ABSTRACT

WU, ZHEJUN. Four-Dimensional Object-Space Data Reconstruction Using Spatial-Spectral Multiplexing. (Under the direction of Michael Kudenov.)

This thesis investigates the data reconstruction problem of a four-dimensional scene based on the Spatial-Spectral Multiplexing (SSM) technique. The SSM technique is used in a hyperspectral imaging system for the application of passive remote sensing. The thesis first introduces the current development of the remote sensing field and its associated applications. The SSM technique, in particular, provides unique advantages over other systems in use for developing longitudinal spatial coherence holograms because it can enable snapshot measurements of a coherence function. Additionally, this technique can enable access to interference patterns that can be decomposed to access the phase information that is present (but inaccessible) in the pupil of a conventional lens. Finally, due to this phase-sensitivity, the technique can also enable an alternative interference-based method for passively sensing distance without the need for stereoscopic co-boresighted objectives or active illuminators. In this thesis, the concept is applied to the field of hyperspectral imaging because it can serve as a feasible problem to ascertain methods of data reduction and reconstruction. Since the algorithm is based on the SSM system, the thesis illustrates its optical model and its related modulation model. Such a model yields a challenging data reconstruction problem.

The goal of data reconstruction in the context of SSM and hyperspectral imaging is to use measurements from a given measurement matrix to recover the scene’s object locations and object spectra. This can be formulated as an underdetermined linear system, thereby producing an infinite number of potential solutions. The research work in this thesis proposes to solve the inverse problem by incorporating several heuristic prior assumptions and formulate an optimization from a probabilistic perspective. Based on this notion, this thesis proposes two categories of reconstruction methods: non-parametric and parametric. In particular, the thesis shows that by incorporating B-splines to reparameterize the object spectra, the number of unknown variables is highly reduced in the linear system, so it can achieve a much more accurate and efficient reconstruction for large scale data.

In order to validate the algorithm, three kinds of synthetic ground truth scene objects are generated, with which the modulation model can simulate the measurements and the measurement matrix based on the SSM technique. To demonstrate the accuracy and efficiency of this algorithm, the thesis compares the non-parametric method and parametric method based on several reparameterization methods at various data scale. Besides, the Poisson noise with different SNRs is also added onto the measurements before optimization in order to test the algorithm’s robustness. To be specific, this reconstruction algorithm is shown to be effective
on both smoothly varying and point cloud objects for the SSM system. It can achieve a 0.2\% overall MSE without noise, and 0.3\% overall MSE with noise (SNR greater than 30). The algorithm can handle a quite large data scale with the reparameterization method. The results of this thesis show the potential of applying such reconstruction methods on real-life scenes in the future.
Four-Dimensional Object-Space Data Reconstruction Using Spatial-Spectral Multiplexing

by

Zhejun Wu

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APPROVED BY:

__________________________  __________________________
Michael Escuti             Edgar Lobaton

__________________________
Michael Kudenov
Chair of Advisory Committee
BIOGRAPHY

Zhejun Wu was born and grew up in Shanghai, an eastern coastal city of China. In her path of learning, she has cultivated great interests in science and engineering since her high school. She majored in Electrical and Electronics Engineering as an undergraduate student in Shanghai Jiao Tong University, where she also completed her first master’s degree majoring in Instrument Science and Engineering. After that she came to North Carolina State University to pursue her second master’s degree, majoring in Electrical Engineering. During her study in NCSU, she was also an research assistant in the Optical Sensing Lab, under the advisement of Dr. Michael Kudenov. She has published papers in both of her graduate research experiences.
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Chapter 1

Introduction

This section will first introduce the development, applications and current limitations of the remote sensing. Then it will analyze some hyperspectral imaging techniques that are widely used to improve remote sensing technology, and will also propose a Spatial-Spectral Multiplexing (SSM) technique, which can be used in hyperspectral imaging. Based on these theoretical backgrounds, the section will introduce the modulation model of the SSM system, which is also the optical model for this reconstruction algorithm. After that, it will briefly describe the research purpose and give an outline of the proposed methods. Additionally, an overview of the following content will be presented in the end of this chapter.

1.1 Background

1.1.1 Development in the remote sensing field

The term “remote sensing” was first used in 1960s by a scientist of U.S. Navy’s Office of Navals Research[4]. There are various definitions for remote sensing. Generally speaking, remote sensing is a scientific technology of acquiring information from an object, an area or any phenomenon and analyzing the acquired data[17]. The devices or sensor systems used to obtain these useful data are not directly in contact with the objects need to be identified or measured[20].

It has been well-known that the sensors could be divided into active and passive based on the source of energy. Remote sensing that relies on solar and terrestrial energy belongs to the passive sensor systems. Taking the example of the solar energy, which accounts for the vast majority of the non-human-made sources, the sunlight is reflected, emitted or transmitted from the objects and recorded by the passive devices. There are also remote sensing systems working with active sensors[15], which emit the energy sources such as radar, sonar and laser by themselves[13]. These remote sensing systems provide human-made energy sources to scan the object then measure the returned energy[8]. According to the wavelength region, there are three
categories: (1) visible and reflective infrared, (2) thermal infrared and (3) microwave remote sensing[14]. The former two types are passively detected, while the microwave type can be based on either passive or active sensors. The algorithms and techniques provided in this thesis are aimed to be applied for passive, visible and reflective infrared remote sensing.

One of the most straightforward passive sensors for remote sensing are eyes[10]. Human beings have been eager to know about the earth and space since a long time ago. Not limited to their eyes, humans began to make use of external tools, such as telescopes, to visualize the area outside the field of view. The invention of photography was another milestone in the history of remote sensing. Using the photography technology, humans tried to observe the ground and take photographs at a greater height on the mountain or in a balloon. During this initial period, remote sensing were still referred as aerial photography[22]. Afterwards, the technique of aerial photography steadily evolved due to the improvement in platforms, camera hardwares and photo interpolation techniques. The rapid growing of interests in remote sensing started after the successful launch of artificial satellites. To date, there are three main platforms in remote sensing: ground-based, airborne and space borne[14]. As the name implies, they are classified by the height from the earth’s surface. The remote sensors are mounted on these platforms to detect the electromagnetic energy reflected from the earth’s surface, which can be mapped to the prior knowledge of some features[16]. Multispectral images are one of the most common images acquired by the remote sensors. In recent years, further improvement in sensors and satellites has greatly increased the quality of data and the accuracy of mapping by providing us not only multispectral images but also hyperspectral images[25].

Remote sensing has played an increasingly vital role in numerous applications fields. These include crop monitoring and type classification in the field of agriculture; flood, landslides and volcanic activity observation; geological structure mapping and natural resources exploration. There are also many promising applications in the field of hydrology, oceanography, glaciology, forest, climate, urban, military and meteorology[11]. With remotely sensed images and data, human beings have widely broadened the field of view and range of cognition.

These applications, however, can still be limited by the quality of image data, especially the image resolution and sampling frequency. There are some important properties of the remote sensing data: spatial resolution/sampling, radiometric resolution, spectral resolution/sampling and temporal resolution/sampling. The resolution means the maximum discrimination of a measurement, and the sampling refers to the sample frequency to collect the data. This thesis will mainly focus on the spatial and spectral resolution/sampling, where the spectral resolution is also known as the bandwidth or the wavelength interval recorded by the detector. The spatial resolution decides the minimum size of objects that can be resolved.
1.1.2 The SSM technique

In order to surpass the limit of image quality, some new imaging techniques have been proposed to acquire a higher spectral resolution and a larger spatial coverage for remote sensing. For example, the multispectral imaging makes it possible to obtain the spectral features of many bands simultaneously[24]; hyperspectral imaging[7] makes it possible to acquire continuous and complete spectral information of each spatial location across the scene[27]. The most important part of these imaging spectroscopy methods are the imaging sensors have very high spectral resolution. Using such hyperspectral imaging techniques, a conventional way to obtain the three-dimensional(3D) spatial and spectral information of a scene is to use temporal scanning to scan the entire scene with a two-dimensional(2D) sensor. However, this method can only be applied on static objects[18]. Other non-scanning methods, such as the snapshot approach, have also achieved major advances. Snapshot spectral imaging can record a 3D object-space datacube by multiplexing the high-dimensional signal onto the 2D sensor[28].

With the same motivation, the SSM theory was also presented in [29]. The pivotal idea behind the SSM technique is that it can encode angular information of a scene onto the incident power spectrum. To be specific, the SSM technique uses the two-beam interference, generated from a spectrally-resolved interferometer, to represent every object point and to modulate the angular spectrum (incident angle) of each object point onto a spectral carrier frequency using the Fourier basis function. This means that the resultant channeled spectrum contains both phase and amplitude information, such that the spatial and spectral content of the scene can be reconstructed. Additionally, the SSM technique enables phase retrieval to be achieved from the pupil of an imaging system, without the use of a lens. Based on this SSM technique, the spatial-spectral multiplexer can be agnostic, and includes interferometers such as Fabry Perot Etalon (FPE), Michelson Interferometer (MI), MachZehnder (MZ) interferometer, etc.

Compared with the other state-of-the-art hyperspectral imaging techniques, the advantages of SSM are: first, it can encode four-dimensional(4D) scene information using 3D spectrometer measurements, which can reduce the amount of data needed during the measurement. Second, the SSM technique uses a 3D spectrometer array to sample the pupil of the spatial-spectral modulator, so that we do not need to form images before reconstruction, which provides a more computational imaging process. The limitation of SSM is that it has not achieved much improvement in the spatial resolution in hyperspectral imaging.

1.1.3 Optical model of SSM systems

The measurements of SSM systems are simulated through modulation. The 2D modulation is needed to encode the object information onto the 2D spectrometer array for a 4D object-space data reconstruction. A 3D object can also be reconstructed by simulating the 1D modulation,
because the system only need a 1D spectrometer array to sample the pupil. Since the SSM technique uses two-beam interference to modulate the spatial information onto the power spectrum, there are two point sources for each spatial location of the object in this model. As shown in Figure 1.1, the red spot is one of the actual object sampling points. The blue and green ones are the two simulated point sources. Figure 1.1 illustrates the modulation process for each of the $y - z$ planes in the 2D case.

Figure 1.1: Two dimensional modulation model

The 4D object is in the space of the Cartesian product of the 3D spatial domain and the 1D spectral domain, which can be expressed by a 4D function $f(\sigma, \theta_{xz}, \theta_{yz}, z)$. Here, $\sigma$ is the wavenumber, $\theta_{xz}$ and $\theta_{yz}$ are the incident angles in $x - z$ and $y - z$ planes, and $z$ is the distance from the object sampling point to the pupil plane in $z$ direction. For notational simplicity, the following thesis will also use $f$ to denote the 4D object function $f(\sigma, \theta_{xz}, \theta_{yz}, z)$. The measurement with the 2D-spatial pupil plane $g(\sigma, x_p, y_p)$ is simulated by the dot product between the measurement matrix $H$ and object $f$, as shown in Eq. 1.1.

$$g(\sigma, x_p^i, y_p^j) = \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{K} H(\sigma, x_p^i, y_p^j, \theta_{xz}^n, \theta_{yz}^m, z^k) \cdot f(\sigma, \theta_{xz}^n, \theta_{yz}^m, z^k),$$

(1.1)

where $i$, $j$, $m$, $n$ and $k$ are the indexes of the variables. Assuming a perfect spherical wave, the channeled spectrum $H(\sigma, x_p^i, y_p^j, \theta_{xz}^n, \theta_{yz}^m, z^k)$ is computed as follow:

$$H(\sigma, x_p^i, y_p^j, \theta_{xz}^n, \theta_{yz}^m, z^k) = \frac{1}{2} [1 + \cos(2\pi\sigma(r_1 - r_2))].$$

(1.2)

The two beams pass through the interferometer and are combined to generate the channeled
spectrum. Therefore, the measurement matrix $H$ is made of the channeled spectra between each object point and each detector location. In Eq. 1.2, $r_1$ and $r_2$ are the distances from the two object point sources to the spectrometer, which can be computed by

$$r_1 = \sqrt{((x_{ip}^i - z_k \cdot \tan(\theta_{xz}^n))^2 + ((y_{jp}^j - z_k \cdot \tan(\theta_{yz}^m))^2 + (z_k - \text{OPD}(\sigma/2))^2}, \quad (1.3)$$

$$r_2 = \sqrt{((x_{ip}^i - z_k \cdot \tan(\theta_{xz}^n))^2 + ((y_{jp}^j - z_k \cdot \tan(\theta_{yz}^m))^2 + (z_k + \text{OPD}(\sigma/2))^2}. \quad (1.4)$$

Here the Optical Path Difference (OPD) has either a positive sign or a negative sign depending on which one of the two point sources is being measured. Note that the OPD can be defined differently according to the specific spatial-spectral multiplexer to be used. Details of this SSM model can also be found in [29]. Such a model motivates the need of object reconstruction; that is, given the simulated measurements $g$ and the measurement matrix $H$, the goal is to estimate the object $f$.

This Eq. 1.1 is essentially a linear equation. However, the number of pupil planes is smaller than the number of sources. Therefore the SSM system is an underdetermined linear system, which means the algorithm do not have enough information to uniquely solve for the spectrum of the object $f$. Therefore, to optimally recover $f$, some additional appropriate assumptions need to be incorporated into the object scenes. These assumptions will be further described in Sec. 3.1.1.

## 1.2 Four-dimensional scene reconstruction using SSM

This section illustrates the problem going to be worked on and the research purpose. As introduced in the background section, remote sensing relies on images with very fine spatial and spectral resolutions. The hyperspectral imaging is a widely used and rapidly growing imaging technique in the remote sensing field. The problem needs to be solved is to develop an reconstruction algorithm for an optical system using a new SSM hyperspectral imaging technique. Different from other hyperspectral imaging methods, the SSM system provides a more computational way to obtain the spatial and spectral information of a scene and stores much less amount of measurements. The method that the SSM technique uses to acquire the image information is computational imaging as it encodes the scene information onto a 2D spectrometer array, which is used to sample the pupil plane. The idea is that instead of producing complete sub-images with the same level of spatial dimensions as the object, the SSM system only needs to record the multiplexed information with the same dimensions as the detector plane.

Using the multiplexer technique, the research purpose of this thesis is to develop an algorithm to reconstruct 4D object-space data from the measurements acquired by the 2D pupil.
planes. The 4D object-space data can be a specific scene with 3D-spatial and 1D-spectral information, as what we can perceive from the world around us. In the experiments, the scenes are indicated by point clouds. The requirements of this reconstruction algorithm are: (a) to work on different realistic objects, including extended, random points and point clouds sources; (b) to deal with relative large data sets (roughly 15 Gigabytes) within an acceptable running time; (c) to achieve high accuracy in cases of both with and without Poisson noise.

1.3 Methodological design of the reconstruction algorithm

This section gives an outline of the methodologies. There are two main challenges in the reconstruction algorithm: (a) since the SSM system only stores a small amount of multiplexed information, the object from a highly underdetermined linear system (the number of unknown variables is larger than the number of linear equations) needs to be recovered, and a direct pseudo-inverse will produce meaningless results; (b) the size of the 4D object is extremely large. A non-parametric representation of the object will lead to a large-scale linear system that is numerically difficult to solve. In the experiments, a simple $1000 \times 10 \times 10 \times 10$ object with a non-parametric representation will cost 100 Megabytes storage, and the reconstruction can be extremely time consuming.

To handle the first challenge, the algorithm proposes several heuristic prior assumptions that can adapt to the realistic 4D scenes, and with these priors the algorithm can uniquely recover the object. Moreover, an optimization model is constructed to minimize the least square reconstruction error with the constraints of the aforementioned heuristic assumptions. To deal with this second challenge, the algorithm proposes to reparameterize the 4D object data with certain basis functions to reduce the number of unknown variables in the linear system. Several parametric models based on the eigenfunctions of Gaussian kernels and basis splines are tested and compared. Additionally, a memory-efficient numerical method is also applied to accelerate the optimization process.

1.4 Overview of chapters

After having a general idea of the background and its associated application, the following four chapters of this thesis will describe the technical details of the reconstruction algorithm. Chapter 2 will give the experiments materials, parameter settings of the SSM model and describe the experimental setup to test the reconstruction accuracy. The object models for the reconstruction tests include: 3D-spatial piecewise-smooth spiral curves, multiple random points and a 3-bar scene model. Besides, the experimental setup for measuring the effect of Poisson noise on the reconstruction algorithm will also be mentioned in Sec. 2.4. Chapter 3 will be
focused on the non-parametric data reconstruction method. It will introduce the specific definition of the realistic assumptions on the objects and will also propose the optimization model for the linear system. Afterwards, Chapter 4 will elaborate on the parametric data reconstruction method. It covers two reparameterization models and the comparison of their characteristics and performances. Sec. 4.3 will compare the above two main data reconstruction methods, analyze the accuracy and computational efficiency as well as the robustness of the algorithm. Finally, Chapter 5 will conclude the work and contributions of this thesis, as well as discuss some existing problems and potential further work.
Chapter 2

Experimental materials and setup

Before giving the technical details of the proposed reconstruction methods, this chapter demonstrates the experimental materials and the parameter settings of the SSM model. In this work, the experimental materials for validating the reconstruction algorithms are several synthetic scenes. The main reason for using synthetic scenes in the experiment is that the use of SSM systems, including the optical model and the reconstruction methods, is still in its preliminary stage. With the manually constructed ground truth objects in synthetic scenes, I can exactly validate the reconstruction algorithm and draw insights from the results. Therefore, a common experimental setup is to first construct a ground truth synthetic scene (a synthetic object $f$) and to use the SSM optical model to produce its measurements in the detector. Finally, the object will be reconstructed, denoted as $f^*$, from the measurements and compare the reconstructed object $f^*$ with the ground truth object $f$ to validate the reconstruction algorithm’s accuracy.

Three kinds of objects will be designed to be recovered by the reconstruction algorithm. In order to test the generality of the algorithm, these objects have very different properties. To test the algorithm robustness, the noise will be added to the measurements before the reconstruction. Sec. 2.4 will describe the adding noise procedure and the associated experiment setup in detail.

2.1 Three-dimensional piecewise spiral curves

The first type of synthetic object is a 3D-spatial spiral curve, which is generated from the following steps:

(a) Construct a spiral curve in the 3D-spatial space;
(b) Randomly break the spiral curve into four intervals;
(c) Randomly set different radii to the four spiral pieces;
(d) Shift the four intervals apart along the $z$-axis;
(e) Smooth the object using a 3D Gaussian filter.

In this way, it becomes spatially smoothed 3D spiral curves with some random radii. Figure 2.1 (a) shows the piecewise spiral curve object, where \( x \) and \( y \) donate the \( \theta_{xz} \) and \( \theta_{yz} \) dimension of the object \( f(\sigma, \theta_{xz}, \theta_{yz}, z) \) in Eq. 1.1. The corresponding 3D datacube is then generated and Figure 2.1 (b) shows three 2D cross-sectional slices. Each cross-sectional plane in Figure 2.1 (b) is labeled as a corresponding color as in the Figure 2.1 (a).

Due to the way of constructing the curve and the 3D Gaussian smoothing, the object is spatially smooth almost everywhere in the 3D-spatial space except for those breaking points (step (b)). The next chapter will show that this spatial smoothness nature plays a very important role in this reconstruction algorithm. The 3D-spatial space is known as the hyperplane, and the spectrum on every spatial hyperplane is the same, which is set to be smoothly varying from 0.1 to 1. This 1D spectrum corresponds to the \( \sigma \) dimension in \( f \).

2.2 Multiple random points

The second type of object is a random point cloud. To construct this kind of object in spatial space, it needs to define a fixed number of points in the 3D space. Assume the 3D datacube has \( N \) voxels and \( M \) points need to be generated in it, a random sample without replacement is performed to draw \( M \) positions from the \( N \) candidates.

Then different smoothly varying spectra are set to each of these points. The spectrum of a point is assumed to be one of the four functions sampled from a Gaussian Process \( GP(m, K) \). A Gaussian process (GP), written as \( X \sim GP(m, K) \), defines a distribution, in which the random continuous function \( X \) is distributed with mean function \( m \) and covariance function \( K \). Since a GP defines a probability distribution on the space of continuous functions, the spectral intensity functions can be sampled from it. Here, a very important property of GP is used: every point in the continuous input space (domain of \( X \)) is associated with a normally distributed random variable; every finite collection of those random variables has a multivariate normal distribution, whose covariance matrix can be derived using the covariance function \( K \). Therefore, if a function measured on \( N \) discrete points in \( \sigma \) needs to be sampled, a \( N \)-dimensional multivariate Gaussian has to be defined, whose mean is the discrete realization of \( m \). Each entry of the covariance matrix is computed by \( K(x, x') \), where \( x \) and \( x' \) are two of the \( N \) discrete points. Here, the covariance function is

\[
K(x, x') = \exp\left(-\frac{||d||^2}{2l^2}\right),
\]

where \( d = x - x' \), \( l \) is the characteristic length-scale parameter. Finally, the functions are sampled from that multivariate Gaussian. After the four spectral intensity functions are generated, they are scaled to be positive and normalize them to the range of \([0.2, 1.2]\). The spectrum of each
(a) 3D-spatial spiral piecewise-smooth object

(b) The 2D cross-sectional planes of the spatially smoothed spiral curve object associated with the fifth position in the other dimension

Figure 2.1: 3D-spatial piecewise spiral object
spatial location (voxel) is randomly assigned with one of the four spectral intensity functions. Figure 2.2 shows four randomly generated spectral intensity functions with $l = 3 \times 10^5$. Such a large $l$ can ensure the functions’ smoothness.

![Figure 2.2: Spectral intensity functions sampled from a Gaussian Process](image)

Figure 2.3 shows the ground truth of three 2D cross-sectional planes of a synthetic point cloud with 100 sampled points in a 3D spatial datacube. The spectra of the points are color-coded with four different colors, corresponding to the four spectral intensity functions.

Compared to the spiral curve object, this point cloud object is more difficult to be accurately reconstructed due to the following three facts.

(a) The object is more complicated.

There are about 10 points in each 2D cross-sectional planes, but in previous spiral object, there are only 1 or 2 points per plane. Thus what needs to reconstruct is a mazy-like pattern.

(b) The object is not spatially smooth.

No Gaussian filter is applied to the point cloud object, meaning the intensity at each location is independent to others so that smoothness constraints can not be added between neighboring locations.

(c) The object has more complex spectra.

A randomly selected spectral intensity function is added for each spatial location, whereas in the spiral curve object, the spectral intensity on each spatial hyperplane is the same.
2.3 A specific scene

This section presents a further more complicated and specific object, which is a specific scene that consists of three bar objects. The shape of the bar is also represented using a point cloud. As shown in Figure 2.4 (a) and (b), the three bars are placed diagonally in the spatial 3D space. Each bar contains 32 sample points divided into $4 \times y$ planes. The spectral intensity on each spatial location is generated using the same method as the previous point cloud object. Figure 2.4 (c) shows three cross-sectional planes.

To adequately demonstrate the bar shape of the point cloud, the spatial resolution of the object needs to be increased. Moreover, the sampling points have been doubled: 288 sampling points each $\sigma$ in the spectrum direction compared to the previous objects (100 sampling points each $\sigma$). The detector size and detector resolution are also increased to improve the condition of determination. All these facts lead to a much larger linear system with a huge number of variables to be estimated. The later chapter will show later that a simple non-parametric representation of the object will be computationally challenging, thereby motivating the parametric reconstruction methods.

2.4 Experimental setup

To setup the experiments, there ground truth objects $f(\sigma, \theta_{xz}, \theta_{yz}, z)$ (the object models described above) are designed. All of these three kinds of objects are put in the same spatial location in the modulation model with the incident angle from $45^\circ$ to $55^\circ$ and the distance of $z$ from 1m to 4m. They have the same range of wavelength $\frac{1}{\sigma}$ from 650nm to 1000nm. While the object’s spatial resolution and spectrometer’s detector resolution are set to be different so as to adapt to each model. For the object described in Sec. 2.1 and Sec. 2.2, the object resolution is
(a) The 3-bar object shown as a point cloud using MeshLab (this is the actual form of data the algorithm deals with)
(b) The 3-bar object shown as a mesh using MeshLab (this is a conceptual illustration of the object)
(c) The 2D cross-sectional planes of the spatially smoothed spiral curve object associated with the fifth position in the other dimension

Figure 2.4: Specific scene of three bars shown by point clouds
1000 × 10 × 10 × 10, which means there are 1000 sampling points in σ and 10 sampling points in each spatial dimension. For the 3-bar object, the number of sampling points in σ is reduced and more points are sampled in the 3D space, so that the object resolution is 100 × 17 × 17 × 17.

Then the corresponding measurements \( g(σ, x_p, y_p) \) and the measurements matrix \( H(σ, x_p, y_p, /\theta_{xz}, \theta_{yz}, z) \) need to be simulated. The spectrometer detector is a squared 2D plane, of which the side length varies according to the object type. The squared detector is set to be 5 cm for the spiral curve object, and 25 points are sampled within the detector (5 points for each spatial dimension), and have 25 equations for each wavelength σ. Thus the measurement resolution is 1000 × 5 × 5. The measurement matrix \( H \) is a 1000 × 5 × 10 × 10 × 10 6-dimensional matrix. Since the random point cloud object is more complex, the detector size is increased to be 10 × 10 cm², so that \( g \) has a 1000 × 10 × 10 resolution and \( H \) matrix becomes 1000 × 10 × 10 × 10 × 10. For the most challenging 3-bar scene object, the measurements \( g \) becomes 100 × 20 × 20 with a detector of 15 cm side length, so that \( H \) is 100 × 20 × 20 × 17 × 17 × 17.

There are various noise in imaging systems, including photon noise, dark noise, read noise, etc. It has been shown that the photon noise[3] accounts for the majority of the total noise. Since the Poisson distribution can best model the photon noise, Poisson noise is added onto the measurements \( g \) before the reconstruction, and compare the reconstructed result to the one without noise to test the robustness of the algorithm. The mean of the Poisson distribution, which is the square of the Signal Noise Ratio (SNR), ranges from 5 to 30 in the tests.

2.5 Conclusion

This chapter introduced three types of ground truth objects that are going to be used to validate the reconstruction algorithm. It described the methods to generate these objects and their characteristics respectively. The spiral curves and the random point clouds will be used in both the non-parametric and parametric data reconstruction, while the 3-bar object will only be used in the parametric model due to its large number of variables and model complexity. It also gave the parameter settings of the SSM system for generating these ground truth objects, including the detector size of the spectrometer, the range of incident angles and the range of the spatial distance from the detector. Besides, it specified the spatial and spectral resolution for each kind of objects and their corresponding measurements. Additionally, it gave the range of SNR for the Poisson noise used in the robustness tests.
Chapter 3

Non-parametric data reconstruction

With the given optical model and the sensor measurements, the goal is to reconstruct the object scene. This chapter introduces the technical details of the data reconstruction methods. Data reconstruction is a widely studied topic in pattern recognition [2]. Methods like deep learning and Bayesian inference have achieved success in many machine learning fields to solve the reconstruction and inverse problems. Considering the fact that the reconstruction system is an ill-conditioned inverse problem, it is proposed to use two optimization-based data reconstruction approaches. This chapter will introduce the first one: non-parametric data reconstruction method, with which all of the spatial/spectral variables in the object \( f \) need to be optimized. Then this chapter will describe an iterative numerical method used for solving such a large-scale optimization problem. Finally, it will present the reconstructed results for the spiral curves and the random point cloud object and analyze the existing problems of the non-parametric optimization.

3.1 Posterior optimization for an underdetermined linear system

As mentioned in Sec. 1.1.3, the object to be recovered is the 4D data. In the discrete setting, the simplest way to represent such an object is to have one variable for each frequency component of each spatial location; that is, \( f \) is represented by a discrete 4D data (dimension ordered by \((\sigma, x, y, z)\)) with a dimension of \( M \times N \times N \times N \). In other words, \( f \) is regarded as an \( M \times N \times N \times N \) dimensional variable. Such a representation is called the non-parametric representation of the 4D data.

As discussed before, the algorithm is dealing with an underdetermined linear system with more unknowns than the number of equations, so the object \( f \) cannot be uniquely computed in a closed form. Therefore, \( f \) is viewed as a high-dimensional random variable and the algorithm
tries to estimate it from a probabilistic perspective. The algorithm uses an optimization-based method to estimate \( f \) instead of directly solving the linear system by matrix inverse. The method is to formulate a probability distribution on the unknown object \( f \) and perform an Maximum-a-Posteriori (MAP) analysis [21]. The measurement matrix \( H \) can be generated based on the SSM system, and the measurements \( g \) can be simulated from the synthetic ground truth object. These two given data can be the entry of the optimization method to estimate the object. To be specific, the goal is to model the probability distribution \( p(f|g, H) \), which indicates the probability of observing \( f \) given the measurement matrix \( H \) and the measurements \( g \). By the Bayes’ rule,

\[
p(f|g, H) \propto p(g, H|f)p(f). \tag{3.1}
\]

The posterior is proportional to the product of a likelihood term \( p(g, H|f) \) and a prior term \( p(f) \). The goal then is to find the mode (minimizer of the negative logarithm) of \( p(f|g, H) \). Due to the fact that the system is an under-constraint linear system, some proper prior distributions have to be imposed on the \( f \) variable.

### 3.1.1 Heuristic prior assumptions

As mentioned above, the underdetermined linear system on \( f \) may lead to an infinite number of solutions. Therefore, a key step in the Bayesian analysis is to have a proper prior distribution \( p(f) \) to regularize the recovered object. There are two heuristic assumptions to construct \( p(f) \): the spatial smoothness assumption and the spectral smoothness assumption.

The *spatial smoothness* assumption refers to the fact that in almost all situations, the intensity of an object across the spatial domain tends to be smoothly varying rather than randomly changing. This idea has been extensively applied to feature detection, image segmentation tasks [6] and image denoising [23]. In this case, it is assumed that the intensity of each frequency component of the 4D object is varying smoothly in the spatial domain (\( x\text{-}y\text{-}z \) coordinates). In other words, the spectral intensities of neighboring locations have strong correlation, meaning in each spatial hyperplane (the 3D space formed by \( x\text{-}y\text{-}z \)), the spectral intensity in one location will not change too much with respect to its neighbors in any of the three spatial (\( x, y \) and \( z \)) directions. In order to model this fact, a Gaussian distribution is imposed on the object’s three spatially directional derivatives:

\[
p(\nabla_x f) \propto \mathcal{N}(0, I), \quad p(\nabla_y f) \propto \mathcal{N}(0, I), \quad p(\nabla_z f) \propto \mathcal{N}(0, I), \tag{3.2}
\]

where \( \nabla_q f \) is a vector containing spatial derivatives of spectral intensities for all spatial locations and all frequency components. \( \nabla_x, \nabla_y \) and \( \nabla_z \) are the three spatially directional derivative operators associated with the three orthogonal spatial directions \( \theta_{xz}, \theta_{yz} \) and \( z \) in the 4D space.
Discrete computation of such operations will be given at the end of this section. The zero-mean Gaussian encourages small spatial derivatives, thereby allowing spatially smoothness estimation of the object.

The spectral smoothness assumption means that for each spatial location (discretized location of the object), its 1D spectrum should also vary smoothly across all $\sigma$ planes (frequency components). Similar to the spatial smoothness situation, one possible way is to impose a Gaussian distribution on the spectral derivatives of the object.

\[ p(\nabla_{\sigma} f) \propto \mathcal{N}(0, I), \tag{3.3} \]

where $\nabla_{\sigma} f$ is a vector containing spectral derivatives for all spatial locations. Assuming the spatial smoothness and spectral smoothness can be modeled independently, the prior distribution $p(f)$ follows

\[ p(f) \propto p(\nabla_x f)p(\nabla_y f)p(\nabla_z f)p(\nabla_{\sigma} f). \tag{3.4} \]

The negative logarithm of the prior distribution $p(f)$, also known as the data regularization term, has the following form:

\[ -\log(p(f)) \propto (\lambda_1 \| \nabla_x f \|^2 + \lambda_1 \| \nabla_y f \|^2 + \lambda_1 \| \nabla_z f \|^2 + \lambda_2 \| \nabla_{\sigma} f \|^2), \tag{3.5} \]

where $\lambda_1$ and $\lambda_2$ can be viewed as the weighting factors between spatial smoothness and spectral smoothness, and they are essentially related to the Gaussian distributions’ standard deviations. Normally the two weighting factors should be different, because the spatial smoothness and spectral smoothness are not commensurable. The spatial smoothness priors may be disabled by setting $\lambda_1$ to zero for the random point cloud and 3-bar scene objects, because point clouds inherently do not satisfy the spatial smoothness assumption. The spectral smoothness constraint is always in use for all objects, but $\lambda_2$ may be set to different values for different spectral smoothness levels and different spectral resolutions. For example, a larger $\lambda_2$ is needed to model a higher smoothness level of the spectral intensity function with a larger $l$ in Eq. 2.1.

### 3.1.2 Likelihood

Given $g$ and $H$, the estimated $f$ should satisfy the linear relationship in Eq. 1.1. This corresponds to a likelihood distribution that penalizes deviation from that linear equation. Rewriting Eq. 1.1
in a convolution format, the negative logarithm of the likelihood distribution can be defined as

\[ p(g, H | f) \propto \exp(- \sum_{\sigma, x_p, y_p} \| g(\sigma, x_p, y_p) - H(\sigma, x_p, y_p) \otimes f(\sigma) \|^2) \] (3.6)

\[- \log(p(g, H | f)) = \sum_{\sigma, x_p, y_p} \| g(\sigma, x_p, y_p) - H(\sigma, x_p, y_p) \otimes f(\sigma) \|^2 + \text{constant.} \] (3.7)

Note that this term is often referred to as the data matching term, because it encourages the estimated \( f \) to better explain the observed data \( g \) and \( H \). The following text slightly abuses this notation by denoting the linear relationship simply as \( g - Hf \).

\[ g(\sigma, x_p, y_p) - H(\sigma, x_p, y_p) \otimes f(\sigma) \rightarrow g - Hf \] (3.8)

### 3.1.3 MAP analysis

Maximizing the posterior of Eq. 3.1 is equivalent to minimize its negative logarithm, which corresponds to the following objective function:

\[ \arg \min_f - \log(p(f|g, H)) \] (3.9)

\[- \log(p(f|g, H)) = \| g - Hf \|^2 + (\lambda_1 \| \nabla_x \|^2 + \lambda_1 \| \nabla_y \|^2 + \lambda_1 \| \nabla_z f \|^2 + \lambda_2 \| \nabla_\sigma f \|^2). \] (3.10)

Here only the L2 norm is used in the data matching and regularization term. This naturally corresponds to the form of the Gaussian logarithm. The Gaussian distribution is preferred in many other applications due to its many elegant properties. One particular reason is that the quadratic term of L2 norm is easy to optimize numerically. However, it is known that for some specific situation, L1 norm is more desirable due to the fact that it can induce sparse solutions, which might be a more appropriate assumption for some applications. The downside of using the L1 norm is that numerical methods for L1 optimization are generally not well developed. This application chooses to use the L2 norm instead of the L1 norm.

**Discrete computation of directional derivatives.**

In the discrete situation, the derivatives are approximated by computing finite difference in the four directions as illustrated in the following. Using the non-parametric presentation of the object, \( \nabla_v f \) is computed by

\[ \nabla_v f = [\nabla_v f(0, 0, 0), ..., \nabla_v f(i, j, k, p), ..., \nabla_v f(M, N, N, N)], \] (3.11)
where \( v \) can be either one of \( \{\sigma, x, y, z\} \)

\[
\nabla_\sigma f(i, j, k, p) = f(i + 1, j, k, p) - f(i, j, k, p) \\
\nabla_x f(i, j, k, p) = f(i, j + 1, k, p) - f(i, j, k, p) \\
\nabla_y f(i, j, k, p) = f(i, j, k + 1, p) - f(i, j, k, p) \\
\nabla_z f(i, j, k, p) = f(i, j, k, p + 1) - f(i, j, k, p).
\]

(3.12) (3.13) (3.14) (3.15)

3.2 The L-BFGS numerical method

After having an objective function for the data reconstruction model, the next step is to find an appropriate numerical optimization method. It can be seen that the above objective function Eq. 3.9 is quadratic. If the data dimension is small, the optimality condition can be directly written down as another linear system with the linear matrix operator being the Hessian matrix. However, if the data dimension is large, methods that require computing or inversing the Hessian of the objective function will be too time-consuming. Since the ultimate goal is to recover a 4D point cloud dataset with more than 10GB variables, it will be too expensive to directly compute the Hessian.

There are some popular quasi-Newton methods such as symmetric rank-one (SR1), Davidon-Fletcher-Powell (DFP) and Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm, which can approximate Hessian with faster progress [26]. An advanced BFGS algorithm, Limited-memory BFGS (L-BFGS) [19] is applied, which is especially proposed for optimizing large scale problems with only a limited computer memory. Instead of computing and storing a dense \( n \times n \) approximation to the inverse Hessian (\( n \) being the number of variables in the problem), L-BFGS stores only a few vectors that represent the Hessian approximation implicitly. Thus, it can achieve fast and accurate computational results for this problem. It used to cost 1.5 hours to run a preliminary 3D object with the data scale of only 80 MB. Using L-BFGS, it can reduce the running time of 80 MB data to 5 minutes, which is a huge improvement.

3.3 Reconstruction results

This section presents two experimental results using the non-parametric posterior optimization method. The first experiment is to verify the necessity of both spatial and spectral smoothness assumptions using the spiral curves object with a lower spectral resolution (100 spectral sampling points per location). The second one is to analyze the reconstruction results of the spiral curves and random point clouds with a higher resolution (1000 spectral sampling points per location).
Validation tests of the spatial and spectral smoothness assumptions

In order to validate the two prior assumptions, the algorithm first recovers the spiral curves without using any regularization terms, meaning that it simply performs a matrix inverse (pseudo-inverse) to the underdetermined linear system. Figure 3.1 shows six cross-sectional planes (three spatial cross-sectional planes in Figure 3.1(a) and three spectral cross-sectional planes in Figure 3.1(b)) of the ground truth synthetic object for reference. Note that, Figure 3.1 shows the ground truth for all the rest reconstruction results of the spiral curves object. Figure 3.2 shows the reconstruction results by pseudo-inverse for the same six cross-sectional planes as in Figure 3.1. It can be seen that without any spatial and spectral regularization, a direct matrix inverse does not yield meaningful results. Neither spatial nor spectral planes can be recovered to assemble the ground truth.

Then Figure 3.3 shows the reconstruction results assuming spatial smoothness only. It can be seen that the three spatial cross-sectional planes are recovered much better. However, the spectra are still problematic due to the lack of constraints.

However, when assuming spectral smoothness only, it can achieve more accurate results in both the spatial and spectral cross-sectional planes as shown in Figure 3.4. That means the spectral prior alone is more effective than the spatial prior alone.

If using both spatial and spectral assumptions, the all six cross-sectional planes can be very well recovered as shown in Figure 3.5, which proves the importance of the two prior assumptions used in this method. The accuracy of the algorithm is quantified with two kinds of Mean Square Errors (MSE). One MSE is the overall MSE, which is computed by averaging the squared intensity error over all spatial locations and frequency components. The other one is sparse point MSE, which is computed only on those locations with object sampling points. Besides MSEs, the running time is another important aspect needs to be investigated. The goal of this algorithm is to achieve high accuracy with limited computational time. The overall MSE of the final experiment (with both prior assumptions) is shown in Figure 3.5 is 0.85% (0.85% of the maximum intensity in the normalized object). The sparse point MSE is 10.08%. Since the 4D object is extremely high-dimensional, the running time of the optimization was about 1.5 minutes on an i7-3770k 3.50GHz Intel CPU processor.

Reconstruction results of high resolution objects

When using the high resolution $1000 \times 10 \times 10 \times 10$ object, the system have much more unknown variables in spectral dimension. As shown in Figure 3.6, the non-parametric method cannot reconstruct well even with both prior assumptions. The overall MSE (0.92%) and the sparse point MSE (10.17%) are both larger than the low resolution results and the running time is also much longer: 6 minutes.

When it goes to the random point cloud, the object is no longer spatially smooth. Therefore, the algorithm removes the spatial smoothness constraint and only optimizes with the data
(a) Three spatial cross-sectional planes of the ground truth spiral curves object

(b) Three spectral cross-sectional planes of the ground truth spiral curves object

Figure 3.1: The ground truth for the spiral curves object
(a) Reconstruction results of the three spatial cross-sectional planes

(b) Reconstruction results for the three spectral cross-sectional planes

Figure 3.2: Reconstruction results for the low resolution spiral curves object without using any prior assumptions
Figure 3.3: Reconstruction results for the low resolution spiral curves object using the non-parametric method with the spatial smoothness assumption
(a) Reconstruction results of the three spatial cross-sectional planes

(b) Reconstruction results for the three spectral cross-sectional planes

Figure 3.4: Reconstruction results for the low resolution spiral curves object using the non-parametric method with the spectral smoothness assumption
Figure 3.5: Reconstruction results for the low resolution spiral curves object using the non-parametric method with both spatial and spectral smoothness assumptions.
Figure 3.6: Reconstruction results for the spiral curves object using the non-parametric method with both spatial and spectral smoothness assumptions.
matching and spectral smoothness terms. Figure 3.7 shows the ground truth for the random point cloud object and Figure 3.8 gives the reconstruction results.

The overall MSE of this experiment is 1.73%, and it also takes nearly 15 minutes on the optimization.

### 3.4 Conclusion

This chapter described a posterior optimization model and gave the derivation of the objective function for the non-parametric presentation of the 4D data. It also presented the reconstruction results using this non-parametric optimization approach and showed the importance of the two prior assumptions being incorporated. The MSEs indicated that the algorithm can work well on both spatially smooth and non-smooth objects with relative lower resolution in the spectrum. However, the remaining problem is that the optimization with the non-parametric data representation cannot perform well on the objects with a higher resolution and it is too inefficient. Usually 100 iterations in L-BFGS require more than 15 minutes to reconstruct an object with the scale of 380 MB (this is only $\frac{1}{30}$ of the largest possible scale in this thesis). With this in mind, a more computationally efficient method will be introduced in the next chapter to handle this problem.
Figure 3.7: The ground truth for the random point cloud object

(a) Three spatial cross-sectional planes of the ground truth random point cloud object

(b) Three spectral cross-sectional planes of the ground truth random point cloud object
Figure 3.8: Reconstruction results for the random point cloud object using the non-parametric method with the spectral smoothness assumption
Chapter 4

Parametric data reconstruction

It has been noticed that a non-parametric representation (one variable per frequency and per location) of $f$ will contain a large number of variables. This leads to a high-dimensional optimization problem that is computationally intensive even with the Hessian approximation method. To handle this situation, this chapter further explores the spectral smoothness idea mentioned in Sec. 3.1.1 with the reparameterization trick. Given a function $f$ represented in a canonical domain, the reparameterization of $f$ refers to the procedure of transforming its representation in some other domains, such as the Fourier domain, the Wavelet domain [5] and the Reproducing-Kernel-Hilbert-Space (RKHS) [1]. The intention of such reparameterization tricks is that the function can be compactly represented in the new domain with much fewer coefficients; that is, after projecting $f$ into the new coordinate system (not necessarily orthonormal), the projection coefficients are mostly zero or near-zeros. Therefore, the data components associated with the near-zero coefficients can be directly discarded, and the reconstructed function with the limited number of coefficients can still highly approximate the original $f$.

Therefore, given a 4D $f$, the idea is to reparameterize the spectrum of each spatial location to reduce the number of variables in the spectral direction. As mentioned before, in most cases, the spectra are smoothly varying and they can be compactly represented in many other domains. The reparameterize spectra are particularly chosen instead of reparameterizing spatial components because in some cases, e.g., the point cloud, the spatial components are not smooth.

This chapter will investigate two reparameterization methods: RKHS and B-spline. Again, the method will still incorporate the previous prior assumptions, the posterior optimization framework and the L-BFGS numerical method. Such methods are called as parametric methods because they work with the data form after reparameterization. Besides the spiral curves and the random point cloud object, this chapter will use the parametric method to reconstruct the most challenging 3-bar object and show the effect of noise with different SNRs on all three kinds of objects. Finally, it will give a detailed comparison of the time and accuracy performance of
both non-parametric and parametric methods with different objects of different scales.

4.1 The RKHS reparameterization model

To reduce the number of variables, the first model applied to reparameterize the function is the RKHS space spanned by the eigenfunctions of a Gaussian kernel. The idea is that the eigenfunctions sorted by the eigenvalues show the nature of multi-scale smoothness, where the first eigenfunction is a very smooth function and the last eigenfunction is a fast oscillating function. This leads to a fact that when projecting a smooth function into the RKHS space, the projection coefficients decay very fast. In other words, it can just approximate a smooth function using the very few coefficients associated with the first few eigenfunctions [1]. The Gaussian kernel $K$ is given in Eq. 2.1. By Mercer’s theorem, $K$ may be written in terms of the eigenvalues and continuous eigenfunctions

$$K(x, x') = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \phi_i(x')^T,$$

(4.1)

where $\lambda_i$ and $\phi_i(x)$ are eigenvalue and eigenfunction pairs. The RKHS is spanned by linear combinations of these eigenfunctions[12].

Therefore, for each spatial location, the spectral function $f(\cdot, x, y, z)$ can be then approximated as:

$$f(\cdot, x, y, z) \approx \Phi \alpha(x, y, z),$$

(4.2)

where the vector $\alpha(x, y, z) = [\alpha_0, ..., \alpha_N]$ is the coefficients vector for the first $N$ eigenfunctions, and $\Phi$ represents a matrix of the first $N$ eigenfunctions. With this reparameterization, $\alpha$ is the new representation of a spectrum in the RKHS space. Typically $N$ is much smaller than the dimension in $\sigma$, thereby enabling reduction in the number of variables. For example, 4.1 shows the result of projecting a simple linear function onto the first 1000 eigenfunctions. It can be seen that the projection coefficients are decaying very fast, which means the first few coefficients with their eigenvectors can precisely approximate the original function.

Therefore, if reparameterizing the spectrum of each location only using the first 50 eigenfunctions, only a 50-tuple vector for each object location needs to be recovered, whereas in the previous non-parametric representation the spectrum is a 1000-tuple vector. Now the objective function in Eq. 3.10 is changed to optimize the $\alpha$ vectors for every spatial location. With a little abuse of notation, now $\alpha$ is used to represent the reparameterization vectors for all spatial locations and use $\alpha(x, y, z)$ to represent the $N$-tuple alpha vector for a particular spatial location $(x, y, z)$. Therefore, $\alpha = [\alpha(0,0,0), ..., \alpha(x,y,z), ..., \alpha(N,N,N)]$. With such a
Figure 4.1: A linear function is projected onto the first 1000 eigenfunctions. The projection coefficients are shown in the above figure.
reparameterization, the likelihood term becomes
\[ \| g - H\Phi \alpha \|^2. \] (4.3)

The spectral smoothness assumption is inherently given by the RKHS reparameterization: it turns out that the way for measuring a function’s smoothness in the RKHS space is to use the so-called \( k \)-norm. Assume \( \beta \) is an RKHS reparameterization coefficient vector, its \( k \)-norm is defined as
\[ \| \beta \|_k = \sum_{i=1}^{N} \frac{\beta_i^2}{\lambda_i}, \] (4.4)
where the set of \( \{\lambda_i\} \) is the corresponding first \( N \) eigenvalues. The spatial smoothness, on the other hand, is then measured by the \( k \)-norm of the difference between the two \( \alpha \) vectors of neighboring spatial locations. It is defined that
\[ \| \nabla \alpha(x, y, z) \|_k = \| \nabla_x \alpha(x, y, z) \|_k + \| \nabla_y \alpha(x, y, z) \|_k + \| \nabla_z \alpha(x, y, z) \|_k, \] (4.5)
where \( \nabla_x \alpha(x, y, z) \) is the subtraction of the two \( \alpha \) vectors between neighboring locations in the \( x \) direction, i.e.
\[ \nabla_x \alpha(x, y, z) = \alpha(x + 1, y, z) - \alpha(x, y, z), \] (4.6)
and the same rule applies for \( \nabla_y \alpha(x, y, z) \) and \( \nabla_z \alpha(x, y, z) \). Therefore, the overall objective function becomes
\[ \arg \min_{\alpha} \| g - H\Phi \alpha \|^2 + \lambda_1 \sum_{x,y,z} \| \nabla \alpha(x, y, z) \|_k + \lambda_2 \sum_{x,y,z} \| \alpha(x, y, z) \|_k, \] (4.7)
The same L-BFGS method can be applied here to optimize the \( \alpha \) variables. However, the dimension of \( \alpha \) is much smaller than that of the non-parametric form.

4.2 The B-spline reparameterization model

The second model used is to reparameterize the spectrum of the object using B-spline coefficients. The smooth nature of the spectrum allows us to adopt a low-order polynomial approximation with a limited number of control points (knots). Similar to the previous RKHS idea, instead of optimizing over a high-dimensional non-parametric domain associated with \( f \), it only needs to optimize over a lower-dimensional B-spline coefficient space.

A B-spline of order \( n \) is a piecewise polynomial function of degree \(< n \)[9]. The places where the pieces meet are known as knots, which can also be regarded as control points. Any spline function (piecewise polynomial) of given degree can be expressed as a linear combination of
B-splines of that degree. Assume the $B(x)$ is a spline with degree $n$ and there are $K$ interior knots, the following equation shows the B-spline reparameterization:

$$B(x) = \sum_{i=0}^{K+n} \alpha_i B_{i,n}(x), \ x \in [t_0, t_{N+1}],$$

(4.8)

where $n$ is the degree of B-spline, $K$ is the number of interior knots, and $\alpha_i$ are the control point coefficients. The $i^{th}$ B-spline basis function can be defined recursively using the Cox-de Boor recursion formula:

$$B_{i,0}(x) = \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

(4.9)

$$B_{i,k}(x) = \frac{x-t_i}{t_{i+k}-t_i} B_{i,k-1}(x) + \frac{t_{i+k+1}-x}{t_{i+k+1}-t_{i+1}} B_{i,k-1}(x).$$

(4.10)

Usually, more knots should be put in regions where the functions are changing more rapidly. Since in this problem the spectrum is smoothly varying, the knots can be equidistantly placed. For any given set of knots, the B-spline is unique, Figure 4.2 shows a series of B-spline functions with 10 knots and an order of 4. Each different color of the polynomial in the figure represents a different basis spline.

![Figure 4.2: B-spline functions with order 4 and 10 knots](image-url)
Therefore, for each spatial location, the 1D spectrum function of $f$ at any given local can be then approximated as a linear combination of B-splines. Similar to the previous RKHS formulation, $B$ is used to denote a matrix storing the $N$ B-splines and use $\alpha(x, y, z)$ to denote the B-spline coefficients of a particular spatial location (just like in the previous section $\alpha(x, y, z)$ is used to denote RKHS coefficients). Then Eq. 4.8 can be rewritten as

$$f(\cdot, x, y, z) \approx B\alpha(x, y, z), \quad (4.11)$$

Again, the following text will use $\alpha$ to denote the set of B-spline coefficients for all spatial locations. Since the method is using a limited number of knots for generating the B-splines, the dimension of $\alpha$ is much smaller than the non-parametric representation, and the number of unknown variables to be estimated is highly reduced.

Now the objective function in Eq. 3.10 has to be rewritten in terms of B-spline coefficients $\alpha$. This part follows exactly to the previous section except that the RKHS coefficients are replaced by B-spline coefficients. For the likelihood term, it is obvious that Eq. 4.11 and Eq. 3.8 yield a composite linear relationship, which is denoted by $HB\alpha$. For the spatial smoothness term, the directional derivatives of B-spline coefficients should be penalized instead of spectral intensities. Lastly, the B-spline reparameterization already implicitly guarantees the spectral smoothness by constraining the 1D spectrum to be a low-order polynomial. However, an additional regularization $\|\alpha(x, y, z)\|^2$ is put to penalize too large B-spline coefficients to avoid overfitting. The objective function becomes

$$\arg\min_{\alpha} \|g - HB\alpha\|^2 + \lambda_1 \sum_{x,y,z} \|\nabla\alpha(x, y, z)\|^2 + \lambda_2 \sum_{x,y,z} \|\alpha(x, y, z)\|^2. \quad (4.12)$$

### 4.3 Reconstruction results

This section will provide the reconstruction results for three different experiments with their goals being (a) to compare the RKHS and B-spline models (b) to test the robustness of the algorithm under Poisson noise and (c) to perform a thorough comparison between non-parametric and parametric methods. 

**Comparison tests of the RKHS and B-spline reparameterization models**

Since the emphasis of this chapter is the reparameterization trick, the first validation test is to compare the performance of two parametric methods: RKHS and B-spline. Here the experiment uses the spiral curves object and assumes both spatial and spectral smoothness. The following Figure 4.3 shows the reconstruction results using the RKHS reparameterization model with 10 eigenfunctions, with the overall MSE 1.33%, the sparse points MSE 10.45%, and the running time 2.5 minutes. The ground truth object can be referred to Figure 3.1. It
can be seen that the spatial cross-sectional planes can be almost recovered, but RKHS can not accurately represent the spectral smoothness. Therefore, RKHS is not an appropriate orthogonal basis for the reparameterization purpose.

![Reconstruction results of the three spatial cross-sectional planes](image1)

(a) Reconstruction results of the three spatial cross-sectional planes

![Reconstruction results for the three spectral cross-sectional planes](image2)

(b) Reconstruction results for the three spectral cross-sectional planes

Figure 4.3: Reconstruction results for spiral curves object using the RHKS parametric method

While, as shown in Figure 4.4, B-spline performs well in both spatial and spectral reconstruction, because the low-order polynomial functions can well represent the continuous spectral smoothness feature. The overall MSE for the results in Figure 4.4 is 0.005% (sparse points MSE
7.97%), and the running time is 2 minutes for 100 L-BFGS iterations, which are all better than the results from RKHS model. Thus the thesis will use the B-spline reparameterization model for the following experiments.

Then the B-spline reparameterization method is used to recover the random point cloud object. Figure 4.5 shows the reconstruction results, while the ground truth can be found in Figure 3.7. The overall MSE for the result without noise is 0.14% (sparse point MSE 3.62%) and the time cost is 7 minutes.

Due to the reduction in the number of unknown variables, the large data scale of the 3-bar scene object can be handled using the reparameterization trick. Figure 4.6 gives the ground truth of the six cross-sectional planes of the 3-bar object and Figure 4.7 shows the six corresponding reconstruction results. The overall MSE is 0.12% (sparse point MSE 10.82%), and running time are 40 minutes for 100 iterations.

To visualize the sparse point MSE more intuitively, the ground truth and recovered spectral intensity on one spatial location from the results in Figure 4.6 and Figure 4.7 are plotted in Figure 4.8. The two spectral intensity functions are very close to each other, where the blue line denotes the ground truth spectrum and the green line is the recovered one.

**Robustness tests with Poisson noise**

The second experiment is to validate the robustness of the B-spline-based parametric data reconstruction algorithm. As described in Sec. 2.4, the measurements are polluted with Poisson noise before the optimization. Taking the example of the 3-bar object, the algorithm use the ground truth object to simulate the measurements and add Poisson noise with SNR = 30. It uses the B-spline-based parametric method to recover the 3-bar object in Figure 4.6. Figure 4.9 shows the reconstruction results with Poisson noise (SNR = 30). The MSE for the result with noise is 0.26%. It can be seen that visually the recovered object still looks reasonable and the MSE value is only increased slightly, so it can be claimed that this algorithm gives equally reasonable results in dealing with the Poisson noise at this SNR level. The running time is also similar.

To give an overall view of the robustness tests, Figure 4.10 summarizes the MSE of different SNR settings (ranging from 5 to 30) for all three objects using the B-spline-based parametric optimization. It can be seen that the performance of the method drops with lower SNR. However, but the algorithm can still achieve reasonably good MSE at the level of SNR = 30.

**Comparison tests of non-parametric and parametric methods**

Finally, this section will illustrate the advantages of the B-spline-based data reconstruction method by comparing the accuracy and efficiency with the non-parametric method. The above reconstruction figures can give us a qualitative comparison, while the following figures and tables will provide a quantitative results.

Besides the fact that the reparameterization trick can save more computer memory, the
Figure 4.4: Reconstruction results for spiral curves object using the B-spline parametric method
Figure 4.5: Reconstruction results for the random point cloud object using the B-spline parametric method
Figure 4.6: The ground truth for the 3-bar object
Figure 4.7: Reconstruction results for the 3-bar object using the B-spline parametric method
most critical advantage is its higher computational efficiency. The parametric method can be much more time-consuming at any data scale level as shown in Figure 4.11. The vertical axis means the running time with the unit of seconds and the horizontal axis represents different data scales. The gap between the running time becomes even larger when the data scales up.

Although the parametric method costs less running time and saves memory, it can provide even better reconstruction results as the non-parametric method. Figure 4.12 shows the overall MSEs for different data scales using these two methods.

### 4.4 Conclusion

Considering both of the limited computer memory and high-dimension object datasets, the algorithm proposed two reparameterization methods to reduce the number of variables in the posterior optimization. One is the RKHS model, which uses the first few eigenfunctions of a Gaussian kernel to approximate a smooth function. The other one is the B-spline model, which reparameterizes a function by some low-order polynomials. According to the experiment results, the B-spline model is better at modeling the continuous spectra. Then it demonstrated the reconstruction results of three kinds of objects using this B-spline-based method with and without the Poisson noise. In the end, it compared the parametric method with the non-parametric method from the aspects of accuracy and efficiency to prove the advantage of using reparameterization.
Figure 4.9: Reconstruction results for the 3-bar object using the B-spline parametric method with Poisson noise
Figure 4.10: The overall MSEs of recovering three objects from noised measurements with different SNR using the B-spline parametric method.
Figure 4.11: The time costs of recovering object with different data scales using the non-parametric and parametric methods
Figure 4.12: The overall MSEs of recovering object with different data scales using the non-parametric and parametric methods
Chapter 5

Conclusion and discussion

This thesis investigated a reconstruction algorithm for 4D object-space data using the SSM technique and validated the algorithm using synthetic ground truth objects. The SSM technique is a hyperspectral imaging technique applied for far-field target detection in passive remote sensing systems.

The first problem encountered is that the SSM system is an underdetermined linear system. The system has much fewer number of equations from the optical and modulation model than the number of unknown variables in the scene object, so it cannot uniquely compute the unknown variables in a closed form. Therefore, the algorithm models the problem from a probabilistic point of view. The algorithm has proposed two proper prior distributions on the object scene and formulated a posterior optimization to solve the inverse problem. The second challenge is the optimization efficiency. When trying to reconstruct the 4D scene with much larger data scale (more than 10 GB), one problem is the computer memory is not large enough to handle the optimization, and the computational time is also too slow; the other problem is that with limited detector size this system becomes further ill-posed for such a large scale object scene. Thus the data is reparameterized by projecting it into some other domain and using only a few projection coefficients. In particular, much fewer variables are used to represent the continuous spectra of the object. Such a reparameterization trick can reduce the optimization memory consumption by 95%. Through the experiments on several different reparameterization models, it proves that the B-spline reparameterization can best reconstruct the object with efficient computation.

The novelty of the algorithm are: (a) it proposes to incorporate proper prior assumptions to the object scene so that it can solve effectively the underdetermined linear system by an inverse optimization (b) it proposes to use RKHS basis functions and B-spline polynomials to represent the continuous frequency components of the highly sampled spectra, thereby producing more efficient and accurate reconstruction.
Based on the optical model of the SSM system, the ground truth 4D object-space data used to validate the algorithm are three kinds of synthetic scenes: smooth spiral curves, random point clouds and a specific 3-bar point cloud. The performance of this algorithm should be tested from three aspects: accuracy (overall MSE and sparse point MSE), efficiency (optimization time) and robustness (with Poisson noise). The overall MSE is less than 0.3%, which means it can reasonably determine the location and the general shape of the objects. While the relatively higher sparse point MSE (about 11%) means that it is still needed to further improve the object intensity estimation on those specific point locations. Using the parametric method, the algorithm can reduce the optimization memory by 95% with respect to the non-parametric method, which gives more possibilities to the reconstruction of more complex real-life scenes with larger data scale in the future work. Besides, the experiments have shown that our algorithm can handle Poisson Noise effectively.

To conclude, this data reconstruction algorithm can perform well for the SSM hyperspectral imaging system. It has potential to be applied to other optical systems that can be formulated as an underdetermined system with given measurements and measurement matrices. The future work mainly includes: (a) to investigate more on the sparse point cloud object to increase the reconstruction accuracy on the point locations; (b) to apply this method on more complex point cloud objects; (c) to extend our method to other image compression inverse problems.
REFERENCES


