ABSTRACT

EPPERSON, KATHLEEN ANNE. Preservice Elementary Teachers’ Solution Strategies and Justifications When Generating Rules for Figural Growing Patterns. (Under the direction of Dr. Allison McCulloch).

The purpose of this study is to examine how eight preservice elementary teachers generalize three different figural growing patterns during a clinical interview. The study intends to add to a small, but growing body of research that addresses what algebraic knowledge preservice elementary teachers have when they enter the classroom, particularly in respect to their functional thinking and beliefs regarding valid justifications.

The participants represent a voluntary sample of preservice elementary teachers from a southeastern university in the United States who had completed their methods coursework and were preparing to enter the classroom as student teachers. The data used in this study consisted of the participants’ recording sheet and video-taped interviews, in which the participants generalized patterning problems.

The conceptual framework for this study was created from the literature on how figural growing patterns are generalized (Walkowiak, 2013; Markworth, 2012; Rivera & Becker, 2005; Rivera & Becker 2009; Stump, 2001; Lannin, 2002; Billings, 2008) and what constitutes as a valid justification (Lannin, 2005; Stylianides, Stylianides, & Shilling-Traina, 2013; Simon & Blume, 1996).

The analysis of the data revealed that when generalizing, the preservice elementary teachers relied on a mixture of strategies (recursive and explicit) and reasoning (numerical

and figural). All of the participants were successful at identifying additional instances for each pattern and most were successful at creating an explicit rule for each pattern. However, those that struggled relied more heavily on recursive strategies followed by numerical reasoning. When stating why a rule would work every time, only a few of explanations were deductive. Justifications were overwhelmingly either inductive or lacked any attempt to form a convincing argument. Additionally, data reveal that almost all participants reasoned deductively at some point when generalizing, but did not necessarily see this as an important part of their justification.

Preservice Elementary Teachers' Solution Strategies and Justifications When Generating Rules for Figural Growing Patterns

by
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BIOGRAPHY

Kathleen Epperson was born in Richmond, Virginia and raised in Concord, North Carolina. She attended the University of North Carolina at Wilmington, where she earned her Bachelors in Elementary Education. After teaching for four years in Cabarrus County, she went back to school full-time to earn her Masters in Mathematics Education at North Carolina State University.

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CHAPTER 1

INTRODUCTION

Current research reveals that students today require a vastly different approach to learning mathematics than in the past. The continuing advancement towards technology and globalization requires that students not simply know how to do mathematics, but be able to think mathematically (Gardner, 2009; Richardson, Berenson, & Staley, 2009). Understanding how to engage with mathematical concepts is becoming increasingly more important than the traditional emphasis on simply producing the right answer (Boaler, 2015). This shift in the teaching and learning of mathematics requires changes in curriculum, professional development and teacher preparation programs to align with new values. This is particularly true in regards to the mathematical content strand of algebra (Blanton & Kaput, 2011).

For many years, algebra was postponed until a formal class in high school, when it was believed students were ready to grapple with the abstract ideas embedded in its study. Even being in school until high school did not guarantee a student exposure to algebra. It was often seen as an enrichment course in math, with concepts that were only available to students who excelled in traditional math classes and whom teachers believed could handle the complexity (Kilpatrick & Izsack, 2008). Students who were not exposed to algebra lost a chance to develop necessary mathematical skills, often along with the chance to pursue higher education. However, even when students were exposed to algebra, they often found the concepts difficult to grasp (Booth, 1988). Algebra was presented in a way that involved an emphasis on symbol manipulation and arithmetic methods, without much regard for how students understood or thought about the concepts they were learning (Kilpatrick & Izsack, 2008; Kaput, 2008). Kaput (2008) claims “this superficial approach to algebra had led to both teacher alienation and high student failure” (p. 6). He argues that while there is a core aspect of algebra that involves a focus on symbol manipulation and arithmetic methods, the traditional view has mistakenly allowed this to become the sole focus. There is another core aspect that focuses on building generalizations and “reasoning about more general relationships and their forms” (p.12, Kaput, 2008) that deserves attention and Kaput (2008) is not alone in this dynamic view. Recent attempts to shift the focus in algebra to creating and understanding generalizations prior to symbol manipulation have steadily grown. Instead of waiting until high school, many believe that elementary students should be exposed to algebraic ideas, also known as *early algebra* (Blanton & Kaput, 2011, Stump, 2011; Yackel, 1997; Richardson et al., 2009; Markworth, 2012). This movement should not be misinterpreted as a push to move the formal algebra standards in elementary school. Instead, instruction of early algebra encompasses conceptual algebraic ideas that are widely viewed as appropriate and necessary to expose to students in Pre-K through 5th grade (Richardson et al., 2009). The need for early algebra is recognized by NCTM (2000), which advocates for opportunities for students to understand patterns, analyze mathematical structures and use mathematical models to represent relationships in the elementary curriculum. Blanton & Kaput (2011) assert that as a part of early algebra, elementary students should be exposed to “experiences in building, expressing, and justifying mathematical generalizations” (p.7) from the moment they step into the classroom. These experiences are key in fostering elementary students’ *functional thinking*. Functional thinking is defined as the ability to create and recognize relationships between two or more quantities through mathematical representations and effectively communicate this relationship (Blanton & Kaput, 2011; Markworth, 2012). Research has documented that students as young as Pre-K have functional thinking abilities (Blanton & Kaput, 2011; Yackel, 1997). In fact, not only are elementary students capable of doing this kind of work, but “functional thinking is a critical part of mathematical development, and introducing informal ideas about functions in elementary grades allows students the time and space to develop a more complex understanding than they might otherwise have if they first encounter functions in secondary school” (Blanton, 2008, p.39). Blanton & Kaput (2011) echoed these results, finding that students who are exposed to functional thinking in elementary school are more prepared to grapple with similar topics in higher level algebra, than students who lack this experience.

Statement of Problem

While more and more educators are aware of exposing elementary students to early algebra, more improvement is needed, specifically in regards to nurturing elementary students’ abilities to thinking functionally. It is important that elementary teachers are prepared to design and plan activities that embody early algebra ideas and one way to do this is through the use of figural growing patterns. While there is a growing body of research exploring how students generalize figural growing patterns and ways in which a classroom teacher effectively (or ineffectively) supports their generalizing (Markworth, 2012; Driscoll, 1999; Rivera & Becker, 2009; Walkowiak, 2014; Stump, 2011; Friel & Markworth, 2009), there are fewer studies that explore the knowledge that preservice elementary teachers have surrounding figural growing patterns. A handful of studies address this topic, but more are necessary to extend understanding in how to best prepare preservice elementary teachers to utilize figural growing patterns as they prepare to enter the classroom.

This thesis adds to the small body of research that reports on preservice elementary teachers' abilities to generalize figural growing patterns. The present study will explore how eight preservice elementary teachers generalize two figural growing patterns in a one on one clinical interview.

Definitions of Terms

It is necessary to define several terms that will be used throughout the thesis. First, the tasks utilized in the research are referred to as *figural growing patterns*. Figural growing patterns are defined as patterns that show the functional relationship spatially by use of either pictures, figures or objects which provides visual cues that can "explain and support pattern generalization" (Friel & Markworth, 2009; Rivera & Becker, 2005). Second, generalization strategies will be labeled by the type of reasoning used, either numerical or figural (Rivera & Becker, 2005). *Numerical reasoning* involves using relationships between numeric values to find a pattern. *Figural reasoning* focuses on the relationships that can be seen in the figural representations. If numerical reasoning is used, it will be specified whether or not a numerical representation was used or created to assist with reasoning. A *numerical representation* refers to written numbers used in any form. At times, the numerical representations will be specified as either *tabular*, representing both the independent and dependent values or as a *list*, showing only the dependent values. Lastly, to avoid confusion, *preservice teachers* will be used to describe university students in teacher preparation programs and *students* will be used to describe children in grades 12 and below. Any reference to *teachers* represents current practitioners in grades 12 and below.

Organization of Paper

The following chapter contains a review of the literature in regards to current research related to growing patterns and functional thinking. The focus then shifts to consider these same ideas specifically in regards to figural growing patterns. The chapter continues with an overview of research on how students and teachers utilize figural growing patterns in general. The chapter concludes by providing a synopsis of recent efforts to expose preservice teachers to figural growing patterns. The specific research questions of interest are at the beginning of the Chapter 3. Next, the methodology of the study is outlined and followed by the results of the clinical interviews. The paper closes with a discussion of the results.

CHAPTER 2

BACKGROUND

Using patterns to teach mathematics is not a new idea (Orton & Orton, 1999). In fact, many consider mathematics to be “the science of patterns” (Smith, 2003, p.137). However, merely generalizing and finding the answer to a pattern itself is not the goal. Instead, patterns should be seen as opportunities to reason and search for meaning in the broader context of numerical relationships (Stump, 2011). With effective instruction, patterns can begin to open the door to complex mathematical ideas, such as functional thinking.

Research on Generalizing Growing Patterns

Most research focused on elementary students’ conceptions of functional thinking revolves around generalizing patterns, both numerical and figural. The results overwhelming show that when students are attempting to generalize these patterns there are two dominant strategies that emerge: *recursive* and *explicit*. Recursive generalization highlights the variation between successive values in a sequence. Explicit generalization evaluates the relationship between a particular value of the sequence and its position in the sequence (Lannin, 2002; Wilkie & Clarke, 2014; Blanton & Kaput, 2001; Warren & Cooper, 2008). Students at the elementary level who are growing in their functional thinking should engage in both of these processes, and are often documented using them together to generalize patterns (Walkowiak, 2014; Wilkie & Clark, 2014; Driscoll, 1999; Warren & Cooper, 2008). Lannin (2005) suggests that while using these strategies, students should be challenged to consider contexts in which each type of generalization may be more appropriate. This “helps students develop a deeper understanding of the advantages and disadvantages of generalizing recursively and explicitly” (p.347).

While both types are useful, explicit generalization best supports functional thinking because it highlights the relationship between the independent and dependent values (Smith, 2003; Markworth, 2012; Driscoll, 1999). Advancing past recursive generalization to explicit generalization serves as stepping stone for elementary students, who ultimately will need to generalize and abstractly represent advanced functions (Carraher, Martinez & Schliemann, 2008). However, despite the necessity of this transition, students often favor using just the recursive approach. This ignores the relationship between the input and the output and does not promote functional growth (Rivera & Becker, 2009). Walkowiak (2014) cites a study designed to measure how well 60,000 incoming freshmen could analyze functions, that found that 70 percent were successful at extending the pattern recursively, but only 15 percent could create an explicit rule. Rivera & Becker (2005) cited their study of 30,000 eighth grade students’ ability to complete tasks involving linear functions, which found that only twenty percent of the students could generalize explicitly. Therefore, it is important that explicit generalization is emphasized in the classroom and modeled as the ultimate tool for finding functional relationships as soon as these patterns are introduced in the classroom.

While elementary students should be acknowledging the explicit relationship, this does not mean that they are expected use the terms independent and dependent variable.

However, they should be encouraged to refer these ideas informally, which can be done by using the terms "input" and "output" when describing a pattern (Blanton, 2008). Furthermore, some research has found that not drawing attention to the explicit relationship at all can have consequences. Blanton (2008) notes that the exclusion of the relationship between independent and dependent variables in the thinking and writing of younger students can "limit the depth of functional thinking that students attain" (p. 34). When this relationship is not highlighted, students tend to attribute values only to the sequence and do not realize that the term number is more than just a label (Carraher et al., 2008; Rivera & Becker, 2005).

This issue can also be compounded as elementary students begin to work with variables in relation to functions. Students in the lower grades often only use variables to represent a static unknown value, and this can be perpetuated by a strictly recursive view of a function (Stump, 2011). They often see the variables as placeholders for the sequence, instead of a representation of the relationship (Rivera & Becker 2005; English & Warren, 1998). The result of this misconception is that students tend to ignore the function as an entity in and of itself, which can reduce or arguably eliminate their need to search for an explicit generalization (Driscoll, 1999; Richardson et al., 2009).

Although conceptual understanding of the independent and dependent variable is important, elementary students do not need to use formal notation to convey this idea (Carraher et al., 2008; Rivera & Becker, 2009). Adversely, the assumption that correct use of notation always indicates functional thinking is misleading (Lannin, 2005). Research results have shown a range of representations from students who are developing their functional thinking through patterns. This includes, but is not limited to, describing relationships verbally (Walkowiak, 2014; Billings, 2008; Lannin, 2002), explaining the relationship through t-charts (Blanton & Kaput, 2011), utilizing student invented representations and symbols (Rivera & Becker, 2009) or formal notation (Walkowiak, 2014; Rivera & Becker, 2009; Blanton & Kaput, 2011). The consensus from these studies, and others, is that more emphasis should be on the thinking behind the chosen representation and not just on the type of representation.

It is also important to consider how many representations students use and how fluidly they can move between them. Numerical and symbolic representations of patterns are often too abstract to be supported on their own, especially at the elementary level. In order to develop a deeper understanding of the relationship between numbers, students need to see functions in multiple ways (Blanton & Kaput, 2011; Wilkie & Clark, 2014; Rivera & Becker, 2005; Friel & Markworth, 2009). This can come in many forms including “pictures, function tables, graphs, manipulatives, mathematical symbols, or even children’s natural language” (Blanton, 2008, p.39). While each representation describes the same functional relationship, they accentuate the relationship in a different way, allowing students to see and consider different qualities of a pattern. This can serve varied learning needs and it encourages students to make connections between and synthesize the presented ideas, enhancing their understanding of the function (Imre & Akkoc, 2012; Rivera & Becker, 2005)

The benefits of multiple representations are clear and so are the drawbacks of allowing one type of representation to dominant. Growing patterns are often introduced as a numerical representation in the elementary school classroom. This is a crucial representation and there is a place for them, such as in activities like Guess-My-Rule. However, when used exclusively, these activities tend to mask the relationships between numbers. This promotes guess-and-check type strategies which are also encouraged by the use of the word 'Guess' in the name (Markworth, 2012). While guess-and-check is a strategy commonly promoted, it often ignores the algebraic aspects of the pattern and has a tendency to lead to incorrect conclusions about generalizing and justifying (Rivera & Becker, 2005; Lannin, 2002). One way to encourage functional thinking and to complement strictly number patterns with the use of figural growing patterns (Rivera & Becker, 2005).

Research on Generalizing Figural Growing Patterns

Teaching through figural growing patterns is consistent with how elementary students generally learn (Carraher et al., 2008; Bruner, 1966). They engage concretely and pictorially with ideas before being asked to work with abstract ideas. Figural growing patterns allow students to visually experience the structure and relationships that are hidden when strictly numerical representations are used to convey a relationship. In order to understand these benefits, it is important not just to consider the use of figural growing patterns, but also how students reason when generalizing them. If a student uses the figures given to count the totals for each case, generate a numerical representation and then begin to generalize, they are using *numerical reasoning*. Their focus is on finding a relationship between the numeric values that the pattern creates. On the other hand, if a student recognizes the spatial aspects of a figure and draws on these to generalize, they are using *figural reasoning.* Their focus is to use design of the figure to find a relationship (Rivera & Becker, 2005).

Research has revealed specific benefits of using figural growing patterns to cultivate students' functional thinking abilities. When students work with figural growing patterns they have an easier time making sense of the pattern and are more successful at generalizing and justifying (Becker &Rivera, 2005; Billings, 2008; Markworth, 2012; Friel & Markworth, 2009).

There are 4 key reasons why this happens.

1. In figural growing patterns, it is easy to see what part of the pattern is growing and what is staying the same.

Lannin (2002) demonstrated this when working with students who were finding the number of stickers it took to cover each face of a rod as the length increased by one cube, as shown in Figure 1. Students were able to use the figure directly to discover that there were always 2 cubes on the end that had 5 stickers, while the cubes in the middle (which each had 4 stickers) were increasing by one each time. In another study, Markworth (2012) gave an example of an expanding hexagon pattern that was presented to students. Many of the students who were able to successfully find an explicit relationship saw the center hexagon as a constant piece that was a part of each figure, while the square tiles stemming from the hexagon were increasing each time (see Figure 2). In another study, Rivera & Becker (2009) found benefits in using figural growing patterns and specifically teaching students to attend to what stayed the same and what changed (see Figure 3). The use of figural representations in all these studies allowed students to generate explicit ideas about how the pattern was growing.

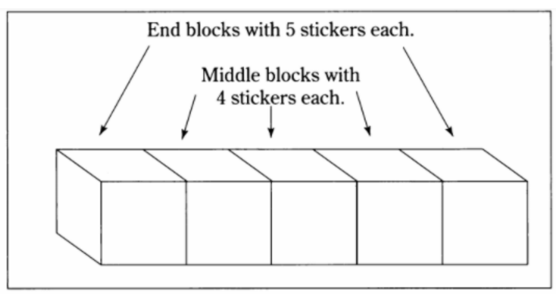


Figure 1: Cube and Stickers Growing Pattern (Lannin, 2002)

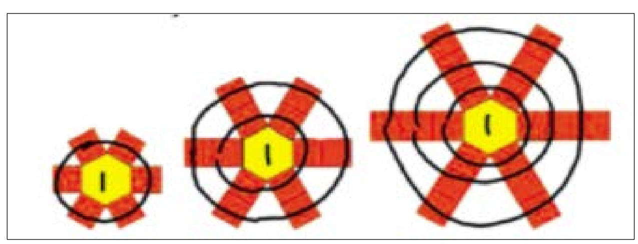


Figure 2: Expanding Hexagon Pattern (Markworth, 2012)

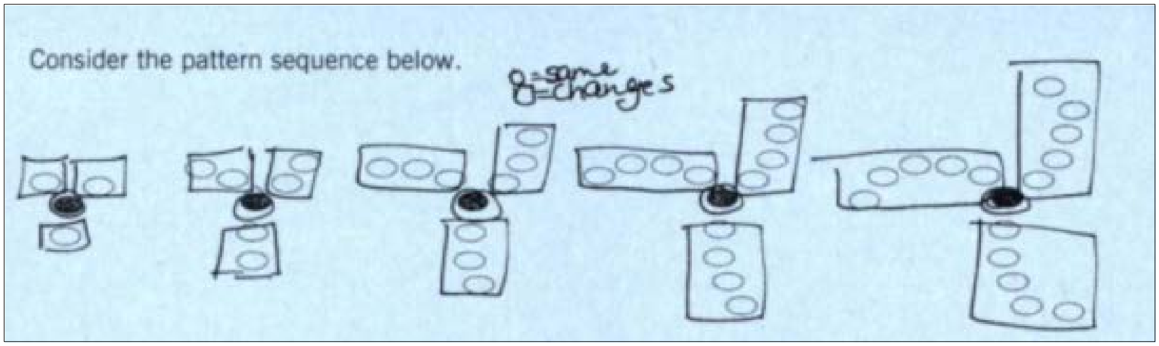


Figure 3: Growing Pattern (Rivera & Becker, 2009)

2. Figural growing patterns connect directly to the rule or formula that is formed. The visualization of the figural growing pattern "can be the analytical process itself which concludes with a general formal solution" (Hershkowitz, Arcavi & Bruckheimer, 2011, p.262). In continuing with Lannin (2002)'s example, students were able to extend the commonalities and changes they saw happening with the addition of each cube to create a rule that stated "To find the number of middle blocks [sic], subtract 2 from the length of the rod and multiply that number by 4. Then, add 10 to that total because the 2 blocks on the end have 5 stickers each" (p. 354). Billings (2008) also found this when working with a young student who used the strategic placement of dots in the pattern shown to him to extend to a general relationship "between the number of dots and the figure number" (p.306). Stump (2011) asserts that the being able to explain how a rule was generated from the original context is an important component of the problem-solving process. What is clear from these examples and others is that young students are more successful in communicating their

functional thinking if they have a concrete pictorial representations in which they can reference (Whitin & Whitin, 2014; Walkowiak, 2014).

3. Figural growing patterns allow students to see patterns in more than one way.

When chosen appropriately, figural growing patterns lend themselves to being seen in multiple ways (Wilkie & Clarke, 2014; Warren & Cooper, 2008; Imre & Akkoc, 2012). This has two clear benefits. First, it encourages students to think flexibly about the pattern, as they discover multiple ways to describe the functional relationship. Then, they can compare attributes and identify which view is more useful in a given context (Markworth, 2012). Specifically, as they develop their functional thinking, students learn to distinguish which patterns will lead to explicit algebraic rules and which will not (English & Warren, 1998; Lee, 1996). Secondly, as students' progress, multiple ways of seeing the pattern leads naturally to the creation of equivalent generalizations. This is another foundational idea that elementary school children can begin to build on (Friel & Markworth, 2009; Rivera & Becker, 2009: Wilkie & Clarke, 2014; Carraher et al., 2008; Rivera & Becker, 2005).

4. Students can establish meaningful justifications for their rule by referencing the figural growing pattern.

The concrete nature of figural growing patterns allows students a platform upon which to construct a meaningful justification (Lannin, 2002; Whitin & Whitin, 2014; Richardson et al., 2009; Markworth, 2012). Many assert that creating the correct algebraic expression for any given pattern is not enough. This type of expression can only be considered an informed conjecture. If the claim is intended to be generalized for all cases, there must be proof (Driscoll, 1999; Stump, 2011; Rivera & Becker, 2009; Lannin, 2002; Lannin, 2005). This is done through a convincing argument by using the figures. Of course, students are not expected to present a formal mathematical argument in elementary school (Richardson et al., 2009). Instead, at the elementary school level a valid figural argument would be one that requires that all parts of generalization (formula, rule, statement, etc.) be tied back to the original problem, in this case, the figure. Further, the explanation should be absent of particulars (Lannin, 2002; Driscoll, 1999; Lannin, 2005; Whitin & Whitin, 2014; Markworth, 2012). Multiple studies found that students who utilized the figural aspects of patterns were more likely to provide valid justifications for their rules than students who utilized the numbers (Lannin, 2005; Warren & Cooper, 2008; Richardson et al., 2009). Finally, the accessibility of justifications in figural growing patterns to elementary students reinforces the logical and reasoned origins of mathematics as a whole. This teaches students to hold themselves and others accountable for validity in mathematical claims (Driscoll, 1999; Simon & Blume, 1996).

These four key benefits support the claim that the exploration of figural growing patterns by elementary students builds a much needed foundation for advanced functional thinking (Walkowiak, 2014; Markworth, 2012; Warren & Copper, 2008; Friel & Markworth, 2009; Billings, 2008). While there are other ways to promote functional thinking in elementary students, figural growing patterns will be the focus of this thesis.

Issues with Figural Growing Pattern Implementation in the Classroom

Despite the expanding body of research that suggests the use of figural growing patterns as means to develop functional thinking in elementary grades, they are often not utilized by teachers. Much of this can be attributed to lack of curriculum resources that emphasis this type of thinking (Blanton & Kaput, 2005). While it is important to ensure that teachers have the materials they need to present effective instruction, curricular changes on their own are insufficient. In order to affect real change, teachers must be effectively prepared to teach functional thinking through figural growing patterns (Blanton & Kaput, 2011). Current research shows that often teachers who do utilize figural growing patterns, do so in a way that does not utilize the four benefits discussed in the previous section.

In fact, research shows that when figural growing patterns are introduced, they are not adequately supported by teachers, which results in considerable difficulty for students (Lannin, 2005). The main reason this happens is the tendency of teachers to over emphasize numerical reasoning, even though the patterns are presented in figural forms (Rivera & Becker, 2005). Often teachers see the figures as a means for counting and generating a numerical representation which they use to generalize; never returning to consult the original figures given (Markworth, 2012). Not surprisingly, students' struggles with figural growing patterns, often mirror the struggles of teachers. Most often when a figural representation is presented with a pattern, students use it a counting tool and then use only the numeric values to generalize going forward (Imre & Akkoc, 2012; Friel & Markworth, 2009). Students do not see the figure in and of itself as a means by which to generalize. And without a teacher who models this type of thinking, the use of figural growing patterns offers no benefit.

Unfortunately, the emphasis on strictly numerical reasoning that perpetuates most classrooms allows explicit generalization abilities only to the few students who can perceive the relationship between the input and the output on their own (Rivera & Becker, 2005). This leaves most students to struggle aimlessly. Indeed, the process of manipulating numbers to find a pattern may unintentionally cause added mystery and stress. Often students who successfully determine the patterns cannot explain why or how the strategy works (Rivera & Becker, 2009). Without an effective support, such as figures, students may find it difficult to appreciate the structure within the pattern. Due to this, many students resort to the guess and check method to identify an explicit rule. While this strategy is also popular among teachers, it is often void of algebraic thinking (Imre & Akkoc, 2012; Lannin, 2005; Driscoll, 1999).

Problems can also arise when teachers do not carefully consider what pattern they will use and what questions will be asked to elicit functional thinking. One issue arises when teachers choose a figural growing pattern that does not clearly relate back to the relationship between the input and the output. Friel & Markworth (2009) refer to these as non-transparent patterns. Non-transparent patterns require that “something more needs to be done before students are able to see a possible function rule from the available clues” (Rivera, 2007, p.72). Teachers should carefully consider whether students are ready to explore non-transparent figural growing patterns before introducing them.

Another thing to consider is whether the figural growing pattern involves a real-world context that may impact how students generalize. Lannin (2005) noted a scenario used with students that involved rows in a theater that "added two each time" (p.243) directly suggested for students to use a recursive strategy, narrowing their approach to generalizing. Carraher et al. (2008) also noted this issue for a problem where guests arrived as time passed. These types of figural growing patterns still have their place in instruction, but teacher should carefully consider when to use each and provide appropriate questions to help shift students from recursive to explicit strategies (Wilkie & Clarke, 2014).

There are many examples of questions that elicit functional thinking from students as they are generalizing a figural growing pattern. When completing the activity where students had to identify how many stickers would be needed to cover each side on rods of different lengths, Lannin (2005) noted that the context and question set up forced "students to move beyond using drawing and counting strategies toward identifying a general relationship that exists in the situation" (p.343). Many figural growing pattern activities begin by drawing attention to what is changing in the pattern by asking students to consider or draw the next stage. The questions progressively ask for cases farther and farther away and culminate by asking for a general rule for the n^{th} term to encourage explicit generalization (Friel & Markworth, 2009; Markworth, 2012). However, even with guiding questions, students may have trouble identifying what parts of the pattern are growing and what parts are staying the same. This is especially true when they are just beginning to work with figural growing patterns. One way to assist students, particularly when first introducing figural growing patterns, is to choose patterns that differentiate the constant with a different color or shape like those shown in Figure 2 and Figure 3 (Friel & Markworth, 2009).

Even if the activity context and questions inadvertently promote recursive or numeric thinking, proper teacher support can overcome this. Friel & Markworth (2009) suggest beginning figural growing pattern tasks by asking questions that draw attention specifically to the visual cues in the figure, delaying the urge to immediately begin generalizing or manipulating numbers. In the same respect, the results of many studies encourage teachers to ask questions like “How might you extend this pattern?” and “What stays the same and what changes in your pattern?” (Rivera & Becker, 2009; Walkowiak, 2014; Stump, 2011). Carraher et al. (2008) documented an instance when students failed to recognize the relationship between the input and the output in a figural growing pattern and the teacher effectively used discussion and guiding statements to elicit the concept. Without proper emphasis through questioning, these ideas may not have been revealed by the students.

Issues in instruction also arise even after students are successful at finding a generalization, because they regularly do not provide a valid justification (Lannin, 2005). A critical issue is often students are not asked to prove that their rule will always work. If this is a result of a belief that elementary students are not developmentally prepared to construct these arguments, the belief is misplaced. Stylianides et al. (2013) cites an entire body of research, ranging from elementary classroom studies to psychological examination, from small case studies to large scale international data collections, that all conclude elementary students are capable of reasoning and proof. He suggests that if there is a deficit in a student's abilities to reason and proof, this may reflect the lack of appropriate classroom instruction and proper emphasis on justification instead of a lack of student readiness or ability.

Even when a justification is required in the classroom, confusion can occur if students are not clear on what a sound justification involves. Sometimes students mistake the generalizing process with justifying and simply explain how they generalized when asked to justify their rule (Simon and Blume, 1996). While the processes of generalizing and justifying are usually similar, it is important that students make a distinction between the two. In particular, using specific examples from the pattern is not appropriate when justifying. In fact, the most common error among students, also modeled by teachers, is the use of a few cases to generalize proof for the entire function (Becker & Rivera, 2009; Lannin, 2002; Richardson et al., 2009; Lannin, 2005). This is not acceptable for a valid proof because it verifies only the correctness of a specific case, and fails to explain the mathematical relationship that is responsible for establishing the rule for all cases.

While a justification should be rooted in general validity, how this is accomplished and what is deemed an acceptable form of proof is as much a social process as it is a mathematical one (Simon and Blume, 1996; Lannin, 2005). Dialogue in a classroom highlighted by Lannin (2005) dictates conversations of students sharing empirical evidence as proof with each other and forming an overall consensus that this was an acceptable form of justification. Without proper guidance from a teacher, this type of thinking will dominate the classroom and create misconceptions in formal algebra. “It is the teacher’s job to promote the establishment of a classroom mathematics community in which mathematical validation and understanding are seen as appropriate and important foci and to endeavor to make mathematical ideas problematic in ways that students are likely to see a need for deductive proof and proofs that explain” (Simon and Blume, 1996, p. 9).

It is clear that in order to encourage students’ functional thinking in respect to growing figural growing patterns, they need the support and guidance from a teacher who has access to their own functional thinking skills and realizes the importance of eliciting these ideas from their students. In her work with teachers, Stump (2011) focused on providing teachers with experiences aimed at enhancing their functional thinking capacity, which included work with figural growing patterns. Teachers were encouraged to focus on their own understanding, share ideas with their colleagues and ultimately consider how they could apply what they have learned with their students. At the conclusion of the activities, though many teachers still needed to continue developing their ideas about generalizations, their reflections revealed that they had gained many important insights. They reflected on a renewed vigor for the use of patterns in the classroom with an emphasis on allowing students to explore and explain why patterns work. While exposing teachers to these ideas may not guarantee an increase in functional thinking for their students, a complete lack of this knowledge guarantees no increase (Blanton, 2008).

Preservice Teachers' Knowledge in Regards to Figural Growing Patterns

Teacher knowledge on this topic is clearly important and efforts to provide teachers with more support and knowledge with respect to growing patterns and functional thinking is crucial (Blanton & Kaput, 2011; Stump, 2011). Along with this idea, it is important to consider how functional thinking, specifically in regards to figural growing patterns, is addressed in elementary teacher preparation programs, including how preservice elementary teachers' reason about and generalize figural growing patterns themselves.

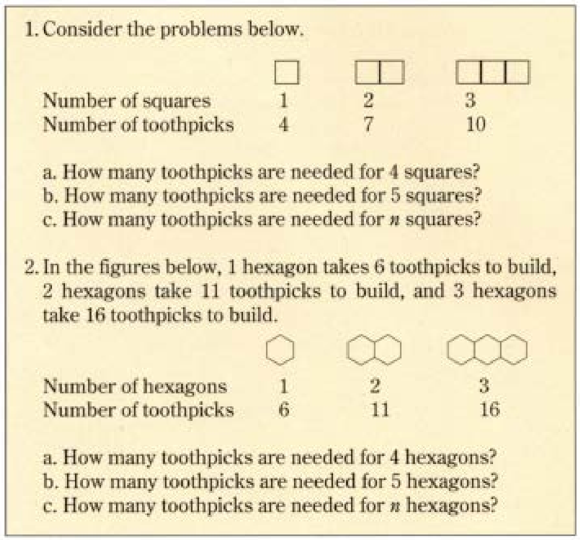
There are a few studies that begin to answer these inquiries. One such study, conducted by Imre & Akkoc (2012), was designed to see how three preservice teachers generalized growing patterns and how they supported others who were working with growing patterns. The study focuses on three voluntary participants who were in their last year of an elementary mathematics teacher program. It spanned 10 weeks and included the participants planning micro-teaching activities and administering them to their classmates at the beginning and end of the course. In between the two micro-teaching activities, the preservice teachers participated in numerous activities aimed at improving their own pattern generalizing strategies and providing more insight into how to help students understand patterns. The structure of the course was as follows in Figure 4.

Week	Course content
1	First micro-teaching activity
2	Watching the videos of the first micro-teaching activity and interviewing
3	Questioning the observation of the first component of PCK: knowledge of students' understanding of and difficulties with particular topics
4	Questioning the observation of the second component of PCK: knowledge of strategies and representations for teaching particular topics
5 and 6	Questioning the observation of both components of PCK
7	Questioning the observation of both components of PCK of generalising number patterns
8	Watching videos of two mentors' lessons and questioning their approaches
9	Second micro-teaching activity
10	Watching the videos of the second micro-teaching activity and interviewing

Figure 4: Preservice Teacher Course Structure (Imre & Akkoc, 2012)

Overall, the participants shifted their emphasis from recursive to explicit strategies from their first lesson to their second. The preservice teachers also asked more questions that encouraged their classmates to think about the relationship between the case number and the total number of objects in the case. Despite their growth, the researchers noted that there were still areas that did not improve. One of the most notable was that the preservice teachers failed to use the picture representations in effective ways. While the researchers had suggested ways the preservice teachers could use these representations more effectively in the follow-up interviews, a lack of modeling by the mentor teacher's lesson may have contributed to the lack of growth in this area.

In another study, Rivera & Becker (2005) drew conclusions from task based interviews of forty-two elementary and middle school preservice solutions for two figural growing patterns (see Figure 5). The interviews were conducted voluntarily by students participating in an introductory mathematics course for elementary teachers.



1. Consider the problems below.

Number of squares	1	2	3
Number of toothpicks	4	7	10

a. How many toothpicks are needed for 4 squares?
b. How many toothpicks are needed for 5 squares?
c. How many toothpicks are needed for *n* squares?

2. In the figures below, 1 hexagon takes 6 toothpicks to build, 2 hexagons take 11 toothpicks to build, and 3 hexagons take 16 toothpicks to build.

Number of hexagons	1	2	3
Number of toothpicks	6	11	16

a. How many toothpicks are needed for 4 hexagons?
b. How many toothpicks are needed for 5 hexagons?
c. How many toothpicks are needed for *n* hexagons?

Figure 5: Patterns from Task Based Interviews (Rivera & Becker, 2005)

The interviews revealed that the preservice teachers used numerical reasoning to generalize more frequently than figural reasoning. While this was the result the researchers expected, what surprised them was only five of the twenty-six preservice teachers who used a numerical representation with a recursive strategy were able to create some version of the explicit rule 3n+1 and 5n+1. And none of these five were able to connect their rule back to the context of the figures. To these preservice teachers “the numbers 3 and 1 had no other value for them except that they were instrumental in generating all the numbers in the sequence" (Rivera & Becker, 2005, p.20). One student successfully found an explicit rule with a trial and error strategy, using information from the figures and from the numbers. While he was correct, he was unable to explain why his rule worked, stating for his rule 3n+1 that he did not "know what the 3s are there for" (Rivera & Becker, 2005, p.201). Students who focused on recursive rules also had trouble. Another surprising finding was how often the preservice teachers incorrectly used the variable n to formalize their findings. The most common mistake was the use of an equation with n to formalize a recursive generalization; for example, citing n+3 as the rule for the first pattern.

The rest of the preservice teachers, sixteen out of forty-two, employed figural reasoning more often than numerical. Rivera & Becker (2005) found that these rules clearly reflected how the perceived the figures and thus these preservice teachers were much more successful in connecting their rule back to the figure for their justification. For example, one participant justified her rule 1+3n by explaining "You're trying to make a full square with 4 toothpicks and if you already have one side then you would be adding 3 more onto it depending on the number of squares that you wanna make 'cause [sic] that's how many you would put, that's how many 3s you would add on" (Rivera & Becker, 2005, p.201).

A similar study was conducted by Alajmi (2016), who used task-based interviews with preservice teachers (K-12), asking them to generalize figural growing patterns. They worked with three linear, three quadratic and three exponential patterns, but only the linear results from the preservice elementary teachers will be discussed here. The results come from six randomly selected preservice elementary teachers who were in their third or fourth year in their program. Over half of their generalizing strategies involved looking for explicit rules, while recursive made up only 12 percent of the strategies used. Overall, the preservice elementary teachers saw recursive strategies useful for smaller values of n, but preferred to look for explicit rules to generalize for larger values.

While these results were encouraging, 1 out of 4 strategies that the preservice elementary teachers used was coded as incorrect or no response. The participants had the most trouble with the linear pattern presented in Figure 7. The brief overview of their strategies for this problem highlights multiple instances of guessing and checking to generalize with little to no connection to the problem context. Only two of the six preservice elementary teachers were able to provide an explicit rule. It should be noted that the rule for this pattern is (3(n-1)+7) which is considered a non-transparent figural growing pattern, and therefore more difficult that the other figural growing patterns referred to in previous research.

Seeking to find out how preservice elementary teachers' thinking developed over time, Richardson et al., 2009 et al. (2009) designed a whole-class study that took place over 3 weeks with twenty-five preservice elementary teachers. A team of researchers taught and collected data during each class session throughout the course of the study. It took place during the preservice elementary teachers' only mathematics method course. The following patterns in Figure 7 were utilized for teaching and data collection.

Theater Seat

In a theater there are 7 seats in the first row. Each row after the first row contains 3 more seats than the row before it. Below is a diagram of the first 3 rows in the theater.

□□□□□□□□□□□□□

□□□□□□□□□□

□□□□□□□

a) How many seats are there in the 4^{th} row? Explain how you determined this.
b) How many seats are there in the 10^{th} row? Explain how you determined this.
c) How many seats are there in the 20^{th} row? Explain how you determined this.
d) How many seats are there in the 150^{th} row? Explain how you determined this.
e) Write a rule that would allow you to calculate the number of seats in any row. Explain your rule.

Figure 6: Patterns from Task Based Interview II (Alajmi, 2016)

Week 1. Write a rule that could be used to find the perimeter of a square pattern block train that is *n* blocks long. Then justify your rule with an explanation.

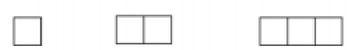


Fig. 1. Task given on week 1 of the teaching experiment.

Week 2. Write a rule that could be used to find the perimeter of a triangle pattern block train that is *n* blocks long. Then justify your rule with an explanation.

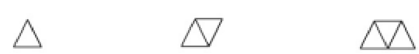


Fig. 2. Task given on week 2 of the teaching experiment.

Week 3. Write a rule that could be used to find the perimeter of a hexagon pattern block train that is *n* blocks long. Then justify your rule with an explanation.

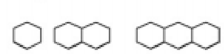


Fig. 3. Task given on week 3 of the teaching experiment.

Figure 7: Tasks from Course (Richardson et al., 2009)

Only two out of the twenty-five failed to generate any explicit rules during the three weeks. The researchers chose to focus on the results from the other twenty-three preservice elementary teachers in their study. They did not aggregately report the generalization strategies, but instead highlighted interesting findings. The first two examples that were shared described students in week 1 and week 2 who created a tabular representation of values from the figure and used the numbers to create explicit rules. In both instances, it was noted how the preservice elementary teachers could not offer a justification for their rule. A shift in how the preservice elementary teachers generalized is highlighted after whole-class discourse during week 2. In this discussion, the preservice elementary teachers made connections between the figure and different rules that were created. Moving into the third pattern, the researchers noted "the pattern block representation received more attention from the preservice elementary teachers than the tabular representations" they had previously been valuing in weeks 1 and 2 (Richardson et al., 2009, p.192). However, despite this, it was noted that there were still preservice elementary teachers who are unable to create valid justifications.

Overall, the study reaffirmed the importance of figural reasoning when preservice elementary teachers are generalizing and justifying figural growing patterns. One of the two implications for teaching figural growing patterns to preservice teachers was an emphasis on selecting tasks "that encourages students to base their reasoning on a physical structure or concrete model rather than on numeric thinking" (Richardson et al., 2009, p.197).

Figural growing patterns have clear benefits for students, teachers and preservice teachers alike. They illuminate the parts of a function that are growing and staying the same which then connects directly to an explicit rule and allows a platform to form a valid justification. Figural growing patterns can also be interpreted in multiple ways, which allows students to create and compare equivalent generalizations. Introducing elementary students to these ideas through figural growing patterns, can better ensure their readiness for formal algebra.

In order to promote effective use of these patterns, it is important to consider what preservice elementary teachers know in regards to them. While the studies overviewed in this chapter reveal important information about preservice elementary teachers' knowledge in relation to figural growing patterns, more like them are needed. This study intends to continue to fill the gap on how preservice elementary teachers generalize and justify figural growing patterns. It will analyze how preservice elementary teachers generalize and justify figural growing patterns and make connections between these processes.

CHAPTER 3

METHODOLOGY

Much research has been done to assess how elementary students generalize figural growing patterns and subsequently argue for their use in elementary classrooms. However, less is known about what knowledge preservice elementary teachers possess in regards to these types of patterns. This study attempts to fill that gap. The purpose of this section is to describe the research questions, context for the study and how the data will be analyzed.

Research Questions

This study investigates how eight preservice elementary teachers generalize two different figural growing patterns during a clinical interview. The intent is to observe how preservice elementary teachers generalize and whether or not they utilize the four benefits of figural growing patterns outlined in the previous chapter. The specific questions of interest are:

1) What strategies and reasoning do preservice elementary teachers use to generalize figural growing patterns?
2) How do preservice elementary teachers justify their rules?
3) How does the preservice elementary teachers' first attempt to find a rule vary from their second attempt to find an additional rule?

Context for the Study

Sixty students who were seniors in an elementary education program were invited to participate, eight volunteered. The purpose of this section is to introduce the background of the participants, and describe how the eight clinical interviews were designed, conducted and then analyzed.

Participants. The eight participants in this study were enrolled in an elementary teacher preparation program at a southeastern university in the United States. They were interviewed near the end of the fall semester of their senior year, just before their full-time, culminating student-teaching experience in the following spring semester. The participants were all female. The participants had completed two required mathematics methods courses during the previous year as juniors. The first course emphasized mathematical concepts important in the earlier elementary grades, particularly developing number sense in young students and additive concepts. Special attention was paid to how to utilize different forms of assessment and instructional units to maximize learning in elementary mathematics. The second class, exposed the preservice elementary teachers to concepts in upper elementary grades, notably ideas around multiplicative reasoning. Consideration was also given to discerning effective instructional practices, questioning techniques and ways to diversify the elementary mathematics classroom.

In addition to the mathematics methods courses, this particular teacher preparation program places a special emphasis on Science, Technology, Engineering and Math (STEM).

The preservice elementary teachers' abilities in these areas are heavily considered when they apply to be in the program and if accepted these areas are continually nurtured during their time in the program. Those accepted into the program must take 27 hours of courses in the STEM areas, including courses in calculus, physics and design thinking. Other content courses are integrated with the STEM.

In specific reference to their core mathematics content background, all participants had previously taken Geometry, Algebra and Precalculus in high school. Additionally, four students had taken at least one AP math course in high school. During their college career, seven of the students took Calculus I and II for Elementary majors. Additional math courses taken by all the participants were introductory courses. Out of the eight preservice elementary teachers who participated, six noted that they felt adequately prepared to teach math at the elementary level, one felt very prepared and one felt somewhat prepared.

Sources of data. For this study clinical interviews were the sole source of data. Participants were interviewed one on one with the researcher, in a private setting. Participants were presented with two figural growing pattern tasks (see Appendix A). Figural Pattern #1 is shown below in Figure 8. The participants were informed that Figural Pattern #1 represented tables being set up at a party and each red dot represented one person who can sit at the table. They were asked to answer the following questions on a recording sheet:

1. How many guests can sit when there are 4 tables?
2. How many guests can sit when there are 10 tables? 25 tables?

3. Can you create a rule that will allow you to find the number of people for any number of tables?

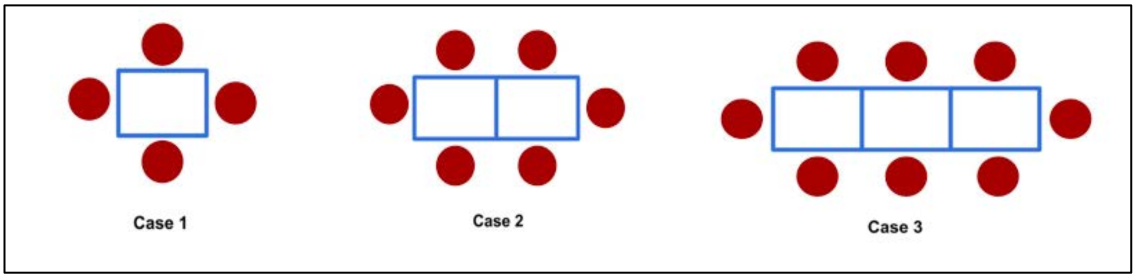


Figure 8: Figural Pattern #1 (Adapted from Friel & Markworth, 2009)

While the participants were solving the researcher quietly observed and did not interfere. Once the participant completed the recording sheet they were asked to explain how they solved each question. Clarifying questions were asked by the researcher to provide additional information that would be necessary to analyze the participants' functional thinking and their use of the figural growing pattern while generalizing. All participants were then asked to justify any rule they created.

Upon completion of the first figural growing pattern, participants were presented with a second figural growing pattern, which is shown below in Figure 9. No background information was given for this pattern. They were asked to answer the following questions on a recording sheet:

1. How many dots will be in the next case?

2. How many dots will be in the 10th case? The 25th?
3. Can you create a rule that will let you find the number of dots for any case?

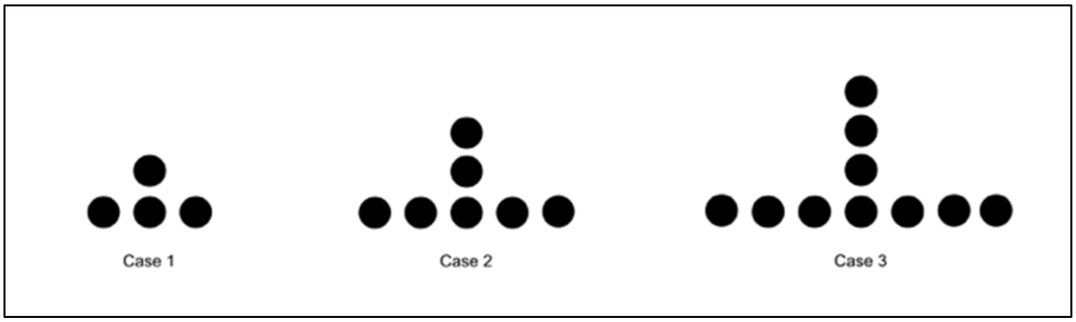


Figure 9: Figural Pattern #2 (Adapted from Friel & Markworth, 2009)

The interview process for the second task was the same as the first, except the questions on the recording sheet were adjusted to align with the context of the new pattern.

Finally, participants were asked to revisit one task of their choice and find an alternate rule for the growing pattern. If participants were successful at finding a new way to represent the pattern rule they were asked to elaborate on their discovery and provide a justification. If participants were unsuccessful they were asked to clarify whether they believed there was another rule that they could not find or they believed no other existed.

Design of the interview protocol. The three questions included with each of the figural growing pattern tasks were specifically chosen. Even though recent research suggests prompting students to describe the figure and how they think it is growing before generalizing to promote figural reasoning (Rivera & Becker, 2009; Friel & Markworth, 2009; Walkowiak, 2014), the preservice teachers in this study were not asked to do so before beginning. The intention of the interviews was not to teach a reasoning or solving strategy, but merely observe how the preservice teachers generalized. Only questions traditionally seen in figural growing pattern tasks were used so as not to promote figural reasoning, but instead observe whether or not preservice teachers utilized it on their own. First, asking the participants to consider the next case falls in line with the format of tasks presented in the research (Friel & Markworth, 2009; Rivera & Becker, 2009; Lannin, 2002). This encourages the participants to consider what is changing from case to case. Next, asking the participants to consider the 10th and 25th case is intended to push participants to notice the generalities of the pattern (Friel & Markworth, 2009; Richardson et al., 2009; Lannin, 2002). The third question is designed to encourage participants to formalize their generalization and effectively communicate their ideas. After the participants completed the two tasks, they were asked to find a different rule for either of the tasks. This question is asked to reveal whether or not the participants are flexible in how they view the pattern, one of the benefits of figural growing patterns noted in Chapter 2.

While other types of figural growing patterns exist two simple linear patterns were specifically chosen for this study. Figural Pattern #1 and Figural Pattern #2 displayed the simplified explicit rules of 2n+2 and 3n+1 respectively. These figural growing patterns were specifically chosen for three reasons. First, this matches the types of patterns elementary students have been asked to generalize in previous classroom research (Blanton & Kaput, 2011; Billings, 2008; Yackel, 1997; Markworth, 2012). Secondly, past research with preservice elementary teachers has overwhelming favored using linear patterns (Richardson et al., 2009; Rivera & Becker, 2005; Imre & Akkoc, 2012). In designing tasks for preservice elementary teachers in a mathematic methods course, Richardson et al. (2009) specifically chose not to use quadratic or exponential fearing that they "would present a problem to the novice in terms of expressing generalization" (p.196). Lastly, this research was designed to serve as a starting point to exploring small sample of preservice elementary teachers' functional thinking in regards to figural growing patterns without having previously been assessed in this area. For these reasons, it seemed best to start with the most basic form of linear figural growing patterns to serve as a baseline assessment.

Data collection procedure. The interviews were filmed and the participants' recording sheets were collected upon completion as an additional source of information. All participants completed the three tasks and the interviews lasted between twenty and forty minutes. The variance in length can be attributed to the amount of time it took a participant to complete the tasks and the extent at which the researcher probed the participants thinking to gain understanding. At the beginning of the task the participants were informed that they could use any solving method of their choosing, as the researcher stated "You may write on the paper or use any tools on the table to solve." The tools provided on the table were: snap cubes, counters, extra paper, pencil, pen, highlighter, ruler and sticky notes. The concrete materials (snap cubes and counters) were provided based on generalizing strategies utilized by students and preservice elementary teachers in the research. Richardson et al. (2009) specifically found that the use of concrete materials consistently led to more algebraic thinking than other strategies. Multiple types of writing utensils were also provided based on research where students identified separate parts of the figures on their paper to correlate with how they viewed the figure in their mind (Marthworth, 2012).

Conceptual framework. The framework for this study draws on the connection between how figural growing patterns are generalized and subsequently how those generalizations are justified. While both a generalization and a justification can be analyzed on their own, looking at the relationship between them reveals additional information about each in its own regard. This is because the generalization strategies and reasoning inform the justification created and the justification created is a reflection of the strategies and reasoning used to generalize (Lannin, 2002; Stump, 2011; Driscoll, 1999; Rivera & Becker, 2009; Markworth, 2012).

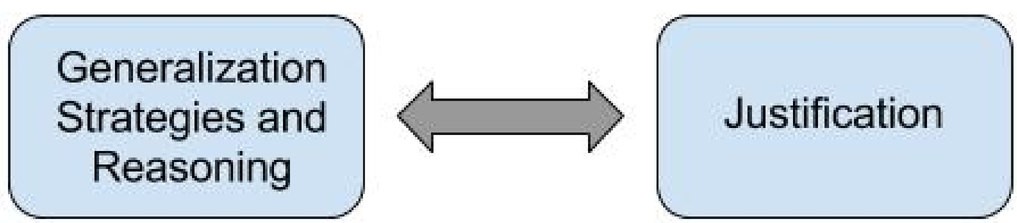


Figure 10: Conceptual Framework

Analysis of Data

During the first phase, the transcripts and videos were repeatedly viewed to establish what strategies the preservice elementary teachers used to generalize and aspects of the pattern they relied on when generalizing. The strategies that participants chose were coded first as either recursive or explicit as defined in the previous chapter. If participants used either strategy it was coded only once per pattern solving attempt. The codes do not reflect how often a strategy was used within one solving attempt. Participants' reasoning was coded figural if they relied on the figure to generalize or numerical if they relied on a numeric representation, in line with the research from the previous chapter. When the participants used either reasoning it was coded in reference to the strategy for which it was observed. It was also noted whether or not an additional figural representation was created by the participant in order to analyze how these representations were used. Mixed approaches by participants were marked with multiple codes. Table 1 describes the codes used.

In the second phase of analyzing, the focus was shifted to justifications made by the participants once a generalization was found. The coding scheme was adapted from Lannin (2005, p. 236) and qualifications outlined by Simon & Blume (2003). These were used to assess each justification. Table 2 describes the codes used. Finally, comparisons were made between the participants first attempt to determine a rule was compared with their second attempt according to the coding schemes in Table 1 and Table 2.

After coding all the interviews, a detailed overview of each individual interview was written. The code table, transcripts, recording sheets and original videos were all used to write each overview. The goal of each overview was to provide a general description of what occurred during the interview, specifically highlighting the data from the interview that was used to generate the codes.

Table 1: Generalizing Strategies

Strategy	Reasoning	Example
DR- Created a representation of the figure	**N**- Used the new representation only to count the total	Participant: “So this first one I just drew 4 tables and put a person in each and found 10 guest could sit at 4 tables.”
	F- Used the new representation to visualize a pattern	Participant: “Whenever I was counting how many people were at the 10 tables I realized I was just counting 10 and 10 again.”
RC-Recursive Generalization: Referenced how the pattern was growing from case to case	**N-** Saw the pattern growing through a numerical representation	Participant: “The second one for 10 tables I just like listed like 4, 5, 6, 7, 8, 9, 10 tables and it went up by 2 for each one and then went all the way to 25. There is probably an easier way to do that, but that was easiest for me.”
	F- Saw the pattern growing through a figural representation	Participant: “Well, I just added 3 because they added 3, like they added 1 to each point (pointing to additional dots in each figure).”
EX- Explicit Generalization: Referenced the relationship between the input and the output	**N**- Saw the relationship through a numerical representation	Participant: “Well, I tried to think of, first I was thinking about multiples that involve the number of tables, like multiplication facts that involve the number of tables and the number of people that sit around the tables.”
	F- Saw the relationships through a figural representation	Participant: “Um, for the second one (referring to the question asking about 10 and 25 tables) I realized there was a pattern, so, like for example, for 3 tables there would be 3 people on each side and then two at the end.”

Table 2: Justification Strategies

Code	Strategy	Examples
N	No Justification: The justification given made no attempt to convince others why the rule worked or was a justification for a different rule.	Participant's Rule: You would go up by two people from each table Participant's Justification: Yeah, because that seems to be the trend for the first 3 tables (pointing to figures) and the first 25 tables (pointing to numerical list she created). So, it just would.
NI	Numerical Induction: The justification given used specific calculation examples of the rule.	Participant's Rule: 3n + 1 Participant's Justification: "Well I tested it out on 4 different ones, 3 of which were pictured (points to cases 1-3 that were given) and one the one that was in number 1 (reference to case 4) and it worked for all those."
FI	Figure Induction: The justification given connected specific examples back to the figure.	Participant's Rule: (case # x 2) + (case # + 1) Participant's Justification: "Umm, because, the number like, the dots on the outside… there are always two groups that are the same number as the case number. So, I always know, like if I did case number 30, there would be 30 dots on the outside of this group going up on each side and there would be 31 going up and it worked for each of the other cases too."
DE	Deduction: The justification was an argument that highlighted the general nature of the pattern and avoided use of any particulars.	Participant's Rule: To get to the next case you add 2 more dots horizontal and 1 more dot going vertical Participant's Justification: Yes, because if you are continuing to add 1 to the side and 1 more up then you're going to have to add 2 because there's 1 on each side and then 1 more above. So, every horizontal line, if you're putting the same amount going out from the central point, then you're going to add 1 here and add 1 here (points to end dots in figure 3). And then 1 more going up, you're going to add 1 here (points to top dot in figure 3).

CHAPTER 4

FINDINGS

This chapter describes the findings that came from the data analysis. The sources of data were the video recorded clinical interviews and the recording sheets, as stated in Chapter 3. Summaries of each participant's work on each task are included in Appendix C. The chapter will begin with a description of the generalizing strategies for all participants' first and second attempts of Figural Pattern #1. This is followed by the generalizing strategies of all the participants for the first and second attempts of Figural Pattern #2. Finally, the justification strategies for all attempts of Figural Pattern #1 and then Figural Pattern #2 will be described.

Generalizing Strategies

Table 3 and Table 4 show the aggregate results of the interviews by how the participants' work was coded. Two additional codes were used to include additional information that was relevant when considering the results of the interviews. A code was used to indicate if a participant did not create a rule when generalizing on their own. If a prompt by the researcher, in the form of a question, encouraged the participant to create a rule, the code (*) is used. If a prompt by the researcher in the form of indication of a calculation error allowed the participant to formalize a rule the code (^) is used. The code (X) indicates that a participant used a recursive or explicit strategy, but it was unclear whether they used figural or numerical reasoning.

Table 3: Participants' Generalizing Strategies for Figural Pattern #1

	First Rule				*Second Rule*			
	DR	**RC**	**EX**	**Rule**	**DR**	**RC**	**EX**	**Rule**
PST-1	N, F		F	EX				
PST-2	N, F		F	EX				
PST-3	N	N		RC			F	EX*
PST-4		N		RC			N, F	EX
PST-5	N	X	X	EX				
PST-6		F	F	EX			F	EX
PST-7	N	N		RC			N	EX
PST-8			F	EX				

Table 4: Participants' Generalizing Strategies for Figural Pattern #2

	First Rule				*Second Rule*			
	DR	**RC**	**EX**	**Rule**	**DR**	**RC**	**EX**	**Rule**
PST-1	F	N,F	N,F	EX	F		F	EX
PST-2	F		F	EX		F	N	EX
PST-3		N	X	EX		F	N	EX
PST-4		F		RC^			N	EX^
PST-5	N	F		RC			N	
PST-6		F	F	EX				
PST-7	N	F		RC			N	EX
PST-8	F	F	F	EX	X	F		RC

Figural pattern #1. The following is an overview of how participants solved Figural Pattern #1.

Use of created figural representations. In generalizing Figural Pattern #1 for the first time, five participants drew out 4 tables to find the total number of guests. PST-1 chose to draw out only 4 tables, noting “I could have done 10 but I didn’t want to draw out 25 tables.” Not wanting to draw out an excessive number of tables was the main reason participants did not continue with this strategy. PST-1 looked back at the figures for another way to generalize. While she does not fully draw out 10 and 25 tables, she uses the outline of a table to demonstrate the explicit pattern she noticed through the figures (see Figure 11).

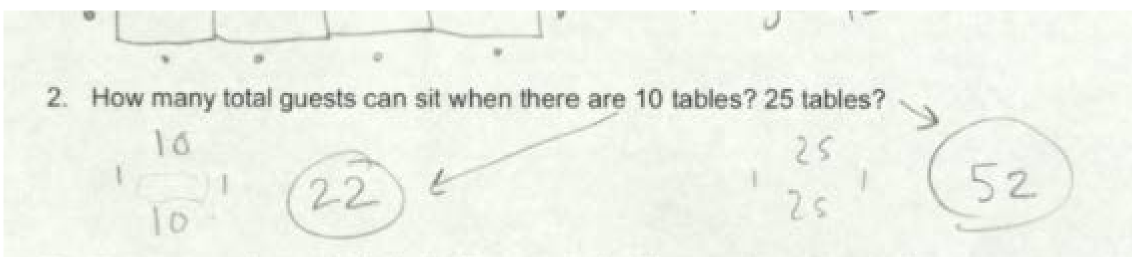


Figure 11: PST-1’s Representation of 10 and 25 Tables

PST-3 and PST-5 used the totals from tables 1-4 to create a numerical representation. PST-2 and PST-7 decided to draw 10 tables as well. PST-2 began by constructing a representation of the 4 tables out of unifix cubes and then counted the spaces around the cubes to find the total number of guests that could sit at 4 tables. Then, she drew out 10 tables and counted around the drawing to find the total. PST-2 noticed an explicit pattern as she was counting the total number of guests and switched to this strategy. PST-7 used her findings to begin looking for an explicit rule within her numerical representation.

Strategies and reasoning used when generalizing. When considering all generalizing strategies, five participants used a recursive strategy at some point during Figural Pattern #1. PST-3, PST-4 and PST-7 ultimately used this strategy to create a recursive rule. PST-4 noticed that the total number of guests “just kept going up by 2”, so she added 2 repeatedly to find how many guests would sit at 4 and 10 tables. Instead of continuing this strategy, she switched to a multiplicative recursive strategy to make generalizing more efficient. She started with an initial number of tables and counted how many tables were in between the initial number of tables and the final number of tables. She then multiplied this value by 2, since she saw there would be 2 guests at each table, to calculate how many guests were added. She ultimately created the rule “2 times the number of added tables added to the start point”. While her strategy was more efficient, it was not considered explicit because it neglected the direct relationship between the number of guests and the number of tables. PST-3 and PST-7 each created a written recursive rule that acknowledged every time a table was added, the number of guests increased by 2. Both participants recognized that there was a more efficient way to generalize, though PST-3 was initially satisfied with her rule, explaining “There is probably an easier way to do that, but that was easiest for me.” On the other hand, PST-7 continued to look for another pattern in the numbers even after she recorded her first rule. It was clear that she did not think this rule was sufficient. She later referred to what she wrote for her first rule as “kind of a sentence of just getting my thoughts onto paper.”

PST-3 did not originally advance past a recursive strategy, but during the interview when she was explaining her rule, she stated that she would not want to use it to find the number of seats at 1,000 tables. This followed in line with why the other participants stated that they advanced from a recursive strategy to an explicit one. PST-5 created a list up to 10 tables, by adding 2 each time. After going through the list PST-5 came to the conclusion that “you just double the number and add 2; that would be the people on the end” and when prompted, she was able to connect all parts of her rule back to the figure. However, it was unclear whether she relied on the numeric list she generated or the figures to formulate her rule.

While PST-6 noticed that the addition of a table added 2 seats, she did not attempt to use this to generate a numerical representation, but instead visualized how the number of guests at each table related to the number of tables. To find the total number of guests at 4 tables, she visualized adding an additional table to the figure of case 3. She imagined 4 dots on each side of the table and 2 on the end and recorded this as $4 + 4 + 2 = 10$. She followed the same line of thinking to find the total number of seats in case 10 and 25, creating the equations $10 + 10 + 2 = 22$ and $25 + 25 + 2 = 52$. From this thinking she generated the rule $2n + 2$, with n representing the number of tables. This quick shift from recursive to explicit strategies through figural reasoning led her to create and use an explicit rule earlier than PST-7 and PST-5.

There were three participants that did not use recursive strategies and instead relied on explicit generalization to find a rule. Both PST-1 and PST-2 drew additional figures to help them find the total number of guests. PST-1, upon seeing she needed to find the total for larger tables, looked back at the figures to find a pattern to solve for the total number of guests at 10 and 25 tables. She “realized there was a pattern, so like for example, for 3 tables there would be 3 people on each side and then 2 at the end.” She continued, “I knew the pattern was the number of tables; there would be that many people on each side and then two extra people on the end and that's how I got the rule, number of tables times 2 plus 2.” PST-2 came to the same conclusion as she was constructing 10 tables, noticing she “was just counting 10 and 10 again (pointing to alternate sides of the table she drew on her paper)”. They used these observations to create an explicit rule. Instead of drawing out additional tables, PST-8 visualized what the additional tables would look like in her mind. This allowed her to also see the explicit relationship immediately. Almost right she wrote $4 + 4 + 2 = 10$ in response to the first question on the recording sheet. She noticed that each table always had 2 dots on the end and the same number of dots across the top and bottom, which corresponded directly with the case number. She created similar equations for 10 and 25 tables and used this idea to generate her rule of $2n + 2$ = total # of people at tables, with n = # of tables.

Additional rule created. Four participants created an additional rule for this pattern. All of the additional rules created were explicit. PST-4 and PST-7 both created a rule by looking for a number pattern within the table of values they generated. PST-7 looked at the relationship between the case number and the total number of dots to find a number pattern that would help her generalize. She found success with “times 2 plus 2” for case 4. She tried this new strategy to make sure it would also work with the given cases. She then tried 10 x 2 + 2 and got 22, which was the same answer she got when drawing out 10 tables. When asked to connect her rule back to the figure she asserted that the plus 2 came from the 2 guests on the end of the table but could not explain the times 2 part of her rule. She stated “I’m not exactly sure where that comes in, but I just know the way I found it was by looking at the 4, 6, 8 (pointing to numbers she wrote above tables), looking at the pattern and that’s kind of how I found that number.” PST-4 attempted to use the “multiplication facts that involve the number of tables and the number of people that sit around the tables.” She was looking for a direct relationship between the input and output, but could not find a familiar number pattern. Then, she noticed that if she accounted for the 2 seats at the end with plus 2 that she could just double the number of tables “because each table adds 2”. This thinking was used to create the rule 2(n)+2 for n = # of tables.

PST-3 and PST-6 used only the figures to generate their explicit rule. PST-3 originally utilized number strategies to create a recursive rule, but when pushed to solve for 1,000 tables she studied the figure and recognized that for “2 tables you have 2 on each top and bottom (pointing to figure), 3 tables you have 3 on each top and bottom (pointing to figure) plus 2… so 1,000 would be 1,000 (motioning to top of a table) and 1,000 (motioning to bottom of a table) and then 2 (motioning to outside of a table) so 2,002." PST-6 was the only participant that created two explicit rules for this pattern. When looking at the figures again, she saw that "each side of the table fits 1 person and if you just looked at the tables knowing that each side could hold potentially 1 person, so each table has the potential to hold 4 people and so if you have a table, multiply it for the 4 people that could sit there." From this thinking, she generated 4n. Then she noticed that "each of those lines represents 2 seats lost (pointing to lines on inside of tables that show where 2 tables have been put together). One seat on this table, one seat on this table (pointing to either side of a middle line) ... So, I thought, if there's 3 tables (points to figure 3) there's going to be 2 lines (pointing to lines on inside of tables), 4 tables there's 3 lines or like 3 meeting points of those tables." She used this thinking to generate (n-1)*2 which is subtracted from her original 4n.

Figural pattern #2. The following is an overview of how participants solved Figural Pattern #2.

Use of created figural representations. For Figural Pattern #2, five participants chose to draw additional cases in order to solve for specific case values. PST-5 and PST-7 drew out case 4 to find the total number of dots in the case. PST-1 drew case 4 to verify the pattern she saw happening in the first 3 figures. PST-2 and PST-8 originally drew out the figure based on the recursive pattern that they saw, but both recognized a new way to view the pattern through their drawings and shifted from recursive to explicit strategies.

Strategies and reasoning used when generalizing. On the first generalizing attempts, seven of the participants used a recursive strategy for Figural Pattern #2. Two of the seven mentioned recognizing the consistent growth in the numerical representations they generated. PST-3 was the only participant who utilized just a numerical strategy to find a recursive rule. She noticed that the total number of dots in the figures was increasing by 3 and used this information to determine the total for figure 4. When prompted if she saw the pattern through the numbers or the figures, she stated the numbers, but added she now recognized 1 dot being added to each end of the figures. When she went to find the total number of dots for case 10 and 25, instead of continuing with this rule she constructed the equation, (10x3)+1=31, for the total number in case 10 and (25x3)+1=76 for case 25. It is unclear from the interview how she generated this equation, but later in the interview she was unable to successfully connect her rule back to the figures:

> Researcher: Do you see this rule (pointing to (c x 3) + 1) in the pictures?
>
> Participant: Umm… yeah, because the case here is 1 (pointing to case 1) multiplying it by 3, so that is these 3 (pointing to the 3 outside dots) and then 1 in the middle just stays the same. Case 2, multiplying it by 3.... That’d be the plus 1 (pointing to 1 dot in the middle of case 2) ... hmm, actually maybe I don’t see it.

Researcher: It's fine, I'm just curious.

Participant: Yeah, I don't think I see it.

PST-1 was the only participant to discuss seeing the pattern recursively through both a numerical and figural representation. She originally counted and recorded the total number of dots in each case, noticing that the total in each case increased by 3. While she was satisfied that her finding answers question 1, she turned her attention to what was happening in the figures. She noticed the total number of dots in the top column was the same as the case number. This left her to create a list continually adding 2 to find out the number of dots across the bottom row. She used the list that she generated to solve for the number of dots on the bottom row of case 10 and 25. Finding that there were 21 dots on the bottom of case 10 and 51 dots on the bottom of case 25, she concluded that the bottom row was 2 times the case number plus 1 for case 10 and 25. She compiled this knowledge into a written rule of case # + case #*2 + 1.

The five other participants, who reasoned recursively, spoke only of how they saw the pattern growing in the figures. All but PST- 5 visualized the pattern by observing that one dot was being added to each end of the figure and all added 3 to the total number of dots in case 3 to find the total number of dots for case 4. PST-5 noticed "on the bottom it's adding 2 each time and whatever the case number is that's how many you are adding to the top". She explained how she saw this observation happening for cases 1 - 3, and from these observations she began a tabular representation that reflected this pattern (see Figure 12).

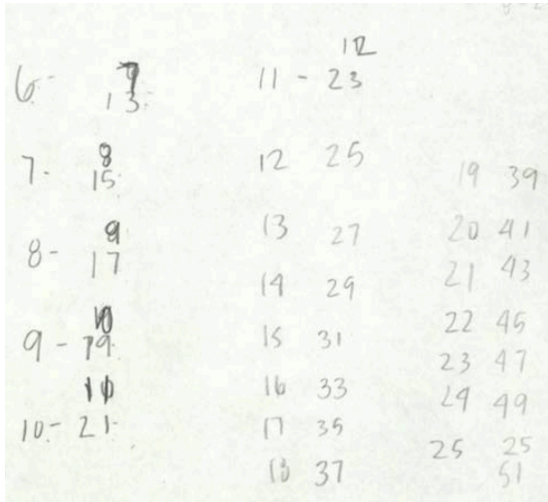


Figure 12: PST-5’s Tabular Representation of Figural Pattern #2

PST-4 also generated a tabular list to find the total number of dots in case 10 and case 25. Both participants made addition mistakes while completing these tables that they were unaware of. They were convinced that there was, in the words of PST-5, “an easier way to get from the case number to whatever it is”, but admitted that they could not find it. Because of this, both were hesitant to record their recursive generalization as a rule. PST-5 eventually did, but PST-4 did not. PST-4 tried to employ the same multiplicative technique she used in Figural Pattern #1, but was unable to find a consistent rule that provided the same answer as the values in her list (some of which were incorrectly added). She was visibly puzzled and the following excerpt highlights her confusion:

> Researcher: I know you attempted to use the rule that you did here (referring to rule in task 1), do you think that something like that should work and you just didn't find it or do you think this (referring to rule in task 1) wouldn't work?
>
> Participant: I think that something like that should work, maybe not even the same exact idea, but I think that there is a rule you could create to do this that would work but I don't... I just got like frustrated so that's why I was like whatever but, I think there is one that would work… because you're doing the same thing every time and it doesn't change so… I don't know…

It is only when her calculation error is pointed out that she can create a multiplicative recursive rule, which she recorded as her first rule.

PST-7 realized that "each case grows by 1 dot on top, to the right and to the left" and recorded this observation under the given figures. She used this finding to create case 4 and count the total number of dots to answer question 1. Although she eventually recorded this finding as a rule, she was hesitant to use it to solve for the total number of dots in case 10 and 25. She stated, "I knew that wasn't substantial enough, in order to find patterns without listing it". She eventually did create a list to find the total number of dots in case 10 and 25, but continued looking for a number pattern that will help her generate a "better" rule.

While PST-6 and PST-8 originally visualized the figure growing recursively, after deciding that they needed a more effective way of generalizing, they returned to the figure and found a different way to visualize it. Both noted how there was a central dot and then three sets of dots that moved outward from the central dot and represented in the figures on the recording sheet (see Figure 13 and 14). They used this reasoning to generate the total number of dots in case 10 and 25 and form their general rule of 3n+1.

PST-2 was the only participant who initially visualized the figure in an explicit way. PST-2 saw the dots in two groups: a column and a row with two distinct parts on either side of the column, noting that the parts of the row were made up of the same number of dots as the case number and the number of dots in the column was one more than the case number. She drew out case 4 using this strategy and counted to find a total of 13 dots. She then does case 10 in her head thinking that "there would be 10 dots on each side, so that's 20, plus 11 (motioning upwards to demonstrate middle column) is 31." The same reasoning was followed for the case 25. She used this reasoning to write out the rule "add the case number to itself, or multiply by 2, then add the case number plus one to the previous answer." She was also the only participant whose initial generalizing strategy for both patterns involved visualizing the figure explicitly.

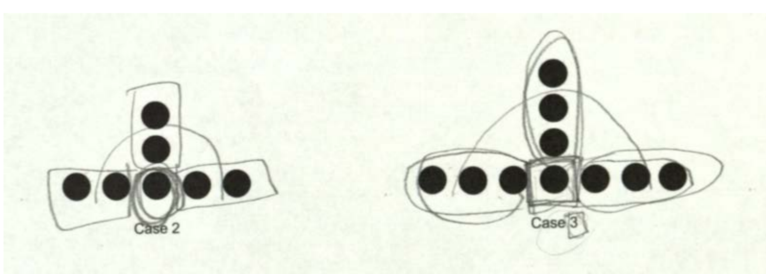


Figure 13: PST-6's Use of the Figural Representation of Figural Pattern #2

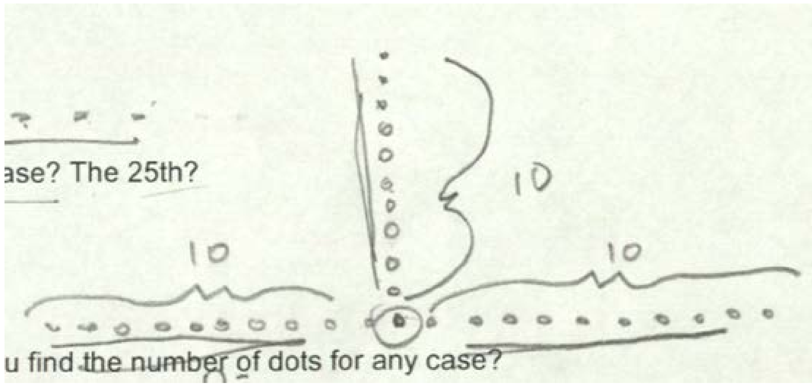


Figure 14: PST-8's Figural Representation of Figural Pattern #2

Additional rule created. Seven participants attempted to find a second way to generalize Figural Pattern #2. All but one were successful, thought PST-4 was only able to do so after her calculation error was pointed out. PST-5 attempted to find a rule for Figural Pattern #2 by searching for a number pattern in her tabular representation, but found that "some of them are times 3 plus 1 and some are times 3 plus 2". She could not find a rule that fit all the values she has generated and she failed to realize that this was due to multiple mistakes within her tabular representation. PST-7 used a similar approach, but was able to correctly create a list of outputs from the first figure to the twenty-fifth and generate a rule. She noticed a relationship between 10 and 31 recalling "Once I saw 31, I wanted to say to myself, okay I know that 10 times 3 is 30 and then I would only have to add 1 more. And then I checked, does that work for all those up here (pointing to generated list)." She used this finding to create the generalized rule to multiply the number of the case by 3 and add 1.

When prompted to connect her rule back to the figure, she summarized by referencing to the figures given to show that the 3 represented the outer ring of new dots. She explained that the n came from the case number and the addition of 1 came from the base dot.

PST-3 and PST-8 created a recursive rule, both having originally created an explicit rule. Returning to the pattern they noticed in the figure when generalizing the first time, both chose to formalize it as a second rule. PST-3 noted that her new rule to add a circle to each side was time consuming. PST-8 was convinced there was a way to make her new rule of adding 2 dots to the bottom and 1 dot to the top explicit, but she was not sure how. The other four participants created explicit rules, relying on a mixture of figural and numerical representations. PST-2 created a t-chart, noting that each time 3 is being added and connected this observation back to what was happening in the figures. However, she does not see this as a rule and continues looking for an equation. She used the values from the table to generate a new explicit rule, which she could not connect back to the figure. PST-4 and PST-7 both found an explicit relationship in the numerical representations they generated. They both specifically noticed the number pattern in case 10. PST-7 recalled “Once I saw 31, I wanted to say to myself; okay I know that 10 times 3 is 30 and then I would only have to add 1 more.” PST-1 redrew case 1-3 and shaded in the dots to indicate how she was grouping the figure in her mind. She noticed that the parts of the row were “two groups of the same number, of the case number”. She showed this thinking through her representation of the figure (see Figure 15). She used the generalization of those 3 cases to create her rule.

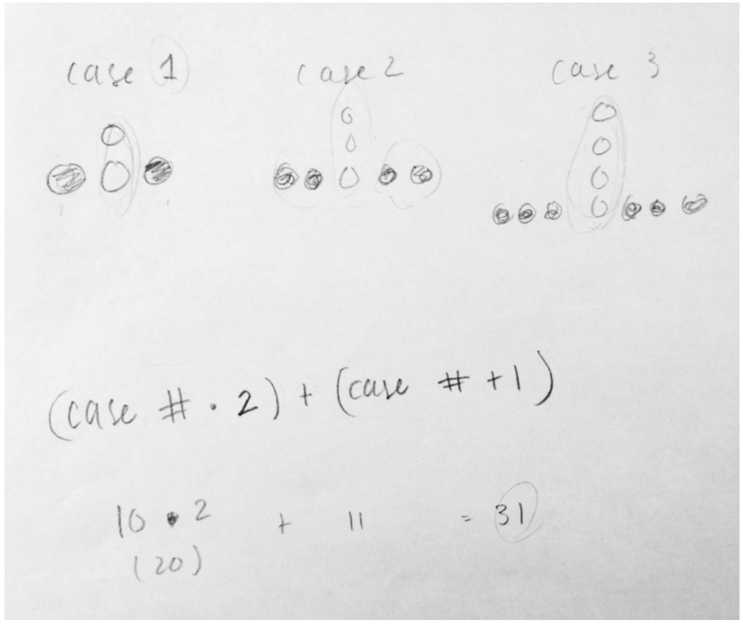


Figure 15: PST-1’s Representation of Figural Pattern #2

Justifications for Generalizations

Table 5 shows the aggregate results by how they were coded. Four additional codes were added to include additional information that was relevant when considering the results of the interviews. In order to indicate what type of justification a participant provided, they were coded in the correlating column with an (R), indicating that the rule they were justifying was recursive, or with an (E), indicating the rule they were justifying was explicit. For validity purposes (*) was used to code justifications that students gave after they stated uncertainty about whether or not their rule was always true and (^) to indicate if the justification was unclear based on the interview.

Table 5: Participants' Justification Strategies

Pattern:	*#1*				*#2*				*#1 Again*				*#2 Again*			
	N	**NI**	**FI**	**D**	**N**	**NI**	**FI**	**D**	**N**	**NI**	**FI**	**D**	**N**	**NI**	**FI**	**D**
PST-1			E	E		E		E							E	E
PST-2			E			E								E		
PST-3	R				E				E						R	
PST-4				R*	R				E					E^		
PST-5			E			R										
PST-6		E		E		E				E						
PST-7	R						R	R		E			E*			
PST-8				E	E											R

Figural pattern #1. For Figural Pattern #1, PST-3 did not attempt to explain convincingly why her rule would work, she just stated "that seems to be the trend for the first 3 tables and the first 25 tables, so… it'd be the same for if you wanted like 1,000." PST-7 was asked to justify her first rule, but explained "I guess I wrote that because of the 4, 6, 8 thing I saw, but I don't know that the wording is best." She switches the conversation to her second rule and does not make another attempt justify her first rule.

The other six participants were able to provide some type of proof for their rule. PST-6 began her justification by calculating specific examples of her equation, but then concluded with a deductive explanation of how the general aspects of the figure connected back to the equation. PST-2 and PST-5 used only figural induction to prove their rule, which were both some word form of the explicit rule 2n + 2. They each gave a singular example to support their justification; one demonstrated for 5 tables while the other participant demonstrated for 50 tables. Their examples referenced the general aspects of the figure, as PST-2 explained "you can only have one person at each table so you know that when they are all lined up 50 people can be on one side and 50 people can be on the other", but their explanations were specific only to the case they provided. PST-1 used a mixture of figural induction and deduction. After creating an equation form of explicit rule 2n + 2, she justified her rule by first explaining how it applied to the figural representations that were given. She went on to make statements that highlighted the general, instead of the particular; and explained "So there will always be 2 people on the end, since the tables are pushed together. And there will always be one spot on each table, like considering the number of tables that there are (circles table number with pencil)". This second explanation was an appropriate and valid justification for her rule.

Finally, two participants used only deduction to justify their rule. PST-8 created the explicit rule 2n + 2 and explained that her rule would work every time "as long as you are still keeping your people at the end of the table and sitting 2 people on each side". PST-4, who created the rule "2 times the number of added tables added to the start point," was at first unsure about stating that her rule would always work, saying "I get scared saying that because I just always feel like there is a loophole in math rules… so yes, but maybe not." She continued to ponder her thoughts out loud and eventually came to the conclusion that "each side is going to fit, like each table adds 2 people and so yeah it would."

Four participants created two rules for this pattern. PST-4, who previously created a recursive rule and justified it deductively, creates an explicit rule, 2(n)+2 but is unable to produce a justification of why it would work. She stated the only way to really prove it was to draw out the particular case you were trying to prove. PST-3 did not use any examples to explain her new rule of "multiply the number of tables by 2 and then add 2 more." She simply stated that it worked for the three cases she tried, but she was unsure how to explain it. PST-7's justification strategy was coded numerical induction after she stated her rule would work "because 25 times 2 is 50 plus 2, so we are doubling the number of tables and then we're adding 2 to get that 52 number." PST-6, who successfully produced a deductive argument in her first rule, justifies her second rule would work for all cases because she tested on a few cases. No participants used figural induction or deduction when justifying this pattern a second time.

Figural pattern #2. Three participants' responses were coded 'No Justification' when generalizing this pattern for the first time. PST-8 and PST-3 used the recursive aspects of the pattern in an attempt to justify their explicit rule. While they were correct in stating things like "you're just adding 1 more dot on the left, right and top" this explanation did not relate back to the relationship between the case number and the total number of dots and therefore failed to support their generated rules. PST-4, who created the rule "3 times the amount between cases added to the start point," felt that her rule was not developed enough to work every time. She stated that in order for her rule to work every time she would need to form a "more mathematical rule." When pressed to explain what she meant by mathematical, she explained "I don't know, easily accessible to anyone that read it and I think this one (pointing to her rule of 2n+ 2 for task 1) would work every time essentially had I done that."

Four participants used numerical induction to justify their rule. PST-6 created the rule of 3n + 1 and was sure it would always work, because she tested her rule for the first 4 cases. When asked to justify her rule, PST-2 restated how she generalized for case 10 and case 25 and concluded her justification by saying "I guess that would be a rule for any of the case numbers." In a similar attempt, PST-5 referenced the figures she had drawn for case 4 and case 5, but merely restated the numbers she had generated and assumed the rule would work "if you do that every time." PST-1 used a combination of numerical induction and deduction. To justify the first part of her rule, 2 plus 1, she gave calculation examples from case 10 and case 25. She concluded her justification saying "And then the number of dots on top are just representative of or like equal the same number as the case number."

PST-7 stated "I always know that it's growing by 3" and tied specific examples of her recursive rule back to the figure in her justification, pointing to the three dots that were added in cases 1-3. No participants were successful in forming a deductive justification on their first attempt at generalizing Figural Pattern #2.

Six participants found an alternate rule for Figural Pattern #2. PST-7 stated that she thought her rule would work, but she did not know how to check. PST-2 gave the calculations of case 25 and 30 as proof and added “so, yeah, I guess it does work.” PST-4 double checked to make sure that case 14 would work, but was not asked to explain why. It is unclear if she saw this as a complete justification, therefore this result is excluded from the analysis. PST-1 first checked to see if the number pattern she generated from case 10 and case 25 worked for cases 1-3. She then started her justification explaining that “the dots on the outside… there are always 2 groups that are the same number as the case number” and then showed why the rule would work for case 30. She concluded her explanation saying “and it worked for each of the other cases too.” She only tied part of her rule back to the figure and then proceeded to use a few cases to justify all cases. PST-3 explained that the figures given were always adding 1 one each side, but did not generalize this statement to include all cases. PST-8 was the only participant to create a deductive argument for Figural Pattern #2. She generated a recursive rule and gave a generic example of how the figure would grow.

CHAPTER 5

DISCUSSION

The purpose of this study was to examine how preservice elementary teachers generalize and justify figural growing patterns presented to them in a clinical interview. The rationale for this study draws from several points highlighted in the literature surrounding the need to explore what functional thinking abilities preservice elementary teachers possess in regards to figural growing patterns.

This chapter is divided into five sections: limitations, addressing the research questions, implications, future research and conclusion.

Limitations

First, since this study is case based, the significance of the aggregate results and findings are limited. Generalizations from this study to a different or larger population of preservice elementary teachers cannot be made, but can only be used for comparison among similar studies. Second, the conclusions of the participants' knowledge in regards to generalizing figural growing patterns are only based on one interview. These findings can only be considered a piece in a larger picture of each participant's functional thinking and justification abilities in regards to figural growing patterns. Additional work with these patterns would provide a more comprehensive view of the preservice elementary teachers' knowledge.

Addressing the Research Questions

This section will address the research questions from this study**.**

Research question #1. What strategies and reasoning do preservice elementary teachers use to generalize figural growing patterns?

Recursive versus explicit strategies. Over all the attempts to generalize both patterns, of which there were twenty-seven, there were thirty-five different strategies coded. Explicit strategies were coded twenty times and recursive strategies were coded fifteen times. The use of recursive strategies by many was no surprise due to the fact that the first question in each task asked for the total people or dots in the next figure. While all participants were coded for recursive strategies at some point, it was clear that using an explicit strategy was a priority. Participants either started generalizing with an explicit strategy or intentionally re-adjusted their strategy after initially thinking about the pattern recursively. Only twice did a participant revert from using an explicit strategy back to a recursive strategy and this occurred only when they were asked by the researcher to find an additional rule. Focusing on individual preferences, all participants were coded for both strategies at least once. Five out of the eight participants were coded for recursive and explicit strategies equally. The remaining three participants were coded more often for explicit strategies, revealing that they did not always consider the recursive aspects of the patterns. The results of this study suggest that both recursive and explicit strategies play a role in generalizing, but explicit strategies were ultimately favored by the participants.

In line with the generalizing strategies, the rules generated by the participants in this study were overwhelmingly explicit, accounting for nineteen out of twenty-six total rules. All eight participants were able to generate at least one explicit rule during the clinical interview. Since the ultimate goal of the figural growing patterns in this context was to display functional thinking, this result was promising. It displayed the participants' abilities to find and represent functional relationships. These results are similar to Richardson et al. (2009)'s study, where twenty-three out of twenty-five preservice elementary teachers were able to produce explicit rules at some point while generalizing multiple patterns. While all participants in this study were all able to create at least one explicit rule, it should be noted that there were many different ways that participants found and represented their findings. One difference was the type of reasoning they used to generalize.

Numerical versus figural reasoning. While the participants were generalizing, they were coded for numerical reasoning twenty times and the figural reasoning twenty-eight times. Six participants relied on a mixture of both forms of reasoning. Out of those six, two participants were coded more often for figural reasoning and four were coded more often for numerical reasoning. The remaining two participants were coded only for figural reasoning. This shows that half of the participants favored numerical reasoning and half favored figural reasoning. These findings are different than the results from the Rivera & Becker (2009)'s study, who found that in all their preservice teacher participants preferred to use numerical reasoning more than figural reasoning.

Overall, there were no obvious correlations between the codes for reasoning and for generalizing strategies. There were, however, a few interesting findings when partitioning the data into certain subsets. First, at some point when generalizing Figural Pattern #2, all of the participants used a recursive strategy they saw through figural reasoning. As they continued to generalize, three were successful at re-visualizing the figures to find an explicit rule, but five chose a different route. Instead trying to find a different way to view the figures, the participants switched to numerical reasoning and manipulated numbers to look for an explicit rule. This revealed that while figural reasoning was displayed by all eight participants during the recursive strategy it did not always translate to the explicit strategy. This result is discussed more in depth in the next section.

Second, when creating an explicit rule for Figural Pattern #1, for which there were nine attempts, six participants were coded for figural reasoning and one for numerical reasoning. Of the remaining two participants, one was coded for both figural and numerical reasoning and one of the participant's reasoning strategies was unclear. Figural reasoning was clearly preferred for explicit strategies in Figural Pattern #1. When solving for an explicit rule in Figural Pattern #2, four participants were coded for figural reasoning and five were coded for numerical reasoning. There was also one attempt that included a mixture of reasoning and one where reasoning was unclear. There was not a dominant reasoning that was coded for explicit strategies in Figural Pattern #2 pattern. While it cannot be stated with certainty why this difference in reasoning preference occurred between Figural Pattern #1 and Figural Pattern #2, this could be attributed to the actual lay out of the figural designs. Figural Pattern #1 was displayed in a way that drew attention to the dots on the end, which directly represented the constant in the explicit rule. All eight participants recognized that those dots were accounted for by +2 in the rule 2n+2. Figural Pattern #1 was also within a real world context, which made it familiar and may also explain why the preservice teachers were more likely to reason with the figures. In contrast, Figural Pattern #2 was not embedded with a familiar context. Furthermore, the constant in Figural Pattern #2 for the rule 3n+1 was less obvious and two participants were unable to explain how the +1 of their rule was connected to the figure.

Transitioning from recursive to explicit: figural versus numerical reasoning.

Markworth (2012) highlights the importance of being able to flexibility visualize the pattern in more than one way. This helps students differentiate between useful and unhelpful strategies and can assist in the transition from recursive to explicit generalization. In Figural Pattern #1, four participants began with recursive strategies and attempted to transition to an explicit rule. Two participants re-visualized the figure to find the explicit rule. One participant relied on her numerical representations and it was unclear whether the last participant used figural or numerical reasoning to make this transition. All participants were successful at creating an explicit rule, regardless of the type of reasoning they utilized. However, this was not the case in Figural Pattern #2.

In Figural Pattern #2, there were nine attempts to transition from recursive to explicit reasoning. The three participants, who all returned to the figure to look for a different relationship, were successful at finding an explicit relationship. The other five participants' did not re-visualization the figure, but instead relied on solely numerical representations generated from their recursive view of the pattern. Of these five attempts, two participants, PST-4 and PST-5, were unable to create an explicit rule. Both created a tabular representation to represent how the figures were growing, but made mistakes in their calculations without realizing it. Therefore, when they relied only on this to generalize, they were unable to find an explicit rule that worked for all the values they had generated. They both acknowledged that there should be a direct relationship between the input and the output, but they admitted that they could not find it.

This highlights potential drawbacks of relying on recursive strategies to create a strictly numerical representation followed by an attempt to explicitly generalize using only this representation. This same progression also caused issues for the preservice teachers in Becker & Rivera (2009). In PST-5's case, it was only when she went back and examined the figure she created for case 5 that she realized her recorded total was incorrect. It seems referring back to the figure itself can reveal errors that might go unnoticed when relying only on the numerical representations. Unfortunately, there were also multiple other calculation errors in her table that PST-5 failed to notice. Because recursive strategies do not address the relationship between the independent and dependent variables and numerical reasoning can mask it, there was no way for her to verify her calculated total with each figure total, unless she drew each figure out and counted to confirm the total.

This is connected to a similar finding in Richardson et al. (2009) who identified an important transition that the preservice elementary teachers made when working with the final figural growing pattern in their study. The researchers were able to observe how the preservice elementary teachers transitioned from thinking about the figures concretely to thinking about them abstractly. This shift happens when “preservice teachers kept the features of the pattern blocks trains in their minds and no longer relied primarily on a visual, concrete representation of the train” (pg.195, Richardson et al., 2009). This went hand and hand with noticing the explicit aspects of the figure, over the recursive aspects. The results of this study suggest that using recursive strategies to generate a numerical representation and following that with numerical reasoning in an attempt to find an explicit rule can prevent this transition.

PST-4 was initially unable to create an explicit rule for Figural Pattern #2 that fit her tabular representation. She attempted to use a recursive strategy that she had successfully used in for Figural Pattern #1, but was unsuccessful in this attempt because she had miscounted one of the cases. It was only when the researcher pointed out the error that she was able to find an explicit relationship. While it cannot be stated why one tabular representation resulted in errors and the other did not, the tables have two distinct differences. Her table for Figural Pattern #1 ranges from case 4 to 10, while the table from

Figural Pattern #2 ranges from case 4 to 25. She attempted this strategy for much larger cases in Figural Pattern #2 than she did in Figural Pattern #1. It should also be noted that the recursive difference in Figural Pattern #1 was 2, making the dependent values consecutive even numbers. The recursive difference in Figural Pattern #2 was 3, which may be less obvious pattern. It seems PST-4 found more success with the recursive number strategies when the outputs were smaller and followed an easily recognizable number pattern.

Of all the attempts to transition from recursive to explicit strategies that were coded, all of the participants who relied on the figural aspects of the pattern were successful, while two of the participants who relied on numerical aspects were unable to find an explicit rule. This supports the research that numerical representations alone often do not suffice and that figural representations allow for seeing what part of the pattern is growing and what pattern is staying the same, which can be a necessary tool for transitioning between recursive and explicit strategies (Lannin, 2002; Markworth, 2012; Rivera & Becker, 2009).

Research question #2. How do preservice elementary teachers justify their rules?

Ways of justifying. Before even providing a justification, not all the participants were confident that their rule could be applied to any case. Two out of the eight participants stated this uncertainty for one of their rules. PST-3's comments about her first rule revealed a general distrust of stating absolutes in mathematics because "there's always a loophole." PST-7 was uncomfortable with her rule being questioned, revealing that she was not quite certain if she could validate it for all cases. While their results were included with the rest of the justifications, these comments revealed concerns before these participants even attempted to form a justification.

Out of the twenty-five justifications that were given by the preservice elementary teachers, eleven did not involve the figure, while fourteen did. Of the fourteen justifications that involved the figure, eight used deductive reasoning at some point, but only five of those were used to form a complete deductive argument. And out of these five deductive justifications, only three were for explicit rules, all of which were generated for Figural Pattern #1. The two recursive deductive arguments were for Figural Pattern #2. This means only five of the twenty-five justifications involved the participant sharing a complete generalized description of how the figure was evolving and how this connected back to their rule. Interestingly, the full interviews revealed that this was not due to the participants' inability to do so. For example, when PST-2 was generalizing Figural Pattern #2, she explained how she saw the relationship between the case number and the total number of dots in a case through the figures that were given. She even dictated this relationship in her drawing of case 10. However, when asked to justify her rule she was satisfied with giving two examples to prove her rule was true for all cases. Similarly, PST-6, who was the only participate to create two explicit rules for Figural Pattern #1, generated her second explicit rule directly from a visualization of the figures. However, when asked to justify she did not see the need to employ this reasoning and instead checked her equation with a few cases and was satisfied with her explanation. Failing to provide a general description of how the figure was growing as their justification, when it was clear that participants had the ability to do this, happened over and over again. At some point during the interview for eighteen of the twenty-five rules that were coded for justifications, participants were able to connect all the parts of their rule back to what was happening in the figure, yet only five times was this seen as a necessary part of their proof. This is in line with the research reviewed in Chapter 2 that found empirical evidence as proof was the most common type of justification for teachers and students (Becker & Rivera, 2009; Lannin, 2002; Richardson et al., 2009; Lannin, 2005).

At the outset of the study, there was an anticipation of a clear connection between the students who generalized predominantly by using the figures and their ability to create a valid justification. This was reflected in studies that showed participants who used a figural reasoning approach were more likely to create a valid justification (Lannin, 2005; Warren & Cooper, 2008; Richardson et al., 2009). While the five participants who successfully created a deductive argument generalized by utilizing the aspects of the figural representation, the majority of times participants were coded for figural reasoning they were no more successful at justifying their rule that those who were coded for numerical reasoning. Overall, whether a participant was coded for figural or numerical reasoning, it was clear that the participants did not see a need to provide a full deductive argument.

Another surprising result of the study was the inconsistent ways that the participants justified their rules. Three of the eight participants used both inductive and deductive statements when forming their generalizations. Two participants were unable to construct any type of argument for some of their rules, yet were able to reason deductively for others. PST-3 was unable to form any justification for her first three rules, yet created an inductive justification through the figures for her second generalization of Figural Pattern #2. Only two participants were consistent and they both utilized induction. There was also no clear distinction between the types of justification attempts for the Figural Pattern #1 and Figural Pattern #2 and it was not clear from the study why such variety of justification strategies were displayed.

Language use in valid versus invalid justification. There was a clear divide in the type of language that was used when the participants formed a clear deductive justification compared to when their justifications fell short of valid. The explanations that were coded 'no justification' were filled with phrases of uncertainty. Three of the seven attempts used the phrase "I guess" and one participant stated "I don't think it would change." The other common approach was to state a claim without any support, with phrases such as "It just would" and "It works for these."

The attempts to justify through induction also included these types of phrases. After demonstrating a few examples, participants claimed "It worked for all these", "I tested it on a couple", and "If you do that every time". Of the twelve times induction was used, participants used a phrase indicating over generalization in seven instances. Like with the participants who did not create a justification, the phrase "I guess" or "I think" were also common among those that favored induction, being used in five separate justifications. In contrast, when the participants reasoned deductively or successfully created a deductive argument, they tended to speak in absolute terms, using phrases such as "There will always be", "So every", and "If you are continuing to add." Their language conveyed the certainty required by a valid justification, and these phrases were clearly lacking from other participants' attempts to justify.

Research question #3. How does the preservice elementary teachers' first attempt to find a rule vary from their second attempt to find an additional rule?

First and second generalizing strategies. This study revealed significant differences in generalizing strategies between the participants' first attempts to generalize a pattern and their second attempt to generalize the same pattern. Looking at the sixteen first attempts, recursive reasoning was coded for twelve attempts and explicit reasoning was coded in ten. When participants were asked to find another way to generalize the pattern, explicit strategies were the dominant choice. Of the participants' eleven attempts to find a second way to see the pattern, explicit strategies were coded ten times and recursive strategies were coded only three. Six of the eleven attempts to reason explicitly came from participants whose original rule was recursive. This reveals that asking participants to solve for the next case, case 10 and then case 25 was not sufficient enough to force them to move past a recursive strategy. Asking the participants to generalize the pattern in a different way pushed them to look for an explicit relationship and five of the six were successful in finding an explicit rule.

While the participants overwhelming utilized explicit reasoning when looking for a second pattern, it is important to also note what aspects of the pattern they used to reason explicitly. During the first attempt to generalize for both patterns, the participants who created an explicit rule favored the figural aspects of the pattern to do so; in eight of the eleven strategies. Adversely, during the second attempt, the participants who created an explicit rule favored the numerical aspects of the pattern; seven out of the eleven strategies. While they were successful at finding a second rule, it was not because they viewed the figure differently.

Conceptions on explicit versus recursive reasoning. When asked to compare their first rule to their second, participants regularly highlighted the benefits of reasoning explicitly. PST-3 stated that her explicit rule was “definitely simpler. You just plug in the numbers and find the number of dots.” When looking back at Figural Pattern #2 to find another rule PST-5 stated that “it would be easier if there was an equation to use to solve.” In fact, all six recursive rules that were originally created led to the participant’s attempt to find an explicit rule the second time. It was clear they understood the benefits of explicit strategies.

Though used often, recursive generalization was not the favored strategy and some participants hesitated to even consider formalizing it into a rule. When generalizing the second rule again, PST-2 noted that she saw a recursive pattern but commented that in order to make a rule she would “have to create like an equation with that n letter that I’m not good at. Like an algebraic equation." While PST-4 recorded a recursive rule, she regarded it as less "mathematically concrete rule" than her explicit rule. When referring to her first recursive rule, PST-7 was reluctant to justify it and at one point called it "kind of a sentence just getting my thoughts on paper." While explicit generalization best supports functional thinking, recursive generalization still has its place in figural growing patterns. It is not clear why some participants seemed to hold a diminished view of recursive strategies.

Implications for Mathematics Teacher Educators

Drawing from the results of this study, suggestions can be made for teacher preparation and professional development. First, this study highlights the importance of continuing to establish justification as a necessary part of generalizing in mathematics. While this is a norm for real world mathematics, it is often overlooked in the classroom (Lannin, 2005). Despite the math courses taken in high school and college by the preservice elementary teachers, they accepted empirical evidence as proof more often than not. It is unclear why this misconception is present, but it implies that preservice elementary teachers may benefit from classroom instruction on justification while in university and professional development once they enter the classroom. It is important that preservice elementary teachers are aware of the importance of justification and are clear on what constitutes as a valid justification so they have a basis to form the same expectations in their classroom.

Second, preservice elementary teachers may also benefit from solving figural growing patterns in a setting where they can compare their solving strategies with others and discuss the benefits of figural growing patterns. The preservice elementary teachers in this study did not always use the figures in a way that utilized the benefits discussed in Chapter 2. They did not consistently use the patterns to see what was growing and staying the same and they rarely went back to refer to the figures when justifying. They also tended to down play recursive strategies, but at the same time many expressed frustration when attempting explicit strategies. Allowing preservice elementary teachers a chance to see how others solve and also experiences that display the usefulness of these types of patterns may reveal these benefits more clearly.

Future Research

There are multiple areas of interest from the results of this study that should be explored in future studies. First, this study revealed an inconsistency in the participants' abilities to justify compared to how they chose to form a valid justification. Some previous studies, such as Rivera & Becker (2005) and Richardson (2009), briefly overviewed how preservice elementary teachers justified figural growing patterns. However, more in depth analysis, such as was done in this study, are needed to draw comprehensive conclusions on how preservice elementary teachers justify figural growing patterns and what motivates their justification decisions. Questions of interest, aside from repeating Research Question #2 in this study, may be:

1) What language do preservice elementary teachers use to justify figural growing patterns and how does the choice of language reflect their success in forming a valid justification?
2) Do the abilities of preservice elementary teachers to justify align with their construction of a justification when generalizing figural growing patterns?

Second, it is not clear from this study why some participants chose to continue using figural reasoning to re-visualize the figure when changing from recursive to explicit strategies and why others switched to numerical reasoning. More studies that focus on the transition from recursive to explicit strategies and how these are connected to figural and numerical reasoning are needed. Some questions of interest may be:

1) How do the figures used in figural growing patterns assist or hinder the transition from generalizing recursively to generalizing explicitly?
2) How does the type of reasoning (figural or numerical) effect the transition from recursive to explicit generalization?

Conclusion

Despite the limitations of this study, the results contribute to a growing body of research that reports on how preservice elementary teachers generalize and justify figural growing patterns. While the participants used a mixture of strategies, it was clear that they favored creating explicit rules. This was preferable since these rules highlight the functional relationship present in the patterns. While the creation of these types of rules is note-worthy, so is the reasoning by which they arrived at their rules. Many participants recognized the value of figural reasoning, which was revealed by their use of the figural growing patterns when generalizing. They attended to the aspects of the figure that were changing and staying the same and often used these findings to create rules. While all participants utilized figural reasoning at some point, there were still some that preferred to use numerical reasoning to find explicit rules. While this is an important skill, in these particular patterns, it does not utilize the benefits of figural growing patterns and often masks the relationship between the independent and dependent variable. Using figural reasoning to find explicit rules should be the preferred method to make figural growing patterns useful and meaningful.

While all participants were able to create at least one explicit rule, this study revealed that there were significant gaps between the preservice elementary teachers' ability to justify and the justifications they provided. The ability to formulate and defend a valid justification is an integral part of mathematics at all levels that cannot be overlooked. It was clear that the preservice teachers had the ability to justify their explicit rules, but often chose not to. The language used in their justifications also reveals important insights. On one hand, some participants were uncomfortable stating that their rule would work every time and on the other hand, some were comfortable stating that it would always work without providing adequate support for their claim. Understating or overstating the significance of their rules was common. It was clear that the participants' conceptions of what constitutes a valid justification was lacking. Since generalizing figural growing patterns and justifying the generalizations go hand and hand, a crucial benefit of figural growing patterns was missed by the participants.

As research continues to unveil what knowledge preservice elementary teachers possess, considering how they generalize and justify figural growing patterns effectively can provide valuable information that supports the early algebra movement.

REFERENCES

Alajmi, A. H. (2016). Algebraic generalization strategies used by kuwaiti pre-service teachers. *International Journal of Science and Mathematics Education, 14(8)*, 1517-1534.

Blanton, M. L., (2008). *Algebra and the elementary classroom*. Portsmouth, NH: Heinemann.

Blanton, M. L. & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization* (5-23). Heidelberg, Germany: Springer Berlin Heidelberg.

Bednarz, N., Kieran, C., & Lee, L. (1996). *Approaches to algebra: perspectives for research and teaching*. New York, NY: Springer.

Billings, E., (2008). Exploring generalization through pictorial growth patterns. In C. Greenes & R. Rubenstein (Eds.), *Algebra and Algebraic Thinking in School Mathematics: 70th NCTM Yearbook* (pp. 279–293). Reston, VA: National Council of Teachers of Mathematics.

Boaler, J. (2015). *Mathematical mindsets: unleashing students' potential through creative math, inspiring messages and innovative teaching*. San Francisco, CA: Jossey Bass.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra*, *K-12* (20-32). Reston, VA: National Council of Teachers of Mathematics.

Bruner, J. (1966). *Toward a theory of instruction*. New York, NY: Norton.

Cai, J., Swee F. N., & Moyer, J. C. (2011). Developing algebraic thinking in earlier grades: some insights from international comparative studies. In J. Cai & E. Knuth (Eds.), *Early algebraization* (25-41).

Heidelberg, Germany: Springer Berlin Heidelberg.

Carpenter, T. P., Franke, M. L. & Levi, L. (2003). *Thinking mathematically, integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.

Carraher, D.W., Martinez, M. V., & Schliemann, A. D., (2008). Early algebra and mathematical generalization. *ZDM, 40(1),* 3-22.

Driscoll, M., (1999). *Fostering Algebraic Thinking: A Guide for Teachers 6-10.* Portsmouth, NH: Heinemann.

English, L.D. & Warren, E.A., (1998). Introducing the variable through pattern exploration. *The Mathematics Teacher, 91(2)*, 166–170.

Friel, S., & Markworth, K., (2009). A framework for analyzing geometric pattern tasks. *Mathematics Teaching in the Middle School, 15(1),* 24-33.

Gardner, H. (2009). *Five minds for the future*. Waterton, MA: Harvard Business Review.

Imre, S. Y., & Akkoc, H., (2012). Investigating the development of prospective mathematics teachers' pedagogical content knowledge of generalizing number patterns through school practicum. *Journal of Mathematics Teacher Education, 15(3)*, 207-227.

Hershkowitz, R., Arcavi, A., & Bruckheimer, M. (2011). Reflections on the status of natural and visual reasoning- the case of the matches. *International Journal of Mathematical Education in Science and Technology*, 32(3), 255-265.

Kaput, J. J. (2008). What is algebra? what is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton, *Algebra in the Early Grades* (5-16). New York, NY: Lawrence Erlbaum Associates/National Council of Teachers of Mathematics.

Kaput, J. J., Carraher, D.W., & Blanton, M. L., (2008). *Algebra in the early grades.* Portland, OR: Reference and Research Book News.

Kilpatrick, J., & Izsak, A. (2008). A history of algebra in the school curriculum. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and Algebraic Thinking in School Mathematics* (3-18). Reston, VA: National Council of Teachers of Mathematics.

Lannin, J.K., (2002). Developing algebraic reasoning through generalization. *Mathematics Teaching in the Middle School, 8(7)*, 342-348.

Lannin, J.K., (2005). Generalization and justification: the challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning, 7(3),* 231-258.

Lee, L., (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran, & Lee, L. (eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (87-106). Dordrecht, Netherlands: Kluwer Academic Publishers.

Markworth, K. (2012). Growing patterns: seeing beyond counting. *Teaching Children*

Mathematics, 19(4), 254-262.

Morris, A. K. (2007). Factors affecting preservice teachers' evaluations of the validity of students' mathematical arguments in classroom contexts. *Cognition and Instruction, 25(4),* 479-522.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Orton, A., Orton, J., & Roper, T. (1999). Pictorial and practical contexts and the perception of pattern. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 120–136). London: Cassell.

Richardson, K., Berenson, S., & Staley, K. (2009). Prospective elementary teachers use of representation to reason algebraically. *The Journal of Mathematical Behavior*, *28(2-3),* 188-199.

Rivera, F. D. (2007). Visualizing as a mathematical way of knowing: understanding figural generalization. *Mathematics Teacher, 101 (1),* 69-75.

Rivera, F. D., & Becker, J. R. (2005). Figural and numerical modes of generalizing in algebra. *Mathematics Teaching in the Middle School, 11*, 198-203.

Rivera, F. D., & Becker, J.R. (2009). Algebraic reasoning through patterns. *Mathematics Teaching in the Middle School, 15(4),* 212-221.

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: a study of prospective elementary teachers. *Journal of Mathematical Behavior*, 15, 3-31.

Smith, E. (2003). Stasis and change: integrating patterns, functions and algebra throughout the k-12 curriculum. In J. Kilpatrick, G. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (136-150). Reston, VA: National Council of Teachers of Mathematics.

Stephens, A., Blanton, M., Knuth, E., Isler, I., & Gardiner, A. M. (2015). Just say yes to early algebra! *Teaching Children Mathematics*, *22(2)*, 92-101.

Stump, S. L. (2011). Patterns to develop algebraic reasoning. *Teaching Children Mathematics*, *17(7)*, 410-418.

Stylianides, G. J., Stylianides A. J., & Shilling-Traina, L. N. (2013). Prospective teachers' challenges in teaching reasoning-and-proving. *International Journal of Science and Mathematics Education*, 11, 6, 1493-1490.

Walkowiak, T. (2014). Elementary and middle school students' analyses of pictorial growth patterns. *The Journal of Mathematical Behavior, 33*, 56-71.

Whitin, D. J., & Whitin, P. (2014) Building squares and discovering patterns. *Teaching Children Mathematics, 21(4)*, 210-219.

Warren & Cooper (2008). Patterns that support early algebraic thinking in elementary school. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and Algebraic Thinking in School Mathematics* (PAGE #s). Reston, VA: National Council of Teachers of Mathematics.

Wilkie, K. J. & Clarke, D. M. (2014). Developing students' functional thinking in algebra through different visualizations of a growing pattern's structure. *Mathematics Education Research Journal, (28)2,* 223-243.

Yackel, E. (1997). A foundation for algebraic reasoning in the early grades. *Teaching Children Mathematics, 3(6),* 276-280.

Zazkis, R., & Liljedahk, P. (2002). Generalization of patterns: the tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49(3)*, 379-402.

APPENDICES

Appendix A

Participant Survey:
(Researcher reads aloud from this survey and dictates answers. Participant should not write or be identified anywhere on the survey.)

Researcher: The purpose of this survey is to learn a little bit about your math background. Your name will not be written anywhere on this form and your answers will not be reported individually or associated with your performance on the figural patterns. Findings in this survey, if reported, will only be reported aggregately to express the overall math experiences of the entire research group. If you do not wish to respond to a question on this survey you may decline to answer.

What math classes did you take in high school?

What math classes have you taken during your college career (at any University)?

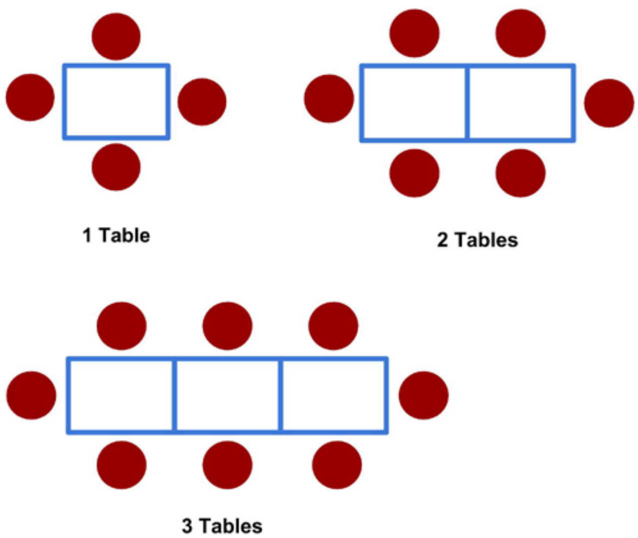
How prepared do you feel to teach math at the elementary level?
Very Prepared = 4
Adequately Prepared = 3
Somewhat Prepared = 2
Not prepared = 1

Appendix B

Participant Recording Sheet:

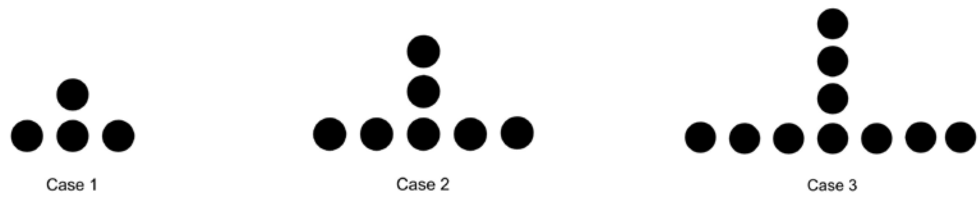
Figural Pattern #1

A big party is being planned and everyone will sit at rectangle-shaped tables. The tables will be put together in one long line as shown below. Each side of the table fits one person.



1. How many total guests can sit when there are 4 tables?
2. How many total guests can sit when there are 10 tables? 25 tables?
3. Can you create a rule that will allow you to find the number of people for any number of tables?

Figural Pattern #2



1. How many dots will be in the next case?
2. How many dots will be in the 10th case? The 25th?
3. Can you create a rule that will let you find the number of dots for any case?

Appendix C

Overview of Preservice Teacher 1 (PST-1)

For Figural Pattern #1, PST-1 drew 4 tables out and counted to find the total number of people. Upon reading question 2 on the recording sheet, she decided she did not want to take the time to draw out more tables. She looked back at the figures to find a pattern to solve for the total number of people at 10 and 15 tables. As she looked back at the first four figures she “realized there was a pattern, so like for example, for 3 tables there would be 3 people on each side and then 2 at the end.” She continued, “I knew the pattern was the number of tables; there would be that many people on each side and then two extra people on the end and that's how I got the rule, number of tables times 2 plus 2.” She was able to generate a rule based off what she saw happening in the figures.

When asked if her rule would work every time, PST-1 began her justification noting that the rule she created worked for 1, 2 and 3 tables. She also noted how the figure connected back to the rule for each table. She then began to explain more generally that “there will always be 2 people on the end (points to 2 dots on the end of the table). And there will always be one spot on each table, considering the number of tables there are.” Her justification connected all the components of her rule back to the figures.

When completing Figural Pattern #2, PST-1 began by counting the total number of dots in each case, noticing that “in these 3 cases it went up by 3 each time.” She added 3 to the total of case 3 to find the amount of dots in case 4. She found that the 3 being added from case to case specifically correlated with 2 dots being added to the bottom row of the case and 1 dot to the top column of the case. PST-1 drew a picture of case 4 to make sure this strategy would produce the same the number of dots as her first answer. Choosing not to draw out case 10 or 25, the participant studied the figures further. She noticed the total number of dots in the top column was the same as the case number. This left her to create a list continually adding 2 to find out the number of dots across the bottom row. She used the list that she generated to solve for the number of dots on the bottom row of case 10 and 25. Finding that there were 21 dots on the bottom of case 10 and 51 dots on the bottom of case 25, she concluded that the bottom row was 2 times the case number plus 1 for case 10 and 25. She compiled this knowledge into a written rule of 'case # + case #*2 + 1'. When asked why this would always work, she checked to make sure it worked for the 3 cases that were given. She also added "I guess the number of dots on the bottom are always 2 times the case number plus 1," but she never directed this statement back to the specific parts of the figure. It should be noted that the researcher did explicitly ask her to connect her rule back to the figure during the interview.

When she solved Figural Pattern #2 again, she was able to see the pattern in a different way. Instead of visualizing it as a bottom row and a top column, she saw a column and then two distinct parts of a row adjacent to either side of the column. She noticed that the parts of the row were "two groups of the same number, of the case number". She showed this thinking through her representation of the figure.

After she noticed the pattern, she demonstrated how her new rule would work for case 1, 2 and 3 by connecting the figures back to the equation for each. She then used the new rule to see if case 10 would generate the same answer that she got for case 10 with her first rule. When asked if her rule will work every time, she stated that “the dots on the outside… there are always 2 groups that are the same number as the case number” and then explained why the rule would work for case 30, concluding her explanation saying “and it worked for each of the other cases too.”

Overview of Preservice Teacher 2 (PST-2)

PST-2 began by constructing a representation of the four tables out of unifix cubes and then counted the spaces around the cubes to find the total number of guests that could sit at 4 tables. Then, the participant drew out 10 tables and counted around the drawing to find the total. As she was doing this she noticed she “was just counting 10 and 10 again (pointing to alternate sides of the table she drew on her paper)”. She used this conclusion to solve for the total number of guests at 25 tables without drawing out the table, noting that “it would be 25 and 25 and 2 (motions to correlate values with outline of imaginary table) and that’s when the pattern hit me”. This allowed her to immediately write 52 guests in reference to 25 tables. She generalized this by writing the rule ‘multiply the number of tables by 2, then add 2’. To justify that her rule would always work, she gave 50 tables as an example, noting that since there can only be one person at each table “50 people can be on one side and 50 people can be on the other (motions to form 2 sides of imaginary table), that’s multiplying by 2 and then there is only 1 person on the ends, that's adding 2 (motions to 2 people on the end of imaginary table)." She clearly connected the specific parts of her rule back to the context of 50 tables.

For Figural Pattern #2, PST-2 began by visualizing the dots in two groups: a column and a row with two distinct parts on either side of the column, noting that the parts of the row were made up of the same amount of dots as the case number and the number of dots in the column was one more than the case number. She drew out case 4 using this strategy and counted to find a total of 13 dots. She then does case 10 in her head thinking that "there would be 10 dots on each side (motions to form imaginary figure with hands) so that's 20, plus 11 (motioning upwards to demonstrate middle column) is 31." The same reasoning was followed for the case 25. She used this reasoning to write out the rule 'add the case number to itself, or multiply by 2, then add the case number plus one to the previous answer'. In response to being asked how she knows her rule will work every time, she restated her reasoning for case 10 and case 25, then remarked "And I guess that would be a rule for any of the case number, for the pattern to go on forever." She used the two specific examples of her rule to justify why all cases in the pattern would work.

When asked to solve one of the patterns in a different way, PST-2 created a t-chart for Figural Pattern #2 and noticed that the figures "are all adding 3 dots each time because you add 1 to the top, 1 to the side and 1 to the side (referring to end dots on figure 3)". She continued thinking aloud looking for how the case numbers and the total number of dots were connected through this pattern. She contended that she needed to find a "little equation to figure out the number of dots." When asked to clarify what she meant, she stated "I'd have to create like an equation with that 'n' letter that I'm not good at. Like an algebraic equation for it." Her attention is drawn back to her first rule for Figural Pattern #2, which is written in word form not equation form, in the following excerpt:

> Researcher: So, this I noticed though, you have a rule and you don't necessarily have an equation. If I asked you with this rule how many would be in 100, you could tell me that couldn't you, with your rule?
>
> Participant: Yes, so case number times 2 plus case number times 1 (writes (cn*2) + (cn+1) on paper). Is that a rule?
>
> Researcher: You tell me.
>
> Participant: So, case number 30 times 2 is 60 plus case number 30 plus 1, so 60 plus 31 is 91 (solves on paper). That is a little equation I guess.
>
> Researcher: So, did you generate this? How did you generate this? (referring to equation)
>
> Participant: From this sentence (circling original written rule "add the case number to itself, or multiply by 2, then add the case number plus one to the previous answer"). So, you multiply the case number by 2 and then add the case number plus 1 to the answer.

Through the discussion, PST-2 realized that even though she did not originally create a rule in equation form, she was easily able to create an equation based on her written rule. After she established the new equation, the participant stated that her rule also works for the table. When asked to explain what she meant, she used the equation to calculate the total number of dots for case 2, to verify that the table and equation show the same thing. PST-2 was prompted again to find additional rule, since the equation she created was the same as her written rule. She looked back to the t-chart (shown below) and noticed a number pattern that allowed her to create the rule (cn*3) + 1. To prove that this rule would always work, she demonstrated the equation for case 30 and case 25 and stated that it also worked for the values in the t-chart. Finally, she is asked if she can connect this new rule to the figure, to which PST-2 replies “Not really. I just kinda got that by looking at the table.”

Overview of Preservice Teacher 3 (PST-3)

First, PST-3 drew out 4 tables and counted to find the total number of guests. She went back to label the total number of guests for the figures shown on the recording sheet. While doing this, she noticed a pattern in the output values. She concluded that the total number of guests went up by 2 each time. Next a list was created that showed the total number of guests at 4 tables, 5 tables, 6 tables, etc. to 25 tables. She stated, “There is probably an easier way to do that, but that was easiest for me.” Therefore, she noted that her method of solving was not the most efficient, but did not attempt to look for another way.

She formalized her finding by creating a rule that said “you would go up by 2 people from each table.” When asked to justify, she explained:

Participant: Yeah, because that seems to be the trend for the first 3 tables (pointing to figures) and the first 25 tables (pointing to numerical list she created). So, it just would. It’d be the same for if you wanted like 1,000.

Researcher: So, if, since you posed this, I’m curious. If I asked how many people could sit at 1,000 tables, how would you figure?

Participant: Well, I wouldn’t do it this way (pointing to numerical list), umm…

Researcher: Is there another way?

Participant: I’m sure there is… I’m trying to like… umm… okay, so 2 tables you have 2 on each top and bottom (pointing to figure), 3 tables you have 3 on each top and bottom (pointing to figure) plus 2… so 1,000 would be 1,000 (motioning to top of a table) and 1,000 (motioning to bottom of a table) and then 2 (motioning to outside of a table) so 2,002.

PST-3 used this reasoning to create an additional rule: “to find the number of people at a given number of tables, you would multiply the number of tables by 2 and then add 2 more.” She claimed her rule would work every time, because it worked for the first 3 figures that were given. When pressed to explain why it worked for those 3 tables, she uses the dots in the figure to explain:

Participant: There's 1 on each side (pointing back to figure 1 and referring to 1 dot on top and 1 dot on bottom), so when you increase the number of tables you increase the amount that could be on the top and bottom (pointing to figure 2's top and bottom dots). So, this one (referring to figure 2) you could have 2 on top and 2 on bottom. This one (pointing to figure 3) you have 3 on top, 3 on bottom.

Researcher: So, each time you add a table, you…

Participant: You add 1 one top and 1 on bottom.

In Figural Pattern #2, PST-3 counted the dots to determine the total per figure. She noticed that the total number of dots in the figures was increasing by 3 and used this information to determine the total for figure 4. When prompted if she saw the pattern through the numbers or the figures, she stated the numbers, but added she now recognized 1 dot being added to each end of the figures. When she solved for case 10 and 25, instead of continuing with this rule she constructed the equation, (10x3)+1=31, for the total number in case 10 and (25x3)+1=76 for case 25. Then, she created the equation 4x3+1=13 and wrote it beside her original answer of 13 in question 1. Finally, as her rule she wrote "you would multiply the case by 3 and then add one more." It is unclear from the interview how she generated these equations. When asked if her rule will work every time she replies "Yeah… um, based on the given examples, these are adding 3 and I don't think it will change. So, you're just adding 1 more dot on the left, right and top (pointing to outside dots in case 3)."

Researcher: I agree with your written rule… so you switched from equation form to written form. I'm curious if there is a way to take your written rule and put it into an equation the way you were representing (shown below). Does that make sense?

Participant: Oh, yeah. Okay… so if the case is c, you multiply the case number times 3, add 1 and that gives you the number of dots.

Researcher: Do you see this rule (pointing to (c x 3) + 1) in the pictures?

Participant: Umm… yeah, because the case here is 1 (pointing to case 1) multiplying it by 3, so that is these 3 (pointing to the 3 outside dots) and then 1 in the middle just stays the same. Case 2, multiplying it by 3.... That'd be the plus 1 (pointing to 1 dot in the middle of case 2) ... hmm, actually maybe I don't see it.

Researcher: It's fine, I'm just curious.

Participant: Yeah, I don't think I see it.

When asked to solve in a different way, PST-3 responded that in the figure she saw "you just add a circle on each side" to move from one case to the next. She was not sure if her rule was a different one or a variation of the same. When encouraged to justify her rule, she pointed to the figures and responded "that's just the pattern that I see in these, adding 1 on each side, so there's no reason it wouldn't work." When comparing the two rules, she noted the second rule would be more time consuming.

Overview of Preservice Teacher 4 (PST-4)

First, she noticed that the total number of guests was going up by 2, so she added repeatedly to find the total number of guests at 4 tables and at 10 tables. For 25 tables, she reasoned that since she knew 22 was the total number of guests at 10 tables, she only needed to figure out how many seats would be added from 10 tables to 25 tables. She originally performed the calculation ‘15+22=37’ but realized that 27 cannot be the total number of seats because “it was an odd number and none of these are odd numbers (referring to total number of guests at tables 1-10)”. Since she knew 2 guests could sit at each table, she corrected her calculation to find that there would be 30 additional guests. Therefore, the total number of guests at 25 tables would be the 22 guests from 10 tables and the 30 additional guests from 15 more tables, for a total of 52 guests. From this thinking she created a rule that read “2 times the number of added tables added to the start point”. When asked if her rule would work every time she responded, “Yes... but I get scared about saying that because I just always feel like there is a loophole in math rules and that is something that is like ingrained in me so yes, but maybe not.” Though she was not sure if she could justify her rule, when pushed to explain why she thought it might work every time, the participant reasoned that it should work since each table always adds 2 guests.

When she attempted Figural Pattern #2 after looking at it for a while, PST-4 stated that she could not find a rule, but she shared what she had solved so far in the pattern. She originally recognized that 3 dots were continually being added to the figure, noticing particularly that they were added to each end point. She then created a list, recording the case number and the total number of dots. She stopped multiple times to erase and recalculate, noticing places where she added incorrectly. However, she did not notice one mistake and her final total for case 25 was off by a dot. She tried to employ the same multiplicative technique she used in Figural Pattern #1, but was unable to find a consistent rule that provided the same answer as the values in her list (some of which were incorrectly added). She was visibly puzzled and the following excerpt highlights her confusion:

> Researcher: I know you attempted to use the rule that you did here (referring to rule in task 1), do you think that something like that should work and you just didn't find it or do you think this (referring to rule in task 1) wouldn't work?
>
> Participant: I think that something like that should work, maybe not even the same exact idea, but I think that there is a rule you could create to do this that would work but I don't... I just got like frustrated so that's why I was like whatever but, I think there is one that would work… because you're doing the same thing every time and it doesn't change so… I dunno…

When asked to create a different rule for one of the patterns, she looked back at Figural Pattern #1 and attempted to use the "multiplication facts that involve the number of tables and the number of people that sit around the tables". She was looking for a direct relationship between the input and output, but could not find a familiar number pattern. Then, she noticed that if she accounted for the 2 seats at the end with plus 2 that she could just double the number of tables “because each table adds 2”. This thinking was used to create the rule “2(n)+2 for n = # of tables”. When asked if her rule would work every time, she concluded that it should but to “really prove it” she would need to draw out a figure for each case she was trying to prove.

The discussion then returned to Figural Pattern #2 and the participant was shown where her calculation error occurred. She was able to correct her mistake and create the rule “3 times the amount between cases added to the start point”. The following excerpt shows the conversation that ensued after the participant was asked if her rule would work every time:

> Participant: Um, well like a more, I guess mathematically concrete rule, like my second rule here (pointing to 2n + 1 in Figural Pattern #1) would be better and would work every time, but this like depends on where you start, I guess, but yeah essentially if I re-wrote it to be a more mathematical rule it would work every time.
>
> Researcher: What do you mean by more mathematical?
>
> Participant: Like, I feel like I didn’t necessarily make a new rule (Figural Pattern #1), I just edited this rule to be more, I dunno, easily accessible to like anyone that read it and I think this one (referring to Figural Pattern #2) would work every time essentially had I done that. Like edited this rule.

Upon thinking about this more, she eventually created the rule “3(n)+1” by noticing number patterns in case 10 and case 14. She checked to make sure case 14 would work. When prompted to see if this rule connected back to the figure she explained that the center dot is “like your automatic 1; your 1 every time you are adding”. She demonstrated this thought further by drawing case 0 and explaining that it would only include the original dot (shown below). She then went on to explain how 3n connected back to cases 0-4, showing that the case number is the same as the “extra dots each way” and “you are multiplying by 3 because there are 3 ways”. When prompted, she was clearly able to connect both parts of her rule back to the figure.

Overview of Preservice Teacher 5 (PST-5)

PST-5 began solving by drawing out the next table in the sequence. After creating the drawing, she generated a list of the number of tables with the corresponding number of guests for tables 1-4. She recorded the total number of guests at 4 tables on the recording sheet under question 1 and then continued her list up to 10 tables by adding 2 each time. After going through the list PST-5 concluded that “you just double the number and add 2; that would be the people on the end”. She was convinced her rule would work every time and explained, “If I were to just draw 5 tables (draws and labels 5 tables), there would be 12 people because there’s 2 people per table and 1 person on each end.” When prompted, she is able to connect the specific parts of her rule back to the figure:

> Participant: Um, yeah because you could say you double each table (pointing to dots on either side of a table in figure 2), like even if you were to just double the number (puts fingers over end dots in figure 2 to hide end dots) it would be 2 times 2 is 4 and then you would add 2 (removes fingers from dots) so they would be the 2 on the end.

Researcher: So, you’re saying this add 2 is?

Participant: The 2 on the end each time.

Researcher: And then the double?

Participant: Is table times people on each side of the table (pointing to dots on the top and bottom of figure 1).

Later in the interview, she created an equation to represent this rule. Once again, she connected the specific parts the equation back to the figure after being prompted. When asked why she thought her rule would work every time, she calculated the total number of guests at 25 tables as an example to justify.

For her explanation of Figural Pattern #2, PST-5 began by stating that she noticed “on the bottom it’s adding 2 each time and whatever the case number is that’s how many you are adding to the top”. She explained how she saw this observation happening for cases 1 - 3, and from these observations she began a tabular representation. She then applied this visual strategy to draw case 4 and 5. When she drew case 4, she initially drew 9 dots across the bottom and an additional 4 dots on the top. However, when she went back to recount she erased the top most dot and recorded the total number of dots in the table as 12, instead of 13. Later, during the interview when she was explaining her thinking, she caught her mistake and redrew the top most dot. She then corrected the total number of dots on her recording sheet in response to question 1, but forgot to amend it in her tabular representation.

Moving to case 6, she wrote out the number of dots that would be on the top and bottom of the figure, instead of drawing out the actual figure. She explained a few of the examples she had written and while explaining noticed multiple calculation errors, which she then corrected. She questioned whether the pattern she has discovered could be considered a rule, but decided to record what she noticed as her answer to question 3. Referring to how she constructed her drawings of case 4 and 5 to justify her rule, she stated that "if you do that every time" the rule would work.

When asked to find an additional rule for either pattern, she struggled. She attempted to find a rule for Figural Pattern #2 by searching for a number pattern in her tabular representation, but found that "some of them are times 3 plus 1 and some are times 3 plus 2". She could not find a rule that fit all the values she has generated and she failed to realize that this was due to some of the values being incorrect. After expressing that she was unsure what the rule was, she was asked to clarify whether she thought there should be another rule:

> Participant: Maybe not. I mean, I feel like there should be some equation way to get there.
>
> Researcher: What do you mean by equation way?
>
> Participant: Like, an easier way to get from the case number to whatever it is than having to like write out each number, whatever it is. So like times 3 plus 1.
>
> Researcher: Oh, so what I think I hear you saying is there should be a way to use whatever the case number is to find the [total]...

Participant: Yeah, but you either have to draw it out (pointing to drawings of case 4 and 5 in pattern 2) or write the two numbers (pointing to numerical "drawings"), instead of just getting from this (pointing to 'case 1') to this (pointing to drawing of case 1)... Like it would be easier if there was an equation to use to solve, because you could do times 3 plus whatever, but it doesn't work for each one.

Researcher: So, you're saying, do you think there is an equation and you didn't find it or do you think one doesn't exist for this pattern?

Participant: Uh, I think there is one, but I can't figure it out.

Since the participant did not originally produce an equation for Figural Pattern #1 either, she was prompted to clarify if she thought there was an equation way to solve it. The interview concluded with the participant changing her written rule from Figural Pattern #1 into the equation 2x+2 using x to represent the number of tables.

Overview of Preservice Teacher 6 (PST-6)

PST-6 began solving by noticing what happened to the figures as a table was added. She noted that when a new table was joined to the existing tables, one person was added on each side. To solve for the total number of guests at 4 tables, she visualized adding an additional table to the figure of case 3. She imagined 4 dots on each side of the table and 2 on the end and recorded this as '4 + 4 + 2 = 10'. She followed the same line of thinking to find the total number of seats in case 10 and 25, creating the equations '10 + 10 + 2 = 22' and '25

+ 25 + 2 = 52'. From this thinking she generated the rule '2n + 2', with n representing the number of tables. After she shared her rule, she was asked if the rule would work every time:

Participant: Yes, I mean you have 3 examples, oh shoot well… (checking with figures), yeah, we have 3 examples that it works with. (writing) '2 times 2 plus 4 is 6 which is that (pointing to figure 2) and then 1 times 2 plus 2 is 4 which is that (pointing to figure 1)'. So, you have 3 examples there and then it's also like, going back to the table each table for sure has 2 people, always using that table, so for every table there is 2 people (draws rectangles around each table pair in 3 table figures). And then there's always going to be 2 end people (circles end points in 3 table figure). So, there's really no reason why it would change.

Researcher: So how does that connect back with your rule?

Participant: Well for each table, if there's 'n' number of tables, you're multiplying by 2 to represent the 2 people that are using that table (circles 2 corresponding table dots in 3 table figure) and then adding 2 on the end because that will never change (pointing to end dots in 3 table figure). There is always going to be 2 people on the end no matter how many tables you do.

As the participant began to solve Figural Pattern #2, she noticed an "outer ring" was being added each time, resulting in 3 additional dots for each figure. She used this logic to solve for the total number of dots in case 4, by calculating '10 + 3 = 13'. Then, upon seeing that she must solve for the total number of dots in case 10 and case 25, she decided to find a way that did not require her to repeatedly add 3. She looked back at the figures to see if there was another way. This led her to see that there was a “dot of origin, I don’t know how to describe it but, (circling center dot in case 3) and then the number of dots that flanks each side relates to the case number (circling outer dots in case 3)”. She was able to visualize the figure in a different way to find another pattern. From this, she solved for the total number of dots in case 10 and case 25. She developed the rule ‘3n +1’ to find the total number of dots for the nth case.

When asked if her rule would work every time, she concluded that it would since she tested it out on four different cases. When asked to connect her rule back to the figure, she explained, “So like in this instance (points to case 3), n is 3, so have your 1 origin one which is this one right here (circles +1 in rule), that’s what that 1 represents. And then you have 3 groups (circling in case 3) 1 group, 2 group, 3 groups of that number. So, like this case 3 and you have 3 groups of 3. So, like with case 2, the 3 groups, in this case n is 2 so you have 3 groups of 2 (circling in case 2) that flank that 1 middle one.” While explaining she manually connects each part of her rule back to the specific parts of the figures in case 2 and case 3 (see Figure 12).

PST-6 returned to pattern 1 when asked to find a different way to solve one of the patterns. When looking at the figures again, she saw that “each side of the table fits 1 person and if you just looked at the tables knowing that each side could hold potentially 1 person, so each table has the potential to hold 4 people and so if you have a table, multiply it for the 4 people that could sit there." From this thinking, she generated 4n. Then she noticed that "each of those lines represents 2 seats lost (pointing to lines on inside of tables that show where 2 tables have been put together). One seat on this table, one seat on this table (pointing to either side of a middle line) ... So, I thought, if there's 3 tables (points to figure 3) there's going to be 2 lines (pointing to lines on inside of tables), 4 tables there's 3 lines or like 3 meeting points of those tables." She used this thinking to generate (n-1)*2 which is subtracted from her original 4n. When asked if her rule would work every time, she affirmed that it would since she tested the equation out for two cases, compared the answers to what she got using her first rule and they were equivalent. She added that if necessary, she could test more figures she knew the answer for to prove her new rule worked.

Overview of Preservice Teacher 7 (PST-7)

To solve Figural Pattern #1, PST-7 drew out 4 tables and counted how many guests would be seated. Next, she looked at question 2 and 3 and decided that she wanted to find an overall rule. She counted and recorded the total number of guests for tables 1-3, noticing and then later recording that "the number increases by two each time a table is added". Though she ultimately recorded this finding, considering it her first rule, she continued to look for a second rule. She referred to what she wrote for her first rule as "kind of a sentence of just getting my thoughts onto paper". It was clear that she did not think this rule was sufficient. She drew out 10 tables and counted to find the total number of guests. PST-7 was continually looking at the relationship between the case number and the total number of dots to find a number pattern that would help her solve. She found success with ‘times 2 plus 2’ for case 4. She tried this new strategy to make sure it would also work with the given cases. She then tried ‘10 x 2 + 2’ and got 22, which was the same answer she got when drawing out 10 tables. She decided to write ‘number of tables times 2 plus 2’ as a second rule, adding “I’m not really sure why it’s those numbers, I just know that that was the pattern that I found.” Upon reading over her first rule again during the interview, she decided “I guess I should say it doubles the number (writing ‘The number of tables doubles and then add two’).”. She did not erase her first rule, but this action again emphasized that she did not value it as highly as her second. When asked later in the interview if her rule would work every time:

> Participant: I think so. Because, it works with all the ones that I tried….?
>
> Researcher: That’s a reason.
>
> Participant: Yeah, I mean I don’t know. Being questioned makes me want to double guess it, but I think it would work every time.
>
> When asked if her rule connects back to the figure:
>
> Participant: So… like I said with this one (points to task 2) it just goes back to it because we are talking about the number of tables (points to ‘2 tables’) and when I have 2 we’re using that in the equation. Right? Yeah. And then… so… (writes 2 x 2 = 4 + 2 = 6 at bottom of paper) so for this one we used the number of the tables, which is 2 and we timesed it by 2 and we get 4 and we add 2 more and get 6 and 6 is the amount of dots (circles all dots in 2 tables). And adding 2 is because of these end

guys I think (circles dots on the ends of 2 tables). And so, we originally had 4 when we multiplied these 2 so we had to add on these extra.

Researcher: Okay, so you're saying that's the 2 on the end (points to +2) and that top 2 (points to the first 2 in 2 x 2) comes from the number of tables (points to label '2 tables'). What does the multiply by 2 come from (points to the second 2 in 2 x 2)?

Participant: That's a good question. Um, I'm not exactly sure where that comes in, but I just know the way I found it was by looking at the 4, 6, 8 (pointing to numbers she wrote above tables), looking at the pattern and that's kinda how I found that number.

For Figural Pattern #2, PST-7 realized that 'each case grows by 1 dot on top, to the right and to the left' and recorded this observation under the given figures. She used this finding to create case 4 and count the total number of dots to answer question 1. Although she eventually recorded this finding as a rule, she was hesitant to use it to solve for the total number of dots in case 10 and 25. She stated, "I knew that wasn't substantial enough, in order to find patterns without listing it". She eventually did create a list to find the total number of dots in case 10 and 25, but continued looking for a number pattern that will help her generate a "better" rule. Then, she noticed a relationship between 10 and 31 recalling "Once I saw 31, I wanted to say to myself, okay I know that 10 times 3 is 30 and then I would only have to add 1 more. And then I checked, does that work for all those up here (pointing to generated list).” She used this finding to create the generalized rule to “multiply the number of the case by 3 and add 1”.

She explained that her initial rule of adding 3 each time would always work by giving a description of how the figure was growing. She noted that the dot in the center was “kind of the base” and that 3 dots were continually added each time on either side of the base dot. For her second rule, she was less sure of how to verify it. PST-7 asserted that it must work because it worked for case 1, 2, 3, 10 and 25 and she could not “think of anything that would make it not work”. When prompted to connect her rule back to the figure, she summarized by referencing to the figures given to show that the 3 represented the outer ring of new dots. She explained that the ‘n’ came from the case number and the addition of 1 came from the base dot. Finally, she was asked to create an additional rule for one of the patterns. She chose pattern 1, saying:

> Participant: So, I don’t know if this way counts, but it’s just the inverse of, well I just thought of it because I know that multiplication and division are inverse of each other. So, if you only knew you had 10 guests and you divided it by 2 you would get the number 5 and minus 1 equals 4 tables. So, that’s your way of going from guests to tables rather than what I was doing was tables to guests in that question (circling question 1).
>
> Researcher: Would that rule work every time? How do you know?
>
> Participant: I think so because, it’s the inverse.

Overview of Preservice Teacher 8 (PST-8)

Almost right away the participant wrote '4 + 4 + 2 = 10' in response to the first question. She noticed that each table always had 2 dots on the end and the same number of dots across the top and bottom, which corresponded directly with the case number. She created similar equations for 10 and 25 tables and used this idea to generate her rule of '2n + 2 = total # of people at tables', with 'n = # of tables'. She returned to the given figures and wrote their corresponding equations below. PST-8 then counted the total of dots, wrote it above each figure and checked the counted total with the equation total. When asked if the rule she generated would always work, she responded "yes" and explained "because… if your tables are consistent, if there's the same number of people around every table then as long as you add the next table, this equation, as long as you plug it in for the 'n' then you should be able to get the total number of people at all the tables. It's hard to explain why it's the same for all of them. I guess just because the number of people at the tables are consistent every time. So, I mean, if you took out the 2 people on the end, you would just take out the plus 2 and 2n equals the number total people at the table." Instead of referring to specific examples to justify her rule, she refers to the general aspects of the figure that are changing and staying the same.

For Figural Pattern #2, she originally visualized the pattern as a row across the bottom and a column going up, observing that the row across the bottom increased by 2 dots and the column was increased by 1 dot. She used this strategy to draw the 4th figure.

However, she felt drawing would not be effective with larger numbers, so she tried to view the pattern in a different way. She noticed that “if you counted from the middle, like you said this was your middle point (pointing to middle dot in case 2) then there’s 2 over on each side. Then (pointing to middle dot in case 3) there’s 3 over on each side. So, the same across each way (points to left side of case 3, then right side) as up (points up case 3).” Upon this discovery, she drew case 10 to represent her thinking (see Figure 13).

She then formalized this pattern into the rule ‘n(3) + 1 = # of dots’ for any case and ‘n = case’. She explained “Case times 3, because you’re going across, up, across (motions to corresponding parts in case 3). That’s 3 different times you're having to move from your central dot, your central point; plus 1 because you have to add your dot (points to middle dot in case 2). And then that’s your total number of dots for any of your cases.” She maintained that her rule will work as long the pattern continues. When pressed to clarify what she meant by “the pattern” she referred to one being added to each end of the figure.

She chose to continue looking at this pattern when asked to find an additional way to solve. She returned to the original pattern that she noticed, adding 2 dots to the bottom and 1 dot to the top and made this her second rule. Her justification included a description of the figure in which she explained, “you’re going to have to add 2, because there’s one on each side and then 1 more above”. She assessed that this rule would be harder to use on large numbers, which she had previously stated as her reason for abandoning this pattern the first time she solved. However, upon returning to this pattern again, she spent more time considering a way to utilize what she noticed to make solving more efficient. She was looking for a way to represent what she noticed in a rule that related the case number to the total number of dots, but could not formulate how to do it:

> Participant: I'm sure there is and I'm trying to think… umm, I'm struggling to see it. I'm sure there is.
>
> Researcher: What makes you sure that there is?
>
> Participant: Because it's a continuous pattern so I feel like there has to somehow be a rule that would make this work because I mean… I dunno. Maybe not.
>
> Researcher: So, you think there may be one, but you're not sure. That's fine.