rectangles adjacent to W , these rectangles form the configuration shown in Diagram (i) of Figure 2.7. Thus, by Theorem 2.3.2, we have that $b_1 <_P a_1$. If $a_1 <_P b_2$, then there exists a sequence of x_k s such that $a_1 <_P x_1 <_P \dots <_P x_l <_P b_2$. Since $b_1 <_P a_1$, and $b_2 <_P a_j$, for each $k \in [l]$ we have that $b_1 <_P x_k <_P a_j$. Thus each rectangle x_k is contained either in the region above W and left of the line containing

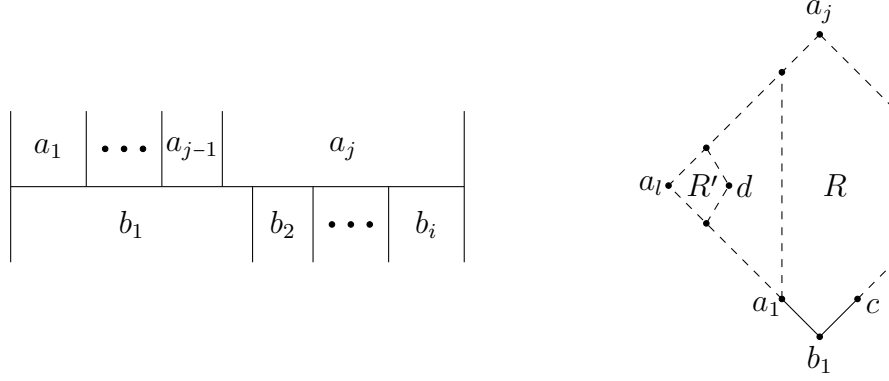


Figure 2.14: Illustrations for the proof of Lemma 2.5.10.

the left side of rectangle a_j or below W and right of the line containing the right side of rectangle b_1 . But in P , no rectangle in the first of these regions covers a rectangle in the second of these regions. We see by this contradiction that $a_1 \not\prec_P b_2$. Since $b_1 \prec_P b_2$ and $a_1 \not\prec_P b_2$, there exists some c such that $b_1 \prec_P c$ and $c \neq a_1$. By Theorem 2.3.2, rectangle c is adjacent to the right side of rectangle b_1 . Since rectangles b_1 , a_1 and c form a configuration shown in Diagram (ii) or (iii) of Figure 2.12, we have that $a_1 \vee c = a_j$ (as shown in the proof of Theorem 2.1.12). This implies that b_1 and a_j are contained in a shared region R of the embedded poset. Observe that for each $k \in [i]$, the lower-left vertex of rectangle b_k is strictly below and left of the upper-right vertex of rectangle b_i so $b_k \prec_P b_i \prec_P a_j$. Similarly, for each $l \in [j]$, we have that $a_l \prec_P a_j$.

For a contradiction, assume that there exists a label of a rectangle adjacent to W that is not contained in the boundary of R . We consider the case in which some a_l is not contained in the boundary of R , as illustrated in the right diagram of Figure 2.14. Since $a_l \prec b_1$ in numerical order, a_l is contained in the left connected component of the interval (b_1, a_j) . Since a_l is not contained in the left boundary of R , the element a_l is contained in the left boundary of some other region, R' . Let d denote an element contained in the right boundary of R' . The planarity of the embedding of P implies that d satisfies $b_1 \prec_P d$. Thus $d \not\prec_P b_1$, implying that no interior points of rectangle d are strictly left of and below the upper-right corner of rectangle b_1 . Additionally, $a_l \not\prec_P d$ so no interior points of rectangle d are strictly right of and above the lower-left corner of rectangle a_l . Since d and a_l are contained respectively in the right and left boundaries of R' , we have that $a_l \prec d$ in numerical order. This implies that rectangle d is contained in the section

of the diagonal rectangulation D below the horizontal line containing W and right of the vertical line containing the right side of rectangle b_1 . Thus $b_1 < d$ in numerical order. However, this contradicts the assumption that P is embedded naturally in the plane. We conclude that each a_l for $l \in [j]$ is contained in the left boundary of R . A similar argument demonstrates that each b_k for $k \in [i]$ is contained in the right boundary of R . Thus, the claim holds.

Since W is horizontal, $b_1 - 1 = a_j$, implying that the arrow of R points to the left. By assumption, σ respects the arrows of R so each b_k occurs before every a_l in σ , i.e. for every horizontal wall, σ satisfies the first condition of Lemma 2.5.5.

A virtually identical argument demonstrates that if W is vertical, and on both sides of W there are at least two adjacent rectangles, then σ satisfies the second condition of the lemma. \square

We can now prove Theorem 2.1.16.

Proof of Theorem 2.1.16. Let P be a Baxter poset, X be the set of linear extensions of P that respect the arrows of P and let σ be the Baxter permutation that is a linear extension of P . By Lemma 2.5.4, the Baxter permutation σ is in X . By Lemma 2.5.10, each element of X satisfies the properties given in Lemma 2.5.5. However, by Lemma 2.5.9, only one linear extension of P satisfies these properties so $X = \{\sigma\}$. \square

Chapter 3

The Hopf Algebra of Generic Rectangulations

In this chapter, we shift our focus to generic rectangulations, rectangulations in which no four rectangles share a vertex.

In [17], Reading describes a map γ from permutations of $[n]$ to generic rectangulations of size n . The fibers of γ define an \mathcal{H} -family of lattice congruences on the right weak order on S_n . This map γ restricts to a bijection between certain pattern avoiding permutations, which we call 2-clumped permutations, and generic rectangulations. Since each fiber of γ contains a unique 2-clumped permutation and this 2-clumped permutation is the minimal element (with respect to the right weak order) of the fiber, the set of all 2-clumped permutations forms a basis for a Hopf subalgebra of the Malvenuto-Reutenauer Hopf algebra of permutations. We call this Hopf sub-algebra Cl^2 and use Cl_n^2 to denote the set of 2-clumped permutations of size n . As described in Section 1.5, the product and coproduct operations in Cl^2 , which we denote respectively by \bullet_{Cl^2} and Δ_{Cl^2} , can be defined using the corresponding operations in the Malvenuto-Reutenauer Hopf algebra and then eliminating elements not in the Hopf algebra of 2-clumped permutations. In this section we provide an alternate description of these operations. Specifically, we describe the operations in the Hopf algebra of generic rectangulations, a Hopf algebra that is isomorphic to Cl^2 .

3.1 Results

Let gRec denote the Hopf algebra of generic rectangulations that is isomorphic to Cl^2 via γ and let \bullet_{gR} and Δ_{gR} respectively denote the product and coproduct operations in gRec . We denote the set of all generic rectangulations of size n by gRec_n . Given two generic rectangulations $R_1 \in \text{gRec}_p$ and $R_2 \in \text{gRec}_q$ we will describe $R_1 \bullet_{\text{gR}} R_2$ as the sum of the elements in an interval of a lattice on gRec_{p+q} . This is analogous to the description of the product in the Malvenuto-Reutenauer Hopf algebra as a sum of all elements in an interval of the right weak order on S_{p+q} . The first main result of this chapter will describe this lattice on gRec_n in terms of the combinatorics of generic rectangulations. Before providing this description, we explain the relationship between this lattice and the right weak order on S_n . The fibres of the map γ from S_n to gRec_n define a lattice congruence on the right weak order. The natural isomorphism from the quotient of the right weak order on S_n (modulo this congruence) to the set of generic rectangulations defines a lattice structure on gRec_n . Reusing notation, we also let gRec_n denote this partial order on generic rectangulations of size n . In our description of the lattice gRec_n , we use two types of local moves, called generic pivots and wall slides, illustrated by the five diagrams in Figure 3.1.

The right two diagrams of Figure 3.1 show wall slides.

Definition 3.1.1. Given a vertical wall W of R , a *vertical wall slide* switches the order of two walls incident to the interior of W . Let W_l and W_r be walls of R incident to the interior of W such that W_l extends to the left of W , wall W_r extends to the right of W and no other wall incident to W has endpoint between the endpoints of W_l and W_r . A wall slide performed on W_l and W_r switches their relative orders along W and results in a new generic rectangulation. Similarly, a *horizontal wall slide* switches the order of two walls incident to a horizontal wall W and results in a new generic rectangulation. If W_u is incident to W , extending up from W , and W_d is incident to W , extending down from W , such that no other walls incident to W have endpoints between the endpoints of W_u and W_d , then switching the order of W_u and W_d on W is a horizontal wall slide.

The precise definition of a generic pivot is more complicated than that of a wall slide.

Definition 3.1.2. We call an edge that can participate in a generic pivot a *pivotable edge*. A *generic pivot* replaces a pivotable vertical (or horizontal) edge of a generic rectangulation with a distinct horizontal (or vertical) edge resulting in a new generic rectangulation.

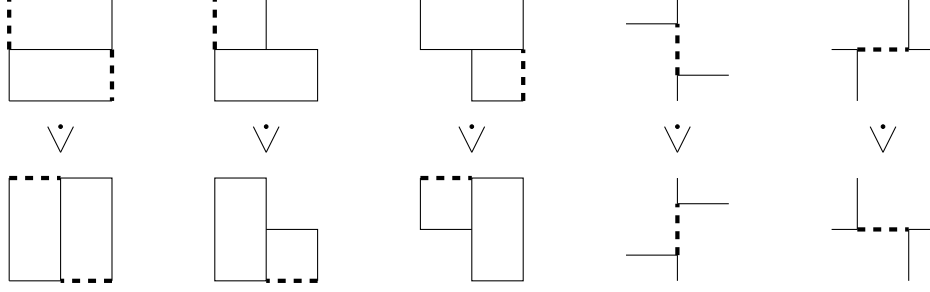


Figure 3.1: Every cover relation in gRec_n is obtained by performing one of the local changes shown in the figure on a generic rectangulation. If the configuration is to participate in the illustrated move, no edge of the generic rectangulation can have an endpoint in the interior of a dashed segment.

The left three diagrams of Figure 3.1 illustrate the three types of generic pivots. In each case, a segment separating two rectangles is removed and replaced with a segment that produces a distinct generic rectangulation. The dashed segments of each diagram indicate edges to which no additional segments of R may be incident.

If a segment of R is incident to a dashed edge, then the edge separating the two rectangles is *not* pivotable. In this case, a wall slide or sequence of wall slides must move the edge(s) incident to the dashed segments before the generic pivot can occur. When a generic pivot is performed, the new edge introduces new vertice(s) along some wall(s) of R and these vertice(s) must be placed with respect to the other vertices already on that wall so that no edges are incident to dashed segments in the new rectangulation.

We now state our first main result of this chapter.

Theorem 3.1.3. *Let R_1 and R_2 be generic rectangulations of size n . Then $R_1 < R_2$ in gRec_n if and only if :*

- R_1 and R_2 are related by a generic pivot such that the pivoted edge is vertical in R_1 , or
- R_1 and R_2 are related by a single wall slide as shown in the two rightmost diagrams of Figure 3.1.

Example 3.1.4. Figure 3.2 shows several examples of the cover relations described in Theorem 3.1.3. The map γ provides a labeling of each rectangle in these rectangulations

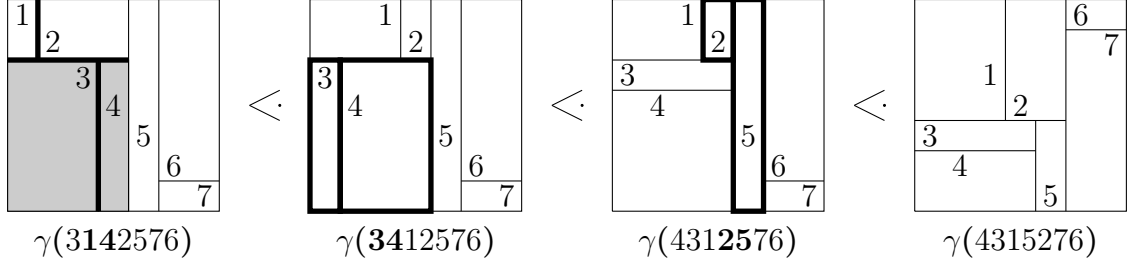


Figure 3.2: A sequence of cover relations in gRec_7 .

by an element of $[7]$. In the first rectangulation of Figure 3.2, a generic pivot cannot be performed on the edge separating the shaded rectangles since the edge separating rectangles 1 and 2 is incident to the interior of the upper segment (or top) of rectangle 3. Performing a horizontal wall slide on the bold edges of the first rectangulation of the sequence, we obtain the second rectangulation. A generic pivot can then be performed on the edge separating rectangles 3 and 4 in the second rectangulation of the sequence to obtain the third rectangulation. To obtain the fourth rectangulation of the sequence, a generic pivot is performed on the edge between rectangles 2 and 5. Performing the pivot introduces a new vertex along the wall separating rectangles 5 and 7. To avoid having an edge incident to the right side of rectangle 5 in the fourth rectangulation (as is disallowed in Figure 3.1), the left vertex of the edge separating rectangles 6 and 7 is placed above the right vertex of the edge separating rectangles 2 and 5. This is possible because, before performing the generic pivot on the edge separating rectangles 2 and 5, the edge between rectangles 6 and 7 can be moved up without changing the equivalence class of the generic rectangulation.

Having described the lattice gRec_n , we use this lattice to describe \bullet_{gR} , the product operation in the Hopf algebra gRec .

Definition 3.1.5. Given generic rectangulations R_1 and R_2 , let $R_1 R'_2$ denote the *horizontal concatenation* of R_1 and R_2 . This is a generic rectangulation obtained by first placing R_1 adjacent to R_2 so that the right side of R_1 coincides with the left side of R_2 . The resulting figure is rescaled so that the outer boundary of $R_1 \cup R_2$ is a square and wall slides are performed on the shared wall so that all edges extending left from the shared wall are below all edges extending right from the shared wall. Let $R'_2 R_1$ denote

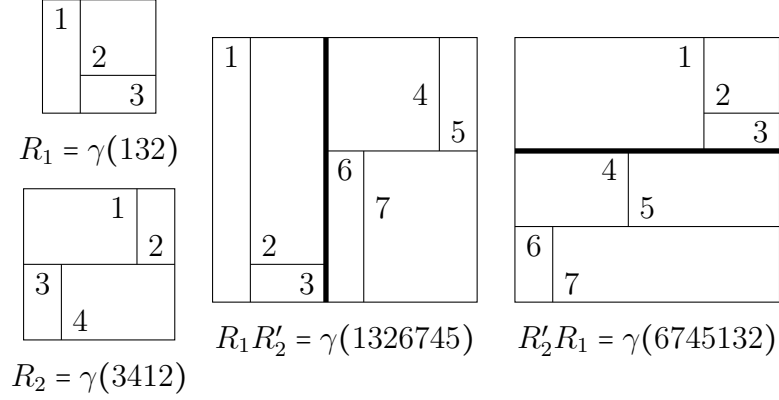


Figure 3.3: The horizontal concatenation $R_1 R'_2$ and the vertical concatenation $R'_2 R_1$ of generic rectangulations R_1 and R_2 . In each concatenation, the wall shared by R_1 and R'_2 is bolded.

the *vertical concatenation* of R_1 and R_2 which is obtained by placing R_1 adjacent to R_2 so that the top of R_2 coincides with the bottom of R_1 , rescaling, and then performing wall slides along the shared wall so that all edges extending down from the wall are left of all edges extending up from the wall.

Examples of a horizontal and a vertical concatenation are shown in Figure 3.3. The numbering of the rectangles in the figure again comes from the map γ , defined in Section 3.3. We denote the horizontal and vertical concatenations by $R_1 R'_2$ and $R'_2 R_1$ respectively because this notation mimics the notation used for related permutations. Specifically, in Section 1.3, the product of two permutations $\sigma \in S_p$ and $\psi \in S_q$ in the Malvenuto-Reutenauer Hopf algebra is described as the sum of the elements of the interval $[\sigma\psi'_{[p]}, \psi'_{[p]}\sigma]$ in the right weak order. In Section 1.5, the product of $\sigma \in Av_p^\Theta$ and $\psi \in Av_q^\Theta$ in the Hopf algebra Av^Θ is given by the sum of the elements of the interval $[\sigma\psi'_{[p]}, \pi_\downarrow(\psi'_{[p]}\sigma)]$ in the lattice Z_{p+q}^Θ .

Our next main result is the following theorem.

Theorem 3.1.6. *Let R_1 and R_2 be generic rectangulations of size p and q respectively such that $p + q = n$. Then*

$$R_1 \bullet_{gR} R_2 = \Sigma[R_1 R'_2, R'_2 R_1]$$

where the summation denotes the sum of all elements of gRec_n in the interval $[R_1 R'_2, R'_2 R_1]$.



Figure 3.4: Configurations that good paths avoid.

To describe Δ_{gR} , the coproduct in gRec , we require several additional definitions.

Definition 3.1.7. Let R be a generic rectangulation and \mathcal{P} be a path from the top-left corner to the bottom-right corner of R , consisting of down and right steps which are edges of R . We say that \mathcal{P} is a *good path* if it meets the following two conditions:

- The interior of no vertical segment of \mathcal{P} contains vertices v and v' of R such that vertex v is the upper-left vertex of a rectangle of R , vertex v' is the lower-right vertex of a rectangle of R and v is below v' .
- The interior of no horizontal segment of \mathcal{P} contains vertices h and h' of R such that vertex h is the lower-right vertex of a rectangle of R , vertex h' is the upper-left vertex of a rectangle of R and h is left of h' .

The left diagram of Figure 3.4 illustrates the configuration described in the first condition of the definition and the right diagram of the figure illustrates the configuration described in the second condition.

Example 3.1.8. A good path in a generic rectangulation is shown as the darkened path in the upper-left diagram of Figure 3.5. In this rectangulation, the path traveling from the upper-left corner of S to the lower-right corner of S , passing above rectangles 1, 3, 4, 5, and 8, and below the remaining rectangles is *not* a good path. The lower-right vertex of rectangle 4 and the upper-left vertex of rectangle 7, both lying on the interior of a single vertical segment of the path, violate the second condition in the definition of a good path.

Let p denote the number of rectangles below a good path \mathcal{P} and q the number of rectangles above \mathcal{P} . As in Section 1.4, let $R_l(\mathcal{P})$ consist of the edges of S together with the edges of R strictly below \mathcal{P} and $R_u(\mathcal{P})$ consist of the edges of S together with the

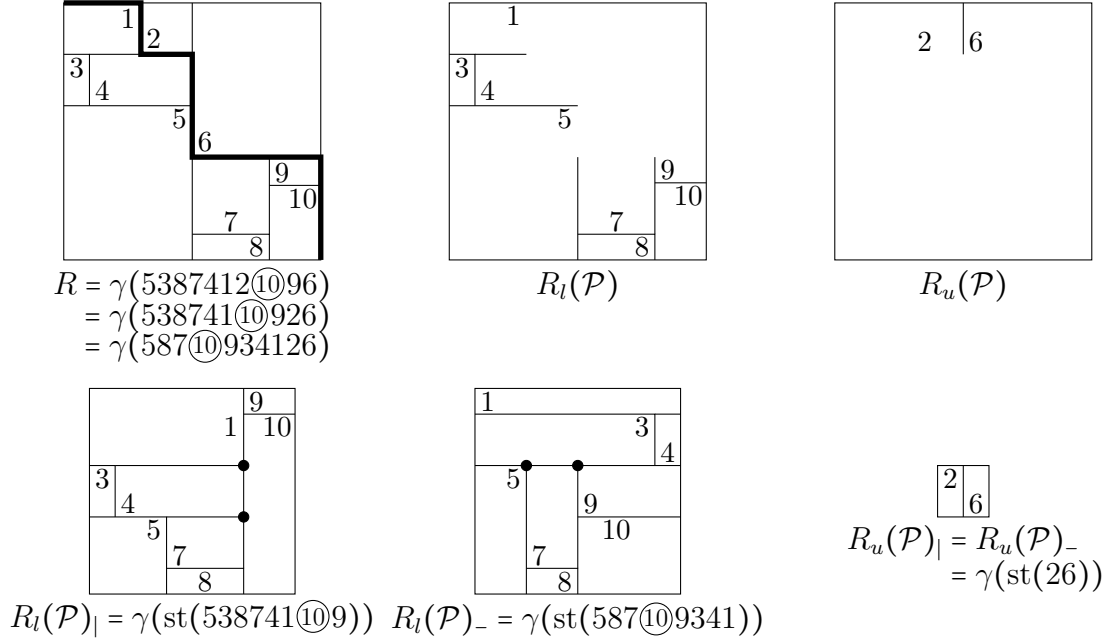


Figure 3.5: Given a good path \mathcal{P} in a generic rectangulation, we construct the vertical and horizontal completions of $R_l(\mathcal{P})$ and $R_u(\mathcal{P})$.

edges of R strictly above \mathcal{P} , as shown in the example in Figure 3.5. We will construct, from $R_l(\mathcal{P})$, two generic rectangulations, $R_l(\mathcal{P})_|$ and $R_l(\mathcal{P})_-$, elements of gRec_p , respectively called the *vertical* and *horizontal completions* of $R_l(\mathcal{P})$. Similarly, from $R_u(\mathcal{P})$, we will construct the vertical completion $R_u(\mathcal{P})_|$ and horizontal completion $R_u(\mathcal{P})_-$, both elements of gRec_q .

The vertical completion $R_l(\mathcal{P})_|$ is constructed using the following four steps:

- Each open horizontal edge of $R_l(\mathcal{P})$ (i.e. each horizontal edge of $R_l(\mathcal{P})$ whose right endpoint lies on \mathcal{P} in R) is extended to the right by ϵ .
- Each open vertical segment of $R_l(\mathcal{P})$ is extended upwards until it meets one of the horizontal edges extended in the previous step or the upper edge of S .
- Every horizontal edge extended in the first step is further extended to the right until the extension meets the interior of some vertical edge or the right side of S . Call each new vertex constructed in this step a *constructed vertex*.

- Along each vertical wall W , wall slides changing the order of a vertex of $R_l(\mathcal{P})$ and a constructed vertex are performed until the resulting order meets one of the following conditions: the set of constructed vertices is immediately above the uppermost vertex that is the right endpoint of an edge in $R_l(\mathcal{P})$, or if no vertex of $R_l(\mathcal{P})$ meets this condition, then wall slides are performed until the constructed vertices are below all other vertices on W .

Example 3.1.9. In the example shown in the lower left diagram of Figure 3.5, the extension of all open horizontal edges of $R_l(\mathcal{P})$ in the first step of the construction of $R_l(\mathcal{P})_l$ prevents the extension of the left edge of rectangle 7 above the bottom edge of rectangle 4. In the final step of the construction, wall slides are performed to place the edge separating rectangles 9 and 10 above the constructed vertices. After these wall slides, the constructed vertices (which are enlarged for emphasis) are immediately above the right endpoint of the edge between rectangles 7 and 8.

The vertical completion $R_u(\mathcal{P})_l$ is similarly constructed, extending horizontal edges to the left rather than to the right, vertical edges down rather than up, and performing slides along each vertical wall W containing constructed vertices so that constructed vertices are immediately below the lowermost vertex that is the left endpoint of an edge in $R_u(\mathcal{P})$ or, if no such vertex exists, so that the constructed vertices are above all other vertices on W .

The constructions of the horizontal completions are similar. To construct $R_l(\mathcal{P})_-$:

- Extend upwards by ϵ every open vertical edge of $R_l(\mathcal{P})$.
- Extend to the right each open horizontal edge of $R_l(\mathcal{P})$ until the edge meets a vertical edge.
- Further extend each vertical edge extended in the first step until the extension meets the interior of some horizontal edge or the top of S . Call the new vertices constructed in this step constructed vertices.
- Perform wall slides along each horizontal wall W containing the constructed vertices, changing the order of a constructed vertex and a vertex of $R_l(\mathcal{P})$ in each wall slide, until all constructed vertices are immediately to the right of the rightmost vertex that is the upper endpoint of an edge in $R_l(\mathcal{P})$, or if no vertex of $R_l(\mathcal{P})$ meets

this condition, until the constructed vertices are to the left of all other vertices on W .

Example 3.1.10. An example of $R_l(\mathcal{P})_-$ is shown in the middle diagram of the lower row of Figure 3.5. Notice that in this diagram, unlike in $R_l(\mathcal{P})_-$, the horizontal wall between rectangles 3 and 5 and the horizontal wall between rectangles 3 and 1 is extended until it reaches the right side of S . Since no edges of $R_l(\mathcal{P})$ extend down from the horizontal wall W between rectangles 3 and 5, in the final step of the construction, wall slides are performed until the constructed vertices (again enlarged for emphasis) are to the left of the other vertex on W .

We construct $R_u(\mathcal{P})_-$ by extending vertical segments downward, horizontal edges to the left, and performing wall slides along horizontal walls containing constructed vertices so that all constructed vertices are immediately to the left of the leftmost vertex that is the lower endpoint of an edge in $R_u(\mathcal{P})$ or, if no such vertex exists, so that the constructed vertices are right of all other vertices on W .

Theorem 3.1.11. *Let $R \in \text{gRec}_n$,*

$$I_{\mathcal{P}} = \sum [R_l(\mathcal{P})_+, R_l(\mathcal{P})_-] \text{ and } J_{\mathcal{P}} = \sum [R_u(\mathcal{P})_+, R_u(\mathcal{P})_-].$$

where the summations respectively denote the sum of all elements of gRec_p in the interval $[R_l(\mathcal{P})_+, R_l(\mathcal{P})_-]$ and the sum of all elements of gRec_q in the interval $[R_u(\mathcal{P})_+, R_u(\mathcal{P})_-]$. Then

$$\Delta_{gR}(R) = \sum_{\mathcal{P} \text{ is good}} I_{\mathcal{P}} \otimes J_{\mathcal{P}}.$$

3.2 The Hopf Algebra of 2-Clumped Permutations

In [17], Reading proves that generic rectangulations are in bijection with 2-clumped permutations. To define k -clumped permutations, and in particular the 2-clumped permutations needed in this section, we first define a descent.

Definition 3.2.1. A pair σ_i, σ_{i+1} of some $\sigma \in S_n$ is a *descent* of σ if $\sigma_i > \sigma_{i+1}$. For every descent of σ , we define a *clump* to be a maximal set of consecutive values $a, a+1, \dots, b$ with $\sigma_{i+1} < a < b < \sigma_i$ such that in σ either all elements of $\{a, a+1, \dots, b\}$ occur to the left of the descent or all elements of $\{a, a+1, \dots, b\}$ occur to the right of the descent. A permutation σ is a *k -clumped* permutation if every descent of σ has at most k associated clumps.

Example 3.2.2. The pair 92 is a descent of the permutation 167439285. Four clumps are associated with this descent, $\{3, 4\}$, $\{5\}$, $\{6, 7\}$, and $\{8\}$. The permutation 167439285 is k -clumped for any $k \geq 4$ because four clumps are associated with the descent 92 and fewer clumps are associated with any other descent of the permutation.

Permutations that avoid the patterns $\{2-31, 31-2\}$ are 0-clumped permutations. Every descent $\sigma_i \sigma_{i+1}$ in a 0-clumped permutation satisfies $\sigma_i - \sigma_{i+1} = 1$. There is a bijection between 0-clumped permutations in S_n and compositions of n . To find the composition of n that corresponds to the 0-clumped permutation $\sigma = \sigma_1 \cdots \sigma_n$, use σ to record a sequence of pluses and commas. Specifically, if $\sigma_i > \sigma_{i+1}$, then the i^{th} entry of the sequence is a plus. Otherwise, the i^{th} entry of the sequence is a comma. For example, the permutation 217654398 corresponds to the sequence $+, +, +, +, +$. Inserting a 1 between each pair of consecutive entries of this sequence, we obtain $1+1, 1+1+1+1+1, 1+1$ or the composition $2, 5, 2$. In [14], twisted Baxter permutations, permutations that avoid the patterns 2-41-3 and 3-41-2, are shown to be in bijection with diagonal rectangulations. The twisted Baxter permutations are exactly the 1-clumped permutations. The permutations considered in this chapter avoid scrambles of the patterns 2-4-51-3 and 3-51-2-4 and are called 2-clumped permutations. For $m, n \in \mathbb{Z}_{\geq 0}$, let Cl_n^m denote the subset of S_n containing all m -clumped permutations. Define V to be the dashed sequence of all even natural numbers strictly between 1 and $m+3$ listed in numerical order such that all adjacent entries are separated by a dash. Define V^C to be the analogous dashed sequence of all odd natural numbers strictly between 1 and $m+3$. Then $\sigma \in Cl_n^m$ if and only if $\sigma \in S_n$ that avoids all scrambles of the pattern $V-(m+3)1-V^C$ and the pattern $V^C-(m+3)1-V$. The union of the elements of Cl_n^m for all $n \geq 0$ forms a basis for a Hopf algebra that we call the Hopf algebra of m -clumped permutations [16, Corollary 1.4, Theorem 9.4].

Define $\pi_{\downarrow}^m : S_n \rightarrow Cl_n^m$ by $\pi_{\downarrow}^m(\sigma) = \psi$ if and only if ψ is the minimal element with respect to the right weak order on S_n that can be obtained from σ using a sequence of adjacent cliff transpositions of scrambles of the patterns $V-(m+3)1-V^C$ and $V^C-(m+3)1-V$. Such a unique minimal element exists because the map π_{\downarrow}^m defines a lattice congruence on the right weak order in which $\pi_{\downarrow}^m(\psi) \neq \psi$ if and only if ψ contains an occurrence of a scramble of $V-(m+3)1-V^C$ or a scramble of $V^C-(m+3)1-V$ [16, Theorem 9.3]. Every congruence class of a lattice congruence on the right weak order is an interval.

Having described the basis elements in the Hopf algebra of m -clumped permutations, we now focus on the Hopf algebra of 2-clumped permutations and describe the opera-

tions \bullet_{Cl^2} and Δ_{Cl^2} . Let $x \in \text{Cl}_p^2$ and $y \in \text{Cl}_q^2$. Specializing the equation for the product given in Section 1.5 to the Hopf algebra of 2-clumped permutations, we obtain:

$$x \bullet_{\text{Cl}^2} y = \sum [xy'_{[p]}, \pi_{\downarrow}^2(y'_{[p]}x)]$$

where the summation denotes the sum of all elements of the right weak order restricted to Cl_{p+q}^2 . We observe that $y'_{[p]}x \in \text{Cl}_{p+q}^2$ so $\pi_{\downarrow}^2(y'_{[p]}x) = y'_{[p]}x$. We will use the following corollary to prove Theorem 3.1.6 in Section 3.5.

Corollary 3.2.3. *Let $x \in \text{Cl}_p^2$ and $y \in \text{Cl}_q^2$. Then*

$$x \bullet_{\text{Cl}^2} y = \sum [xy'_{[p]}, y'_{[p]}x].$$

We now define terms necessary to describe Δ_{Cl^2} . Given a sequence $a = a_1 \cdots a_n$, of distinct natural numbers, recall that we define the standardization of a , denoted by $\text{st}(a)$, to be the unique permutation $x = x_1 \cdots x_n \in S_n$ that respects the ordering of the entries of a . That is, $x_i < x_j$ if and only if $a_i < a_j$.

Definition 3.2.4. Let $x \in \text{Cl}_n^2$. We say that a subset $T \subseteq [n]$ is *good with respect to x* if there exists some permutation $x' = x'_1 \cdots x'_n \in S_n$ such that $\pi_{\downarrow}^2(x') = x$ and $T = \{x'_1, \dots, x'_{|T|}\}$. Given a good set T such that $|T| = p$ and $q = n - p$, let x_{\min} be the minimal element of the right weak order on S_n such that $\pi_{\downarrow}^2(x_{\min}) = x$ and the first p entries of x_{\min} are the elements of T . Let x_{\max} be the maximal element of the right weak order on S_n such that $\pi_{\downarrow}^2(x_{\max}) = x$ and the first p entries of x_{\max} are the elements of T .

Notice that x_{\min} depends on both x and the selected set T which is good with respect to x . Define $x_{\min}|_T$ to be the ordering of the elements of T as they appear in x_{\min} . The ordering of the elements of T as they appear in x_{\max} is denoted by $x_{\max}|_T$. Letting $T^C = [n] - T$, we similarly define $x_{\min}|_{T^C}$ and $x_{\max}|_{T^C}$. The following theorem, which will be used to prove Theorem 3.1.11 in Section 3.5, is a specialization of [13, Theorem 1.3].

Theorem 3.2.5. *Given $x \in \text{Cl}_n^2$,*

$$\Delta_{\text{Cl}^2}(x) = \sum_{T \text{ is good}} I_T \otimes J_T$$

where I_T is the sum of the elements in the interval $[\text{st}(x_{\min}|_T), \pi_{\downarrow}^2(\text{st}(x_{\max}|_T))]$ of the

right weak order on S_p restricted to Cl_p^2 and J_T is the sum of elements in the interval $[\text{st}(x_{\min}|_{T^C}), \pi_{\downarrow}^2(\text{st}(x_{\max}|_{T^C}))]$ of the right weak order on S_q restricted to Cl_q^2 .

3.3 The Map from Permutations to Generic Rectangulations

Having defined 2-clumped permutations, we now describe the map γ from permutations to generic rectangulations which restricts to a bijection between 2-clumped permutations and generic rectangulations. The map $\gamma : S_n \rightarrow \text{gRec}_n$ is described in [17, Section 3] in two parts: we first make use of the map ρ from S_n to the set dRec_n of diagonal rectangulations of size n described in Section 2.2, and then we perform wall slides to obtain an element of gRec_n .

Let $x = x_1 \cdots x_n \in S_n$. To find $\gamma(x)$, first construct $\rho(x)$. Then, for each interior wall W of $\rho(x)$, record a subsequence σ_W of x consisting of the labels of rectangles adjacent to W . For each wall W of $\rho(x)$, we temporarily label the vertices on W using the rectangles adjacent to W (as described in Section 2.5 and below), and then use σ_W and the labeling to determine which wall slides should be performed to obtain $\gamma(x)$. Every vertex on an interior wall W is either the lower-right vertex or the upper-left vertex of some rectangle. Note that no vertex of a diagonal rectangulation is both the lower-right vertex of a rectangle and the upper-left vertex of a rectangle. Thus the labeling described below will result in a total ordering of the entries of σ_W . If the vertex is the lower-right vertex of some rectangle x_i , then label the vertex with x_i . Otherwise, the vertex is the upper-left vertex of some rectangle x_j and we label the vertex with x_j . If W is a vertical wall, we perform wall slides so that the bottom to top order of the labeled vertices on W coincides with σ_W . Since each vertical wall slide switches the order of a wall that extends to the *left* of W and a wall that extends to the *right* of W , we explain why it is always possible to perform a sequence of wall slides so that the bottom to top order of the vertices agrees with σ_W . Each vertex on W that is the lower-right vertex of some rectangle is the endpoint of an edge extending to the left of W and each vertex of W that is the upper-left vertex of some rectangle is the endpoint of an edge extending right of W . By the construction of $\rho(x)$, the subsequence of σ_W consisting of the lower-right corner vertices on W and the subsequence of σ_W consisting of upper-left corner vertices on W both agree with the bottom to top ordering of these vertices along W , so it is possible to perform a

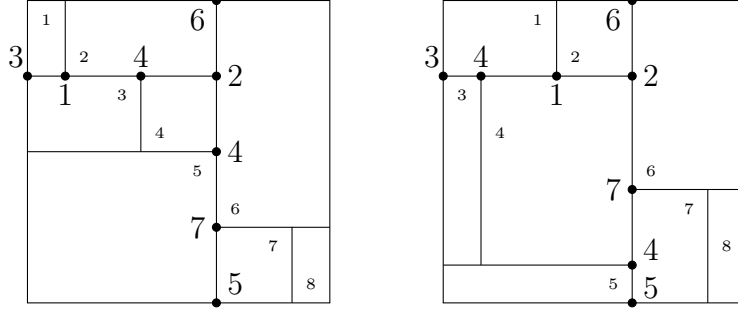


Figure 3.6: The left illustration shows $\rho(53417286)$. The right illustration shows $\gamma(53417286)$.

sequence of wall slides to obtain the desired vertex order. If W is a horizontal wall, we perform wall slides so that the left to right order of the labeled vertices on W coincides with σ_W . A similar argument shows that the desired vertex order can be obtained by some sequence of horizontal wall slides. In either case, because σ_W records the ordering of the walls along W in $\gamma(x)$, we call σ_W the wall shuffle of W .

Example 3.3.1. The diagonal rectangulation that results from applying ρ to the permutation 53417286 is illustrated in the left diagram of Figure 3.6. Note that the small labels along the diagonal of the square are used in the construction of $\rho(x)$. The larger labels (labeling the enlarged vertices of the rectangulation) are used to obtain $\gamma(x)$ from $\rho(x)$. To find $\gamma(x)$ from $\rho(x)$, we consider the wall shuffle corresponding to every interior wall of $\rho(x)$. Since a wall slide cannot be performed along any wall with fewer than two rectangles adjacent to each side, we only need to examine the walls with at least two rectangles adjacent to each side. There are two such walls in $\rho(x)$. First consider the vertical wall W between rectangle 5 and rectangle 7. For this wall, $\sigma_W = 54726$. We label the vertices along W as illustrated in the left diagram of Figure 3.6. To make the ordering of the labels along W in $\gamma(x)$ agree with σ_W , the wall slide switching the order of vertices labeled 7 and 4 is performed. Next consider the horizontal wall W' of $\rho(x)$ between rectangle 1 and rectangle 3. For this wall, $\sigma_{W'} = 3412$. We label the vertices along W' as illustrated in the left diagram of Figure 3.6. Since the left to right order of these vertices in $\rho(x)$ is 3142, we perform a wall slide switching the order of the vertices labeled 1 and 4 along W' to obtain $\gamma(x)$, shown in the right diagram of the figure.

Additional examples of the map γ are shown in Figures 3.2 and 3.3. The generic

rectangulations in both figures are labeled with $\gamma(x)$ where x is the unique 2-clumped permutation such that $\gamma(x)$ is the desired rectangulation. In Figure 3.2, the bold entries in each permutation are transposed to find the next permutation in the sequence. Examining the permutation 3142576 associated with the leftmost generic rectangulation, we see that a generic pivot cannot be performed on the edge separating the shaded rectangles because the 3 and 4 are non-adjacent and there exists no permutation $x \in S_7$ such that the 3 and 4 are adjacent in x and $\gamma(x) = \gamma(3142576)$.

The theorem below is a rephrasing of a more general result from [16, Section 2].

Theorem 3.3.2. *Given a generic rectangulation R , the fiber $\gamma^{-1}(R)$ forms an interval in the right weak order.*

Using the construction in the proof of [17, Proposition 4.2], we define ψ , the inverse of the restriction of γ to the set of 2-clumped permutations. To demonstrate that γ is a surjective map, that proof begins with an arbitrary generic rectangulation R and the associated diagonal rectangulation D . A permutation x is constructed, entry by entry, so that $\rho(x) = D$ and each wall shuffle of R is a subsequence of x . Let T_{i-1} be the partial diagonal rectangulation obtained after completing the first $i-1$ steps in the construction of $\rho(x)$. In the proof of [17, Proposition 4.2], the requirement that $\rho(x) = D$ is translated into the requirement that (in D) the left side and bottom of rectangle x_i are contained in T_{i-1} for all $i \in [n]$. We say that $x_1 \cdots x_i$ *respects the wall shuffles of R* if there exists no $x_j \in [n] - \{x_1, \dots, x_i\}$ such that x_j precedes some element of $\{x_1, \dots, x_i\}$ in a wall shuffle of R . The requirement that each wall shuffle of R is a subsequence of x is equivalent to the requirement that $x_1 \cdots x_i$ respects the wall shuffles of R for all $i \in [n]$. Using these equivalences, to show that γ is surjective, the proof of [17, Proposition 4.2] demonstrates that for all $i \in [n]$ there exists some $x_i \notin \{x_1, \dots, x_{i-1}\}$ such that the left side and bottom of rectangle x_i are contained in T_{i-1} and $x_1 \cdots x_i$ respects the wall shuffles of R . In this construction, each time an entry of x is selected, there may be a choice. We define $\psi(R)$ be the permutation obtained by choosing the minimum possible entry at each step. We will prove the following proposition.

Proposition 3.3.3. *The map $\psi : \text{gRec}_n \rightarrow Cl_n^2$ is the inverse of the restriction of γ to 2-clumped permutations.*

To prove Proposition 3.3.3, we will use the following proposition, which appears as part of [17, Proposition 2.2].

Proposition 3.3.4. *A permutation y is the minimal element of the right weak order such that $\gamma(y) = R$ if and only if y is a 2-clumped permutation.*

Proof of Proposition 3.3.3. Let $R \in \text{gRec}_n$. To prove the proposition, it suffices to demonstrate that $\psi(R) \in Cl_n^2$, or equivalently, by Proposition 3.3.4, that $\psi(R)$ is the minimal element of the right weak order mapping to R under γ . Let $\psi(R) = p = p_1 \cdots p_n$ and $x \in S_n$ such that $x \leq p$ in the right weak order. Then there exists some $i \in [n-1]$ such that $x = p_1 \cdots p_{i-1} p_{i+1} p_i p_{i+2} \cdots p_n$ and $p_{i+1} < p_i$. Since $x_j = p_j$ for all $j \in [i-1]$, and p_i is the smallest entry of any permutation starting with $p_1 \cdots p_{i-1}$ and mapping to R under γ , we have that $\gamma(x) \neq R$. By Theorem 3.3.2, the permutation p is the minimal element of the right weak order such that $\gamma(p) = R$. \square

3.4 The Lattice of Generic Rectangulations

In this section, we prove Theorem 3.1.3. To do so, we rely on results about diagonal rectangulations from [14] and results about generic rectangulations from [17]. Recall that we call each element of Cl_n^1 a twisted Baxter permutation and that the map $\rho : S_n \rightarrow \text{dRec}_n$ restricts to a bijection between Cl_n^1 and dRec_n [14, Theorem 6.1]. The right weak order on S_n modulo the fibers of ρ is a lattice on the set of twisted Baxter permutations. Applying ρ to the elements of this lattice results in a lattice of diagonal rectangulations of size n which, reusing notation, we call dRec_n .

To describe the cover relations of dRec_n , we define diagonal pivots. Diagonal pivots and generic pivots are closely related.

Definition 3.4.1. Diagonal rectangulations D and D' are related by a *diagonal pivot* if and only if they are related by a local change shown in one of the three leftmost diagrams of Figure 3.1, where the dotted segment of each diagram is ignored.

In this thesis, we call each of these local moves a diagonal pivot to emphasize that they are performed on diagonal rectangulations rather than generic rectangulations. The reader should note that this differs from the definition of a diagonal pivot given in [14], where the move illustrated in the leftmost diagram of Figure 3.1 is called a diagonal pivot and the other two local moves which we also call diagonal pivots are instead called vertex pivots. The cover relations in dRec_n are described in [14, Theorem 7.1]:

Theorem 3.4.2. *Two diagonal rectangulations D and D' of size n have $D < D'$ in dRec_n if and only if they are related by a diagonal pivot such that the pivoted edge is vertical in D .*

The following is a restatement of [17, Theorem 4.5, part (3)] and, using the definition of the right weak order, is a corollary of Proposition 2.2.2.

Theorem 3.4.3. *Assume that $x < y$ in the right weak order. Then $\rho(x) = \rho(y)$ if and only if $y = y_1 \cdots y_n$ and $x = y_1 \cdots y_{i-1} y_{i+1} y_i y_{i+2} \cdots y_n$ are related by an adjacent cliff transposition of the pattern 2-41-3 or the pattern 3-41-2 in which some subsequence $y_j y_i y_{i+1} y_k$ of y is an occurrence of the pattern 3-41-2 or the pattern 2-41-3.*

The analogous result, which is a corollary of Proposition 2.5.7 also holds for generic rectangulations [17, Proposition 4.3].

Theorem 3.4.4. *Assume that $x < y$ in the right weak order. Then $\gamma(x) = \gamma(y)$ if and only if x and y are related by an adjacent cliff transposition of a scramble of the pattern 2-4-51-3 or the pattern 3-51-2-4 in which y contains a scramble of the pattern 2-4-51-3 or a scramble of the pattern 3-51-2-4.*

The following is a specialization of a more general result [16, Proposition 2.2] to the case of 2-clumped permutations and generic rectangulations.

Proposition 3.4.5. *Let $y \in Cl_n^2$. Then $R \in \text{gRec}_n$ is covered by $\gamma(y)$ in the lattice of generic rectangulations of size n if and only if there exists some permutation $x \in S_n$ with $\gamma(x) = R$ such that $x < y$ in the right weak order on S_n .*

The following proposition is a specialization of [19, Prop 9-5.4]:

Proposition 3.4.6. *Given distinct $R_1, R_2 \in \text{gRec}_n$, we have that $R_1 < R_2$ in gRec_n if and only if there exist $x_1, x_2 \in S_n$ such that $\gamma(x_1) = R_1$ and $\gamma(x_2) = R_2$ with $x_1 < x_2$ in the right weak order on S_n .*

In light of Proposition 3.4.5, the next proposition is one direction of Theorem 3.1.3.

Proposition 3.4.7. *Let $x \in S_n$ and $y \in Cl_n^2$ such that $x < y$ in the right weak order. Then $\gamma(x) = R_1$ and $\gamma(y) = R_2$ are related by a generic pivot or wall slide shown in Figure 3.1 with the bottom diagram corresponding to R_1 and the top diagram corresponding to R_2 .*

Proof. Let $y = y_1 \cdots y_n$. Since $x \prec y$ in the right weak order on S_n , we have that $x = y_1 \cdots y_{i-1} y_{i+1} y_i y_{i+2} \cdots y_n$ with $y_{i+1} < y_i$. By Proposition 3.3.4, $\gamma(x) \neq \gamma(y)$. We consider two cases: $\rho(x) = \rho(y)$ and $\rho(x) \neq \rho(y)$.

First assume that $\rho(x) = \rho(y)$. Since $\gamma(x) \neq \gamma(y)$, rectangulations R_1 and R_2 differ by wall slides. Every wall shuffle of R_1 is a subsequence of x , so interchanging two elements of x to obtain y changes the order of at most two elements of any wall shuffle. Suppose first that more than one wall shuffle of R_1 differs from the corresponding wall shuffle of R_2 . Specifically, assume that the adjacent pair $y_{i+1}y_i$ appears in two or more wall shuffles of R_1 , so rectangles y_{i+1} and y_i are adjacent to at least two shared walls. Since $\rho(x) = \rho(y)$, the corresponding wall shuffles of R_2 contain adjacent pair $y_i y_{i+1}$, implying that rectangles y_i and y_{i+1} are on opposite sides of those walls. Two rectangles can be adjacent to opposite sides of at most one vertical wall and at most one horizontal wall. If these simultaneously occur, then the rectangles are part of a group of four rectangles that share a single vertex, contradicting the assumption that R_1 is a generic rectangulation. Thus rectangles y_i and y_{i+1} share a single wall, implying that exactly one wall shuffle of R_1 differs from the corresponding wall shuffle of R_2 . If the shared wall W is horizontal, then since $y_{i+1} < y_i$ and the label of each rectangle above W is smaller than the label of each rectangle below W , rectangle y_{i+1} is above W and rectangle y_i is below W . Switching their order in x to obtain y results in the horizontal wall slide shown in the far right diagram of Figure 3.1. Similarly, if the shared wall W is vertical, since the label of each rectangle to the left of W is smaller than the label of each rectangle to the right of W , rectangle y_{i+1} is left of W and rectangle y_i is right of W . Switching their order results in the vertical wall slide illustrated in Figure 3.1.

Now assume that $\rho(x) \neq \rho(y)$. Theorem 3.4.2 implies that $\rho(x)$ and $\rho(y)$ are related by a diagonal pivot such that the pivoted edge is vertical in $\rho(x)$. If there exist a, b with $y_{i+1} < a, b < y_i$ such that a occurs to the left of position i in y and b occurs to the right of position $i + 1$ in y , then x contains the subsequence $ay_{i+1}y_ib$ which is an occurrence of the pattern 2-14-3 or the pattern 3-14-2 and y contains the subsequence $ay_i y_{i+1}b$. By Theorem 3.4.3, this implies that $\rho(x) = \rho(y)$, contradicting our initial assumption, so this cannot occur. We now consider three remaining cases. In this proof, it will be convenient to use the correspondences established in the proof of [14, Theorem 7.1] between each case and a specific diagonal pivot.

Case 1: $y_i = y_{i+1} + 1$. In this case, $\rho(x)$ and $\rho(y)$ are related by the *diagonal* pivot shown

in the leftmost diagram of Figure 3.1. We will show that this implies that R_1 and R_2 are related by the *generic* pivot shown in the leftmost diagram of Figure 3.1. Let W_1 denote a wall of $\rho(x)$ (or equivalently a wall of R_1) that is adjacent to neither rectangle y_i nor rectangle y_{i+1} and W_2 the corresponding wall of $\rho(y)$ (or equivalently R_2). Since W_2 is also not adjacent to either rectangle, $\sigma_{W_1} = \sigma_{W_2}$. Thus the wall shuffles of R_1 and R_2 differ only on walls adjacent to the union of rectangles y_i and y_{i+1} . We consider each of the wall shuffles of R_1 containing y_i or y_{i+1} . In R_1 , the wall shuffle associated to the pivoted edge is $y_{i+1}y_i$ and in R_2 it is y_iy_{i+1} . Now examine $\sigma_{W_{1b}}, \sigma_{W_{1a}}, \sigma_{W_{1l}}$, and $\sigma_{W_{1r}}$, the wall shuffles of the walls below, above, to the left, and to the right of the union of rectangles y_i and y_{i+1} in R_1 . We compare these wall shuffles with $\sigma_{W_{2b}}, \sigma_{W_{2a}}, \sigma_{W_{2l}}$, and $\sigma_{W_{2r}}$, the corresponding wall shuffles in R_2 , to demonstrate that they differ exactly as shown in Figure 3.1. Since $\sigma_{W_{1b}}$ and $\sigma_{W_{1a}}$ contain both y_i and y_{i+1} , and since y_i and y_{i+1} are adjacent in x , they are also adjacent in $\sigma_{W_{1b}}$ and $\sigma_{W_{1a}}$. The wall shuffle of a horizontal wall records the ordering of the left edges of rectangles below the wall and the right edges of rectangles above the wall so the adjacency of y_{i+1} and y_i in these wall shuffles implies that no edge of R_1 is adjacent to the interior of the bottom of rectangle y_i or the top of rectangle y_{i+1} . Similarly, in R_2 no edge is adjacent to the interior of the left side of rectangle y_{i+1} or the right side of rectangle y_i . Since rectangle y_{i+1} is not adjacent to W_{2b} and only y_i and y_{i+1} are switched in y , wall shuffle $\sigma_{W_{2b}}$ is obtained by removing y_{i+1} from $\sigma_{W_{2b}}$. Using the same argument, we see that: wall shuffle $\sigma_{W_{1a}}$ is obtained by removing y_i from $\sigma_{W_{1a}}$, wall shuffle $\sigma_{W_{2l}}$ is obtained by inserting y_i immediately before y_{i+1} in $\sigma_{W_{1l}}$, and wall shuffle $\sigma_{W_{2r}}$ is obtained by inserting y_{i+1} immediately after y_i in $\sigma_{W_{1r}}$. Thus the wall shuffles of R_1 and R_2 differ exactly as shown in the leftmost diagram of Figure 3.1 and no walls are adjacent to the interior of any dashed segment.

Case 2: $y_i > y_{i+1} + 1$ and every a with $y_{i+1} < a < y_i$ occurs to the right of position $i + 1$ in y . In this case, $\rho(x)$ and $\rho(y)$ are related by the *diagonal* pivot shown in the second diagram of Figure 3.1. Now consider R_1 and R_2 . As in Case 1, only wall shuffles containing y_i or y_{i+1} are effected by interchanging y_i and y_{i+1} in x to obtain y . Again, examining each wall shuffle of R_1 and relating it to the corresponding wall shuffle of R_2 , we see that R_1 and R_2 are related as shown in the second diagram of Figure 3.1.

Case 3: $y_i > y_{i+1} + 1$ and every a with $y_{i+1} < a < y_i$ occurs to the left of position i in y . In this case, $\rho(x)$ and $\rho(y)$ are related by the *diagonal* pivot shown in the third diagram of Figure 3.1, and this case is handled like Case 2. \square

The other direction of Theorem 3.1.3 follows from the following sequence of propositions.

Proposition 3.4.8. *Let $y \in S_n$ and $y_i y_j y_k y_l y_m$ be an occurrence of the pattern 3-5-1-4-2 in y . If every y_p satisfying $y_m < y_p < y_l$ occurs before y_j in y , then y contains an occurrence of the pattern 3-51-4-2.*

Proof. Since every y_p such that $y_m < y_p < y_l$ occurs before y_j in y , each entry of y between y_j and y_k is either greater than y_l or less than y_m . If every entry of y between y_j and y_k is greater than y_l , then the subsequence $y_i y_{k-1} y_k y_l y_m$ is an occurrence of the pattern 3-51-4-2 in y . Otherwise, let y_q denote the first entry of y between y_j and y_k such that $y_q < y_m$. In this case, $y_i y_{q-1} y_q y_l y_m$ is an occurrence of the pattern 3-51-4-2. \square

Proposition 3.4.9. *Let $R_1, R_2 \in \text{gRec}_n$ such that R_1 and R_2 are related by a single generic pivot as shown in the leftmost diagram of Figure 3.1 with the lower illustration corresponding to R_1 and the upper illustration corresponding to R_2 . Then $R_1 < R_2$ in gRec_n .*

Proof. Let R_1 and R_2 be generic rectangulations as described in the proposition and E be the horizontal edge of R_2 that is pivoted to form R_1 . Let $\psi(R_2) = y = y_1 \cdots y_n$, the unique element of Cl_n^2 such that $\gamma(y) = R_2$. Label the rectangles directly below and above E rectangle y_i and rectangle y_j respectively.

Let T_i be the partial diagonal rectangulation obtained after the first i steps in the construction of $\rho(y) = D_2$. By the definition of $\psi(R_2)$, entry y_{i+1} is the smallest element of $\{y_{i+1}, \dots, y_n\}$ such that the left side and bottom of rectangle y_{i+1} are contained in T_i and $y_1 \cdots y_{i+1}$ respects the wall shuffles of R_2 . To show that $R_1 < R_2$ in gRec_n , we first demonstrate that $i + 1 = j$.

To reach that goal, we will begin by showing that the bottom and left side of rectangle y_j are contained in T_i . Diagram (i) of Figure 3.7 illustrates a possible configuration of rectangles y_i and y_j with respect to T_{i-1} in D_2 . Since R_2 is a *generic* rectangulation, the wall containing E is E itself. Rectangulations R_2 and D_2 differ only by a sequence of wall slides and no wall slides can be performed along E so the top of rectangle y_i and the bottom of rectangle y_j coincide in D_2 . Thus the bottom of rectangle y_j is contained in T_i . To demonstrate that the left edge of rectangle y_j is contained in T_i , assume for a contradiction that this is not the case (as illustrated in Diagram (i) of Figure 3.7). Then there exists some rectangle y_p not contained in T_i , such that the right side of rectangle y_p is

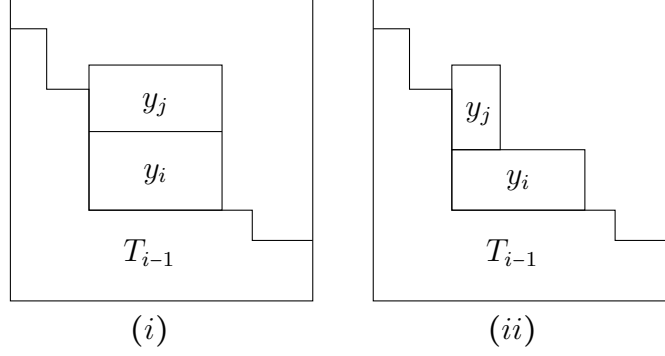


Figure 3.7: Illustrations for the proofs of Propositions 3.4.9-3.4.10.

adjacent to the left side of rectangle y_j and the bottom of rectangle y_p is contained in T_i . In y , the entry y_p occurs after y_i but before y_j . However, after wall slides are performed to obtain $\gamma(y) = R_2$ from D_2 , this implies that the lower-right corner of rectangle y_p is contained in the interior of the left side of rectangle y_j , contradicting the assumption that E is a pivotable edge in R_2 .

Now we show that adding rectangle y_j to the partial rectangulation immediately after y_i respects the wall shuffles of R_2 . Let W_l, W_r, W_b and W_a be the walls respectively to the left of, to the right of, below, and above rectangle y_j in Diagram (i) of Figure 3.7. Since only rectangles y_i and y_j border W_b , following y_i immediately by y_j in y respects this wall shuffle. If there is some y_p between y_i and y_j in σ_{W_l} , then rectangle y_p is on the left side of W_l and in R_2 the bottom right vertex of rectangle y_p is contained in the interior of the left side of rectangle y_j , contradicting the assumption that E is pivotable. The analogous argument shows that y_i and y_j are adjacent in σ_{W_r} . Now consider W_a . If rectangle y_j is the lower leftmost rectangle on W_a , then y_j is the first entry of σ_{W_a} so following y_i immediately by y_j in y respects the wall shuffle of W_a . Otherwise, the upper-left vertex of rectangle y_j coincides with a vertex of T_{i-1} . This case is illustrated in Figure 3.8. Let rectangle y_l be the rectangle contained in T_{i-1} whose upper-right vertex coincides with the upper-left vertex of rectangle y_j , let rectangle y_p be the leftmost rectangle not contained in T_{i-1} such that the bottom of rectangle y_p is contained in W_a , and let rectangle y_k be the rightmost rectangle such that the bottom of rectangle y_k is contained in W_a . If following y_i immediately by y_j does not respect σ_{W_a} , then y_p precedes y_j in y . Since rectangle y_k is the final rectangle above and adjacent to W_a , entry y_k follows y_j in y .

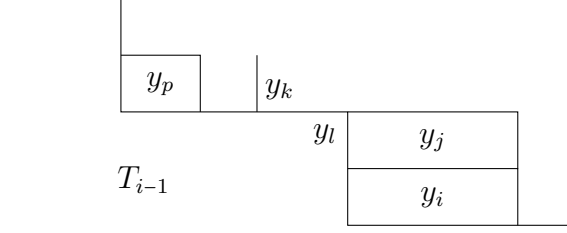


Figure 3.8: An illustration used in the proof of Proposition 3.4.9

Note that $y_p < y_k < y_l < y_j < y_i$ so $y_l y_i y_p y_j y_k$ is an occurrence of the pattern 3-5-1-4-2 in y . Every rectangle y_q with label satisfying $y_k < y_q < y_j$ is in T_{i-1} because the label y_q is on the diagonal of the square S between labels y_k and y_j . Thus every such y_q precedes y_i in y . By Proposition 3.4.8, permutation y contains a 3-51-4-2 pattern, contradicting the assumption that $y = \psi(R_2)$.

We have shown that the bottom and left side of rectangle y_j are contained in T_i and adding rectangle y_j to T_i immediately after rectangle y_i respects the wall shuffles of R_2 . Next we demonstrate that y_j is the smallest of $\{y_{i+1}, \dots, y_n\}$ with these properties. Assume that there is some $y_p \in \{y_{i+1}, \dots, y_n\}$ with these properties such that $y_p < y_j$ in numerical order. As demonstrated in the previous paragraph, rectangle y_p is not adjacent to W_l , the left wall of rectangle y_j which is also the left wall of rectangle y_i . Since $y_p < y_j$ in numerical order, rectangle y_p contains a label above and to the left of the label for rectangle y_j so rectangle y_p shares no walls with rectangle y_i . Since the addition of rectangle y_p to the partial rectangulation after rectangle y_i respects the wall slides of R_2 , this implies that the addition of rectangle y_p to the partial rectangulation immediately before rectangle y_i also respects the wall slides of R_2 . Because $y_p < y_j$ and the left and bottom sides of rectangle y_p are contained in T_i , the left and bottom sides of rectangle y_p are also contained in T_{i-1} . However, since $y_p < y_i$, this contradicts our choice of y_i as the i th entry of $\psi(R_2)$, i.e. rectangle y_p could have been added to T_{i-1} instead of rectangle y_i . Thus $j = i + 1$. Observing that $\gamma(y_1 \cdots y_j y_i \cdots y_n) = R_1$ completes the proof. \square

Proposition 3.4.10. *Let $R_1, R_2 \in \text{gRec}_n$ such that R_1 and R_2 are related by a single generic pivot as shown in the second diagram from the left in Figure 3.1 with the lower illustration corresponding to R_1 and the upper illustration corresponding to R_2 . Then $R_1 < R_2$ in gRec_n .*

Proof. Let D_2 denote the diagonal rectangulation associated with R_2 . As in the proof of

Proposition 3.4.9, let E denote the horizontal edge of R_2 that is pivoted to form R_1 and let permutation $y = y_1 \cdots y_n = \psi(R_2)$. Label the rectangle directly below E with y_i and the rectangle directly above E with y_j . Let W_l, W_r, W_b , and W_a refer to the walls respectively to the left of, to the right of, below, and above rectangle y_j in D_2 . As in the proof of Proposition 3.4.9, we demonstrate that $i + 1 = j$.

Diagram (ii) of Figure 3.7 shows a possible configuration of rectangles y_i and y_j with respect to T_{i-1} in D_2 . In D_2 , as in R_2 , the upper-left vertex of rectangle y_i and the lower left vertex of rectangle y_j coincide. Additionally, as in R_2 , the lower-right vertex of rectangle y_j is contained in the interior of the top of rectangle y_i in D_2 . To see why the second statement is true, note that performing a wall slide to switch the relative locations of the lower-right vertex of rectangle y_j and the upper-right vertex of rectangle y_i results in a rectangulation which is not diagonal. Thus the bottom of rectangle y_j is contained in T_i . Arguments identical to those used in the proof of Proposition 3.4.9 show that the left edge of rectangle y_j is also contained in T_i and that adding rectangle y_j immediately following rectangle y_i respects σ_{W_l} and σ_{W_a} . Since rectangle y_j is the lowermost rectangle on the left side of W_r , the wall shuffle σ_{W_r} begins with y_j . If y_j does not immediately follow y_i in σ_{W_b} , then rectangles y_i and y_j are not in the configuration shown in the second diagram of Figure 3.1. Specifically, if y_j does not immediately follow y_i in σ_{W_b} , then there exists some rectangle y_p whose left side is adjacent to rectangle y_i and whose top is contained in W_b . Performing wall slides to obtain R_2 from D_2 , the lower-right corner of rectangle y_j is not contained in the interior of the top of rectangle y_i . Thus $y_1 \cdots y_i y_j$ respects the wall shuffles of R_2 . Again using the argument from the proof of Proposition 3.4.9, we see that y_j is the smallest element of $\{y_{i+1}, \dots, y_n\}$ such that the walls of the corresponding rectangle are contained in T_i and whose selection respects the wall shuffles of R_2 so $i + 1 = j$. The proof is completed by observing that $\gamma(y_1 \cdots y_{i+1} y_i \cdots y_n) = R_1$. \square

We now describe four maps that will be used to complete the proofs of Theorem 3.1.3 and Theorem 3.1.11. Let rf_\setminus be the automorphism of generic rectangulations of size n that takes a generic rectangulation R to the generic rectangulation R' obtained by reflecting R about the upper-left to lower-right diagonal of the square S . Let rf_\swarrow be the automorphism that takes a generic rectangulation R to the generic rectangulation R' obtained by reflecting R about the lower-left to upper-right diagonal of S . Let $\text{rp} : S_n \rightarrow S_n$ denote the map on permutations that reverses the positions of entries in the one-line notation for a permutation. Let $\text{rv} : S_n \rightarrow S_n$ denote the map on permutations that re-

verses the values of the permutation, replacing each entry x_i of the permutation x with $n + 1 - x_i$. For example, $\text{rp}(34521) = 12543$ and $\text{rv}(34521) = 32145$.

The maps rp and rv are antiautomorphisms of the right weak order on S_n . As noted in [14, Remark 6.5, Remark 6.10], $\text{rf}_\setminus \circ \rho = \rho \circ \text{rp}$ and $\text{rf}_\setminus \circ \rho = \rho \circ \text{rv}$. Since applying rf_\setminus to an arbitrary generic rectangulation reverses each wall shuffle, the wall shuffles of rectangulation $\text{rf}_\setminus \circ \gamma$ agree with the wall shuffles of rectangulation $\gamma \circ \text{rp}$. Thus $\text{rf}_\setminus \circ \gamma = \gamma \circ \text{rp}$. Additionally, given a generic rectangulation R with wall shuffle $\sigma_W = x_{i_1} \cdots x_{i_p}$, the wall shuffle of the corresponding wall W' in $\text{rf}_\setminus(R) = R'$ is $\sigma_{W'} = (n + 1 - x_{i_1}) \cdots (n + 1 - x_{i_p})$. Thus the wall shuffles of $\text{rf}_\setminus \circ \gamma$ agree with the wall shuffles of $\gamma \circ \text{rv}$ so $\text{rf}_\setminus \circ \gamma = \gamma \circ \text{rv}$.

Lemma 3.4.11. *The map rf_\setminus is an antiautomorphism of the lattice of generic rectangulations.*

Proof. Let $R_1, R_2 \in \text{gRec}_n$ such that $R_1 < R_2$ in gRec_n . By Proposition 3.4.6, there exist $x_1, x_2 \in S_n$ such that $\gamma(x_1) = R_1$, $\gamma(x_2) = R_2$, and $x_1 < x_2$ in the right weak order on S_n . Because rp is an antiautomorphism of the right weak order, we have that $\text{rp}(x_1) > \text{rp}(x_2)$ in the right weak order. Since $\gamma(x_1) \neq \gamma(x_2)$, and $\text{rf}_\setminus \circ \gamma = \gamma \circ \text{rp}$, we have that $\gamma(\text{rp}(x_1)) \neq \gamma(\text{rp}(x_2))$. Again applying Proposition 3.4.6, we obtain $\gamma(\text{rp}(x_1)) > \gamma(\text{rp}(x_2))$ in gRec_n . Since $\gamma \circ \text{rp} = \text{rf}_\setminus \circ \gamma$, we conclude that $\text{rf}_\setminus(R_1) > \text{rf}_\setminus(R_2)$ in gRec_n . An identical argument shows that if $\text{rf}_\setminus(R_1) > \text{rf}_\setminus(R_2)$ in gRec_n , then $R_1 < R_2$ in gRec_n . \square

Proposition 3.4.12. *Let $R_1, R_2 \in \text{gRec}_n$ such that R_1 and R_2 are related by a single generic pivot as shown in the center diagram of Figure 3.1 with the lower illustration corresponding to R_1 and the upper illustration corresponding to R_2 . Then $R_1 < R_2$ in gRec_n .*

Proof. Let R_1 and R_2 be generic rectangulations as described in the proposition. Generic rectangulations $\text{rf}_\setminus(R_1)$ and $\text{rf}_\setminus(R_2)$ meet the conditions described in Proposition 3.4.10 (with the lower diagram of Figure 3.1 corresponding to R_2 and the upper diagram of Figure 3.1 corresponding to R_1) so $\text{rf}_\setminus(R_1) > \text{rf}_\setminus(R_2)$ in gRec_n . Thus by Lemma 3.4.11, $R_1 < R_2$ in gRec_n . \square

Proposition 3.4.13. *Let $R_1, R_2 \in \text{gRec}_n$ such that R_1 and R_2 are related by a single wall slide as shown in the fourth or fifth diagram of Figure 3.1 with the lower illustration corresponding to R_1 and the upper illustration corresponding to R_2 . Then $R_1 < R_2$ in gRec_n .*

Proof. First assume that R_1 and R_2 differ by a single vertical wall slide as shown in the fourth diagram of Figure 3.1. Let W_1 and W_2 respectively denote the walls in R_1 and R_2 on which the wall slide occurs. Let $\psi(R_2) = y = y_1 \cdots y_n$. We wish to find some j such that interchanging y_j and y_{j+1} in y results in a permutation x with $\gamma(x) = R_1$. Let $\sigma_{W_2} = y_{w_1} \cdots y_{w_i} y_{w_{i+1}} \cdots y_{w_f}$ be the wall shuffle of W_2 and $\sigma_{W_1} = y_{w_1} \cdots y_{w_{i+1}} y_{w_i} \cdots y_{w_f}$ be the wall shuffle of W_1 as illustrated in Figure 3.9. To prove that $R_1 \prec R_2$ in gRec_n , we will show that y_{w_i} and $y_{w_{i+1}}$ are adjacent in y and that switching their locations in y results in a permutation x such that $\gamma(x) = R_1$. Using the definition of the map ρ , we observe that $y_{w_{i+1}} < y_{w_1} < y_{w_1} + 1 = y_{w_f} < y_{w_i}$. Let $a_1 \cdots a_l$ be the sequence of elements between y_{w_i} and $y_{w_{i+1}}$ in y . Let a_m be the last element of the sequence satisfying $y_{w_{i+1}} < a_m < y_{w_1}$, if such an entry exists. If rectangle a_m were not adjacent to W , then by the definition of $\rho(y)$, rectangle $y_{w_{i+1}}$ would also not be adjacent to W . Thus, rectangle a_m must be adjacent to W . However, this implies that a_m occurs between y_{w_i} and $y_{w_{i+1}}$ in σ_{W_2} , a contradiction. Now let a_m be the first element of the sequence $a_1 \cdots a_l$ satisfying $y_{w_f} < a_m < y_{w_i}$. Then, by the definition of ρ , the left side of rectangle a_m is contained in W . This implies that a_m occurs between y_{w_i} and $y_{w_{i+1}}$ in σ_{W_2} , again a contradiction. Thus every element of the sequence $a_1 \cdots a_l$ must be less than $y_{w_{i+1}}$ or greater than y_{w_i} . Let a_m denote the first element of the sequence that satisfies $a_m < y_{w_{i+1}}$, if such an element exists. In this case, (taking $a_0 = y_{w_i}$ if $m = 1$) we reach a contradiction since $y_{w_1} a_{m-1} a_m y_{w_{i+1}} y_{w_f}$ is an occurrence of the 3-51-2-4 pattern in y . Thus $a_m > y_{w_i}$ for all m . However, this is also impossible since if $a_l \geq y_{w_i}$, then the subsequence $y_{w_1} y_{w_i} a_l y_{w_{i+1}} y_{w_f}$ of y forms a 2-4-51-3 pattern in y . Therefore, y_{w_i} and $y_{w_{i+1}}$ are adjacent in y . Let $x = y_1 \cdots y_{w_{i+1}} y_{w_i} \cdots y_n$. Since $y_{w_1} y_{w_i} y_{w_{i+1}} y_{w_f}$ is an occurrence of the pattern 2-41-3 in y , by Theorem 3.4.3, we have that $\rho(x) = \rho(y)$. Now consider the wall shuffles of x and y . Switching the order of y_{w_i} and $y_{w_{i+1}}$ in y to obtain x switches their order in the wall shuffle associated with W_2 so $\sigma_{W_1} = y_{w_1} \cdots y_{w_{i+1}} y_{w_i} \cdots y_{w_f}$. Every other wall shuffle of R_2 is unchanged in $\gamma(x)$ since $\rho(x) = \rho(y)$ and rectangles x_{w_i} and $x_{w_{i+1}}$ are adjacent to no other shared wall. Thus $\gamma(x) = R_1$.

Now assume that R_1 and R_2 differ by a single horizontal wall slide such that the lower illustration of Figure 3.1 corresponds to R_1 and the upper illustration corresponds to R_2 . By the definition of rf_\setminus , generic rectangulations $\text{rf}_\setminus(R_1)$ and $\text{rf}_\setminus(R_2)$ differ by a single vertical wall slide such that $\text{rf}_\setminus(R_1)$ contains the configuration shown in upper illustration of the fourth diagram of Figure 3.1 and $\text{rf}_\setminus(R_2)$ contains the configuration in the lower illustration. By the first part of this proof, $\text{rf}_\setminus(R_2) \prec \text{rf}_\setminus(R_1)$. Thus, by

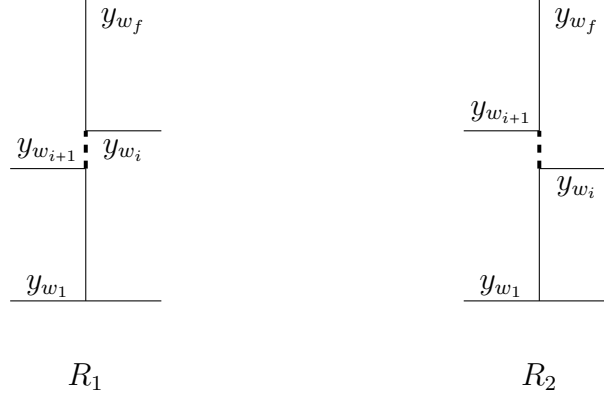


Figure 3.9: Diagrams used in the proof of Proposition 3.4.13. In each diagram, y_{w_1} is the lowest rectangle on the left side of W and y_{w_f} is the uppermost rectangle on the right side of W . No additional edges of R_1 or R_2 may be adjacent to the dashed segments.

Lemma 3.4.11, we have that $R_1 \prec R_2$. □

3.5 The Product and Coproduct

In this section, we prove Theorems 3.1.6 and 3.1.11.

Proof of Theorem 3.1.6. Let $x \in Cl_p^2$ and $y \in Cl_q^2$ such that $\gamma(x) = R_1$ and $\gamma(y) = R_2$. Corollary 3.2.3 states that $x \bullet_{Cl_2} y = \sum [xy'_{[p]}, y'_{[p]}x]$ where the summation denotes the sum of all elements of the interval $[xy'_{[p]}, y'_{[p]}x]$ in the lattice of 2-clumped permutations of size $p + q$. Applying the bijection γ to this equation, we obtain $\gamma(x) \bullet_{gR} \gamma(y) = R_1 \bullet_{gR} R_2 = \sum [\gamma(xy'_{[p]}), \gamma(y'_{[p]}x)]$, where the summation denotes the sum of all elements of the interval $[\gamma(xy'_{[p]}), \gamma(y'_{[p]}x)]$ in $gRec_n$. Applying γ to $xy'_{[p]}$ and $y'_{[p]}x$ results in the generic rectangulations $R_1 R'_2$ and $R'_2 R_1$ respectively. □

To prove that the coproduct in $gRec$ is given by Theorem 3.1.11 requires more work.

Applying γ to the equation in Theorem 3.2.5 and noting that $\gamma(\pi_1^2(y)) = \gamma(y)$ for any permutation y , we first obtain the following corollary:

Corollary 3.5.1. *Suppose $R \in gRec_n$ and $x \in Cl_n^2$ such that $\gamma(x) = R$. Then*

$$\Delta_{gR}(R) = \sum_{\substack{T \text{ is good} \\ \text{with respect to } x}} I_T \otimes J_T$$

where I_T is the sum of elements in the interval $[\gamma(\text{st}(x_{\min}|_T)), \gamma(\text{st}(x_{\max}|_T))]$ in gRec_p and J_T is the sum of elements in the interval $[\gamma(\text{st}(x_{\min}|_{T^c})), \gamma(\text{st}(x_{\max}|_{T^c}))]$ in gRec_q .

Theorem 3.1.11 will follow from Corollary 3.5.1, and Lemmas 3.5.4, 3.5.8, and 3.5.11. In the proof of Lemma 3.5.4, we will demonstrate that for any $x \in Cl_n^2$ such that $\gamma(x) = R$ there is a natural correspondence between sets that are good with respect to x and good paths in R . Then, in the proofs of Lemmas 3.5.8 and 3.5.11, we will show that for each good set T and corresponding good path \mathcal{P} , we have $\gamma(\text{st}(x_{\min}|_T)) = R_l(\mathcal{P})|_l$, $\gamma(\text{st}(x_{\max}|_T)) = R_l(\mathcal{P})_-$, $\gamma(\text{st}(x_{\min}|_{T^c})) = R_u(\mathcal{P})|_l$, and $\gamma(\text{st}(x_{\max}|_{T^c})) = R_u(\mathcal{P})_-$.

Example 3.5.2. The upper left diagram in Figure 3.5 shows the generic rectangulation R obtained by applying γ to the 2-clumped permutation $x = 5387412\textcircled{10}96$. The set $T = \{1, 3, 4, 5, 7, 8, 9, 10\}$ is good with respect to x . For this good set, $x_{\min} = 538741\textcircled{10}926$ and $x_{\max} = 587\textcircled{10}934126$. In this example, we see that the rectangulation $\gamma(\text{st}(x_{\min}|_T))$ coincides with the construction of $R_l(\mathcal{P})|_l$, the rectangulation $\gamma(\text{st}(x_{\max}|_T))$ coincides with the construction of $R_l(\mathcal{P})_-$, the rectangulation $\gamma(\text{st}(x_{\min}|_{T^c}))$ coincides with the construction of $R_u(\mathcal{P})|_l$, and the rectangulation $\gamma(\text{st}(x_{\max}|_{T^c}))$ coincides with the construction of $R_u(\mathcal{P})_-$.

To prove Theorem 3.1.11, we first make the following helpful observations about good sets. Given $x \in Cl_n^2$ such that $\gamma(x) = R$, let P be the partial order on $[n]$ such that the permutation $x' \in S_n$ is a linear extension of P if and only if $\gamma(x') = R$. We call P the *good set poset* of R . For each generic rectangulation, a good set poset exists because of the more general, well-known result given in Proposition 2.4.3. Since each fiber of γ forms an interval in the right weak order, for each generic rectangulation a good set poset exists.

The order ideals of the good set poset P correspond exactly to the sets that are good with respect to x . For each good set T , let $P|_T$ denote the order ideal of P consisting of the elements of T . The minimal linear extension of $P|_T$ is $x_{\min}|_T$. Similarly, the minimal linear extension of $P|_{T^c}$ is $x_{\min}|_{T^c}$, the maximal linear extension of $P|_T$ is $x_{\max}|_T$, and the maximal linear extension of $P|_{T^c}$ is $x_{\max}|_{T^c}$. To better understand the good sets associated with x , we describe the poset P . Although the good set poset is defined by a property that holds for adjacency posets of diagonal rectangulations, we note that the good set poset cannot be obtained from a generic rectangulation in the same way that an adjacency poset is obtained from a diagonal rectangulation (by declaring $x <_P y$ if rectangles x and y are adjacent with rectangle x is left of or below rectangle y , and then

taking the transitive closure of those relations). Because of the relationship between the maps ρ and γ , the construction of a good set poset from a generic rectangulation is a modification of the construction of an adjacency poset from a diagonal rectangulation. The proof of the next result makes use of that relationship.

Lemma 3.5.3. *Let r_i and r_j be rectangles of a generic rectangulation R with n rectangles, and P be the good set poset of R . If r_i comes before r_j in some wall shuffle of R , then $r_i <_P r_j$. Taking the transitive closure of these relations gives all of the relations in P .*

Proof. Given two permutations x and x' in S_n , we have that $R = \gamma(x) = \gamma(x')$ if and only if $\rho(x) = \rho(x')$ and the wall shuffles of $\gamma(x)$ are the same as the wall shuffles of $\gamma(x')$. Let $\rho(x) = D$ and define the poset Q on $[n]$ by declaring $r_i <_Q r_j$ if:

- In D , the right edge of rectangle r_i and the left edge of r_j intersect in their interiors,
- In D , the top edge of rectangle r_i and the bottom edge of rectangle r_j intersect in their interiors, or
- In some wall shuffle of R , the entry r_i precedes r_j

and then taking the transitive closure. The first two bullets in the definition of Q ensure that if x and x' are linear extensions of Q , then $\rho(x) = \rho(x')$. The third item ensures that the wall permutations of x and x' agree. By the definition of γ , the permutation x' is a linear extension of Q if and only if $\gamma(x') = \gamma(x)$. Thus to prove the lemma, it suffices to demonstrate that $r_i <_P r_j$ if and only if $r_i <_Q r_j$.

Since the condition for $r_i <_P r_j$ is identical to the final condition for $r_i <_Q r_j$, we have that $r_i <_P r_j$ implies $r_i <_Q r_j$. For the other direction, first assume that in D the right edge of rectangle r_i intersects the interior of the left edge of rectangle r_j (so $r_i <_Q r_j$) along some vertical wall W . As illustrated in the left diagram of Figure 3.10, since D is a *diagonal* rectangulation, each of the edges extending to the left of the wall is above each of the edges extending to the right of the wall. This implies that either rectangle r_i is the lowermost rectangle on the left side of the vertical wall separating the two rectangles (shown as the darker shaded region in the diagram) or rectangle r_j is the uppermost rectangle on the right side of the wall (shown as the lightly shaded region). This implies that r_i is the first entry of σ_W or r_j is the final entry of σ_W so in either case, $r_i <_P r_j$. Similarly, as illustrated in the right diagram of Figure 3.10, if the top edge of rectangle r_i intersects the interior of the bottom edge of rectangle r_j along some horizontal wall W

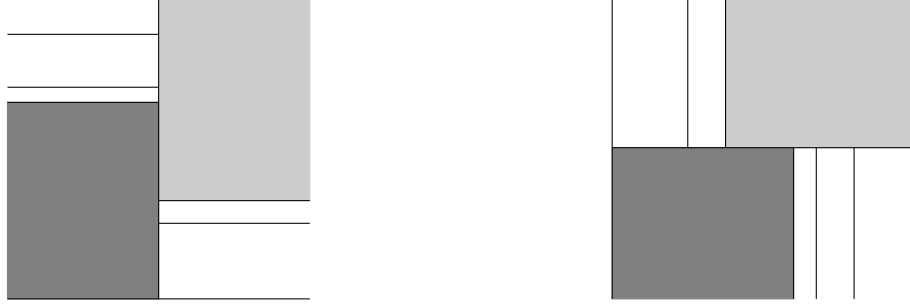


Figure 3.10: An illustration used in the proof of Lemma 3.5.3.

in D , then we again see that r_i precedes r_j in σ_W so $r_i <_P r_j$. Since the final condition for $r_i <_Q r_j$ is identical to the condition for $r_i <_P r_j$ and any relationship that comes from the transitive closure in Q also holds in P , we have that $r_i <_Q r_j$ implies $r_i <_P r_j$. \square

Lemma 3.5.4. *Let $x \in Cl_n^2$ such that $\gamma(x) = R$. The set T is good with respect to x if and only if the union of the rectangles of R labeled by elements of T are exactly the rectangles below some good path \mathcal{P} in R .*

Proof. Let $x \in Cl_n^2$ such that $\gamma(x) = R$, the poset P be the good set poset of R , and $T = \{t_1, \dots, t_p\}$ be a good set with respect to x (i.e. an order ideal of P). If $T = \emptyset$, then the path \mathcal{P} passing above and left of the rectangles of R labeled by elements of T travels down the left side and then across the bottom of the square S . This is a good path in R .

Now suppose that $T \neq \emptyset$. Let R_T denote the set of rectangles of R labeled by elements of T . To show that R_T is the set of rectangles below some good path, we will show that:

- R_T contains the bottom, left vertex of S ,
- R_T is a connected set with no interior holes, and
- the path \mathcal{P} starting at the top, left corner of S , traveling along the left edge of S until it reaches the boundary of R_T , tracing the upper right boundary of R_T , and then traveling along the bottom of S to the bottom right corner of S is a good path.

Since x can be obtained from any permutation x' such that $\gamma(x') = R$ by a sequence of adjacent cliff transpositions of scrambles of the patterns 2-4-51-3 and 3-51-2-4, the

first entry of x is also the first entry of x' . Thus by the definition of γ , some rectangle of R_T contains the bottom, left vertex of S .

If R_T is not connected, has an interior hole, or \mathcal{P} contains a left or up step, then the left side or bottom of some rectangle $t_i \in R_T$ intersects the boundary of some rectangle u such that $u \in [n] - T$. The two leftmost diagrams of Figure 3.11 illustrate these cases. In each of the diagrams of Figure 3.11, the shaded rectangles are contained in R_T . If the left side of rectangle t_i intersects the right side of rectangle u along a vertical wall W (as illustrated in the leftmost diagram of Figure 3.11), then the lower-right vertex of rectangle u is below the upper-left vertex of rectangle t_i on W . Note that the lower-right vertex of rectangle u is not necessarily contained in the left side of rectangle t_i as shown in the diagram, but it is necessarily below the upper-left vertex of rectangle t_i . Thus u precedes t_i in σ_W , contradicting the assumption that T is an order ideal of P . Similarly, if the bottom of rectangle t_i intersects the top of rectangle u along a horizontal wall W (as illustrated in the second diagram of Figure 3.11), then the upper-left vertex of rectangle u is left of the lower-right vertex of rectangle t_i on W . This also contradicts the assumption that T is an order ideal of P .

To complete the argument, we show that \mathcal{P} meets the two conditions for a good path. Assume, for a contradiction, that the interior of a vertical segment of \mathcal{P} contains vertices v and v' of R such that v is the upper-left vertex of a rectangle u with $u \notin T$, vertex v' is the lower-right vertex of a rectangle $t_i \in R_T$ and v is below v' . This configuration is illustrated in the third diagram of Figure 3.11. The thick segment in the diagram is contained in \mathcal{P} . Since the upper-left vertex of rectangle u occurs below the bottom right vertex of rectangle t_i along their shared wall, entry u precedes t_i in the associated wall shuffle, contradicting the assumption that T is a good set. Using the same reasoning, we conclude that the configuration illustrated in the rightmost diagram of Figure 3.11 also does not occur along \mathcal{P} , that is, the interior of no horizontal segment of \mathcal{P} contains vertices h and h' of R such that h is the lower-right vertex of a rectangle $u \notin T$, h' is the upper-left vertex of a rectangle $t_i \in R_T$ and h is left of h' . Thus the upper right border of R_T determines a good path in R .

Next we show that given any good path \mathcal{P} in R , the labels of the set of rectangles below and to the left of \mathcal{P} , denoted by T , form a good set. It is enough to demonstrate that T is an order ideal of P , the good set poset of R . For a contradiction, assume that $u \notin T$, $t_i \in T$, and u precedes t_i in σ_W , some wall shuffle of R . First let W be a vertical



Figure 3.11: Diagrams for the proof of Lemma 3.5.4.

wall. If rectangles t_i and u are on the same side of W or rectangle u is on the left side of W while rectangle t_i is on the right side of W , then \mathcal{P} passes to the right of rectangle t_i and then to the left of rectangle u , or below u and then above t_i . Thus \mathcal{P} contains a left step or an up step, a contradiction. If rectangle t_i is left of W and rectangle u is right of W , since the upper-left corner of rectangle u is below the lower-right corner of rectangle t_i , we have that \mathcal{P} contains a left step or violates the first condition of a good path. When W is a horizontal wall, in each case we again reach a contradiction by showing that \mathcal{P} contains a left or up step, or violates the second condition of a good path. \square

For every good path \mathcal{P} of a generic rectangulation R , in the constructions of $R_l(\mathcal{P})_l$, $R_l(\mathcal{P})_-$, $R_u(\mathcal{P})_l$, and $R_u(\mathcal{P})_-$, the rectangles inherit a labeling (using the elements of T) from the labeling of R . To simplify notation, in what follows, we do not standardize these labels. In particular, when we refer to a permutation x such that $\gamma(x) = R_l(\mathcal{P})_l$, this permutation x will be an ordering of the elements of T rather than an ordering of $\{1, \dots, |T|\}$. To use x to construct $R_l(\mathcal{P})_l$, we label the diagonal of S with the elements of T written in increasing order along the upper-left to bottom-right diagonal of S and then construct $\gamma(x)$ as usual. Additionally, we define the good set poset P' of $R_l(\mathcal{P})_l$ to be the partial order on T such that x is a linear extension of P' if and only if $\gamma(x) = R_l(\mathcal{P})_l$.

Definition 3.5.5. Given a set T that is good with respect to $x \in Cl_n^2$ such that $\gamma(x) = R$, we say that an ordering $t = t_1 \cdots t_{|T|}$ of the elements of T *respects the ordering of the good set poset P of R* if and only if there exists $x' = x'_1 \cdots x'_n \in S_n$ such that $x'_1 \cdots x'_{|T|} = t$ and x' is a linear extension of P (or equivalently $\gamma(x') = R$). If some linear extension t of a poset P' respects the ordering of the good set poset P of R then we say that P' is *compatible* with P .

Lemma 3.5.6. *Let R be a generic rectangulation, \mathcal{P} be a good path in R , poset P be the good set poset of R , and P' be the good set poset of $R_l(\mathcal{P})_l$. Then P' is compatible with P .*

Proof. Let T be the good set corresponding to \mathcal{P} (which exists by Lemma 3.5.4). Assume that P' is *not* compatible with P so there does not exist a linear extension of P' that respects the ordering of P . Since T is a good set with respect to R , there exists an ordering of the elements of T that respects the ordering of P . If none of these orderings is a linear extension of P' then there exist $r_j <_{P'} r_i$ such that $r_i <_P r_j$. Below we show that this cannot occur by demonstrating that if $r_i, r_j \in T$ such that $r_i <_P r_j$ then $r_i <_{P'} r_j$.

To show that $r_i <_P r_j$ implies $r_i <_{P'} r_j$, it suffices to prove that if $r_i <_P r_j$ then $r_i <_{P'} r_j$. Assume that $r_i, r_j \in T$ and $r_i <_P r_j$. By Lemma 3.5.3, this implies that r_i immediately precedes r_j in some wall shuffle σ_W of R . We consider cases and make use of the construction of $R_l(\mathcal{P})_l$.

If rectangles r_i and r_j are on the same side of W , then the rectangles are adjacent, with rectangle r_i left of or below rectangle r_j . Assume that rectangles r_i and r_j are both above a horizontal wall W . Since r_i and r_j are in T , path \mathcal{P} passes above both rectangles, so the bottom of both rectangles and the edge separating them are contained in $R_l(\mathcal{P})$. Thus in $R_l(\mathcal{P})_l$, rectangles r_i and r_j are adjacent to a horizontal wall and r_i precedes r_j in that wall shuffle. Therefore $r_i <_{P'} r_j$. If rectangles r_i and r_j are both right of a vertical wall, the argument is similar. Now assume that rectangles r_i and r_j are both below a horizontal wall W . Since r_i immediately precedes r_j in σ_W , no vertical edge extends from the top of rectangle r_i . Thus either path \mathcal{P} contains no part of the top of rectangle r_i or \mathcal{P} contains the tops of rectangles r_i and r_j . If \mathcal{P} contains no part of the top of rectangle r_i , then the top of rectangle r_i , part or all of the top of rectangle r_j , and the edge separating rectangles r_i and r_j remain in $R_l(\mathcal{P})$. To construct $R_l(\mathcal{P})_l$, the remaining portion of the top of rectangle r_j is extended until it meets a vertical wall. Rectangles r_i and r_j are adjacent to the horizontal wall containing this extension. Thus, r_i precedes r_j in this wall shuffle of $R_l(\mathcal{P})_l$ so $r_i <_{P'} r_j$. If \mathcal{P} contains the tops of rectangles r_i and r_j , then \mathcal{P} contains points left of the upper-left corner of rectangle r_i (as shown in the left diagram of Figure 3.12) or the upper-left corner of rectangle r_i is a vertex of \mathcal{P} (as shown in the right diagram of Figure 3.12). In these illustrations, two possible locations of \mathcal{P} are darkened. The dotted segment of the second diagram may or may not be present in R . In either of these cases, in $R_l(\mathcal{P})_l$ rectangle r_j extends to the top of S and rectangle r_i is adjacent to the wall W' containing the left side of rectangle r_j . Since rectangle r_j is the uppermost rectangle on the right side of W' , the final entry of $\sigma_{W'}$ is r_j . Thus r_i precedes r_j in $\sigma_{W'}$ and so $r_i <_{P'} r_j$. If rectangles r_i and r_j are both adjacent to the left

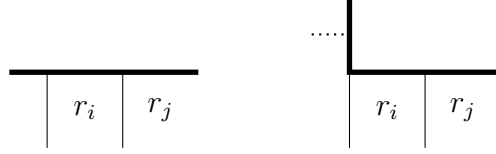


Figure 3.12: Configurations of rectangles r_i and r_j used in the proof of Lemma 3.5.6.

side of a vertical wall W , since r_i and r_j are adjacent in σ_W , no edge of R extends from the right side of rectangle r_i . Thus, regardless of the location of \mathcal{P} , in $R_l(\mathcal{P})|_1$ the right sides of rectangles r_i and r_j are contained in a single vertical wall and rectangle r_i remains below rectangle r_j . Therefore $r_i <_{P'} r_j$.

Now consider the case where rectangles r_i and r_j are on opposite sides of W . If W is horizontal, then rectangles r_i and r_j are in one of the two leftmost configurations shown in Figure 3.13. Since r_j immediately follows r_i in σ_W , no other edge can be adjacent to the dashed segment in the second diagram. If rectangles r_i and r_j are in the first configuration of Figure 3.13, then, regardless of the location of \mathcal{P} , the left edge of rectangle r_i and the horizontal edge between the rectangles remain in $R_l(\mathcal{P})$. Thus by construction, rectangles r_i and r_j remain adjacent to W in $R_l(\mathcal{P})|_1$ with the upper-left vertex of rectangle r_i to the left of the lower-right vertex of rectangle r_j so $r_i <_{P'} r_j$. If rectangles r_i and r_j are in the second configuration of Figure 3.13, then we consider two cases. First, if some part of the right side of rectangle r_i is contained in $R_l(\mathcal{P})$ then some part of the top of rectangle r_j is also contained in $R_l(\mathcal{P})$. Thus in $R_l(\mathcal{P})|_1$, rectangles r_i and r_j remain adjacent to the extension of W with the lower-right vertex of rectangle r_i to the left of the upper-left vertex of rectangle r_j . Therefore $r_i <_{P'} r_j$. If the right side of rectangle r_i is contained in \mathcal{P} , then the dashed segment and the top of rectangle r_j are also contained in \mathcal{P} . In $R_l(\mathcal{P})|_1$, the left edge of rectangle r_j is extended to the top of S and the bottom of rectangle r_i is extended to meet this vertical edge. Thus rectangles r_i and r_j are adjacent to opposite sides of a vertical edge of $R_l(\mathcal{P})|_1$ with the bottom right vertex of rectangle r_i below the top left of rectangle r_j . Therefore $r_i <_{P'} r_j$.

If rectangles r_i and r_j are on opposite sides of a vertical wall W , then they form one of the configurations shown in third or fourth diagram of Figure 3.13. If they form the configuration shown in the third diagram, then the bottom edge of rectangle r_i and the edge between the rectangles remain in $R_l(\mathcal{P})$. Thus in $R_l(\mathcal{P})|_1$, rectangles r_i and r_j are adjacent to the extension of W with the bottom right vertex of rectangle r_i below the

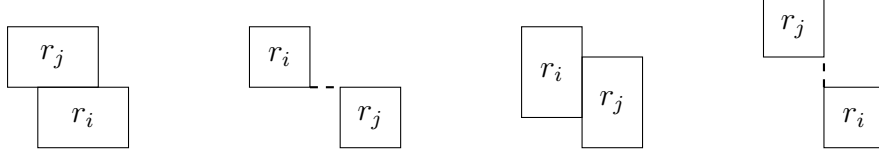


Figure 3.13: Additional illustrations used in the proof of Lemma 3.5.6.

top left vertex of rectangle r_j so $r_i <_{P'} r_j$. If rectangles r_i and r_j form the configuration shown in the final diagram of Figure 3.13 and some part of the right side of rectangle r_j is not contained in \mathcal{P} , then some part of the top of rectangle r_i is also not contained in \mathcal{P} . In $R_l(\mathcal{P})|$, the top edge of rectangle r_i remains below the bottom edge of rectangle r_j so $r_i <_{P'} r_j$. If instead the right side of rectangle r_j is contained in \mathcal{P} , then the top of rectangle r_i is also contained in \mathcal{P} . In the construction of $R_l(\mathcal{P})|$, the extension of the left side of rectangle r_i is stopped by the ϵ extension of the bottom edge of rectangle r_j . Thus in $R_l(\mathcal{P})|$, rectangles r_i and r_j are adjacent to the horizontal wall containing the extension of the bottom edge of rectangle r_j with the upper-left vertex of rectangle r_i left of the lower-right vertex of rectangle r_j so $r_i <_{P'} r_j$. \square

Lemma 3.5.7. *Let $R \in \text{gRec}_n$, let P be the good set poset of R , and let \mathcal{P} be a good path in R . Let $\tilde{R} \triangleleft R_l(\mathcal{P})|$ in $\text{gRec}_{|T|}$ and \tilde{P} be the good set poset of \tilde{R} . Then \tilde{P} is not compatible with P .*

Proof. Let T be the good set corresponding with good path \mathcal{P} . Again to simplify notation, we label each rectangle of $R_l(\mathcal{P})|$ using the label (which is an element of T) inherited from R . By labeling the upper-left to lower-right diagonal of the square S with the elements of T (in numerical order), we also obtain a labeling of the rectangles of \tilde{R} by the elements of T . Let P' be the partial order on T such that x is a linear extension of P' if and only if $\gamma(x) = R_l(\mathcal{P})|$. Let \tilde{P} denote the partial order on T such that x is a linear extension of \tilde{P} if and only if $\gamma(x) = \tilde{R}$. To show that \tilde{P} is not compatible with P , we demonstrate that no linear extension of \tilde{P} respects the ordering of P or equivalently that there exist r_i, r_j in T satisfying $r_j <_{\tilde{P}} r_i$ such that $r_i <_P r_j$.

Since $\tilde{R} \triangleleft R_l(\mathcal{P})|$ in $\text{gRec}_{|T|}$, by Theorem 3.1.3 a wall slide or generic pivot is performed on $R_l(\mathcal{P})|$ to obtain \tilde{R} . First assume that rectangles r_i and r_j of $R_l(\mathcal{P})|$ form a configuration illustrated in one of the three leftmost upper diagrams of Figure 3.1 with $r_i <_{P'} r_j$ and that the edge E which is pivoted to obtain \tilde{R} is completely contained

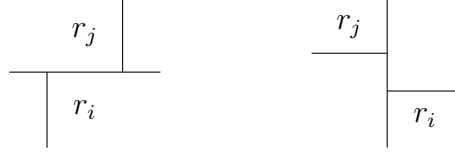


Figure 3.14: An illustration used in the proof of Lemma 3.5.7.

in $R_l(\mathcal{P})$. Since E is completely contained in $R_l(\mathcal{P})$, rectangles r_i and r_j form this same configuration in R and we have that $r_i <_P r_j$. Pivoting E to obtain \tilde{R} , we see that r_j precedes r_i in a wall shuffle of \tilde{R} so $r_j <_{\tilde{P}} r_i$. Thus, in this case, \tilde{P} is not compatible with P .

We next consider the cases in which $R_l(\mathcal{P})|_1$ and \tilde{R} differ by a wall slide. If they differ by a horizontal wall slide, then in $R_l(\mathcal{P})|_1$ rectangles r_i and r_j form the configuration shown in the left diagram of Figure 3.14. By the construction of $R_l(\mathcal{P})|_1$ (since no new vertical edges extending upward from a horizontal walls are created), the lower-right vertex and some portion of the right side of rectangle r_j are contained in $R_l(\mathcal{P})$. Additionally, since the upper-left vertex of rectangle r_i is left of the lower-right vertex of rectangle r_j and \mathcal{P} is a good path, the left side of rectangle r_i is contained in $R_l(\mathcal{P})$. Thus rectangles r_i and r_j form the same configuration in R , implying that $r_i <_P r_j$. Performing a wall slide to obtain \tilde{R} from $R_l(\mathcal{P})|_1$, we see that $r_j <_{\tilde{P}} r_i$. We conclude that in this case, \tilde{P} is not compatible with P . If rectangulations $R_l(\mathcal{P})|_1$ and \tilde{R} differ by a vertical wall slide, then in $R_l(\mathcal{P})|_1$ rectangles r_i and r_j form the configuration shown in the second diagram of Figure 3.14. If the lower-right vertex of rectangle r_j were a constructed vertex, in the final step of the construction of $R_l(\mathcal{P})|_1$, a wall slide would be performed to move the vertex below the upper-left vertex of rectangle r_i . Thus, this configuration of rectangles would not appear in $R_l(\mathcal{P})|_1$. Therefore the lower-right vertex of rectangle r_j is a vertex of $R_l(\mathcal{P})$. Because \mathcal{P} is a good path, the upper-left vertex of rectangle r_i is also a vertex of $R_l(\mathcal{P})$ so this configuration of rectangles r_i and r_j appears in R . Thus $r_i <_P r_j$ and $r_j <_{\tilde{P}} r_i$, implying that \tilde{P} is not compatible with P .

Finally, we consider the effect of performing a generic pivot on a horizontal edge E of $R_l(\mathcal{P})|_1$ such that E is not completely contained in $R_l(\mathcal{P})$. There are two cases to consider: either E is a new edge of $R_l(\mathcal{P})|_1$ (in other words, no points of E are contained in $R_l(\mathcal{P})$) or E is the extension of some edge E' of R .

First consider the case where no points of E are contained in $R_l(\mathcal{P})$. By the construc-

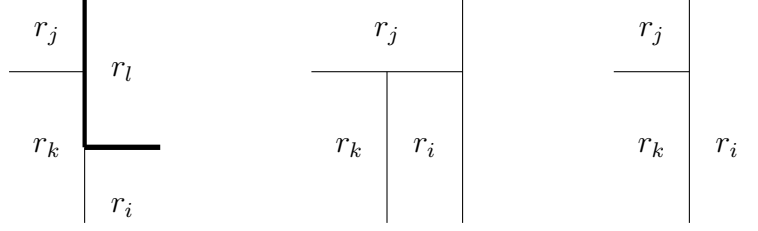


Figure 3.15: Diagrams used in the proof of Lemma 3.5.7.

tion of $R_l(\mathcal{P})|$, edge E results from a configuration in R as shown in the leftmost diagram of Figure 3.15. In the diagram, a subset of the rectangles of R are labeled r_i, r_j, r_k, r_l , and a portion of a good path \mathcal{P} is shown as a darkened segment. The wall shuffle of the vertical wall shown contains the subsequence $r_k r_i r_j r_l$ so $r_i <_P r_j$. To obtain $R_l(\mathcal{P})|$, we remove \mathcal{P} , extend the bottom of rectangle r_j by ϵ to the right, extend the left side of rectangle r_i upwards until it hits the extension of the bottom of rectangle r_j and then extend the bottom of rectangle r_j further until it reaches the extension of some vertical wall or the right side of S . If necessary, we then perform wall slides along vertical walls, as described in the definition of $R_l(\mathcal{P})|$, but these wall slides do not affect the configuration of rectangles r_i, r_j , and r_k in $R_l(\mathcal{P})|$ (shown in the center diagram of Figure 3.15). Let E be the edge of $R_l(\mathcal{P})|$ that separates rectangles r_i and r_j . Performing a generic pivot on E to obtain \tilde{R} results in the configuration shown in the rightmost diagram of Figure 3.15. In \tilde{R} , the lower-right vertex of rectangle r_j is below the upper-left vertex of rectangle r_i along their shared wall, so r_j precedes r_i in this wall shuffle. Thus $r_j <_{\tilde{P}} r_i$, implying that \tilde{P} is not compatible with P .

Now consider the case where E is the extension of some edge E' of R . This means that one endpoint v_0 of E' is contained in $R_l(\mathcal{P})$ and the other is on \mathcal{P} . In R , let rectangle r_i be below E' and rectangle r_j be above E' . Thus the upper-left vertex of rectangle r_i is left of the lower-right vertex of rectangle r_j on the wall of R containing E' . This implies that $r_i <_P r_j$. In $R_l(\mathcal{P})|$, rectangles r_i and r_j are adjacent to E with rectangle r_i below rectangle r_j . By the construction of $R_l(\mathcal{P})|$, the right endpoint of E is the final vertex on the horizontal wall containing E . So that E can be pivoted to obtain \tilde{R} from $R_l(\mathcal{P})|$, in $R_l(\mathcal{P})|$ rectangles r_i and r_j must form one of the configurations shown in Figure 3.16. However, in both cases, pivoting E results in a rectangulation \tilde{R} in which r_j precedes r_i in a vertical wall shuffle. Thus $r_j <_{\tilde{P}} r_i$. Therefore, regardless of the position of the

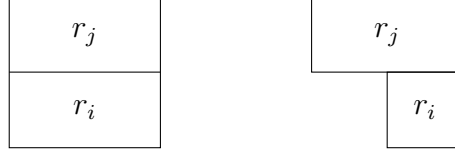


Figure 3.16: Configurations of r_i and r_j in $R_l(\mathcal{P})|_I$ which allow for a generic pivot to be performed on the edge separating the rectangles.

generic pivot or wall slide used to obtain \tilde{R} from $R_l(\mathcal{P})|_I$, the poset \tilde{P} is not compatible with P . \square

Lemma 3.5.8. *Let $R \in \text{gRec}_n$ and $x \in Cl_n^2$ such that $\gamma(x) = R$. For each set T that is good with respect to x and corresponding good path \mathcal{P} , we have that $\gamma(\text{st}(x_{\min}|_T)) = R_l(\mathcal{P})|_I$.*

Proof. Again let P be the good set poset of R and P' be the good set poset of $R_l(\mathcal{P})|_I$ (where P' is a poset on T). Let

$$Y = \{y \in S_n \mid \gamma(y) = R \text{ and } \{y_1, \dots, y_{|T|}\} = T\}.$$

The set of all permutations that map to R under γ and the set of all permutations whose first $|T|$ entries are the elements of T each form a nonempty interval in the right weak order on S_n . Since T is a good set, the intersection of these intervals is nonempty. Thus, since the right weak order is a lattice, the elements of Y form an interval in this lattice. By definition, the minimal element of Y is x_{\min} .

Let

$$X = \{x' \in S_n \mid \gamma(x') = R \text{ and } x'_1 \cdots x'_{|T|} \text{ is a linear extension of } P'\}.$$

By Lemma 3.5.6, the set X is non-empty. Note that $X \subseteq Y$. To prove the lemma, we wish to show that $x_{\min} \in X$.

To obtain a contradiction, assume that $x_{\min} \notin X$. Thus, there exists some $y \in Y$ such that $y \notin X$ and y is covered by an element of X . Since $y \in Y$ and $y \notin X$, we have that $\gamma(\text{st}(y|_T)) \neq R_l(\mathcal{P})|_I$. Then $\gamma(\text{st}(y|_T))$ is some \tilde{R} such that $\tilde{R} \prec R_l(\mathcal{P})|_I$ in $\text{gRec}_{|T|}$. By Lemma 3.5.7, the good set poset of \tilde{R} is not compatible with P . This implies that $y \notin Y$, a contradiction. \square

In what follows, we will consider distinct rectangulations and good sets simultaneously. To identify the rectangulation and good set used in each case, we will use the notation $x_{\max}(R, T)$, $x_{\min}(R, T)$, $R_l(\mathcal{P}(R, T))$, and $R_u(\mathcal{P}(R, T))$ where R indicates the rectangulation of interest, $x \in Cl_n^2$ such that $\gamma(x) = R$, and T is a set that is good with respect to x . We will also make use of the maps rf_\setminus , rf_\swarrow , rp , and rv . Given a generic rectangulation R , recall that $\text{rf}_\setminus(R)$ is the reflection of R about the upper-left to lower-right diagonal of the square S and $\text{rf}_\swarrow(R)$ is the reflection of R about the lower-left to upper-right diagonal of S . Given a permutation $x = x_1 \cdots x_n$, recall that $\text{rp}(x) = x_n \cdots x_1$ and $\text{rv}(x) = (n+1-x_1) \cdots (n+1-x_n)$.

Lemma 3.5.9. *Let R be a generic rectangulation, let $x \in Cl_n^2$ such that $\gamma(x) = R$, let $T = \{t_1, \dots, t_p\}$ a set that is good with respect to x , and let $T' = \{n+1-t_1, \dots, n+1-t_p\}$. Then $x_{\max}(R, T)|_T = \text{rv}(x_{\min}(\text{rf}_\swarrow(R), T')|_{T'})$.*

Proof. Let P be the good set poset of R and P' be the good set poset of $\text{rf}_\swarrow(R)$. Since each wall shuffle $\sigma_W = x_{i_1} \cdots x_{i_s}$ of R corresponds to a wall shuffle $\sigma_{W'} = (n+1-x_{i_1}) \cdots (n+1-x_{i_s})$ of $\text{rf}_\swarrow(R)$, we have that $x_i < x_j$ in P if and only if $n+1-x_i < n+1-x_j$ in P' . Because T is a good set with respect to R , this implies that T' is a good set with respect to $\text{rf}_\swarrow(R)$. The order ideal $P|_T$ is isomorphic to the order ideal $P'|_{T'}$. To find $x_{\max}(R, T)|_T$ an entry at a time using $P|_T$, at each step we consider the elements that have not yet been selected and are only greater than elements that have already been selected. From this collection of elements, we choose the numerically largest value. Analogously, to find $x_{\min}(\text{rf}_\swarrow(R), T')|_{T'}$ using $P'|_{T'}$, we select the numerically smallest value from the candidate elements at each step. Constructing $x_{\max}(R, T)|_T$ and $x_{\min}(\text{rf}_\swarrow(R), T')|_{T'}$ simultaneously, at each step the numerically largest candidate element of P coincides with the numerically smallest candidate element of P' under the poset isomorphism. Thus applying rv to $x_{\min}(\text{rf}_\swarrow(R), T')|_{T'}$ we obtain $x_{\max}(R, T)|_T$. \square

Lemma 3.5.10. *Let $R \in \text{gRec}_n$, let $x \in Cl_n^2$ such that $\gamma(x) = R$ and let T be a set that is good with respect to x . Then $x_{\min}(R, T)|_{T^C} = \text{rp}(x_{\max}(\text{rf}_\setminus(R), T^C)|_{T^C})$.*

Proof. Let P be the good set poset of R and let P' denote the good set poset of $\text{rf}_\setminus(R)$. Since applying rf_\setminus to R reverses each wall shuffle, we have that P' is dual to P . The set T is an order ideal of P so the set T^C is a dual order ideal of P . Thus T^C is an order ideal of P' . Let $u_i, u_j \in T^C$ such that $u_i < u_j$ in numerical order. Then u_i precedes u_j in $x_{\max}(\text{rf}_\setminus(R), T^C)|_{T^C}$ if and only if $u_i <_{P'} u_j$. Equivalently, u_i follows u_j in

$\text{rp}(x_{\max}(\text{rf}_{\setminus}(R), T^C)|_{T^C})$ if and only if $u_i <_{P'} u_j$. Entry u_i follows u_j in $x_{\min}(R, T)|_{T^C}$ if and only if $u_j <_P u_i$. Since P and P' are dual posets, the result follows. \square

Lemma 3.5.11. *Let $R \in \text{gRec}_n$ and $x \in Cl_n^2$ such that $\gamma(x) = R$. For each set T that is good respect to x and corresponding good path \mathcal{P} , we have $\gamma(\text{st}(x_{\max}|_T)) = R_l(\mathcal{P})_-$, $\gamma(\text{st}(x_{\min}|_{T^C})) = R_u(\mathcal{P})_+$, and $\gamma(\text{st}(x_{\max}|_{T^C})) = R_u(\mathcal{P})_-$.*

Proof. We use Lemma 3.5.8 together with the maps rf_{\setminus} , $\text{rf}_{/}$, rp and rv to prove the equalities of this lemma.

To prove the first equality, we define $T' = \{n + 1 - t_i \mid t_i \in T\}$. We now describe the manipulations that appear in (3.1a)-(3.1d) below. Using Lemma 3.5.9, we obtain (3.1a). Since $\text{st} \circ \text{rv} = \text{rv} \circ \text{st}$ and $\gamma \circ \text{rv} = \text{rf}_{/} \circ \gamma$, we obtain (3.1b) from (3.1a). By Lemma 3.5.8, we have that (3.1c) follows. Finally, since the constructions of the vertical and horizontal completions of R_l are related by reflection about the bottom-left to upper-right diagonal, we obtain the desired result.

$$\gamma(\text{st}(x_{\max}(R, T)|_T)) = \gamma(\text{st}(\text{rv}(x_{\min}(\text{rf}_{/}(R), T')|_{T'}))) \quad (3.1a)$$

$$= \text{rf}_{/}(\gamma(\text{st}(x_{\min}(\text{rf}_{/}(R), T')|_{T'}))) \quad (3.1b)$$

$$= \text{rf}_{/}(R_l(\mathcal{P}(\text{rf}_{/}(R), T'))_+) \quad (3.1c)$$

$$= R_l(\mathcal{P}(R, T))_- \quad (3.1d)$$

To prove the second and third equalities of the lemma, we first use Lemma 3.5.10 (see (3.2a) and (3.3a)). For (3.3a), we apply the involution rf_{\setminus} to make use of the equation in Lemma 3.5.10. To obtain (b) from (a) in both manipulations we note that $\text{st} \circ \text{rp} = \text{rp} \circ \text{st}$ and $\gamma \circ \text{rp} = \text{rf}_{\setminus} \circ \gamma$. Then (3.2c) follows from (3.2b) by applying the first result of this lemma. We obtain (3.2d) since the construction of the horizontal completion of R_l and the construction of the vertical completion of R_u are related by reflection about the upper-left to bottom-right diagonal.

$$\gamma(\text{st}(x_{\min}(R, T)|_{T^C})) = \gamma(\text{st}(\text{rp}(x_{\max}(\text{rf}_{\setminus}(R), T^C)|_{T^C}))) \quad (3.2a)$$

$$= \text{rf}_{\setminus}(\gamma(\text{st}(x_{\max}(\text{rf}_{\setminus}(R), T^C)|_{T^C}))) \quad (3.2b)$$

$$= \text{rf}_{\setminus}(R_l(\mathcal{P}(\text{rf}_{\setminus}(R), T^C))_-) \quad (3.2c)$$

$$= R_u(\mathcal{P}(R, T))_+ \quad (3.2d)$$

By Lemma 3.5.8, we have that (3.3c) follows from (3.3b). Since the construction of the vertical completion of R_l and the construction of the horizontal completion of R_u are related by reflection about the upper-left to bottom-right diagonal, the final equality of this lemma follows.

$$\gamma(\text{st}(x_{\max}(R, T)|_{T^C})) = \gamma(\text{st}(\text{rp}(x_{\min}(\text{rf}\setminus(R), T^C)|_{T^C}))) \quad (3.3a)$$

$$= \text{rf}\setminus(\gamma(\text{st}(x_{\min}(\text{rf}\setminus(R), T^C)|_{T^C}))) \quad (3.3b)$$

$$= \text{rf}\setminus(R_l(\mathcal{P}(\text{rf}\setminus(R), T^C))|_l) \quad (3.3c)$$

$$= R_u(\mathcal{P}(R, T))_- \quad (3.3d)$$

□

Lemma 3.5.11 completes the proof of Theorem 3.1.11.

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