

DAMPING EFFECT OF NON-LINEAR RESPONSE OF GRAVELS ON ROOF OF BURIED REINFORCED CONCRETE STRUCTURES

Greg Mertz¹ and Tom Houston¹

¹ Senior Engineer, Costantino & Associates, USA

ABSTRACT

Many buried structures incorporate a layer of granular material supported by the roof of the structure. The weight of the gravel, combined with the self weight of the roof, may lead to a fundamental frequency of the roof structure in the frequency range of peak spectral acceleration in the input seismic ground motion resulting in large values of In-Structure Response Spectra (ISRS) which must be accommodated in the design of roof supported Systems, Structures, and Components (SSCs).

Current code guidance and standard engineering practice for reinforced concrete structures is to use 4% damping when developing ISRS. For a concrete roof structure, this results in a SDOF Fourier amplification factors on the order of $1/(2 \times 0.04) = 12.5$, further exacerbating the challenge of designing roof supported SSCs. For the case of roof structures that support a significant thickness of gravel material, additional damping in the system may be mobilized through non-linear behaviour within the supported gravels and possible sliding behaviour at the gravel/concrete interface. These additional contributions to damping of the structural response may lead to significant reductions in seismic demand on SSC supported by the roof.

This paper quantifies the structural damping appropriate for the concrete roof system supporting overlying gravel materials and examines the effect of the combined gravel-concrete roof damping on the response.

INTRODUCTION

The vibration of a roof beam or slab, with a constant mass, subject to vertical motion is well understood. However, if a portion of the mass supported by the slab lifts off the slab, then, the behavior becomes more complex and is not well understood.

The peak response of a simple span, one way roof slab, subject to a sinusoidal input with a 1g acceleration is $\approx 1.25/(2\zeta)$ where ζ is the damping ratio. For a typical uncracked slab with 4% damping, the peak acceleration in the center of the slab is roughly 15 times the peak of the sinusoidal input motion. Consequently, at input motions above 0.07g, the peak acceleration in the center of the span will exceed 1g and loose material, such as gravel supported by the slab, may lift off the slab.

Seismic input motion is typically characterized as a broadband motion and the peak response of a simple span structure is roughly $1.25 \times Sa(fn)$, where $Sa(fn)$ is the spectral acceleration of the input motion at the natural frequency fn . For a typical uncracked slab with 4% damping and a natural frequency near 3.5 Hz the spectral acceleration using the RG 1.60 spectral shape is 3.33 times the input motion. Thus, for peak seismic input acceleration above 0.24g, items on the slab may lift off the slab during a seismic event.

The purpose of this paper is to investigate the influence of the soil mass lifting off a roof slab on the roof response.

Granular Material Behavior

The interaction between a granular material and a plate (or slab) subject to vertical vibrations is quite complex, Faraday (1831). Experiments with glass boxes filled with uniform sized glass beads are used to explore

different behavior regimes, Wassgren (1996) and Eshuis (2007). At low levels of shaking, the granular material behaves as a solid and the motion of the granular material follows the box base input motion.

As the acceleration increases above 1g the granular material behaves as a ‘bouncing bed’ and the granular mass detaches from the box base. When the granular material is airborne, it is mildly fluidized. After becoming airborne, the fluidized granular material impacts the supporting box base and regains its solid behavior.

At accelerations generally above 5g, currents form in the fluidized granular material and the airborne material will undulate. The vertical seismic response for typical design basis earthquakes is generally below 5g for well supported slabs. The slab response may be above 5g’s when beyond design basis events are considered or when the slab input motion is amplified by the motion of a supporting system.

Sidewall friction has a strong effect on the glass box experiments. As the depth of granular material in the box increases, the sidewall friction forces increase and the transition from solid to ‘bouncing bed’ behavior occurs at accelerations larger than 1g. This observation has implications for the behavior of a roof slab supporting granular material, as the friction force on stair wells and piping that projects above the roof may inhibit the ‘bouncing bed’ behavior. The friction forces acting on piping and other appurtenances may be considerable.

Very thin layers of granular material can become airborne at lower accelerations and behave as a granular gas. Thin layers of material have negligible mass and their responses are usually not important when considering the seismic response of a building structure.

Buried Roof Behavior

A roof slab covered with a granular material or soil, will remain in contact with the soil when the vertical acceleration of the supporting structure is less than 1g. Note that the soil may slide laterally with respect to the roof as the roof vibrates, which is discussed below.

As the vertical response exceeds 1g, the soil becomes airborne with an initial upward velocity. The soil continues on a ballistic trajectory until gravity causes it to fall and the soil impacts the roof. Note that the glass box experiments discussed above indicate that the airborne soil is mildly fluidized. Thus, the soil impact with the roof will be less severe than the impact expected between two solid bodies.

ANALYSIS METHODOLOGY

The behavior of a roof overlain by a granular soil is investigated using the simple plane strain model shown in Figure 1. The roof is represented by a one-way, simple span slab spanning 60 feet between support centers. The roof slab is 4 feet thick with elastic material properties based on uncracked concrete with a 4000 psi compressive strength. The roof slab bears on 4 foot thick walls and the vertical reactions are distributed over the width of the wall. The roof is modeled using 1 ft square plane strain elements.

Granular soil over the roof is idealized as a 10 foot thick elastic solid layer. The granular soil stiffness is consistent with a shear wave velocity of 1000 fps, Poisson’s ratio of 0.25, and unit weight of 120 pcf. The soil weight is 1.2 klf, which is twice the roof slab dead weight of 0.6 klf. The soil is also modeled using 1 ft square plane strain elements.

The damping ratio for an uncracked concrete structure is typically specified to be 4%, ASCE 43. In this study, the change in response is highlighted by considering the nonlinear behavior of contact and sliding on the soil to structure interface. Thus, a nominal damping ratio of 2% is used in this study. For the sine sweep analyses, the damping is assumed to be mass proportional and the α term is calibrated to each individual

frequency of base motion. For the seismic time history analysis, both mass and stiffness proportional damping terms are used and the coefficients are fit to 2% damping at frequencies of 3 Hz and 15 Hz. At 4.7 Hz, the modeled Rayleigh damping is 1.6%, causing the response to be overstated by a factor of 1.26. The same damping terms are applied to the slab and soil materials.

The contact formulation between the soil and roof slab uses a penalty function to apply compressive forces normal to the roof. The soil-structure interface cannot transmit tension and the soil can separate from the roof slab. In the current analysis, impact when the soil falls back on the roof is elastic, with a coefficient of restitution of 1.0.

Traction on the soil-slab interface parallel to the roof are transferred by friction. The coefficient of friction is assumed to be $\mu = 0.3$, and at large sliding velocities is reduced to $\mu = 0.2$.

A robust nonlinear finite element material model capable of representing the observed behavior of granular materials subject to vertical vibrations is not readily available. Thus, the soil is idealized as an elastic solid as a first step in understanding the behavior of the coupled soil-slab system.

Idealizing the soil with solid elements will overestimate the contribution of the soil to the soil-slab system stiffness, overestimating the natural frequency of the soil-slab system and underestimating the deformations. Additionally, idealizing the soil as a solid will over estimate the magnitude of impact forces as the soil falls back onto the roof slab. It is postulated that the impact of a mildly fluidized granular material would dissipate a portion of the impact energy through rearranging grains and compacting the granular material in the impact zone. It is further postulated that the impact force between the granular material would have a longer duration than the hard impact of an elastic solid. The net result would be a lower impact force applied over a longer duration, with less impact energy, which would excite a narrower frequency range than the hard impact.

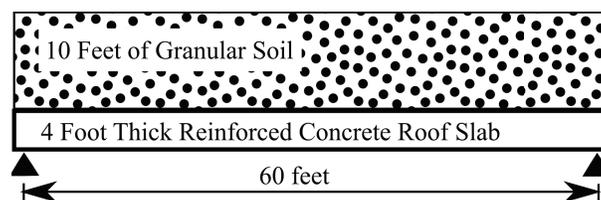


Figure 1: Simplified Roof Model

Soil structure interaction (SSI) analyses are often performed to characterize seismic response of structures embedded in soil. These analyses generally treat the soil as a visco-elastic solid and are typically performed in the frequency domain using the SASSI analysis software. Typically, buried structures are evaluated using one of the following three analysis assumptions. For simplicity, these assumptions will be referred to as SSI assumptions in the text below.

- 1. Slab Stiffness + Soil Mass:** The stiffness of the soil is neglected and the soil mass is added to the roof mass. The nominal structural damping value is applied to the roof slab.
- 2. Combined Stiffness with Lateral Slip:** The soil stiffness is included and the connection of the soil and structure are assumed to transmit normal forces only. Lateral slip on the soil-slab interface is allowed. The soil is typically given the free-field strain compatible properties.
- 3. Combined Stiffness w/o Lateral Slip:** The soil stiffness is included and the connection of the soil and structure are assumed to transmit both normal forces and tractions without slip. The soil is typically given the free-field strain compatible properties.

The elastic analysis methods employed in frequency domain analysis assume that normal forces are transmitted in both tension and compression.

Static deformations and natural frequencies for these three assumptions are calculated using variants of the plane strain model in Figure 1 and are summarized in Table 1.

Under gravity loading, the normal force between the roof slab and soil is $Nu=1.2$ ksf. Using a coefficient of friction of 0.3 results in a shear capacity of $\mu Nu = 0.36$ ksf. Checking the horizontal shear stress between the slab and soil using VQ/I yields a horizontal shear stress, under gravity loading, of 4.3 ksf, which exceeds the friction capacity by a factor of more than 10. Thus, the third case where tractions are transmitted between the soil and slab is not physically realizable, except for small amplitude oscillations. Thus, sliding will occur at the soil-slab interface for any earthquake of significance. As seen below, the sliding which occurs on this plane is beneficial and can reduce the seismic response.

The elastic moduli of the slab and soil indicate that the concrete slab is $n \approx 55$ times stiffer than the soil. Including the bending and shear stiffness of a 10 foot thick elastic solid soil slab has a relatively small (22%) reduction in deformation and a corresponding increase in natural frequency, as observed by comparing the first two cases in Table 1. If we were to rigorously model the degradation of soil properties, the result would be a natural frequency between 3.0 and 3.36 Hz. Thus, using a solid soil model and including contact with sliding on the soil-slab interface captures a significant portion of the response of the roof slab.

STEADY STATE RESPONSE

Transfer functions for the three common elastic SSI assumptions are calculated in the frequency domain using SASSI and are shown in Figure 2. Note that the control motion is specified at the slab support points. Soil supported directly by the slab is considered however the impedance of the soil supporting the roof slab is neglected.

At the natural frequency of each case, the peak response is roughly $1.25/(2\zeta) = 31$ times the input. Thus, an elastic system with a input motion greater than $1/31=0.03g$ will have a resonant peak response, greater than $1g$, which will cause vertical separation between the soil and roof slab.

The nonlinear steady state response of the soil-slab model in Figure 1 is determined by imposing a sinusoidal vertical displacement on the supports. The peak support acceleration associated with the support displacements is varied from 0.01g to 0.50g to simulate nearly elastic response through highly nonlinear response. The frequency of the support acceleration is varied from 1 to 7 Hz to highlight the first mode response of the roof girder. The input time history consists of a 64 cycle sine wave. The first 16 cycles gradually increase in amplitude and the remaining 48 cycles are at a constant amplitude. The steady state response is extracted from the last 31 cycles of response. The nonlinear response is calculated using Abaqus Explicit.

Center span displacements for the last two cycles are shown in Figure 3 for base input having a 3.5 Hz frequency and peak accelerations of 0.25g and 0.50g. Note that the slab and soil responses oscillate around the static displacement, $\Delta_{Static} = -1.06$ inch, while the support input displacement oscillates around zero. In both cases the soil separates from the slab, becomes airborne and impacts the slab. For the 0.50g case,

Table 1: Steady State Response Characterization for Common SSI Analysis Assumptions

Case	Static Deflection (inch)	Natural Frequency (Hz)
1. Slab Stiffness + Soil Mass	1.37	3.00
2. Combined Stiffness with Lateral Slip	1.06	3.36
3. Combined Stiffness w/o Lateral Slip	0.53	4.70

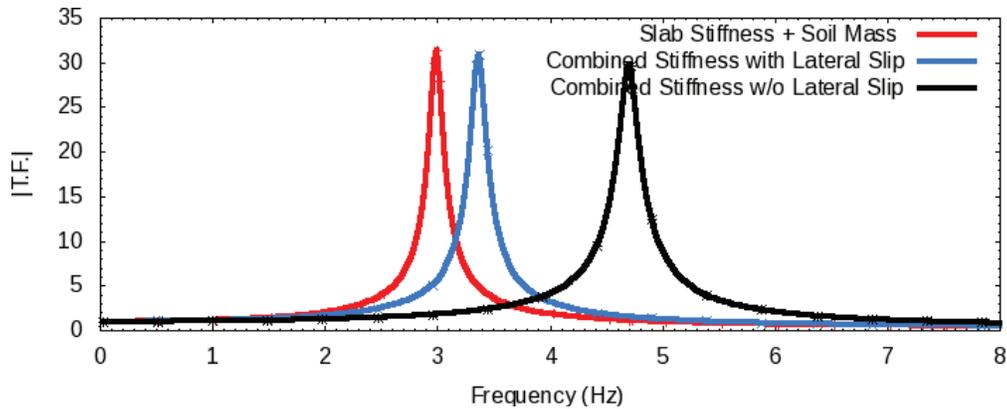


Figure 2: Transfer Functions for Common SSI Analysis Assumptions, 2% Damping

the soil impacts the slab as the slab is moving upward and the resulting impact causes the slab to change direction and move downward. The displacement response of the slab is characterized as having a strong response at 3.5 Hz with superimposed high frequency response. The soil-slab impact for the 0.25g case is less severe, thus the high frequency component is reduced.

The glass box experiments indicate that the granular material in the ‘bouncing bed’ is ‘mildly fluidized’ and we anticipate that the collision of the falling ‘mildly fluidized’ granular material and the slab will not be as severe as the indicated by the elastic solid soil element impact used in this study. Note that the momentum transfer will be similar, but the impact will occur over a longer time, with lower impact force, and will have less high frequency content than the current elastic solid soil model indicates. Thus, we believe that the current model bounds the slab response amplitude.

Careful study of Figure 3 indicates minor variations in the displacement response between cycles. Thus, the actual response is pseudo-steady state. The maximum displacement over the last 31 cycles of response is used to calculate transfer functions below.

The lateral slip between the soil and slab near the supports is shown in Figure 4 for the 4.6 Hz, 0.01g sinusoidal input and the 3.5 Hz, 0.50g sinusoidal input. Note that there is lateral slip in both cases and both

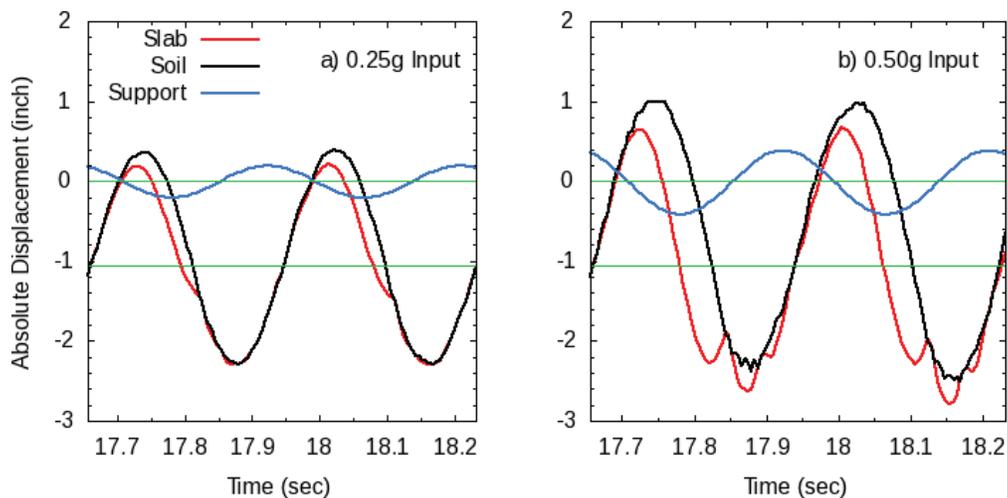


Figure 3: Typical Center Span Steady State Displacements, 3.5 Hz

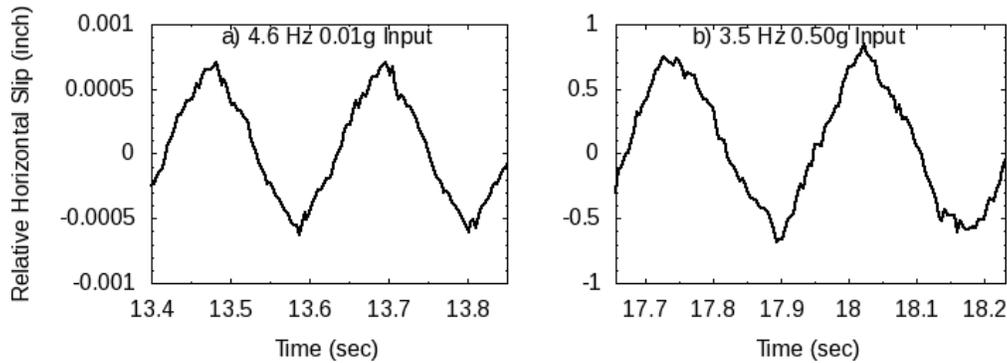


Figure 4: Steady State Slip Between Soil and Slab Near Supports

cases generate frictional damping. Additionally there is roughly a factor of 1000 difference in the magnitude of slip for the two cases but only a factor of 50 difference in input motion. Thus, the potential for frictional damping is much larger at the higher input motion levels.

Transfer functions for the nonlinear responses to 5 magnitudes of sinusoidal input, shown in Figure 5, are generated by dividing the peak roof displacement by the magnitude of input displacement. The elastic transfer functions based on the three common SSI analysis assumptions are shown by the thin black lines. The elastic transfer functions have a peak amplitude of 31, as shown in Figure 2.

The maximum vertical liftoff between the soil and slab is shown in Figure 6 for each of the 5 cases of sinusoidal input. Recall that contact is represented by a penalty function; thus, small amounts of liftoff are interpreted as contact.

Recall that one of the SSI assumptions is that the soil and roof slab act together without slip on their interface. This SSI case has a natural frequency of 4.7 Hz, as shown in Table 1. The steady state response for a 0.01g sinusoidal motion has a peak response at 4.6 Hz and a peak amplitude of 15.7 times the input motion. Thus, the tiny amount of lateral sliding shown in Figure 4a, provides sufficient friction damping to reduce the peak response from 31 to 15.7 times the input motion, which is roughly a factor of 2 reduction. Note that the magnitude of sliding in Figure 4a is not large enough to significantly reduce the natural frequency from the coupled soil-slab value.

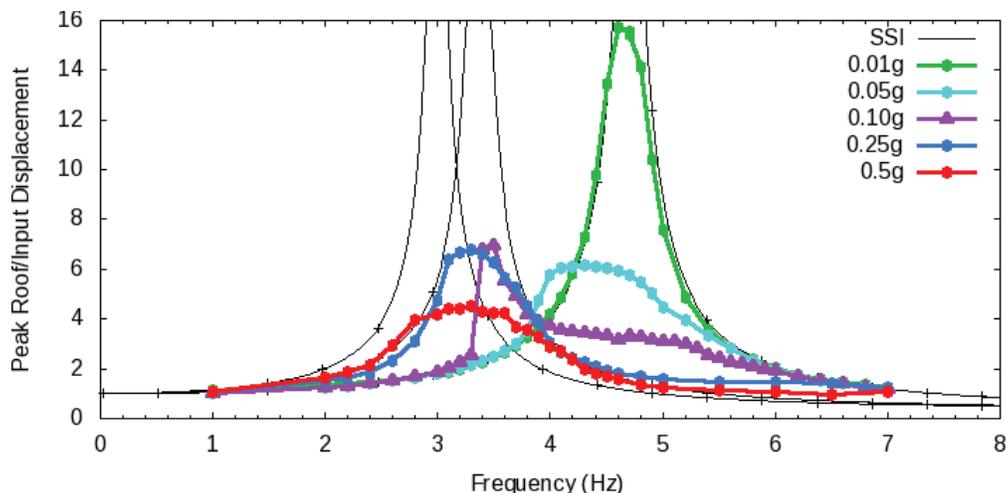


Figure 5: Transfer Functions including Sliding and Contact

Table 2: Summary of Steady State Transfer Function Peak Amplitude

Case	Effective Frequency (Hz)	Peak TF Amplitude	Viscous Damping
Nonlinear 0.01g	4.6	15.7	0.04
Nonlinear 0.05g	4.3	6.1	0.10
Nonlinear 0.10g	3.5	7.0	0.09
Nonlinear 0.25g	3.3	6.8	0.09
Nonlinear 0.50g	3.3	4.5	0.14

Increasing support motion to 0.05g results in additional sliding and the peak amplification is 6.1 times the base input at 4.3 Hz. Note that the liftoff in Figure 6 for the 0.05g base motion is < 0.001 inch, which is negligible. Thus, the reduction in response is due primarily to frictional sliding between the soil and roof slab. Additionally, the peak response occurs at 4.3 Hz, compared to 4.7 Hz for the SSI case, and the peak response near 4-4.2 Hz exceeds the SSI case. Thus, the slip for the 0.05g base motion is large enough to cause a shift in the natural frequency.

Increasing the steady state input to 0.10g, the amplitude in the 4 to 4.7 Hz region ranges from 3.7 to 3.3 which indicates a significant increase in frictional damping compared to the 0.05g and 0.01g cases.

However, the slip between the roof slab and soil for 0.10g input, is large enough that the two components are beginning to act independently and the response is approaching the natural frequency of the intermediate SSI case, 3.36 Hz. The peak response amplitude for 0.10g input is 7 and occurs at 3.5 Hz. The maximum separation between the soil and slab is 0.004 inches which is incipient liftoff. Note that at 3.3 Hz the sliding is too small to support response at the intermediate SSI mode while at 3.4 Hz the sliding is large enough to support response at the intermediate mode and there a dramatic jump in response between 3.3 and 3.4 Hz.

Sliding between the soil and slab for both the 0.25g and 0.50g input cases is large enough that these cases respond at the intermediate SSI mode natural frequency. The peak response amplitude for these two cases is 6.8 and 4.5 times the base input and both occur at 3.3 Hz. As shown in Figure 6, the amount of liftoff, or separation, for both cases is significant. Note that the peak liftoff amplitude occurs at a frequency that is slightly lower than the peak displacement amplification.

The peak transfer function amplitudes for the steady state responses are summarized in Table 2. The amplitude of equivalent viscous damping, ζ , is obtained by setting the transfer function amplitude equal to $1.25/(2\zeta)$. For moderate magnitudes of sinusoidal input, the response is approximated by a 10% damped system with frequencies between 3.3 and 4.6 Hz.

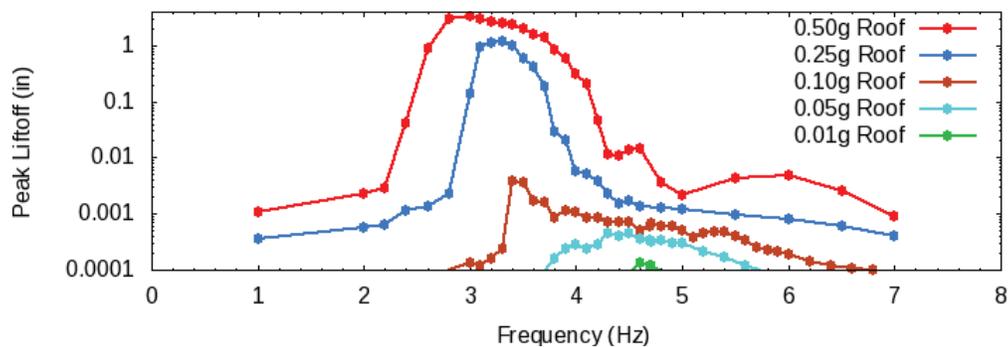


Figure 6: Maximum Vertical Distance between Soil and Slab, Center Span, Steady State

SEISMIC RESPONSE

The seismic response of a single acceleration time history having a spectral shape of a RG 1.60 spectrum is calculated in this section for different peak ground accelerations. Note that nonlinear response calculations can be sensitive to the phasing of a ground motion and consequently the response should be based on the average of multiple input time histories based on recorded ground motions. The results of this limited study provide an indication of the expected response magnitude considering sliding and contact between the soil and slab, not final design results.

Note that the input time history has a strong motion duration of $T_{SM} = T_{75\%} - T_{5\%} = 13$ seconds. Using a 3 Hz lower bound natural frequency, the duration of the strong motion input corresponds to 39 cycles, which is sufficiently long to achieve steady state response.

Peak relative displacements of the slab with respect to the support are calculated for the three SSI analysis assumptions using SASSI. The peak displacements are presented in Table 3. The nonlinear response, including sliding and contact between the soil and roof slab are calculated using Abaqus Explicit and the peak relative displacements are also summarized in Table 3.

Effective natural frequencies for the nonlinear case are extracted from the steady state response and are provided in Table 3. For comparison purposes, the calculated displacements, Δ_C , are normalized to a 1g input, $\Delta_{1g} = \Delta_C / \text{Max}(\ddot{x}_{Base})$, and normalized again to an effective frequency of 3.36 Hz, $\Delta_N = \Delta_{1g} (fn_{eff}/3.36 \text{ Hz})^2$. Differences in the peak displacements between the Elastic SSI cases are due to the slight change in target spectra between 4.7 and 3 Hz and, more importantly, differences between the target spectra and loosely fit time history in this frequency range. On the average, the elastic deformation is 4.8 inches at 3.36 Hz.

The normalized nonlinear deformation begins at 4.59 inches for a 0.01g input motion and decreases to 1.62 inches for a 0.25g input motion. Comparing the SSI and 0.25g nonlinear responses there is factor of 3 reduction in response.

In this range, sliding between the slab and soil result in a significant reduction in response as seen by comparing the deformations normalized to 3.36 Hz. Sliding also reduces the effective frequency which results in larger displacements for a given base acceleration. Thus, the difference in displacements normalized to 1g are not as large as the displacements normalized to 3.36 Hz. Between base accelerations of 0.25g and 0.75g, the normalized slab displacement increases from 1.62 to 2.22 inches. This increase is believed to be due to impact.

The vertical relative displacement between the soil and slab, or liftoff, at the slab center is shown in Figure 7.

Table 3: Relative Slab Seismic Displacement

Case	Frequency (Hz)	Relative Displacement (in)		
		Δ_C	Δ_{1g}	Δ_N
1. SSI Slab Stiffness + Soil Mass	3.00	6.53	6.53	5.20
2. SSI Combined Stiffness with Lateral Slip	3.36	4.51	4.51	4.51
3. SSI Combined Stiffness w/o Lateral Slip	4.70	2.45	2.45	4.80
Nonlinear 0.01g	4.6	0.024	2.45	4.59
Nonlinear 0.05g	4.3	0.105	2.10	3.43
Nonlinear 0.10g	3.5	0.174	1.74	1.89
Nonlinear 0.25g	3.3	0.419	1.68	1.62
Nonlinear 0.50g	3.3	1.08	2.16	2.08
Nonlinear 0.75g	3.3	1.73	2.30	2.22

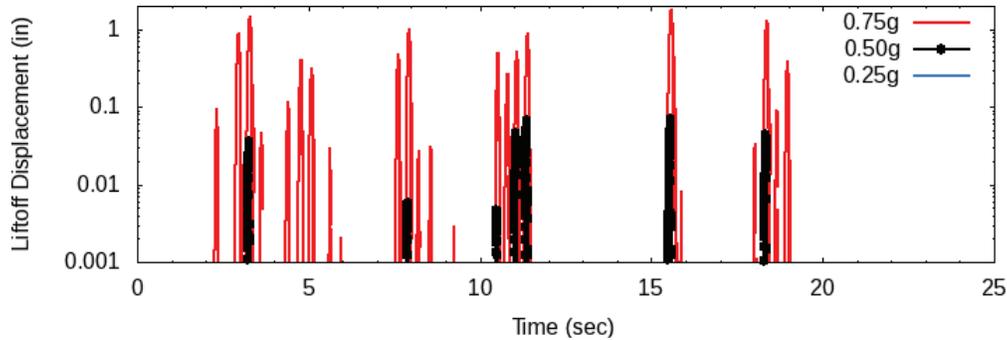


Figure 7: Vertical Distance between Soil and Slab, Center Span, Seismic Input

For the 0.25g case, the liftoff is negligible and frictional damping between the slab and soil are primarily responsible for the reduction in response compared to the elastic models. As the input acceleration is increased to 0.50g, there are 7 liftoff events during the earthquake with a maximum separation of 0.1 inches. As the input acceleration is increased to 0.75g, there are 24+ liftoff events during the earthquake with a maximum separation of 1.7 inches. The increased slab displacement for the 0.50g and 0.75g seismic cases are believed to be due to the impact forces from the soil impacting the slab. As shown in the steady state results in Figure 3, these impacts can increase the slab displacement.

Note that the observed increase in displacement for the 0.50g and 0.75g cases is different than trends observed in the steady state response, Figure 5. This suggests that the irregular spacing of impacts in the seismic case may cause larger displacements than steady state impacts and bears further investigation.

Five percent damped ISRS are shown in Figure 8. Each of the spectra is normalized to 1g input. The elastic SSI analyses, with 2% structural damping, have peak spectral accelerations near 38 to 45g.

Recall that the nonlinear time history analyses use Rayleigh damping which is 2% damped at 3 and 15 Hz. At 4.7 Hz, the Rayleigh damping term is 1.6% which causes the response to be overestimated compared to the 2% damping used in the elastic frequency domain analyses. Thus, the normalized peak response spectral acceleration for the 0.01g case is slightly larger than the elastic SSI solution.

Increasing the magnitude of input base motion from 0.01 to 0.05, 0.10 and 0.25g indicates a decreasing

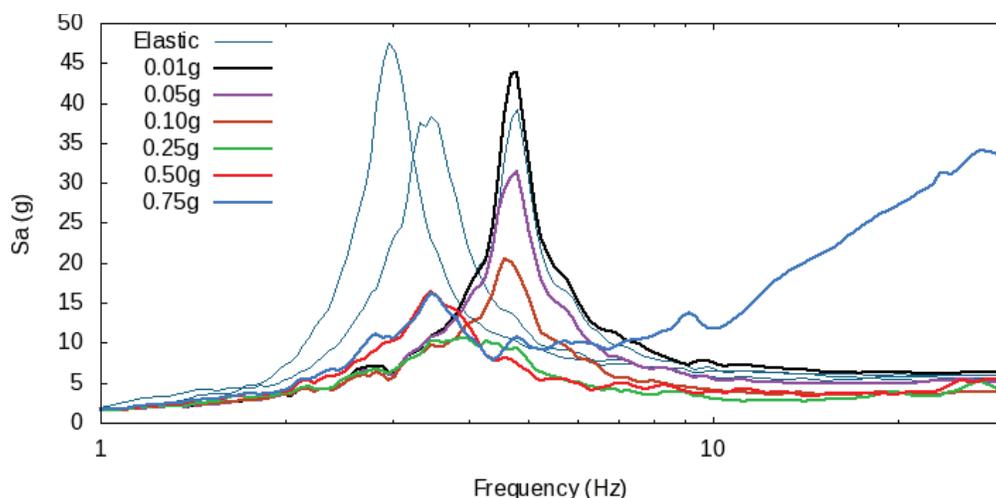


Figure 8: Center Roof 5% Damped ISRS, Normalized to 1g Input

amplitude of normalized peak spectral acceleration and a gradual transition from peak response at 4.7 to 3.3 Hz. Further increasing the magnitude of the input motion to 0.50g and 0.75g results in a similar response at 3.3 Hz. Comparing the peak spectral amplitudes of the SSI results, $S_a=38.3$, to the nonlinear results at 0.25g and 0.5g, $S_a=10.5$ and $S_a=16.4$, indicates a factor of 2.3 to 3.6 reduction in peak spectral acceleration.

Note that the 0.75g case has significant high frequency energy in the ISRS. Analytically, this energy is caused by the repeated large amplitude impacts shown in Figure 7. Note that we believe that the current analytical solution over-estimates the magnitude of the impact force because we are considering impact between two elastic solids with a coefficient of restitution of 1.0.

CONCLUSIONS

This is a limited study to understand how granular soils interact with a buried roof slab during a seismic event. For the evaluated structure, sliding between the roof slab and granular soil dissipated significant energy and reduced the roof response by a factor of up to 3.6.

Slipping between the roof slab and soil also reduces the vertical stiffness and causes significant reductions in natural frequency for moderate amounts of seismic input. On the high side, the natural frequency is bounded by an elastic solid soil that is attached to the roof slab without lateral slip on the soil-slab interface. In this study, the natural frequency on the low side is bounded by a case with combined soil and slab stiffness which allows lateral slip on the soil-slab interface. Allowing slip on the soil-slab interface resulted in a 30% reduction in the natural frequency.

The current analysis recognized that the contribution of soil stiffness is relatively small compared to the slab stiffness and neglects the nonlinear degradation in soil properties. Completely neglecting the soil stiffness reduces the natural frequency an additional 12% which is the lower bound of the soil-slab response.

The current analysis treats the soil as an elastic solid and we believe that it overestimates the impact force when the elastic solid lifts off and then impacts the roof slab. Future work will focus on refining the effect of impact of mildly fluidized granular soil on the roof slab.

An original goal of this work was to develop an effective damping for the soil-slab system that could be used in an equivalent elastic analysis. For small to moderate amounts of nonlinear behavior, the transfer function amplitudes are consistent with 10% or more damping. However, the shape of the transfer functions are not consistent with the shape of elastic transfer functions. Analysis results using an envelop of elastic models with a range of frequencies are currently inconclusive.

The current analysis indicates that it is feasible to use nonlinear analysis to demonstrate significant reductions in buried roof response compared to an elastic SSI analysis.

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