ABSTRACT

CHANDLER, KAYLA CHRISTINE. Examining How Prospective Secondary Mathematics Teachers Notice Students’ Thinking on a Paper and Pencil Task and a Technology Task. (Under the direction of Dr. Karen Hollebrands.)

Attending to student thinking is critical to effective teaching; therefore, it is important to understand how prospective teachers attempt to do so. Teacher noticing is one area of research that has shown promise in helping researchers learn about what teachers focus on when considering students’ thinking. However, little work has been done to understand how prospective secondary mathematics teachers (PSMTs) notice students’ thinking in general and how they notice while students work on tasks that utilize technology. Such an examination could help the field begin to understand how PSMTs handle the practice of making sense of students’ work through noticing and lead to insights about how to better prepare them for the classroom. Thus, this study sought to understand how PSMTs notice students’ mathematical thinking while considering a paper and pencil task and a technology task.

This exploratory multi-case study examined how four PSMTs noticed students’ thinking on two different mathematical tasks; each participant represented an individual case. Over a six-week period, each PSMT participated in two task-based interviews and considered six artifacts of practice, three from a paper and pencil task and three from a technology task, to respond to noticing prompts. Data included video recordings and transcripts of interviews, PSMT’s written responses to the noticing prompts for each artifact of practice, and researcher notes and memos. Data analysis began after the first round of interviews and was on-going throughout the project. Both within case and cross case analyses were used to consider the ways in which PSMTs were noticing on both tasks.
Research has shown examining how teachers notice can reveal some ideas about how they are making sense of students’ work. In this study, examining how the PSMTs were noticing revealed: (a) the PSMTs were focused on mathematics while noticing, (b) the PSMT’s conceptions of what it means to understand a concept and what constitutes evidence of that understanding, (c) the PSMTs had difficulty articulating what they were noticing, especially when it came to making connections with how they would respond to students’ thinking, and (d) the PSMTs, when responding, all wanted to leverage technology to have students further explore the mathematics in the technology task.
Examining How Prospective Secondary Mathematics Teachers Notice Students’ Thinking on a Paper and Pencil Task and a Technology Task

by

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DEDICATION

First, I dedicate this work to God; without Him it would not have been possible. Second, I dedicate this work to my husband, Michael, for your constant support, encouragement, and love throughout this process. Thank you for believing in me! Third, I dedicate this work to my family. Emma, I pray I will continue to be inspired by your learning and excitement for new things. Mom and Dad, thank you for instilling in me the value of education and pursuing my passion. And Mom, thanks for talking to me on my drives to and from Raleigh; Greg, thanks for letting me have her! James and Debra, thank you for encouragement and perspective as Michael and I embarked on this journey together. Finally, I dedicate this work to the colleagues I met during this time who provided guidance, support, and laughter and who I am happy to call friends. They say it takes a village to raise a child; well, I believe it takes a village to obtain your doctorate! I am forever grateful to you all.
BIOGRAPHY

Kayla Christine Chandler was born in Elizabeth City, NC on April 27, 1986. She is the daughter of Tommy Sullivan and Elizabeth Cody. She was raised in Currituck County and graduated from Currituck County High School in 2004. In May 2006, she graduated from College of the Albemarle with an Associate in Science.

In the fall of 2006, Kayla moved to Greenville, NC, enrolled at East Carolina University, and began attending Opendoor Church, where she met her husband. She graduated from East Carolina University in May 2009 with a Bachelor of Science in Mathematics Education, a Bachelor of Arts in Mathematics, and a Minor in Business Administration. In 2009, she was hired at Greene Early College High School in Snow Hill, North Carolina and also began taking graduate courses at East Carolina University. She completed her Master of Arts in Education (MAEd) in Mathematics Education at East Carolina University in July 2010. During this time, she also served as a graduate assistant for the Mathematics, Science, and Instructional Technology Education Department at East Carolina University. After completing her MAEd, she continued teaching at Greene Early College High School for two more years. During this time, she also began teaching mathematics education courses at East Carolina University part-time and married Michael Chandler on August 26, 2011.

In August 2012, she enrolled at NC State to pursue her doctoral degree in Mathematics Education. While there, Kayla served for four years as a graduate research assistant for the Preparing to Teach Mathematics with Technology (PTMT) project under the direction of Dr. Hollylynne Lee, Dr. Karen Hollebrands, and Dr. Allison McCulloch. She
also served one year as a teaching assistant teaching the *Teaching Mathematics with Technology* course.

On June 19, 2016, Mr. and Mrs. Chandler welcomed their first child, Emma Rose Chandler. At this time, Kayla accepted a position teaching mathematics at Greene Central High School while she worked to complete her dissertation. In the future, Kayla hopes to obtain a faculty position where she can continue to examine the ways in which teachers make sense of students’ thinking with both prospective and practicing teachers.
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And finally, to my husband, Michael, thank you for all of the sacrifices you made so that I could pursue my dream. Thank you for reminding me of God’s faithfulness at each step along the way and for always pushing me to give my best. I love you with all of my heart.

We made it!
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CHAPTER 1: INTRODUCTION

What does it take to be an effective classroom teacher? This question has plagued researchers for decades. This question has also been the basis for research aimed at understanding teacher knowledge and the practices of teaching. While there are many ways to measure effective teaching, such as through student understanding, engagement, and motivation, many researchers have measured teacher effectiveness by examining impacts on student achievement. However, more recently, efforts to reform mathematics teaching have considered that teacher effectiveness is practice-based. In other words, teacher effectiveness is related to the work of teaching and the various practices in which teachers engage (Ball, 2001; Ball & Cohen, 1999).

Efforts to reform mathematics teaching recommend teachers to be able to engage in specific practices. One of these practices is to understand how students are thinking about mathematics and use this information to guide instructional decisions (Association of Mathematics Teacher Educators (AMTE), 2017; National Council of Teachers of Mathematics (NCTM), 1991, 2000, 2014). In fact, NCTM (2014) considers eliciting and using evidence of students’ thinking to make instructional decisions one of the key practices of effective teaching and learning. Similarly, AMTE (2017) claims that one indicator of a well-prepared beginning teacher is analyzing mathematical thinking. Specifically, they state, “Well-prepared beginning teachers of mathematics analyze both written and oral productions related to key mathematical ideas and look for and identify sensible mathematical reasoning, even when that reasoning may be atypical or different from their own” (AMTE, 2017, p. 10).
Teachers are also expected to appropriately utilize technology to teach mathematics (Conference Board of the Mathematical Sciences (CBMS), 2012; NCTM, 2000, 2014). In particular, teachers can make use of *mathematical action technologies* (T. P. Dick & Hollebrands, 2011) which “can perform mathematical tasks and/or respond to the user’s actions in mathematically defined ways” (p. xii). In some ways, students’ externalize their thinking as they make decisions about how to interact with these tools (e.g., Arzarello et al., 1998). Therefore, technology can provide another venue for teachers’ to interpret and make sense of students’ thinking.

One way in which teachers make sense of students’ thinking is through teacher noticing (van Es & Sherin, 2002). Teacher noticing is the professional practice of “attending to particular events in an instructional setting” (Sherin, Jacobs, & Philipp, 2011, p. 5) and “making sense of events in an instructional setting” (p. 5). Mathematics teacher noticing then, is the professional practice of attending to a students’ mathematical approach to a problem, interpreting the students’ mathematical understanding, and determining how to respond based on the students’ thinking (Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2002).

The purpose of this study was to examine the ways in which prospective secondary mathematics teachers (PSMTs) were noticing students’ work on a paper and pencil task and on a technology task, as little work has been done to understand how PSMTs notice in general and how they notice when tasks utilize technology. Some have argued that technology requires students to be more explicit about their thinking because they need to interact with external objects provided on a digital device (Noss & Hoyles, 1996). Such an examination could help the field begin to understand how PSMTs handle the practice of
making sense of students’ work through noticing and lead to insights about how to better prepare them for the classroom. Research has suggested that teachers who engage in noticing students’ mathematical thinking are better able to make sense of students’ work and their understandings (Jacobs et al., 2010; Leatham, Peterson, Stockero, & Van Zoest, 2015). Thus, the practice of noticing students’ mathematical thinking is an important practice for effective teaching as it enables teachers to make sense of students’ work.

**Definition of Terms**

Prior to discussing the purpose of the study and research questions, it is necessary to define several terms used in this study to ensure they are consistently interpreted throughout.

**Artifact**
- Artifacts of practice – concrete representations of materials, examples, or incidents that make the practices of teaching available for teachers’ examination (e.g., samples of student work and video examples; Ball & Cohen, 1999)

**Mathematics teacher noticing**
- Mathematics teacher noticing – the professional practice of attending to a student’s mathematical approach to a problem, interpreting the student’s mathematical understanding, and determining how to respond based on the student’s thinking (Jacobs et al., 2010; van Es & Sherin, 2002)

**Prospective teacher**
- Prospective teacher – student pursuing a teaching license through a teacher education program at an institution

**Prospective mathematics teacher (PMT)**
- Prospective mathematics teacher (PMT) – student pursuing a teaching license in mathematics through a teacher education program at an institution

**Prospective secondary mathematics teacher (PSMT)**
- Prospective secondary mathematics teacher (PSMT) – student pursuing a teaching license in mathematics for grades 9 – 12 through a teacher education program at an institution

**Student**
- Student – a student in grades K – 12
Teacher noticing – the professional practice of “attending to particular events in an instructional setting” (Sherin et al., 2011, p. 5) and “making sense of events in an instructional setting” (p. 5)

**Purpose of the Study and Research Questions**

“No task is more fundamental to teaching than figuring out what students are learning. Paradoxically, no endeavour [*sic*] is more difficult” (Ball, 1997b, p. 769).

Because making sense of students’ work is critical to effective teaching, it is important to understand how prospective teachers attempt to do so. Teacher noticing is one area of research that has shown promise in helping researchers learn about how teachers, both prospective and in-service, make sense of students’ work (e.g., Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2008). However, there is a gap in the literature regarding how prospective secondary mathematics teachers notice; rather, much of the work in this area has been focused on how prospective elementary teachers notice students’ mathematical thinking. Also, little work has been done to understand how teachers make sense of students’ work while students complete tasks that utilize technology (Wilson, Lee, & Hollebrands, 2011). In other words, there is very little literature that considers how secondary mathematics teachers notice when the students are working on a task that uses technology. Therefore, engaging prospective secondary mathematics teachers (PSMTs) in the practice of noticing on tasks where students do not use technology, and also in activities where students are using technology, could help the field begin to understand how PSMTs handle the practice of making sense of students’ work and lead to insights about how to better prepare them for the classroom. Thus, this study sought to understand how PSMT’s
notice students’ thinking while considering mathematical tasks that do and do not use technology. As such, the following research questions were answered:

1. How do PSMTs notice students’ mathematical thinking when they examine artifacts of practice from a paper and pencil mathematics task?
2. How do PSMTs notice students’ mathematical thinking when they examine artifacts of practice from a technology task?
3. What similarities and differences exist in the ways PSMTs notice students’ mathematical thinking in the two different types of tasks?

**Overview of Methodological Approach**

To answer the research questions, this qualitative study utilized an exploratory multi-case study approach (Yin, 2009). In particular, four PSMTs enrolled in a secondary mathematics education program at a Southeastern University in the fall of 2015 were selected as cases for this study; each PSMT represented an individual case. Over a six-week period, each PSMT participated in two task-based interviews and considered six artifacts of practice, including student work and video clips, three from a paper and pencil task and three from a technology task, to respond to noticing prompts. Data included video recordings and transcripts of interviews, PSMT’s written responses to the noticing prompts for each artifact of practice, and researcher notes and memos. Data analysis began after the first round of interviews and was on-going throughout the project. Both within case and cross case analyses were used to consider the ways in which PSMTs were noticing on both tasks.
Organization of the Dissertation

This dissertation is organized into six chapters. In Chapter 2, the study is situated within the existing body of literature by providing an account of how teacher effectiveness has been considered over time, how this led to considering teacher knowledge, and ultimately to understanding how teachers make sense of students’ thinking through noticing. In this chapter, the framework for the study is also described. Chapter 3 includes a refined version of the research questions to consider each aspect of noticing (i.e., attending, interpreting, and responding) and a detailed account of the methodology for the study, including the context, research design, and data collection and analysis procedures. Chapter 4 includes an individual case analysis while Chapter 5 includes the cross case analysis. In Chapter 6, a discussion of the findings, limitations, and implications of the study are presented.
CHAPTER 2: LITERATURE REVIEW

The purpose of this chapter is to situate the study within the larger body of existing research. The chapter details a historical perspective of research about teaching effectiveness, which has proven to be multifaceted and the foundation for understanding the practices of teaching. One important piece of teaching effectiveness includes the various components of the knowledge needed for teaching mathematics. Within this body of knowledge are skills and practices that are central to teaching mathematics, such as making sense of students’ work. And, one way researchers have investigated how teachers make sense of students’ work is through teacher noticing. Thus, the literature surrounding these topics is considered in this chapter.

Early Research on Teaching Effectiveness

What does it take to be an effective classroom teacher? This question has plagued researchers for decades. This question has also been the basis for research aimed at understanding teacher knowledge and the practices of teaching. While there are many ways to measure effective teaching, such as through student understanding, engagement, and motivation, many researchers have measured teacher effectiveness by examining impacts on student achievement. In fact, this idea has been explored from at least three different perspectives within mathematics education research, each representing a phase of research on teaching effectiveness (Ball, 1991; Charalambous, 2008). Although work began much earlier, the first perspective, production-function studies, gained popularity in the 1960s as researchers began examining what characteristics (e.g., number of mathematics courses, scores on certification exams) might make teachers effective (Begle, 1979; Coleman et al., 1966). Similarly, work on the second perspective, process-product studies, began in the
1950s but did not gain attention until the 1970s (Rosenshine, 1979). It was during this time that research shifted to study teacher behaviors (e.g., questioning, time spent on task) and their relation to student performance (Brophy, 1986; Medley, 1979; Rosenshine, 1979). However, as early as the 1980s, some researchers began to consider teachers’ thinking and decisions (i.e., cognitive processes and orientations) as a way to understand their practice (Aubrey, 1997; Shroyer, 1981; Thompson, 1984). Each of these phases of research was important in the development of research that was foundational in establishing the importance of teacher knowledge, so it is important to highlight the findings and limitations of each in order to understand how this line of research has evolved.

**Production-function studies.** Based on the assumption that the more a teacher knows, the more effectively he or she will teach, researchers in the 1960s began exploring potential connections between teacher characteristics and student performance, an area of research known today as *production-function studies* (Charalambous, 2008). In particular, researchers considered if proxy variables, such as the number of courses taken, years of teaching experience, scores on certification exams, and philosophical orientations, influenced student outcomes (Hill, Sleep, Lewis, & Ball, 2007). Among the seminal reports in this area were the Coleman report (Coleman et al., 1966), *Equality of Educational Opportunity*, and a meta-analysis of empirical works by Begle (1979).

Coleman and colleagues (1966) used a multiple-choice questionnaire to obtain scores for teacher knowledge and used these scores to predict student performance in reading and mathematics. While they found that there was a relationship between teachers’ scores and student achievement, some researchers questioned the validity of these findings
since the instrument used did not include items specifically focused on mathematics (Hill et al., 2007).

Begle (1979) provided a review of empirical work on teacher effectiveness for both teaching in general and in mathematics. From the general studies, Begle noted that the majority of research had been completed at the elementary level and that variables and procedures used to measure teacher effectiveness varied greatly across the research. Overall, correlations between measures of teacher effectiveness and student achievement were low for the non-subject specific studies. Begle also discussed the findings of the National Longitudinal Study of Mathematical Abilities. In this study, 112,000 students of all grade levels from over forty states were followed in an effort to identify teacher characteristics associated with student performance (Ball, 1991, p. 2). However, the researchers found that teacher characteristics were not strongly related to student achievement (Ball, 1991; Begle, 1979). In fact, the commonly held belief that the more a teacher knows, the more effectively one can teach was challenged by this study’s findings. For example, the number of credits in mathematics methods courses had a significant positive relationship to student achievement only 24% of the time, and 6% of the time this relationship had negative effects (Begle, 1979, p. 43). From these findings, Begle (1979) concluded:

These numerous studies have provided us no promising leads. We are no nearer any answers to questions about teacher effectiveness than our predecessors were some generations ago. What is worse, no promising lines of further research have been opened up. Evidently our attempts to improve mathematics education would not profit from further studies of teachers and their characteristics. Our efforts should be pointed in other directions. (p. 54-55)
Although many educational production-function studies were conducted, the validity of findings claiming a relationship between teacher characteristics and student achievement have been questioned by many researchers (Ball, 1991; Begle, 1979; Hill et al., 2007). Some of the practices included using instruments to measure teacher knowledge that did not directly focus on mathematics, assuming it is reasonable to measure a teachers’ knowledge by proxy variables, and a lack of considering classroom practice (Hill et al., 2007). As a result of such criticisms and findings that teacher characteristics are not related to student achievement, research shifted to explore another perspective.

**Process-product studies.** Hoping to find what sets effective teachers apart, researchers began exploring teacher behaviors during instruction (Medley, 1979). Researchers considered relationships between behavior (process) and student achievement (product), hence the name *process-product studies* (Charalambous, 2008; Medley, 1979). Common behaviors studied were generic and included questioning, time spent on task, praise and feedback given to students, and the approach to presenting instructional material (Ball, 1991; Medley, 1979; Rosenshine, 1979). Further, most of the studies in this area of research focused on mathematics and reading at the elementary level (Ball, 1991). Overall, these studies found that student achievement was positively impacted when instruction was presented in an orderly and structured environment that was teacher-centered (Brophy, 1986; Medley, 1979; Rosenshine, 1979).

As a result of this work, researchers identified behaviors associated with effective teaching, where effectiveness was measured by gains in student performance on summative assessments (Medley, 1979; Rosenshine, 1979). For example, researchers found: classes of an effective teacher were organized as a large group with little time devoted to independent
work or small groups; effective teachers asked factual, low-level questions and provided immediate feedback to students; and effective teachers presented content in a carefully sequenced manner and in a lecture format (Brophy, 1986; Medley, 1979; Rosenshine, 1979).

However, some researchers have disputed the findings of this line of research pointing out that the research was narrowly focused because it mostly considered elementary teachers and did not aim to address issues related to teachers’ lack of knowledge in mathematics (e.g., Aubrey, 1997; Confrey, 1986). Aubrey (1997) noted that the researchers of process-product studies were actually educational psychologists and had not typically been trained as educators. Thus, Aubrey argued that this could be why these researchers did not consider the content being presented as important, but instead focused on how instruction was delivered. Further, Confrey (1986) questioned the discouragement of using cognitively demanding instruction and challenged process-product researchers to address the fact that not all students learn in the prescribed format. Still, research in this area persisted, and for some, as they continued to spend time observing classrooms and analyzing data, they “became increasingly appreciative of the complexity of classrooms and of the job of teaching” (Ball, 1991, p. 4). As more studies were completed, it became evident that finding a prescriptive approach to effective teaching might not be realistic (Ball, 1991). This realization led to yet another shift in research.

**Research on teacher thinking and decisions.** As researchers began to appreciate the complexities of classrooms and teaching, they turned their focus away from behaviors of teachers to understanding teachers’ thinking and decisions. Clark and Yinger (1979) summarize this shift nicely:
A relatively new approach to the study of teaching assumes that what teachers do is affected by what they think. This approach, which emphasizes the processing of cognitive information, is concerned with teachers’ judgment, decision-making, and planning. The study of the thinking processes of teachers—how they gather, organize, interpret, and evaluate information—is expected to lead to understandings of the uniquely human processes that guide and determine that behavior. (p. 231)

From this perspective, researchers began thinking of teaching as the practices associated with “thought and decision-making which takes place before, during and after interactions with children” (Aubrey, 1997, p. 15). Thus, researchers began to focus on why teachers were enacting the behaviors witnessed in classrooms and the knowledge teachers used to make those decisions (Aubrey, 1997; Ball, 1991). As a result, studies on teacher thinking and decision making in specific subject areas became important as researchers began to consider the role of subject matter knowledge in teaching (Aubrey, 1997).

Thompson (1984) explored how three middle school teachers’ conceptions (i.e., their beliefs, views, and preferences; p. 105) about mathematics and teaching affected their practice. From the study, Thompson found that two teachers had very different conceptions while one teacher fell somewhere in between. Thompson based his conclusions about the relationship between teacher knowledge and practice on these two teachers who represented different ends of the spectrum. One of these teachers, Lynn, viewed mathematics as a procedural subject where knowledge is transferred from the teacher to the student; her teaching reflected this view as she focused on computations and did not encourage students to think conceptually. On the other hand, Kay was a reflective practitioner who believed students needed to engage in discovering mathematics and making connections in order to
learn the subject. Consistent with this belief, Kay’s classroom was one that encouraged exploration and emphasized problem solving. Thus, Thompson found that what the teachers believed about mathematics and the teaching of mathematics influenced their instruction. Further, Thompson concluded that the relationship between teachers’ conceptions and their instructional practice “is a complex one” (p. 124).

Another study considered critical moments in upper elementary mathematics classrooms (Shroyer, 1981). Shroyer (1981) found that when unexpected student difficulties arose, teachers engaged in four cognitive processes to determine how to proceed: interpreting, goal setting, searching for elective actions, and evaluating the effectiveness of the actions (p. 248-249). Thus, teachers attempted to interpret the situation, assessed how it fit within the lesson’s goals, and considered how to respond. Once the teacher responded, he/she evaluated the effectiveness of the approach to the problem. However, Shroyer found that in some instances, teachers did not know how to respond to student difficulties due to a lack of subject matter knowledge. In fact, Shroyer concluded that mathematical knowledge played an important role in each of the cognitive processes employed by teachers when unexpected moments occurred.

By examining the thoughts and decisions of teachers, studies such as Thompson (1984) and Shroyer (1981) highlighted the idea that teachers’ conceptions and subject matter knowledge are important factors in understanding effective mathematics teaching. In response to findings in this area of research, Shulman (1986) challenged researchers to continue to explore connections between teacher knowledge and teaching and to re-conceptualize teacher knowledge. Shulman’s revolutionary ideas laid the foundation for the conceptualization of teacher knowledge, as we know it today.
Teacher Knowledge

During his Presidential Address at the 1985 annual meeting of the American Educational Research Association, Lee Shulman shared his thoughts on teacher knowledge (Shulman, 1986). Shulman (1986) recounted the history of research on teacher knowledge and argued that a focus on content was needed in the literature. Shulman noted that researchers should be striving to understand how teachers handle the practice of teaching and how they acquire teacher knowledge. In other words, Shulman argued that researchers should focus on the integration of content knowledge and pedagogical knowledge and not just on the processes of teaching.

Shulman went on to describe three categories of content knowledge: subject matter knowledge, pedagogical content knowledge, and curricular knowledge. Subject matter knowledge refers to the knowledge a teacher has about the content of the subject and the structure of the subject as well as knowledge to explain propositions within the subject and why they are important. Pedagogical content knowledge includes the subject matter knowledge needed to teach, such as the most appropriate approach to teaching a particular topic, which examples to use and why, and an understanding of how students approach the topic (e.g., student strategies, student difficulties, etc.). Curricular knowledge refers to a teachers’ knowledge of the organization of the subject’s curriculum both within and across grade levels as well as knowledge of appropriate materials for teaching.

Another notable aspect of Shulman’s address was his vision for assessments of teacher knowledge. According to Shulman, such an instrument would:

Measure deep knowledge of the content and structures of a subject matter, the subject and topic-specific pedagogical knowledge associated with the subject matter,
and the curricular knowledge of the subject….It would not be a mere subject matter examination. It would ask questions about the most likely misunderstandings of…[particular topics] and the strategies most likely to be useful in overcoming those difficulties….It would be much tougher than any current examination for teachers. (p. 10)

Shulman’s thoughts on teacher knowledge, particularly those of pedagogical content knowledge, and what assessments of teacher knowledge should look like revolutionized research on teacher knowledge, particularly content knowledge and pedagogical content knowledge. Following his remarks, mathematics education researchers began investigating teachers’ content knowledge for a variety of subject specific topics, such as operations with fractional numbers (e.g., Ball, 1990) and proportional reasoning (e.g., Sowder, Philipp, Armstrong, & Schappelle, 1998). Overall, these studies provided empirical evidence to support Shulman’s argument that teachers need a deep understanding of the subject matter they are teaching as well as pedagogical content knowledge regarding how to appropriately support student learning (Charalambous, 2008).

However, these studies also raised some questions about the actual practice of teaching mathematics (Charalambous, 2008). For example, how does teacher knowledge inform responses to students’ questions or misunderstandings, or how does teacher knowledge influence a teacher’s instructional decisions, were some of the questions left unanswered by previous research. As a result, Ball, Lubienski, and Mewborn (2001) aimed to shift the focus of mathematics education research from “one about…what teachers know [emphasis added] to one about teaching and what it takes [emphasis added] to teach” (p. 452).
Mathematical knowledge for teaching. In an effort to better understand the knowledge necessary for the work of teaching mathematics and how this knowledge is developed, Ball and her colleagues launched the *Mathematics Teaching and Learning to Teach* (MTLT) project and the *Learning Mathematics for Teaching* (LMT) project (Ball, Thames, & Phelps, 2008). Through these projects, the researchers built on Shulman’s work to further conceptualize teacher knowledge. In particular, the MTLT project “focused on the work teachers do in teaching mathematics” (Ball et al., 2008, p. 390), and the LMT project developed instruments to measure content knowledge for teaching mathematics.

From the work surrounding these projects, Ball and her colleagues developed a subject-specific and practice-based theory, known as the *mathematical knowledge for teaching* (Ball et al., 2008). The mathematical knowledge for teaching (MKT) framework, as shown in Figure 1, builds off of Shulman’s (1986) work by further breaking down subject matter knowledge and pedagogical knowledge into sub-domains. According to Ball and colleagues (2008), subject matter knowledge (SMK) consists of common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). Pedagogical content knowledge (PCK) contains knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC).
Common content knowledge is defined as “the mathematical knowledge and skill used in settings other than teaching” (Ball et al., 2008, p. 399). This means teachers should be able to do the mathematics they teach and recognize errors in student responses or curriculum materials. Teachers must also use appropriate language and terminology associated with the content they are teaching. Some examples of situations requiring CCK include a teacher knowing that a square is a rectangle, how to calculate the derivative of a function, and that there are two possible definitions of a trapezoid, each of which affects a hierarchical classification of quadrilaterals.

Specialized content knowledge refers to mathematical knowledge that is specific to the practice of teaching (Ball et al., 2008). SCK includes tasks such as selecting appropriate examples, providing mathematical explanations, and being able to analyze common errors. Ball and colleagues (2008) describe this knowledge as being “unpacked” in such a way that the teacher can make “features of particular content visible to and learnable by students” (p.
400). For example, SCK is required to understand the advantages and disadvantages of using geometric versus algebraic representations to develop the concept of area.

“Horizon knowledge is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al., 2008, p. 403). An example of HCK would include knowing that multiplying polynomials is connected to the distributive property.

Knowledge of content and students includes knowledge about students and mathematics (Ball et al., 2008). KCS includes skills such as anticipating common approaches and difficulties students will have for particular problems. For instance, knowing that students have difficulty identifying the hypotenuse of a right triangle requires KCS. Interpreting students’ thinking is also another skill of KCS; in this case, teachers must use knowledge of both the content and students to make sense of thinking as expressed by students.

Knowledge of content and teaching includes knowledge about teaching and mathematics (Ball et al., 2008). Knowing how to organize an instructional unit and select appropriate examples that build upon one another are skills that require KCT. KCT also helps the teacher make decisions about whether or not to pursue an unexpected line of student thinking or if it would be more appropriate to explore at a later time. An example of a task that utilizes KCT is selecting and sequencing problems that would allow you to have a class discussion about multiple strategies to simplify rational expressions.

Knowledge of content and curriculum includes knowledge about mathematics and the structure of the curriculum (Ball et al., 2008). This includes the idea that teachers understand within and across grade level connections in the curriculum. In this sense,
algebra teachers would know how the topics they are teaching are related to ideas in
calculus and would use this knowledge to build a foundation students can build from later.
Further, KCC also contains knowledge of instructional materials available and how these
materials relate to the curriculum.

**Research on mathematical knowledge for teaching.** Since the development of the
MKT framework, Ball and colleagues have worked closely with practicing teachers,
mathematics educators, psychometricians, and other experts to develop instruments to
measure MKT, specifically in the CCK and SCK domains, of elementary teachers in three
areas: numbers and operations; patterns, functions, and algebra; and geometry
(Charalambous, 2008, p. 45). More recently, the group has been working to develop items to
assess teachers’ KCS and KCT and to extend their work to other mathematics topics and
grade levels (i.e., middle grades) (Charalambous, 2008, p. 45). From this work, Ball and
colleagues have been able to collect empirical evidence to support the notion that teachers’
MKT is multidimensional and related to students’ performance (e.g., Hill, Rowan, & Ball,
2005). Studies in this area have also shown that professional development can increase
MKT (e.g., Hill & Ball, 2004). Thus, the work of Ball and her colleagues transformed
conceptions about teacher knowledge and what it means to teach.

**Technological pedagogical content knowledge.** Another branch of teacher
knowledge that built off of Shulman’s (1986) work describing PCK is centered around
*technological pedagogical content knowledge* (TPACK, previously TPCK; Koehler &
Mishra, 2005). Koehler and Mishra (2005) developed the TPACK framework to explore
“teachers’ understanding of the complex interplay between technology, content, and
pedagogy” (p. 132). Just as teachers are expected to have content knowledge and
pedagogical knowledge, Koehler and Mishra argue that in order for technology to effectively be used in the classroom, teachers must have an understanding of the tool and its relationship to practice. Niess (2006) took the argument a step further by stating that teachers need TPACK to be prepared to teach mathematics according to standards set by the National Council of Teachers of Mathematics (2000), the National Research Council (2001) and the International Society for Technology in Education (2000, 2002).

Using these ideas, the TPACK framework was developed to include three areas of knowledge: content, pedagogy, and technology. The domains of content and pedagogy are aligned with Shulman’s (1986) work; content knowledge refers to knowledge of the subject matter that is taught and pedagogical knowledge comprises the practices and methods of teaching. Technological knowledge includes knowledge of classroom technologies (e.g., document cameras, calculators) and general technologies that might be used in classrooms (e.g., computers, videos). However, the central aspect of the TPACK framework is not just the inclusion of three domains of knowledge, but on the relationships between them, as seen in Figure 2.

![Figure 2. Technological Pedagogical Content Knowledge. This figure represents TPACK as described by Mishra and Koehler, 2006, p. 1025.](image)
Here, the overlapping region between content (C) and pedagogy (P) represents pedagogical content knowledge (PCK), which includes a subject specific understanding of various approaches to delivering instruction as well as knowledge about how students learn the subject. The overlapping region between content (C) and technology (T) represents technological content knowledge (TCK) which is “useful for describing teachers’ knowledge of how a subject matter is transformed by the application of technology” (Koehler & Mishra, 2005, p. 134). An example of TCK would be a teachers’ understanding of how the concept of probability can be developed through the use of simulations. The overlapping region between technology (T) and pedagogy (P) represents technological pedagogical knowledge (TPK), which refers to knowledge about “how technology can support pedagogical goals” (Koehler & Mishra, 2005, p. 134). Knowing that a particular software program facilitates students’ problem solving would be an example of TPK. And, the very center of all three overlapping regions, technological pedagogical content knowledge, represents the interaction of all three components. Mishra and Koehler (2006) described TPACK well:

[It] is the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students’ prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones. (p. 1029).
Mishra and Koehler went on to point out that effectively integrating technology into instruction requires the teacher to consider the “complex relationships” between content, pedagogy, and technology (p.1029).

Niess (2006) also contributed to research related to TPACK. Niess agreed with Mishra and Koehler (2006) that an understanding of content, pedagogy, and technology are all needed to effectively implement technology in the mathematics classroom. However, Niess took the argument a step further by stating that teachers need TPACK to be prepared to teach mathematics according to standards set by the National Council of Teachers of Mathematics (2000), the National Research Council (2001) and the International Society for Technology in Education (2000, 2002). Thus, Niess (Niess, 2005) provides four areas of focus for helping teacher educators develop teachers’ TPACK:

1. an overarching conception of what it means to teach a particular subject integrating technology in the learning; 2. knowledge of instructional strategies and representations for teaching particular topics with technology; 3. knowledge of students’ understandings, thinking, and learning with technology in a particular subject; 4. knowledge of curriculum materials that integrate technology with learning in the subject area. (p. 511)

**Research on teachers’ technological pedagogical content knowledge.** Since the development of the TPACK framework, researchers have explored teachers’ TPACK in a variety of ways. For example, Lee and Hollebrands (2008) used materials specifically created to develop teachers’ understandings of content, technology, and pedagogy in an integrated manner and to explore the impact such efforts had on developing prospective teachers’ TPACK. Using a pre/post instrument to assess each component of TPACK, Lee
and Hollebrands found that the materials did have positive effects on prospective teachers’ TPACK.

Another study by Özgün-Koca, Meagher, and Edwards (2010) explored the development of secondary prospective teachers’ TPACK as they designed and implemented technology-based lessons. They found that as a result of participating in these activities, prospective teachers experienced two shifts: one from thinking about technology as a way to reinforce mathematics to seeing technology as a tool to help develop mathematical ideas, and the other from a learner of mathematics to a teacher of mathematics.

Despite the work that has been done, researchers have struggled to measure teachers’ TPACK (e.g., Lee & Hollebrands, 2008). As a result, a group of researchers have developed and validated an instrument to measure some components of prospective secondary mathematics teachers’ TPACK, namely TK, PK, CK, and TPACK (Zelkowski, Gleason, Cox, & Bismark, 2013). However, measuring PCK, TPK, and TCK proved more difficult and results were not found to be valid. The researchers also noted that generating self-reports of knowledge in these areas is difficult for prospective teachers. Yet, they were among the first to develop an instrument for measuring mathematics teachers’ TPACK. Other researchers have utilized qualitative methods to examine portions of TPACK, such as PCK, by focusing on a particular practice. One practice that is often difficult for teachers to utilize in the classroom involves making sense of students’ work.

Making Sense of Students’ Work

The practice of making sense of students’ work is challenging for teachers (Ball, 2001; Fennema, Carpenter, & Franke, 1992; Schifter, 2001). In fact, Ball (1997a) argues, “no matter what age one’s students are or what topic one’s curriculum presents, probing
students’ thinking is an inherently complex task” (p. 733). Determining what students think is not always easy since this requires teachers to “enter the child’s mind” (Ginsburg, 1997). Further, within any mathematics classroom, students may approach problems in a variety of, and sometimes unexpected, ways (Ball, 2001; Schifter, 2001). Students’ thinking does not always make sense to adults, so determining if there is mathematical validity to a student’s approach is often difficult (Jacobs & Philipp, 2010; Warfield, 2001). Attempts to understand students’ work are further complicated because teachers lack a cohesive knowledge structure about student thinking which limits their ability to make instructional decisions (Carpenter et al., 1988; Fennema et al., 1992).

Researchers agree that making sense of students’ work is a practice that requires a teacher to draw upon his/her MKT, specifically KCS, a subset of PCK (Ball & Cohen, 1999; Carpenter, Fennema, & Franke, 1996; Niess, 2005). And, when the students’ work includes a task that utilizes technology, a teacher must utilize his/her TPACK to make sense of the students’ work (Niess, 2005). Thus, teachers need a deep knowledge of mathematical content as well as an understanding of how students think about and learn this content (Ball et al., 2008; Carpenter et al., 1996; Shulman, 1986). Further, the knowledge teachers require to adequately make sense of students’ work may be different from how they have previously experienced the teaching and learning of mathematics (Ball & Cohen, 1999). Thus, there is consensus that teachers need pedagogical content knowledge which can inform decisions that promote student learning (Ball & Cohen, 1999; Feiman-Nemser, 2001; Shulman, 1986). In fact, Feiman-Nemser (2001) argues “to teach in ways that are responsive to students’ thinking, [teachers] must also learn how to elicit and interpret students’ ideas and to generate appropriate pedagogical moves as a lesson unfolds” (p. 1028). Because of the
influence of teachers’ knowledge on instruction, researchers hypothesize that teaching and learning mathematics can be improved by cultivating teachers’ knowledge of how students think about specific concepts (Carpenter et al., 1996; Feiman-Nemser, 2001; Hill, Ball, & Schilling, 2008). Thus, numerous studies emerged that were aimed at improving teacher knowledge to explore the impact on how teachers make sense of students’ work (e.g. Barnett, 1998; Carpenter et al., 1996, 1988; Davies & Walker, 2005; Kazemi & Franke, 2004).

**Cognitively guided instruction.** One research team in particular explored how exposing teachers to research on students’ thinking of particular mathematical topics impacted the teacher’s practice of making sense of students’ work. Members of the Cognitively Guided Instruction (CGI) program explored how elementary teachers assessed and predicted students’ thinking and compared their findings with research to create a framework for understanding and interpreting students’ thinking (e.g. Carpenter et al., 1988). From this work, a research-based model of children’s thinking was developed by the CGI program and shared with its participants (Carpenter et al., 1996; Fennema et al., 1992; Franke & Kazemi, 2001). In at least one experiment, CGI teachers were exposed to research about how children develop particular concepts (i.e., addition and subtraction) and trained on how to apply this knowledge in the classroom when making instructional decisions (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989).

From this work, researchers started to realize that CGI teachers attempted to attend to individual students’ thinking by focusing on interactions with students (Fennema et al., 1992). In fact, Kazemi and Franke (2004) note that CGI teachers “began to notice that the student work did not speak for itself” (p. 217), thus requiring more effort to draw out what
the students were thinking. Other researchers who had been utilizing student work as a means of exploring student thinking also noted similar findings (Schifter, 2001; Sleep & Boerst, 2012). This recognition prompted teachers to make sense of students’ work in new ways, such as explicitly asking students to explain their thinking (Kazemi & Franke, 2004) and taking notes of individual student’s thinking during instructional time and interviews (Kazemi & Franke, 2004; Schifter, 2001; Sleep & Boerst, 2012). Other researchers encouraged teachers to focus on communicating with students through journaling (Gordon & Macinnis, 1993) or letter exchanges (Crespo, 2000).

**Teacher Noticing**

Through the recognition of the need for teachers to focus on making sense of students’ work, another branch of research emerged. In research on teacher noticing, the focus is on how teachers handle in-the-moment interactions with students. Although teacher noticing can encompass many observable actions in the classroom, researchers agree that teacher noticing includes “attending to particular events in an instructional setting” (Sherin et al., 2011, p. 5) and “making sense of events in an instructional setting” (p. 5). Also, although each teacher notices different things based on his/her experiences, beliefs and environment, research has shown teachers can improve their practice of noticing (Jacobs, Lamb, Philipp, & Schappelle, 2011; Sherin & Han, 2004; Sherin et al., 2011; van Es & Sherin, 2008). According to van Es and Sherin (2002), learning to notice involves: (a) identifying what is important in a teaching situation, (b) making connections between specific events and broader principles of teaching and learning, and (c) using what one knows about the context to reason about a situation.
Early efforts in mathematics education research related to teacher noticing revealed that when teachers were asked to notice, they did not start out focusing on students’ mathematical thinking (Jacobs et al., 2011; Sherin & Han, 2004; Sherin et al., 2011; van Es & Sherin, 2008). Rather, they began by making general, non-mathematical observations, such as how many times students were out of their seats (Crespo, 2000; Kazemi & Franke, 2004; van Es & Sherin, 2008). Research with PMTs has noted that PMTs had a difficult time providing evidence to support claims of student understanding and that PMTs struggled to explain how their decisions of how to respond connected back to students’ thinking (Simpson & Haltiwanger, 2016; Tyminski, Land, Drake, Zambak, & Simpson, 2014). However, through targeted efforts to engage teachers in the practice of noticing students’ mathematical thinking, such as through video clubs focused on learning to notice, researchers found that teachers, over time, were better able to attend to students’ thinking and make sense of students’ mathematical understandings (Fennema et al., 1996; Jacobs et al., 2010; Stockero, 2014). Yet, much of the work in this area has utilized interventions to examine if teachers can improve in their practice of noticing. Interventions have mainly included having teachers consider students’ written work and the use of video clubs where teachers examine, discuss, and analyze actual mathematics lessons (Schack et al., 2013; Stockero, 2014; van Es & Sherin, 2008). During the intervention, teachers learned how to notice through the guidance of a researcher asking questions aimed at improving this skill. Further, these studies often included groups of teachers who were learning to notice together. Thus, this work has mostly centered on using video examples in professional development or teacher education courses to help prospective and practicing teachers begin
to realize what they *should* notice with respect to the teaching and learning of mathematics (Jacobs et al., 2011; Sherin & Han, 2004; Sherin et al., 2011; van Es & Sherin, 2008).

Narrowing the scope even more, Jacobs, Lamb, and Philipp (2010) examined what they refer to as a “specialized type of noticing” (p. 172) which they call “professional noticing of children’s mathematical thinking” (p. 172, emphasis in original). Professional noticing of children’s mathematical thinking encompasses “three interrelated skills: attending to children’s strategies, interpreting children’s understandings, and deciding how to respond on the basis of children’s understandings” (p. 172). Thus, they were able to explore and compare how a range of teachers (prospective through very experienced) attended to students’ thinking both prior to and after sustained professional development (Jacobs et al., 2010, 2011).

Building on previous work, researchers have more recently begun to capitalize on the idea of teacher noticing to examine the potential impact of teachers’ in-the-moment decisions on student learning (Davies & Walker, 2005; Davis, 1997; Leatham et al., 2015; Stockero & Van Zoest, 2013; Thames & Ball, 2013). Such in-the-moment opportunities have been referred to by many names: “potentially powerful learning opportunities” (Davis, 1997, p. 360), “significant mathematical instances” (Davies & Walker, 2005, p. 275), “crucial mathematical hinge moment[s]” (Thames & Ball, 2013, p. 31), and “pivotal teaching moment[s]” (Stockero & Van Zoest, 2013, p. 127). Similarly, Leatham and his colleagues (2015) have identified three critical characteristics such moments must possess: student thinking, significant mathematics, and pedagogical opportunities. Thus, Leatham and his colleagues refer to these important instances as “Mathematically Significant Pedagogical Opportunities to build on Student Thinking (MOSTs)” (p. 90). In each of these
cases, researchers are attempting to develop frameworks for “helping teachers notice and learn to build on high-leverage student ideas to enhance [students’] mathematical understanding” (Stockero & Van Zoest, 2013, p. 127). The hope is that by utilizing such frameworks, we can begin to improve teacher education and better understand which moments, when acted upon properly, provide the greatest impact on student learning (Leatham et al., 2015; Stockero & Van Zoest, 2013).

It is also important to note that the researcher was only able to locate one study that considered PMT’s noticing on a technology task (Wilson et al., 2011). In this study, the researchers considered how PMT’s made sense of students’ work on statistical problems where the students were using TinkerPlots, a dynamic statistical program. They found that through examining student work and a video clip, that PMTs made sense of students’ work by describing what the students did on the task, comparing students actions with their own, inferring about student thinking, and restructuring their own knowledge. Thus, a review of the literature revealed a gap in understanding how PSMTs notice students’ thinking and in noticing on technology tasks.

**Framework**

The professional noticing of children’s mathematical thinking framework by Jacobs, Lamb, and Philipp (2010) was used to guide this study. In this framework, Jacobs and her colleagues proposed that teachers could make sense of and learn from children’s thinking by attending to their strategies, interpreting their understandings, and then deciding how they should respond based on the children’s understandings. According to the framework, attending focuses on how teachers “attend to a particular aspect of instructional situations; the mathematical details in children’s strategies” (p. 172). The interpreting piece of the
framework is an attempt to examine “the extent to which the teacher’s reasoning is consistent with both the details of the specific child’s strategies and the research on children’s mathematical development” (p. 172). Jacobs and colleagues were careful to note that they did not expect a teacher to fully grasp the child’s mathematical understandings on the basis of a single problem. The last piece, responding, is aimed at determining “the extent to which teachers use what they have learned about the children’s understandings form the specific situation and whether their reasoning is consistent with the research on children’s mathematical development” (p. 173). Thus, the goal is not to determine the single best response, nor is the goal to enact the response. Rather, the intention is to consider how potential responses connect back to students’ thinking from the problem. Finally, Jacobs and colleagues note that the components of the framework do not necessarily occur in seclusion but are interrelated. In other words, how a teacher chooses to respond is often the result of analyzing what the student did and potentially understands. The professional noticing of children’s mathematical thinking framework was an appropriate choice for this study because it allowed the researcher to examine how individual PSMTs were noticing students’ thinking.
CHAPTER 3: METHODOLOGY

As previously noted, the goal of this study was to explore how prospective secondary mathematics teachers (PSMTs) notice students’ mathematical thinking. Specifically, this study explored the ways in which four PSMTs noticed students’ mathematical thinking while considering artifacts from a paper and pencil task and a technology task. PSMTs were given these artifacts of practice from the two different geometry tasks, centered on the same mathematical idea, with the purpose of examining to what features of the artifacts they attended and what inferences about students’ mathematical thinking were made. Based on the professional noticing of children’s mathematical thinking framework (Jacobs et al., 2010), the framework for this study, the research questions were refined to include specific actions associated with noticing students’ mathematical thinking. Thus, this study considered how PSMTs noticed students’ thinking by answering the following research questions.

1. How do PSMTs notice students’ mathematical thinking when they examine artifacts of practice from a paper and pencil mathematics task?
   a. In what ways do PSMTs attend to students’ mathematical thinking?
   b. In what ways do PSMTs interpret students’ mathematical thinking?
   c. In what ways do PSMTs decide to respond to students’ mathematical thinking?

2. How do PSMTs notice students’ mathematical thinking when they examine artifacts of practice from a technology task?
   a. In what ways do PSMTs attend to students’ mathematical thinking?
   b. In what ways do PSMTs interpret students’ mathematical thinking?
c. In what ways do PSMTs decide to respond to students’ mathematical thinking?

3. What similarities and differences exist in the ways PSMTs notice students’ mathematical thinking in the two different types of tasks?

To answer these questions, this study used a qualitative exploratory multi-case approach. This chapter details the research design, the artifacts of practice, the participant selection, and the data collection and analysis procedures. Also included are discussions about how the study addresses validity, reliability, and ethical issues. Finally, a subjectivity statement is included in an effort to maintain transparency about potential factors contributing to the researcher’s position.

**Research Design**

According to Creswell (2013), one of the reasons for doing qualitative research is “because a problem or issue needs to be explored” (p. 47, emphasis in original). Often, this means that a group or population needs to be studied to better understand a complex situation (Creswell, 2013). Thus, a researcher will personally go out into the field to collect data in its natural setting and analyze that data in an effort to understand the participants’ perspective on the problem or issue of interest (Merriam, 1998).

One method of qualitative inquiry is case study (Creswell, 2013). The term case study has been defined in a variety of ways, yet there is agreement that the purpose of a case study is to gain a holistic understanding of a particular situation and the meanings for those involved (Merriam, 1998; Sturman, 1994; Yin, 2009). Further, case studies might be used to explore a unique case or when the researcher wants to know about a particular issue in general and selects a case to investigate the issue (Stake, 1995). At times, the researcher
may decide he/she can gain further insight into the issue by examining multiple perspectives, so the researcher selects several cases, known as a multi-case or collective case study.

Another important part of any case study is to define its boundaries. Since defining the case determines the scope and unit of analysis for the study, it is a vital component of a case study (Creswell, 2013; Stake, 1995; Yin, 2009). Louis Smith is credited for describing a case as a *bounded system* and for providing insight into how to establish the boundaries of the case (Stake, 1995). Stake (1995) is careful to note that not everything is a case; rather, a case is “a specific, a complex, functioning thing” (p. 2). Thus, a case might be an individual, a program, or a school, but not a relationship or policy (Stake, 1995). Contextual factors also come into play when defining a case (Creswell, 2013; Yin, 2009). In other words, it is important to ensure that the case is defined by parameters such as time, place, and activity (Creswell, 2013; Stake, 1995). This helps to establish how the topic of study (the case) is distinguished from others, essentially indicating what will and will not be studied in the research project (Baxter & Jack, 2008; Yin, 2009).

The goal of this study was to understand how PSMTs make sense of students’ thinking on mathematical tasks by using the professional noticing of children’s mathematical thinking framework (Jacobs et al., 2010). As noted in Chapter 2, little is known about how PSMTs engage in this practice, and even less is known about how they interpret and respond to students’ thinking on tasks that utilize technology. Thus, a qualitative approach was appropriate to explore how PSMTs make sense of and plan to respond to students’ thinking.
More specifically, this study employed an exploratory multi-case study methodology (Yin, 2009). In other words, this study utilized multiple cases and was exploratory in nature, meaning the purpose was to understand a phenomenon of interest, particularly PSMT’s noticing. As such, PSMTs at a Southeastern University in the fall of 2015 were invited to participate in this study in order to explore this issue. Originally, PSMTs enrolled in their final methods course were invited to participate in the study, as these PSMTs were about to enter their full-time student teaching semester and might provide the best insight into how PSMTs who were about to complete their coursework were noticing. However, there was not enough interest, so the researcher broadened the invitation to include PSMTs who had completed or were currently enrolled in the program’s *Teaching Mathematics with Technology* course. These participants would have been exposed to at least one course in which activities related to teacher noticing occurred. In all, eight invited PSMTs agreed to participate. Thus, each participant represented a potential individual case bound by time (fall of 2015), place (secondary mathematics education program at a Southeastern University), and activity (noticing students’ mathematical thinking). Multiple cases were used in an attempt to increase understanding of the issue and generalizability across the cases (Creswell, 2013; Yin, 2009). Following data analysis, four of the eight PSMTs were selected to represent interesting and unique cases; more details about the selection process are provided later in this chapter.

In the study, six artifacts of practice were used to simulate real teaching experiences; there were three artifacts of practice for each task. PSMTs were asked to respond to prompts as they considered students’ thinking by examining samples of student work, videos of students working together, and videos of students’ participation in classroom discussions.
that were all from the same task. After an initial analysis of these responses, the researcher interviewed PSMTs individually to probe for more insight into their perspectives on students’ thinking. These interviews were videotaped and later transcribed, and researcher notes were also taken. The use of multiple data sources allowed for a rich description of PSMTs noticing of students’ mathematical thinking for each research question and facilitated within and cross case analyses.

**Artifacts of Practice**

As stated in the literature review, researchers often use artifacts of practice (Ball & Cohen, 1999) to make the practices of teaching available for teachers’ examination. For this study, the researcher utilized two different tasks, one paper and pencil task and one technology task as described below, with high school students to generate artifacts of practice. Thus, from each task there was a sample of student work, a video of a group of students working on the task, and a video of the teacher facilitating a class discussion about the task. This section details the selection of the tasks, the generation and selection of the artifacts of practice, and provides a description of each artifact of practice.

**Selection of tasks**. The following tasks were selected to generate the artifacts of practice because they are cognitively demanding and had the potential to encourage student discourse as students had to negotiate meanings and come to agreement on a solution (Bauersfeld, 1995; Stein & Smith, 1998). In the paper and pencil task, as shown in Figure 3, students had the opportunity to use *procedures with connections* (Stein & Smith, 1998) as they recalled information about tangent lines to circles to reason that the triangles shown are similar. In the technology task, as shown in Figure 4, students were *doing mathematics* (Stein & Smith, 1998) as they explore relationships between distances of pre-image and
image points and distances between these points and the center of dilation. In particular, this task took advantage of the affordances of dynamic geometry by encouraging the students to reason about and verify these relationships dynamically. Students were also asked to justify why the relationship is true. In both tasks, students worked with a partner or group, articulated their thinking, and reasoned collaboratively in an attempt to arrive at a solution.

Figure 3. Paper and pencil task. This figure represents the paper and pencil task adopted from the Illustrative Mathematics “G-SRT Tangent Line to Two Circles” task (https://www.illustrativemathematics.org/content-standards/tasks/916).

Figure 4. Technology task. This figure represents the technology task adapted from the Illustrative Mathematics “G-SRT Dilating a Line” task (https://www.illustrativemathematics.org/content-standards/tasks/602).
It is also important to note that each task was centered on the same mathematical idea: similarity. This concept was chosen because of the researcher’s familiarity with the topic and because exploring properties of similarity, especially through transformations, is a different approach to considering this idea. Typical approaches to teaching similarity often do not include its connections to transformations. While each task approached this concept in a different way, students were still required to draw upon their knowledge of similar triangles while reasoning through the task. Further, it did not make sense to use the same task, with different students, for both the paper and pencil task and the technology task. Rather, the tasks needed to be different in order to capitalize on the dynamic affordances of the technological environment (T. P. Dick & Hollebrands, 2011). Thus, for the technology task, it was important to select a task that allowed for students to dynamically discover a relationship through constructing, measuring, and dragging. If this same task had been completed using paper and pencil, the students would have verified the relationship for the case constructed on paper and would likely have had a more difficult time deriving a generic solution.

**Description of the paper and pencil task.** For the paper and pencil task, students were given the task shown in Figure 3. In order to solve this task, students had to consider what they know about lines that are tangent to circles to establish that the triangles shown are in fact similar. In other words, students had to recognize that the line segment, $\overline{CD}$, is tangent to circles $A$ and $B$ at points $D$ and $E$, respectively. Then, they had to recall that a line tangent to a circle is perpendicular to the circle’s radius at the point of intersection. Students then had to use this information to reason that $\angle ADC$ and $\angle BEC$ are congruent right angles. Further, $\angle ACD$ and $\angle BCE$ are the same angle, and are therefore congruent also. At this
point, students had to use the angle-angle similarity postulate to establish that\[ \triangle ADC \sim \triangle BEC. \]Once they figured out that the triangles were similar, they had to apply their knowledge of similar triangles to use proportions to solve for the length of the unknown side.

**Description of the technology task.** In this task, as shown in Figure 4, students were first asked to conjecture about what would happen to line segments under a dilation. Then, using *The Geometer’s Sketchpad* (GSP), students performed the dilation and were directed to consider certain distances. Students were asked to manipulate the sketch by dragging points and reason about what relationship is present between the distances. At this point, students had to rely on their prior knowledge of operations to recognize that the ratio between each of the given distances is two. Students could then make the connection that perhaps the ratio of these distances was two because that was the scale factor for the dilation. Finally, to prove this conjecture was true, students had to draw upon prior knowledge of similar triangles to show that the distance between two image points, say \( A'B' \) is twice that of the distance between their corresponding pre-image points, or \( AB \).

**Generating the artifacts of practice.** To generate the artifacts of practice, the researcher located a high school teacher who was willing to use these tasks with two of her Advanced Functions and Modeling (AFM) classes. The researcher specifically sought out students in their fourth high school mathematics course since at this point in their learning they should have been exposed to the background knowledge needed to complete the tasks. A couple of days prior to collection of the artifacts, the teacher and the researcher met to discuss the tasks and anticipate the various strategies students might employ. During this conversation, the teacher and the researcher also came up with ideas for how the teacher
might help any students that might get stuck and how the teacher could facilitate a class
discussion for each task. Thus, the goal for the implementation of the task was that the
researcher would play the role of an observer while the teacher facilitated the lesson.
Therefore, the teacher determined, based on her knowledge of her students, in which class
she would implement each task. Further, the teacher also made recommendations of which
groups the researcher should videotape since the researcher would only be videotaping four
groups from each class.

**Paper and pencil task.** The class where the paper and pencil task was used lasted for
55 minutes and consisted of 20 students who sat in five groups of four. The researcher
arrived early and set up to videotape the four groups the teacher suggested. Upon the
students’ arrival, the teacher introduced the researcher and explained to the students that the
researcher was there to collect information about how they solve a task to share with people
going to school to become mathematics teachers. The teacher reminded the students that it
was important for them to share their thinking aloud as they worked through the task
together in their groups. After ensuring no one had questions, the teacher gave the students
the task, told them to work with their group of four, and asked them to get started. While
they worked, the researcher monitored the video recording equipment and took notes about
how each group was approaching the problem. The teacher circulated the room checking in
with each group to answer and ask questions and listen to the students’ reasoning. After
about 30 to 35 minutes, the teacher and the researcher briefly spoke to discuss the students’
strategies. The teacher and the researcher talked about which groups would be important to
include within the class discussion. After about five more minutes, the teacher instructed the
students to stop working on the task so they could talk about it. During this transition, the
researcher set one of the video cameras to record the class discussion. The teacher then proceeded to lead a class discussion in which she asked each group to verbally share from their tables how they approached the problem. While the teacher did attempt to sequence the groups in order of the sophistication of their strategies, she ended up including each group within the discussion. After hearing from each of the groups, the teacher thanked them for their participation, instructed them to leave their work, and dismissed them from class stating that they would talk more about the problem tomorrow. As the students left the room, the researcher collected the student work for the groups that were video recorded and made notes to help with organizing the artifacts by groups.

**Technology task.** The class where the technology task was used lasted for 55 minutes and consisted of 12 students who sat in two groups of four and two groups of two. The researcher arrived early and set up to videotape the four groups the teacher suggested. As the students arrived, the teacher instructed them to get out one laptop computer for each pair of students. She also told them to go to the class webpage and download the *Geometer’s Sketchpad* file they would need for the day. According to the teacher, these students were not familiar with GSP and had only used it a few times in the past. Once class began, the teacher introduced the researcher and explained to the students that the researcher was there to collect information about how they solve a task to share with people going to school to become mathematics teachers. The teacher reminded the students that it was important for them to share their thinking aloud as they worked through the task together with their partner. The teacher then informed the students that the problem they would consider today would utilize a program called *The Geometer’s Sketchpad*. Using the file they downloaded, they should follow the prompts and answer the questions. After ensuring
no one had questions, the teacher gave the students a hard copy of the task in case they wanted it, told them to work through the task with their partner, and asked them to get started. While they worked, the researcher monitored the video recording equipment and took notes about how each group was approaching the problem. The teacher circulated the room checking in with each group to answer and ask questions and listen to the students’ reasoning. After about 30 to 35 minutes, the teacher and the researcher briefly spoke to discuss the students’ strategies. The teacher and the researcher talked about which groups would be important to include within the class discussion. After about five more minutes, the teacher instructed the students to stop working on the task so they could talk about it. During this transition, the researcher set one of the video cameras to record the class discussion. The teacher then proceeded to lead a class discussion in which she asked each group to bring their computer to the front of the room to display their work on the projector for the class. For each group that came up to share, the teacher asked them to explain how they approached the task and what interesting things they noticed. At various points within the discussion, she also instructed some groups to drag points around the screen to illustrate what they had done while completing the task. While the teacher did attempt to sequence the groups in order of the sophistication of their strategies, she did end up including each group within the discussion. After hearing from each of the groups, the teacher thanked them for their participation, instructed them to leave their written work and laptops, and dismissed them from class stating that they would talk more about the problem tomorrow. As the students left the room, the researcher collected the written student work and technology files for the groups that were video recorded and made notes to help with organizing the artifacts by groups.
Selecting the artifacts of practice. Once the artifacts had been collected, the researcher organized the files according to groups for each task. The researcher then went through the videos from each recorded group and edited them down so that the focus was specifically on the students working through the task. This resulted in four group video clips that were approximately 20 minutes each for the paper and pencil task and four group video clips that ranged from 12 to 18 minutes each for the technology task. The videos of the whole class discussions from both tasks were also edited to eliminate down time (e.g., students setting up to share for technology task). The videos of the whole class discussions were about five and seven minutes each for the paper and pencil task and technology task, respectively.

The next step was to determine which single group from each task would be selected for use with prospective teachers as the group video clip. To facilitate in making this decision, the videos of each group of students for each task were analyzed according to recommendations by Sherin, Linsenmeier, and van Es (2009). In this work, Sherin and her colleagues describe how they characterized video clips of student mathematical thinking and implemented these clips with teachers. Specifically, they considered the extent in which the video clip offered windows into students’ mathematical thinking (i.e., evidence of students’ thinking as observed through verbal explanations and written work), the depth of students’ mathematical thinking (i.e., how substantial the mathematical ideas are that students consider while engaging with the task), and the clarity of students’ thinking (i.e., whether the students’ thinking was made clear or required interpretation by the viewer). For each of these dimensions, they assigned a score of low, medium, or high based on the rubric shown in Figure 5. Then through examining the discussions that resulted between teachers with
each video clip, Sherin and her colleagues identified which categorizations of video clips promoted productive discussion of students’ mathematical thinking amongst teachers. They found that the following three combinations of video clip categorizations yielded more productive discussions with teachers: (a) high windows, high depth, high clarity; (b) high windows, high depth, low clarity; and (c) high windows, low depth, and low clarity. And, through their analysis, Sherin and her colleagues reasoned that it is the relationship between these dimensions that makes the difference between less and more productive discussions. Thus, they argue that video clips categorized according to one of the three combinations described above have the greatest potential for promoting discussion about students’ thinking from the video clip.

Therefore, in order to select clips that offered prospective teachers opportunities to discuss students’ thinking, each group video clip from each artifact (i.e., four video clips of groups from the paper and pencil task and four video clips of groups from the technology task) was categorized for each of the three dimensions as described by Sherin et al. (2009) (i.e., windows, depth, and clarity) according to the extent each was present (i.e., high, medium, or low). The researcher and another mathematics educator completed this video clip analysis for all eight group videos, and a third mathematics education expert coded a random subset of four of the group videos (two from each task). The results of these analyses are shown in Tables 1 and 2. As the tables show, there was agreement amongst the experts for 3 out of the 4 paper and pencil group videos and 2 out of the 4 technology group videos with respect to windows. With respect to depth, there was agreement amongst the experts for 3 out of the 4 paper and pencil group videos and all 4 of the technology group
videos. And, there was agreement amongst the experts for 1 out of the 4 paper and pencil group videos and 3 out of the 4 technology group videos with respect to clarity.

Figure 5. Rubric for characterizing video clips of students’ mathematical thinking (Sherin et al., 2009, p. 216).

Table 1


<table>
<thead>
<tr>
<th>Group</th>
<th>Expert 1</th>
<th>Expert 2</th>
<th>Expert 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1</td>
<td>L</td>
<td>H</td>
<td>M</td>
</tr>
<tr>
<td>Group 2</td>
<td>L</td>
<td>H</td>
<td>L</td>
</tr>
<tr>
<td>Group 3</td>
<td>M</td>
<td>M</td>
<td>M</td>
</tr>
<tr>
<td>Group 4</td>
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<td>M</td>
<td>L</td>
</tr>
<tr>
<td>Group 4</td>
<td>M</td>
<td>M</td>
<td>M</td>
</tr>
</tbody>
</table>
Table 2

*Expert Analysis of the Group Video Clips from the Technology Task.*

<table>
<thead>
<tr>
<th></th>
<th>Windows</th>
<th>Depth</th>
<th>Clarity</th>
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<td><strong>Group 1</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expert 1</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td>Expert 2</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td><strong>Group 2</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Expert 1</td>
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<td>L</td>
<td>L</td>
</tr>
<tr>
<td>Expert 2</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td><strong>Group 3</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expert 1</td>
<td>M</td>
<td>M</td>
<td>H</td>
</tr>
<tr>
<td>Expert 2</td>
<td>M</td>
<td>M</td>
<td>H</td>
</tr>
<tr>
<td>Expert 3</td>
<td>H</td>
<td>M</td>
<td>H</td>
</tr>
<tr>
<td><strong>Group 4</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expert 1</td>
<td>M</td>
<td>M</td>
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<tr>
<td>Expert 2</td>
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<td>M</td>
<td>L</td>
</tr>
<tr>
<td>Expert 3</td>
<td>L</td>
<td>M</td>
<td>M</td>
</tr>
</tbody>
</table>

After each expert had independently analyzed the videos, the researcher and the other expert who also analyzed each video met to discuss our categorizations. Rather than try to reach agreement for each video within each category, the expert and the researcher decided to look for combinations of codes that aligned with Sherin et al.’s recommendations for promoting discussion. Although the participants in the study would not be discussing the videos with each other, they would be discussing students’ mathematical thinking with the researcher. Thus, it seemed appropriate to select the group from each task whose video provided the greatest potential for promoting discussion.

In reviewing the combinations noted by Sherin et al., it was apparent that the ideal clip would be rated as high for windows into students’ thinking. Thus, to begin the initial process of narrowing the video clips for each task, the expert ratings for windows were considered first. But, for each task, the highest ratings given for windows into students’ thinking were medium; there was only one group (group 3 from the technology task) that was classified as having high windows by one expert. After acknowledging the
classifications could have been low as compared to Sherin and her colleagues, the expert and researcher decided to proceed with selecting a video with the highest classification received for windows. From the paper and pencil task, there were three groups rated as medium for windows. As such, group 1 was eliminated from consideration for the paper and pencil task. For the technology task, there was only one group (group 3) rated by one expert as high for windows. The other two experts rated this group as medium for windows. There was also one other group (group 4) that was rated as medium for windows. Thus, groups 1 and 2 were eliminated from consideration for the technology task.

Next, the depth and clarity of students’ mathematical thinking were considered simultaneously. From the paper and pencil task, groups 2, 3, and 4 were considered, and from the technology task, groups 3 and 4 were considered. In reviewing the classifications for these groups with respect to depth and clarity, it was apparent that there was one group from each task (group 3 for each task) that had been classified as having high clarity. Further, each of these groups was regarded as medium (paper and pencil group) or medium-low (technology group) with respect to depth. Thus, in an effort to provide the participants in this study with the greatest windows and clarity to students’ thinking, group 3 from each task was selected. While others might make a case for selecting different groups, the expert and researcher felt it was important to select groups with the highest possible windows, depth, and clarity combination in an effort to provide ample opportunities for the participants in this study to analyze students’ thinking.

**Descriptions of the artifacts of practice.** Once the artifacts of practice for each task had been chosen, descriptions of each were crafted so they could be shared with participants in the study. The descriptions of the six artifacts of practice are provided here below. Details
about the order in which participants received the artifacts are provided later in this chapter in the Data Collection section.

**Paper and pencil task.** Based on the analysis previously described, artifacts from a particular group (group 3) who worked on the paper and pencil task were selected. From this group, the student work for each of the four students was collected and a video of the students working on the task was edited to about a 12-minute clip. The video of the class discussion for the paper and pencil task was also included as an artifact of practice. Details about each specific artifact and the descriptions of each artifact for the paper and pencil task, as they were presented to the participants, are provided here below.

**Student work.** The student work for the paper and pencil tasks was generated from a group of high school juniors and seniors (who have each completed Math 1, 2, and 3) who worked on a mathematical task. The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion about the task.

The student work for the paper and pencil task consists of four PDFs of scanned student written work. Each PDF is titled Student 1, Student 2, Student 3, and Student 4. The student number corresponds to the number assigned to the student based on the order in which they appear in the group video clip, which is the second artifact you will review. All four of these students did work together on the task. (Student work from the paper and pencil task can be found in Appendix A.)
Group video clip. In this video, you will see a group of high school juniors and seniors (who have each completed Math 1, 2, and 3) working on a mathematical task. The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion about the task.

The video focuses on Student 1, Student 2, Student 3, and Student 4 who were working together as a group (see Figure 6). But, at times, you might hear comments from other students in the class. Note that the video has been edited down to about 12 minutes. Thus, time spent by the teacher setting up the task, times where the students were not on task, or times where the students were quiet, were intentionally cut out for this video. Students are numbered according to the order in which they appear in the group video transcript. (The transcript of the group video clip for the paper and pencil task can be found in Appendix B.)

Figure 6. This figure shows the orientation of the group and camera placement for the group video from the paper and pencil task.
Video clip of class discussion. In this video, you will see a class of high school juniors and seniors (who have each completed Math 1, 2, and 3) discussing their work on a mathematical task. The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion (that lasted about five minutes) about the task. *Note: The video slowly fades in; the teacher started talking before the researcher had the camera completely adjusted. The layout of the classroom, group compositions, and location of students who speak in the video is shown in Figure 7. Students are numbered according to the order in which they appear in the class discussion transcript. (The transcript of the video clip of the class discussion for the paper and pencil task can be found in Appendix C.)

Figure 7. This figure shows the layout of the classroom and camera placement for the class discussion video from the paper and pencil task.
Technology task. Based on the analysis previously described, artifacts from a particular group (group 3) who worked on the technology were selected. From this group, the student work for each of the two students was collected and a video of the students working on the task was edited to about a 12-minute clip. The video of the class discussion for the technology task was also included as an artifact of practice. Details about each specific artifact and the descriptions of each artifact for the technology task, as they were presented to the participants, are provided here below.

Student work. The student work for the technology task was generated from a group of high school juniors and seniors (who have each completed Math 1, 2, and 3) who worked on a mathematical task. The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion about the task.

The student work for the technology task consists of a PDF of the scanned written student work from the group, a PDF of the Geometer’s Sketchpad file from the group, and the actual Geometer’s Sketchpad file from the group. There were only two students in this group and they did work together on the task. (Student work from the technology task can be found in Appendix D.)

Group video clip. In this video, you will see a pair of high school juniors and seniors (who have each completed Math 1, 2, and 3) working on a mathematical task in The Geometer’s Sketchpad (GSP). The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. The students had little to no
prior experience with GSP. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion about the task.

The video focuses on Student 1 and Student 2 working together as partners (see Figure 8; Students 3 and 4 were also partners and sat across from Students 1 and 2). But, at times, you might hear comments from Student 3, Student 4, or others in the class. Note that the video has been edited down to about 12 minutes. Thus, time spent by the teacher setting up the task, times where the students were not on task, or times where the students were quiet, were intentionally cut out for this video. Students are numbered according to the order in which they appear in the group video transcript. (The transcript of the group video clip for the technology task can be found in Appendix E.)

![student placements](image)

*Figure 8.* This figure shows the orientation of the group and camera placement for the group video from the technology task.
*Video clip of class discussion.* In this video, you will see a class of high school juniors and seniors (who have each completed Math 1, 2, and 3) discussing their work on a mathematical task in *The Geometer’s Sketchpad* (GSP). The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. The students had little to no prior experience with GSP. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion about the task. While the actual class discussion lasted about 13 minutes, the video you will see has been edited down to about seven minutes. Thus, time spent by the students setting up their computers to share with the class was intentionally cut for this video. *Note: There was an issue with the camera, so the first part of the video is audio only where the teacher sets up the class discussion. In the piece following this, you will see just the screen capture of the group who is talking (Student 1 and Student 2’s group). The camera issue also affected the first part of what they shared. You will see the video pick back up to look at the entire class at about 1:52. At this point, Student 1 and Student 2 are still at the front of the room. The layout of the classroom, group compositions, and location of students who speak in the video is shown in Figure 9. Students are numbered according to the order in which they appear in the class discussion transcript. Each student only worked with the person sitting next to him or her. (The transcript of the group video clip for the technology task can be found in Appendix F.)
 Participant Selection

Participants for this study were recruited from a secondary mathematics education program at a Southeastern University in the fall of 2015. In this program, aside from general elective courses and standard required courses, students must complete mathematics and statistics courses, mathematics education courses, and a full-time student teaching experience either at the undergraduate or graduate level in order to obtain a secondary licensure. In particular, one of the mathematics education courses is titled *Teaching Mathematics with Technology*. In this course, PSMTs become familiar with methods, tools, and resources for teaching mathematics with technology. At the time of the study, the course included a geometry component and a data analysis and probability component. Each
component comprised about half of the course and focused on helping PSMTs become acquainted with specific tools including *The Geometer’s Sketchpad* (GSP), *Fathom*, and *TinkerPlots*. Another aspect of this course was the inclusion of tasks in which PSMTs were asked to considering students’ thinking on technology tasks. These activities usually began with PSMTs working through the technology task for themselves and then answering questions related to students’ thinking while considering a video clip and student work. Therefore, this program provided an ideal context for considering how PSMTs notice students’ thinking on tasks. Only students who had completed or were currently enrolled in the *Teaching Mathematics with Technology* course were invited to participate in the study, as these PSMTs would have some background knowledge related to considering students’ thinking on technology tasks. Other courses in the program also incorporated activities that allowed PSMTs to analyze students’ thinking; however, these experiences were not considered when selecting participants for the study.

As previously mentioned, eight of the invited students agreed to participate and were compensated for their time. Thus, each participant represented a potential individual case bound by time (fall of 2015), place (secondary mathematics education program at a Southeastern University), and activity (noticing students’ mathematical thinking). During data analysis, it became apparent that there were some PSMTs who represented unique cases and others who were similar to each other in the ways they noticed students’ mathematical thinking. Thus, in an attempt to best illustrate the range of noticing found, four PSMTs were selected as the cases for this study. When determining which PSMTs to select as cases, a variety of factors were considered. First, variation in the ways PSMTs responded to the noticing prompts in terms of attending, interpreting, and responding were
considered. The researcher also considered if there were interesting differences within a particular PSMT’s responses between the two different contexts (i.e., paper and pencil versus technology). The order of the tasks and artifacts the PSMTs received was also taken into consideration, and it was determined that selecting two PSMTs who received the paper and pencil task first and two PSMTs who received the technology task first was appropriate since the assignment of these tasks and artifacts had been randomized. Finally, two of the eight participants were MAT students, and during data analysis, it was apparent that these two PSMTs represented unique cases, albeit for different reasons. Thus, selecting these two PSMTs provided the opportunity to have half of the cases represent how MAT students notice students’ thinking and half of the cases represent how undergraduate students notice students’ thinking. In particular, one of the MAT PSMTs, Chase\textsuperscript{a}, represented a case whose noticing classifications during analysis were consistently lower than the other PSMTs. The other MAT PSMT, Mary, represented a case whose noticing classifications during data analysis with respect to the noticing frameworks were the highest of all eight participants. Thus, when considering which of the remaining six PSMTs to select, the researcher looked for interesting cases and also for any PSMTs who were similar in their noticing of students’ thinking. And, since both of the MAT PSMTs had received the technology task first, the researcher looked at the undergraduates who received the paper and pencil task first. From the four PSMTs who received the paper and pencil task first, it was apparent that one PSMT, Susan, represented a unique case because of her differences in scores between the two different contexts (i.e., paper and pencil vs. technology) and her consistently low scores with respect to responding. Another PSMT, Nicole, was more representative of about half of

\textsuperscript{a} Pseudonyms were used for all cases.
all eight of the PSMTs as she had consistent scores regardless to which aspect of noticing she was considering (i.e., attending, interpreting, or responding), the artifact type, or context. Thus, in all, four of the eight participants were selected to represent the range of noticing exhibited as well as take into account the differences in educational level (i.e., MAT vs. undergraduate) and first task received (i.e., paper and pencil vs. technology). A summary of the selected cases is provided in Table 3. A more thorough description of each case is provided in Chapter 4.

Table 3

Summary of Selected Cases

<table>
<thead>
<tr>
<th>Participant</th>
<th>Educational Level</th>
<th>Student Teaching Semester</th>
<th>First Task Received</th>
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<td>Chase</td>
<td>MAT</td>
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<td>MAT</td>
<td>Fall 2016</td>
<td>Technology</td>
</tr>
<tr>
<td>Susan</td>
<td>Undergraduate</td>
<td>Spring 2017</td>
<td>Paper and Pencil</td>
</tr>
<tr>
<td>Nicole</td>
<td>Undergraduate</td>
<td>Spring 2017</td>
<td>Paper and Pencil</td>
</tr>
</tbody>
</table>

Data Collection and Analysis

Data collection. In order to capture how PSMTs notice students’ mathematical thinking, PSMTs considered six artifacts of practice over a six-week period (Ball & Cohen, 1999). Thus, for each task (i.e., paper and pencil task and technology task), PSMTs considered three artifacts of practice. First, they considered the sample of student work, then the video of a group of students working together, and then the video of the teacher leading a class discussion. For each artifact, the PSMT was asked to provide a written response to the following three noticing prompts: 1) Please describe in detail what you think the
students did in response to this problem, 2) Please explain what you learned about these students’ mathematical understandings, and 3) Pretend you are the teacher of these students. What problem or problems might you pose next and why? (Adapted from Jacobs, Lamb, & Philipp, 2010).

During the first week of the study, which was about halfway through the fall semester, the researcher met with each participant individually to gather background information and to conduct a task-based interview using one of the tasks (Goldin, 2000). The purpose of having the PSMT complete the task-based interview was to allow the PSMT to become familiar with the task and to give the researcher insight into how the PSMT thought about the mathematics within the task. The assignment of this task was randomized. In other words, four of the PSMTs received the paper and pencil task and the other four the technology task. For the interview, PSMTs were asked to work through the task while sharing his or her thinking out loud. For the paper and pencil task, PSMTs were provided with paper, a pencil, a calculator, a straightedge, a protractor, and a compass. For the technology task, PSMTs were provided with the above items and a computer with GSP. The interview protocols for each task, which included questions about background information, can be found in Appendices G and H; questions were written to encourage PSMTs to share verbalize their thinking. All initial interviews were videotaped and transcribed, and the researcher typed notes in a Word document. All work on each task was collected and screen capture software was used to record PSMT’s interactions with GSP; data sources related to the PSMT’s work on the tasks were captured in the event the PSMT made reference his/her own experience with the task when noticing students’ thinking.
After the task-based interview, each PSMT received the artifact of student work from the task they just completed. Thus, the four PSMTs who completed the paper and pencil task received the student work from the paper and pencil task and the four PSMTs who completed the technology task received the student work from the technology task. All artifacts were shared electronically with individual PSMTs via a password-protected location. PSMTs were only granted access to the artifacts needed each week and previously reviewed artifacts were removed. PSMTs were asked to consider the artifact of practice and provide a typed response to the noticing prompts within a week. Each PSMT provided each written response using the survey link provided for that week; the survey form asked PSMTs to indicate their unique identifier and provided a text box where the PSMT could respond to each noticing prompt. The form also included a space where PSMTs could upload other documents or images relevant to their written responses. This password-protected survey tool allowed the researcher to quickly organize the PSMT’s responses for review. The following week, the researcher interviewed the PSMT to probe for insight into his/her perspective on the noticing task (Patton, 2002). The researcher generated questions for the interview based on each PSMT’s written response. For example, one PSMT stated in his response, “Still these students understand what a dilation is more or less,” so in the interview, the researcher asked the PSMT what was meant by “more or less” and how he knew the students understand what a dilation is. The researcher also asked what evidence from the artifact of practice supported his claim. At the end of the interview, the researcher provided the next artifact for the PSMT. This process continued until each PSMT had considered all three artifacts of practice for a particular task. At that time, the PSMT worked through the second task (i.e., If the PSMT started with the technology task, he/she would
now move on to consider the paper and pencil task and vice versa.). Then, the PSMT received the sample of student work from this task and continued in the same manner as before until all artifacts of practice were considered. Thus, the PSMT was given one artifact at a time and had one week to consider the artifact and respond to the noticing prompts. In the end, each PSMT considered all six artifacts of practice. Figure 10 details this trajectory for a PSMT who was first assigned the paper and pencil task. A PSMT who was first assigned the technology task followed the same sequence but with the technology task first. All seven interviews were videotaped and transcribed, and the researcher made electronic memos in a Word document after each interview.
Figure 10. This figure illustrates the data collection process for a PSMT who was first assigned the paper and pencil task.
Summary of data sources. In case studies, a variety of data sources should be used to gather evidence to describe the phenomenon of interest (Creswell, 2013; Merriam, 1998). For this study, data sources included videos of participants’ task-based interviews (for reference), written responses to the noticing prompts for each artifact, and videos of follow up interviews about each artifact. Researcher notes from the interviews and memos from continual data analysis also provided additional insight.

Data analysis. For this study, there were five phases of data analysis. The first three phases were in preparation for the within and cross case analyses. The first phase occurred during data collection while the second and third phases occurred after data collection was complete. Following these analyses, the within and cross case analyses, phases 4 and 5, respectively, were conducted. This section provides details about each phase of the data analysis.

Phase 1. In qualitative research, data should be analyzed throughout data collection (Merriam, 1998). Thus, Merriam (1998) suggests that after each data collection activity, a “rudimentary analysis” (p. 165) be performed. During this time, the researcher has the opportunity to review the data and make note of things to ask or do in the next data collection activity. The researcher can also do some preliminary coding and make memos about potential themes that are recognized in the data. As more data is collected, the researcher should constantly be comparing the data (Strauss & Corbin, 1998) to see how instances within the data might relate.

To facilitate ongoing data analysis, ATLAS.ti, a qualitative analysis software program, was used to organize and analyze the data. This program provided an organized filing system where the researcher coded and annotated segments of information as well as
visually inspected relationships among codes throughout the data collection process. Thus, as each new piece of data was collected, it was added to the database and a preliminary analysis was performed.

During this phase of analysis, the main goal was to inspect the participants’ written responses to the noticing prompts to identify comments where clarification was needed. Thus, each week, the researcher and another expert reviewed each participant’s written responses for a particular artifact and generated probing questions for the follow up interview accordingly. Then, immediately following each interview, the researcher made memos about interesting ideas or comments that were shared. Throughout data collection, the researcher also made notes of patterns and potential themes that appeared either for an individual participant or across participants. Conducting this rudimentary analysis provided the researcher the opportunity to take advantage of clarifying participants’ comments and maintain a general sense of how data collection was going.

**Phase 2.** A common approach to analyzing participants’ responses to noticing tasks is to categorize the responses according to the amount of evidence that is present with respect to the particular category (i.e., attending, interpreting, responding) (L. Dick, 2013; Jacobs & Philipp, 2010; Schack et al., 2013). For instance, when considering participants’ responses to an attending prompt, the researcher looks for evidence that the teacher has given attention to the children’s strategies and the mathematically significant details used within the strategy. When considering how participants interpret children’s understandings, the researcher is looking for evidence that the teacher’s interpretations are consistent with the children’s strategies and make sense mathematically. When considering teachers’
responses about how to respond, the researcher looks for evidence that the teacher utilized the children’s mathematical understandings when determining the appropriate response.

Thus, during the second phase of data analysis, the researcher considered classifications from the literature for each facet of noticing (i.e., attending, interpreting, and responding) as well as the data from this study to develop frameworks for categorizing participants’ responses. First, the researcher reviewed the literature to examine how other researchers had classified participant responses to noticing prompts and looked for commonalities between them (L. Dick, 2013; Jacobs & Philipp, 2010; Schack et al., 2013). It was evident that researchers agreed there were distinct indicators that separated responses to noticing prompts into different categories. For instance, with attending, L. Dick (2013) noted the participant might correctly note the students’ strategy, might incorrectly note the students’ strategy, or might fail to mention the mathematics. Schack and colleagues (2013) noted participants might provide an elaborate response completely describing the students’ approach, a salient response describing the majority of the students’ approach, a limited response describing an aspect of the students’ approach which also included assumptions about the students’ actions or understanding, or an inaccurate response describing an incorrect interpretation or lacking an interpretation of the students’ approach. Using these indicators and the question proposed by Jacobs and Philipp (2010) regarding participants’ responses to the attending prompt (i.e., To what extent is there evidence the PSMT is attending to the mathematically significant details of the students’ approach to the problem?), the researcher developed the Attending Framework shown in Table 4. Following this same process, the researcher developed the Interpreting Framework by considering the indicators noted by L. Dick (2013) and Schack et al. (2013) and the question proposed by
Jacobs and Philipp (2010) regarding participants’ responses to the interpreting prompt (i.e., *To what extent does the PSMT provide evidence to support interpretations of students’ mathematical understandings?*). Similarly, the Responding Framework was developed by also considering the indicators noted by L. Dick (2013) and Schack et al. (2013) and the question proposed by Jacobs and Philipp (2010) regarding participants’ responses to the responding prompt (i.e., *To what extent is there evidence the PSMT is using students’ mathematical understandings when determining how to respond?*). These frameworks and examples for each category are presented in Tables 4, 5, and 6.

Once the frameworks were developed, each participant’s responses to the noticing prompts for the student work and video artifacts from both tasks were categorized. (It was evident during data collection that responses to the noticing prompts for the class discussion video clips were all vague, so they were not considered during any part of the data analysis.) Therefore, responses to the first noticing prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were categorized according to the Attending framework (Table 4), responses to the second noticing prompt (i.e., Please explain what you learned about these students’ mathematical understandings.) were categorized according to the Interpreting framework (Table 5), and responses to the third noticing prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were categorized according to the Responding framework (Table 6).
## Table 4

**Attending Framework**

<table>
<thead>
<tr>
<th>Categorization</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>PSMT completely describes students’ approach to the problem in detail with specific evidence from artifact. This includes describing the details about each step of the approach.</td>
<td>The students found the length of segment $AB$ to be 8, “because the distance from $A$, from $B$, from center to out is 3 and from $A$ to center out is 5” (lines 22-23, 39-41). They then tried to determine how to relate the different segments to each other by identifying triangles. They identified triangle $EBC$, but determined that it was not a right triangle and that they did not have enough information to use the Pythagorean theorem (lines 90-95). They then identified triangle $EBA$ and applied the Pythagorean theorem despite it not being a right triangle (lines 107-129). They solved for “c” (lines 128-129) which they did not interpret to be segment $AE$ (line 157). They then tried to recall a relationship for 30, 60, 90 triangles (lines 159-213). With the teacher’s guidance, they settled on applying the Pythagorean theorem to triangle $ADE$ and found the length of segment $DE$ to be “about 7” (line 354). Next, they tried to figure out how the segments they had already determined related to the question (lines 411-418) but they seemed to be unable to find any connections. The teacher then intervened and indicated that they had incorrectly applied the Pythagorean theorem (line 448), and helped them realize that $ADE$ was a right triangle due to the definition of a tangent (lines 474-516).</td>
</tr>
<tr>
<td>3</td>
<td>PSMT completely describes students’ approach to the problem in detail with some specific evidence from artifact. This includes describing each step of the approach but only providing details for some of the steps.</td>
<td>The students first make a prediction that the dilation makes the line bigger by making the scale two times bigger. They followed the instructions, and with the help of the teacher were able to create the desired dilation with a scale factor of 2. Then they labeled the points $A'B'$ and $C'D'$. Then they measured the distance of $AB$ and $A'B'$. They observed that the dilation produced the line with a measure twice as big as the original one. They performed the same action on $CD$ and $C'D'$ and witnessed the same thing. The teacher suggested that they drag the points around and the students say it still looks like the length is doubled. They then confirm their observation by saying the measurements prove the observation when you move the lines around. The teacher asked them to draw a proof, and they claimed they had a proof drawn through the technology tool and showed the teacher how when you move any point around the measurements are still double regardless. The teacher then asked them to move point $P$ and the students suggested that it doesn’t really do anything. The teacher then asked them to measure the line from $P$ to $A$ and from $P$ to $A'$. The student says “it is basically still giving you double.” Then the students move $B$ and say that when you move $B$ it doesn’t really affect the measurement of $P$ to $A$ and $P$ to $A'$.</td>
</tr>
<tr>
<td>PSMT provides an overview of the students’ approach to the problem with very little to no specific evidence from artifact.</td>
<td>The students made predictions based on their knowledge of dilation. They proved their predictions by measuring the lengths of the segments AB and CD per the instructions and proving their relationships.</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>There is no mention of student strategies or attention to detail.</td>
<td>No examples of an attending categorization of 0 were present in the data for this study.</td>
<td></td>
</tr>
</tbody>
</table>

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Table 4 Continued

<table>
<thead>
<tr>
<th>Level</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>PSMT describes students’ approach to the problem in less detail but still with some specific evidence from artifact. This includes mentioning each step of the approach but only providing details for some of the steps.</td>
<td>The students started off by defining the line AB. Then they moved toward getting additional values by exploring information given in the problem. Such as when Student 4 asks about the meaning of collinear. They also started bouncing ideas off of each other throughout the video. For example the discussion about DE. They then worked with the concept of right triangles and calculating side lengths by trying to use the Pythagorean theorem and the 30-60-90 ratio. They also brought in trigonometric functions in attempts to come to a solution. When the teacher checked in, they went back to the Pythagorean Theorem. Once they had the values for AE and DE, they tried to determine the length of BC by interpreting the line and the components around it.</td>
</tr>
<tr>
<td>1</td>
<td>PSMT provides an overview of the students’ approach to the problem with very little to no specific evidence from artifact.</td>
<td>The students started off by defining the line AB. Then they moved toward getting additional values by exploring information given in the problem. Such as when Student 4 asks about the meaning of collinear. They also started bouncing ideas off of each other throughout the video. For example the discussion about DE. They then worked with the concept of right triangles and calculating side lengths by trying to use the Pythagorean theorem and the 30-60-90 ratio. They also brought in trigonometric functions in attempts to come to a solution. When the teacher checked in, they went back to the Pythagorean Theorem. Once they had the values for AE and DE, they tried to determine the length of BC by interpreting the line and the components around it.</td>
</tr>
<tr>
<td>0</td>
<td>There is no mention of student strategies or attention to detail.</td>
<td>No examples of an attending categorization of 0 were present in the data for this study.</td>
</tr>
</tbody>
</table>
```
<table>
<thead>
<tr>
<th>Categorization</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>PSMT provides evidence to support each interpretation of what the students understand mathematically. PSMT mentions most of the opportunities for making interpretations.</td>
<td>They don’t know the notation for the magnitude of a segment, as they incorrectly refer to it as the absolute value of the segment (line 6). They understand that the radius of a circle is constant (lines 22-23, 39-41). They are familiar with the Pythagorean theorem – they know the formula, and they know that $a$ and $b$ represent the legs and $c$ represents the hypotenuse (lines 128-129). However, they do not seem to know that this only applies to a right triangle (lines 113-116). There also seems to be a misconception that all right triangles have the measures of 30, 60, 90 or 45, 45, 90 (lines 177, 185, 381, 393).</td>
</tr>
<tr>
<td>2</td>
<td>PSMT provides evidence to support some of interpretations of what the students understand mathematically. PSMT mentions some of the opportunities for making interpretations.</td>
<td>I learned that the students understood that the dilation would result in a bigger line segment, by saying that the scale of $AB$ and $CD$ would be larger. The student manipulated the tool to see that the relationship between the original line segments and their dilated counterparts stays the same when moving a point on one of the line segments. The student also measured the distance from point $P$ to $A$ and point $P$ to $A'$ which made me think the student knew the relationship should also apply to the point the line segments are centered around. In regards to language, the student used the term “distance” to describe the lengths of $AB$, $A'B'$, $CD$, and $C'D'$.</td>
</tr>
<tr>
<td>1</td>
<td>PSMT does not provide evidence to support interpretations of what the students understand mathematically or PSMT’s interpretation is hard to follow. Further, PSMT missed opportunities for making interpretations.</td>
<td>Students understand what a dilation is and how it affects a line dilated in proportion to a point. They understand that when given a whole number of the dilation that increases the lines.</td>
</tr>
<tr>
<td>0</td>
<td>PSMT is not focused on interpreting what the students understand mathematically but instead is focused on something else (describing what the students did, evaluating their efforts, assuming they had trouble with the problem because they do not have problem solving skills).</td>
<td>Some of this is speculation. 1) So I noticed several times that the students would read the number of the length of a segment out very deliberately. I mention this because when I do the problem or watch them work it's like I don't even see the number, except maybe as part of a ratio. Somehow measuring this length is very important to the students in making the problem real. I'm tempted to take a ruler and put it on the screen to see if the length really is 11.16 cm. Maybe just having a number makes them feel more comfortable, a concrete answer to show the teacher? 2) I noticed the students would move the points around rather hesitantly, maybe a one inch shift here or there, not really testing the boundaries of how far you can move the points. I also noticed that the students separately measured CD and C'D'. Do they realize that moving the points of AB is the same really as measuring a new line segment? Why are there two line segments here? These observations make me think that the students don't understand that moving the points creates a generalization of the original rule, perhaps they don't understand why a generalization would be valuable. 3) The students seem to me almost too content. I agreed with the teacher’s decision to prod them to think harder, to make their statements clearer, and to consider different questions that can be examined in the problem. 4) With regards to content, I think the students could identify a dilation if they saw one and guess a dilation factor but not express a dilation algebraically or find the dilation by hand with a ruler. They lack the key point that given any point X on a figure, P, X, and X' (its image) are collinear.</td>
</tr>
</tbody>
</table>
Table 6

Responding Framework

<table>
<thead>
<tr>
<th>Categorization</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>The PSMT poses a potentially useful problem or question and provides a rationale with evidence specifically connected to student thinking.</td>
<td>No examples of a responding categorization of 3 were present in the data for this study.</td>
</tr>
<tr>
<td>2</td>
<td>The PSMT poses a potentially useful problem or question and provides a rationale with some evidence indicating student thinking was considered.</td>
<td>Since they were able to identify the triangles $ABE$ and $ADE$, I would ask the students to identify other triangles in the figure, specifically ones that included segment $BC$. I would then ask them if they could find any relationships between any of these triangles that they had identified. Also, asking them what type of relationships triangles can have. This will hopefully lead them to see the similar triangles $ADC$ and $BEC$. After which I would ask them what similar triangles meant.</td>
</tr>
<tr>
<td>1</td>
<td>The PSMT poses a potentially useful problem or question and provides a vague or unclear rationale that is not linked to student thinking from the problem.</td>
<td>I would ask them why they measured $PA$ and $PA'$ since they measured those distances as well. I would ask the students to discuss the relationships between the other points, like $AD$ and $A'D'$ to let students further explore the definition of dilation. I would also ask the students for their prediction if the scale factor was a half and -2, so that students understand how the scale factor affects the dilation.</td>
</tr>
</tbody>
</table>
**Phase 3.** During the third phase of analysis, the researcher coded the participant’s responses to each artifact and interview using *a priori* codes that had been developed prior to the start of the study (Miles & Huberman, 1994). These codes were derived from the literature on how teachers notice students’ thinking on elementary mathematics tasks and on technology tasks and are described in the next section. Open coding (Strauss & Corbin, 1998) was also used, as additional codes needed to be added based on themes that emerged from the data as noticing literature at the secondary level and that considers technology was limited. And, throughout this process, the researcher revisited previous codes and memos to check for consistency. If needed, previous codes were refined to align with the evolving coding scheme.
While assigning codes to the data, the researcher went sentence-by-sentence through the written responses to the noticing prompts and the interview transcripts. Codes were assigned to segments of written responses or interview transcripts in an effort to capture each complete thought from the PSMT regarding that particular code. Thus, some segments that were coded might have only been a single sentence, while others might have been comprised of several sentences. It is also important to note that when necessary, multiple codes might have been assigned to the same written segment. It is also important to note that if a thought appeared in the PSMT’s written response and the same thought appeared in the interview transcript, both instances were coded using the appropriate descriptive code. Although some might consider this “double counting,” there was no clear alternative, thus the researcher chose to consistently code all of the data. This decision also ensured that the salient ideas that appeared in the written responses and again appeared in the interview data were brought to light. Thus, frequency counts of the descriptive codes for each PSMT were used in an effort to create a picture of the ways in which each PSMT was noticing. The next section provides details about how the descriptive coding scheme evolved.

Coding the data. After examining the literature on how teachers elicit and interpret students’ thinking, an initial coding scheme was developed. Using the professional noticing of children’s mathematical thinking framework, there are three activities teachers do when noticing students’ thinking: attend, interpret, and respond (Jacobs et al., 2010). In fact, these three activities were evident in the noticing prompts PSMTs were required to answer. The first prompt (i.e., Please describe in detail what you think the students did in response to this problem.) required the PSMT to give attention to the actions of students during the task. The second prompt (i.e., Please explain what you learned about these students’ mathematical
understandings.) encouraged PSMTs to make sense of students’ thinking. Finally, the third prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) asked PSMTs to consider how they might react based on the students’ thinking. Thus, these three activities provided the overarching structure for classifying PSMTs noticing of students’ thinking during this coding process.

However, PSMTs are likely to attend, interpret, and respond in different ways. When attending, PSMTs might describe the students’ approach to solving the problem or compare the students’ approach to the problem to his/her own approach (Wilson et al., 2011). When interpreting, PSMTs might infer what the students are thinking based on observable actions (Wilson et al., 2011). When thinking about how to respond to students’ thinking, PSMTs could pose new problems that encourage reflection among students or prompt students to look for alternate solutions (Fraivillig, Murphy, & Fuson, 1999). Thus, these additional descriptive codes provided a better understanding of how each PSMT engaged in attending, interpreting, and responding.

Using the classifications noted above, the initial coding scheme shown in Table 7 was developed. Thus, when reviewing written responses and transcripts of interviews, these descriptive codes were used and additional descriptive codes were added as necessary to depict instances that did not fit within this original coding structure. While most of these codes evolved based on the data, at times the literature was revisited to seek out specific ways the data might be coded. In particular, while coding participant responses for the third noticing prompt (i.e., “Pretend you are the teacher of these students. What problem or problems might you pose next and why?”), it was evident that the PSMTs were attempting to make decisions about either what types of questions they would ask the students about
their work on the current task or what types of examples they might choose as follow up problems. Thus, the researcher reviewed the literature relating to these two categories (i.e., questioning and choice of example) to find codes that could better describe the data. To facilitate classifying the PSMT’s types of questions, categories from Boaler and Humphreys’ (2005) work were used. In this work, Boaler and Humphreys describe how they examined many examples of teaching and sought to develop a list to describe the different types of questions they witnessed. The complete list of question types from Boaler and Humphreys (2005, p. 37), with examples from this study, can be seen in Table 8. To describe the PSMT’s choices of examples, codes from Zodik and Zaslavsky (2008) were used. Zodik and Zaslavsky studied teachers’ choice of examples, both planned and spontaneous, across fifteen different seventh through ninth grade classes comprising fifty-four different lessons. From this work, they were able to generate a list to describe teachers’ considerations when choosing examples. The list of codes used from Zodik and Zaslavsky (2008), with examples from this study, can be seen in Table 9; note that some codes (e.g., avoiding cases) were not from Zodik and Zaslavsky but were added based on the data in this study. Finally, while coding the data, it was apparent that the best way to attend to the uses of and references to technology within the data was to assign a descriptive code that specifically accounted for the mention of technology. Therefore, the context codes from the list of initial codes were eliminated, as there was no longer a need to distinguish between each case of descriptive codes.
**Table 7**

*Initial Codes and Examples*

<table>
<thead>
<tr>
<th>Noticing Code</th>
<th>Descriptive Code</th>
<th>Context</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Attend</td>
<td>Describing</td>
<td>Without technology</td>
<td>The students used GSP to measure the distances and calculate their ratios.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>With technology</td>
<td></td>
</tr>
<tr>
<td>Comparing</td>
<td></td>
<td>Without technology</td>
<td>To show the triangles were similar, the students used the angle-angle similarity postulate. I showed the triangles were similar by using the side-angle-side postulate.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>With technology</td>
<td></td>
</tr>
<tr>
<td>Interpret</td>
<td>Inferring</td>
<td>Without technology</td>
<td>The students do not understand the properties of similar triangles.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>With technology</td>
<td></td>
</tr>
<tr>
<td>Respond</td>
<td>Encourages reflection</td>
<td>Without technology</td>
<td>I would ask the students to look for connections between this task and one that considers a similar idea.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>With technology</td>
<td></td>
</tr>
<tr>
<td>Prompts for alternate solutions</td>
<td></td>
<td>Without technology</td>
<td>I would ask the students to find other ways to solve the problems.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>With technology</td>
<td></td>
</tr>
</tbody>
</table>

*Note.* For the sake of brevity, examples provided here illustrate comments that might be made in reference to technology; similar examples could have been constructed without reference technology.

**Table 8**

*Classifications of Teacher Questions*

<table>
<thead>
<tr>
<th>Question Type</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
</table>
| Gathering information, checking for a method, leading students through a method | Wants direct answer, usually wrong or right  
Rehearses known facts or procedures  
Enables students to state facts or procedures | I would then ask what else we could do to find the length of BC? What similarities do we see between the two triangles?  
They have congruent corresponding angles and are the same shape, so then they must be similar triangles and thus we can use a ratio to find the length of BC. |
| Inserting terminology                              | Once ideas are under discussion, enables correct mathematical language to be used to talk about them | I would want to discuss... vocabulary...I noticed that these students predicted that the dilation would "make the scale larger." I would like to know what they were referring to by "the scale" and help them understand how to be more specific. |
| **Table 8 Continued** |
|------------------------|-------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| Exploring mathematical meanings and relationships | Points to underlying mathematical relationships and meanings  
Makes links between mathematical ideas | *How do you know with absolute certainty that the examples you didn’t try will also be a ratio of 1:2?*

| Probing, getting students to explain their thinking | Clarifies student thinking  
Enables students to elaborate their thinking for their own benefit and for the class | *I still don’t think they realized the relationship, but moving P around moves the images around and so, maybe start to ask them, “Well, why are the images moving if you’re just moving point P?” and they start to see a relationship or start to realize that, I think.*

| Generating discussion | Enables other members of class to contribute and comment on ideas under discussion | *I would have liked the class to hear how Student 3’s group used the scale of 3 to 5 to solve for other pieces, which they presumably were attempting to do.*

| Linking and applying | Points to relationships among mathematical ideas and mathematics and other areas of study or life. | No examples of linking and applying were present in the data for this study.  
Examples from Boaler and Humphreys include:  
“*In what other situations could you apply this? Where else have we used this?”* (p. 37)

| Extending thinking | Extends the situation under discussion, where similar ideas may be used | *Then I would ask if the same thing happens for a figure that’s a different shape, such as a triangle.*

| Orienting and focusing | Helps students focus on key elements or aspects of the situation in order to enable problem solving | *What are we trying to find again?*

| Establishing context | Talks about issues outside of math in order to enable links to be made with mathematics at later point | *I want a problem that uses application and context as a way to help students familiarize themselves. Maybe a worksheet about constructing a house or table, which needs to be twice the size. You could even have a fun question about what happens to the area of figures, which are dilated and talk about why.*

---

*Note.* The question types and descriptions are directly from Boaler and Humphreys (2005, p. 37) and the examples provided are from the data in this study unless otherwise noted.
Table 9

*Teachers’ Considerations When Choosing Examples*

<table>
<thead>
<tr>
<th>Consideration</th>
<th>Description</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Start with a simple or familiar case</td>
<td>Gradually increases the sequence of examples by varying just one thing at a time or increasing in complexity or difficulty</td>
<td><em>I would ask them to redraw the problem without the circles and reorient it such that point C is at the top, to see if this will help them see that the triangles are similar.</em></td>
</tr>
<tr>
<td>Attend to students’ errors</td>
<td>Draws on prior experience or assumptions to select examples based on common errors students make or difficulties students might encounter</td>
<td><em>I believe asking them to predict what would happen if the scale factor were a half would help correct their misconception of dilation that it only means to ‘make bigger.</em></td>
</tr>
<tr>
<td>Draw attention to relevant features</td>
<td>Deliberate attempt to reduce “the noise” of an example Structured variation - changes just one thing at a time to draw attention to that piece Occurs when looking at a sequence of examples</td>
<td><em>I’d definitely do them at different times like keep the same center of dilation and look at different scale factors or like keep the same scale factor and look at different centers, so I think I talked about that a little bit last time too, just to understand that nothing else is changing, we’re just changing this one factor, so how is that one factor affecting the dilation?</em></td>
</tr>
<tr>
<td>Convey generality by “random” choice</td>
<td>Use “random” numbers to get to the general concept</td>
<td><em>They were just kind of arbitrary things that I picked, and I liked that 1.5 was less than 2, and then 3 is greater than 2 so they could see the difference between if it was, you know, smaller than the figure or bigger than the other figure, or the new figure.</em></td>
</tr>
</tbody>
</table>
| Include uncommon cases               | Selecting exceptional or under-represented cases in mathematics             | No examples of including uncommon cases were present in the data for this study. An example from Zodik and Zaslavsky is when teachers explained their choice of including \( \sqrt{0} \) and \( \sqrt{1} \) while teaching the definition of a square root.  

“…for example, the special status of 1 and 0 as the only numbers that are invariant under integer powers, that is,  

\[
0^0 = 0 \Rightarrow \sqrt{0} = 0, \quad \text{or} \quad 1^1 = 1 \Rightarrow \sqrt{1} = 1
\]

(p. 177).
Keep unnecessary work to a minimum | Selecting examples that spend too much time on technical work instead of understanding (Getting lost in procedures) Highlights relevant part of example and does not go into details | No examples of keeping unnecessary work to a minimum were present in the data for this study. An example from Zodik and Zaslavsky is when a teacher explains why she chose $\frac{1}{7}$ to show the period of a rational number.

“The teacher explicitly said that she chose $\frac{1}{7}$ instead of $\frac{1}{17}$ or $\frac{1}{19}$, because $\frac{1}{7}$ had a long enough period, and it was unnecessary to add the work needed for obtaining periods of $\frac{1}{17}$ or $\frac{1}{19}$ that are much longer” (p. 177).

Avoiding cases | Avoiding certain cases of examples for a purpose (e.g., does not feel students can handle the math or teacher not sure about the math). | I think a negative scale factor would be interesting as well for them. Um, although, considering what they understand of dilation, I wouldn’t, because that would be stretching them a little bit too much, I think.

Drawing from experience to select example | Making decisions about what example to use or how to respond based on prior experiences with students or with the task. | I would have them dilate by a scale factor of a half. I tutor math and so, that’s like one of the first fractions you introduce students to is half. You can easily break something in half, it’s just…it’s a common fraction that you hear even when you’re not doing fractions; you just know what half means.

Note. The question types and descriptions are directly from Zodik and Zaslavsky (2008, pp. 173–178) and the examples provided are from the data in this study unless otherwise noted.

a, b This code was added based on the data in this study.

**Phase 4.** According to Merriam (1998), there are two stages of analysis in a multiple case study: within case analysis and cross case analysis. The goal of the within case analysis is to generate a detailed description of the case and its context (Creswell, 2013). Thus, the data analyses from phases 1 through 3 laid the foundation for the case analysis. In phase 4, the researcher reviewed the data and selected the cases. Then, a within case analysis for each selected case was conducted to develop a picture of how the PSMT noticed students’ thinking on each mathematical task. To do this, findings from each artifact associated with a
task are summarized and explicated using illustrative instances directly from the data. This process was completed for both the paper and pencil task and the technology task.

**Phase 5.** Then, a cross case analysis was conducted in an attempt to “build abstractions across cases” (Merriam, 1998, p. 195). In other words, the researcher looked broadly across the cases to try to identify similarities and differences between participants for each task, as well as comparing across tasks. Specifically, the researcher first considered the ways in which the PSMTs were noticing within a task by evaluating the classifications of PSMT’s written responses according to the noticing frameworks (i.e., Tables 4, 5, and 6) and then according to the descriptive codes. Next, the researcher considered how the PSMTs were noticing across both tasks. This analysis again included comparing the classifications of written responses as well as descriptive codes. However, this time, only the three most frequently occurring codes from each task were considered for each aspect of noticing. This decision was made because codes beyond these typically appeared less often and for fewer PSMTs. These findings were then used to describe the ways in which the PSMTs were similar and different in their attending, interpreting, and responding across tasks. Both the within case analysis and cross case analysis are organized by research question.

**Validity and Reliability**

Validity and reliability are measures used in research to ensure accuracy and rigorousness (Creswell, 2013; Merriam, 1998). In education, this is particularly important as research must be conducted in a valid and reliable way in order to have any effect on practice or theory (Merriam, 1998, p. 199). As a result of these expectations, various techniques are often used to establish validity and reliability in qualitative research (Creswell, 2013; Merriam, 1998).
Validity. According to Creswell (2013), validity is a measure of the accuracy of the research findings. Creswell goes on to provide eight validation strategies that can be incorporated into research: prolonged engagement, triangulation, peer debriefing, negative case analysis, clarifying researcher bias, member checking, rich description, and external audits. Ultimately, he recommends that at least two strategies be used in a research study. Thus, this study used a variety of techniques to ensure validity. In particular, this study utilized data triangulation, peer debriefing, rich description, and clarification of researcher bias.

Data triangulation involves using multiple sources of data (Creswell, 2013). First, participants’ typed responses to the noticing prompts for each of the six artifacts of practice provided initial insight into how each PSMT notices students’ mathematical thinking on tasks. After each artifact of practice was reviewed, the researcher interviewed each PSMT individually to probe for further information about how he/she was making sense of students’ thinking. Each interview was videoed and transcribed verbatim by a paid transcriptionist for analysis. These sources of information, along with researcher notes and memos, comprised the study’s data sources. Altogether, the data from the various sources can be corroborated and any weaknesses should be compensated for by other data (Patton, 2002).

Peer debriefing is an external check by a peer who is willing to question the researcher about the methods and interpretations that are made (Creswell, 2013, p. 251). For this study, peer-debriefing sessions with an expert in the field were held weekly during data collection and at least once a month during data analysis. This expert asked questions to
ensure the researcher was being honest and considering the data from a variety of meanings and interpretations. A typed record of notes from these meetings was kept.

Rich description involves providing detailed accounts of the context, themes, and cases in the study (Creswell, 2013). According to Creswell (2013), rich description “enables readers to transfer information to other settings to determine whether the findings can be transferred” (p. 252). Thus, a rich description of each case is provided. Also, direct quotes are used to illustrate important ideas that emerged.

By clarifying researcher bias in the study, the reader will have a better understanding of how past experiences, biases, and orientations may impact my interpretation of the data (Creswell, 2013; Merriam, 1998). Therefore, the researcher has provided a subjectivity statement in an effort to be transparent about her background as both an educator and a researcher.

**Reliability.** Reliability, in the traditional sense, “refers to the extent to which research findings can be replicated” (Merriam, 1998, p. 205). However, in qualitative research, this idea is often problematic as studies are not easily replicable since they have unique contexts (Merriam, 1998). Thus, the goal for qualitative researchers is to establish that the results are consistent with the data (Merriam, 1998). Merriam (1998) recognizes three ways qualitative researchers can establish reliability: being open about the investigator’s position, triangulating the data, and establishing an audit trail. As such, this study includes a subjectivity statement to establish my position as the researcher. Also, multiple sources of data were used in an effort to strengthen the consistency of results. Further, data collection and analysis procedures are carefully noted in the study. This
combined with the use of researcher memos provide an audit trail for the authentication of the findings.

**Ethical Issues**

In any research study, it is important to consider the ethical issues that might arise (Creswell, 2013; Merriam, 1998). Therefore, consent was obtained from the NC State University Institutional Review Board to conduct this study. Participants were informed of the purpose of the study and had to sign a legal consent form prior to participating. There were no risks for the participants in this study and pseudonyms were used to protect each PSMT’s identity. While participation in the study was voluntary, each participant received a stipend for his or her involvement in the study. Finally, data was stored electronically on a password-protected location as required by the university.

**Subjectivity Statement**

While teaching high school mathematics, I taught in a one-to-one laptop district. I also taught at an Early College High School where access to technology and other resources was not an issue. So, early in my career, I fell in love with the affordances of technology to teach mathematics. As a result, I attended many professional development sessions related to teaching math with technology. When using technology in my classroom, I found that students were not only more engaged in doing math but they were also making connections they had not made before without the technology. Needless to say, technology, particularly dynamic geometry software, easily became a staple in my classroom.

During this time, I also had the privilege of beginning my work as an adjunct instructor for East Carolina University. There, I have worked with elementary and middle grades PSMTs in both face-to-face and online content courses related to geometry, discrete
mathematics, and mathematical modeling. It was in this position that I first began realizing that many PSMTs are not comfortable with the mathematics content they will be teaching, much less mathematics content beyond what they will be required to teach.

It was during these first few years as an educator that I also became interested in knowing more about how teachers, both prospective and practicing, make pedagogical decisions about what is best for their students. How do math teachers decide how to present content? How do they decide when a lesson should incorporate technology? How do they decide when a concept lends itself to discovery? How do they know if their students are really learning? These questions, and more like them, fueled a passion inside of me to know more about teaching and learning mathematics and ultimately, led me back to graduate school.

In graduate school, I began working for a project developing materials to prepare teachers to teach middle and secondary mathematics with technology. During my time on the project, the team was working to develop facilitator’s guides for the geometry materials, to develop an algebra module and its facilitator’s guide, to maintain a national community of shared materials between technology using mathematics teacher educators, to research mathematics teacher educators beliefs and practices with technology, and to research students’ uses of dragging (in dynamic geometry software) for examining geometric representations of functions. While working on this project, I also served as a graduate teaching assistant to teach a course titled Teaching Mathematics with Technology at NC State University. Additionally, I have been working with colleagues on some external research to develop a framework to facilitate teachers in evaluating and designing tasks that utilize dynamic geometry software.
It was from this work and my previous experiences that I became interested in teacher knowledge and wanted to know more about how teachers make sense of students’ work. However, I first realized that it was important to be able to articulate my own philosophy of education. By doing so, I would not only be aware of my own positions, but I would also be able to share them with others in the spirit of being transparent. From reading literature on teaching and learning mathematics and reflecting on my own experiences, it was clear to me that I agree with the ideas behind reformed mathematics. Mathematics is not a subject to be memorized but one to be explored and investigated. Rather than transferring knowledge to others, educators should provide a variety of experiences (e.g., use of technology, manipulatives, modeling eliciting activities, etc.) for learners to build on their current knowledge. I also believe that there is a social component to learning meaning that learners should be given the opportunity to work together to solve problems and make connections about relationships in mathematics. Thus, my philosophy does not pertain to one age group; I believe it applies for any educator or learner of mathematics.

When it was time to decide what I would like to investigate for this study, I immediately knew I wanted to work with PSMTs. Although I have had experience working with prospective teachers at all levels, I felt I could best relate to PSMTs since I was once one myself. Further, I knew from the literature that there has not been much work with PSMTs in my area of interest around teacher noticing. Therefore, I decided to do an exploratory multi-case study where I facilitated interactions between the participants and myself to learn about how they engage in teacher noticing. I was confident I would be able to, using knowledge from my background working with prospective teachers, establish a
rapport with each PSMT such that he/she felt comfortable sharing his/her perspectives on students’ thinking.

**Chapter Summary**

This chapter provided the details about the context and design of this exploratory multi-case study. In all, eight PSMTs from a secondary mathematics education program agreed to participate in this study and four of these eight were selected as cases. Qualitative research methods were used to collect and analyze the data to answer the questions for the study. The next chapter presents the findings for the four selected cases.
CHAPTER 4: CASE ANALYSIS

In this chapter, an analysis is presented of each case. For each, a description of the case is provided and followed by details about how the PSMT was noticing on the paper and pencil and technology tasks. Recall that for each artifact, the PSMT provided a written response for each of the noticing prompts (i.e., attending, interpreting, responding). These written responses were categorized according to the noticing frameworks (i.e., Tables 4, 5, and 6) and then coded using the descriptive codes. Follow up interviews were only coded using the descriptive codes. Thus, for each case, the framework categorization, summary tables of descriptive codes for all pieces of an artifact (i.e., student work or group video clip), and a summary of findings from the case are presented in accordance with the research questions.

Chase

As a child, Chase shared he and his family moved often. He had the opportunity to live in Chicago, Florida, Maryland, Philadelphia, and North Carolina. Prior to returning to school to become a high school mathematics teacher, Chase held a Master’s degree in Mathematics and was a professional chess player. Chase had also tutored for Sylvan Learning Center and found that he enjoyed tutoring children in both mathematics and in chess. These experiences led Chase to pursue his teaching licensure through the Master of Arts in Teaching (MAT) program at his local university.

Chase’s background with mathematics courses included taking algebra, geometry, pre-calculus, and calculus A/B while in high school. Chase noted he failed pre-calculus the first time through. As an undergrad, Chase reported he took approximately eight to ten mathematics courses (e.g., linear algebra, numerical analysis, differential equations) and
found he especially enjoyed abstract algebra. As a graduate student, Chase shared his specialties were computer algebra and real analysis and that he was interested in topology. Chase also shared that he had very few statistics courses and that he did not do well in the few he did have. Chase said he was comfortable with high school mathematics topics, and because of his MAT program, he had become more adept utilizing technology in mathematics. At the time of the study, Chase reported he was somewhat comfortable utilizing The Geometer’s Sketchpad for the teaching and learning of mathematics but that he was more comfortable using GeoGebra, a similar dynamic mathematics tool. Chase also noted despite his comfort level with these tools, that he still had moments of frustration when the software would not do what he desired.

At the time of the study, Chase had already completed the Teaching Mathematics with Technology course and was enrolled in the semester prior to his student teaching experience. By random assignment, Chase completed and received artifacts for the technology task first followed by the paper and pencil task. However, here, discussion of the tasks is ordered based on the research questions in the study.

**Paper and Pencil Task**

This section includes a summarized account of how Chase approached the paper and pencil task during the task-based interview. His work on the task is also included in Figure 11. Following this description, details about and analysis of Chase’s responses for each artifact from the paper and pencil task will be discussed in order to answer research question 1.

Chase’s first attempts on the task included labeling the given information and the realization that to find $BC$, you need to first find $AC$. He looked for ways he might make use
of triangle $ABE$ or the law of sines and cosines to find $AE$, but pointed out that he did not know any angles to pursue this approach.

![Figure 11](image.png)

*Figure 11.* Chase’s work on the paper and pencil task.

Chase’s work on the task also included an algebraic approach. Chase centered the figure on a coordinate plane using point $A$ as the origin. He drew in circles and other pieces from the original figure to obtain the lower image shown in Figure 11. His goal was to use equations for circles $A$ and $B$ along with equations for the $x$-axis and the tangent line, $\overline{DC}$, to find $AC$. 
Towards the end of his work on the task, Chase assumed triangle $\triangle ADC$ was similar to triangle $\triangle BEC$ and made use of a proportion to find $BC$, as shown in Figure 11. Chase noted this approach was “so much easier” and that it took him much longer than it should have to solve the problem. Chase also pointed out that he needed to prove the triangles were similar by showing that two pairs of angles were “the same.” He said angle $C$ was clearly the same in both triangles and that $AD$ and $BE$ are parallel because otherwise $DC$ would not be able to be tangent to the circles. At this point, the interview was ended due to time. Chase reiterated he needed a better proof that $AD$ and $BE$ are parallel and believed that he would eventually be able to prove the triangles were similar.

Throughout his work on the task, Chase continually monitored his own thinking and evaluated whether or not his strategies were productive in moving him towards a valid answer. Further, Chase’s perseverance and need for mathematical proof of the soundness of his approach were hallmarks of his time on the task. These characteristics also became evident as Chase made sense of student work, as described in the next section.

**Attending.** To answer research question 1a, Chase’s responses to the attending prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were considered. This section provides an analysis of how Chase attended to students’ thinking for both the student work and the group video clip from the paper and pencil task in his written responses and during interviews.

Recall from Chapter 3 that the written responses were classified according to the noticing frameworks (i.e., Tables 4, 5, and 6). It is important to note that the classifications of written responses to the noticing prompts were made prior to any descriptive coding of the data. Further, the classifications of written responses were based on the extent to which
evidence was present that the PSMT was attending to the mathematical details of the students’ approach to the problem, interpreting the students’ mathematical understandings, and choosing to respond based on students’ thinking from the task. To assign descriptive codes, data was reviewed sentence-by-sentence and broken into segments of data that were focused on the same thought. Thus, segments of data might have been a single sentence or multiple sentences depending on the context. Also, a segment of data might have been assigned multiple descriptive codes if necessary. It is also important to note that if a thought appeared in the PSMT’s written response and the same thought appeared in the interview transcript, both instances were coded using the appropriate descriptive code. Although some might consider this “double counting,” there was no clear alternative, thus the researcher chose to consistently code all of the data. This decision also ensured that the salient ideas that appeared in the written responses and again appeared in the interview data were brought to light.

Thus, it should be noted that it was possible for the PSMT to have a low classification of his/her written response according to one of the noticing frameworks, yet still have descriptive codes that seemed to contradict this classification. For example, in a written response for attending, a PSMT could have provided a brief overview of the students’ approach to the problem without giving specific details, which would have been classified as a 1. Just considering this classification, it would not appear the PSMT engaged in actions indicating he/she was attending. But, during the descriptive coding of this written response, it was still possible that the code describing would have appeared, an action that indeed indicated the PSMT was attempting to discuss what the students did in the problem. The distinction is that the noticing framework was measuring the extent evidence of the
noticing action was present in the written response, while the descriptive coding simply noted the presence of any indication the PSMT was attempting to discuss the students’ approach.

In Table 10, the categorizations of Chase’s written responses to the attending prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the attending prompt on the paper and pencil task. The frequency counts of the descriptive codes are shown in an effort to create a picture of the ways in which Chase was noticing.

Table 10

*Chase’s Attending on the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Attending Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>1</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>11</td>
</tr>
<tr>
<td>Reflecting</td>
<td>4</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>3</td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>2</td>
</tr>
<tr>
<td>Discussing beliefs</td>
<td>1</td>
</tr>
</tbody>
</table>

Summary. Throughout his responses and interviews regarding the attending prompt for the artifacts from the paper and pencil task, Chase mostly focused on describing what the students did in the problem. However, Chase’s descriptions often lacked specific details about the students’ mathematical approach. For example, during the interview regarding the group video clip, Chase described that he saw the students write things like sine, cosine, and tangent down and that the students used a saying, “Oscar has a heap of apples” to do this.
He also mentioned when the students said things like, “What was the thing we learned, like cosine and all of that, couldn’t we use some of that?” that is an example of “them like…they’re trying to apply things they know.” In this case, Chase was describing what the students were doing: they were using a verbal statement to help them record the trigonometric functions they had previously learned. Such instances of Chase describing the students’ work and actions were readily present in his responses for the paper and pencil task.

Chase also showed evidence of reflecting throughout his consideration of the artifacts associated with the paper and pencil task. In one interview, in order to provide Chase an opportunity to expand upon his responses to see if he might provide more specific details about how the students approached the problem, the researcher asked Chase if there was anything else he wanted to add about what the students did in the problem. Chase responded,

Um…I’m never exactly sure how to answer that question. Like I don’t wanna write down every single thing they did…I try and get a sense of…they did Pythagorean theorem, they labeled it, the Pythagorean theorem didn’t really work. They started to realize the Pythagorean theorem couldn’t work after a while. Um…no I’m good. This response was coded as reflecting since Chase noted he was not sure how to answer the question and stated that he did not want to write down everything the students did. Chase revealed in this excerpt that his goal in answering the attending prompt was to try to provide an “overall sense” (his words) of what the students did in the problem. While attending, giving an overview of the students’ approach appeared to be sufficient and specific details did not seem to be viewed as necessary.
**Interpreting.** To answer research question 1b, Chase’s responses to the interpreting prompt (i.e., Please explain what you learned about these students’ mathematical understandings.) were considered. This section provides an analysis of how Chase interpreted the students’ thinking for both the student work and the group video clip from the paper and pencil task in his written responses and during interviews. In Table 11, the categorizations of Chase’s written responses to the interpreting prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the interpreting prompt on the paper and pencil task.

Table 11

*Chase’s Interpreting on the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Interpreting Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>0</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>10</td>
</tr>
<tr>
<td>Reflecting</td>
<td>9</td>
</tr>
<tr>
<td>Evaluating</td>
<td>7</td>
</tr>
<tr>
<td>Describing</td>
<td>4</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>3</td>
</tr>
<tr>
<td>Discussing students’ language</td>
<td>3</td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>2</td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>1</td>
</tr>
<tr>
<td>Discussing group/pair dynamics</td>
<td>1</td>
</tr>
<tr>
<td>Inferring; non-mathematic</td>
<td>1</td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** While interpreting students’ understanding, Chase did share some concepts he thought the students understood. When Chase was *inferring about student*
understanding, he was stating what he believed the students understood mathematically. However, Chase did not consistently provide evidence to support his claims about what the students understood. When he was asked to provide evidence, his answers were often vague, but if he was asked specifically by the researcher to show where something he referred to occurred, he did so. Thus, it is important to note that one can make inferences about student understanding without providing evidence to support one’s claims. For example, Chase stated, “These students clearly understand the Pythagorean theorem. They want to use it as a tool to solve problems;” however, he did not provide insight into why he believed the students understood the Pythagorean theorem. Chase’s statements regarding students wanting to use the Pythagorean theorem as a tool to solve problems might have been an indication of student understanding to him, but this was not clear based solely on his written response.

During a follow up interview, the researcher asked Chase what evidence he had that the students understood the Pythagorean theorem. Chase responded, “They applied it correctly twice.” Chase also shared that he believed the students wanted to use the Pythagorean theorem as a tool to solve problems because “they just started using it” without knowing how it was going to help them in the problem. To Chase, applying the Pythagorean theorem correctly was sufficient evidence of understanding, even if it was applied when it might not be useful in solving the problem.

Chase was also asked to clarify what, for him, was the difference between familiarity and mastery of concepts, as Chase had stated the students had “some familiarity though not mastery with special triangles and trig functions.” Chase shared that the students knew what special right triangles were, but that they did not know how to use them. He said, “So to be
familiar means to have heard it before, um, and to know something like, this is what it is. Mastery is, this is what it is, and this is how to use it, and this is how to make it um…to be like artful with it, to know like when to apply it and when not to apply it….” Through this statement, Chase discussed his conceptions of understanding. He revealed that for him there were two categories of understanding: familiarity and mastery. Familiarity meant the individual could recall the concept or formula while mastery meant the individual knew how and when to utilize the concept or formula.

While interpreting, Chase also spent much of his time reflecting and evaluating. An example of when Chase was reflecting was when he shared the idea that the students did not seem to know how to approach the problem. Chase said that he has found students often just perform a lot of calculations without “having a plan in their head of like, ‘okay, I’m gonna do this calculation, then I’m gonna do this calculation and that’s gonna give me this and that’s gonna help me get to where I want to go.” He stated that this means students end up wasting time and making errors because they are just plugging numbers into their calculators without thinking about how to solve the problem. Chase noted that finding and evaluating different approaches to problems was “a complicated thing.” He continued, “There is no clear way to teach this. Perseverance and a wide breath of knowledge help but teaching creativity and resourcefulness in problem solving is a difficult but important goal.” Here Chase revealed his thoughts about the importance of utilizing an appropriate approach to a problem and how after viewing the students’ work on the problem, he realized teaching students how to determine an appropriate approach was likely a challenging goal. In fact, in Chase’s reflective comments while interpreting, Chase often imagined himself as the teacher and provided insight into what he would do to combat various issues for students.
While not quite as prominent, *evaluating* was another descriptive code that frequently appeared while Chase was interpreting. An exemplar of this was when Chase shared, “They don’t have a good grasp of working backwards or the desire to make everything an algebraic equation.” In this statement, Chase evaluated the students ability to work backwards and implied their work might have been better had they utilized an algebraic equation. The tone of this statement suggested Chase had an expectation the students failed to meet.

**Responding.** To answer research question 1c, Chase’s responses to the responding prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were considered. This section provides an analysis of how Chase planned to respond to students’ thinking for both the student work and the group video clip from the paper and pencil task in his written responses and during interviews. In Table 12, the categorizations of Chase’s written responses to the responding prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the responding prompt on the paper and pencil task.

Table 12

*Chase’s Responding on the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Responding Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
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<td>Student Work</td>
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<tr>
<td>Group Video Clip</td>
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</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Reflecting</td>
<td>5</td>
</tr>
<tr>
<td>Orienting and focusing</td>
<td>4</td>
</tr>
<tr>
<td>Drawing from experience to select example</td>
<td>3</td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>3</td>
</tr>
</tbody>
</table>
Table 12 Continued

<table>
<thead>
<tr>
<th>Establishing context</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Exploring mathematical meanings and relationships</td>
<td>2</td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>2</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of similarity</td>
<td>2</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of dilation</td>
<td>1</td>
</tr>
<tr>
<td>Extending thinking</td>
<td>1</td>
</tr>
<tr>
<td>Needs time or unsure how to respond</td>
<td>1</td>
</tr>
<tr>
<td>Starting with a simple or familiar case</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** While responding to students’ thinking on the paper and pencil task, it was apparent Chase was reflecting and orienting and focusing. Another interesting code that appeared was drawing from experience to select example. In fact, when providing some example lessons he might utilize, Chase drew from his experience twice to select examples. The first time was when he described his choice to use the Incan temple as a context for a problem involving similar triangles based on a lesson he had seen a friend use. The second was when Chase described what an example of a lesson that teaches students to work backwards might look like. He shared:

I saw this guy like do a lesson on that once. Um…the question was like a ‘you have a pond…and there’s lily pads in the pond and the first day there’s one lily pad and the second day there’s two, four, and so on right, and you’re trying to figure out which day is the pond half full.’ And…you know…most kids rush in and they’re starting with like, “Okay, there’s 2,000 lily pads in first day one to…right, I gotta like figure this out” and some smart kids says, “The day right before the last day,
right” the…in the problem, the lily pad pond is filled up in a month and they say, you know, “29 days right?” um…yeah…

In each of these instances, Chase used something he had seen before as a basis for his example or lesson idea.

Chase also had the idea of creating a lesson to help students know which “pathway” to use when solving a problem. He discussed that a way to have students focus on the important aspects of the problem is to first “envision the solution and then…you wanna have a clear idea of path that’s getting you towards that solution before you start like crunching numbers.” He continued:

I talked in there about how you could have like maybe a lesson that has different approaches and you could talk, ‘Like okay, that’s a good approach. That’s gonna work. That’s a good approach. That’s gonna work. This is a good approach. Which one is the best approach? Which one is the clearest? Which one will let you check your work? Which one uses the fanciest tools? Is there like something aesthetically pleasing to do it?

In this excerpt, Chase confirmed that he believed such a lesson would help the students become aware of which approaches were available and teach them which approach was most appropriate. Through this lesson, Chase’s goal was orienting and focusing the students on the relevant information to enable problem solving.

Another idea that surfaced frequently with Chase while responding was that students would benefit from having a “procedural” (his words) lesson that was focused on how to solve problems involving similar triangles. Chase also desired to incorporate a “conceptual” (his words) lesson to help the students understand why there are relationships between the
corresponding sides of similar triangles. However, throughout the descriptions of how he would respond, Chase did not explicitly provide a rationale related to the students’ thinking on the problem. Rather, Chase spent a good amount of time reflecting and discussing more generally how he would handle instruction beyond this task. This idea was best illustrated in an interview. Chase provided detail about his intentions for a “procedural” lesson. He described that if the students knew this problem dealt with similar triangles then they would have been able to solve it quickly. As he drew Figure 12, he noted:

I would probably do something like I would draw an angle there and an angle here and maybe some angles here, maybe some side lengths here would be….and then they would be like, “Oh, okay, well we can see if that if we know this angle, we can find out this angle and then we know they’re the same and then we can set up some sort of ratio and we can find something we’re missing” and they need to do that like three times.

![Figure 12](image)

*Figure 12. Chase’s representation of dilations using matrices.*

Through this approach, Chase described that he would have lead the students to realize that these triangles were similar in order to know the procedure to solve the problem. He also included that he would have the students practice this procedure a few times. Chase went on
to describe how he would make use of a “demonstration” problem that was “part way done for them” as an introduction to the task. Both of these instances were coded as gathering information, checking for a method, leading students through a method as Chase was focused on having the students recall (or learn) and appropriately apply a procedure to solve problems that involve missing parts of similar triangles.

Chase spent another part of this interview reflecting on his idea of utilizing a “conceptual” lesson, which also revealed some conceptions about his own understanding of similarity, as shown in the interview excerpt below.

Researcher (R): Okay, um, you also stated, “You might work on underlining the conceptual foundation of similar shapes.” So what do you mean by this, what is the conceptual foundation?

Chase (C): So I don’t know, I have been thinking about this like…why is it true that like there has to be a relationship between $\overline{AC}$ and $\overline{BC}$ that has to? Like I mean, what I’m thinking is that there’s content knowledge, which is missing here, right?

R: For you, or for the students?

C: For the students on similar triangles. And I want to sort of build this content knowledge up, but I don’t wanna just like tell them this is similar triangles, obey a ratio, I wanna give them a like a reason why, um, and I’m honestly not that sure about the reason why other than it just like…it makes sense that the wider you make the angle and keeping the sides is, or at lease one side the same. Now, it just makes sense that the wider you make the angle, the more the…I don’t know…I’m not saying this very well.

R: That’s okay. You’re still thinking about that a little bit…
C: Yeah, I don’t…I know there has to be a relation between like $\overline{AD}$ and $\overline{BE}$ because angles are the same, but I’m not sure how to say that beyond I just know there has to be one.

R: Okay, so it sounds like when you talk about working on the underlining conceptual foundation, you mean that there’s some content knowledge they’re missing that you need to help them gain?

C: Yes.

Through the excerpt above, Chase revealed that the students needed to explore mathematical meanings and relationships through his idea of a “conceptual” lesson. Chase shared that he desired to help the students understand why there has to be a relationship between $\overline{AC}$ and $\overline{BC}$; however, he stated that he himself was still unsure about the reasoning why. Thus, through his reflections, Chase not only shared what he wanted for students but also disclosed conceptions of his own understanding of similarity.

**Technology Task**

This section includes a summarized account of how Chase approached the technology task during the task-based interview. Following this description, details about and analysis of Chase’s responses for each artifact from the technology task will be discussed in order to answer research question 2.

Throughout this task, Chase followed the prompts provided and continually made conjectures. For example, after dilating the figure, Chase said if you made a line through $P$ and $A$ then the line would also go through $A'$ and that the same was true if you drew a line from $P$ to $B$, it would go through $B'$. Chase also frequently tested his conjectures. For instance, after noting that the distance between points $P$ and $A$ is half the distance from $P$
and \( A' \), Chase dragged points \( P, A, \) and \( A' \) around to see if this relationship held true. Then, in part D, he recorded the following: “So I notice that the line \( PA' \) goes through \( A \) no matter where we put \( P \) or the line segment \( AB \). Point \( A \) bisects \( PA' \). I think if we changed the scale factor this would change. The same rule applies with the other points.” Chase went on to share that dilations are a multiplicative relationship that can be represented by a scalar, \( k \), times a matrix to determine the position of the image, as shown in Figure 13.

![Figure 13. Chase’s representation of dilations using matrices.](image)

Chase also consistently commented that he wanted to prove his conjectures even prior to reading part F of the task, which did ask for proof of observations that were made. When Chase reached part F of the task, he noted he had several conjectures he needed to prove. Chase recalled the idea of representing dilations using matrices in an attempt to prove: (a) that \( PA' \) goes through \( A \) no matter where \( P \) or \( AB \) is positioned; and (b) that the distance between point \( A' \) and the origin is a scalar factor times the distance between point \( A \) and the origin. He recorded the following under part F in his sketch:

Assume without loss of generality (I hope) that \( P \) is at the origin \((0, 0)\). Then for a set of points that define a figure. Dilation can be performed by scalar multiplication on these points. Dilation by scale factor \( k \), is \( k(x_1, y_1), k(x_2, y_2) \). Point \( A = \)
\((x_1, y_1) \rightarrow A' = k(x_1, y_1) \rightarrow A' = (kx_1, ky_1) \rightarrow \sqrt{[(x_1)^2 + (y_1)^2]}\) distance from \(A\) to origin, \(\sqrt{[(kx_1)^2 + (ky_1)^2]} \rightarrow k\sqrt{[(x_1)^2 + (y_1)^2]}\). Which is a proof cause I say so about the relation between the distances from part D.

Through his work on this task, Chase revealed he was focused on an algebraic approach when thinking about the dilation. It was also evident that while Chase was comfortable making and testing his conjectures in the software program, he desired mathematical proof for each conjecture he made. Unfortunately due to time, the interview was stopped before Chase had the opportunity to provide proof for each of his observations.

**Attending.** To answer research question 2a, Chase’s responses to the attending prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were considered. This section provides an analysis of how Chase attended to students’ thinking for both the student work and the group video clip from the technology task in his written responses and during interviews. In Table 13, the categorizations of Chase’s written responses to the attending prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the attending prompt on the technology task.

<table>
<thead>
<tr>
<th>Attending Framework Categorization</th>
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<tbody>
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<td>Student Work</td>
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<td>Group Video Clip</td>
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<th></th>
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<tbody>
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<tr>
<td>Reflecting</td>
<td>5</td>
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<tr>
<td>Discussing students’ language</td>
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Table 13

*Chase’s Attending on the Technology Task*
Table 13 Continued

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<tr>
<td>Discussing group/pair dynamics</td>
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</tr>
<tr>
<td>Discussing PSMT’s language</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>1</td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>1</td>
</tr>
</tbody>
</table>

Summary. Throughout his responses and interviews regarding the attending prompt for the artifacts from the technology task, Chase mostly focused on describing what the students did in the problem. For the student work artifact, Chase provided an overview of the approach and reflected on his expectations of what he thought the students did in the problem. After reviewing the group video clip of the technology task, Chase provided a very descriptive written response of what the students did in the problem and included some evidence from the artifact. This included providing descriptions of the students’ conjectures, actions, and recorded observations. Overall, Chase’s attending responses for the group video clip were more detailed than those he provided for the student work from the technology task.

While describing the students’ work, Chase also carefully distinguished between the use of language of the students and himself. For example, Chase noted the students’ use of the word “scale” within the context is questionable. Additionally, he pointed out when he referenced the students’ observation regarding the length of the image that the use of the word image was his own and not that of the students. Chase revealed that when describing students’ work, he paid close attention to students’ language usage in an attempt to interpret what the students actually meant when their language was imprecise. Chase noted, “I tend to think they’ve understood what they’re saying, even if they didn’t say it right.” In other
words, to Chase, an unclear use of terminology did not necessarily discredit students’ understanding.

Chase also pointed out things that he noticed the students did not do in the problem. In these cases, Chase was reflecting. For instance, Chase shared the students did not make the observation that $P$, $A$, and $A'$ were collinear, an observation he himself noted while completing the task. Another of Chase’s observations was non-mathematical in nature. This occurred when Chase mentioned the group/pair dynamics; he pointed out that the students did not disagree with each other at any point on the task. He shared: “I wanted more emotion out of them when they were doing the task. I don’t know, they should be like, I don’t know, excited or arguing or I don’t know, something.” Through this statement, it was evident that Chase was reflecting on his expectations of what should happen between students while working on a mathematics problem, one of which included the presence of argumentation.

**Interpreting.** To answer research question 2b, Chase’s responses to the interpreting prompt (i.e., Please explain what you learned about these students’ mathematical understandings.) were considered. This section provides an analysis of how Chase interpreted the students’ thinking for both the student work and the group video clip from the technology task in his written responses and during interviews. In Table 14, the categorizations of Chase’s written responses to the interpreting prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the interpreting prompt on the technology task.
Table 14

Chase’s Interpreting on the Technology Task

<table>
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<tr>
<td>Describing</td>
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</tr>
<tr>
<td>Inferring about student understanding</td>
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<td>Inferring about student action</td>
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<td>Evaluating mathematical correctness</td>
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<td>Evaluating</td>
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<td>Discussing conceptions of understanding</td>
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<tr>
<td>Comparing to self</td>
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</table>

**Summary.** While interpreting students’ understanding for the technology task, Chase mostly reflected, described, and made inferences. Chase’s reflective thoughts revealed his perspective regarding students’ understanding. To Chase, if the students did not do what he expected, he concluded the students did not understand the concept. This idea surfaced during the follow up interview for Chase’s responses to the student work. For Chase, the “proof” the students provided in the student work for part F of the task did not constitute a proof. He indicated in order for these students to have a mathematical proof, they needed to include specific measurements, clearly state the relationship between a pre-image and its image, and include a statement regarding the relationship holds regardless of the position of the points. To Chase, this would “be a proof for high school and not a proof for college.” A proof for college would need to “include a mathematical derivation of the lengths, and then
reasons why…and then some sort of like algebraic thing showing that the lengths have to obey the ratio involved.” Therefore, Chase evaluated the mathematical correctness of the students’ proof stating, “even though they make a good…reasonable argument in favor of their conclusions being true…they don’t have a clear expression of what they’re trying to say, and they don’t have a full proof argument.” And, for Chase, because the students did not provide what he considered mathematical proof of their observations, he determined the students did not understand what was meant by a proof.

Chase also shared that he based what the students understood on what they said (or did not say) and whether or not they made wrong conclusions. Further, for Chase, there seemed to be a degree of understanding for the students. For example, Chase stated that the students “understand dilation, more or less.” Chase revealed that because these students do not make any wrong conclusions and because they say the same thing in two different ways (i.e., “the distance of $AB$ is half of the distance of $A'B'$” and “the distance of $CD$ is double the distance of $C'D'$”), that he believed they understood dilation. (Note: Chase cited the students here; their language was imprecise. For example, rather than state the distance between points $A$ and $B$ or the length of $AB$, the students referred to $AB$ as “the distance of $AB$”. ) The only aspect of dilation he believed the students did not understand was the relationship of the position of the image to its pre-image “because they didn’t talk about it.” Still, Chase was convinced the students understood dilations; he stated: “I mean they understand the important parts of it and there might be this like one detail about like lines going through in the position that they don’t understand, which is fine.” To Chase, this meant that he believed the students understood what he considered the most important
aspects of the concept (i.e., dilation) but that they still might have some knowledge gaps (i.e., that the center, image, and pre-image are collinear).

Finally, Chase seemed to have difficulty connecting what he viewed in the artifacts back to the students’ mathematical thinking. An example of this was revealed during the interview regarding the group video clip. During the interview, Chase was asked to clarify some of his written statements and to provide evidence to support his claims. For instance, Chase had mentioned in his written response that the students were focused on number during the task. In the following excerpt, the researcher asked Chase to clarify what he meant by that statement and what the students’ focus revealed to him about their understanding.

R: Okay, so you mentioned the students were focused on number while working on the task. So, for example, you mentioned, “When you work on the task you don’t even see number, but they were very”…I think your direct words were…“they deliberately mentioned number.” So what evidence do you have that they’re focused on number?

C: Well they just read it out a few times in the video, I don’t know, let’s see in here, but um, they read out like the length of it, they would be like “All right, it’s 11.175 or 7.531,” they weren’t like uh, (looked through transcript and pointed to line 267) oh yeah, “A gives it a distance 8.28 after you move everything around.” Or I think they did it again, “Or had a distance of 7.04, or had a distance of 14.08” (pointed to lines 218-219 in the transcript) they say…they say the whole thing, um.

R: So what did that tell you about their understanding?

C: Uh, not a whole lot, that’s a very natural, normal thing to do. Um, I think it tells
me though, I think it tells me that having a number makes them feel comfortable. They are like “Okay, I’ve measured this, now I understand it because I know how long it is, that’s sort of what an answer looks like to me, it looks like a number written down and not like a relation between things, and so, now I have an answer, I’m ready to go on to the next thing” sort of thinking, I don’t know.

In this portion of the excerpt, Chase described what the students did in the video that prompted him to say they were focused on number. Chase pointed out specific evidence from the transcript to show that the students read the measurements aloud. However, when Chase was asked what this focus on number revealed about the students’ understanding, Chase reflected on what this focus told him about how the students felt during the problem rather than what this meant they understood mathematically. He also noted that the students simply saw the lengths as numbers and not as part of a “relation between things,” and that because they had an “answer” they understood “it” (referring to the dilation). As Chase finished with “I don’t know,” it became apparent that Chase was unsure of how what he heard and saw in the student video was connected to the students’ mathematical thinking.

**Responding.** To answer research question 2c, Chase’s responses to the responding prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were considered. This section provides an analysis of how Chase planned to respond to students’ thinking for both the student work and the group video clip from the technology task in his written responses and during interviews. In Table 15, the categorizations of Chase’s written responses to the responding prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the responding prompt on the technology task.
Table 15

*Chase’s Responding on the Technology Task*

<table>
<thead>
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<th>Responding Framework Categorization</th>
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<td>Establishing context</td>
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<td>Using a by-hand approach</td>
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<td>Discussing beliefs</td>
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<td>Discussing PSMT’s understanding of dilation</td>
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<tr>
<td>Discussing PSMT’s understanding of geometry</td>
<td>1</td>
</tr>
<tr>
<td>Starting with a simple or familiar case</td>
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</table>

**Summary.** While responding to students’ thinking for the technology task, Chase provided some potentially useful ideas for what to do next with the students. For instance, Chase proposed students could make connections between the algebraic and geometric representations of dilations by using a familiar figure in the coordinate plane. Chase described this process in the excerpt below while he drew the image in Figure 14.
Figure 14. Chase’s triangle on the coordinate plane.

C: …I thought about doing like this sort of a task where it would be like a triangle here and then, like this might be the center of dilation and it would be like dilated out to a bigger triangle. Um…and then, I would have them sort of say where specifically these points are. I did this triangle because I thought it would be easy to notice that these points, they did…they did easily find that, you know, this is gonna be some point A, where this is (a, 0), it’s gonna be (2a, 0), something like that right?

R: Okay.

C: So they’re going to easily be able to find where this point is…this is gonna be…this is scale factor $k$, right? And then, they’ll be able to have…they could actually talk about the area involved here if they, we wanted to do that it would be interesting, and they would be able to have like an algebraic…now it’s easy to talk about the lengths right, because it’s $2a – 0$, you know, over $a – 0$ or something like that right?

R: So what gave you this idea?

C: What gave me this idea? Spit-balling ways to try and do it, you want to make it easy for them to get at the algebra where there’s like…with arbitrary points there’s a
lot of, a lot of different variables floating around, expression is a little more complicated. You have to have a square root for the distance formula. I think this makes it simpler and thus more accessible.

Here, Chase suggested that by using a figure the students were already familiar with, beginning to generate the algebraic representation of the dilation might be less complex and therefore was more accessible. Chase also shared he believed it was important for one to make as many connections between mathematical topics as possible because “you will remember things better if they’re connected to different parts of your brain and it’ll be more interesting to you.” Through this idea, Chase was attempting to have students make connections between mathematical ideas and improve their mathematical understanding.

While responding to the artifacts for the technology task, Chase also proposed some ideas that were not necessarily focused on mathematics or appropriate considering the mathematics from the problem. For example, Chase included an idea for having students generate a smiley face using dilations of just one circle. Chase shared he intended for the smiley face task to help students become more familiar with the software tool and have a better understanding of how the center of dilation and image were related. Here, Chase’s focus was not entirely mathematical; he desired for the students to better learn GSP. He also did not fully describe how he envisioned this task would help students better understand the relationship between the center of dilation and the image. And, while this task included the concept of dilations, the manner in which dilations was considered was vastly different from how they were considered in the technology task. In the technology task, similar triangles were used to discuss the properties of dilations.
Lastly, while responding, Chase did mention using both technology and a by-hand approach with students. When using technology as a tool, Chase proposed that students could extend their thinking by applying dilations to a different figure (i.e., circles). He also envisioned students could explore mathematical relationships by considering an algebraic operation to describe the effect of a dilation on a set of coordinates in GSP. Using a by-hand approach, Chase again considered how students could apply their knowledge of dilations in a new way by generating several dilations on a whiteboard. Further, Chase believed students using a by-hand approach would not necessarily learn anything different about dilations but that this approach would “make it very tangible” for students and that they would remember dilations better. However, Chase noted an affordance of using technology: “the technology can do this flexibility thing where you can change the lengths and observe the relation…I don’t think you can do that by hand.”

**Discussion of Chase**

The analysis of Chase’s written responses and interviews revealed Chase was noticing in a variety of ways. To summarize these findings, concise answers to each research question are provided below.

**Research question 1.** For this case, the researcher considered the question, “How did Chase notice students’ mathematical thinking when he examined artifacts of practice from a paper and pencil mathematics task?” In other words, in what ways did Chase attend, interpret, and respond to students’ mathematical thinking?

When attending to students’ thinking on the paper and pencil task, Chase provided an overview to the students’ approach with little evidence (classified as a 1) in his written responses for both the student work and group video clip. Also when attending, more than
half of the total count of all descriptive codes (i.e., 11 out of 21) from attending indicated that Chase was describing the student’s work. Another 19% and 14% of the total count of all descriptive codes for attending revealed Chase was reflecting and inferring about student action, respectively. For interpreting, Chase’s written responses were not focused on interpreting the students’ thinking (a 0) for each artifact. When interpreting students’ thinking, Chase was inferring about student understanding in about one-quarter of the total count of all descriptive codes (i.e., 10 out of 42) for interpreting. The descriptive coding also revealed that Chase was reflecting (i.e., 9 out of 42) and evaluating (i.e., 7 out of 42) while interpreting. For responding, Chase’s written response for the student work artifact did not include a rationale that was connected to students’ thinking on the problem (a 0), and for the group video clip, Chase’s written response included a vague rationale (a 1). While responding to students’ thinking on the paper and pencil task, the descriptive codes indicated that Chase was reflecting in 5 out of 27 instances of the total count of all descriptive codes for responding and orienting and focusing in 4 out of 27 instances of the total count.

**Research question 2.** For this case, the researcher considered the question, “How did Chase notice students’ mathematical thinking when he examined artifacts of practice from a technology task?” In other words, in what ways did Chase attend, interpret, and respond to students’ mathematical thinking?

When attending on the technology task, Chase’s written response for the student work included an overview to the students’ approach with little evidence (categorized as a 1). For the group video clip, Chase completely described the students’ approach to the problem and included some evidence from the artifact (a 3) in his written response. About
half of the total count of descriptive codes for the attending prompt indicated Chase was describing. Also, in 5 out of 24 instances of the total count of descriptive codes for attending, Chase was reflecting and in 3 out of 24 instances of the total count, Chase was discussing students’ language. While interpreting students’ thinking on the technology task, Chase’s written response for the student work lacked evidence to support claims that were made regarding students’ thinking (a 1) and his written response for the group video clip was not focused on interpreting students’ thinking (a 0). According to the total count of descriptive codes for interpreting, Chase spent about one-quarter of the time (i.e., 11 out of 43) reflecting and an equal amount of time describing and inferring about student understanding (i.e., 8 out of 43 each). For responding, Chase’s written response for the student work included a potentially useful problem with a vague rationale (a 1), whereas his written response for the group video clip included both appropriate and inappropriate responses without rationales (a 0). The total count of descriptive codes revealed that when responding, Chase was reflecting in 6 out of 27 instances of the total count of descriptive codes and selected examples that would have students exploring mathematical meanings and relationships in 5 out of 27 instances of the total count.

**Mary**

Mary grew up in Dubai and attended college in Oklahoma. While in high school and college, Mary enjoyed tutoring students. In college, Mary had the opportunity to have a teaching experience and found that she really loved teaching. However, she was already close to finishing her undergraduate studies in engineering. Thus, Mary completed her undergraduate degree and obtained a job as a structural engineer in Oklahoma. After about
five years, Mary moved and decided to pursue her teaching licensure through the Master of Arts in Teaching (MAT) program at her local university.

Mary’s background with mathematics courses included taking algebra, geometry, applied mathematics, and calculus while in high school. In college, Mary recalled taking calculus two and three, linear algebra, and differential equations. As a graduate student, Mary noted she had not yet been enrolled in a mathematics class but instead was taking mathematics education and general education courses. Mary reported being comfortable with high school algebra and geometry. She also stated she was rusty with calculus topics and did not like statistics. At the time of the study, Mary was enrolled in a teaching mathematics with technology course and reported that because of that course, she was comfortable teaching geometry using technology. She described that she was very familiar with The Geometer’s Sketchpad and had tried using GeoGebra, a similar dynamic mathematics tool, but that she was still not comfortable with that program.

At the time of the study, Mary had already completed the technology course but still had three semesters to complete prior to her student teaching experience. By random assignment, Mary received artifacts for the technology task first followed by the paper and pencil task. However, here, discussion of the tasks is ordered based on the research questions in the study.

**Paper and Pencil Task**

This section includes a summary of how Mary approached the paper and pencil task during the task-based interview. Her work on the task is also included in Figure 15. Following this description, details about and analysis of Mary’s responses for each artifact from the paper and pencil task will be discussed in order to answer research question 1.
After reading the problem, Mary labeled the given radii and labeled $BC$ as $x$. Then, Mary stated she saw two similar triangles and wrote that triangle $ADC$ was similar to triangle $BEC$ on her paper. When the researcher asked how she knew those triangles were similar, Mary stated, “Well they look similar…” When the researcher asked what was needed in order to prove two triangles were similar, Mary said you need three congruent sides or side-angle-side. Mary said she knew these triangles were similar but that she just could not remember why. At this point, the researcher asked if the triangles were similar, what would you do to find $BC$? Mary stated that she would set up a proportion and then wrote the following proportion on her paper:

$$\frac{AD}{EB} = \frac{AC}{BC}$$  

(1)
When the researcher asked how she determined this was the proportion to use, Mary stated she knew this because of the properties of similar triangles. She said she used \( AD \) and \( BE \) because they were given and, since \( BC \) was unknown, she had to use \( AC \) since it corresponded to \( BC \). She then went on to substitute what she knew into the proportion and solved for \( x \), as shown in Figure 15. Thus, Mary stated, the magnitude of \( BC \) was 12.

After finding the magnitude of \( BC \), Mary again stated she knew the triangles were similar, but she still was not sure how to show that they were. She stated that there is side-angle-side and side-side-side or maybe something with tangents going to the same point (as she points to \( C \)). So, while Mary finished the task confident in her response that the magnitude of \( BC \) was 12, she left the interview unsure of how to prove triangle \( ADC \) was similar to triangle \( BEC \).

**Attending.** To answer research question 1a, Mary’s responses to the attending prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were considered. This section provides an analysis of how Mary attended to students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 16, the categorizations of Mary’s written responses to the attending prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the attending prompt on the paper and pencil task.
Table 16

Mary’s Attending on the Paper and Pencil Task

<table>
<thead>
<tr>
<th>Attending Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>4</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>25</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>21</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>14</td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>3</td>
</tr>
<tr>
<td>Withholding assumptions</td>
<td>3</td>
</tr>
<tr>
<td>Discussing chronology</td>
<td>2</td>
</tr>
<tr>
<td>Discussing group/pair dynamics</td>
<td>1</td>
</tr>
<tr>
<td>Reflecting</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** Throughout her responses and interviews regarding the attending prompt for the artifacts from the paper and pencil task, it was evident Mary was focused on describing what the students did in the problem. In fact, describing accounted for just over one-third of the total count of descriptive codes for Mary’s attending on the paper and pencil task. Another characteristic of Mary’s attending was the inclusion of evidence to illustrate what she was describing. An exemplar of how Mary described students’ work and utilized evidence was her response to the attending prompt for the group video clip from the paper and pencil task.

The students found the length of segment $AB$ to be eight, “because the distance from $A$, from $B$, from center to out is three and from $A$ to center out is five” (lines 22-23, 39-41). They then tried to determine how to relate the different segments to each other by identifying triangles. They identified triangle $EBC$, but determined that it was not a right triangle and that they did not have enough information to use the
Pythagorean theorem (lines 90-95). They then identified triangle $EBA$ and applied the Pythagorean theorem despite it not being a right triangle (lines 107-129). They solved for “c” (lines 128-129), which they did not interpret to be segment $AE$ (line 157). They then tried to recall a relationship for 30-60-90 triangles (lines 159-213). With the teacher’s guidance, they settled on applying the Pythagorean theorem to triangle $ADE$ and found the length of segment $DE$ to be “about seven” (line 354).

Next, they tried to figure out how the segments they had already determined related to the question (lines 411-418) but they seemed to be unable to find any connections. The teacher then intervened and indicated that they had incorrectly applied the Pythagorean theorem (line 448), and helped them realize that $ADE$ was a right triangle due to the definition of a tangent (lines 474-516).

In her response about what the students did in the problem, Mary described each action the students performed while working through the task. She also provided direct evidence of her descriptions by citing the specific line numbers where these instances occurred.

While attending on the paper and pencil task, Mary also made inferences about student actions in about one-quarter of the total count of descriptive codes. The excerpt below includes an exemplar from one of Mary’s interviews where she was inferring about student actions based on considering the student work artifact.

Researcher (R): Okay. Why do you think Students 3 and 4 were writing things about trigonometric functions in 30-60-90 triangles?

Mary (M): Oh, um, well they presented…they drew a little table on 30-60-90. I saw it on Student 1 and 2 as well, um, and I wasn’t sure what that was, um, I’m guessing that’s what they were thinking because they had SOHCAHTOA written down and
then they had that, so I think they were trying to relate them trigonometrically, um, not successfully, but you know.

R: Okay. You mentioned that “The students did not determine the magnitude of $\overline{BC}$, but they do have some numbers on their papers, so for instance they have $b = 6.9$, $c = 8.54$.” So what do these numbers represent for the students?

M: Um, well, I think they were…what they were doing was, you know, they saw a triangle so they started to think of different things that related to triangles so they thought of Pythagorean theorem and also SOHCAHTOA and so, what they did was they found $c$ which is $\overline{AE}$ through the Pythagorean theorem, um, using $\overline{AB}$ and $\overline{BE}$ and then, they found the other segment, $\overline{DE}$, using um, Pythagorean theorem as well, um…using that first, um, segment $c$ as a hypotenuse, um, so yeah, that’s kind of what I determined.

In this exchange, Mary revealed that she was making some assumptions about what the students did in the problem based on what she saw written on their papers. Mary concluded that the students were trying to use trigonometry because she saw SOHCAHTOA and 30-60-90 written on their papers. And, she not only presumed the students had used the Pythagorean theorem, but also to which triangles they had applied the theorem.

**Interpreting.** To answer research question 1b, Mary’s responses to the interpreting prompt (i.e., Please explain what you learned about these students’ mathematical understandings.) were considered. This section provides an analysis of how Mary interpreted the students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 17, the categorizations of Mary’s written responses to the interpreting prompt are shown, as well as
frequency counts of each descriptive code that appeared in the data for the interpreting prompt on the paper and pencil task.

Table 17

*Mary’s Interpreting on the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Interpreting Framework Categorization</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>3</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Citing evidence</td>
<td>8</td>
</tr>
<tr>
<td>Inferring about student understanding</td>
<td>5</td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>4</td>
</tr>
<tr>
<td>Describing</td>
<td>2</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>2</td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of radius</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** While interpreting students’ understanding on the paper and pencil task, Mary notably provided shorter responses than she did when she was attending. Although her responses to the interpreting prompt were brief, Mary consistently focused on *citing evidence*, as found in 8 of the 23 instances of the total count of descriptive codes for interpreting on the paper and pencil task. One example, which highlights her attention to evidence, was her written response about the group video clip. In her response, it was apparent that Mary was providing evidence to support her claims by including specific line numbers from the video transcript as shown below.

They don’t know the notation for the magnitude of a segment, as they incorrectly refer to it as the absolute value of the segment (line 6). They understand that the
radius of a circle is constant (lines 22-23, 39-41). They are familiar with the Pythagorean theorem – they know the formula, and know that \( a \) and \( b \) represent the legs and \( c \) represents the hypotenuse (lines 128-129). However, they do not seem to know that this only applies to a right triangle (lines 113-116). There also seems to be a misconception that all right triangles have the measures 30-60-90 or 45-45-90 (lines 177, 185, 381, 393).

While interpreting, Mary also made **inferences about student understanding**. These inferences typically also included statements that **evaluated mathematical correctness**. An exemplar from the interview about the student work artifact illustrates this.

R: Okay, you also mentioned “The students are familiar with the Pythagorean theorem and that they incorrectly applied it to the triangle \( ABE \).” What does it mean to be familiar with something?

M: Mm-hum. Um, to know the formula and um, to kind of be able to say that’s, you know, side \( a \), that’s side \( b \), and that’s \( c \); um, however, they didn’t um, they didn’t show that this was a right triangle, triangle \( ABE \), and so, they’re only familiar with it, they’re not, you know, they don’t know it for sure.

R: So that would have been my next question, is being familiar the same as understanding it?

M: No, I don’t think so.

R: So how are they different?

M: How are they different? Um, being able to recognize when it’s applicable I would say is part of really understanding when you can and can’t use Pythagorean theorem, and how it applies. Um, yeah, so that’s one thing that they missed is that
you can’t just apply Pythagorean theorem to any triangle, um, it has to be a right triangle.

In this exchange, Mary revealed that she believed the students were familiar with the Pythagorean theorem, but that they did not understand the theorem. To Mary, understanding meant one knew the formula and the appropriate situation in which to apply it. On the other hand, familiarity meant one just knew the formula but applied it incorrectly. Thus, for Mary, her evaluation that the students incorrectly used the Pythagorean theorem revealed something to her about their understanding. She concluded that because these students applied the Pythagorean theorem to a triangle that was not a right triangle, they were only familiar with the theorem and that they didn’t “know it for sure.”

**Responding.** To answer research question 1c, Mary’s responses to the responding prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were considered. This section provides an analysis of how Mary planned to respond to students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 18, the categorizations of Mary’s written responses to the responding prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the responding prompt on the paper and pencil task.
Table 18

Mary’s Responding on the Paper and Pencil Task

<table>
<thead>
<tr>
<th>Responding Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>2</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>6</td>
</tr>
<tr>
<td>Starting with a simple or familiar case</td>
<td>4</td>
</tr>
<tr>
<td>Attending to student errors</td>
<td>1</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>1</td>
</tr>
<tr>
<td>Exploring mathematical meanings and relationships</td>
<td>1</td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>1</td>
</tr>
<tr>
<td>Reflecting</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** While responding to students’ thinking on the paper and pencil task, Mary again shared less than she did while attending. Further, the descriptive codes revealed that Mary typically chose to respond to students’ thinking in two ways: by gathering information, checking for a method, leading students through a method or by starting with a simple or familiar case. For both artifacts, Mary shared that she wanted to help students realize that the problem was dealing with similar triangles. Mary envisioned helping the students start with something more familiar, such as considering two similar triangles drawn separately and leading the students to realize the triangles were similar. Another idea Mary had was to redraw the figure from the problem without circles and to reorient the triangles such that point $C$ was at the top. In the following excerpt, Mary shared her intentions for this approach.
R: You mentioned, “You would have the students redraw the picture.” So I was wondering if you could draw the picture to illustrate how you would want students to redraw it?

M: Okay. So…(drawing the triangles as shown in Figure 16) five, three, this is not to scale.

R: That’s okay.

M: But yeah, so then maybe that will jog their memory in them then they can start to see that those are two similar triangles.

R: Okay, so you said that if they don’t see that they’re similar triangles by reorienting the picture, that you would start to ask them, “What can be said about the angles?” to help lead them to the angle-angle similarity postulate of similar triangles.

So how do you envision this scenario playing out? What would you ask them?

M: Mm-hum. Well, maybe I’d be like, “Okay, so uh, these are both right angles,” and then um, you know, start to ask them, “Well, what can we say about the other angles, how do those relate?” Um, for example, “Angle $C$ is common to both

\[ \text{Figure 16. How Mary wanted students to redraw the picture.} \]
triangles, can we say that those are congruent, like um, and so then we have two angles for triangle $CEB$ and triangle $CED$.” Um, you know, “Does that…do you recall any relationships that you can then, um, come up with based on two congruent angles for two triangles?”

The exchange shared here was representative of the ideas Mary communicated for how she would respond to the students based on their thinking. It was evident Mary planned to lead the students to recognize the similar triangles by asking questions that warranted a direct answer and required students to recall known facts. By starting with a modified figure that would be more familiar to students and by asking a series of these types of questions, Mary hoped to guide the students to realize that the triangles in the original problem were similar. Mary revealed that she believed if the students knew the triangles were similar then they would be able to solve the problem.

**Technology Task**

This section includes a summarized account of how Mary approached the technology task during the task-based interview. Following this description, details about and analysis of Mary’s responses for each artifact from the technology task will be discussed in order to answer research question 2.

Throughout the task, Mary followed the directions and responded to the prompts. At the beginning of the task, Mary had two predictions: (a) That the dilation would create an image of $AB$ and $CD$ that is two times as long as the pre-image, and (b) That the image segments would be twice as far from point $P$. After completing the dilation, Mary used measures and dragging to confirm her predictions. For example, Mary measured the distance between points $A'$ and $B'$ and noted that it was 14.08. Then she measured the
distance between points $A$ and $B$ and noted that it was 7.04. She said she noticed that $A'B'$ was twice as long as $AB$. After dragging $A'$ around the screen, Mary noted the relationship held true.

When Mary read part F of the problem, which asked for a proof of her observations, Mary utilized the calculate and measure features. For instance, Mary calculated $AB \times 2$ and found this was the same as $A'B'$; she performed a similar calculation for $CD$. Mary also measured the distances between $P$ and the pre-image and image points and compared these calculations. One example of this is when she measured the distance between $P$ and $A$ and stated it was 5.77. Then, she measured the distance between $P$ and $A'$ and said it was 11.55. After calculating $PA \times 2$ she noted that $PA'$ was the same. Mary reported she was confident in this response and proceeded to type the following in response to part F of the task: “I calculated $AB \times 2$, and found it to be equal to $A'B'$, and I calculated $CD \times 2$ and found it to be equal to $C'D'$. Also, image points $A', B', C'$, and $D'$ are twice the distance from point $P$ as compared to the distance of the pre-image points $A, B, C$, and $D$ from point $P$.”

Throughout her work on the technology task, Mary illustrated she was comfortable using the measure and calculate features of the software program. It was also evident that, for Mary, empirical evidence was convincing enough to be considered proof of her observations.

**Attending.** To answer research question 2a, Mary’s responses to the attending prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were considered. This section provides an analysis of how Mary attended to students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 19, the categorizations of
Mary’s written responses to the attending prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the attending prompt on the technology task.

Table 19

*Mary’s Attending on the Technology Task*

<table>
<thead>
<tr>
<th>Attending Framework Categorization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
</tr>
<tr>
<td>Group Video Clip</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
</tr>
<tr>
<td>Citing evidence</td>
</tr>
<tr>
<td>Inferring about student action</td>
</tr>
<tr>
<td>Inferring about student understanding</td>
</tr>
<tr>
<td>Discussing conceptions of proof</td>
</tr>
<tr>
<td>Evaluating</td>
</tr>
<tr>
<td>Inferring about technology</td>
</tr>
<tr>
<td>Reflecting</td>
</tr>
<tr>
<td>Withholding assumptions</td>
</tr>
<tr>
<td>Comparing one student/group to another</td>
</tr>
<tr>
<td>Comparing to self</td>
</tr>
<tr>
<td>Discussing group/pair dynamics</td>
</tr>
<tr>
<td>Discussing students’ language</td>
</tr>
</tbody>
</table>

**Summary.** Throughout her responses and interviews regarding the attending prompt for the artifacts from the technology task, Mary mostly focused on describing what the students did in the problem. In fact, *describing* accounted for about 32% of the total count of descriptive codes for Mary’s attending on the technology task, almost double that of the second most frequently appearing descriptive code, *citing evidence* at 17% of the total count. However, Mary’s attending responses also revealed a variety of other descriptive
codes. This was evident in her response to the attending prompt from the group video clip, as shown below.

The students used their understanding of dilation to make a prediction. Student 1 defined dilation to mean ‘make bigger’ (line 16). After making their prediction, the students then followed the instructions for parts B through E. They had little difficulty figuring out how to dilate the objects in GSP per the instructions, but the teacher was able to help them. However, they did not drag any of the points in the figure as they were instructed to do so in parts D and E. This may be due to their limited understanding of dilation. Since their prediction was limited to a change in size, their observations were limited to that change in size. Hence, when the students noted that the measurements for parts D and E supported their prediction, the students felt that they were finished with the task (video 8:45). It was only after some direction from the teacher that the students began to further investigate the relationship of the objects to the point $P$.

While Mary described what the students did in the problem, she only provided evidence for a couple of her statements (e.g., citing the time in the video when the students felt they were finished with the task). While attending, Mary also made assumptions about why the students performed certain actions and what the students understood. For example, Mary inferred that the students failed to drag the points as instructed “due to their limited understanding of dilation.” Such comments were characteristic of Mary’s attending responses on the technology task.

**Interpreting.** To answer research question 2b, Mary’s responses to the interpreting prompt (i.e., Please explain what you learned about these students’ mathematical
understandings.) were considered. This section provides an analysis of how Mary interpreted the students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 20, the categorizations of Mary’s written responses to the interpreting prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the interpreting prompt on the technology task.

Table 20

*Mary’s Interpreting on the Technology Task*

<table>
<thead>
<tr>
<th>Interpreting Framework Categorization</th>
<th>Student Work</th>
<th>Group Video Clip</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Citing evidence</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Discussing students’ language</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Reflecting</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Describing</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of dilation</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Evaluating</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Withholding assumptions</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

*Summary.* While interpreting on the technology task, the descriptive codes revealed that Mary was *inferring about student understanding, citing evidence, and discussing conceptions of understanding*. Mary also had notably less descriptive codes for interpreting as compared to attending; this was likely because Mary provided briefer responses during
the discussions about interpreting. For example, Mary’s response for the interpreting prompt from the student work on the technology task was one sentence: “They understand that a dilation means that objects are scaled up or down by a specified scale factor from a given point and the distance of the objects from the given point are also scaled up or down by the same scale factor.” In this statement, Mary inferred the students understood dilations affect both the size of the image and the distance of the image from the center of dilation. But, Mary offered no further interpretations of the students’ mathematical understandings.

Although Mary did not make additional assumptions about the students’ understanding, she did reveal something about her own conceptions of understanding during the follow up interview for this artifact. When the researcher asked Mary what it meant for an individual to understand a particular idea, Mary shared that, in this case, “because their predictions were correct and then they were able to prove it, that shows understanding of the definition of dilation.” Mary went on to say that, in general, if students “can answer questions correctly that’s usually a good indication” that they understand the concept. Thus, Mary indicated she evaluated mathematical understanding based on whether or not students could provide a correct answer.

Responding. To answer research question 2c, Mary’s responses to the responding prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were considered. This section provides an analysis of how Mary planned to respond to students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 21, the categorizations of Mary’s written responses to the responding prompt are shown, as well as
frequency counts of each descriptive code that appeared in the data for the responding prompt on the technology task.

Table 21

Mary’s Responding on the Technology Task

<table>
<thead>
<tr>
<th>Responding Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>1</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Exploring mathematical meanings and relationships</td>
<td>12</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of dilation</td>
<td>8</td>
</tr>
<tr>
<td>Starting with a simple or familiar case</td>
<td>5</td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>3</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>2</td>
</tr>
<tr>
<td>Extending thinking</td>
<td>2</td>
</tr>
<tr>
<td>Attending to student errors</td>
<td>1</td>
</tr>
<tr>
<td>Avoiding cases</td>
<td>1</td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>1</td>
</tr>
<tr>
<td>Drawing from experience to select example</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about teacher</td>
<td>1</td>
</tr>
<tr>
<td>Probing, getting students to explain their thinking</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** While responding to students’ thinking for the technology task, Mary’s focus was on having students *explore mathematical meanings and relationships* around the concept of dilation. Often, Mary’s ideas led to a discussion that connected her choices of how she would have students explore dilation back to her own understanding of dilation. An exemplar of Mary’s responding occurred in an interview about the group video clip; the excerpt is included below.
R: You mentioned that you would have them explain the purpose of point \( P \) by dragging point \( P \) around the screen. Why would you have them drag point \( P \)?

M: Because I think, they actually did it in the video, that once the teacher kind of pushed them to do that. And even then, I still don’t think they realized the relationship, but moving \( P \) around moves the images around. And so, maybe start to ask them, “Well, why are the images moving if you’re just moving point \( P \)?” and they start to see a relationship or start to realize that, I think.

R: Okay, so what is the purpose of point \( P \)?

M: Um, well, it’s the center point of dilation and so um, you can’t really have a dilation without a center point and so, that relationship sets up where the images will be, or center point \( P \) sets up where the images will be because \( A' \) will end up being twice the distance from point \( P \) relative to \( A \).

It is evident in this excerpt that Mary envisioned helping the students better understand the role of the center of dilation through dragging. She thought if the students could see how the pre-image and image points moved in relation to \( P \), the center of dilation, that maybe they would start to realize that \( P \) was also an important aspect of the dilation. Mary wanted the students to expand their understanding of dilation from how the scale factor determined the distance between certain points to include that the location of image points were also dependent on the center of dilation. Through her explanation of what she would have the students do to further explore dilations, Mary inevitably discussed her own understanding of dilation by stating why it was important for the students to drag point \( P \) and what she wanted them to realize from that exercise.
Discussion of Mary

The analysis of Mary’s written responses and interviews revealed Mary was noticing in a variety of ways. To summarize these findings, concise answers to each research question are provided below.

**Research question 1.** For this case, the researcher considered the question, “How did Mary notice students’ mathematical thinking when she examined artifacts of practice from a paper and pencil mathematics task?” In other words, in what ways did Mary attend, interpret, and respond to students’ mathematical thinking?

When attending to students’ thinking on the paper and pencil task, Mary’s written responses completely described the students’ approach and included specific evidence from the artifacts (classified as 4) for both the student work and group video clip. The descriptive codes also revealed that while attending, Mary was most often describing and citing evidence. In fact, she cited evidence in almost as many instances as she was describing (i.e., 21 versus 25 of the total count of descriptive codes, respectively). For interpreting, Mary’s written responses included evidence to support her claims about students’ thinking (a 3) for each artifact. When interpreting students’ thinking, Mary notably shared less than she did while she was attending. However, her responses while interpreting for the paper and pencil task were still focused on citing evidence, as shown in 35% of the total count of descriptive codes for interpreting. She was also inferring about student understanding (i.e., 5 out of 23 instances of the total count of descriptive codes) and evaluating mathematical correctness (i.e., 4 out of 23) while interpreting. For responding, Mary’s written response for the student work artifact included a potentially useful problem with evidence indicating students’ thinking was considered (a 2), but for the group video clip, Mary’s rationale for how she
would respond to students was vague (a 1). While responding to students’ thinking on the paper and pencil task, Mary again shared less than she did while attending. And, for responding, the descriptive codes indicated that Mary was *gathering information, checking for a method, and leading students through a method* in 6 out of 15 instances of the total descriptive codes for responding and *starting with a simple or familiar case* in 4 out of 15 instances of the total count.

**Research question 2.** For this case, the researcher considered the question, “How did Mary notice students’ mathematical thinking when she examined artifacts of practice from a technology task?” In other words, in what ways did Mary attend, interpret, and respond to students’ mathematical thinking?

When attending on the technology task, Mary’s written response for the student work provided an overview to the students’ approach with little evidence from the artifact (categorized as a 1) and her written response for the group video clip described the students’ approach with some evidence from the artifact (a 2). Close to one-third of the total count of descriptive codes for the attending prompt indicated Mary was *describing*. Also, in 7 out of 41 instances of the total count of descriptive codes for attending, Mary was *citing evidence*. While interpreting students’ thinking on the technology task, Mary’s written response for the student work artifact lacked evidence to support her interpretations (a 1), whereas her written response for the group video clip included evidence to support some of her inferences (a 2). For interpreting, there was about half the number of descriptive codes than there were for attending (i.e., 21 versus 41 instances of the total count). According to the descriptive codes for interpreting, Mary was focused on a variety of things; however, *inferring about student understanding* (i.e., 5 out of 21 instances of the total count) and
citing evidence (i.e., 4 out of 21) appeared most often. For responding, Mary’s written response for the student work posed a useful problem but included a vague rationale (a 1). On the other hand, her written response for the group video clip provided a rationale with some evidence to indicate student thinking was considered when determining how to respond (a 2). The descriptive codes revealed that when responding, Mary selected examples that would have students exploring mathematical meanings and relationships in 12 out of 38 instances of the total count. Mary also often discussed her understanding of dilation while responding (i.e., 8 out of 38).

Susan

Susan, a native of North Carolina, was inspired to be a teacher because of a history teacher she had in high school. According to Susan, she loved how this history teacher was connected to his students. She described him as engaged and active in the lives of his students by attending extracurricular activities and building relationships with them. Susan also noted that in high school, she realized a lot of people did not like math or seem to understand it. Susan, on the other hand, enjoyed math and understood it. This led to her tutoring her peers and wanting to help others see the importance of math in the world. Thus, Susan shared that her high school experiences prompted her to pursue a degree in mathematics education so that she could teach high school mathematics.

Susan’s background in mathematics courses was broad. In middle school, Susan took algebra and geometry. In high school, Susan completed algebra two, pre-calculus, AP calculus, and AP statistics. In college, as a triple major in mathematics, mathematics education, and communications, Susan had a plethora of mathematics courses. Some she recalled were linear algebra, modern algebra, Euclidean geometry, math analysis one,
differential equations, calculus two and three, discrete mathematics, a course on probability, statistics for engineering, and statistics for behavioral sciences. Even in collegiate mathematics courses, Susan stated she often helped her peers understand the material. And, while Susan shared she did not always enjoy college level math, she did find that the idea of persevering through a math problem to arrive at a solution has helped her become more passionate about teaching mathematics. Susan reported being most comfortable with geometry. In fact, she described geometry as her favorite and stated she was most excited to teach that topic. Susan stated she was comfortable with other high school mathematics topics such as algebra and was becoming more comfortable with statistics. Susan also shared she had been working as a teacher’s assistant for an eighth grade mathematics class for four years and that she had recently started observing a pre-calculus class. According to Susan, these experiences contributed to her comfort level for teaching high school mathematics. However, Susan reported she was not as comfortable teaching mathematics with technology. She stated that she did have a class focused on this, but that at the time, she did not have enough time to devote to becoming truly comfortable with the technology. She shared that instead, she only did what was required for the course and now would like to take time to become more comfortable using technology to teach mathematics. She also stated that since she finished the course that her comfort level in using tools, like The Geometer’s Sketchpad, had decreased because she had not been using them.

At the time of the study, Susan had already completed the teaching with technology course, and she still had a year before she would have her student teaching experience. By random assignment, Susan completed and received artifacts for the pencil and paper task
first followed by the technology task. Here, discussion of the tasks is ordered based on the research questions in the study.

**Paper and Pencil Task**

This section includes a summarized account of how Susan approached the paper and pencil task during the task-based interview. Her work on the task is also included in Figure 17. Following this description, details about and analysis of Susan’s responses for each artifact from the paper and pencil task will be discussed to answer research question 1.

![Figure 17. Susan’s work on the paper and pencil task.](image)

Throughout her work on the paper and pencil task, Susan did not hesitate to share her thinking. She consistently wondered aloud, such as when, upon reading the problem, she asked aloud to herself if the problem told her that these were similar triangles and then immediately answered no. Susan stated that she knew similarity was based on proportions and if she could find that triangles $ADC$ and $BEC$ were similar then she would have:
\[
\frac{3}{5} = \frac{BC}{AC},
\]  \hspace{1cm} (2)

(It is important to note that the notation within the proportion Susan wrote was incorrect due to the inclusion of the segment symbol; however, to best illustrate Susan’s approach and thinking, the symbols she used were included here.) This type of out loud thinking was characteristic of Susan as she worked on the task. She talked the researcher through which lengths she knew and why. After sharing that she did not know the length of all of \(AC\), Susan exclaimed, “Oh! I can label this whole thing ‘\(x\)’ (referring to \(BC\)) because it’s my unknown.” Then, as shown in Figure 17, she substituted what she had found into her equation to get:

\[
\frac{3}{5} = \frac{x}{8+x},
\]  \hspace{1cm} (3)

However, Susan shared she was hesitant to continue on this line of thinking because she had not proven the triangles were similar yet. After considering how she might prove corresponding angles were congruent, Susan mentioned she was still unsure if showing all three angles were congruent would be sufficient justification that the triangles were similar. So, operating under the assumption the triangles were similar, Susan found the magnitude of \(BC\) to be 12 as shown in her work in Figure 17. Susan also noted that a student would probably just assume that the triangles were similar because they look similar but that she knows that’s not mathematically acceptable. (Later data would reveal that such comments were typical of Susan; she was frequently drawing from her experiences while noticing.) After finishing the problem, Susan shared she was pretty confident in her answer but if she were able to prove that the triangles were similar, she would be more confident in her response.
**Attending.** To answer research question 1a, Susan’s responses to the attending prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were considered. This section provides an analysis of how Susan attended to students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 22, the categorizations of Susan’s written responses to the attending prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the attending prompt on the paper and pencil task.

Table 22

*Susan’s Attending on the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Attending Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>1</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>36</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>17</td>
</tr>
<tr>
<td>Withholding assumptions</td>
<td>5</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>4</td>
</tr>
<tr>
<td>Reflecting</td>
<td>3</td>
</tr>
<tr>
<td>Inferring about teacher</td>
<td>2</td>
</tr>
<tr>
<td>Inferring about understanding</td>
<td>2</td>
</tr>
<tr>
<td>Probing, getting students to explain their thinking</td>
<td>2</td>
</tr>
<tr>
<td>Comparing one student/group to another</td>
<td>1</td>
</tr>
<tr>
<td>Discussing chronology</td>
<td>1</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>1</td>
</tr>
<tr>
<td>Evaluating</td>
<td>1</td>
</tr>
<tr>
<td>Inferring non-mathematic</td>
<td>1</td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>1</td>
</tr>
</tbody>
</table>
Summary. While attending, Susan did share her thoughts about what the students were doing in the problem, but she did not provide specific evidence to support her claims. And, as shown in the descriptive coding, Susan’s responses and interviews regarding the attending prompt for the artifacts from the paper and pencil task were focused on describing. In fact, almost half of the total count of descriptive codes (i.e., 36 out of 77) were describing. The descriptive codes also revealed that Susan was inferring about student action in almost one-quarter of the total count of descriptive codes (i.e., 17 out of 77). An example that illustrates how Susan was describing and making inferences about student action occurred during the interview regarding the student work from the task. An excerpt is included here below.

Researcher (R): So you say, and you said this a little bit even now in your explanation that they “knew things” about right triangles or you said, “I know that they knew that there were right triangles in order to be able to use Pythagorean theorem here.” So how do you know that they “knew” things about right triangles?

Susan (S): Yeah, so, I’m not convinced that they know that these for sure are right triangles, just like I had trouble with determining if they were for sure similar triangles last week. What I do know is that they wrote in right angles on the triangles to mark right triangles, and then proceeded to write a whole bunch of other relationships that right triangles have. So they drew in a hypotenuse to, you know, make the triangle DEA and the triangle ABE even though they didn’t draw in the right triangle or the right angle at B. They went straight into the Pythagorean theorem, for both of them, and they were like, “this is how I’m gonna get my side lengths.” And then they drew this little right triangle, 30-60-90 triangle diagram to
find the relationships there, which is another thing, like if you’re just, you know, spit-balling right triangle stuff that those are the kinds of things that you’re going to throw down on your paper.

In this excerpt, Susan was describing when she discussed things the students did on their papers, such as when she mentioned “they wrote in right angles on the triangles” or “then they drew this little right triangle, 30-60-90 triangle diagram.” But, in many instances of her describing in this excerpt, Susan attempted to explain why the students performed these actions. For example, Susan reasoned the students drew in the 30-60-90 diagram “to find the relationships there.” Such comments from Susan were characteristic of her attending on the paper and pencil task.

Interpreting. To answer research question 1b, Susan’s responses to the interpreting prompt (i.e., Please explain what you learned about these students’ mathematical understandings.) were considered. This section provides an analysis of how Susan interpreted the students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 23, the categorizations of Susan’s written responses to the interpreting prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the interpreting prompt on the paper and pencil task.
Susan’s Interpreting on the Paper and Pencil Task

<table>
<thead>
<tr>
<th>Interpreting Framework Categorization</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>1</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>17</td>
</tr>
<tr>
<td>Describing</td>
<td>13</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>6</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>3</td>
</tr>
<tr>
<td>Reflecting</td>
<td>3</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>2</td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>2</td>
</tr>
</tbody>
</table>

**Summary.** While interpreting students’ understanding, Susan’s foci were *inferring about student understanding* and *describing*. In fact, in Susan’s written response to the interpreting prompt about the student work, she was solely focused on making inferences about student understanding. Her response is included here below.

They understood that they needed to find more information about the triangles to determine the length of $BC$. They knew that the length of radii are the same all the way around the circle. They understood that the Pythagorean theorem can allow for students to find distances when they only have the length of two sides. They also recognized that you can draw in a hypotenuse to create triangles that are not obviously there and allows you to find more numbers to assign distances. They also understood that there are certain relationships between sides that exists when you have certain angles.
This response provided many assumptions about what the students know. For example, Susan suggested the students knew all radii are the same and that they know Pythagorean theorem can help find a missing side when two out of three are known. However, Susan did not provide insight into why she believed the students had this knowledge. And, in the follow up interview, rather than provide details to support her claims about student understanding, Susan actually revealed more about her conceptions of understanding. For Susan, understanding indicated that a student not only knew the rule but could also apply it. Thus, while interpreting, Susan made many inferences regarding student understanding of particular concepts without sharing evidence to justify her assertions.

Another aspect of Susan’s interpreting was describing. Almost all of Susan’s describing occurred during the follow up interview for the group video clip (i.e., 12 out of 13 instances of the total count of descriptive codes). In this interview, the researcher attempted to have Susan provide specific evidence for claims she made, such as why she stated that the students knew they could use rules to make sense of the mathematics. During her response, Susan was describing instances from the video to summarize what happened, rather than citing evidence from the transcript (i.e., 11 versus 2 instances of the descriptive codes, respectively). For example, Susan recalled, “With the SOHCAHTOA, they did not explore the angles and how to use that to find side length. It sounded like one guy was suggesting that maybe that was gonna be a thing they were going to do and he put it on his paper.” Here, Susan described how one student wrote down SOHCAHTOA on his paper and indicated that the students did not use this information. Descriptive statements like this were typical of Susan’s interpreting on the paper and pencil task.
**Responding.** To answer research question 1c, Susan’s responses to the responding prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were considered. This section provides an analysis of how Susan planned to respond to students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 24, the categorizations of Susan’s written responses to the responding prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the responding prompt on the paper and pencil task.

Table 24

*Susan’s Responding on the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Responding Framework Categorization</th>
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<tr>
<td>Group Video Clip</td>
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</table>

<table>
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<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>16</td>
</tr>
<tr>
<td>Probing, getting students to explain their thinking</td>
<td>8</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>7</td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>6</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>4</td>
</tr>
<tr>
<td>Inferring non-mathematic</td>
<td>2</td>
</tr>
<tr>
<td>Orienting and focusing</td>
<td>2</td>
</tr>
<tr>
<td>Reflecting</td>
<td>2</td>
</tr>
<tr>
<td>Discussing beliefs</td>
<td>1</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
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</tr>
<tr>
<td>Drawing from experience to select example</td>
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</tr>
<tr>
<td>Encouraging reflection</td>
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</tr>
<tr>
<td>Inferring about student understanding</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about teacher</td>
<td>1</td>
</tr>
</tbody>
</table>
Summary. In her written replies to the responding prompt for the technology task, Susan did not provide a rationale for the follow up questions she intended to ask students. And, although the descriptive coding revealed that Susan was describing in 16 instances of the total count of descriptive codes, 14 of those instances occurred during the follow up interview for the group video clip, as well as all of the instances of the citing evidence codes. In that interview, the researcher asked Susan to provide evidence for the development of the questions she planned to ask students. While citing evidence from the transcript, Susan also described what was happening in those occurrences. For example, in her written response, Susan stated she would ask, “What rule have we learned that can prove that those angles you marked as right angles, angle \( ADE \) and angle \( BEC \), are right angles?” In the follow up interview, when she was asked what prompted this question, she shared the following.

Umm… Line 113 when Student 3 says, ‘Ah, is that a right triangle?’ and then, like their little discussion after that whether or not it is a right triangle; that led me to believe that they didn’t have a full understanding of what made the right angles and why they were right angles.

Here, Susan provided a specific line from the transcript to support her decision to ask her question and she also went on to continue describing what happened from this point on in the video regarding the students’ discussion of whether or not the triangle was a right triangle. Thus, it is important to note that without the researcher’s question for evidence, there would only have been two instances of descriptive codes for describing and zero for citing evidence.
In considering the other descriptive codes that appeared most often aside from describing and citing evidence, it was found that Susan focused on probing, getting students to explain their thinking, and gathering information, checking for a method, and leading students through a method. One exemplar of questions where Susan was probing students’ thinking was in her reply to the responding prompt, shown below.

My first question would be, how did you know that those angles are ninety degrees? Are you sure? Why? Then I would ask, what are you going to do with the length of $DE$ when you find it? What are we trying to find again? How do you plan on using the relationship of the 30-60-90 triangle?

In these questions, Susan desired to learn more about what the students were thinking by asking them for clarifications. For instance, she wanted to know why the students thought the angles they labeled were right angles and what they planned to do with the length of $DE$ once they found it. In contrast, Susan’s reply to the responding prompt for the group video clip was focused more on gathering information, checking for a method, and leading students through a method. Her written response is below.

I would ask, "What rule have we learned that can prove that those angles you marked as right angles, angle $ADE$ and angle $BEC$, are right angles? What do you plan to figure out after finding the value of line segment $DE$? Are there any relationships between the two triangles that might help you?"

Here, Susan planned to ask for a specific rule to show that certain angles are right angles. She also asked for students to recall what “relationships” there might be between the triangles. Both of these questions were geared at leading students down a particular path towards the correct solution. The other question, which asked what students planned to do
with the length of $DE$, was a probing question to clarify students’ thinking. Thus, when considering Susan’s typical choices for how she would respond to students’ thinking on the paper and pencil task, probing and leading questions were her most common choice.

**Technology Task**

This section includes a summarized account of how Susan approached the technology task during the task-based interview. Following this description, details about and analysis of Susan’s responses for each artifact from the technology task will be discussed in order to answer research question 2.

Throughout her work on the technology task, Susan revealed her conceptions of proof and her own understanding of dilation. She began by predicting that $AB$ and $CD$ would become twice as long as their original length “since a dilation by a whole number meant you stretch, or multiply, in this case, their lengths by a factor of 2.” She also noted that if the scale factor were one-half, the dilation would shrink their lengths by one-half. Susan then followed the directions to complete the dilation, measures, and record her observations. Upon reading part F, which asked for a proof of her observations, Susan shared that she did not want use just the numbers of the measurements because those were just examples and not really a proof. When the researcher asked what she considered a proof, Susan responded that a proof was something that was definitively true in all cases. She continued explaining that if she were to say, as she was dragging a point, that the image is always twice the length of the original, then she would have to go through every single case to show it was always true. In other words, she would have to do an exhaustive proof, which she stated is not timely or mathematically sound. The other way to prove it, she said, was to use rules or relationships that we already know or can observe. At this point, the
researcher asked Susan if she could prove any of her observations. Susan stated that she might visually try to prove that \( A'B' \) was twice as long as \( AB \). She then proceeded to drag \( AB \) on top of \( A'B' \), aligning \( B, B', \) and \( P \). After copying \( AB \), she placed the copied segment on top of \( A'B' \) to show that two \( AB \)s would fit on \( A'B' \). When the researcher asked Susan if she considered this a proof or not, Susan responded that at this point, she did not want to say that this was a proof. She said she had found in her many proof classes that her first thoughts were not always proofs. However, she believed this was useful in helping her start to articulate a proof and this approach was better than just dragging the points around to show examples. Susan left the interview stating that she would have to continue thinking about how to generate a proof for her observations.

**Attending.** To answer research question 2a, Susan’s responses to the attending prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were considered. This section provides an analysis of how Susan attended to students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 25, the categorizations of Susan’s written responses to the attending prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the attending prompt on the technology task.
### Table 25

**Susan’s Attending on the Technology Task**

<table>
<thead>
<tr>
<th>Attending Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>1</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>16</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>12</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>4</td>
</tr>
<tr>
<td>Inferring about student understanding</td>
<td>4</td>
</tr>
<tr>
<td>Evaluating</td>
<td>3</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>2</td>
</tr>
<tr>
<td>Discussing students’ language</td>
<td>2</td>
</tr>
<tr>
<td>Reflecting</td>
<td>2</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about teacher</td>
<td>1</td>
</tr>
<tr>
<td>Withholding assumptions</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** Susan’s written responses for the attending prompt from the technology task were very different for the two artifacts. For the student work artifact, Susan provided a very brief response with very little detail. She stated, “The students followed the given instructions and found that the dilation of 2 makes the new line segment twice as long. They determined that they could prove this was the case by observing that as they move the line around, the ratios remain the same, 1:2.” In contrast, Susan’s written response for the group video clip was vastly different as it was more thorough and referenced some evidence from the artifact, as shown below.

Student 1 knew that a dilation meant you get bigger and then guessed that the "factor of 2" part meant that you get bigger by 2, and thus you multiply by 2 (line 12). Then both students went through all of the instructions of the task, not applying much
thought. When they made the measurement they observed that it "basically doubles every time." Then when they got to the prove section, they said, "yes, because of the measurements." (lines 184-186) Then the teacher came over to challenge them to show her that visually, and they did not believe that it could be done (or should be done, maybe): "no, yeah..." and then there was no further movement in figuring out how to do it visually. (lines 197-200) Then the teacher took a different approach and asked them to measure the distances between $P$ and $A$ and $P$ and $A'$ to maybe get them to notice the 1:2 relationship that the points have with the center in addition to with the line segments. They did notice that the distance from $P$ to $A$ was half of the distance from $P$ to $A'$ but they did not predict it when she asked, and they did not pursue the idea any further in the video so they did not ever use the information to prove it.

This response was also characteristic of Susan’s attending on the technology task. It was evident in this written response and during both interviews that Susan focused on describing what the students did in the problem, and when asked, provided evidence to support her claims.

**Interpreting.** To answer research question 2b, Susan’s responses to the interpreting prompt (i.e., Please explain what you learned about these students’ mathematical understandings.) were considered. This section provides an analysis of how Susan interpreted the students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 26, the categorizations of Susan’s written responses to the interpreting prompt are shown, as well as
frequency counts of each descriptive code that appeared in the data for the interpreting prompt on the technology task.

Table 26

Susan’s Interpreting on the Technology Task

<table>
<thead>
<tr>
<th>Interpreting Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
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<tr>
<td>Group Video Clip</td>
<td>3</td>
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</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>10</td>
</tr>
<tr>
<td>Discussing conceptions of proof</td>
<td>3</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>3</td>
</tr>
<tr>
<td>Evaluating</td>
<td>3</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>1</td>
</tr>
<tr>
<td>Comparing one student/group to another</td>
<td>1</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>1</td>
</tr>
<tr>
<td>Describing</td>
<td>1</td>
</tr>
<tr>
<td>Discussing beliefs</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s language</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** While interpreting students’ understanding for the technology task, Susan’s responses (written and verbal) were briefer than while she was attending, as evidenced in the descriptive codes (i.e., 22 versus 48 instances of the total count of descriptive codes, respectively). Also while interpreting, Susan had a clear focus of *inferring about student understanding*. An exemplar of this was her written response to the interpreting prompt for the student work artifact, shown below.

The students understand that dilations mean you multiply by the scale factor, in this case, 2. I know this because their prediction was correct that it would make the scale
two times bigger, which they then wrote did happen when they dilated the line segments. They do not seem to have a firm grasp of the necessary arguments for proving a mathematical statement, since they stated that we do have proof that they are always in a 1:2 ratio, which they saw by dragging around the point. I know this because that would require them to exhaust the possibilities of places it could be dragged in order to prove by those examples.

In this response, Susan shared that she believed the students understood dilations and that they did not understand what it meant to prove a mathematical statement. In each case, Susan provided justification for how she made conclusions regarding the students’ understanding. Such statements were common for Susan’s interpreting on the technology task.

**Responding.** To answer research question 2c, Susan’s responses to the responding prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were considered. This section provides an analysis of how Susan planned to respond to students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 27, the categorizations of Susan’s written responses to the responding prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the responding prompt on the technology task.
Table 27

Susan's Responding on the Technology Task

<table>
<thead>
<tr>
<th>Responding Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
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</tr>
<tr>
<td>Group Video Clip</td>
<td>0</td>
</tr>
</tbody>
</table>

**Descriptive Coding**

<table>
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<tr>
<th>Description</th>
<th>Count</th>
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<td>7</td>
</tr>
<tr>
<td>Evaluating</td>
<td>2</td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>2</td>
</tr>
<tr>
<td>Discussing beliefs</td>
<td>1</td>
</tr>
<tr>
<td>Extending thinking</td>
<td>1</td>
</tr>
<tr>
<td>Needs time or unsure how to respond</td>
<td>1</td>
</tr>
<tr>
<td>Reflecting</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** While responding to students’ thinking for the technology task, Susan’s main focus was to have the students *explore mathematical meanings and relationships*. Susan desired to have the students better understand dilations by exploring the relationship between the center, image, and pre-image points and to also provoke a deeper need in the students for a mathematical proof that the length of the lines of the image were twice the length of those in the pre-image. Some typical questions she shared throughout her written responses and interviews to elicit exploration among students are listed here below.

- How do you know with absolute certainty that the examples you didn't try will also be a ratio of 1:2?
- Is there any other evidence you could use to prove that a dilation of 2 will be a ratio of 1:2 no matter where you drag the lines?
• Why might that relationship (between the center and points) be important to prove your prediction? Would it (the dilation) work if we didn't have a center?

Discussion of Susan

The analysis of Susan’s written responses and interviews revealed Susan was noticing in a variety of ways. To summarize these findings, concise answers to each research question are provided below.

Research question 1. For this case, the researcher considered the question, “How did Susan notice students’ mathematical thinking when she examined artifacts of practice from a paper and pencil mathematics task?” In other words, in what ways did Susan attend, interpret, and respond to students’ mathematical thinking?

When attending to students’ thinking on the paper and pencil task, Susan’s written response for the student work lacked evidence from the artifact (classified as 1) and her response for the group video clip described the students’ approach to the problem with some evidence (a 2). Also when attending, almost half of the total count of descriptive codes (i.e., 36 out of 77) indicated that Susan was describing the student’s work. Another 22% of the total count of descriptive codes revealed Susan was inferring about student action while attending. For interpreting, Susan’s written responses lacked evidence to support her claims about students’ understandings (a 1) for each artifact. When interpreting students’ thinking, Susan was inferring about student understanding in about 28% of the total count of descriptive codes (i.e., 17 out of 46). The descriptive coding for interpreting also revealed that Susan was describing in 13 out of 46 instances of the total count of descriptive codes, 12 of which occurred during one interview when the researcher attempted to elicit evidence for Susan’s claims. For responding, Susan’s written responses included potentially useful
problems but lacked clear rationales that were linked to students’ thinking from the problem (a 1) for both the student work and group video clip. While responding to students’ thinking on the paper and pencil task, the descriptive codes indicated that Susan was describing in 16 out of 53 instances of the total count of descriptive codes and citing evidence in 7 out of 53 instances. However, it was found that 14 instances of the total count of the describing codes and all of the instances of the citing evidence codes occurred during one interview when the researcher asked Susan for specific evidence to support her claims. Other descriptive codes for responding indicated that Susan was probing, getting students to explain their thinking (i.e., 8 out of 53 instances of the total count) and gathering information, checking for a method, and leading students through a method (i.e., 6 out of 53).

**Research question 2.** For this case, the researcher considered the question, “How did Susan notice students’ mathematical thinking when she examined artifacts of practice from a technology task?” In other words, in what ways did Susan attend, interpret, and respond to students’ mathematical thinking?

When attending on the technology task, Susan’s written response for the student work provided an overview of the students’ approach and lacked evidence from the artifact (categorized as 1). On the other hand, Susan’s written response for the group video clip completely described the students’ approach to the problem with some specific evidence from the artifact (a 3). While attending on these two artifacts, Susan was describing in 16 out of 48 instances of the total count of descriptive codes and citing evidence in 12 instances of the total count of descriptive codes. While interpreting students’ thinking on the technology task, Susan’s written response for the student work included evidence to support some of the interpretations she made (a 2) and her written response for the group video clip
provided evidence that supported each inference she made regarding students’ mathematical understandings (a 3). According to the descriptive codes for interpreting, Susan was mostly inferring about student understanding (i.e., 10 out of 25 instances of the total count of descriptive codes). For responding, Susan’s written responses lacked rationales (a 0) for both artifacts. The descriptive codes revealed that when responding, Susan was focused on asking questions that required the students to explore mathematical meanings and relationships (i.e., 7 out of 15 instances of the total descriptive codes). Finally, Susan notably had briefer responses while interpreting and responding on the technology task than she did while attending.

Nicole

Nicole grew up in Illinois and then moved to North Carolina towards the end of elementary school. In high school, Nicole had a teacher she enjoyed who inspired her to become a mathematics teacher. Nicole described this teacher as someone who challenged her and motivated her. Nicole said this teacher was a great supporter and was always there for her. In fact, it was this high school teacher who pushed Nicole towards mathematics and is the reason she chose to pursue her high school licensure.

In high school, Nicole took algebra, geometry, calculus and statistics. Nicole, a statistics major and math education major, admittedly enjoyed statistics more than other topics. In college, Nicole had taken calculus two and three, foundations of advanced mathematics, and a couple of statistics courses. Nicole stated she was comfortable teaching high school mathematics, but she was less comfortable teaching mathematics with technology.
At the time of the study, Nicole was enrolled in a course about teaching mathematics with technology. Nicole mentioned that she was enjoying the course but that she still had not had enough experience with the technology to truly be comfortable with it yet. Nicole also had three semesters to complete before her student teaching experience. At random, Nicole completed and received artifacts for the paper and pencil task first followed by the technology task. Here, discussion of the tasks is ordered based on the research questions in the study.

**Paper and Pencil Task**

This section includes a summarized account of how Nicole approached the paper and pencil task during the task-based interview. Following this description, details about and analysis of Nicole’s responses for each artifact from the paper and pencil task will be discussed in order to answer research question 1.

After reading the problem, Nicole’s first attempt included redrawing triangles $\triangle ADC$ and $\triangle BEC$ and considering how she might use $BD$ to help eventually find $BC$. Stating that she was unsure if she could do that, she drew quadrilateral $ADEB$ to the side and drew in $BD$ (see Figure 18). After labeling $AD$ 5 and $BE$ 3, Nicole stated that she was not sure if angle $CAD$ was a right angle or not. The researcher asked if she was able to convince herself angle $CAD$ was a right angle how that would help. Nicole said it would help her in finding the length of $BC$; however, the researcher could not get Nicole to provide more insight into this line of thinking.
Nicole’s next attempt on the problem included using measurement. Once she labeled the radii in the circles from the given problem, she went on to explain, by drawing a new picture, where each radius fell on $\overline{AC}$ (see Figure 19). Nicole then used a ruler to measure the diameter of circle $A$, which she found to be just over 3 cm, and then compared that length to $\overline{BC}$. She noted that the diameter of circle $A$ was longer than the length of $\overline{BC}$. She then measured the diameter of circle $B$, which she found to be about 2 cm, and compared that to the length of $\overline{BC}$ noting that it was pretty close. She double-checked this measurement and recorded that circle $B$ had diameter of about 2 cm. She then measured from point $B$ out 2 cm towards $C$ and marked this point (see Figure 20). Then, she used the radius of circle $B$ at 1 cm to measure from the 2 cm point out towards $C$ to mark this length. She noted that this length of 1 cm goes to $C$. Next, she labeled the measured parts of $\overline{BC}$ as 6 and 3, respectively, since 6 was the diameter of circle $B$ (representing the first marked length) and 3 was the radius of circle $B$ (representing the second marked length). Then, Nicole said that $\overline{BC}$ was 12 by adding together $3 + 6 + 3$ as noted in Figure 20. While Nicole provided an answer for the length of $\overline{BC}$, she left the interview reporting she was unsure if her solution was correct since it had been a while since she did a problem like this.
Attending. To answer research question 1a, Nicole’s responses to the attending prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were considered. This section provides an analysis of how Nicole attended to students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 28, the categorizations of Nicole’s written responses to the attending prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the attending prompt on the paper and pencil task.
Table 28

Nicole’s Attending on the Paper and Pencil Task

<table>
<thead>
<tr>
<th>Attending Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>2</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>16</td>
</tr>
<tr>
<td>Comparing one student/group to another</td>
<td>2</td>
</tr>
<tr>
<td>Discussing chronology</td>
<td>2</td>
</tr>
<tr>
<td>Evaluating</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about student understanding</td>
<td>1</td>
</tr>
<tr>
<td>Reflecting</td>
<td>1</td>
</tr>
<tr>
<td>Withholding assumptions</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** Throughout her responses and interviews regarding the attending prompt for the artifacts from the paper and pencil task, Nicole’s focus was on describing what the students did in the problem. In fact, 64% of the total count of descriptive codes were describing. Both of Nicole’s written responses provided insight into the students’ approach to the problem. In interviews, Nicole clarified and elaborated on these descriptions. A typical example of Nicole’s describing was her written response to the attending prompt for the student work artifact, shown below.

The students started off by defining the line $AB$. Then they moved toward getting additional values by exploring information given in the problem. Such as when student 4 asks about the meaning of collinear. They also started bouncing ideas off of each other throughout the video. For example the discussion about $DE$. They then worked with the concept of right triangles and calculating side lengths by trying to use Pythagorean theorem and the 30-60-90 ratio. They also brought in trigonometric
functions in attempts to come to a solution. When the teacher checked in, they went back to the Pythagorean theorem. Once they had the values for $AE$ and $DE$, they tried to determine the length of $BC$ by interpreting the line and the components around it.

In this response, Nicole summarized the steps the students took when working through the problem. During the follow up interview, the researcher asked Nicole to further explain some of the statements she made. An example is included in the excerpt below.

Researcher (R): You mentioned that students started bouncing ideas off of each other throughout the video. What ideas were they bouncing off of each other?

Nicole (N): So like they started asking each other questions in terms of like what certain terms meant, like collinear and like throughout the task they were trying to figure out what they could do with what they had at that point. So they started talking about like right triangles and like 30-60-90 ratios, and they were trying to figure out what would work. And they’d be like…like there was one student who was like, “Well, we couldn’t do that” and they kind of like, you know, went off of each other what they knew together.

Here, Nicole described that the students were working together to come up with ideas to solve the problem; however, she did not provide too many details. Rather, she provided an overview and spoke more generally about what students were doing. Such responses were characteristic of Nicole’s attending on the paper and pencil task.

Interpreting. To answer research question 1b, Nicole’s responses to the interpreting prompt (i.e., Please explain what you learned about these students’ mathematical understandings.) were considered. This section provides an analysis of how Nicole
interpreted the students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 29, the categorizations of Nicole’s written responses to the interpreting prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the interpreting prompt on the paper and pencil task.

Table 29

*Nicole’s Interpreting on the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Interpreting Framework Categorization</th>
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</thead>
<tbody>
<tr>
<td>Student Work</td>
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<td>Group Video Clip</td>
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<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>15</td>
</tr>
<tr>
<td>Describing</td>
<td>6</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>3</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>2</td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of right angles</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of similar triangles</td>
<td>1</td>
</tr>
<tr>
<td>Evaluating</td>
<td>1</td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>1</td>
</tr>
<tr>
<td>Reflecting</td>
<td>1</td>
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</tbody>
</table>

*Summary.* While interpreting on the paper and pencil task, the descriptive codes revealed that Nicole was *inferring about student understanding* in about 47% of instances of the total count of descriptive codes for interpreting. An example of Nicole’s inferring about student understanding was her written response to the group video clip, shown below.
I believe they understand a circle's radius as they were able to apply that in the problem. All students seemed knowledgeable about the Pythagorean theorem and could apply it. However, they did not seem to understand if the triangles were right triangles. They also talked about the 30-60-90 triangles, but again, had confusion on how to use the rule. Lastly, student three knew the definition of collinear.

Here, Nicole shared that the students understood what radii of a circle meant, the Pythagorean theorem, and that one student knew the definition of collinear. Nicole also shared that there were likely some concepts the students did not understand as well, such as if the triangles were right triangles and how to apply rules related to 30-60-90 triangles. And, as revealed in the previous interview, Nicole made assumptions about understanding based on whether or not the students could correctly apply the rule or concept.

The codes for interpreting also revealed that Nicole was describing in about 19% of the total count of descriptive codes (i.e., 6 out of 32 instances). These instances of describing were found during interviews when Nicole was asked to elaborate on her written responses. To illustrate this, an excerpt of the follow up interview for the group video clip artifact is included below.

R: You stated, “However, they did not seem to understand if the triangles were right triangles.” So why did you say this?

N: ‘Cause I know at one point they were asking…let me check…let me try to find the line number…um…there was one point where they’re like, “Well, that’s not 90”, and they were like questioning it. Um…and at the end like even as she was like, “Well, you didn’t know that that one was right.” But, there was also a point in the task where they were like, “Well, is it 90?”
Here, Nicole described how the students were having a discussion about whether or not an angle measured ninety degrees. Based on this conversation, Nicole concluded that the students did not know for sure if the triangles were right triangles or not since they were never able to confidently state whether or not the angle in question was in fact ninety degrees. Nicole often used such descriptive statements when justifying her claims regarding student understanding.

**Responding.** To answer research question 1c, Nicole’s responses to the responding prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were considered. This section provides an analysis of how Nicole planned to respond to students’ thinking for both the student work and the group video clip from the paper and pencil task in her written responses and during interviews. In Table 30, the categorizations of Nicole’s written responses to the responding prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the responding prompt on the paper and pencil task.

Table 30

*Nicole’s Responding on the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Responding Framework Categorization</th>
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</table>

<table>
<thead>
<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Probing, getting students to explain their thinking</td>
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</tr>
<tr>
<td>Describing</td>
<td>2</td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>2</td>
</tr>
<tr>
<td>Comparing one student/group to another</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 30 Continued

| Exploring mathematical meanings and relationships | 1 |
| Inferring about student understanding            | 1 |
| Inserting terminology                            | 1 |
| Starting with a simple or familiar case          | 1 |
| Reflecting                                       | 1 |

**Summary.** While responding to students’ thinking on the paper and pencil task, Nicole was *probing, getting students to explain their thinking* in one-third of the total count of descriptive codes. An example to illustrate how Nicole planned to probe students’ thinking was her written response to the student work artifact. Nicole wanted to ask the students a couple of clarifying questions in order to better understand their thinking. During the follow up interview, the researcher asked Nicole to consider how the students might answer her questions in an attempt to gain insight into Nicole’s reasoning for the questions she chose to ask. An excerpt is included here below.

R: I want you to think about how students would respond to the questions that you posed. You have the question, “Why did you include $3 + x$ when trying to find the length of the remaining line segment?” How do you think a student would respond to that?

N: I hope it would like challenge them to think about it because when looking at…like student 4 had a hard time, like I didn’t have enough work I guess from their point to see where they were getting the actual three from. And so, I want them to kind of think about where they…or like explain where they came from and hopefully…if they would feel comfortable enough sharing what they thought and
how they came about it using the three.

R: Okay, the second question you asked was, “How do you think these different approaches will help you find $BC$?” So, before you tell me how you think students would respond, what are the different approaches that you were referencing in that question?

N: So like, they started using the Pythagorean theorem to find sides here (triangle $ABE$), but we had yet to make progress on this side (triangle $ADE$), and so, I wasn’t sure like what their thoughts were and how they were going to use it on the other half and get that line segment.

Nicole went on to share that by asking students why they have the Pythagorean theorem and other things, like the trigonometric functions, listed on their paper she “was trying to figure out how…[the students] thought…[these approaches] would come into play.” Thus, Nicole’s purpose in asking questions like the ones found in her written response for the student work was to have students explain their thinking. Such questions were the most common response to students’ thinking for Nicole.

**Technology Task**

This section includes a summarized account of how Nicole approached the technology task during the task-based interview. Following this description, details about and analysis of Nicole’s responses for each artifact from the technology task will be discussed in order to answer research question 2.

At the beginning of the technology task, Nicole predicted that nothing would happen to the segments, but that perhaps the dilation would make a circle around $P$ with a radius of two and that maybe the circle would touch $\overline{AB}$ at $B$. After following the instructions to
perform the dilation and measure the indicated distances, Nicole pointed out that the lengths of $\overline{AB}$ and $\overline{A'B'}$ changed as she dragged points around. Nicole noted that $\overline{CD}$ and $\overline{C'D'}$ behaved similarly. When Nicole was asked to prove her observations in part F of the task, she typed: “Record initial length, drag the segment and then record the new length to compare this.” The researcher asked Nicole what was she comparing in her statement under part F. She responded that you would compare the measurements between the segments, such as with $AB$ and $A'B'$. She shared that when $\overline{AB}$ was dilated, $\overline{A'B'}$ was twice $AB$ and that when $AB$ changed, that $A'B'$ also changed so that it was always twice $AB$. At the end of the task, Nicole reported being somewhat confident in her efforts on the technology task, as this was the first time she had ever used GSP.

**Attending.** To answer research question 2a, Nicole’s responses to the attending prompt (i.e., Please describe in detail what you think the students did in response to this problem.) were considered. This section provides an analysis of how Nicole attended to students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 31, the categorizations of Nicole’s written responses to the attending prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the attending prompt on the technology task.
Table 31

Nicole’s Attending on the Technology Task

<table>
<thead>
<tr>
<th>Attending Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>2</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>23</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>8</td>
</tr>
<tr>
<td>Reflecting</td>
<td>2</td>
</tr>
<tr>
<td>Discussing chronology</td>
<td>1</td>
</tr>
<tr>
<td>Discussing students’ language</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>1</td>
</tr>
</tbody>
</table>

**Summary.** It was evident that while attending on the technology task, Nicole’s focus was *describing*. About 64% (i.e., 23 out of 36) of the total count of descriptive codes for attending were categorized as *describing*. The coding also revealed that in about 22% of the total count of descriptive codes (i.e., 8 out of 36), Nicole was *citing evidence*. An exemplar that illustrates Nicole’s attending on the technology task was her written response for the group video clip, shown below.

The students started off by reading up to part A. Then stated that dilate means “getting bigger” and that if they dilated by two they would make it two times bigger [11-13]. Next, the students played around on the sketchpad for some time trying to perform the dilation. After the teacher visited their table they were able to do this [104]. They then labeled $A'$, $B'$, $C'$, and $D'$. After this was completed, the students calculated the distance between points $A$ and $B$ and points $A'$ and $B'$. Once they completed this, they noticed that the distance was doubled and recorded that [144]. Once that was finished they then began to calculate the distance between points $C$...
and $D$ and point $C'$ and $D'$. Again, they noticed that the distance was doubled. The teacher then came and told the students to drag the points. The students then dragged the point $A'$ and decided that the distance would still be doubled [172]. Student 1 then started moving point $B$ on the sketchpad while student 2 recorded the relationship. Next, the students talked about how they would prove what found in the previous parts to which they responded using the measurements [186]. The teacher then questioned the students until they moved point $P$, the students noticed that the distance of the lines did not change; however, that $A'B'$ and $C'B'$ would move with $P$ [233]. The teacher then asked the students to calculate the distance between $P$ and $A$ and $P$ to $A'$. The students then did this to find that the distances were double [275-281]. They also found this to be the same if they moved $P$ around. The teacher then asked the students to move $B$ or $B'$. The students found this did not affect the distance from $P$ and $A$ or $P$ and $A'$; however, the distance continued to change for $AB$ and $A'B'$ staying twice the length [308].

In this response, Nicole described each step that the students were taking throughout their work on the task. She also provided direct evidence by providing line numbers from the transcript of the video. The detail illustrated here was typical of Nicole’s attending on the technology task.

**Interpreting.** To answer research question 2b, Nicole’s responses to the interpreting prompt (i.e., Please explain what you learned about these students’ mathematical understandings.) were considered. This section provides an analysis of how Nicole interpreted the students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 32, the
categorizations of Nicole’s written responses to the interpreting prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the interpreting prompt on the technology task.

Table 32

Nicole’s Interpreting on the Technology Task

<table>
<thead>
<tr>
<th>Interpreting Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>2</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>9</td>
</tr>
<tr>
<td>Describing</td>
<td>5</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>1</td>
</tr>
</tbody>
</table>

Summary. While interpreting students’ understanding for the technology task, Nicole notably shared less than she did while attending. This was evidenced not only in more brief written responses but also in the descriptive coding. According to the descriptive codes, Nicole was inferring about student understanding in about 56% (i.e., 9 out of 16) of the total count of descriptive codes and describing in about 31% (i.e., 5 out of 16) of the total count. Her written response for the student work artifact, shared below, provided insight into how Nicole was inferring about student understanding and describing while interpreting on the technology task.

From their responses on the worksheet, I would say that they had an understanding of dilation and how it would affect the distance of the lines by their responses in D and E. They state that the "distance of AB is half the distance of A'B'," and the same
for CD and C'D'. You can also see this through the distances they have recorded for each line. They recorded numbers that made it fairly easy to see the relationship from the dilation, whereas the numbers on the PDF of the GSP file may have been a little more difficult to say that AB is half of A'B'.

In this response, Nicole shared that she believed the students understood dilation because of the responses they provided regarding the measurements of AB, A'B', CD, and C'D'. Nicole also described the responses the students shared, noting that the distances they recorded on the worksheet more clearly revealed the relationship between the lengths of the segments.

**Responding.** To answer research question 2c, Nicole’s responses to the responding prompt (i.e., Pretend you are the teacher of these students. What problem or problems might you pose next and why?) were considered. This section provides an analysis of how Nicole planned to respond to students’ thinking for both the student work and the group video clip from the technology task in her written responses and during interviews. In Table 33, the categorizations of Nicole’s written responses to the responding prompt are shown, as well as frequency counts of each descriptive code that appeared in the data for the responding prompt on the technology task.

Table 33

*Nicole’s Responding on the Technology Task*

<table>
<thead>
<tr>
<th>Responding Framework Categorization</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Student Work</td>
<td>0</td>
</tr>
<tr>
<td>Group Video Clip</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Descriptive Coding</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Probing, getting students to explain their thinking</td>
<td>4</td>
</tr>
<tr>
<td>Extending thinking</td>
<td>3</td>
</tr>
</tbody>
</table>
Table 33 Continued

| Gathering information, checking for a method, leading students through a method | 2 |
| Using technology as a tool | 2 |
| Discussing PSMT’s understanding of dilation | 1 |
| Exploring mathematical meanings and relationships | 1 |
| Inferring about student action | 1 |
| Inferring about student understanding | 1 |

**Summary.** While responding to students’ thinking for the technology task, the descriptive codes revealed that Nicole’s ideas about how to respond to students’ thinking were varied. In other words, there was not one type of response that appeared considerably more than others. The two most frequent responses evidenced in the descriptive codes were *probing, getting students to explain their thinking* and *extending thinking*. Further, Nicole’s written responses lacked a rationale for how she wanted to respond to students.

While *probing, getting students to explain their thinking*, Nicole asked questions such as “Why did you measure the distance between $\overline{PA}$ and $\overline{PA}'$?” and “What do you mean when you say, ‘when you drag the points, they're still the same’?” According to Nicole in an interview, she planned to ask questions like these “just to make sure that they did understand it, um, to like confirm that they know how dilation is going to change the line, but also just to like make sure that they really do understand, and help them if they do not.” Thus, Nicole envisioned such probing questions would enable students to clarify their thinking so that she could address areas of concern.

While choosing how to respond to students’ thinking on the technology task, Nicole also planned to have other tasks aimed to extend students’ thinking. For instance, Nicole
wanted to have students perform dilations on line segments with other scale factors between zero and one and with integers larger than one. Through using different scale factors, Nicole hoped students would realize that images could also get smaller or even larger. Nicole even wanted to have students dilate other objects, such as triangles, to discuss the impacts dilations have on other images. Nicole envisioned that such tasks would help students build on what they already knew about dilations to encompass a better understanding of the transformation.

Discussion of Nicole

The analysis of Nicole’s written responses and interviews revealed Nicole was noticing in a variety of ways. To summarize these findings, concise answers to each research question are provided below.

Research question 1. For this case, the researcher considered the question, “How did Nicole notice students’ mathematical thinking when she examined artifacts of practice from a paper and pencil mathematics task?” In other words, in what ways did Nicole attend, interpret, and respond to students’ mathematical thinking?

When attending to students’ thinking on the paper and pencil task, Nicole’s written responses for both the student work and group video clip described the students’ approach to the problem with some evidence from the artifact (classified as 2). Also when attending, 64% of the total count of descriptive codes (i.e., 16 out of 25) indicated that Nicole was describing the student’s work. For interpreting, Nicole’s written response for the student work included evidence to support each inference she made regarding students’ mathematical understandings (a 3) and her written response for the group video clip included evidence to support some of her inferences (a 2). When interpreting students’
thinking, Nicole was *inferring about student understanding* in about 47% of the total count of descriptive codes (i.e., 15 out of 32). The descriptive coding also revealed that Nicole was *describing* (i.e., 6 out of 32 instances of the total count of descriptive codes) while interpreting. For responding, Nicole’s written response for the student work artifact provided a rationale indicating student thinking had been considered (a 2); her written response for the group video clip included a vague rationale for how she would respond (a 1). While responding to students’ thinking on the paper and pencil task, the descriptive codes indicated that Nicole used questions categorized as *probing, getting students to explain their thinking* in one-third of the total count of descriptive codes (i.e., 5 out of 15).

**Research question 2.** For this case, the researcher considered the question, “How did Nicole notice students’ mathematical thinking when she examined artifacts of practice from a technology task?” In other words, in what ways did Nicole attend, interpret, and respond to students’ mathematical thinking?

When attending on the technology task, Nicole’s written response for the student work provided a description of the students’ approach with little evidence from the artifact (categorized as a 2). On the other hand, her written response for the group video clip completely described the students’ approach to the problem and included specific evidence from the artifact about each step of the students’ approach (a 4). Well over half of the total count of descriptive codes (i.e., 64%, 23 out of 36) for the attending prompt indicated Nicole was *describing*. Also, in 8 out of 36 instances of the total count of descriptive codes for attending, Nicole was *citing evidence*. While interpreting students’ thinking on the technology task, Nicole’s written response for the student work artifact included evidence to support some of her inferences (a 2); however, her written response for the group video clip
lacked evidence to support her claims (a 1). In 56% of the total descriptive codes for interpreting, Nicole was *inferring about student understanding*, and in another 31% of the total count of descriptive codes she was *describing*. For responding, Nicole’s written response for the student work lacked a rationale (a 0) and her written response for the group video clip included an unclear rationale that was not connected to students’ thinking from the problem (a 1). The descriptive codes revealed that when responding, Nicole’s ideas were varied. The two most frequent descriptive codes for Nicole’s responding on the technology task were *probing, getting students to explain their thinking* (i.e., 4 out of 15 instances of the total count of descriptive codes) and *extending thinking* (i.e., 3 out of 15).
CHAPTER 5: CROSS CASE ANALYSIS

The previous chapter shared findings about each individual case with regards to his/her attending, interpreting, and responding on the paper and pencil and technology tasks. The purpose of this chapter is to answer research question three: What similarities and differences exist in the ways PSMTs notice students’ mathematical thinking in the two different types of tasks? Thus, the researcher looked across the cases to determine in what ways the PSMTs were attending, interpreting, and responding in similar and different ways for the two tasks. The results are presented here first by task type and then comparing across both tasks.

**Paper and Pencil Task**

**Attending**

For both artifacts of the paper and pencil task, three of the four cases (i.e., Chase, Susan, and Nicole) had written responses to the attending prompt that were classified as providing an overview or description of the students’ approach with little to no evidence from the artifacts, according to the Attending Framework (Table 4). Mary was the only case who provided a detailed description of what the students did in the problem, including specific evidence, for both the student work and video clip artifacts. Examples of PSMT’s written responses to the attending prompt for the paper and pencil task are included later in this section.

The descriptive codes revealed that while the PSMTs in this study attended in many ways, as evidenced by 18 different codes shown in Table 34, they all had *describing* as the most frequently appearing descriptive code, accounting for about 46% of the total count of descriptive codes for attending. The second most frequently occurring code, *inferring about*
student action was also present for all four cases and the third most frequent code, citing evidence, was only present for one case, Mary. The data related to attending for all four PSMTs also included reflecting and discussing chronology, and three PSMTs were found to be withholding assumptions. The remaining 13 descriptive codes appeared less often in the data and were only present in the data from one or two cases.

Table 34

Summary of Descriptive Codes for Attending for the Paper and Pencil Task

<table>
<thead>
<tr>
<th>Descriptive Code</th>
<th>Number of Occurrences</th>
<th>Number of Cases Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>88</td>
<td>4</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>35</td>
<td>4</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>21</td>
<td>1</td>
</tr>
<tr>
<td>Reflecting</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>Withholding assumptions</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>Discussing chronology</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about student understanding</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Comparing one student/group to another</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Evaluating</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Inferring about teacher</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Probing, getting students to explain their thinking</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Discussing beliefs</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discussing group/pair dynamics</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Inferring non-mathematical</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>193</td>
<td></td>
</tr>
</tbody>
</table>

To illustrate similarities in how the PSMTs were describing while attending a written response from each PSMT is provided below followed by a discussion. Here is Chase’s written response to the attending prompt for student work from the paper and pencil task:
The group labeled the right angles formed by the radii and the tangent lines. The group also labeled the angle measurements they were given. They applied the Pythagorean theorem to find the lengths of several segments but failed to arrive at the correct answer or to address the issue of similar triangles and ratios. I think they thought about using 30-60-90 triangles or the law of sines but didn't really apply either.

Below is Mary’s written response to the attending prompt for student work from the paper and pencil task:

Students 1 through 4 determined the length of $AB$ to be $5 + 3 = 8$, which was based on the given radii of the circles, 5 and 3. The students marked angle $ADC$ to be a right angle. Student 2 also marked angle $BEC$ to be a right angle. The students applied the Pythagorean theorem to triangle $ABE$, as such $8^2 + 3^2 = c^2$. They solved for $c$, which was segment $AE$. They found it to be 8.54 (approx.). Next they applied the Pythagorean theorem to triangle $ADE$, as such $5^2 + b^2 = 8.54^2$. They solved for $b$, which was segment $DE$. They found it to be 6.87 (approximately).

Students 3 and 4 make reference to the trigonometric functions by writing $S = O/H$, $C = A/H$, and $T = O/A$ on their sheets. They may have been trying to recall the values for a 30-60-90 triangle, as they noted 1, square root of 3, and 2. They did not determine the magnitude of segment $BC$.

Below is Susan’s written response to the attending prompt for student work from the paper and pencil task:

The students recognized that they could use the radii of the circles to determine the distance of line segment $AB$. Then they marked what they thought were right angles
in the triangles because they knew things about right triangles. They drew in a hypotenuse from $A$ to $E$ that they solved for using the Pythagorean theorem. Then they found the length of line segment $DE$ using the Pythagorean theorem again. They also wrote in the relationships between side lengths for a 30-60-90 triangle, but it is not clear what they did once they determined those relationships.

Below is Nicole’s written response to the attending prompt for student work from the paper and pencil task:

The students started off by defining the line $AB$. Then they moved toward getting additional values by exploring information given in the problem, such as when Student 4 asks about the meaning of collinear. They also started bouncing ideas off of each other throughout the video, for example, the discussion about $DE$. They then worked with the concept of right triangles and calculating side lengths by trying to use Pythagorean theorem and the 30-60-90 ratio. They also brought in trigonometric functions in attempts to come to a solution. When the teacher checked in, they went back to the Pythagorean theorem. Once they had the values for $AE$ and $DE$, they tried to determine the length of $BC$ by interpreting the line and the components around it.

In each of these written responses, it is evident the PSMTs were describing the students’ approach to the problem. For example, each PSMT discussed how the students labeled things, such as the given radii or right angles, and that the students attempted to use the Pythagorean theorem in their approach. However, it is important to note that just because all of the PSMTs were found to be describing while attending, they may not have all been describing the same things or in the same way. For instance, in the exemplars
above, Nicole is the only PSMT who pointed out that Student 4 asked about the meaning of the word collinear. Other differences included details about how the students applied the Pythagorean theorem; Chase simply stated the students used the Pythagorean theorem while Mary provided the equation the students generated and used. Thus, it is important to remember that although their descriptions might have varied, the common finding was that all PSMTs were describing while attending.

**Interpreting**

According to the Interpreting Framework (Table 5), two of the four cases (i.e., Chase and Susan) either were not focused on interpreting students’ understandings or did not provide evidence to support his/her interpretations of students’ understandings when responding to the Interpreting prompt for both artifacts. The other two cases (i.e., Mary and Nicole) consistently provided at least some evidence to support their interpretations of students’ understandings. A comparison of the written responses of Chase and Mary for the group video clip exemplifies this finding. Here is Chase’s written response to the interpreting prompt:

> These students have an array of tools in their mind and they see solving this problem as a matter of choosing the right tool and applying it to the right information. This is generally a good way of thinking about problems, it's too bad it didn't work for them here. They have a solid understanding of the Pythagorean theorem, even if they messed up using it, and also some familiarity though not mastery with special triangles and trig functions. These students work well together and are interested in this sort of problem. I want to make a special note about language here. Twice there were issues with language, colinear [sic] and secant-tangent lines. There also may
have been some confusion about the word radius. In each case the student was aware
that the language was confusing them and hesitated to use it or sought clarification.
This is a great approach to unfamiliar terminology. Still I might make a specail [sic]
note to make sure to clarify my language use on future problems to ensure
comprehension.

Here is Mary’s written response to the interpreting prompt for the group video clip from the
paper and pencil task:

They don’t know the notation for the magnitude of a segment, as they incorrectly
refer to it as the absolute value of the segment (line 6). They understand that the
radius of a circle is constant (lines 22-23, 39-41). They are familiar with the
Pythagorean theorem – they know the formula, and know that $a$ and $b$ represent the
legs and $c$ represents the hypotenuse (lines 128-129). However, they do not seem to
know that this only applies to a right triangle (lines 113-115). There also seems to be
a misconception that all right triangles have the measures 30-60-90 or 45-45-90
(lines 177, 185, 379, 391).

In Chase’s response, it was evident he was not focused on making interpretations
regarding the students’ mathematical understandings. Rather, Chase discussed multiple
things, including the students’ perspective on problem solving (i.e., “These students have an
array of tools in their mind and they see solving this problem as a matter of choosing the
right tool and applying it to the right information.”), group dynamics regarding how well the
students were working together, and how one might handle unfamiliar terminology with
students. So while Chase did vaguely imply he made some inferences regarding students’
derstandings, such as when he mentioned, “They have a solid understanding of the
Pythagorean theorem, even if they messed up using it,” this was clearly not his focus. On the other hand, Mary made multiple interpretations regarding students’ mathematical understandings, and for each, she provided evidence directly from the artifact in the form of a line number from the transcript.

The descriptive codes for interpreting on the paper and pencil task revealed 17 different descriptive codes, as shown in Table 35. The most frequently occurring code, *inferring about student understanding*, accounted for about 33% of the total count of descriptive codes for interpreting and was present in the data of all four cases. In fact, this code was the most frequently appearing code for Chase, Susan, and Nicole; it was Mary’s second most frequent code. The second overall most frequently occurring code for interpreting, *describing*, was found in the data of all four cases and accounted for about 17% of the total count of descriptive codes. The only other descriptive code that appeared for all four cases was *discussing conceptions of understanding*; this was likely because the researcher asked each PSMT to provide insight as to what “understanding” meant to him/her. The data also revealed that certain codes were present for three of the PSMTs. These included *reflecting*, *citing evidence*, *inferring about student action*, and *evaluating mathematical correctness*. While these codes appeared in the data for three of the cases, they were only all present for Nicole; the other two represented cases varied between the other PSMTs. The remaining 10 codes were only present in the data for one or two cases.
Table 35

Summary of Descriptive Codes for Interpreting for the Paper and Pencil Task

<table>
<thead>
<tr>
<th>Descriptive Code</th>
<th>Number of Occurrences</th>
<th>Number of Cases Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>47</td>
<td>4</td>
</tr>
<tr>
<td>Describing</td>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>Reflecting</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>Evaluating</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Discussing students' language</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Discussing group/pair dynamics</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Inferring: non-mathematical</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of radius</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT's understanding of right angles</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of similar triangles</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>143</td>
<td></td>
</tr>
</tbody>
</table>

To illustrate similarities in how the PSMTs were inferring about student understanding while interpreting, some examples from the follow up interviews of the group video artifact, followed by discussion, are provided here. Below is an excerpt from Chase’s interview:

Researcher (R): So for Question 2, you mentioned “Students have a solid understanding of the Pythagorean theorem.” Based on what you’ve seen, how would you describe the students’ understanding of Pythagorean theorem? In other words, what does solid understanding mean to you?

Chase (C): That’s a good question. Um…they got the $a^2 + b^2 = c^2$ thing in their
head.

R: Okay.

C: Um, they recall, but maybe are not sure about why it needs…there needs to be a
right angle.

R: Okay.

C: Um, they’re sort of tempted to apply it to all triangles. Um, they see it as like a
tool for finding missing side lengths. What they want is two side lengths and then the
third one if they can use to find…yeah…um, actually that seems good…that seems
like more than they just know it, they want, I mean, know it and they want to use it
to find things out.

Here is an excerpt from Mary’s interview:

Researcher: So based on what you’ve seen, can you…what can you tell me about the
students’ understanding of the Pythagorean theorem?

Mary: Um, they know the formula, um, and some of them seem to be familiar that it
has to be a right triangle, um, but I guess they didn’t make that connection or I don’t
know, like at one point someone asked “Is triangle $ABE$ a right triangle?” before
applying Pythagorean theorem and Student 4 says, “No, but let’s do it anyway” kind
of thing, I don’t know. And so, they applied it and so, and they knew to use, you
know, the legs for $a$ and $b$, but they didn’t know that $c$ …or…they do know that $c$ is
the hypotenuse, but then like in this instance they didn’t know that $\overline{AE}$ was the
hypotenuse. Um, so some how, I think, they’re not connecting the visual and the
formula or some how, so yeah.
Here is an excerpt from Susan’s interview:

Researcher: So tell me about Student 3’s understanding.

Susan: By asking, you know, “Is that a right triangle?” talking about EBC. So Student 3 doesn’t think EBC is a right triangle, so Student 3 has not grasped, necessarily, that the tangent line and the radius make a right triangle or a right angle. But, Student 3 understands that just by looking at it you can’t decide what angles the triangle has.

Here is an excerpt from Nicole’s interview:

Researcher: Okay. You also mentioned that, “Student’s talked about 30-60-90 triangles but had confusion on how to use the rule.” What confusion did the students have?

Nicole: So when they first like brought it up they had like…they had the ratio I think, and they were working on how to like set it up so it would be with a right angle and then, once they had it they were like, “Well, okay, now what do we do with it?” So they had, they knew the ratio and they knew that the angles that corresponded to, but they were only sure how to apply it into the triangle or to the triangle, and so, I feel like they had some, like they didn’t quite get it in that area.

In each of these selections, the PSMT provided insight into the interpretations he/she made regarding students’ mathematical understandings. For example, the excerpts from both Chase and Mary revealed how they determined the students understood the Pythagorean theorem. Susan’s piece included a discussion of Student 3’s understanding of the relationship between the radius of a circle and a line tangent to the circle (i.e., they form a right angle); she concluded the student did not understand this relationship. Nicole’s excerpt
revealed how she determined the students knew the 30-60-90 special right triangle ratios. It is important to note that the PSMTs might not be correct in their interpretations; the claims they make about the students’ understandings may not be accurate. However, inferring about student understanding was the most common descriptive code found within the data.

**Responding**

The classifications of the PSMT’s written answers to the Responding prompt (Table 6) for the paper and pencil artifacts were varied. Both of Susan’s responses lacked a rationale for the problems or questions posed. Chase’s response for the student work artifact also lacked a rationale that was connected to students’ thinking on the problem, and for the group video clip, Chase’s written response included a vague rationale. Mary’s and Nicole’s responses were categorized the same for both artifacts. Their responses on the student work artifact included useful problems with some evidence that indicated students’ thinking was considered, and their responses for the group video clip included vague rationales. Examples of PSMT’s written responses to the responding prompt for the paper and pencil task are included later in this section.

The PSMT’s data for Responding revealed 25 different descriptive codes on the paper and pencil task, as shown in Table 36. The three most frequently occurring descriptive codes were *describing* (about 16% of the total count of descriptive codes); *gathering information, checking for a method, leading students through a method* (about 15%); and *probing, getting students to explain their thinking* (about 12%), respectively. Of these three descriptive codes, *gathering information, checking for a method, leading students through a method* was the only code that appeared in the data for all four cases; however, it was not the most frequent code for each case. Rather, each case had a distinct most frequently
occurring code indicating that there was not consistency in how the PSMTs were choosing to respond. The other two most frequently occurring codes were present in the data for two cases, specifically Susan and Nicole. *Reflecting* was the only other descriptive code that appeared in the data for all four cases, and *exploring mathematical meanings and relationships* and *inserting terminology* were present for three cases. The 20 remaining codes were present for only one (i.e., 5 codes) or two (i.e., 15 codes) cases, revealing the very diverse ways in which the PSMTs were responding.

Table 36

*Summary of Descriptive Codes for Responding for the Paper and Pencil Task*

<table>
<thead>
<tr>
<th>Descriptive Code</th>
<th>Number of Occurrences</th>
<th>Number of Cases Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>18</td>
<td>2</td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td>Probing, getting students to explain their thinking</td>
<td>13</td>
<td>2</td>
</tr>
<tr>
<td>Reflecting</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>Orienting and focusing</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Starting with a simple or familiar case</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Drawing from experience to select example</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Exploring mathematical meanings and</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Establishing context</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of similarity</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Inferring non-mathematic</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about student understanding</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of dilation</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Extending thinking</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Needs time or unsure how to respond</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Attending to student errors</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discussing beliefs</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Encouraging reflection</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about teacher</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 36 Continued

<table>
<thead>
<tr>
<th>Comparing one student/group to another</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total</td>
<td>110</td>
<td></td>
</tr>
</tbody>
</table>

To illustrate the various ways in which the PSMTs were responding, an example of a written response from each PSMT is included here below, followed by a discussion. Chase’s written response to the responding prompt for the group video clip from the paper and pencil task was:

So a procedural lesson on similar triangles is appropriate. If these students had that lesson they could have easily solved this problem. This lesson needs several pairs of triangles with some angles and lengths marked and some angles and lengths to be solved for using ratios. There also needs to be demonstration problem. I might also work on underlining the conceptual foundation of similar shapes. I'm not sure how to do this off the top of my head. It feels to me like an intuitive thing. If shapes share angles their sides share ratios of sizes. I need to think more about this sort of conceptual lesson. A lesson where we list possible tools to solve geometric problems and decide where we could apply them might be useful. Finally, I hate teaching vocabulary but it seemed to me that there was a clear need for a little of that here. I've seen people do this in fun ways, maybe a crossword worth bonus points on their homework grade.

Mary’s written response to the responding prompt for the group video clip from the paper and pencil task was:
I would ask them to redraw the problem without the circles and reorient it such that point $C$ is at the top, to see if this will help them see that the triangles are similar. If this is not helpful, I would start to ask them what can be said about the angles, to help lead them to the AA postulate of similar triangles.

Susan’s written response to the responding prompt for the student work from the paper and pencil task was:

My first question would be, “How did you know that those angles are ninety degrees? Are you sure? Why?” Then I would ask, “What are you going to do with the length of $DE$ when you find it? What are we trying to find again? How do you plan on using the relationship of the 30-60-90 triangle?”

Nicole’s written response to the responding prompt for the student work from the paper and pencil task was:

A question I would ask Student 4, is "Why did you include $3 + x$ when trying to find the length of the remaining line segment?" This was the only student who had the "extra" 3 and I would like to see their understanding. I would also ask students "How do you think these different approaches will help you find the length of $BC}?"

Since they spent a great deal of time working with right angles. "How would this help us find the last unknown segment?"

It is evident in the exemplars presented above that there was variation in how the PSMTs were responding. In Chase’s response, he alluded to considering students’ thinking when he makes statements such as “If these students had that lesson they could have easily solved this problem,” and, “Finally, I hate teaching vocabulary but it seemed to me that there was a clear need for a little of that here.” In these statements, Chase suggested he had
considered students’ understanding when determining how to respond. For instance, to
Chase these students lacked the background knowledge needed to be successful on the
problem, so he wanted to remedy that by providing a lesson to facilitate their understanding
of similar triangles. Chase’s idea to incorporate a “procedural” (his words) lesson on similar
triangles is also an example of gathering information, checking for a method, leading
students through a method. In other words, Chase intended through this similar triangles
lesson that he would equip students with the skills needed to recall facts and procedures
when given a problem that incorporated similar triangles. Chase’s written response for the
group video clip also included instances of other descriptive codes, such as inserting
terminology, needs time or unsure how to respond, and reflecting.

Mary’s written response included some evidence that she considered students’
mathematical understandings when determining how to respond. For example, she shared
she would have the students redraw the picture so that the triangles were oriented in a more
familiar way. She believed she needed to do this because for her, the students were having
difficulty recognizing the similar triangles. Mary shared she would continue by asking the
students about “relationships triangles can have” to “lead them to see the similar triangles.”
Thus, Mary’s response indicated that she was gathering information, checking for a method,
leading students through a method while responding. Mary’s response also included an
instance of inserting terminology when she shared she would ask the students “what similar
triangles meant.”

Susan’s written response provided many questions she wanted to ask students;
however, she did not include a rationale for what motivated her questions or why they were
important. Most of the questions Susan included in her written response for this prompt for
student work artifact indicated that she was *probing, getting students to explain their thinking*. For example, asking the students “What are you going to do with the length of $DE$ when you find it?” requires the students to clarify their thinking. Thus, by asking such questions, Susan hoped to gain more insight into what the students were thinking as they worked through the problem. In her follow up interview, Susan shared that by having students answer the questions she formulated, she would have a better grasp of the students’ understandings and would then be able to determine what to do next. Susan’s response also included a question that was classified as *orienting and focusing*: “What are we trying to find again?” Through this question, Susan was attempting to help refocus the students on what the problem was asking and how that might be connected to their approach.

Nicole’s written response provided some evidence she considered students’ thinking. For example, she said she would ask Student 4 about the use of “$3 + x$” on his/her work because this was the only student who wrote that. Nicole wanted to understand why and decided she needed to ask. Thus, in this case, Nicole was *probing, getting students to explain their thinking*. Nicole’s response also included a question classified as *gathering information, checking for a method, leading students through a method*: “How would this help us find the last unknown segment?” This question indicated Susan was attempting to check that the students were on the right path towards finding the correct answer.

Each of the examples provided above reveals some of the similar and different ways in which the PSMTs were responding. Overall, there was not one clear way the PSMTs chose to respond. It is important to note that the instances of *describing* were typically found in the data from follow up interviews. Perhaps this occurred because the researcher was probing the PSMT for a rationale for his/her decision on how to respond. Also, some of the
responses the PSMTs shared were not always necessarily appropriate given the mathematics of the problem.

Summary

The data for the PSMT’s written and verbal responses related to the paper and pencil task revealed some commonalities between cases and differences. While attending, most of the cases included little evidence from the artifacts in their written responses to support their claims of what the students were doing in the problem. Also, the most frequently occurring descriptive code for each case was *describing*, which accounted for about 46% of the overall total count of descriptive codes for attending. While interpreting, the cases were split in the classifications of their written responses. Two of the cases were not focused on interpreting students’ thinking and did not include evidence from the artifacts while two of the cases did include some evidence to support their claims. The overall most frequently occurring descriptive code was *inferring about student understanding*, accounting for about one-third of the total codes for interpreting; this code was the most frequent for three PSMTs. While responding, the PSMT’s written responses were varied in their classifications. Some responses lacked a rationale while others included some evidence to indicate students’ thinking might have been considered. There was also a wide range of descriptive codes that appeared for responding. Of these, *gathering information, checking for a method, leading students through a method* was one of the more frequent overall codes and was present for all four cases.
Technology Task

Attending

For the student work artifact from the technology task, three of the four cases (i.e., Chase, Mary, and Susan) had written responses to the attending prompt that were classified as providing an overview or description of the students’ approach with little to no evidence from the artifacts, according to the Attending Framework (Table 4). Nicole was the only case who described what the students did in the problem using some evidence from the artifact. On the other hand, the written responses to the attending prompt for the group video clip were classified higher for every case according to the Attending Framework. Mary’s written response included a description with some evidence from the artifact; Chase and Susan’s responses included a more detailed description and more evidence from the artifact. Nicole’s response provided a complete description of what the students did in the problem with specific evidence from the artifact. Examples of the written responses are shared later in this section.

The descriptive codes for attending on the technology task revealed that the PSMTs attended in a variety of ways, as evidenced by 19 different codes, as shown in Table 37. The two most frequently occurring codes included describing and citing evidence. In fact, describing was the most frequently occurring descriptive code for each case, accounting for about 42% of the total count of descriptive codes for attending. Citing evidence was the second most frequently occurring code for three of the cases; only Chase differed here with reflecting as his second most recurrent code. Other codes that appeared in the data for all four cases included reflecting, inferring about student action, and discussing students’
language. The remaining descriptive codes were present in the data for just one or two cases.

Table 37

**Summary of Descriptive Codes for Attending for the Technology Task**

<table>
<thead>
<tr>
<th>Descriptive Code</th>
<th>Number of Occurrences</th>
<th>Number of Cases Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Describing</td>
<td>63</td>
<td>4</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>27</td>
<td>3</td>
</tr>
<tr>
<td>Reflecting</td>
<td>11</td>
<td>4</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>Discussing students’ language</td>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>Inferring about student understanding</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>Evaluating</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Withholding assumptions</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Discussing group/pair dynamics</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Discussing conceptions of proof</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about technology</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s language</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Comparing one student/group to another</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Inferring about teacher</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discussing chronology</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>149</td>
<td></td>
</tr>
</tbody>
</table>

The most revealing picture of how the PSMT’s were *describing* is by examining their written responses to the attending prompt for the technology task. Below, the written responses for the student work and the group video clip from each PSMT is shared and followed by a discussion. Here is Chase’s written response to the attending prompt for the student work from the technology task:
The students clearly measured and compared the lengths of the line segments. I notice in the written section that they also measured the length of \(PA\) and \(PA'\) though they didn't write down a conclusion with regards to that. I suspect they moved \(P\) and observed the effects, from their comment about moving the points in part F.

Below is Chase’s written response to the attending prompt for the group video clip from the technology task.

The students conjectured that "the scale will become two times bigger" (Not sure about the use of the word 'scale'). The students first followed the instructions to dilate both \(AB\) and \(CD\). The students measured the lengths of \(AB\), \(A'B'\), \(CD\), \(C'D'\) and concluded that the length of the image (though they didn't use this word) was twice that of the preimage. The students moved points \(A\), \(B\) and \(P\) and observed that this did not effect their conclusion. The students reported the information to their teacher. Upon further questioning by the teacher, the students measured the distance between \(P\) and \(A\) and \(P\) and \(A'\). They concluded that the distance between \(P\) and \(A'\) was twice the distance between \(P\) and \(A\). They moved points \(P\) and \(A'\) to check this. They also moved point \(B\) and found this did not affect the distances involved in the second conclusion. Here is a list of things the students did not do that I noticed, - Notice that \(P\), \(A\) and \(A'\) were colinear [sic]. - Formally answer or ask the teacher what mathematically a dilation is beyond "make bigger." - Disagree with each other at anytime on any matter.

Below is Mary’s written response to the attending prompt for the student work from the technology task.
Both students made predictions based on their knowledge of dilation. They proved their predictions by measuring the lengths of the segments $AB$ and $CD$ per the instructions and proving their relationships.

Below is Mary’s written response to the attending prompt for the group video clip from the technology task.

The students used their understanding of dilation to make a prediction. Student 1 defined dilation to mean ‘make bigger’ (line 16). After making their prediction, the students then followed the instructions for parts B through E. They had a little difficulty figuring out how to dilate the objects in GSP per the instructions, but the teacher was able to help them. However, they did not drag any of the points in the figure as they were instructed to do so in parts D and E. This may be due to their limited understanding of dilation. Since their prediction was limited to a change in size, their observations were limited to that change in size. Hence, when the students noted that the measurements for parts D and E supported their prediction, the students felt that they were finished with the task (video 8:45). It was only after some direction from the teacher that the students began to further investigate the relationship of the objects to the point $P$.

Below is Susan’s written response to the attending prompt for the student work from the technology task.

The students followed the given instructions and found that the dilation of 2 makes the new line segment twice as long. They determined that they could prove this was the case by observing that as they move the line around, the ratios remain the same, 1:2.
Below is Susan’s written response to the attending prompt for the group video clip from the technology task.

Student 1 knew that a dilation meant you get bigger and then guessed that the "factor of 2" part meant that you get bigger by 2, and thus you multiply by 2 (line 11). Then both students went through all of the instructions of the task, not applying much thought. When they made the measurement they observed that it "basically doubles every time." Then when they got to the prove section, they said, "yes, because of the measurements." (lines 179-181) Then the teacher came over to challenge them to show her that visually, and they did not believe that it could be done (or should be done, maybe): "no, yeah..." and then there was no further movement in figuring out how to do it visually. (lines 192-195) Then the teacher took a different approach and asked them to measure the distances between \( P \) and \( A \) and \( P \) and \( A' \) to maybe get them to notice the 1:2 relationship that the points have with the center in addition to with the line segments. They did notice that the distance from \( P \) to \( A \) was half of the distance from \( P \) to \( A' \) but they did not predict it when she asked, and they did not pursue the idea any further in the video so they did not ever use the information to prove it.

Below is Nicole’s written response to the attending prompt for the student work from the technology task.

Looking at the worksheet, I would assume that the students went down the questions in order. Therefore, I believe they started off by making a prediction of what would happen when there was a dilation of 2 at \( P \), which they wrote would make the scale larger. I then believe they made the dilation and labeled \( A' \), \( B' \), \( C' \), and \( D' \). Which
can also be observed in the PDF from the GPS. From this PDF, I can also see that they measured the distance of $AB$, $A'B'$, $CD$, $C'D'$, $PA$ and $PA'$. During this time, I also believed they were able to experiment with moving the points around and exploring with how the distances would change.

Below is Nicole’s written response to the attending prompt for the group video clip from the technology task.

The students started off by reading the up to part A. Then stated that dilate means “getting bigger” and that if they dilated by two they would make it two times bigger [11-13]. Next, the students played around on the sketchpad for some time trying to perform the dilation. After the teacher visited their table they were able to do this [103]. They then labeled $A'$, $B'$, $C'$ and $D'$. After this was completed, the students calculated the distance between points $A$ and $B$ and points $A'$ and $B'$. Once they completed this, they noticed that the distance was doubled and recorded that [139]. Once that was finished they then began to calculate the distance between points $C$ and $D$ and point $C'$ and $D'$. Again, they noticed that the distance was doubled. The teacher then came and told the students to drag the points. The students then dragged the point $A'$ and decided that the distance would still be doubled [167]. Student 1 then started moving point $B$ on the sketchpad while student 2 recorded the relationship. Next, the students talked about how they would prove what found in the previous parts to which they responded using the measurements [181]. The teacher then questioned the students until they moved point $P$, the students noticed that the distance of the lines did not change; however, that $A'B'$ and $C'B'$ would move with $P$ [227]. The teacher then asked the students to calculate the distance between $P$ and $A$
and $P$ to $A'$. The students then did this to find that the distances were double [281]. They also found this to be the same if they moved $P$ around. The teacher then asked the students to move $B$ or $B'$. The students found this did not affect the distance from $P$ and $A$ or $P$ and $A'$; however, the distance continued to change for $AB$ and $A'B'$ staying twice the length [301].

Each response is provided to highlight the similarity between the classifications of their responses; each PSMT’s group video clip response was categorized higher using the Attending Framework than their respective student work responses. For example, it was evident Mary provided a more thorough description of the students’ work for the group video clip; she even included evidence to support her claims in the response. A comparison of the other PSMT’s responses indicated the same finding; each PSMT provided a more detailed response for the group video clip artifact. It is possible the PSMT’s responses included more information about the students’ approach for the group video clip artifact than for the student work due to their unfamiliarity with the technology tool, GSP. The PSMT’s might not have enough knowledge of the functionalities of GSP to decipher what students might have done on the problem by simply examining their work.

Through the responses, it was also apparent the main focus of each PSMT was describing. At times, this describing was general, as in Susan’s response from the student work when she stated, “The students followed the given instructions and found that the dilation of 2 makes the new line segment twice as long.” Other times, the describing was more specific, such as when Chase shared in his written response for the group video clip, “The students measured the lengths of $AB$, $A'B'$, $CD$, $C'D'$ and concluded that the length of the image (though they didn't use this word) was twice that of the preimage.” It is also
important to recall that within their written responses, describing was not the only
descriptive code found. In other words, the PSMTs were not only describing when they
provided their written responses, as evidenced in the data.

**Interpreting**

According to the Interpreting Framework (Table 5), only one case’s written
responses (i.e., Susan) consistently provided at least some evidence to support her
interpretations of students’ understandings. Both Mary and Nicole provided some evidence
for their inferences for one written response, but lacked evidence or had unclear
interpretations in their other written response. Chase was the only case whose written
responses were both classified low according to the Interpreting Framework; Chase’s
responses revealed that he was either not focused on making interpretations of the students’
understandings or that his responses lacked evidence or were difficult to follow regarding
his inferences.

The descriptive codes for interpreting on the technology task revealed 17 distinct
descriptive codes, as shown in Table 38. The most frequently occurring code, *inferring
about student understanding*, accounted for about 30% of the total count of descriptive
codes for interpreting and was present in the data of all four cases. However, this code was
only the most frequent code for three of the cases (i.e., Mary, Susan, and Nicole). The
second overall most frequently occurring code for interpreting, *describing*, was found in the
data of all four cases and accounted for about 14% of the total count of descriptive codes.
The data also revealed that two codes were present for three of the PSMTs. These included
*evaluating* and *citing evidence*. While these codes appeared in the data for three of the cases,
they were only both present for Mary and Susan; the other represented cases were Chase for
evaluating and Nicole for citing evidence. The remaining 13 codes were only present in the data for one or two cases.

Table 38

Summary of Descriptive Codes for Interpreting for the Technology Task

<table>
<thead>
<tr>
<th>Descriptive Code</th>
<th>Number of Occurrences</th>
<th>Number of Cases Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>32</td>
<td>4</td>
</tr>
<tr>
<td>Describing</td>
<td>15</td>
<td>4</td>
</tr>
<tr>
<td>Reflecting</td>
<td>13</td>
<td>2</td>
</tr>
<tr>
<td>Evaluating</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Discussing conceptions of proof</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Evaluating mathematical correctness</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Discussing students' language</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Drawing from experience to make inferences</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Discussing beliefs</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Comparing to self</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of dilation</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Withholding assumptions</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Comparing one student/group to another</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Discussing PSMT’s language</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>105</td>
<td></td>
</tr>
</tbody>
</table>

Examples from the PSMT’s written responses provide a picture of how the PSMT’s were interpreting on the technology task as described above. Those examples are presented here followed by a discussion. Chase’s written response to the interpreting prompt for the group video clip was:

Some of this is speculation, 1) So I noticed several times that the students would read the number of the length of a segment out very deliberately. I mention this because when I do the problem or watch them work it's like I don't even see the
number, except maybe as part of a ratio. Somehow measuring this length is very important to the students in making the problem real. I'm tempted to take a ruler and put it on the screen to see if the length really is 11.16 cm. Maybe just having a number makes them feel more comfortable, a concrete answer to show the teacher?

2) I noticed the students would move the points around rather hesitantly, maybe a one inch shift here or there, not really testing the boundaries of how far you can move the points. I also noticed that the students seperately [sic] measured $CD$ and $C'D'$. Do they realize that moving the points of $AB$ is the same really as measuring a new line segment. Why are there two line segments here? These observations make me think that the students don't understand that moving the points creates a generalization of the original rule, perhaps they don't understand why a generalization would be valuable. 3) The students seem to me almost too content. I agreed with the teachers decision to prod them to think harder, to make their statements clearer, and to consider different questions that can be examined in the problem. 4) With regards to content, I think the students could identify a dilation if they saw one and guess a dilation factor but not express a dilation algebraically or find the dilation by hand with a ruler. They lack the key point that given any point $X$ on a figure $P, X, X'$ (it's image) are colinear [sic].

Mary’s written response to the interpreting prompt for the student work is below.

They understand that dilation means that objects are scaled up or down by a specified scale factor from a given point and the distance of the objects from the given point are also scaled up or down by the same scale factor.

Susan’s written response to the interpreting prompt for the student work is below.
The students understand that dilations mean you multiply by the scale factor, in this case, 2. I know this because their prediction was correct that it would make the scale two times bigger, which they then wrote did happen when they dilated the line segments. They do not seem to have a firm grasp of the necessary arguments for proving a mathematical statement, since they stated that we do have proof that they are always in a 1:2 ratio which they saw by dragging around the point. I know this because that would require them to exhaust the possibilities of places it could be dragged in order to prove by those examples.

Nicole’s written response to the interpreting prompt for the student work is below.

From their response on the worksheet, I would say that they had an understanding of dilation and how it would effect the distance of the lines by their responses in D and E. They state that the "distance of \( AB \) is half the distance of \( A'B' \)," and the same for \( CD \) and \( C'D' \). You can also see this through the distances they have recorded for each line. They recorded numbers that made it fairly easy to see the relationship from the dilation, whereas the numbers on the PDF of the GSP may have been a little more difficult to say that \( AB \) is half of \( A'B' \).

As described earlier, the classifications of the PSMT’s written responses varied on the interpreting prompt. In Chase’s written response, it was evident he was not focused on interpreting students’ understandings. Rather, he made many other observations and only briefly mentioned a couple of things he thought the students understood (i.e., “generalizing a rule” and “identify a dilation”). Mary’s written response provided some ideas about what students might understand about dilations but she did not include evidence as to why she thought the students understood these things. On the other hand, the responses from Susan
and Nicole both provided at least some evidence for why the assumption was made regarding students’ understandings. For example, Susan shared she knew the students understood “that dilations mean you multiply by the scale factor…because their prediction was correct.”

Within each written response above, there were also examples of instances that were coded as *inferring about student understanding*. For example, Nicole shared, “From their response on the worksheet, I would say that they had an understanding of dilation.” Chase claimed, “These observations make me think that the students don't understand that moving the points creates a generalization of the original rule, perhaps they don't understand why a generalization would be valuable.” Susan stated, “They do not seem to have a firm grasp of the necessary arguments for proving a mathematical statement.” And Mary: “They understand that dilation means that objects are scaled up or down by a specified scale factor from a given point and the distance of the objects from the given point are also scaled up or down by the same scale factor.” So, while the PSMT’s did differ in what inferences they made regarding students’ understandings, while interpreting, *inferring about student understanding* was the most prominent descriptive code.

**Responding**

The classifications of the PSMT’s written answers to the Responding prompt for the technology task indicated that the PSMT’s either had inappropriate questions given the mathematics of the problem, did not provide rationales for their proposed problems, or posed a useful problem that included a rationale that was not clearly linked to students’ thinking. Only one case, Mary, provided a written response for one artifact, the group video
clip, which included a rationale with evidence that indicated students’ thinking had been considered.

The PSMT’s data for Responding revealed 24 different descriptive codes on the technology task, as shown in Table 39. The most frequently occurring descriptive code for responding was *exploring mathematical meanings and relationships*, which accounted for just over one-quarter of the total count of descriptive codes (i.e., about 26%) for responding. In particular, this was the most frequently occurring code for Mary and Susan, the second most frequent code for Chase, and only appeared once in the data for Nicole. Only one other code, *extending thinking*, appeared in the data of all four cases. *Discussing PSMT’s understanding of dilation* appeared for three cases (i.e., Chase, Mary, and Nicole). This finding is likely a result of the researcher asking specific questions of the PSMT’s in an attempt to illuminate the PSMT’s understanding of dilation in order to better understand his/her interpretations of the students’ understandings. The 21 remaining codes were present for only one (i.e., 15 codes) or two (i.e., 6 codes) cases, revealing the very diverse ways in which the PSMTs were responding.

Table 39

*Summary of Descriptive Codes for Responding for the Technology Task*

<table>
<thead>
<tr>
<th>Descriptive Code</th>
<th>Number of Occurrences</th>
<th>Number of Cases Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exploring mathematical meanings and relationships</td>
<td>25</td>
<td>4</td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of dilation</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>Extending thinking</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>Reflecting</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>Starting with a simple or familiar case</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Using technology as a tool</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Probing; getting students to explain their thinking</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>Activity</td>
<td>Count</td>
<td></td>
</tr>
<tr>
<td>------------------------------------------------------------------------</td>
<td>-------</td>
<td></td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Discussing beliefs</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Inserting terminology</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Establishing context</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Using a by-hand approach</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Citing evidence</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Evaluating</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Discussing motivation</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Discussing PSMT’s understanding of geometry</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Attending to student errors</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Avoiding cases</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Discussing conceptions of understanding</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Drawing from experience to select example</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Inferring about teacher</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Needs time or unsure how to respond</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Inferring about student understanding</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>95</td>
<td></td>
</tr>
</tbody>
</table>

Although the PSMT’s did not always provide a rationale for their choices of how to respond in their written responses, their responses did include instances of the most frequently occurring descriptive code, *exploring mathematical meanings and relationships.* To illustrate this, examples of the PSMT’s written responses to the responding prompt for the technology task are included here below followed by a discussion. Here is Chase’s written response for the responding prompt for the student work from the technology task:

1. We should talk about dilation in the context of algebraic manipulation. Students should understand that dilation is connected to multiplying. To do this I would construct an exercise where \( P \) is at the origin and the students have to dilate figures and write down the resulting coordinates. Maybe a right triangle with points on the hypoteneus [*sic*] and the right angle at the origin. This will help cement the notion of
dilation in their minds and connect geometry to algebra. 2. We should relate dilation to similar concepts like, translation, reflection, and rotation. Maybe a compare and contrast open-ended exercise or possibly a matching activity where you have to find the transformation that matches an image to a pre-image. This kind of exercise will help students connect the new information to previous knowledge. 3. Finally, these are fairly abstract concepts. I want a problem that uses application and context as a way to help students familiarize themselves. Maybe a worksheet about constructing a house or table which needs to be twice the size. You could even have a fun question about what happens to the area of figures which are dilated and talk about why?

Mary’s written response for the responding prompt for the group video clip from the technology task is below.

I would ask them to explain the purpose of point P by dragging point P around the screen, in order to lead them to see the relationship between point P and the images. The teacher was starting to do this at the end of the video when she sees that they did not already make that observation. I believe asking them to predict what would happen if the scale factor were a half would help correct their misconception of dilation that it only means to ‘make bigger.’ Additionally, I would ask them to write mathematical relationships between the images and pre-images in order to clarify the concept of scale and scale factor.

Susan’s written response for the responding prompt for the student work from the technology task is below.
How do you know with absolute certainty that the examples you didn't try will also be a ratio of 1:2? Is there any other evidence you could use to prove that a dilation of 2 will be a ratio of 1:2 no matter where you drag the lines? What are some other relationships you see between the two lines? Are there any others, do they remain the same distance away from each other or on the same side of each other? Why might that be?

Nicole’s written response for the responding prompt for the group video clip from the technology task is below.

I would ask students what they think will happen to the distance between $P$ and $B$ and $P$ and $B'$ since this is where they leave off in the video. I would also like to ask them if they think it will have a similar relationship as $P$ and $A$ and $P$ and $A'$ to help make the connection that lines and their relationship to $P$ is similar.

In each response above, the PSMT had at least one example of how he/she would have students explore mathematical meanings and relationships. For instance, Chase indicated that he would have students consider the connections between geometric and algebraic representations of dilations. In doing so, Chase believed, as he revealed in the follow up interview, the students would better understand the concept of dilations. Mary decided she wanted to have students explore the role $P$ was playing in the dilation as the center; she hoped such an investigation would help students see that the dilation is not only dependent on the scale factor. All of Susan’s questions in her response were aimed at helping the students make connections concerning the various properties of dilation. Nicole also wanted to have students consider the role of the center $P$ by exploring the distances between the center and the image and pre-image points. So, although the data revealed the
PSMTs responded in a variety of ways, they all did desire to have students *explore mathematical meanings and relationships*, even if the ideas they were having students consider were different.

**Summary**

The data for the PSMT’s written and verbal responses related to the technology task included similarities and differences between the cases. While attending, most of the cases included little evidence from the artifacts in their written responses to support their claims of what the students were doing in the problem for the student work. However, each case’s written response for the group video clip was classified higher indicating that each case was attempting to provide more evidence for that response. Also, the most frequently occurring descriptive code for each case was *describing*, which accounted for about 42% of the overall descriptive codes for attending. While interpreting, the cases were varied in the classifications of their written responses. Their responses ranged from not making interpretations about students’ thinking to including some evidence to support claims about students’ understandings. The overall most frequently occurring descriptive code was *inferring about student understanding*, accounting for about 30% of the total codes for interpreting; this code was present for all four cases and was the most frequent for three PSMTs. While responding, most of the PSMT’s written responses were either inappropriate given the mathematics of the problem or lacked a clear rationale regarding students’ thinking. There was also a wide range of descriptive codes that appeared for responding. Of these, *exploring mathematical meanings and relationships* was one of the more frequent overall codes and was present for all four cases.
Comparing Tasks

After looking at how the cases were attending, interpreting, and responding in similar and different ways within each task, the researcher considered how the PSMTs were noticing across tasks. First, the researcher considered the noticing framework (i.e., attending, interpreting, responding; Tables 4, 5, and 6, respectively) classifications of PSMT’s written responses. Following this evaluation, the researcher then considered the three most frequently occurring codes from each task for each aspect of noticing. Only the top three codes from each task were considered because codes beyond these typically appeared less often and for fewer PSMTs. This section presents those results to describe the ways in which the PSMTs of this study were attending, interpreting, and responding across the two different contexts (i.e., paper and pencil and technology).

Attending

When considering the framework categorizations of the written responses for attending, Chase and Susan were fairly consistent in their responses across tasks, as shown in Table 40. They both lacked details from the artifacts for three of their four written attending responses and each had one response that included some evidence. Nicole’s written responses also had similar classifications; three of her four responses contained some evidence from the artifacts. But, for the technology task, which she received second, the classification of Nicole’s attending response for group video clip was much higher, indicating for that particular response that she completely described the students’ approach in full detail with evidence from the artifact. Mary had the largest contrast in the categorizations of her attending responses. For the technology task, which she received first, Mary’s responses were categorized as low for the Attending framework signaling that she
lacked evidence to support her description of the students’ approach. However, on the paper and pencil task, Mary’s responses were classified much higher. In fact, for both written responses to the attending prompt on the paper and pencil task, Mary received the highest classification revealing that she provided a complete description, with evidence, of the students’ work on the task. Thus, it is possible that for both Nicole and Mary the increases in classifications of their written responses was a result of learning to better attend since the increase for each was seen on their responses for the second task they received.

Table 40

Summary of Written Response Classifications for Attending for Each Task

<table>
<thead>
<tr>
<th></th>
<th>Paper and Pencil Task</th>
<th>Technology Task</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Student Work</td>
<td>Group Video Clip</td>
</tr>
<tr>
<td>Chase</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Mary</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Susan</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Nicole</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

A comparison of the three most frequently occurring descriptive codes related to the attending prompt from each task, summarized in Table 41, revealed that for both tasks, the main focus of the PSMTs while attending was *describing*. In fact, for each task, this code appeared about twice as often in the data as the second most frequently occurring code. *Citing evidence* was also a code that appeared in this comparison; however, this descriptive code was only found in the data for one PSMT on the paper and pencil task versus three PSMTs on the technology task. The remaining recurrent codes again reveal a difference in how all four of the PSMTs were attending within the two different contexts; for the paper
and pencil task, PSMTs were *inferring about student action* and on the technology task they were *reflecting*. It is important to note that these findings do not mean that the PSMTs did not attend in other ways, rather these codes indicated the ways in which the PSMTs were attending most often.

Table 41

*Summary of Most Frequent Descriptive Codes for Attending for Each Task*

<table>
<thead>
<tr>
<th></th>
<th>Paper and Pencil Task</th>
<th>Technology Task</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Descriptive Code</strong></td>
<td><strong>Number of Occurrences</strong></td>
<td><strong>Number of Cases Present</strong></td>
</tr>
<tr>
<td>Describing</td>
<td>88</td>
<td>4</td>
</tr>
<tr>
<td>Inferring about student action</td>
<td>35</td>
<td>4</td>
</tr>
<tr>
<td>Citing evidence</td>
<td>21</td>
<td>1</td>
</tr>
</tbody>
</table>

**Interpreting**

For interpreting, Chase was the most consistent in his written responses across tasks, as shown in Table 42. Mary and Susan both had slightly higher classifications of their responses for their second tasks, paper and pencil and technology, respectively. These higher classifications indicated that both PSMTs were providing more evidence to support their claims about students’ understandings when responding to the interpreting prompt for the artifacts from their second task. It is possible that Mary and Susan provided more evidence in their responses for the second task because they had been asked during interviews from their first task to provide more detail. On the other hand, the categorizations of Nicole’s written responses were somewhat higher for her first task, the paper and pencil task. This could be a result of Nicole’s comfort level with GSP, the software program used
in the technology task; recall Nicole reported having a low comfort level with GSP and was currently enrolled in the teaching mathematics with technology course at the time of the study.

Table 42

Summary of Written Response Classifications for Interpreting for Each Task

<table>
<thead>
<tr>
<th></th>
<th>Paper and Pencil Task</th>
<th>Technology Task</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Student Work</td>
<td>Group Video Clip</td>
</tr>
<tr>
<td>Chase</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mary</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Susan</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Nicole</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

The descriptive coding for written and verbal responses for each task revealed some commonalities in the ways in which PSMTs were interpreting. Table 43 provides a comparison of the most frequently occurring descriptive codes related to interpreting from each task. These findings indicated that PSMTs were interpreting in the same ways regardless of the context. In other words, for both tasks, inferring about student understanding was the most frequently appearing code, followed by describing, and then reflecting. For the paper and pencil task, the third most frequently occurring code also included citing evidence as this code appeared 13 times across three PSMTs in the data as well. Finally, the most frequently occurring code, inferring about student understanding, appeared about twice as often in the data for both tasks as did describing.
Table 43

Summary of Most Frequent Descriptive Codes for Interpreting for Each Task

<table>
<thead>
<tr>
<th>Descriptive Code</th>
<th>Paper and Pencil Task</th>
<th>Technology Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inferring about student understanding</td>
<td>47, 4</td>
<td>Inferring about student understanding</td>
</tr>
<tr>
<td>Describing</td>
<td>25, 4</td>
<td>Describing</td>
</tr>
<tr>
<td>Reflecting &amp; Citing evidence</td>
<td>13, 3</td>
<td>Reflecting</td>
</tr>
</tbody>
</table>

Responding

The classifications of written responses to the responding prompts for three cases, Chase, Mary, and Susan were about the same indicating that these PSMTs were consistent in their replies while responding, as shown in Table 44. In particular, Chase and Susan had responses that indicated they included an inappropriate question given the mathematics of the problem or lacked a clear rationale for the problems posed. Mary’s responses revealed that she too lacked a clear rationale at times, but on other occasions included a rationale with some evidence she considered students’ thinking. Nicole’s classifications were also similar between tasks with one response classified higher on the paper and pencil task revealing she posed a potentially useful problem and provided a rationale that included some connections to students’ thinking from the task.
Table 44

Summary of Written Response Classifications for Responding for Each Task

<table>
<thead>
<tr>
<th></th>
<th>Paper and Pencil Task</th>
<th>Technology Task</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Student Work</td>
<td>Group Video Clip</td>
</tr>
<tr>
<td>Chase</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Mary</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Susan</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Nicole</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

The comparison of the most frequent descriptive codes for responding revealed that the PSMTs were responding in a variety of ways, as shown in Table 45. In fact, no single frequent code appeared within the top three codes on more than one task. For the paper and pencil task, PSMTs there was not much variation in the most common ways in which they were responding. In other words, the three most frequently occurring codes accounted for about 15%, 16%, and 12% of the total descriptive codes for responding from that task. Further, only gathering information, checking for a method, leading students through a method appeared for all four PSMTs further confirming the lack of consistency across the PSMTs in their choice of how to respond. On the other hand, the codes from the technology task revealed that exploring mathematical meanings and relationships was a common choice among all cases. This particular code accounted for about 26% of the total count of descriptive codes for responding from the technology task, while the remaining codes accounted for about 11% and 9%, respectively. Thus, when looking across tasks, the descriptive coding did not reveal a common way in how the PSMTs were responding.
Table 45

**Summary of Most Frequent Descriptive Codes for Responding for Each Task**

<table>
<thead>
<tr>
<th>Descriptive Code</th>
<th>Paper and Pencil Task</th>
<th>Technology Task</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number of Occurrences</td>
<td>Number of Cases Present</td>
</tr>
<tr>
<td>Describing</td>
<td>18</td>
<td>2</td>
</tr>
<tr>
<td>Gathering information, checking for a method, leading students through a method</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td>Probing; getting students to explain their thinking</td>
<td>13</td>
<td>2</td>
</tr>
</tbody>
</table>

**Discussion of Research Question 3**

In order to answer research question 3 (i.e., What similarities and differences exist in the ways PSMTs notice students’ mathematical thinking in the two different types of tasks?) comparisons within and between the two tasks were made to determine how PSMTs were attending, interpreting, and responding. These comparisons revealed some similarities and some differences.

For the written responses for all three facets of noticing, the PSMTs were varied in the classifications of their responses. In other words, for certain aspects of noticing, PSMTs were able to maintain consistency in the categorizations of their written responses, while others varied. For example, while attending, Susan’s written responses were typically classified low, indicating she provided some details about the students’ approach to the problem but with little to no evidence from the artifact. While responding, Susan’s written responses were also classified low because she failed to provide a rationale for how she
planned to respond. But, while interpreting, Susan’s written responses varied between tasks. Her responses on the paper and pencil task were low, characteristic of not including evidence to support her claims about students’ understandings, while her responses on the technology task were generally higher, indicating she provided some evidence to support her assertions. Thus, Susan exemplifies how for certain aspects of noticing, she was able to maintain consistency in the classifications of her written responses (i.e., attending and responding) while she varied in the classifications of her written responses for another aspect of noticing (i.e., interpreting).

The comparison of descriptive coding between tasks revealed some commonalities in the ways that PSMTs were noticing; however, there were no consistent patterns noted that any PSMT followed or that they collectively followed while noticing. For example, there was no consistency found to indicate that the PSMTs started off attending by describing and then moved into a different form of attending. Rather, the descriptive coding revealed overall, that while attending, PSMTs were found to be describing most often. Comparing codes for interpreting revealed that the PSMTs were consistently interpreting in the same ways regardless of the task type. In particular, the code that appeared most often while interpreting was inferring about student understanding; however, describing and reflecting were also common ways PSMTs were interpreting on both tasks. A comparison of the descriptive codes for responding revealed the largest discrepancy in how the PSMTs were noticing. In fact, there was no single code that appeared in the top three most frequent codes of both tasks. These findings suggest that attending and interpreting may be easier aspects of noticing for PSMTs than responding.
It is also important to note that the PSMTs rarely indicated they were drawing upon their own approaches to the tasks while noticing. This was a surprising finding as the researcher expected the PSMTs might use their own experiences with the task as they noticed students’ thinking. In fact, only one PSMT, Susan, referred back to her own work from the task-based interviews. This occurred when she was describing how she would respond based on the students’ thinking from the paper and pencil task. Susan wanted to help the students recognize the similar right triangles and noted that her question, “Are there any relationships between the two triangles that might help you?”, was based on what she “knew from…[her] experience solving the problem.” Here Susan shared with the researcher that her question was a result of her own work on the problem. So although the PSMTs in this study did not explicitly state they considered their own work on the tasks while noticing, it is possible they did and did not share this information with the researcher. As such, no findings indicated that the PSMTs were actively using their own work on the tasks while noticing.
CHAPTER 6: DISCUSSION, LIMITATIONS, AND IMPLICATIONS

Discussion

The purpose of this study was to examine the ways in which PSMTs were noticing students’ work on a paper and pencil task and on a technology task, as little work has been done to understand how PSMTs notice in general and how they notice when tasks utilize technology. Such an examination could help the field begin to understand how PSMTs handle the practice of making sense of students’ work through noticing and lead to insights about how to better prepare them for the classroom. Research has suggested that teachers who engage in noticing students’ mathematical thinking are better able to make sense of students’ work and their understandings (Jacobs et al., 2010; Leatham et al., 2015). Thus, the practice of noticing students’ mathematical thinking is an important practice for effective teaching as it enables teachers to make sense of students’ work.

What can we learn from PSMT’s noticing?

Research has shown that examining how PSMTs notice on tasks can provide insight into how the PSMT is thinking about the students’ thinking (Wilson et al., 2011). Early efforts in mathematics education research related to teacher noticing revealed that when teachers were asked to notice, they did not start out focusing on students’ mathematical thinking (Jacobs et al., 2011; Sherin & Han, 2004; Sherin et al., 2011; van Es & Sherin, 2008). Rather, they began by making general, non-mathematical observations, such as how many times students were out of their seats (Crespo, 2000; Kazemi & Franke, 2004; van Es & Sherin, 2008). However, through targeted efforts to engage teachers in the practice of noticing students’ mathematical thinking, such as through video clubs focused on learning to notice, researchers found that teachers, over time, were better able to attend to students’
thinking and make sense of students’ mathematical understandings (Fennema et al., 1996; Jacobs et al., 2010; Stockero, 2014). Yet, much of the work in this area has utilized interventions to examine if teachers can improve in their practice of noticing. Interventions have mainly included having teachers consider students’ written work and the use of video clubs where teachers examine, discuss, and analyze actual mathematics lessons (Schack et al., 2013; Stockero, 2014; van Es & Sherin, 2008). During the intervention, teachers learned how to notice through the guidance of a researcher asking targeted questions aimed at improving this skill. Further, these studies often included groups of teachers who were learning to notice together.

In contrast, this study considered how PSMTs, without a structured intervention, noticed students’ mathematical thinking. The PSMTs were asked to notice individually and without specific instruction or guidance on what it means to notice. The only exposure these PSMTs had related to noticing were the various tasks they might have encountered within their program where they considered students’ work and videos of students completing tasks. Unlike early research on teacher noticing, these PSMTs were not initially focused on non-mathematical factors. This could be because the PSMTs were asked to first consider students’ work on the task prior to watching a video. It seems natural that a video lends itself to noticing more than just the mathematics task at hand; however, even when these PSMTs watched the video artifacts, their focus from the start of the study was on the mathematics that was present. This consistent focus on the mathematics and attempts to make sense of students’ work could be due to their exposure to noticing-type activities within their program, or it could have been a result of the use of noticing prompts to which PSMTs were asked to respond. By asking the PSMTs specific questions aimed at attending,
interpreting, and responding to students’ thinking, the researcher did likely narrow the PSMT’s focus.

This brings to light another finding of this study. What the PSMTs noticed was at times influenced by the researcher’s prompts. This was especially evident during follow up interviews. For example, during interviews, PSMTs often shared reflections. It was expected that reflecting would be present in the interviews as the researcher was asking the PSMTs to revisit their responses to the noticing prompts. But, what was interesting is that when an instance of reflecting appeared during interviews, it was often found that the PSMT was thinking about implications for his/her own teaching, such as how he/she might help the students combat issues with vocabulary. Aside from this study, no other literature was located to substantiate these findings.

Another idea that emerged during interviews as a result of the researcher’s questioning was discussion of the PSMT’s conceptions of understanding. Often, during written responses to the noticing prompt for interpreting (i.e., Please explain what you learned about these students’ mathematical understandings.), the PSMTs would make statements indicating there may be a degree of understanding for students. Such statements included phrases like “the students are familiar with” or “they recognize” or “they mostly understand.” These statements prompted the researcher to ask each PSMT what “understanding” meant to him/her. Through this question, the researcher learned that for some PSMTs, there were two categories of understanding: familiarity and mastery. Familiarity meant the individual could recall the concept or formula while mastery meant the individual knew how and when to utilize the concept or formula. For others, understanding simply meant the student could answer the question correctly. Simpson and
Haltiwanger (2016) also found that PMTs had a difficult time providing evidence to support claims of student understanding.

It was also apparent throughout the study that the PSMTs were having difficulty clearly articulating their noticing. In fact, without interviews would have been difficult to understand what PSMTs meant solely based on their written responses. Overall, their ability to articulate and support their claims decreased over the noticing prompts. In other words, the PSMTs were able to describe the students approach to each problem (i.e., attending). Through their written responses and interviews, they generally gave a good description of the students’ attempts on the tasks and did not miss many details. When asked for evidence to support their claims, most of the PSMTs could point to something from the artifact to justify their statements. For interpreting, the PSMTs were able to provide some ideas about what mathematics the students might understand. However, when attempting to describe how they determined what the students understood, the PSMTs struggled to provide evidence to support their assertions. Through discussions of what constitutes understanding, some of the PSMTs were able, when asked, to give details for why they believed the students understood or did not understand an idea. Yet, others still had difficulty substantiating their claims. When responding, all of the PSMTs were able to provide some potential follow up questions or tasks. But, when asked for a rationale as to why a particular question or task was selected, the PSMTs had trouble connecting their choices back to the students’ thinking on the task. Perhaps their difficulty with responding could be due to a lack of experience from which to draw upon (Jacobs et al., 2010). It could also be a result of not understanding the mathematics content within the problem at a level deep enough to know what connections could be made to other areas of mathematics (Ball et al., 2008). In
either case, these findings are similar to those of Simpson and Haltiwanger (2016) and Tyminski and colleagues (2014), who also found that PMTs struggled to explain how their decision of how to respond connected back to students’ thinking.

This study also found that the PSMTs did not have much variation in how they were noticing between tasks. Prior to the start of the study, the researcher expected the PSMTs would notice students’ thinking on the technology task in different ways due to the affordances technology offers in making students’ thinking more visible through observable actions, such as dragging (Arzarello et al., 1998). Thus, it seemed reasonable for the researcher to expect the PSMTs might make more claims about students’ understandings or better be able to describe the students’ approach for the technology task. However, this was not the case; there was no apparent difference in the ways the PSMTs were noticing between tasks while they were attending and interpreting. While responding, however, there was a distinct difference in the ways the PSMTs were responding. On the paper and pencil task, the PSMTs varied greatly in how they were choosing to respond to students. But, on the technology task, the PSMTs all had ideas about how they would have students further explore dilations using the technology tool. While they varied in how they would approach this with students, it was apparent that each PSMT wanted the students to engage in mathematical exploration in an attempt to make meaningful connections. This finding is interesting as mathematical action technologies, like GSP, do allow the user more opportunities to explore the mathematics, which the PSMTs recognized and wanted to leverage.

Research has shown examining how teachers notice can reveal some ideas about how they are making sense of students’ work (Jacobs et al., 2010; Wilson et al., 2011). In
this study, examining how the PSMTs were noticing revealed: (a) the PSMTs were focused on mathematics while noticing, (b) the PSMT’s conceptions of what it means to understand a concept and what constitutes evidence of that understanding, (c) the PSMTs had difficulty articulating what they were noticing, especially when it came to making connections with how they would respond back to students’ thinking, and (d) the PSMTs, when responding, all wanted to leverage technology to have students further explore the mathematics in the technology task.

Complexity of Tasks

Cognitively demanding high school mathematics tasks can be complex. There are often many ideas or concepts that require students to make connections and draw upon prior knowledge in order to successfully complete the task. In this study, the students from the artifacts for the paper and pencil task were unsuccessful in determining the length of $\overline{BC}$. The students from the artifacts for the technology task were able to complete the dilation and make some observations; however, they were not able to generate a proof of their observations. While these artifacts provided opportunities for the PSMTs to notice students thinking, it was not uncommon for the PSMTs in this study to have difficulty with various aspects of noticing. Similar findings where novice teachers miss opportunities to notice students’ thinking are prevalent in literature (e.g., Peterson & Leatham, 2009; Stockero & Van Zoest, 2013). In particular, as described above, the PSMTs in this study struggled to provide justifications for claims regarding students’ understandings and to articulate their reasoning for how they chose to respond to students. It is possible that another reason the PSMTs had difficulty articulating responses for certain aspects of noticing was because the students did not do what the PSMT expected. This would be consistent with Ball’s claim
that teachers have trouble making sense of students’ thinking when the students do unexpected things or their ideas are represented in unfamiliar ways (Ball, 2001). Thus, it makes sense to wonder how might these PSMTs have noticed differently if they considered artifacts of tasks where the students were successful in completing the task.

It is also worth considering how PSMTs might notice students’ thinking if they were asked to consider artifacts through a particular lens (e.g., Colestock & Sherin, 2009; Leatham et al., 2015). In other words, how might PSMTs notice if they were asked to notice something particular? And, further, are there certain things we want PSMTs to notice? One group of researchers suggests one lens that might be useful in making sense of students’ work and thinking is to look for MOSTs (Mathematically Significant Pedagogical Opportunities to Build on Student Thinking; Leatham et al., 2015). These opportunities must include student mathematical thinking, be mathematically significant, and contain pedagogical opportunity. By identifying and analyzing MOSTs, the researchers claim that teachers can better identify mathematically important instances of students’ thinking and determine whether to act upon that instance by considering if doing so would promote the students’ mathematical understandings. Thus, it is important to consider what lenses might be helpful for teachers to consider when engaged in the practice of noticing.

**Limitations**

In research, limitations are unavoidable. As such, this study did have limitations that should be considered in light of the findings. First, the participants in this study represented a convenience sample; a selected group of PSMTs was invited to participate because of their location within the program. Thus, the four selected cases were representative of the PSMTs in this study, but they might not be representative of all PSMTs. Also, because this study
was a dissertation, there were time limits and only one researcher. This study was conducted over a six-week period and only utilized artifacts of practice from two tasks. Further, in both tasks, the students were unsuccessful in some way. Thus, it is difficult to say if the results of the study might have revealed different insights into PSMT’s noticing had the study taken place over a longer period of time, used artifacts of practice from more tasks for comparison, and used artifacts of practice from tasks where the students successfully completed the problem. Also, because this study only included one researcher, one person only coded the data. While peer-debriefing sessions with an expert in the field were held weekly during data collection and at least once a month during data analysis in an attempt to validate the findings, it is possible that another researcher could have had different interpretations of the data.

**Implications**

This study was aimed at exploring how PSMTs notice students thinking on a paper and pencil task and a technology task. The findings indicated that the PSMTs were consistent in how they were attending and interpreting students’ thinking. While attending, the PSMTs were describing the students’ approach to the problem; while interpreting, the PSMTs were inferring about student understanding. However, while responding, the PSMTs were more varied in their decision of what to do next with students. Some asked probing questions to clarify students’ thinking while others focused on how they might lead students to a correct solution. The data also revealed that the PSMTs had difficulty articulating what they were noticing and providing clear evidence connected to students’ thinking on the task. These findings have implications for mathematics teacher educators and researchers.
Implications for Mathematics Teacher Educators

Mathematics teacher educators must continue efforts to facilitate PSMTs in the practice of making sense of students’ work. By engaging PSMTs in activities that require them to notice, PSMTs will gain experience in learning to attend to, interpret, and respond to students’ thinking. However, careful attention needs to be given to facilitate PSMTs in learning how to clearly articulate their noticing. The PSMTs in this study came from a program that required them to practice the skill of noticing in at least one course, yet the findings indicate that exposure within one course might not be sufficient in helping PSMTs learn to convey their noticing in a clear way. Thus, efforts must be made to provide PSMTs with more opportunities to engage in noticing. Also, specific guidance on how to notice and the chance to discuss what they are noticing may prove helpful as well since these strategies have been effective for PMTs of other grade levels (e.g., Jacobs et al., 2010; Schack et al., 2013; Stockero, 2014; van Es & Sherin, 2008). This might be especially true when tasks use technology as the PSMTs in this study did not attend or interpret differently on the technology task than on the paper and pencil task. Because of the unique features of mathematical action technologies, like measuring and dragging, PSMTs could notice in different ways by examining how the students are interacting with the technology tool (Arzarello et al., 1998). The findings of this study indicate that PSMTs need support in learning to recognize and interpret the ways students are using these features, rather than just knowing what features are available and what affordances each provides.

Implications for Research

Researchers should continue to examine how PSMTs notice students’ thinking. The findings of this study are promising considering the PSMTs did not have a specific
intervention to improve their practice of noticing. Thus, these findings revealed that for PSMTs with some prior exposure to noticing tasks were able to engage in the practice of noticing but that they need help learning to articulate their responses. As future teachers, PSMTs should be able to describe what they are noticing and how it connects with students’ thinking (AMTE, 2017; NCTM, 2014). Therefore, researchers should explore ways to help PSMTs learn to clearly articulate their noticing so that follow up interviews, like those used in this study, or conversations, are not needed in order to understand what the PSMT is saying about what the students did in the problem, what the students understand, or how the PSMT would respond and why.

One way researchers might continue exploring how PSMTs learn to notice is by offering all of the artifacts from a task at one time rather than one at a time in succession. This would provide the PSMTs the opportunity to consider the artifacts in an environment more closely resembling what a teacher would experience in real time in the classroom. It would also afford the opportunity for PSMTs to make connections between the artifacts to support what they are noticing. The decision to offer the artifacts separately for this study was made based on other research studies, so it would be interesting to explore what differences might have appeared had the PSMTs been given all of the artifacts for a task at one time.

Another way researchers can continue exploring how PSMTs notice is to consider the prompts PSMTs are given or asked. For this study, the researcher used the noticing prompts developed by Jacobs and colleagues (2010). This meant that there was a prompt to target attending, a prompt for interpreting, and a prompt for responding. For this study, the prompts and associated data were considered separately in an effort to examine how PSMTs
were engaging in each aspect of noticing. However, even Jacobs and colleagues note that attending, interpreting, and responding are interrelated practices. Thus, it is important to consider whether PSMTs should be asked questions that target specific actions or prompted with a general question to discuss what they notice. Further, as mentioned above, due to the complexity of tasks, it is important to also consider how these prompts are framed. In other words, do we ask PSMTs to notice something particular? Thus, research needs to examine how the framing of our prompts impacts what PSMTs notice. Perhaps by revising how or what we are asking, PSMTs will better be able to articulate their noticing.

Finally, researchers need to continue to explore how PSMTs notice on a variety of tasks across different contexts (i.e., paper and pencil tasks and technology tasks that utilize various tools). This study was limited to examining how PSMTs noticed on a single pencil and paper task and a single technology task. More research is needed to consider how PSMTs in the same study are given the opportunity to notice on multiple tasks of each type. Researchers should also examine various mathematical topics; this study was again limited to one mathematical concept. Finally, researchers should also give PSMTs the chance to examine artifacts from tasks where students were both successful and unsuccessful in completing the task. By varying these different factors, researchers can gain more insight into how PSMTs notice students’ mathematical thinking.

**Closing Remarks**

This study contributes to the expanding field of teacher noticing by providing insight into how PSMTs notice students’ thinking on a paper and pencil task and on a technology task. Aside from this study, little work has been done to examine how PSMTs notice in general, let alone when tasks utilize technology. While the findings indicated the PSMTs
had difficulty articulating their noticing, the findings also revealed that there were some similarities and differences in the ways in which the PSMTs were noticing. Overall, the PSMTs were consistently describing while attending and inferring about student understanding while interpreting. While responding, however, the PSMTs varied the most in their responses indicating that determining how to respond on the basis of students’ thinking is an aspect of noticing with which PSMTs struggle. For the technology task, however, there was a commonality for all PSMTs when responding; they all wanted to leverage technology to have students further explore the mathematics in the technology task. These exploratory findings provide a foundation from which to build as we continue to explore the noticing of PSMTs and ways to help them learn and practice this important skill.

In particular, to build from this study, I would like to extend my research in two ways. First, I would like to continue exploring how PSMTs notice using artifacts of practice across different contexts. Thus, in a future study, I would include artifacts from at least three different paper and pencil tasks and three different technology tasks. And, rather than give the PSMTs the artifacts from a task one at a time, I would give them the artifacts all at once. For my first study, I would keep the noticing prompts used in this study; however, I would also conduct a second study where the prompts were framed differently. While I am unsure at this moment what those prompts might look like, I imagine they would ask PSMTs to notice something in particular, such as what they were able to notice due to how students were dragging something in GSP.

The second way I would like to extend my research is examining how PSMTs notice in the moment. Recently, educational simulations have been developed to provide support for prospective teachers while they engage in authentic situations. These simulations,
modeled after the medical field, make use of adult actors trained to present a particular issue to the prospective teacher (Dotger, Dotger, & Maher, 2010). These one-on-one interactions between a “student” and a prospective teacher provide a new way of observing how a prospective teacher engages in specific skills of practice. Thus, it would be interesting to explore how PSMTs notice students’ thinking while engaging with the student. Some questions to consider are: How does the PSMT notice when provided access to the student rather than just artifacts?, How does the PSMT make sense of students’ thinking in real time?, and How does the type of noticing (i.e., from artifacts or in real time) influence what the PSMT notices? As questions like these continue to be explored, our understanding of how PSMTs notice can continue to be broadened so we can better support them in preparing for the classroom.
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Society and Mathematical Association of America.


Appendix A

Student Work for Paper and Pencil Task

Student 1

In the picture below, points $A$ and $B$ are the centers of two circles. The two circles touch at a single point on $AB$. Also, points $A$ and $B$ are collinear with point $C$. Finally, $D$ and $E$ lie on the two respective circles and they are also collinear with point $C$. What is $|AC|$? Explain.

$AB = 9$

$8^2 + 9^2 = c^2$

$64 + 81 = c^2$

$c = \sqrt{145}$

$5^2 + b^2 = 8.5^2$

$25 + b^2 = 72.25$

$b^2 = 47.75$

$b = b.9$

Figure 21. Student 1’s work from the paper and pencil task.
In the picture below, points A and B are the centers of two circles. The two circles touch at a single point on $AB$. Also, points A and B are collinear with point C. Finally, D and E lie on the two respective circles and they are also collinear with point C. What is $BE$? Explain.

\[8^2 + 3^2 = c^2\]
\[w^2 + a^2 = c^2\]
\[\sqrt{73} = \sqrt{c^2}\]
\[8.54 = c\]

\[5^2 + b^2 = 8.54^2\]
\[25 + b^2 = 72.25\]
\[-25 - 25\]
\[\sqrt{b^2} = \sqrt{47.25}\]
\[b = 0.87\]

*Figure 22. Student 2’s work from the paper and pencil task.*
Figure 23. Student 3’s work from the paper and pencil task.
Figure 24. Student 4’s work from the paper and pencil task.
Appendix B

Transcript of Group Video Clip for Paper and Pencil Task

Description of artifact: In this video, you will see a group of high school juniors and seniors (who have each completed Math 1, 2, and 3) working on a mathematical task. The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion about the task.

The video focuses on Student 1, Student 2, Student 3, and Student 4 who were working together as a group (see Figure 25). But, at times, you might hear comments from other students in the class. Note that the video has been edited down to about 12 minutes. Thus, time spent by the teacher setting up the task, times where the students were not on task, or times where the students were quiet, were intentionally cut out for this video.

Figure 25. Group arrangement and camera placement of students on group video clip for the paper and pencil task.
Teacher: Alright! Have at it!

Student 1: In the picture below, point A and B are the center of two circles. The circles touch at a single point on line AB. Also, the points A and B are collinear with point C. Finally, D and E lie on two respective circles and are also collinear with point C. What is the absolute value of line BC? Explain.

Student 2: Well, we know line AB...

Student 4: Are the centers.

Student 2: …is.

Student 1: Well, since...

Student 2: …8.

Student 1: …like you know that.

Student 2: It would be 8, right?

Student 1: Yeah line AB is 8 because the distance from A, from B, from center to out is 3 and from A to center out is 5.

Student 4: Yeah, that’s right, that’s right. But what does collinear mean?

Student 1: Yeah, I don’t know.

Student 3: Are you sure it would be 8? Like we would just merge them like that?

Student 4: Yeah…

Student 1: Yeah…

Student 4: …because look…

Student 1: …the whole line.

Student 4: …the radius of this one is 5, so it’s the same acá [Spanish for here]. And then the radius of BE is 3 so it’s the same going this way. The radius is always the same around the circle.

1:00

Student 1: Mm-hmm. Exactly.
Student 3: So AB is eight.

Student 4: What does collinear mean?

Student 1: Hmm.

Student 3: Collinear? They share the same line.

Student 1: Yeah.

Student 2: Mm-hmm.

Student 1: That’s what I was thinking. I was like…

Student 4: So the other side would also be 3 right?

Student 3: Mm-hmm.

Student 1: Yeah, it sh…

Student 3: And then this would be x.

Student 1: So DE would also be 8.

Student 2: It looks shorter.

Student 1: Hm. Well. Hmmm.

Student 3: Uh-uh. Because then it would say AB is congruent…

Student 1: Yeah.

Student 2: Mm-hmm.

Student 3: …to DE. So it’s not.

Student 1: Is there any way to figure out, oh, if only A, D, E, and B was a triangle we could figure out…. We only have two sides, we could figure out the third one, but it’s…

Student 3: Couldn’t we use like the triangle EBC? Or not?

Student 1: We don’t…

Student 4: Oh yeah! You’re right! Yeah, you could do the Pythagorean theorem because you have the height, which is 3. Oh wait, no you couldn’t…
Student 1: Yeah ‘cause it’s…
Student 3: ‘Cause it’s not a right triangle.
Student 1: Yeah
Student 4: Yeah, yeah, yeah, it’s not completely done.
Student 3: Man.
Student 1: Ok. K. Hold on. Two circles…
2:00
Student 4: We could do EBA.
Student 3: E…B…
Student 4: A…and A back to E.
Student 3: Ah, is that a right triangle?
Student 4: No, but if you make the line it could, and you could find the distance of A to E. Y’all know what I’m saying?
Student 3: Mm-hmm.
Student 1: Yeah, ‘cause AB is 8 and then EB is 3.
Student 4: Mm-hmm. So it would be…what would it be? It would be 3 squared…
Student 1: a squared plus b squared equals c squared. So…
Student 3: 3 squared plus 8 squared.
Student 1: Aren’t we looking for the hypotenuse? c is the hypotenuse, so would it be 8 squared plus 3 squared equals c squared?
Student 3: Yeah. *(Inaudible.)*
Student 1: 8…
Student 3: *(Inaudible.)*
3:00
Student 4: 3 squared is nine.
Student 1: Yep. 8 squared is...
Student 4: 64.
Student 1: Yeah.
Student 3: Mm-hmm.
Student 4: And...73? Square root of 73?
Student 1: Yep.
Student 4: ADE.
Student 3: It would be 5. The other one would be...
Student 4: A to E? You don't know what it is.
Student 3: Don’t y'all remember we learned how to solve? This is one-third square root of three or some...thing? Like that? Like if we didn’t know the hypotenuse was we put one side was square root of one-third.
Student 4: Oh! The 80, 90...
Student 3: Yeah, something like that.
Student 4: No, the 30, 60, 90.
Student 1: 30, 60, 90.
Student 4: 45, 45, 90.
Student 3: Yeah, that stuff.
Student 2: Yeah.
Student 1: Yeah, 'cause we know um, angle D can- is a right angle.
Student 3: Oh yeah, so it would be...
Student 1: Mm-hmm, it would be...
Student 3: ...5 and 90?
Student 1: Yeah. Wouldn’t B, wouldn’t A be 60 and B be 30?

Student 4: 4:00

Student 1: Now what do we do for 30, 60, 90 again? I really don’t remember.

Student 4: I think it was…

Student 3: It was 1, 2, 3.

Student 4: I think it was 1, square root of 3, 2.

Student 1: Mmmm.

Student 4: I think it was 1, square root of 3, 2.

Student 3: Inaudible.

Student 1: I think so. I think so. Yeah, I think so.

Student 4: Or…

Student 2: That looks about right.

Student 4: Was it 1, square root of 2, square root of 3? No, no, no. It wasn’t. I think it was…

Student 1: Yeah, I think it was 1, root 3, 2. (pause) I’m trying to see what else we could try.

Student 3: We should just…(inaudible)…other groups.

Student 2: What was that thing we learned, like, cosine and all that? Couldn’t we use some of that?

Student 4: Oh.

Student 1: I have no idea.

Student 3: Some old, some old hoe…

Student 4: Oscar Has…

Student 1: Oscar Has a Heap of Apples
Student 4: …a Heap of Apples. So it’s cosine, tangent.

Student 1: Yep.

Teacher: I’m interested to see what you’re thinking.

Student 4: We were thinkin…

Teacher: Do you have any of these angles?

Student 4: Yeah.

Teacher: I mean, you have this. You have a 90. But do you have a 30 or a 60?

Student 4: Not that we’re 100% sure that it’s a 30 or a 60.

Teacher: I…

Student 3: *Inaudible* …a 30 and a 60.

Teacher: Yeah, I…not necessarily. What if you had a 45, 45, 90?

Student 1: Oh, yeah, true.

Student 4: Oh!

Student 3: Oh, I forgot that.

Teacher: Or a 50, 40, 90?

Student 1: Oh, yeah.

Student 4: We thought Pythagorean theorem, then we thought…

Student 1: Yeah. Pythagorean theorem…

Teacher: Ok well what happened with Pythagorean theorem?

Student 1: We ended up with the square root of 37, 73. And you can’t factor that.

Teacher: Hm.

Student 4: We were hoping to find A to E.
Teacher: That’s interesting. Okay. So you wanted to find A to E, okay. And that was the square root of 73 which you decided wasn’t helpful.

Student 1: Yeah.

Student 4: ‘Cause then if we find that one, it coulda helped us found this one…

Teacher: Ah, you’re right!

Student 4: …DE, and then we could’ve…

Teacher: It could help you find D to E. Um, I, I really like, I really like where you’re going with that. So use that to work back to find DE.

Student 1: Mm-hmm. Mm-hmm.

Teacher: Okay! Well do it. Let’s see what happens!

Student 1: Okay.

Student 4: Okay. So the square root of 73 is 8.5…4. Eight point fifty-four.

Student 1: Mmmk.

Student 4: So it’s kinda like 8.5?

Student 1: Yeah.

Student 4: So A to E is about 8.5?

Student 1: Approximately…8.5.

Student 4: Then you can do the other one.

Student 1: Yeah.

Student 4: From ABE?

Student 1: Yep. Mm-hmm. So it’d be…square root of 7…

7:00

Student 4: Wait, what, what are we finding? We’re finding one of the legs this time.
Student 1: So it would be square root of 5 plus b squared equals 8.5.

Student 4: 5 squared is 25…b squared…8.5 squared is 67.24. Subtract 25.

Student 1: You sure? I thought it was 8.5 squared is 72.25.

Student 3: 8.5 is supposed to be squared?

Student 4: Oh! 8.5? I put 8.2. Yeah.

Student 3: It’s supposed to be squared?

Student 4: Everything is squared in Pythagorean theorem.

Student 3: *Inaudible.*

Student 1: Subtract 25 from both sides…

8:00

Student 3: 72.25?

Student 1: …b squared is equal to…

Student 4: Yeah.

Student 1: 47…

Student 2: 47.25? The square root of 47.25.

Student 1: Yeah, it’s 6.9.

Student 4: So DE is about 7?

Student 1: Yeah.

Student 4: Or should we keep it as a decimal?

Student 1: Decimal is fine. Should be since we have one of them.

Student 4: If it’s vertical…AB and B to C, if they’re vertical angles, wouldn’t it be 180 degrees the whole thing?

Student 3: Mm-hmm

Student 4: And then half of it? So just angle B would be 90?
Student 1: I do believe so.

Student 3: Mmhmm.

Student 4: Do you get what I’m saying?

Student 3: Yeah. That’s what I was trying to remember, like what the angle was.

Student 4: Mmhmm.

Student 1: Hm.

Student 4: And the other ones could be either 60, 30 or 45, 45.

Student 3: Mmhmm.

Student 1: Yeah.

Student 4: But this one looks smaller, so I think it might be…

Student 1: 60, 30?

9:00

Student 4: Yeah. So E would prolly be 60 and C would prolly be 30.

Student 1: Yeah.

Student 4: But then we don’t have everything so there’s nothin’ to put down.

Student 1: Well we have B.

Student 3: Can’t we get this one? Don’t we already have ABE?

Student 4: Yeah.

Student 1: ABE, yes.

Student 3: What are the measurements?

Student 4: 8.5.

Student 1: AE is approximately 8.5. And then AB is 8, and EB is 3.

Student 4: ‘Cause the whole thing, ABC, is a right triangle.
Student 1: Yes, A, ADC is a right triangle.

Student 4: And then that’s just looking for the absolute value of BC. We’re looking for a portion of the hypotenuse.

Student 1: Yes. (Pause) Yeah. I have no idea.

Student 2: If we found EC…

Student 1: We could figure out…

Student 2: …we could figure out

Student 4: We could figure out the whole leg.

Student 1: Yeah. We’re prolly supposed to find EC before we find BC, but…

Student 2: Mmhmm.

Student 4: Yeah. Let’s see…

Student 1: But I have no idea how to figure it out…

Student 4: …how could we find BC?

Student 1: I feel like it has something to do with DE, but…

Teacher: Um, here’s my dilemma. I really like what you did.

Student 3: Mmhmm.

Teacher: But I’m not convinced that ABE was a right angle to begin with.

Student 4: ABE?

Teacher: Not convinced that was a right angle to begin with.

Student 3: EBC. Or BEC.

Student 4: It would be ABE.

Student 2: Did we put it as a right angle?
Student 1: Yeah, we didn’t put ABE as a right angle.

Student 4: We did it to find the hypotenuse.

Student 1: We did A…oh, we did?

Student 4: We did the Pythagorean theorem basically.

Student 2: Oh yeah. Yeah, yeah, yeah, yeah.

Teacher: Um, so I do like ADE as a right angle.

Student 4: Yeah we did that one.

Teacher: How did you know that was a right angle?

Student 3: The D.

Student 1: The squarage.

Student 4: ‘Cause the point, the dot is on the very top.

Teacher: Well, what kind of segment is AD?

11:00

Student 2: It’s…

Student 1: A radius.

Teacher: Thank you! What about DC? What kind of—we’ll pretend it’s a line—is DC?

Student 4: Uh, a s…the s…

Student 3: Ah…secant? Cosecant? Tangent?

Teacher: I mean it’s probably one of those.

Student 1: Cotangent?

Teacher: Do you want, do you want to, do you want to say it more confidently, um, what do you think it is?

Student 1: It’s secant.

Teacher: And how do you know?
Student 1: Because it touches the circle only once.

Teacher: Mmmm, that to me is not the definition of secant line though. Let’s, let’s agree that yes, because this is a radius and this is a?

Student 3: Wait.

Student 4: Line.


Student 3: Tangent line.

Teacher: Anything else that’s going to help you with?

Student 4: But, one question.

Teacher: Mmhmm?

Student 4: If you said ADE is a right triangle…

Teacher: Yes. I mean, I didn’t say that, but it is.

Student 4: Ok, no, yes, you did.

Student 1: You did say it.


Student 4: Okay. Then we basically solved it wrong because we don’t have another leg to go by.

Teacher: Correct. But it was such a great idea! It really was!

Student 4: So it’s not 6.9.

Teacher: It was (inaudible)
Appendix C

Transcript of Class Discussion Video for Paper and Pencil Task

Description of artifact: In this video, you will see a class of high school juniors and seniors (who have each completed Math 1, 2, and 3) discussing their work on a mathematical task. The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion (that lasted about 5 minutes) about the task.

*Note: The video slowly fades in; the teacher started talking before I had the camera completely adjusted.

Figure 26. Layout of classroom, group compositions, and location of students who speak in the video.
Teacher: I was going to ask some of you if you don’t mind sharing what you did. And
I’m going to choose people specifically, no names, um, that I’d like to share
your idea. Okay? Does that make sense? Okay. So, let’s start with any
volunteers. Anybody want to share what they came up with to solve this
problem? (Student 1 raises hand) Go ahead.

Student 1: So we started off with Pythagorean theorem.

Teacher: Okay.

Student 1: And then went on to the 30, 60, 90…

Teacher: Okay.

Student 1: …of the triangle. And then we thought of the sin, cosine, and tangent. And
then it turns out that none of the three were working, so we started all over.
(Laughter in class)

Teacher: Okay, um I, I would think part of that did work, though, um, because you
— and I’m telling, or I’m explaining this for you—so let me ask you.
How did you use Pythagorean theorem? Why do you say that didn’t work?

Student 1: Because we got ABE. We thought that was a right triangle, but it turns out
that it wasn’t, so when we did solve it, if it would’ve been a right triangle we
solved for ADE. But since ADE wasn’t a right triangle, we couldn’t find the
leg of AB.

Teacher: And just to add on ‘cause I know you guys can’t see the picture, um, I liked
that. Did you see triangle ABE isn’t a triangle in the picture. They made it.
They drew that segment in. Um, and I thought that was a really cool idea is to
split it up like that. Um…I remember we had a good idea over here. Does
anybody want to share kinda what you did?

Student 2: Well, we started off um, trying to like look at the sides

Teacher: Mm-hmm.

2:00

Student 2: and segments and trying to figure out if we could find any of the other ones,
like using the radiuses and that kind of stuff. And then, um, we were trying to
think of a formula that could help us find the angles and the sides. And then
we tried 30, 60, 90, and then we, um, we tried that a couple of times with both
triangles. And then we tried to do some, um, we tried to do sine, cosine, and tangent.

Teacher: Okay, so a lot of the same ideas back and forth, right? Okay. Yeah. I did like the ideas from over here. Is somebody willing to share that they came up with from over here?

Student 3: What did we say that amazed you?

Student 4: Oh, we decided to, to, first label everything we knew about the circles and the triangles. We deter-we wrote down everything we knew first and then we decided to split the two triangles. We saw two triangles in the drawing, so we broke them into two and we wrote everything we knew about it.

Teacher: I think that’s a fabulous idea, and, as a matter of fact, had your group paired up with this group, I think we might have gotten somewhere. Do you want to share what it was you did? Mm-hmm.

Student 5: Me?

Teacher: Mm-hmm, yeah you.

Student 5: I don’t remember.

Teacher: About um...?

Student 5: Oh, oh yeah, we put um triangle ABC and BEC to scale, so we took like the 5 AD and compared it to the 3 from BE.

Teacher: And so had you separated the triangles like their group did, I think you would have seen how your segments compared to each other. So we had a lot of 30, 60, 90s which was a great idea. Did anybody actually find an angle other than a 90 in these two triangles? (Some students shake heads). No? Did anybody come up with what they thought the answer was? Okay. Did anybody not share? You guys haven’t shared. Do you want to tell us what you did? You had a really great idea. As a matter of fact, you may have the answer. I don’t know.

4:00

(Student talks amongst themselves for a moment to determine who will share.)

Student 6: Well, what we first tried to do is we were looking at, like, the segments and stuff to see what the possibilities could be. And then we saw that uh, we saw
the radiuses of the um the end points, and then we kinda solved it. Well, 
Mikayla kind of solved it. And, well, we figured that out one of the segments 
was 8 and then from then on we could do Pythagorean theorem, but that didn’t 
work out because we had another missing segment. So the big triangle that 
you can see in the picture, we tried to solve for it but it didn’t work out for us. 
Then I tried doing a little circle but there was still something missing, and I 
think I could have gotten it right, but I had something missing it- if I only had 
that missing detail, I think I would’ve got it.
Appendix D

Student Work for Technology Task

Written Work

Suppose we apply a dilation by a factor of 2, centered at the point P, to the figure below.

A) What do you think happens to AB and CD when we perform the dilation? Make a prediction.

B) Dilate the figure by a factor of 2 with center at point P. To do this, double click on point P (you’ll see it “explode”). Then select AB and CD (by clicking on each blue line segment) and their endpoints (click on each red endpoint). Next, select Transform, then Dilate by Fixed Ratio with Scale Factor of 2. Select Dilate and GSP performs the dilation.

C) Label A', B', C', and D'. To label these points, select the “K” on the toolbar and click on each point once. The labels will automatically appear.

D) Measure the distance A'B' by selecting A' then B' (in that order) and then under Measure, select Distance. Also measure the distance AB. Drag various points in the figure. What do you notice?

E) Measure the distance CD' and the distance CD. Drag various points in the figure. What do you notice?

F) Can you prove your observations in parts D and E?

Yes because of the measurements, and when you drag the points, they’re still the same.

Figure 27. Students’ written work from the technology task.
Suppose we apply a dilation by a factor of 2, centered at the point \( P \), to the figure below.

A) What do you think happens to \( AB \) and \( CD \) when we perform the dilation? Make a prediction. The scale will become two times bigger.

B) Dilate the figure by a factor of 2 with center at point \( P \). To do this, double click on point \( P \) (you’ll see it “explode”). Then select \( AB \) and \( CD \) (by clicking on each blue line segment) and their endpoints (click on each red endpoint). Next, select Transform, then Dilate by Fixed Ratio with Scale Factor of \( \frac{2}{1} \). Select Dilate and GSP performs the dilation.

C) Label \( A', B', C', \) and \( D' \). To label these points, select the “A” on the toolbar and click on each point once. The labels will automatically appear.

D) Measure the distance \( A'B' \) by selecting \( A' \) then \( B' \) (in that order) and then under Measure, select Distance. Also measure the distance \( AB \). Drag various points in the figure. What do you notice? The measured distance doubled the original \( AB \).

E) Measure the distance \( CD' \) and the distance \( CD \). Drag various points in the figure. What do you notice? The distance of \( CD \) is half the distance of \( CD' \).

F) Can you prove your observations in parts D and E? Yes, because of the measurements and when you move the lines around they are still half.

\[
\begin{align*}
AB &= 6.94 \text{ cm} \\
A'B' &= 13.88 \text{ cm} \\
CD &= 13.36 \text{ cm} \\
C'D' &= 30.72 \text{ cm} \\
PA &= 6.98 \text{ cm} \\
PA' &= 13.95 \text{ cm}
\end{align*}
\]

**Figure 28.** Screenshot of student work on the Geometer’s Sketchpad file from the technology task.
Appendix E

Transcript of Group Video Clip for Technology Task

Description of artifact: In this video, you will see a pair of high school juniors and seniors (who have each completed Math 1, 2, and 3) working on a mathematical task in The Geometer’s Sketchpad (GSP). The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. The students had little to no prior experience with GSP. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion about the task.

The video focuses on Student 1 and Student 2 working together as partners (see Figure 29; Student 3 and Student 4 were also partners and sat across from Student 1 and Student 2). But, at times, you might hear comments from Student 3, Student 4, or others in the class. Note that the video has been edited down to about 12 minutes. Thus, time spent by the teacher setting up the task, times where the students were not on task, or times where the students were quiet, were intentionally cut out for this video.

Figure 29. Group arrangement and camera placement of students on group video clip for the paper and pencil task.
Teacher: Ok, so, now you guys can start. So, read your directions and play.

Student 1: Alright, k. Suppose we apply a dilation by a factor of two, centered at the point P to the figure below. What would you think happens to A, lines A and B and lines C and D when we perform the dilations? Make a prediction.

Student 2: So what do you think would happen?

Student 3: What does dilation mean?

Student 1: It means, like, getting bigger. So I’m guessing with the dilation—since it’s asking to make it bigger by two—two times bigger?

Student 3: Okay, so what does dilate mean?

Student 1: Dilate means to make bigger.

Student 3: To make bigger. By a factor of two center the point P to the figure below. What do you think?

1:00

Student 2: So A B…

Student 1: Yeah. What would happen?

Student 3: Well, if we make the scale bigger by a factor of two, wouldn’t that make the items bigger or smaller depending on the graph?

Student 2: Bigger by two.

Student 1: Alright, so then it tells you to dilate the figure by a factor of two, center of point P. It can make this, um, making the scale larger. Okay. So you wanna to say that?

Student 2: Yeah.

Student 1: (while writing) Make the scale…

Student 3: Oh, it’s right there. (pause) Then select AB by clicking on each blue line segment and their endpoints.

2:00

Teacher: Did you guys type it?
Student 3: Whoa, it exploded!

Teacher: Did you type the answer?

Student 3: Yeah.

Teacher: Group one? Ok. I just can’t see it. Sorry.

Student 1: Oh! (pause) Ok, let’s try it again, let’s try it again. (double clicks P) It got bigger!

Student 3: Did it get bigger?

Student 1: Did it get bigger?

Student 2: I guess so. Now select AB (selects segment AB)…

Student 1: Uh huh. And then…double click. OK! And CD and their endpoints.

3:00

Teacher: [to class] Alright, so I’m just going to tell you guys this out, out loud, out front. You can move things around and once you do the, the dilation, with the third step, I think, third part. Oh yeah, Part B, ok? You can start moving things around and seeing what it does. Ok?

Student 1: Alright, so it’s saying dilate it by a ratio of two over one.

Student 2: (to teacher) Do you know where the transform button is?

Teacher: Uh, it’s not a button. (Points to screen). It’s up at the top under your menu. Transform.

Student 1: Oh!

Teacher: So you have to double click…

Student 1: Double click the things again?

Teacher: …the P, right?

Student 1: Give. Nope…

Teacher: No you have to double click the P and select everything else.

Student 2: Yep.
Student 1: Double click, we already did double click the P.

Teacher: Oh, well, you’re going to have to do it again ’cause it looked like it unselected it.

Student 1: Yeah. We have to do it again.

Teacher: Yeah. There you go! Now try transform.

Student 1: Alright. And then it says…

Teacher: Ok.

Student 1: (while Student 2 performs these actions on the computer)…select transform and then dilate with a fixed ratio of one over two. Two over one.

4:00

Student 1: Label AB, the new lines A, B, C, and D.

Student 2: (Labels the new endpoints A', B', C', and D' by clicking each in GSP.) Hold on.

Student 1: Alright, so what we do next- we have to measure the distance from the new A and B. And then under measure we select distance. (while Student 2 performs these actions) So A, B. And then…

Student 2: Under measure…

Student 1: We hit

5:00

Student 3: (talking to his partner – not Student 1 or Student 2) Click the little arrow?

Student 1: Oh, because you’re still, you’re still in there. And then it says you to, uh, select distance.

Teacher: [to Student 3 and his partner – not Student 1 or Student 2] So you can drag it. You can actually move it back to where you can see it. Yeah!

Student 2: Inaudible

Teacher: (unclear who teacher is talking to) Oh, yours popped up right on screen as well.
Student 1: Not the old one. Right?

Student 2: I was just doing this one (pointing to screen).

Student 1: Oh, ok. (pause) What do you, what do we notice? That its basically doubled.

Student 2: Yeah.

Student 1: (to video recorder) Do you want us to write that out? What we saw and the difference? You want us to write that? Alright. (records information on paper; Student 2 records on computer)

6:00

Student 3: Ok, so we did that one. Measure the distance AB by selecting A then B in that order. And then under Measure, Select Distance.

Student 1: Alright C and D. (selecting the points on the computer)

Student 2: Inaudible.

Student 1: (while performing actions on computer) Let’s try that again. C, D. Measure distance. Alright and then…ok. C and D and then we also get that…hold on. No! K.

7:00

Student 1: Distance. Alright, so it looks like it actually is still double though…because…

Teacher: Have you actually dragged the points around? You can click any point you want and drag it. (Student 2 drags points on computer)

Student 1: It’s, it happens when I see it still looks like it. It still look like it’s double.

Teacher: Ok.

Student 1: (records something on paper while Student 2 continues to drag) So (reading from her paper) the measure of CD is half the distance of C'D'.

8:00

Student 1: Alright, you got it? Alright, so it’s basically saying, number F is saying that can we prove these observations in parts D and E. And D and E

Student 2: Yes because of the measurements…
Student 1: (writing) Yes because of the measurements.

Student 2: (while typing) ...and when you move them around they’re still the same thing, like they’re still half.

Student 1: (to teacher) What do you want us to do when we’ve finished?

Teacher: Are you finished?

Student 1: Yeah, we’re finished.

Teacher: Could you show me in picture form F, part F? Is there any way to show me in a picture?

Student 1: No. (pause) Yeah... .

Teacher: I mean I know that. Like...

Student 3: You could draw and arrow to it.

Student 1: We could.

Teacher: So could you go one step further?

Student 1: Can you prove the observations in parts D and E? Yes because the measurements, which if you look here, if you look here, A and B had a distance- uh, the original A and B had a distance of 7.04. Ok. And A' and B' had a distance of 14.08.

Teacher: And when you moved it around, it stayed like that?

Student 1: It stayed, it stayed like that. Basically, if you moved it around at any point...hold on. Let me make sure. (dragging on computer) At any point if you moved it around, it stayed that way regardless.

Teacher: What if you move P?

Student 1: If you move P, it doesn’t really change anything.
Teacher: It doesn’t change the length at all?

Student 1: It doesn’t change the length. It just ch…

Teacher: Interesting. Now, did you measure those distances?

Student 1: I did. And, if you were to look at CD, you got…

Teacher: Wait. How far away…I don’t see that. Let’s try that. Tell me how far away P is from A, and how far away P is from A'. And, because you’re moving P, right?

Student 1: Uh-huh.

Teacher: And it’s making them go away from each other.

Student 1: Yeah.

Teacher: What do you think is gonna happen? If you measure from P to A, and you measure from P to A', what do you think is gonna happen to that distance?

Student 1: Um…

Teacher: You see what I’m saying?

Student 1: Yeah.

Teacher: Measure it. So you’re going to click P and then you’re going to click point A. Oh. Unclick everything.

Student 1: Alright. P (selects P on computer)

Teacher: P and A. (Student 1 selects A) And then go up to measure, distance.

Student 1: Distance. P and A gives you 8.28, after moving everything around.

Teacher: Now find P to A'.

Student 1: Okay. (selecting P) P…

11:00

Student 1: (selecting A') to A'.
Teacher: You can move those lengths around too also to help you out. Mhmmm. And find that one.

Student 1: Distance. It’ll give you 16. It’s basically still giving you double.

Teacher: Hmm. Move P around. (Student 1 drags P) Is it still double?

Student 1: Yes. Nooooo…

Student 2: Yes.

Student 1: Yes.

Teacher: So what if you move…now what happens if you move like B, or B’?

Student 1: Obviously…

Teacher: How does that affect the length from P?

Student 1: Ok so say we move B the length from P…it affects it.

Teacher: You can’t really see both though.

Student 1: No, it doesn’t affect it all really.

Teacher: What do you mean?

Student 1: It does…

Teacher: Slide it around while we can see the measures.

Student 1: Like it’s not affecting it really. It’s still staying the same. Like it’s still double it.

Teacher: Hmm.
Appendix F

Transcript of Class Discussion Video for Technology Task

Description of artifact: In this video, you will see a class of high school juniors and seniors (who have each completed Math 1, 2, and 3) discussing their work on a mathematical task in *The Geometer’s Sketchpad* (GSP). The task was given to students by their regular mathematics teacher. The task was not part of an instructional sequence; it was given to students at the beginning of the school year as part of their review activities. The students had little to no prior experience with GSP. Students were given about 40 minutes to work on the task; the teacher ended the period with a class discussion about the task. While the actual class discussion lasted about 13 minutes, the video you will see has been edited down to about 7 minutes. Thus, time spent by the students setting up their computers to share with the class was intentionally cut for this video.

*Note: There was an issue with the camera, so the first part of the video is audio only where the teacher sets up the class discussion. In the piece following this, you will see just the screen capture of the group who is talking (Student 1 and Student 2’s group). The camera issue also affected the first part of what they shared. You will see the video pick back up to look at the entire class at about 1:52. At this point, Student 1 and Student 2 are still at the front of the room.

![Diagram of classroom layout and group compositions](image)

**Groups:**
- Student 1 and Student 2
- Student 6 and Student 7
- Student 9 and Student 10
- Student 4 and Student 5
- Student 8 and Student 3
- Two unnumbered males

*Figure 30.* Layout of classroom, group compositions, and location of students who speak in the video.
Teacher: Alright, so the plan right now we’re gonna discuss this and I want you to kinda come up and show us what you discovered. Um, and I think I’m going to save you guys for last. So would you like to come up, plug in, and show us something that you found?

Student 1: So what we’ve learned, ‘cause, ya know when Nesbitt (the teacher) came and told us what about the P to the other ones, then Sean (Student 2) was like, they’re same distance, so we proved this. So from PA to A pri- A to prime, PA to A and A’ is the same distance. And…

Teacher: Sean can you play Vanna and show us what she’s talking about?

Student 2: Oh, sorry. I wish I had a yardstick.

Teacher: Yeah, sorry.

Student 2: Alright, P to C will always be the same distance from C to C’, like P to D will always be the same distance from D to D’. They will always be equal. (To Student 1) Like slide, slide it over. PA to A’, they’ll always be the same distance.

~1:00

Student 1: So PA and A, A’, so…(To Student 2) drag our P. And those two will always stay the same while PA’ is measuring the whole distance between both of them.

Student 2: It looks like P to A’ is double so it’s both of them added together.

Teacher: Beautiful. (Pause) Ok. Does anybody have anything to add to that? That’s interesting, right? But that makes sense, doesn’t it? That if you add them together it would be twice as big? It would be double? Um, should it always be like this? Will it always be double? Go ahead, Casey (Student 3).

Student 3: Well, it depends on, um, the factor you’re dilating by.

~2:00

Teacher: Hmm.

Student 3: So, like we dilated by 2. But what if we dilated by, like, ½?

Teacher: Oh!

Student 3: It would be different. It just depends on your factor.
Teacher: That’s a good point. Although, ½ and 2 are pretty similar. Maybe, maybe what if you dilated by 3?

(Several students answer at once.)

Teacher: Mm-hmm. Alright, thank you very much. (Cuts to Student 4 and Student 5.)

Alright, tell us what you found.

Student 4: Well…

Teacher: Even if it’s the same thing they already said. Anything.

(Student 4 and Student 5 start talking at the same time)

Student 4: Go ahead.

Student 5: Basically when we were doing it, um, we actually saw the same thing they did is that they’re basically double the distance. So A, regular A and B were, they like was like 12.7 for us or something like that. And…

Teacher: Do you have the measurements on there?

Student 5: I think we do.

Teacher: You could drag them over.

Student 5: Yeah, oh, so like originally we were (inaudible) as we played around with it was, like, it’s gonna be the same for us no matter what distance we dragged it at or what measurement we made it.

Teacher: What do you mean the same?

Student 5: Oh, it’s, like, still going to be double. Like A’ and B’ would be double the original of just A and B.

Teacher: Okay. And the same is true for that distance from P. Okay.

Student 5: Mm-hmm.

Teacher: Because that’s what they told us, right?

Student 5: Yeah.
Teacher: Okay. (Video cuts) Okay. So distance from P was twice, the lengths of the
segments were twice, and Casey (Student 3) says that’s all because the scale
factor was 2, right? Okay. Alright. Anybody want to add anything? Anything
you noticed about that?

~4:00

Teacher: Alright. Can somebody in here tell me what is different about theirs compared
to everybody else’s?

Student 1: It has a graph!

Student 3: Theirs is a graph.

Teacher: Whaat?? Do you know how you did that or was that an accident?

Student 6: You go up there to graph.

Student 7: And it says show graph.

Student 6: Yeah

Student 7: (Inaudible)

Teacher: Okay. Interesting. Did that help you in any way? Like, was there a purpose to
turning that on or…you just playing?

Student 6: Well, I mean, it looks better than having it blank.

Teacher: Okay. So did you use it at all to help you find measurements?

Student 6: No. No. I mean it’s just, it’s just was visual help I guess.

Teacher: Okay.

Student 8: Could you drag it and measure it from like the points?

Teacher: Yeah, I’m sure you could. Could you drag the, the uh the grid or drag your
points even to the grid to use to measure?

Student 6: Like this?

Teacher: Like that or take the segments to the point, or to the grid, like to measure ‘em.

Student 6: Uhhh, I don’t know. Maybe.
Teacher: Un se…yeah. There you go. So you could put maybe A on -5 or something and used that. And then put the other…or, whatever. Yeah, so put it on -5 and put this one somewhere.

Student 6: Yeah.

Teacher: So you know the length?

Student 6: Yeah.

Teacher: Hmm! And then what if you brought P closer so we could see this other piece come closer?

Student 6: Oops.

Teacher: It didn’t move this at all. Oh yeah it did, it’s moving it. So yeah, so bring it closer to the thing. So you could see how long it was even, almost. Slide it my way. Nevermind. I was going to say slide your P my way.

Student 6: The what?

Student 7: The P.

Teacher: Slide point P

Student 6: Ohhh…

Teacher: Slide it, slide it, uh, yeah and see where, see if you can get it to line up on 10, 10, 10, -10.

Student 6: Oh it’s…

Teacher: No

Student 7: No the line.

Teacher: Your A’, B’, this piece.

Student 7: Yeah.
Teacher: Do you see what I’m talking about? So that you could see that it really is twice. (Cuts to Student 9 and Student 10.) Okay. So show us kinda how you came up with your relationship.

Student 9: So what we, after we did the dilation, we noticed that if we drug this and we lined ‘em up with P….Well I know you can’t really see, but it shows that um A and B and A’ and B’, um, that A’ is double the size of A and B. And we don’t know where we got k’ from but that showed up somehow. I don’t know where that came from.

Teacher: And then did you verify it with actual lengths?

Student 10: Yeah, we did.

Student 9: Yes.

Teacher: Okay.

Student 9: Over here (Scrolls down to show measurements).
Appendix G

Paper and Pencil Task-based Interview Protocol

Materials: Video camera, tape recorder, microphone, computer for interviewer notes, paper, pencil, protractor, straightedge, compass, calculator, copy of task

Introduction and Background (if this is the very first task-based interview):

Interviewer: As you know, I’m Kayla and this interview is part of your agreement to participate in my research study. Please know that I really appreciate you taking time to meet with me today. As previously explained, I am interested in learning how you think about students’ thinking. But, prior to having you consider what it looks like when students work on a task, I would like to gather a little information about your background and then see how you think about the task.

So let’s begin with you telling me a little bit about yourself.

• Where are you from?
• What made you want to be a high school mathematics teacher?
• What mathematics courses did you have in high school? College?
• What education courses?
• What is your comfort level with geometry? Other high school topics?
• What is your comfort level with using technology for teaching and learning mathematics, such as Geometer’s Sketchpad?

Now we are going to transition to let you work through a task. Before you leave, I will give you the first artifact where you will consider students’ thinking on the same task you are completing today. The problem that I will show you today may or may not be familiar to you. Keep in mind that what you do will not have any effect on your grades in your classes and what you say will remain confidential and anonymous.

Introduction (if this is the PSMT’s second task-based interview, meaning they already completed all of the work associated with the technology task):

Interviewer: As you know, this interview is part of your agreement to participate in my research study. Please know that I really appreciate you taking time to meet with me today. As previously explained, I am interested in learning how you think about students’ thinking. But, prior to having you consider what it looks like when students work on a second task, I would like to see how you think about the task first. So, just like with the technology task, you will work through this problem and then before you leave, I will give you the first artifact where you will consider students’ thinking on the same task you are completing today. The problem that I will show you today may or may not be familiar to you. Keep in mind that what you do will not have any effect on your grades in your classes and what you say will remain confidential and anonymous.
Roles:

Interviewer: Because I am interested in how you think about this task, it would really help me if you would talk as much as you can. While you are working and talking, I will probably ask questions to be sure I understand what you are saying. I am asking because I am really interested. I am not asking a question because what you are doing is correct or incorrect, but rather, I just want to be sure I understand what you are saying. Chances are, I probably just missed something you said or did. Likewise, if I ask a question that does not make sense to you, please tell me.

Recording:

Interviewer: I might take a few notes to help me remember. But, because I don’t want to take too many notes, I would like to tape our conversation. The video camera will be focused on what you are doing.

Awareness of materials:

Interviewer: Everything you do today will be on paper. I’ve provided a pencil, a protractor, a straightedge, a compass, and a calculator for you to use as you would like. While you may or may not use all of these materials, they are here when or if you need them.

![Figure 31. The paper and pencil task as it was presented to students and PSMTs.](image)

Interviewer: Please read the task out loud (see Figure 31).

PSMT reads task aloud.

Interviewer: Do you understand what the problem is asking?
If the PSMT says yes, ask him/her to explain to check for understanding.

If the PSMT says no, or seems to not have a clear understanding for our purposes, use probing questions similar to the following:

- What information does the task provide?
- What does the question in the task ask of you?

**Interviewer:** Okay, why don’t you go ahead and see if you can complete this task. Remember to share your thinking aloud as you work through the problem. If you have any questions along the way, please do not hesitate to ask. Remember I may remind you from time to time to share your thinking or to clarify something you have said to make sure that I understand what you are doing. Do you have any questions before you get started?

If the PSMT says no, allow him/her to begin solving the task.

If the student says yes, answer his/her question before he/she begins the task.

If the PSMT seems unsure about how to start or appears to reach an impasse, ask:

**Interviewer:** Can you tell me what you are thinking? How are you using the information provided in the problem?

When the PSMT is finished with the task, ask:

**Interviewer:** Do you feel confident in your response? What did you think about the task?

At the close of the interview, provide the PSMT with the first artifact, the student work from this task. Although the process for the study has already been discussed, remind the PSMT that he/she will use the student work to consider the students’ thinking and respond to the noticing prompts provided. A typed response should be emailed to the researcher within one week. Also explain that at the next meeting, the PSMT will be interviewed about his/her responses to the noticing prompts for this artifact to probe for more insight. After that interview, the PSMT will receive the second artifact for this task, the video of student work.
Appendix H

Technology Task-based Interview Protocol

**Materials:** Video camera, tape recorder, microphone, computer with video screen capture capability and task file, wireless mouse (in case interviewee prefers it to the laptop trackpad), paper, pencil, protractor, straightedge, compass, calculator

**Introduction and Background (if this is the very first task-based interview):**

*Interviewer:* As you know, I’m Kayla and this interview is part of your agreement to participate in my research study. Please know that I really appreciate you taking time to meet with me today. As previously explained, I am interested in learning how you think about students’ thinking. But, prior to having you consider what it looks like when students work on a task, I would like to gather a little information about your background and then see how you think about the task.

So let’s begin with you telling me a little bit about yourself:

- Where are you from?
- What made you want to be a high school mathematics teacher?
- What mathematics courses did you have in high school? College?
- What education courses?
- What is your comfort level with geometry? Other high school topics?
- What is your comfort level with using technology for teaching and learning mathematics, such as Geometer’s Sketchpad?

Now we are going to transition to let you work through a task. Before you leave, I will give you the first artifact where you will consider students’ thinking on the same task you are completing today. The problem that I will show you today may or may not be familiar to you. Keep in mind that what you do will not have any effect on your grades in your classes and what you say will remain confidential and anonymous.

**Introduction (if this is the PSMT’s second task-based interview, meaning they already completed all of the work associated with the paper and pencil task):**

*Interviewer:* As you know, this interview is part of your agreement to participate in my research study. Please know that I really appreciate you taking time to meet with me today. As previously explained, I am interested in learning how you think about students’ thinking. But, prior to having you consider what it looks like when students work on a second task, I would like to see how you think about the task first. So, just like with the paper and pencil task, you will work through this problem and then before you leave, I will give you the first artifact where you will consider students’ thinking on the same task you are completing today. The problem that I will show you today may or may not be familiar to you. Keep in mind that what you do will not have any effect on your grades in your classes and what you say will remain
confidential and anonymous.

Roles:

**Interviewer:** Because I am interested in how you think about this task, it would really help me if you would talk as much as you can. While you are working and talking, I will probably ask questions to be sure I understand what you are saying. I am asking because I am really interested. I am not asking a question because what you are doing is correct or incorrect, but rather, I just want to be sure I understand what you are saying. Chances are, I probably just missed something you said or did. Likewise, if I ask a question that does not make sense to you, please tell me.

Recording:

**Interviewer:** I might take a few notes to help me remember. But, because I don’t want to take too many notes, I would like to video our conversation. The video camera will be focused on what you are doing. Finally, I will use a program on the computer that will record everything you are doing on the computer since the video camera may not show your computer screen or record the sound very well.

Awareness of materials:

**Interviewer:** Everything you do today will be on the computer, but I’ve given you some paper and a pencil in case you would like to draw or write something. I’ve also provided a wireless mouse for you in case you are more comfortable using that instead of the one on the laptop. Finally, you also have a protractor, a straightedge, a compass, and a calculator for you to use as you would like. While you may or may not use all of these materials, they are here when or if you need them.
Figure 32. The technology task as it was presented to students and PSMTs.

Interviewer: Open the GSP file titled “Technology_Task” and turn the computer to face the PSMT. Please read the task out loud (see Figure 32).

PSMT reads task aloud.

Interviewer: Do you understand what the problem is asking?

If the PSMT says yes, ask him/her to explain to check for understanding.

If the PSMT says no, or seems to not have a clear understanding for our purposes, use probing questions similar to the following:
• What information does the task provide?
• What does the question in the task ask of you?
• Do you know how to perform a dilation in GSP?

**Interviewer:** Okay, why don’t you go ahead and see if you can complete this task. Remember to share your thinking aloud as you work through the problem. If you have any questions along the way, please do not hesitate to ask. Remember I may remind you from time to time to share your thinking or to clarify something you have said to make sure that I understand what you are doing. Do you have any questions before you get started?

*If the PSMT says no, allow him/her to begin solving the task.*

*If the student says yes, answer his/her question before he/she begins the task.*

*If the PSMT seems unsure about how to start or appears to reach an impasse, ask:*

**Interviewer:** Can you tell me what you are thinking? How are you using the information provided in the problem?

*When the PSMT is finished with the task, ask:*

**Interviewer:** Do you feel confident in your response? What did you think about the task?

*At the close of the interview, provide the PSMT with the first artifact, the student work from this task. Although the process for the study has already been discussed, remind the PSMT that he/she will use the student work to consider the students’ thinking and respond to the noticing prompts provided. A typed response should be emailed to the researcher within one week. Also explain that at the next meeting, the PSMT will be interviewed about his/her responses to the noticing prompts for this artifact to probe for more insight. After that interview, the PSMT will receive the second artifact for this task, the video of student work.*