ABSTRACT

SULLIVAN, STEVEN MCKAY. Twisted Logarithmic Modules of Free Field and Lattice Vertex Algebras. (Under the direction of Bojko Bakalov.)

Vertex algebras formalize the relations between vertex operators and appear naturally in several areas of physics and mathematics, including 2-dimensional chiral conformal field theory, the representation theory of infinite-dimensional Lie algebras, the construction of the "moonshine module" for the monster group, boson-fermion correspondences, and integrable systems among others. Given an automorphism φ of a vertex algebra V, φ-twisted modules of V provide a rigorous mathematical framework for the "twisted" vertex operators appearing in the applications listed above. They also give rise to representations of orbifold subalgebras. If the automorphism φ is not semisimple, the fields in a φ-twisted V-module involve the logarithm of the formal variable. Another distinguishing feature of such modules is that the action of the Virasoro operator L_0 is not semisimple. We construct examples of such modules, realizing them explicitly on Fock spaces when V is generated by free fields. Specifically, we consider the cases of symplectic fermions (odd superbosons), free fermions, and the $\beta\gamma$-system (even superfermions). In each case we determine the action of the Virasoro algebra.

Free field vertex algebras and the closely related oscillator Lie algebras are constructed using supersymmetric or skew-supersymmetric bilinear forms. Such forms are generally assumed to be even. We show that inhomogeneous supersymmetric bilinear forms, i.e., forms that are neither even nor odd lead to a new type of oscillator superalgebra. We classify such forms on a complex vector superspace up to dimension seven in the case when the restrictions of the form to the even and odd parts of the superspace are nondegenerate.

Given an integral lattice Q, we study the φ-twisted modules of the lattice vertex algebra V_Q where φ is not semisimple. We show that under certain conditions the twisted logarithmic vertex operators on a φ-twisted V_Q-module can be written in terms of the action of the φ-twisted Heisenberg algebra and a certain group on the module. We also show that the classification of such modules reduces to the classification of modules of the corresponding group.

Twisted Logarithmic Modules of Free Field and Lattice Vertex Algebras

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2017

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DEDICATION

For Anna, Paul, and Will with all my love.

BIOGRAPHY

Steven McKay Sullivan was born on December 3, 1986 in Saint George, Utah to Steve and Kathy Sullivan. He was the third and by many accounts the most difficult of their eight children. He developed a love for mathematics and science at a very young age, which he attributes at least in part to the time his father spent teaching him and his mother's constant encouragement. He fondly remembers tagging along to the evening astronomy labs taught by his father and looking through telescopes at the wonders of the universe. In 2009, he married Anna Cieslewicz. Together they attended the University of Utah. They graduated simultaneously, he with his Bachelor of Arts in Mathematics and she with her Masters of Occupational Therapy.

He visited North Carolina State University as a prospective graduate student and instantly fell in love with Raleigh, its people, and its beautiful scenery. As a graduate student he had the opportunity to do research with several amazing people and developed a love for teaching. He is excited to return to Saint George, Utah with Anna and their two sons, Paul and Will, to begin a faculty appointment in the mathematics department at Dixie State University.

ACKNOWLEDGEMENTS

I would like to thank Dr. Bojko Bakalov, whose constant support has been integral to my success as a graduate student. I could not have asked for a better advisor. I also want to thank Dr. Stitzinger, Dr. Jing, and Dr. Misra, along with the many other faculty members in the mathematics department at NC State who have taught me, supported me, and influenced me for good.

Finally I want to thank my wife, Anna. It is through her constant encouragement and support that I have made it this far in my career. Especially while I have worked to finish this dissertation, she has taken on the lion's share of the responsibilities in caring for our two young children on top of her commitments as a pediatric occupational therapist. For that I will forever be grateful to her.

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Chapter 1

Introduction

A *symmetry* is a mapping of a mathematical object onto itself that preserves the structure of the object. Often we learn valuable information about a mathematical object by studying its symmetries. Hence symmetries play a key role in many areas of mathematics and science. The set of symmetries of a mathematical object satisfies the axioms of a group. Thus a strong understanding of groups greatly aids the study of mathematical objects via their symmetries. Such an understanding for finite groups begins with the classification of the fundamental building blocks, known as finite simple groups. The classification of finite simple groups is a remarkable achievement comprising thousands of pages of publications (see [88] for a historical overview). The complete list consists of 18 infinite subfamilies and 26 *sporadic groups* that do not fit into any of the families.

The largest and most complicated sporadic group is known as the *monster group* M. Given a finite simple group such as M, one may pose the natural question whether it is the space of symmetries of some mathematical object. In the early 1980s, Robert Griess answered this question for M when he constructed the Griess algebra V and showed that M is its full automorphism group [47, 48]. Also occurring in the late 1970s and early 1980s was the adoption of vertex operators, which were already in use by physicists in dual resonance theory, for the purpose of studying the representation theory of affine Lie algebras [41, 78]. Frenkel, Lepowsky, and Meurman further developed and used this theory of vertex operators to construct the *moonshine module*, an irreducible graded representation of the monster group [42], thereby shedding some light on the mysterious moonshine conjectures [19]. Their construction made use of so-called *twisted vertex operators*, which are a central focus of this text.

Given the complexity of the monster group, it comes as no surprise that the moonshine module itself has a very complicated structure, namely that of a *vertex algebra*. In 1986, Richard Borcherds defined vertex algebras as a rigorous set of axioms describing the spaces of vertex operators appearing in physics and representation theory [17]. Two years later, Frenkel, Lepowsky, and Meurman added additional axioms, defining a *vertex operator algebra*. They constructed the so-called *monster vertex operator algebra* $V^\natural$, obtaining the twisted vertex operators used in their original construction of the moonshine module by considering the orbifold subalgebra of a certain lattice vertex algebra under an automorphism of order two [43].

Twisted vertex operators have proven useful in many of the original places where vertex operators appear. Given a finite-order automorphism φ of a vertex algebra V, one studies the *φ-twisted V-modules* [10, 28, 39, 43]. As in the case of the monster vertex algebra $V^\natural$, these φ-twisted modules give rise to orbifold subalgebras, which consist of the elements of V fixed by φ (see e.g. [27, 30, 67]).

In the case when the automorphism φ is not of finite order, φ-twisted modules can still be defined, but one is forced to allow logarithms of the formal variable in the twisted fields. The first example of such a module appeared in 2009 in a paper by Adamović and Milas [4]. Only a few weeks later, Huang published a paper formally defining such logarithmic modules [51]. However, the full development of the theory of logarithmic modules was hindered by the lack of a good notion of nth products of logarithmic fields. In 2015, using an alternative definition of nth products and a more general definition of a logarithmic module, Bakalov proved versions of the Borcherds identity and commutator formulas for logarithmic fields. With these tools in hand, he further developed the theory [6], and provided explicit examples of twisted logarithmic modules.

The motivation to study twisted logarithmic modules comes from logarithmic conformal field theory (see e.g. [5, 20, 72]) and from Gromov–Witten theory (cf. [11, 15, 31, 82]). They also look promising for applications to integrable systems and hierarchies of partial differential equations [24, 80]. The purpose of this text is to further develop the theory of twisted logarithmic modules as presented in [6], and to explicitly construct important examples of such modules.

In Chapter 2, we review some basic definitions from the theory of Lie (super)algebras and provide several examples of Lie algebras, including those of finite-dimensional Clifford-Weyl algebras. Then we move on to review infinite-dimensional affine Lie algebras including the special case of the Weyl affinization, and the closely related infinite-dimensional Clifford affinization.

In Chapter 3, we present our first original results of the text. We consider inhomogeneous supersymmetric bilinear forms, i.e., forms that are neither even nor odd. We classify such forms up to dimension seven in the case when the restriction to even and odd parts of the superspace is nondegenerate, and we indicate a possible approach for a complete classification in any finite dimension. We conclude the chapter by providing an example of an oscillator algebra obtained from such a form and consider its subalgebra of quadratic elements.

Chapter 4 is a review of the basic theory of vertex algebras. We begin by giving a definition of a vertex algebra in terms of modes and show how one can use this to arrive at a second definition in terms of quantum fields. Then we consider the cases of affine and free field vertex algebras, including the superbosons and superfermions. Next we briefly review lattice vertex algebras. Finally, we review twisted logarithmic modules of vertex algebras as developed in [6].

In Chapter 5, we present our classification of the twisted logarithmic modules of the four types of free field algebras arising via the Clifford or Weyl affinization of either an even or odd abelian Lie superalgebra, namely the free bosons (previously computed in [6]), symplectic fermions, free fermions, and bosonic ghost system.

In Chapter 6, we initiate the study of twisted logarithmic modules of lattice vertex algebras. Following similar methods to those found in [10] for the case of a finite-order automorphism, we reduce the classification of twisted logarithmic modules of vertex algebras to the question of classifying modules of a certain group. We provide explicit 3 and 4-dimensional examples.

In Chapter 7, we provide a conclusion and discuss the application of Chapters 5 and 6 to orbifolds and other subalgebras. We also mention several future projects, including the possible use of twisted logarithmic modules to construct new boson-fermion correspondences and integrable systems, the application of recently defined logarithmic vertex algebras to logarithmic conformal field theory, and possible applications of vertex algebras to lattice-based cryptosystems.

Chapter 2

Lie (Super)algebras

Often the group of symmetries of a mathematical object with structure takes the form of a *Lie group*, which is a group that also has a compatible structure of a smooth manifold. A Norwegian mathematician named Sophus Lie opened the curtains on Lie theory in the early 1870s when he made the realization that certain differential equations could be integrated using classical methods precisely because of a high-degree of symmetry in the equation, i.e., a large group of transformations which left the equation invariant. Thus the idea of continuous symmetry groups was born. At first he considered only one-parameter groups of transformations. Then in 1873, he introduced the notion of infinitesimal transformation groups, which allowed him to study continuous groups of noncommuting symmetries using a less-complicated linear space (see [12] for a detailed history).

In modern language a continuous symmetry group is known as a Lie group, and the corresponding linear space of infinitesimal transformations is a Lie algebra. Much can be learned about a Lie group by studying its Lie algebra. Lie algebras have proven useful in many areas of mathematics and physics. Thus they have received an enormous amount of attention during the last century.

2.1 Lie algebras

The theory of Lie algebras is interesting in its own right and has many applications. The definition can be given independently of groups as follows:

Definition 2.1.1. A complex *Lie algebra* is a vector space $\mathfrak{g}$ over $\mathbb{C}$ taken together with a binary *bracket* operation $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying

1. (linearity) $[cx+y,z]=c[x,y]+[x,z]$,
2. (skew-symmetry) $[x,y]=-[y,x]$,
3. (Jacobi identity) $[x,[y,z]]=[[x,y],z]+[y,[x,z]]$,

for all $x,y,z\in\mathfrak{g}$ and $c\in\mathbb{C}$.

The first example of a Lie algebra one encounters is generally the vector space $\mathbb{R}^3$ with the Lie bracket defined by the vector cross product:

$$[v,w]=v\times w.$$

Another natural example is $\mathfrak{g}=\mathfrak{gl}(V)$, the space of linear operators on a vector space V together with the commutator bracket:

$$[T,U]=TU-UT, \tag{2.1}$$

where TU is the composition of operators obtained by first applying U and then T. If we assume $\dim V=n<\infty$, then any choice of basis $\mathcal{B}$ for V defines an isomorphism $T\mapsto \big[T\big]_{\mathcal{B}}$ from $\mathfrak{gl}(V)$ to the *general linear Lie algebra* $\mathfrak{gl}(n)$ of $n\times n$ matrices over $\mathbb{C}$ with the bracket (2.1), where the composition TU becomes matrix multiplication.

Definition 2.1.2. A *representation* of a Lie algebra $\mathfrak{g}$ of finite dimension n is a linear map $R\colon\mathfrak{g}\to\mathfrak{gl}(n)$ which is a homomorphism of the Lie bracket:

$$R([a,b])=[R(a),R(b)]=R(a)R(b)-R(b)R(a).$$

Example 2.1.3. The Lie algebra $\mathfrak{sl}(2)\subseteq\mathfrak{gl}(2)$ is the Lie algebra of 2×2 matrices over $\mathbb{C}$ with trace zero. It has basis

$$e=\begin{bmatrix}0&1\\0&0\end{bmatrix},\qquad h=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\qquad f=\begin{bmatrix}0&0\\1&0\end{bmatrix}.$$

One can easily check that the commutation relations are given by

$$[h,e]=2e,\qquad [h,f]=-2f,\qquad [e,f]=h.$$

The *adjoint representation* of $\mathfrak{sl}(2)$ is obtained by letting $V = \mathfrak{sl}(2) \cong \mathbb{C}^3$ as vector spaces and defining the adjoint action of $\mathfrak{sl}(2)$ on V by

$$a \cdot v = [a, v]$$

for all $a \in \mathfrak{sl}(2)$ and $v \in V$. Thus by letting the elements of $\mathfrak{sl}(2)$ act on the basis $\{e, h, f\}$ of V, we can write the map R explicitly as

$$R(e) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R(h) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad R(f) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Example 2.1.4. The 3-dimensional *Heisenberg* Lie algebra $\mathfrak{n}(3)$ has basis $\{X, Y, Z\}$ with commutation relations

$$[X, Y] = Z, \qquad [X, Z] = [Y, Z] = 0. \tag{2.2}$$

In this case, the adjoint representation is not *faithful*, meaning the kernel of the representation map R is nontrivial ($Z \in \ker R$). However, a faithful 3-dimensional representation of $\mathfrak{n}(3)$ on $\mathbb{C}^3$ is given by the matrices

$$R(X) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R(Y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R(Z) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.3}$$

The Heisenberg algebra $\mathfrak{n}(3)$ also has the infinite-dimensional representation $V = \mathbb{C}[x]$, where the operators corresponding to the basis are

$$X = \partial_x, \qquad Y = x, \qquad Z = \mathrm{Id}. \tag{2.4}$$

One can easily check that these operators satisfy the commutation relations (2.2). Such representations involving differential operators on polynomial spaces will be important throughout this text. One advantage of this representation over (2.3) is that it extends to a faithful representation of the universal enveloping algebra of $\mathfrak{n}(3)$, which we discuss in subsection 2.1.2. It is a standard exercise to show that an infinite-dimensional representation such as (2.4) is required to represent Z as the identity operator.

The action of a Lie algebra on its representation may also be written in the language of modules:

Definition 2.1.5. Let $\mathfrak{g}$ be a Lie algebra. An $\mathfrak{g}$-*module* is a vector space V together with a bilinear map $\mathfrak{g} \times V \to V$ often written as $(a, v) \to a \cdot v$ satisfying

$$[a, b] \cdot v = a \cdot (b \cdot v) - b \cdot (a \cdot v), \qquad a, b \in \mathfrak{g},\ v \in V.$$

For any $\mathfrak{g}$-module V, the corresponding representation is given by the map $R\colon \mathfrak{g} \to \operatorname{End} V$ defined by

$$R(a)v = a \cdot v. \tag{2.5}$$

Given any representation V, (2.5) defines the corresponding $\mathfrak{g}$-module structure. Because of this equivalence, we move freely between the language of modules and representations throughout the text.

A subspace S of a Lie algebra $\mathfrak{g}$ is itself a Lie algebra if $[u, v] \in S$ for all $u, v \in S$. Then S is called a *subalgebra* of $\mathfrak{g}$. A subalgebra $I \in \mathfrak{g}$ satisfying $[u, v] \in I$ for all $u \in \mathfrak{g}, v \in I$ is called an *ideal* of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is *nilpotent* if the sequence of subalgebras

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{g}, \\ \mathfrak{g}_k &= [\mathfrak{g}, \mathfrak{g}_{k-1}], \qquad k \geq 1, \end{aligned}$$

satisfies $\mathfrak{g}_k = \{0\}$ for $k \gg 0$ (i.e., there exists $N \in \mathbb{N}$ such that $k > N$ implies $\mathfrak{g}_k = \{0\}$). Given a Lie algebra $\mathfrak{g}$, a *Cartan subalgebra* of $\mathfrak{g}$ is a nilpotent subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ satisfying the condition that if $[u, v] \in \mathfrak{h}$ for all $v \in \mathfrak{h}$, then $u \in \mathfrak{h}$.

Let $\mathfrak{h} \in \mathfrak{g}$ be a Cartan subalgebra. Consider a 1-dimensional representation of $\mathfrak{h}$, or equivalently a linear map $\lambda\colon \mathfrak{h} \to \mathbb{C}$ satisfying $\lambda([u, v]) = 0$ for all $u, v \in \mathfrak{h}$. Let V be a representation of $\mathfrak{g}$. If the space

$$V_\lambda = \{v \in V \colon a \cdot v = \lambda(a)v \text{ for all } a \in \mathfrak{h}\}$$

is nonzero, then λ is called a *weight* of the representation V, and V_λ is called the *weight* *space with weight* λ. A vector $v \in V_\lambda$ is a *weight vector*.

The matrix multiplication AB in $\mathfrak{gl}(n)$ endows it with the structure of an *associative algebra*, i.e., a vector space together with a compatible associative bilinear product. One can check that given any associative algebra A, the commutator bracket (2.1) defines a canonical Lie algebra structure on A.

Similar to Lie algebras, a representation V of an associative algebra is a linear map $R\colon A \to \operatorname{End} V$, where $\operatorname{End} V$ is the space of linear operators on V, such that R satisfies

$$R(ab) = R(a)R(b)$$

for all $a, b \in A$. The representation theory of an associative algebra is closely related to the representation theory of its canonical Lie algebra structure.

A subspace U of an associative algebra A satisfying $ab \in U$ for all $a, b \in U$ is itself an associative algebra known as a *subalgebra*. A subalgebra $I \subseteq A$ is called an *ideal* in A if $ab \in I$ and $ba \in I$ for every $a \in A$ and $b \in I$. Weights, weight spaces, and weight vectors are defined analogously to those of Lie algebras.

2.1.1 Exponentiation of operators

Given a vector space V, a linear operator $a \in \operatorname{End} V$, and any integer $n \geq 0$, the operator $a^n \in \operatorname{End} V$ is defined by applying a to V repeatedly n times:

$$a^n v = a(a(a(\cdots(av)\cdots))).$$

This operator is well-defined and linear for any $n = 0, 1, 2, \ldots$, where we let $a^0 = \mathrm{Id}$ be the identity operator on V. One may also consider the exponential operator

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}. \tag{2.6}$$

However, such an operator does not make sense in every case. If a is *locally nilpotent*, i.e., for every $v \in V$ we have $a^n v = 0$ for $n \gg 0$, then (2.6) is well-defined, as the sum terminates for the action on any vector. For example, the operators corresponding to the Heisenberg elements X, Y, and Z in the 3-dimensional representation of $\mathfrak{n}(3)$ in Example

2.1.4 all square to zero and hence can be exponentiated. For instance

$$\exp(X) = I + X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $V = \mathbb{C}[x]$ is the infinite-dimensional representation of $\mathfrak{n}(3)$ given in Example 2.1.4, then $X = \partial_x$ is locally nilpotent, and

$$\exp(X) = e^{\partial_x}$$

is the well-defined shift operator satisfying

$$e^{\partial_x} f(x) = f(x+1).$$

However, the exponentials of $Y = x$ and $Z = \mathrm{Id}$ are not well-defined on $\mathbb{C}[x]$. One way to fix this issue is to extend the representation to allow formal power series in x: $V = \mathbb{C}[[x]]$. Such issues are important to our discussion in Chapter 6, where we will need to exponentiate the action of a Heisenberg algebra on a certain representation.

Now assume $a \in A$ is such that e^a is well-defined as an operator on V. If $b \in A$ commutes with a, then from (2.6) we have $[b, e^a] = 0$. If $d \in A$ commutes with $[d, a]$, the following power rule holds:

$$[d, a^n] = n[d, a]a^{n-1}, \qquad n \geq 1. \tag{2.7}$$

Then (2.6) and (2.7) imply that if d commutes with $[d, a]$, then

$$[d, e^a] = [d, a]e^a. \tag{2.8}$$

Furthermore, if the operators a, b commute with their commutator $[a, b]$, then

$$e^a e^b = e^{\frac{1}{2}[a,b]+a+b} = e^{[a,b]}e^b e^a. \tag{2.9}$$

2.1.2 Universal enveloping algebra of a Lie algebra

Every Lie algebra $\mathfrak{g}$ embeds in some associative algebra A so that the bracket on $\mathfrak{g}$ is given by the commutator in A. In general this embedding is not unique. The freest such algebra, known as the universal enveloping algebra of $\mathfrak{g}$, has no relations other than those induced by the Lie algebra structure of $\mathfrak{g}$:

Definition 2.1.6. An associative algebra with multiplicative unit is called the *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ and denoted $\mathcal{U}(\mathfrak{g})$ if there exists an embedding $\sigma\colon \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ satisfying

(i) $\sigma([a,b]) = \sigma(a) \circ \sigma(b) - \sigma(b) \circ \sigma(a)$ for all $a, b \in \mathfrak{g}$,

(ii) the *universal property*: given any associative algebra A with unit 1 and a linear map $\sigma'\colon \mathfrak{g} \to A$ satisfying (i), there exists a unique homomorphism of algebras $R\colon \mathcal{U}(\mathfrak{g}) \to A'$ with $R(1) = 1'$ and $R \circ \sigma = \sigma'$. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & \mathcal{U}(\mathfrak{g}) \arrow[d, "R"] \\ & A \end{tikzcd} .$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is unique up to isomorphism. Let

$$T(\mathfrak{g}) = \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots \tag{2.10}$$

be the *tensor algebra*. Then $\mathcal{U}(\mathfrak{g})$ can be constructed as the quotient of $T(\mathfrak{g})$ by the two-sided ideal generated by the elements

$$a \otimes b - b \otimes a - [a,b], \qquad (a, b \in \mathfrak{g}),$$

where $[a,b] \in T(\mathfrak{g})$ is obtained via the inclusion $\mathfrak{g} \hookrightarrow T(\mathfrak{g})$. We will suppress the tensor notation in $\mathcal{U}(\mathfrak{g})$ and write $a_1 \otimes \cdots \otimes a_\ell = a_1 \cdots a_\ell$. We give explicit examples of universal enveloping algebras and computations in such algebras in Chapter 3.

2.2 Superalgebras

The study of Lie superalgebras originated in physics with the idea of supersymmetry, a proposed symmetry of spacetime and the fields of quantum field theory which related bosonic and fermionic particles (see, e.g., [83] for an introduction). Though to date all attempts to experimentally verify supersymmetry have failed, graded Lie superalgebras have proven themselves useful to other areas of physics and mathematics. For good references on Lie superalgebras see [61], [21], and [79].

Definition 2.2.1. A complex vector superspace is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over $\mathbb{C}$. The elements of $V_{\bar{0}}$ are called *even*, and the elements of $V_{\bar{1}}$ are called *odd*. An element which is either even or odd is called *homogeneous*. We let $p(x) = 0$ if $x \in V_{\bar{0}}$ and $p(x) = 1$ if $x \in V_{\bar{1}}$.

Note that each inhomogeneous vector $v \in V$ can be uniquely written as a sum $v = v_{\bar{0}} + v_{\bar{1}}$ of an even and an odd vector, allowing us to write the axioms of a Lie superalgebra in terms of homogeneous elements:

Definition 2.2.2. A complex *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ together with a bilinear bracket operation $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

1. (linearity) $[cx + y, z] = c[x, y] + [x, z]$,
2. (skew-supersymmetry): $[x, y] = -(-1)^{p(x)p(y)}[y, x]$,
3. (Jacobi identity): $[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]]$,

where each of x, y, and z is a homogeneous element of $\mathfrak{g}$. The bracket uniquely extends to inhomogeneous elements via bilinearity. The bracket of two homogeneous elements is again homogeneous, and the parity of the bracket is given by

$$p([x, y]) = p(x) + p(y) \mod 2.$$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace. Then $\operatorname{End} V$ naturally has the structure of a Lie superalgebra, where $A \in \operatorname{End} V$ is even if

$$A(V_{\bar{0}}) \subseteq V_{\bar{0}}, \qquad A(V_{\bar{1}}) \subseteq V_{\bar{1}},$$

and odd if

$$A(V_{\bar{0}}) \subseteq V_{\bar{1}}, \qquad A(V_{\bar{1}}) \subseteq V_{\bar{0}}.$$

Then the bracket of homogeneous operators is given by the *(super)commutator bracket*:

$$[a, b] = ab - (-1)^{p(a)p(b)} ba. \tag{2.11}$$

The commutator is then extended linearly to the inhomogeneous elements. In other words, given $a = a_{\bar{0}} + a_{\bar{1}}$ and $b = b_{\bar{0}} + b_{\bar{1}}$, we have

$$[a, b] = [a_{\bar{0}}, b_{\bar{0}}] + [a_{\bar{0}}, b_{\bar{1}}] + [a_{\bar{1}}, b_{\bar{0}}] + [a_{\bar{1}}, b_{\bar{1}}].$$

Let $m = \dim V_{\bar{0}}$ and $n = \dim V_{\bar{1}}$. Then any choice of homogeneous basis $\mathcal{B}$ for V gives an isomorphism $\operatorname{End} V \to \mathfrak{gl}(m|n)$, where $\mathfrak{gl}(m|n)$ is the set of $(m+n)$-dimensional matrices together with the commutator bracket (2.11).

An *associative superalgebra* is a $\mathbb{Z}_2$-graded associative algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ whose associative product satisfies $A_{\bar{a}} A_{\bar{b}} \subseteq A_{\overline{a+b}}$ where $a+b$ is taken modulo 2. Each associative superalgebra admits a canonical Lie superalgebra structure via the bracket (2.11). The *universal enveloping (super)algebra* $\mathcal{U}(\mathfrak{g})$ of a Lie superalgebra $\mathfrak{g}$ is constructed similarly to that of a Lie algebra. One takes the quotient of the tensor (super)algebra $T(\mathfrak{g})$ (cf. (2.10)) by the two-sided ideal generated by the elements

$$a \otimes b - (-1)^{p(a)p(b)} b \otimes a - [a, b].$$

The natural embedding $\sigma \colon \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ satisfies the universal property (cf. Definition 2.1.6). The parity of elements in $\mathcal{U}(\mathfrak{g})$ is inherited from $\mathfrak{g}$ via the rule $p(ab) = p(a) + p(b)$.

A representation of a Lie superalgebra is defined analogously to that of a Lie algebra with the added assumption that R is an even map. Then the corresponding Lie superalgebra module is defined by the action (2.5).

2.3 Clifford-Weyl Algebras

Many important Lie superalgebras can be constructed as subalgebras of oscillator superalgebras (see, e.g., [34]), which in turn are constructed using central extensions of abelian

Lie algebras. An *exact sequence* of Lie algebras is a sequence

$$\mathfrak{g}_0 \xrightarrow{f_1} \mathfrak{g}_1 \xrightarrow{f_2} \mathfrak{g}_2 \xrightarrow{f_3} \cdots \xrightarrow{f_k} \mathfrak{g}_k$$

of Lie algebras $\mathfrak{g}_i$ and homomorphisms f_i between them such that the image of f_i is exactly the kernel of f_{i+1} for $1 \leq i \leq k-1$. A *central extension* of a Lie algebra $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{c}$ is an exact sequence

$$0 \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{b} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

where $\pi \colon \mathfrak{b} \to \mathfrak{g}$ is the projection.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace. We aim to put a Lie superalgebra structure on $V \oplus \mathbb{C}K$ by declaring that K is an even central element and introducing the bracket

$$[a, b] = f(a, b)K,$$

where $f \colon V \times V \to \mathbb{C}$. Then the bilinearity of the bracket requires that $f(\cdot, \cdot) = (\cdot|\cdot) \colon V \times V \to \mathbb{C}$ is a *bilinear form* on V, i.e.,

$$(cu + v|w) = c(u|w) + (v|w),$$
$$(w|cu + v) = c(w|u) + (w|v),$$

for all $u, v, w \in V$. Furthermore, the skew-supersymmetry of the bracket requires that $(\cdot|\cdot)$ must also be skew-supersymmetric, i.e., for all homogeneous $a, b \in V$

$$(a|b) = -(-1)^{p(a)p(b)}(b|a).$$

We restrict our attention to the case when $(\cdot|\cdot)$ is *nondegenerate*, meaning that for every $u \in V$ there exist v and w in V such that $(u|v) \neq 0$ and $(w|u) \neq 0$. We also assume the restrictions of $(\cdot|\cdot)$ to $V_{\bar{0}}$ and $V_{\bar{1}}$ are nondegenerate. Finally, if we assume additionally that the form is *even*, i.e., $(V_{\bar{0}}|V_{\bar{1}}) = 0$, then we arrive at the following definition:

Definition 2.3.1. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace over $\mathbb{C}$ and $(\cdot|\cdot) \colon V \times V \to \mathbb{C}$ an even nondegenerate skew-supersymmetric bilinear form on V. Define a Lie superalgebra structure on $A = V \otimes \mathbb{C}K$ by declaring K to be an even central element and letting

$$[a, b] = (a|b)K, \qquad a, b \in V.$$

The Lie algebra A is known as a *1-dimensional central extension of* A, and the *Clifford-Weyl* algebra is the quotient $\mathcal{A} = \mathcal{U}(A)/(K - \mathbf{1})$, where $\mathbf{1}$ is the unit of $\mathcal{U}(A)$.

There exists a basis $\{b_1^\pm, \ldots, b_m^\pm\}$ for $V_{\bar{0}} \subseteq \mathcal{A}$ such that

$$[b_i^\pm, b_j^\pm] = 0, \qquad [b_i^-, b_j^+] = \delta_{ij},$$

where δ is the *Kronecker delta*:

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Such a basis can be obtained using a Gram-Schmidt orthogonalization process. The elements $b_i^\pm$ are known as *bosonic oscillators*. Similarly, there exists a *Darboux* basis of *fermionic oscillators* $\{a_1^\pm, \ldots, a_n^\pm\}$ for $V_{\bar{1}}$ satisfying

$$[a_i^\pm, a_j^\pm] = 0, \qquad [a_i^-, a_i^+] = \delta_{ij}. \tag{2.12}$$

For this reason, the Clifford-Weyl algebra $\mathcal{A}$ is called an *oscillator superalgebra*.

We note that the elements $a_i^\pm$ are odd, and thus the brackets in (2.12) are anti-commutator brackets: $[x, y] = xy + yx$. We make no distinction in the notation throughout this text, since the parity of the elements implies whether the commutator or anti-commutator should be used.

Remark 2.3.2. Another way to view $\mathcal{A}$ is as the quotient of the tensor algebra $T(A)$ by the ideal generated by the elements

$$a \otimes b - (-1)^{p(a)p(b)} b \otimes a - (a|b)\mathbf{1}$$

for all homogeneous $a, b \in \mathfrak{h}$.

In Chapter 3, we introduce analogous constructions allowing nontrivial interactions between bosons and fermions, i.e., allowing $(A_{\bar{0}}|A_{\bar{1}})$ to be nonzero. This leads to new oscillator-like algebras.

2.4 Affine Lie (Super)algebras

The most relevant Lie superalgebras to the present text are infinite-dimensional, meaning that any linear spanning set of the algebra contains infinitely many elements. As a first example, we consider infinite-dimensional analogues of Clifford-Weyl algebras, namely the Weyl and Clifford affinizations of abelian Lie superalgebras.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra. The *loop algebra* $\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ is the Lie superalgebra of Laurent polynomials in a formal variable t with coefficients in $\mathfrak{g}$. The parity of elements is given by $p(at^n) = p(a)$, and the brackets by

$$[at^n, bt^n] = [a, b]t^{m+n}.$$

Let $(\cdot|\cdot)$ be a nondegenerate even supersymmetric bilinear form on $\mathfrak{g}$. Furthermore, assume that $(\cdot|\cdot)$ is *invariant*, i.e., $([a, b]|c) = (a|[b, c])$ for all $a, b, c \in \mathfrak{g}$. Then the *affine Lie superalgebra* $\hat{\mathfrak{g}}$ is the 1-dimensional central extension of $\tilde{\mathfrak{g}}$ by an even vector K:

$$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus K$$

with brackets

$$[at^n, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a|b)K, \qquad [at^m, K] = 0. \tag{2.13}$$

The Lie algebra $\hat{\mathfrak{g}}$ has a *triangular decomposition*

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}^+ \oplus \hat{\mathfrak{g}}^0 \oplus \hat{\mathfrak{g}}^-, \tag{2.14}$$

where $\hat{\mathfrak{g}}^+ = \mathfrak{g}[t]t$, $\hat{\mathfrak{g}}^- = g[t^{-1}]t^{-1}$, and $\hat{\mathfrak{g}}^0 = \mathfrak{g}t^0 \oplus \mathbb{C}K$. Given such a decomposition and a representation V of $\mathfrak{g}$, a *highest weight vector* is a weight vector $v \in V$ satisfying $\hat{\mathfrak{g}}^+ v = \{0\}$, and V is a highest weight representation of $\hat{\mathfrak{g}}$ with highest weight v if $V \cong \mathcal{U}(\hat{\mathfrak{g}}^-)v$.

The *(generalized) Verma module* is the highest-weight $\hat{\mathfrak{g}}$-module obtained by letting $\mathfrak{g}[t]$ act as 0 and $K = \kappa\mathrm{Id}$ for some $\kappa \in \mathbb{C}$ on a representation $R = \mathbb{C}$, then inducing this representation to all of $\hat{\mathfrak{g}}$:

$$\widetilde{V}^\kappa = \mathrm{Ind}_{\mathfrak{g}[t]\oplus\mathbb{C}K}^{\hat{\mathfrak{g}}} R. \tag{2.15}$$

The highest weight vector, written as $\mathbf{1} \in \widetilde{V}^\kappa$, is the image of $1 \in R$ in the induced representaton.

2.4.1 Weyl Affinization

In the special case when $\mathfrak{g} = \mathfrak{h}$ is an abelian Lie superalgebra, the affinization $\hat{\mathfrak{h}}$ is called the *Weyl affinization* of $\mathfrak{h}$. The brackets (2.13) become

$$[at^m, bt^n] = m\delta_{m,-n}(a|b)K, \qquad [\hat{\mathfrak{h}}, K] = 0, \qquad (m, n \in \mathbb{Z}), \tag{2.16}$$

and $p(at^m) = p(a)$, $p(K) = 0$. Since the bilinear form $(\cdot|\cdot)$ is even and nondegenerate, there is an orthonormal basis $\{u_1, \ldots, u_k\}$ for $V_{\bar{0}}$ and a Darboux basis $\{u_{k+1}, \ldots, u_{k+2\ell}\}$ for $V_{\bar{1}}$ so that the corresponding Gram matrix, i.e., the matrix with entries $G_{ij} = (u_i|u_j)$, has the block form

$$G = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & J_{2\ell} \end{array}\right], \tag{2.17}$$

where

$$J_{2\ell} = \mathrm{diag}(J, \ldots, J), \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.18}$$

We make the following change of basis for $\mathfrak{h}_{\bar{0}}$:

$$v_i = \begin{cases} \frac{1}{\sqrt{2}}(u_i + \mathrm{i}u_{k-i+1}) & 1 \le i \le \frac{k}{2}, \\ u_i & i = \frac{k+1}{2}, \\ \frac{1}{\sqrt{2}}(u_{k-i+1} - \mathrm{i}u_i) & \frac{k}{2} + 1 < i \le k. \end{cases}$$

Then

$$(v_i|v_j) = \delta_{i+j,k+1}, \qquad 1 \le i, j \le k. \tag{2.19}$$

Using similar methods, we can find a basis $w_1, \ldots w_{2\ell}$ of $\hat{\mathfrak{h}}_{\bar{1}}$ satisfying

$$(w_i|w_j) = \delta_{i+j,2\ell+1} = -(w_j|w_i), \qquad 1 \le i \le j \le 2\ell. \tag{2.20}$$

Let $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^0 \oplus \hat{\mathfrak{h}}^-$ be the triangular decomposition (2.14). The elements of $\hat{\mathfrak{h}}^-$ supercommute. We write their action on the Verma module $\widetilde{V}^1$ as

$$v_i t^{-m} = x_{i,m}, \qquad w_j t^{-m} = \xi_{j,m}, \qquad 1 \le i \le k, \quad k+1 \le j \le k+2\ell, \tag{2.21}$$

where the x's commute among themselves and with all the ξ's, and the ξ's anticommute among themselves. Then

$$\widetilde{V}^1 \cong \mathbb{C}[x_{1,m}, \dots, x_{k,m}]_{m=1,2,\dots} \otimes \bigwedge(\xi_{1,m}, \dots, \xi_{2\ell,m})_{m=1,2,\dots}.$$

Using (2.16), we can explicitly write the action of $\hat{\mathfrak{h}}^+$ on $\widetilde{V}^1$. For example, we compute the action of $v_i t^m$ on $x_{j,n}$ as follows:

$$\begin{aligned}(v_i t^m) x_{j,n} &= (v_i t^m)(v_j t^{-n})\mathbf{1} \\ &= [v_i t^m, v_j t^{-n}]\mathbf{1} \\ &= \delta_{m,n}\delta_{i+j,k+1}\mathbf{1}.\end{aligned}$$

Thus $v_i t^m = \partial_{x_{k-j+1}}$ on $\widetilde{V}^1$. We compute the actions of the other elements similarly. Thus $\hat{\mathfrak{h}}^+$ acts on $\widetilde{V}^1$ by

$$v_i t^m = m\partial_{x_{k-i+1,m}}, \qquad w_j t^m = m\partial_{\xi_{2\ell-j+1,m}}, \qquad w_{\ell+j} = -m\partial_{\xi_{\ell-j+1}},$$

for $1 \leq i \leq k$, $1 \leq j \leq \ell$.

2.4.2 Clifford Affinization

We let $\mathfrak{a}$ be an abelian Lie superalgebra with $\dim \mathfrak{a} = d < \infty$, and $(\cdot|\cdot)$ be a nondegenerate even skew-supersymmetric bilinear form on $\mathfrak{a}$. The *Clifford affinization* of $\mathfrak{a}$ is the Lie superalgebra

$$C_{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}K$$

with commutation relations

$$[at^m, bt^n] = (a|b)\delta_{m,-n-1}K, \qquad [C_{\mathfrak{a}}, K] = 0 \tag{2.22}$$

for $m, n \in \mathbb{Z}$, where $p(at^m) = p(a)$ and $p(K) = 0$. We let $k = \dim \mathfrak{a}_{\bar{1}}$ and $2\ell = \dim \mathfrak{a}_{\bar{0}}$, and we choose a basis $w_1, \dots, w_{2\ell}$ of $\mathfrak{a}_{\bar{0}}$ satisfying (2.20) and a basis $v_1, \dots, v_k$ of $\mathfrak{a}_{\bar{1}}$ satisfying

(2.19). We have a triangular decomposition $C_{\mathfrak{a}} = C_{\mathfrak{a}}^{+} \oplus C_{\mathfrak{a}}^{0} \oplus C_{\mathfrak{a}}^{-}$, where

$$(C_{\mathfrak{a}})^{+} = \{w_i t^m, v_j t^m\}_{m=0,1,2,\dots},$$
$$(C_{\mathfrak{a}})^{-} = \{w_i t^{-m}, v_j t^{-m}\}_{m=1,2,\dots},$$

and $(C_{\mathfrak{a}})^0 = \mathbb{C}K$. The generalized Verma module $\widetilde{V}^1$ for $C_{\mathfrak{a}}$ is defined analogously to that of the Weyl affinization. We label the action of the elements of $C_{\mathfrak{a}}^{-}$ on $\widetilde{V}^1$ as

$$w_i t^{-m} = \xi_{i,m}, \qquad v_j t^{-m} = x_{\ell+j,m}.$$

Then $C_{\mathfrak{a}}^{+}$ acts by

$$w_i t^m = \partial_{x_{2\ell-i+1,m+1}}, \qquad w_{\ell+i} t^m = -\partial_{x_{\ell-i+1,m+1}}, \qquad v_j t^m = \partial_{\xi_{j,m+1}}.$$

2.4.3 The Virasoro Lie Algebra

Many important vertex algebras have the additional structure of a vertex operator algebra, meaning that they have a conformal element whose corresponding field is a Virasoro field. The modes of a Virasoro field satisfy the commutation relations of the Virasoro Lie algebra, which can be constructed as follows.

The space of real polynomial vector fields on the circle can be complexified to obtain the *Witt algebra* (see, e.g., [65]), which may be realized as the following space of differential operators on $\mathbb{C}[z, z^{-1}]$:

$$\mathfrak{d} = \text{span}\left\{d_m = -z^{m+1}\frac{d}{dz}\right\}_{m\in\mathbb{Z}}.$$

Then a short computation yields the following commutation relations in $\mathfrak{d}$:

$$[d_m, d_n] = (m-n)d_{m+n}, \qquad m, n \in \mathbb{Z}.$$

The Virasoro algebra Vir is the unique (up to isomorphism) nontrivial 1-dimensional central extension of $\mathfrak{d}$:

Definition 2.4.1. The Virasoro algebra with central charge C is the infinite-dimensional Lie algebra

$$Vir = \text{span}\{L_m, C\}_{m\in\mathbb{Z}}$$

with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m,-n}C, \qquad [C, L_m] = 0 \tag{2.23}$$

for all $m, n \in \mathbb{Z}$.

Remark 2.4.2. The Virasoro algebra can be realized as a subalgebra of a completion of $\mathcal{U}(\hat{\mathfrak{g}})$ for any affine Lie superalgebra $\hat{\mathfrak{g}}$ where $\mathfrak{g}$ is either simple or abelian using the so-called Sugawara construction. The Clifford affinization also yields such a construction. Though this can be done on the level of Lie (super)algebras, it is more elegantly expressed in the language of vertex algebras (cf. Section 4.2).

Chapter 3

Supersymmetric Bilinear Forms

In Chapter 2, we saw that supersymmetric and skew-supersymmetric bilinear forms can be used to construct interesting and useful algebras. Each bilinear form we considered in Chapter 2 was even. In the current chapter, we consider what happens when we do not assume $(\cdot|\cdot)$ is even. We have submitted the results contained in this chapter for publication under the title *Inhomogeneous Supersymmetric Bilinear Forms* [14].

We let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace and $(\cdot|\cdot)\colon V \times V \to \mathbb{C}$ be an inhomogeneous (i.e. neither even nor odd) skew-supersymmetric bilinear form. Then there exist $a \in V_{\bar{0}}$ and $b \in V_{\bar{1}}$ satisfying $(a|b) \neq 0$. Thus an oscillator-like algebra constructed from $(\cdot|\cdot)$ will have $[a, b] \neq 0$. We note that $p([a, b]) = 1$. So $[a, b]$ cannot be a scalar multiple of the even vector K. Instead we must adjoin a second central element κ satisfying $p(\kappa) = 1$. Then we can set $[a, b] = (a|b)\kappa$. To make this explicit, we consider the extension

$$A = V \oplus \mathbb{C}K \oplus \mathbb{C}\kappa. \tag{3.1}$$

We give this extension the structure of a Lie superalgebra by declaring K and κ to be even and odd central elements respectively and letting

$$[a, b] = \begin{cases} (a|b)K, & p(a) = p(b), \\ (a|b)\kappa, & p(a) \neq p(b), \end{cases} \tag{3.2}$$

for all homogeneous vectors $a, b \in V$. We obtain an oscillator-like superalgebra by letting $\mathcal{A} = \mathcal{U}(A)/(K - \mathbf{1})$.

Though skew-supersymmetric forms are used to construct oscillator algebras, there is a natural one-to-one correspondence between supersymmetric forms on V and skew-supersymmetric forms on V^Π, where V^Π is the superspace obtained by reversing the parity of the elements of V:

$$V^\Pi_{\bar 0} = V_{\bar 1}, \qquad V^\Pi_{\bar 1} = V_{\bar 0}.$$

We will call inhomogeneous supersymmetric bilinear forms whose restrictions to $V_{\bar 0}$ and $V_{\bar 1}$ are nondegenerate *pre-oscillator forms* and the algebras obtained as universal enveloping algebras of central extensions of V^Π *inhomogeneous oscillator algebras*.

Remark 3.0.1. Further relaxing the assumption that the restrictions of the form to even and odd parts are nondegenerate also leads to oscillator-like algebras. An example is the superspace spanned by an even vector v_1 and an odd vector v_2, with the form given by $(v_1|v_1) = (v_1|v_2) = (v_2|v_1) = 1$, $(v_2|v_2) = 0$.

We call a bilinear form $(\cdot|\cdot)$ on a superspace V *reducible* if $V = V_1 \oplus V_2$ is an orthogonal direct sum of subsuperspaces, i.e., if $(V_1|V_2) = 0$. Otherwise $(\cdot|\cdot)$ is *irreducible*. A natural first step in investigating inhomogeneous oscillator algebras is to classify all irreducible pre-oscillator forms on a given superspace V.

Remark 3.0.2. If $V = V_{\bar 0}$, then $(\cdot|\cdot)$ is symmetric, and via $Gram - Schmidt$ orthogonalization we obtain an orthonormal basis for V. Thus V is an orthogonal direct sum of 1-dimensional subspaces. On the other hand, if $V = V_{\bar 1}$, then $(\cdot|\cdot)$ is skew-symmetric and there exists a *Darboux basis* for V, i.e., a basis $\{v_1, \ldots, v_{2\ell}\}$ such that

$$(v_{2i-1}|v_{2j-1}) = (v_{2i}|v_{2j}) = 0, \qquad (v_{2i-1}|v_{2j}) = \delta_{i,j}, \qquad (1 \le i, j \le \ell). \tag{3.3}$$

Thus V restricts to an orthogonal direct sum of subspaces $V_i = \text{span}\{v_{2i-1}, v_{2i}\}$ with form (3.3).

We call two bilinear forms $(\cdot|\cdot)_1$ and $(\cdot|\cdot)_2$ *equivalent* if there exists an even automorphism $\varphi : V \to V$ satisfying

$$(\varphi u|\varphi v)_1 = (u|v)_2 \tag{3.4}$$

for all $u, v \in V$. In Section 3.1, we introduce invariants that help us distinguish between equivalence classes of forms. In Section 3.2, we find representatives of equivalence classes of irreducible forms and use the invariants to prove that they are irreducible and distinct. This allows us to obtain the main result of this chapter: a classification of pre-oscillator forms on complex superspaces of dimension up to 7. In Section 3.3, we discuss a 3-

dimensional example.

3.1 Invariants

In this section, we reduce the question of classification to one of matrices. Given a vector (super)space V and a bilinear form $(\cdot|\cdot)$ on V, a choice of basis $\{v_i\}$ allows us to construct the *Gram matrix* G of $(\cdot|\cdot)$ in this basis by letting

$$G_{ij} = (v_i|v_j).$$

Then if we write any two vectors of V in component form:

$$x = (x_1, \dots, x_d)^T, \qquad y = (y_1, \dots, y_d)^T,$$

we have

$$(x|y) = x^T G y.$$

Throughout the remainder of this chapter, every bilinear form $(\cdot|\cdot)\colon V \times V \to \mathbb{C}$ we discuss will be assumed to be a pre-oscillator form. Then by Remark 3.0.2 we can choose a homogeneous basis for V so that the Gram matrix of the form is a block matrix of type

$$G = \left[\begin{array}{c|c} I_k & B \\ \hline B^T & J_{2\ell} \end{array}\right], \tag{3.5}$$

where $\dim V_{\bar{0}} = k$, $\dim V_{\bar{1}} = 2\ell$, $B \in \mathbb{C}^{k \times 2\ell}$, and $J_{2\ell}$ is the matrix (2.18).

We aim to find invariants that distinguish the equivalence classes of forms on a given superspace V. We assume $(\cdot|\cdot)_1$ and $(\cdot|\cdot)_2$ are equivalent forms on V, and choose bases for V so that the Gram matrices G_i $(i = 1, 2)$ take the form (3.5) with $B = B_i$. If M is the matrix of an even automorphism φ satisfying (3.4), then (3.4) can be written as a matrix equation

$$M^T G_1 M = G_2. \tag{3.6}$$

Since φ is even, M is a block matrix of the form

$$M = \left[\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array}\right].$$

Then (3.6) holds if and only if

$$\begin{aligned} X &\in \mathrm{O}(k) = \{X \in \mathrm{GL}(k) \colon X^T X = I_k\}, \\ Y &\in \mathrm{Sp}(2\ell) = \{Y \in \mathrm{GL}(2\ell) \colon Y^T J_{2\ell} Y = J_{2\ell}\}, \\ B_2 &= X^T B_1 Y. \end{aligned}$$

Hence finding the equivalence class of a bilinear form with Gram matrix (3.5) is equivalent to finding the orbit of B under the right action of $\mathrm{O}(k) \times \mathrm{Sp}(2\ell)$ on $\mathbb{C}^{k\times 2\ell}$ defined by

$$B \cdot (X, Y) = X^T B Y. \tag{3.7}$$

Given $X \in \mathrm{O}(k)$ and $Y \in \mathrm{Sp}(2\ell)$, let $A = X^T B Y$. Then $A^T A = Y^T B^T B Y$ is independent of X. Thus any invariant of the right action of $\mathrm{Sp}(2\ell)$ on $\mathbb{C}^{2\ell\times 2\ell}$ given by

$$C \cdot Y = Y^T C Y$$

is an invariant of $B^T B$ under the action (3.7). Similarly $A J_{2\ell} A^T = X^T B J_{2\ell} B^T X$ is independent of Y. Thus any invariant of the right action of $\mathrm{O}(k)$ on $\mathbb{C}^{k\times k}$ given by

$$C \cdot X = X^T C X \tag{3.8}$$

is an invariant of $B J_{2\ell} B^T$ under the joint action (3.7).

Let us consider the action (3.8). Since $\det(X) = \pm 1$, it is apparent that $\det(X^T C X) = \det(C)$. Thus the determinant of $B J_{2\ell} B^T$ is an invariant of the action (3.7). More generally, we can write

$$X^T (B J_{2\ell} B^T - \lambda I_k) X = (X^T B Y) J_{2\ell} (X^T B Y)^T - \lambda I_k.$$

This implies that the characteristic polynomials of the two matrices $BJ_{2\ell}B^T$ and

$$(X^TBY)J_{2\ell}(X^TBY)^T$$

are equal. Therefore, the characteristic polynomial

$$P_B(\lambda) = \det(BJ_{2\ell}B^T - \lambda I_k)$$

is invariant under the joint action (3.7). We have obtained the following theorem.

Theorem 3.1.1. *Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace with $\dim V_{\bar{0}} = k$ and $\dim V_{\bar{1}} = 2\ell$, and let $(\cdot|\cdot)\colon V \times V \to V$ be an inhomogeneous nondegenerate supersymmetric bilinear form on V with Gram matrix (3.5) in a given homogeneous basis. Then each coefficent of the characteristic polynomial $P_B(\lambda)$ is an invariant for the joint action (3.7) of $\mathrm{O}(k)\times\mathrm{Sp}(2\ell)$ on $B \in \mathbb{C}^{k\times 2\ell}$.*

By a similar argument, the polynomial

$$Q_B(\lambda) = \det(B^TB - \lambda J_{2\ell})$$

is also invariant under the action (3.7). However, it is essentially the same as $P_B(\lambda)$.

Lemma 3.1.2. *With the above notation, we have*

$$Q_B(\lambda) = (-1)^k \lambda^{2\ell-k} P_B(-\lambda). \tag{3.9}$$

Proof. Sylvester's determinant identity (see, e.g., [84]) states that if U and W are matrices of size $m \times n$ and $n \times m$ respectively, then

$$\det(I_m + UW) = \det(I_n + WU).$$

We replace U with $\lambda^{-1}A^{-1}U$, where A is an invertible $m \times m$ matrix, thus obtaining

$$\det(\lambda A + UW) = \lambda^{m-n} \det(\lambda I_n + WA^{-1}U) \det(A).$$

Letting $m = 2\ell$, $n = k$, $A = -J_{2\ell}$, and $U^T = W = B$, we get (3.9). $\square$

Along with the constant terms

$$p_0(B) = P_B(0) = \det(BJ_{2\ell}B^T),$$
$$q_0(B) = Q_B(0) = \det(B^T B),$$

the coefficient of λ^{k-2} in $P_B(\lambda)$ will also be useful in the following section. Let $R_1, \ldots, R_k$ be the rows of B. Then up to a sign, this coefficient is given by

$$\begin{aligned} p_{k-2}(B) &= \sum_{1 \le s_1 < s_2 \le k} \left(R_{s_i} J_{2\ell} R_{s_j}^T\right)^2 \\ &= \sum_{1 \le s_1 < s_2 \le k} \left(\sum_{t=1}^{\ell} \det \begin{bmatrix} b_{s_1,2t-1} & b_{s_1,2t} \\ b_{s_2,2t-1} & b_{s_2,2t} \end{bmatrix} \right)^2. \end{aligned} \tag{3.10}$$

Remark 3.1.3. Though not an invariant, the following observation will be useful in the classification given in the next section. Assume the matrix B satisfies $C^T C = 0$ for every linear combination C of columns of B. Then this same property holds for every matrix in the orbit of B under the action (3.7).

3.2 Classification up to dimension 7

In this section, for each superspace V with $\dim V \le 7$, we find a representative of each equivalence class of irreducible pre-oscillator forms on V. First, we let $b = (b_1, \ldots, b_k)^T \in \mathbb{C}^k$ be a column vector and $q(b) = b^T b$. We aim to find a canonical form for a representative of the orbit of b under the left action Xb of $X \in \mathrm{O}(k)$.

Proposition 3.2.1. *Let $b \in \mathbb{C}^k$ be a column vector. If $q(b) \neq 0$, then under the left action of $\mathrm{O}(k)$, b is in the orbit of*

$$(\sqrt{q(b)}, 0, \ldots, 0)^T, \tag{3.11}$$

where $\sqrt{a}$ is defined to be the unique element γ of

$$\mathbb{C}^+ = \{\gamma \in \mathbb{C} : \operatorname{Re}\gamma > 0 \text{ or } \operatorname{Re}\gamma = 0 \text{ and } \operatorname{Im}\gamma > 0\}$$

satisfying $\gamma^2 = a$. On the other hand, if $q(b) = 0$, then either $b = 0$ or b is in the orbit *of*

$$(1, \mathrm{i}, 0, \ldots, 0)^T. \tag{3.12}$$

Proof. If $q(b) \neq 0$, it is enough to give the proof in the case when $k = 2$, because the general case can be reduced to that. Let $b = (b_1, b_2)^T$ be such that $q(b) \neq 0$. Then the matrix

$$X = \frac{1}{\sqrt{q(b)}} \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix} \in \mathrm{O}(2)$$

satisfies $Xb = (\sqrt{q(b)}, 0)^T$.

Now assume $q(b) = 0$. If $k = 2$, then $q(b) = 0$ implies $b = (b_1, \pm \mathrm{i} b_1)^T$. These two possible forms for b are in the same orbit, so without loss of generality $b = (b_1, \mathrm{i} b_1)^T$. Then $Xb = (1, \mathrm{i})^T$ for

$$X = \frac{1}{2b_1} \begin{bmatrix} b_1^2 + 1 & \mathrm{i}(b_1^2 - 1) \\ -\mathrm{i}(b_1^2 - 1) & b_1^2 + 1 \end{bmatrix} \in \mathrm{O}(2). \tag{3.13}$$

Now suppose $k \geq 3$ and b is nonzero. Then possibly after reordering, we may assume $b_1 \neq 0$. Then

$$b_2^2 + \cdots + b_k^2 = -b_1^2 \neq 0.$$

There exists an $X \in \mathrm{O}(k)$ that leaves b_1 invariant and replaces $(b_2, \ldots, b_k)^T$ with a vector of the form (3.11). Thus we obtain

$$Xb = (b_1, \mathrm{i} b_1, 0, \ldots, 0)^T.$$

Then using an orthogonal transformation that acts as (3.13) on rows 1 and 2 and as the identity on the remaining rows, we obtain (3.12). $\square$

Now we consider the right action of the symplectic group: $B \mapsto BY$ for $B \in \mathbb{C}^{k \times 2\ell}$ and $Y \in \mathrm{Sp}(2\ell)$. Let $C_1, \ldots, C_{2\ell}$ be the columns of B. For $1 \leq i \leq \ell$, we will say columns C_{2i-1}, C_{2i} are *paired*. Using suitable $Y \in \mathrm{Sp}(2\ell)$, we can perform the following elementary operations on paired columns of B.

i. Rescaling by $\lambda \neq 0$:

$$(\ldots, C_{2i-1}, C_{2i}, \ldots) \mapsto (\ldots, \lambda C_{2i-1}, \lambda^{-1} C_{2i}, \ldots).$$

ii. Adding any multiple of a column to its paired column:

$$(\ldots, C_{2i-1}, C_{2i}, \ldots) \mapsto (\ldots, C_{2i-1}, C_{2i} + \lambda C_{2i-1}, \ldots).$$

iii. Switching columns:

$$(\ldots, C_{2i-1}, C_{2i}, \ldots) \mapsto (\ldots, C_{2i}, -C_{2i-1}, \ldots).$$

The following are elementary operations outside of pairs.

iv. Adding a multiple of a column to a column other than its pair:

$$\begin{aligned}&(\ldots, C_{2i-1}, C_{2i}, \ldots, C_{2j-1}, C_{2j}, \ldots)\\ &\qquad \mapsto (\ldots, C_{2i-1} - \lambda C_{2j-1}, C_{2i}, \ldots, C_{2j-1}, C_{2j} + \lambda C_{2i}, \ldots).\end{aligned}$$

v. Switching pairs of columns:

$$\begin{aligned}&(\ldots, C_{2i-1}, C_{2i}, \ldots, C_{2j-1}, C_{2j}, \ldots)\\ &\qquad\qquad \mapsto (\ldots, C_{2j-1}, C_{2j}, \ldots, C_{2i-1}, C_{2i}, \ldots).\end{aligned}$$

From the above discussion, we see that using the orthogonal action, we can always reduce to at most 4ℓ nonzero rows. Also, using the symplectic action we can always reduce so that at most the first $2k$ columns are nonzero. Thus if V has an irreducible bilinear form, its even and odd dimensions must satisfy $\ell \leq k \leq 4\ell$. It follows that to obtain a complete classification of irreducible pre-oscillator forms up to dimension 7, we only need to consider the following cases.

Case $k = \ell = 1$

It is easy to see that B is in the orbit of

$$\mathrm{B}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Case $k = \ell = 2$

If there exists a linear combination $C = \lambda_1 C_1 + \cdots + \lambda_{2\ell} C_{2\ell}$ of columns satisfying $q(C) \neq 0$, then we can use the symplectic action to put C in the first column of B. Then using the orthogonal action and rescaling we obtain

$$B = \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \end{bmatrix}.$$

Then using the symplectic action we eliminate b_{13} and b_{14} followed by b_{12} obtaining a matrix of the form

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & b_{24} \end{bmatrix}.$$

Then again via the symplectic action we get

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{bmatrix}.$$

If $\alpha = 0$, this reduces to B_1. If $\alpha \neq 0$, then this reduces to the case

$$\mathrm{B}_{2,\alpha} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \qquad (\alpha \in \mathbb{C}^+).$$

Finally, assume every linear combination C of the columns of B satisfies $q(C) = 0$. Then using the orthogonal action we obtain

$$B = \begin{bmatrix} 1 & b_{12} & b_{13} & b_{14} \\ \mathrm{i} & \mathrm{i}b_{12} & \mathrm{i}b_{13} & \mathrm{i}b_{14} \end{bmatrix}.$$

Using the symplectic action we eliminate columns 2, 3, and 4. Thus this reduces to the case

$$\mathrm{B}_3 = \begin{bmatrix} 1 & 0 \\ \mathrm{i} & 0 \end{bmatrix}.$$

Notice that this argument also shows that every irreducible matrix B of size $k = 2, \ell = 1$ is in the orbit of $\mathrm{B}_{2,\alpha}$ or B_3. The details of the remaining cases are similar to those already shown, so we omit them and provide the results.

Case $k = 3, \ell = 1$

The matrix B is in the orbit of

$$\mathrm{B}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \mathrm{i} \end{bmatrix},$$

or it can be reduced to B_1, $\mathrm{B}_{2,\alpha}$, or B_3.

Case $k = 3, \ell = 2$

Either the matrix B is in the orbit of

$$\mathrm{B}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \end{bmatrix},$$

or it can be reduced to an orthogonal direct sum involving the previous four irreducible cases.

Case $k = 4, \ell = 1$

Either B is in the orbit of

$$\mathrm{B}_6 = \begin{bmatrix} 1 & 0 \\ \mathrm{i} & 0 \\ 0 & 1 \\ 0 & \mathrm{i} \end{bmatrix},$$

or it reduces to one of B_1, $\mathrm{B}_{2,\alpha}$, B_3, or B_4.

Theorem 3.2.2. *Let* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *be a superspace of dimension* ≤ 7. *Then Table 3.1* *is a complete list up to equivalence of inhomogeneous irreducible supersymmetric bilinear forms on V whose restrictions to $V_{\bar{0}}$ and $V_{\bar{1}}$ are nondegenerate* ($\alpha \in \mathbb{C}^+$):

Table 3.1: Classification of inhomogeneous supersymmetric bilinear forms up to dimension 7.

$\dim V_{\bar{0}}$	$\dim V_{\bar{1}}$	B
1	2	B_1
2	2	$\mathrm{B}_{2,\alpha},\ \mathrm{B}_3$
3	2	B_4
3	4	B_5
4	2	B_6

.

Proof. We have shown that every matrix B corresponding to a bilinear form on a superspace of dimension up to seven is either reducible or in the orbit of one of the representatives listed in the table. In order to complete the proof of the theorem we need to check that the entries of the table are irreducible and distinct, i.e., in different orbits.

The matrix B_1 clearly corresponds to an irreducible form. We note that $\det(\mathrm{B}_3)^2 = 0$ and $\det(\mathrm{B}_{2,\alpha})^2 = \alpha^2$. Thus these matrices are all in distinct orbits. If any of these matrices reduces, it must be in the orbit of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.14}$$

but this matrix has determinant 0 and therefore cannot share an orbit with $\mathrm{B}_{2,\alpha}$. So $\mathrm{B}_{2,\alpha}$ is irreducible for $\alpha \in \mathbb{C}$. That B_3 is not in the orbit of (3.14) follows from Remark 3.1.3.

The matrix B_4 has rank 2 and so reduces only if it is in the orbit of the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & 0 \end{bmatrix} \tag{3.15}$$

for some $\alpha \in \mathbb{C}^+$. But the invariant p_1 (c.f. (3.10)) evaluated on the matrix (3.15) is $\alpha^2 \neq 0$, whereas $p_1(\mathrm{B}_4) = 0$. So B_4 is irreducible. The matrix B_6 has rank 2 and is therefore reducible only if it is in the orbit of B_4 or the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.16}$$

for some $\alpha \in \mathbb{C}^+$. By Remark 3.1.3, B_6 is not in the orbit of B_4 or (3.16). So B_6 is irreducible.

B_5 has rank three and so is reducible only if it is in the orbit of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.17}$$

for some $\alpha \in \mathbb{C}^+$. But the invariant p_1 shows that B_5 is not in the same orbit as (3.17). So B_5 is irreducible. $\square$

Applying our methods to higher dimensions, we have found that the classification seems to become increasingly complicated as $\dim V$ increases. Though it should be possible to extend the classification to dimension 8 or 9 with the methods of this paper, a more general approach will be needed for a complete classification in any dimension. We plan to address this question in the future by using the theory of *θ-groups* (see [22, 62]).

3.3 A 3-dimensional example

Before considering an example of such a form, let us recall a construction of $\mathfrak{osp}(1|2)$ as a subalgebra of a homogeneous oscillator algebra. Assume $k = \ell = 1$ and $(\cdot|\cdot)$ is a nondegenerate, even, skew-supersymmetric bilinear form. Then there exist bases $\{b_1, b_2\}$

of $V_{\bar{0}}$ and $\{a\}$ of $V_{\bar{1}}$ such that the Gram matrix of the bilinear form is

$$G = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

The central extension $A = V \oplus \mathbb{C}K$ has nonzero brackets

$$[b_1, b_2] = K, \qquad [a, a] = K,$$

Then $\mathfrak{osp}(1|2)$ is realized as a subalgebra of the oscillator algebra $\mathcal{U}(A)$ as follows (see, e.g., [34]):

$$\begin{aligned} H &= \frac{1}{4}b_2b_1 + \frac{1}{4}b_1b_2, & F^+ &= \frac{1}{4}\ ab_2 + \frac{1}{4}b_2a, \\ E^+ &= \frac{1}{2}b_2^2, & F^- &= \frac{1}{4}\ ab_1 + \frac{1}{4}b_1a, \\ E^- &= -\frac{1}{2}b_1^2, \end{aligned} \tag{3.18}$$

with brackets

$$\begin{aligned} [H, E^\pm] &= \pm E^\pm, & [E^+, E^-] &= 2H, \\ [H, F^\pm] &= \pm\frac{1}{2}F^\pm, & [F^+, F^-] &= \frac{1}{2}H, \\ [E^\pm, F^\mp] &= -F^\pm, & [F^\pm, F^\pm] &= \pm\frac{1}{2}E^\pm. \end{aligned}$$

For example, to compute $[E^+, E^-]$, we repeatedly make use of the fact that $b_2b_1 = b_1b_2 - 1$ obtaining

$$
\begin{aligned}
E^+E^- &= -\frac{1}{4}b_2^2b_1^2 \\
&= -\frac{1}{4}b_2(b_1b_2 - 1)b_1 \\
&= -\frac{1}{4}b_2b_1b_2b_1 + \frac{1}{4}b_2b_1 \\
&= -\frac{1}{4}(b_1b_2 - 1)(b_1b_2 - 1) + \frac{1}{4}b_2b_1 \\
&= -\frac{1}{4}b_1b_2b_1b_2 + \frac{1}{2}b_1b_2 + \frac{1}{4}b_2b_1 - \frac{1}{4}K \\
&= -\frac{1}{2}b_1(b_1b_2 - 1)b_2 + \frac{1}{2}b_1b_2 + \frac{1}{4}b_2b_1 - \frac{1}{4}K \\
&= -\frac{1}{2}b_1^2b_2^2 + \frac{1}{2}b_1b_2 + \frac{1}{2}b_2b_1 \\
&= E^-E^+ + 2H.
\end{aligned}
$$

We obtain a highest weight representation of $\mathcal{A}$ on a Fock space $\mathcal{F} = \mathbb{C}[x] \otimes \mathbb{C}\xi$ where ξ is an odd indeterminate satisfying $\xi^2 = \frac{1}{2}$. The action of A on $\mathcal{F}$ is given by

$$
b_1 \mapsto \partial_x, \qquad b_2 \mapsto x, \qquad a \mapsto \xi, \qquad K \mapsto \mathrm{Id}.
$$

Now consider the case when the Gram matrix is given instead by

$$
G = \left[\begin{array}{cc|c} 0 & 1 & 1 \\ -1 & 0 & 0 \\ \hline -1 & 0 & 1 \end{array}\right].
$$

This inhomogeneous form gives rise to the central extension (3.1) with nonzero brackets

$$
[b_1, b_2] = K, \qquad [b_1, a] = \kappa, \qquad \{a, a\} = K.
$$

We consider the same elements (3.18) of $\mathcal{A}$ we used to construct $\mathfrak{osp}(1|2)$ in the previous example. Then the brackets become

$$
\begin{aligned}
[H, E^{\pm}] &= \pm E^{\pm}, & [E^-, F^-] &= \kappa E^-, \\
[H, F^+] &= \frac{1}{2}F^+ + \frac{\kappa}{2}E^+, & [E^+, E^-] &= 2H, \\
[H, F^-] &= -\frac{1}{2}F^- + \frac{\kappa}{2}H, & [F^+, F^-] &= \frac{1}{2}H - \frac{\kappa}{2}F^+, \\
[E^+, F^-] &= -F^+, & [F^+, F^+] &= \frac{1}{2}E^+, \\
[E^-, F^+] &= -F^- - \kappa H, & [F^-, F^-] &= -\frac{1}{2}E^- - \kappa F^-.
\end{aligned} \tag{3.19}
$$

Observe that $\kappa^2 = 0$ in $\mathcal{A}$. Thus the brackets of the $\mathfrak{osp}(1|2)$ subalgebra have been modified by elements of the abelian ideal

$$M = \operatorname{span}\{\kappa H, \kappa E^{\pm}, \kappa F^{\pm}\}.$$

We note that $\operatorname{span}\{H, E^{\pm}, F^{\pm}\}$ acts on M as the adjoint representation of $\mathfrak{osp}(1|2)$, and thus we have a Lie superalgebra structure on the vector space $L = \mathfrak{osp}(1|2) \oplus M$ such that the projection $\pi\colon L \to \mathfrak{osp}(1|2)$ is a surjective homomorphism, and the restriction of the adjoint representation of L to M yields the original action of $\mathfrak{osp}(1|2)$ on M. Thus L is an abelian extension of $\mathfrak{osp}(1|2)$ by its adjoint representation viewed as an abelian Lie superalgebra with parities reversed.

Denote by $[\cdot,\cdot]_L$ the bracket on L given by (3.19) and let

$$\gamma\colon \mathfrak{osp}(1|2) \times \mathfrak{osp}(1|2) \to \mathfrak{osp}(1|2)$$

satisfy

$$[a, b]_L = [a, b] + \kappa\gamma(a, b).$$

This extension is trivial if there exists an odd linear map $f\colon \mathfrak{osp}(1|2) \to \mathfrak{osp}(1|2)$ such that

$$\gamma(a, b) = (-1)^{p(a)}[a, f(b)] - (-1)^{(p(a)+1)p(b)}[b, f(a)] - f([a, b]) \tag{3.20}$$

for all $a, b \in \mathfrak{osp}(1|2)$. It is straightforward to check that the following choice of f satisfies (3.20):

$$f(H) = f(E^{\pm}) = 0, \qquad f(F^+) = E^+, \qquad f(F^-) = H.$$

As before, we can represent A on the Fock space $\mathcal{F} = \mathbb{C}[x] \otimes \bigwedge(\xi, \kappa)$ where ξ is odd with $\xi^2 = \frac{1}{2}$ and κ acts as an odd indeterminate that we denote again by κ. Then the action of A on $\mathcal{F}$ is given by

$$b_1 \mapsto \partial_x + \kappa \partial_\xi, \qquad b_2 \mapsto x, \qquad a \mapsto \xi, \qquad K \mapsto \mathrm{Id}, \qquad \kappa \mapsto \kappa.$$

Chapter 4

Vertex Algebras

Vertex algebras are highly nontrivial generalizations of both Lie algebras and associative algebras. In essence, a vertex algebra is a vector space together with bilinear nth products for every integer n satisfying a restrictive set of axioms. Besides their applications to number theory and the representation theory of finite groups and Lie algebras discussed in the introduction of this text, vertex algebras also apply to other areas of mathematics such as: combinatorics [60], algebraic geometry and algebraic topology [40], and integrable systems of differential equations [80]. They also have important applications to physics including conformal quantum field theory [16, 26] and string theory [92].

In the present chapter, we give two equivalent definitions of vertex algebras and recall several of their basic properties. Then we review the examples of Affine vertex algebras, free superbosons, free superfermions, and lattice vertex algebras. Finally, we review portions of the theory of twisted logarithmic modules of vertex algebras as developed in [6]. For more detailed introductions to vertex algebras, see, e.g., [60, 75].

4.1 A basic introduction

Several equivalent definitions of vertex algebras exist. For a detailed exposition of the different definitions and comparisons between them, see [25, Section 1]. We give the two definitions which will be most useful in the sequel.

Definition 4.1.1. A *vertex algebra* is a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over $\mathbb{C}$ with a distinguished *vacuum* element $\mathbf{1} \in V_{\bar{0}}$ and bilinear nth products $a_{(n)}b \in V$, $(n \in \mathbb{Z})$

satisfying

$$
\begin{aligned}
a_{(n)}\mathbf{1} &= \delta_{n,-1}a, && (n \geq -1) \\
\mathbf{1}_{(n)}b &= \delta_{n,-1}b, && (n \in \mathbb{Z}) \\
a_{(n)}b &= 0 && (n \gg 0)
\end{aligned}
$$

and the *Borcherds identity*:

$$
\begin{aligned}
\sum_{i=0}^{\infty}(-1)^i \binom{n}{i} \Big(a_{(m+n-i)}(b_{(k+i)}c) - (-1)^n b_{(k+n-i)}(a_{(m+i)}c) \Big) \\
= \sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c.
\end{aligned}
\tag{4.1}
$$

The Borcherds identity can be thought of as a generalization of the Jacobi identity of a Lie algebra. The *infinitesimal translation operator* $T\colon V \to V$ is defined by

$$
Ta = a_{(-2)}\mathbf{1} \tag{4.2}
$$

for $a \in V$.

Since the nth products are bilinear, for any $n \in \mathbb{Z}$ and any $a \in V$ the map $a_{(n)}\colon V \to V$ defined by $a_{(n)}(b) = a_{(n)}b$ is linear. For any given $a \in V$, it is convenient to arrange all of the *modes* $a_{(n)}$ of a into a formal series

$$
Y(a, z) = \sum_{m \in \mathbb{Z}} a_{(n)} z^{-n-1}. \tag{4.3}
$$

The map $Y(\cdot, z)\colon V \to \mathrm{End}(V)[[z, z^{-1}]]$ is called the *state-field correspondence*. The series (4.3) is an example of a *(quantum) field*, which is a formal distribution

$$
b(z) = \sum_{m \in \mathbb{Z}} b_{(n)} z^{-n-1}, \qquad b_{(n)} \in \mathrm{End}(V)
$$

satisfying the condition that for every $v \in V$, $b_{(n)}v = 0$ for $n \gg 0$. In other words there exists a positive integer M such that $n > M$ implies $b_{(n)}v = 0$. We denote the space of fields on V by $\mathrm{Fie}(V)$.

Recall that $\mathrm{End}\, V$ carries the structure of an associative superalgebra (see Section

2.2). Thus there is a natural Lie superalgebra structure on the *modes* $a_{(n)} \in \operatorname{End} V$ defined by the commutator bracket (2.11). Thus we can define a bracket of fields:

$$\big[a(z), b(w)\big] = \sum_{m,n\in\mathbb{Z}} \big[a_{(m)}, b_{(n)}\big] z^{-m-1} w^{-n-1}. \tag{4.4}$$

We note that the bracket (4.4) encodes all possible brackets between modes of a and modes of b.

Definition 4.1.2. Two fields $a(z) = Y(a, z)$ and $b(z) = Y(b, z)$ are called *(mutually) local* if

$$(z-w)^N \big[a(z), b(w)\big] = 0 \tag{4.5}$$

for $N \gg 0$.

When $a(z)$ and $b(z)$ are local, their bracket $[a(z), b(w)]$ can be conveniently written in terms of the *formal delta-function*

$$\delta(z, w) = z^{-1} \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \in \mathbb{C}\big[\big[z, z^{-1}, w, w^{-1}\big]\big] \tag{4.6}$$

as follows:

$$\big[a(z), b(w)\big] = \sum_{j=0}^{N-1} c^j(w) \partial_w^{(j)} \delta(z, w), \tag{4.7}$$

where $\partial_w^{(j)} = \partial_w^j / j!$ Alternatively, we can express the formal delta-function as

$$\delta(z, w) = i_{z,w} \frac{1}{z-w} - i_{w,z} \frac{1}{z-w},$$

where $i_{z,w}$ is the power series expansion in the domain $|z| > |w|$. For this reason, $\delta(z, w)$ is sometimes denoted in the literature as $\delta(z-w)$. It is straightforward to show that more generally

$$\partial_w^{(j)} \delta(z, w) = i_{z,w} \frac{1}{(z-w)^{j+1}} - i_{w,z} \frac{1}{(z-w)^{j+1}}. \tag{4.8}$$

It is clear from (4.8) that for $0 \leq k \leq j$ the formal delta-function satisfies

$$(z-w)^k \partial_w^{(j)} \delta(z, w) = \partial_w^{(j-k)} \delta(z, w). \tag{4.9}$$

One can also show that for any formal distribution $f(z)$, the following equation holds:

$$\operatorname{Res}_z f(z)\delta(z,w) = f(w), \tag{4.10}$$

where $\operatorname{Res}_z f(z)$ is the coefficient of z^{-1}. Thus the coefficients $c^j(w)$ in (4.7) can be recovered using (4.9) and (4.10) as follows:

$$c^j(w) = \operatorname{Res}_z (z-w)^j \big[a(z), b(w)\big].$$

In general it does not make sense to multiply two fields $a(z) = Y(a,z)$ and $b(z) = Y(b,z)$, since the coefficients on powers of the formal variable z may themselves be infinite sums of operators on V. We make use of a well-defined pairing of fields called the *normally ordered product*:

$$:a(z)b(w): = a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w)a(z)_-,$$

where

$$a(z)_- = \sum_{n\in\mathbb{Z}_{\geq 0}} a_{(n)} z^{-n-1}, \qquad a(z)_+ = \sum_{n\in\mathbb{Z}_{<0}} a_{(n)} z^{-n-1}.$$

Then $:a(z)b(z):$ is a well-defined formal distribution which is a field if $a(z)$ and $b(z)$ are fields.

A second way to encode the coefficients $c^j(w)$ of the bracket of two local fields $a(z)$ and $b(z)$ is via the *operator product expansion* (OPE) of $a(z)$ and $b(z)$ used heavily by physicists:

$$a(z)b(w) = \sum_{j=0}^{N-1} \left(i_{z,w} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + :a(z)b(w): \tag{4.11}$$

Note that the OPE is split into the part that is singular at $z = w$, and the normally ordered product, which is regular at $z = w$. Also note that all of the coefficients $c^j(w)$ are encoded in the singular part of (4.11). Throughout the rest of this text, we will follow the convention of physicists in writing only the singular part of the OPE and suppressing the notation for the $i_{z,w}$ expansion:

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}. \tag{4.12}$$

We have seen that for any $a, b \in V$, the vertex algebra has nth products $a_{(n)}b \in V$ for every integer n. Now we aim to define the nth product of fields in such a way that the state-field correspondence is a homomorphism of all of the nth products, i.e., such that

$$Y(a_{(n)}b, z) = Y(a, z)_{(n)}Y(b, z). \tag{4.13}$$

We define the nth product of two local fields for $n \geq 0$ so that $a(w)_{(n)}b(w)$ is the coefficient $c^n(w)$ in the commutator formula:

$$a(w)_{(n)}b(w) = \mathrm{Res}_z(z - w)^n[a(z), b(w)]. \tag{4.14}$$

Then (4.13) holds for all $n \geq 0$. The nth product of fields can be extended to include negative values of n by expanding the terms of the formula (4.14) in the appropriate domain:

$$a(w)_{(n)}b(w) = \mathrm{Res}_z\Big(i_{z,w}(z - w)^n a(z)b(w) - (-1)^{p(a)p(b)} i_{w,z}(z - w)^n b(w)a(z)\Big). \tag{4.15}$$

Then (4.13) holds for all $n \in \mathbb{Z}$.

One can alternatively obtain the coefficients $c^j(w)$ using the following formula (see, e.g, [65, Lecture 14]):

$$a(w)_{(n)}b(w) = \partial_z^{(N-1-n)}\big((z - w)^N a(z)b(w)\big)\big|_{z=w}, \tag{4.16}$$

where N is chosen so that (4.5) holds, and $n \leq N-1$. For $n \geq N$ one has $a(w)_{(n)}b(w) = 0$. By setting $b(z) = Y(\mathbf{1}, z)$, choosing $N = 0$, and replacing n with $-n-1$ in (4.16), we see that

$$Y(a_{(-n-1)}\mathbf{1}, z) = \partial_z^{(n)}Y(a, z), \qquad n \in \mathbb{Z}_{>0}. \tag{4.17}$$

Recalling that the infinitesimal translation operator T acts by (4.2) and setting $n = 1$ in (4.17), we recover the property of vertex algebras known as *translation covariance*:

$$[T, Y(a, z)] = \partial_z Y(a, z).$$

Thus we arrive at a definition of vertex algebras given in terms of fields:

Definition 4.1.3. A vertex algebra is a triple $(V, \mathbf{1}, Y)$ where $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace, $\mathbf{1} \in V_{\bar{0}}$ is the vacuum vector, and $Y(\cdot, z)\colon V \to \mathrm{Fie}(V)$ is the state-field correspondence map, and the following axioms are satisfied:

1. (vacuum) $T\mathbf{1} = 0$, $Y(a,z)\mathbf{1}\big|_{z=0} = a$,
2. (translation covariance) $\big[T, Y(a,z)\big] = \partial_z Y(a,z)$,
3. (locality) $(z-w)^N\big[Y(a,z), Y(b,w)\big] = 0$ for $n \gg 0$,

where T is the infinitesimal translation operator (4.2).

The following theorem and proposition make the OPE a powerful tool when working with the field definition of a vertex algebra:

Theorem 4.1.4 (Wick theorem)**.** *Let $a^1(z), a^2(z)$ and $b^1(z), b^2(z)$ be two pairs of fields satisfying*

(i) $[[a^i(z)_-, b^j(w)], c^k(z)_\pm] = 0$ *for all* i, j, k, *and* $c = a$ *or* b.

(ii) $[a^i(z)_\pm, b^j(w)_\pm = 0$ *for all* i *and* j.

Following the notation in [60], we let $[a^ib^j] = [a^i(z)_-, b^j(w)]$ be the "contraction" of $a^i(z)$ and $b^j(w)$. Then

$$\begin{aligned}
{:}a^1(z)a^2(z){:}{:}b^1(w)b^2(w){:} = {:}a^1(z)a^2(z)b^1(w)b^2(w){:}& \\
+ (-1)^{p(a^1)p(a^2)}[a^1b^1]{:}a^2(z)b^2(w){:}& \\
+ (-1)^{p(a^1)p(a^2)}(-1)^{p(b^1)p(b^2)}[a^1b^2]{:}a^2(z)b^1(w){:}& \\
+ [a^2b^1]{:}a^1(z)b^2(w){:}& \\
+ (-1)^{p(b^1)p(b^2)}[a^2b^2]{:}a^1(z)b^1(w){:}& \\
+ (-1)^{p(a^1)p(a^2)}[a^1b^1][a^2b^2] + [a^1b^2][a^2b^1].&
\end{aligned} \tag{4.18}$$

A more general form of the Wick theorem is obtained in a similar way [60, Thm 3.3]. In practice, it is much easier to use than the general formula suggests. The right-hand side of (4.18) is not yet written in the standard form of the OPE of the two fields $a(z) = {:}a^1(z)a^2(z){:}$ and $b(z) = {:}b^1(z)b^2(z){:}$. In order to obtain the standard form, one must replace all of the $a^i(z)$ with fields in terms of w using the following formula:

Proposition 4.1.5 (Taylor's formula (see, e.g., [60]))**.** *Let $a(z)$ be a formal distribution. Then in the domain $|z| > |w|$ the following equality in formal distributions in z and w holds:*

$$a(z+w) = \sum_{j=0}^{\infty} \partial_z^{(j)} a(z) w^j. \tag{4.19}$$

By replacing z by w and w by $z-w$, we obtain a second form of Taylor's formula that is useful in calculating OPEs. In the domain $|z-w| < |w|$ we have the following equality of formal distributions in w and $z-w$:

$$a(z) = \sum_{j=0}^{\infty} \partial_w^{(j)} a(w)(z-w)^j. \tag{4.20}$$

This second form of Taylor's formula allows us to rewrite the coefficients of the singular part of the (4.18) strictly in terms of w as required by the OPE (cf. (4.12)).

4.2 Affine and free field vertex algebras

Recall from Section 2.4 that the affine Lie algebra $\hat{\mathfrak{g}}$ is defined using a finite-dimensional Lie algebra $\mathfrak{g}$ with a nondegenerate supersymmetric invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$. In the case when $\mathfrak{g}$ is simple and the $(\cdot|\cdot)$ has been scaled in a particular way, the generalized Verma module $\widetilde{V}^\kappa$ (see (2.15)) has the structure of a vertex algebra [44], which is called the *universal affine vertex algebra* at level κ, and denoted $V^\kappa(\mathfrak{g})$. The vacuum vector is taken to be the highest weight vector $\mathbf{1} \in \widetilde{V}^\kappa$. It is standard practice to identify $a \in \mathfrak{g}$ with $a_{(-1)}\mathbf{1} \in V^\kappa(\mathfrak{g})$. Then the fields

$$Y(a,z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \qquad a \in \mathfrak{g}$$

are local and generate $V^\kappa(\mathfrak{g})$.

Recall from Section 2.4 that the Virasoro algebra has basis $\{L_n, C\}_{n\in\mathbb{Z}}$ and bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3-m)\delta_{m,-n}C, \qquad [C, L_m] = 0$$

for all $m, n \in \mathbb{Z}$. We arrange the generators L_n into the following formal distribution:

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

Then $L(z)$ satisfies the OPE

$$L(z)L(w) \sim \frac{\frac{1}{2}C}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}. \tag{4.21}$$

In particular, $L(z)$ is local with itself. Any vertex algebra V containing a field $L(z)$ satisfying the OPE (4.21) such that C acts on V as $C = c\,\mathrm{Id}$ for some $c \in \mathbb{C}$ is called a *Virasoro field with central charge c*.

Definition 4.2.1. Let V be a vertex algebra and $\omega \in V$ an even vector. If the corresponding field $Y(\omega, z)$ is a Virasoro field with central charge c, then ω is a *conformal vector*. A vertex algebra with a conformal vector ω whose corresponding field $Y(\omega, z)$ has central charge c is called a *conformal vertex algebra* of *rank* c.

As we mentioned in Section 2.4, in the case when $\mathfrak{g}$ is abelian or simple the Virasoro algebra can be constructed as a subalgebra of $V^\kappa(\mathfrak{g})$. Choose a basis $\{v_i\}$ of homogeneous vectors for $\mathfrak{g}$. Let $\{v^i\}$ be the *dual* basis for $\mathfrak{g}$, which is the unique homogeneous basis such that $(v_i|v^j) = \delta_{i,j}$ for $1 \leq i, j \leq d$. We define the vector $\omega \in V^\kappa(\mathfrak{g})$ using the *Sugawara construction*:

$$\omega = \frac{1}{2(\kappa + h^\vee)} \sum_{i=1}^{d} v^i_{(-1)} v_i, \qquad \kappa \neq -h^\vee, \qquad d = \dim \mathfrak{g}, \tag{4.22}$$

where $h^\vee$ is a certain number associated to $\mathfrak{g}$ called the *dual Coxeter number*. In the case when $\mathfrak{g}$ is abelian, we have $h^\vee = 0$. Let

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_{(n)} z^{-n-1} = \sum_{m\in\mathbb{Z}} L_m z^{-m-2}.$$

Using the Wick formula and Taylor's theorem together with the fact that

$$L(z) = Y(\omega, z) = \frac{1}{2(\kappa + h^\vee)} \sum_{i=1}^{d} {:}v^i(z) v_i(z){:},$$

one computes the OPE

$$L(z)L(w) \sim \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w},$$

where $c = \kappa \operatorname{sdim} \mathfrak{g}/(\kappa + h^\vee)$, and $\operatorname{sdim} \mathfrak{g} = \dim \mathfrak{g}_{\bar{0}} - \dim \mathfrak{g}_{\bar{1}}$. Equivalently, the modes L_n satisfy the commutation relations of the Virasoro algebra (2.23). Thus ω is a conformal vector with central charge

$$c = \frac{\kappa \operatorname{sdim} \mathfrak{g}}{\kappa + h^\vee}.$$

For a thorough treatment of universal affine vertex algebras see, e.g., [60, Ch. 4]. Now we will consider the vertex algebras corresponding to the Weyl and Clifford affinizations we introduced in Section 2.4.

4.2.1 Free Superbosons

In this section, we review the definition of free superbosons following [60]. Let $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ be an abelian Lie superalgebra with $\dim \mathfrak{h} = d < \infty$. Let $(\cdot|\cdot)$ be a nondegenerate even supersymmetric bilinear form on $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ be the Weyl affinization (cf. Section 2.4.1) with brackets (2.16). We will use the notation $a_{(m)} = at^m$. The *free superbosons*

$$a(z) = \sum_{m \in \mathbb{Z}} a_{(m)} z^{-m-1}, \qquad a \in \mathfrak{h},$$

have OPEs given by

$$a(z)b(w) \sim \frac{(a|b)K}{(z-w)^2}.$$

The (generalized) Verma module

$$\widetilde{V}^1 = \operatorname{Ind}_{\mathfrak{h}[t] \oplus \mathbb{C}K}^{\hat{\mathfrak{h}}} \mathbb{C}$$

is constructed by letting $\mathfrak{h}[t]$ act trivially on $\mathbb{C}$ and K act as 1. Then $\widetilde{V}^1$ has the structure of a vertex algebra called the *free superboson algebra* and denoted $B^1(\mathfrak{h})$. The commutator (2.16) is equivalent to the following n-th products:

$$a_{(0)}b = 0, \qquad a_{(1)}b = (a|b)\mathbf{1}, \qquad a_{(j)}b = 0 \qquad (j \geq 2) \tag{4.23}$$

for $a, b \in \mathfrak{h}$, where $\mathbf{1}$ is the vacuum vector in $B^1(\mathfrak{h})$.

Recall from Section 2.4.1 that there exist bases $v_1, \dots v_k$ and $w_1, \dots, w_{2\ell}$ for $\hat{\mathfrak{h}}_{\bar{0}}$ and $\hat{\mathfrak{h}}_{\bar{1}}$ respectively such that (2.19) and (2.20) hold. Then let

$$v^i = v_{k-i+1}, \qquad w^j = w_{2\ell-i+1}, \qquad w^{\ell+j} = -w_{\ell-i+1}, \tag{4.24}$$

for $1 \leq i \leq k$, $1 \leq j \leq \ell$. The basis $\{v^i, w^j\}$ is dual to the basis $\{v_i, w_j\}$. Thus via the Sugawara construction (4.22) we obtain a conformal vector

$$\omega = \frac{1}{2}\sum_{i=1}^{d} v^i_{(-1)}v_i + \frac{1}{2}\sum_{j=1}^{2\ell} w^j_{(-1)}w_j. \tag{4.25}$$

In the even case ($\mathfrak{h} = \mathfrak{h}_{\bar{0}}$), the free superbosons are known as the *Heisenberg vertex algebra* or simply *free bosons*. In the odd case ($\mathfrak{h} = \mathfrak{h}_{\bar{1}}$), they are called *symplectic fermions*.

4.2.2 Free Superfermions

Let us review the definition of free superfermions given in [60]. Let $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ be an abelian Lie superalgebra with $\dim \mathfrak{a} = d < \infty$. Let $(\cdot|\cdot)$ be a nondegenerate even skew-supersymmetric bilinear form on $\mathfrak{a}$. Let $C_{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}K$ be the Clifford affinization (cf. Section 2.4.2) with brackets (2.22). The *free superfermions*

$$a(z) = \sum_{m\in\mathbb{Z}} a_{(m)}z^{-m-1}, \qquad a_{(m)} = at^m,$$

have OPEs given by

$$a(z)b(w) \sim \frac{(a|b)K}{z-w}.$$

The (generalized) Verma module

$$\widetilde{V}^1 = \mathrm{Ind}_{\mathfrak{a}[t]\oplus\mathbb{C}K}^{C_{\mathfrak{a}}} \mathbb{C}$$

is constructed by letting $\mathfrak{a}[t]$ act trivially on $\mathbb{C}$ and K act as 1. Then $\widetilde{V}^1$ has the structure of a vertex algebra called the *free superfermion algebra* and denoted $F^1(\mathfrak{a})$. The brackets

(2.22) are equivalent to the following n-th products in $F^1(\mathfrak{a})$:

$$a_{(0)}b = (a|b)\mathbf{1}, \qquad a_{(j)}b = 0 \quad (j \geq 1), \tag{4.26}$$

where $\mathbf{1}$ is the vacuum vector.

There exist bases $w_1, \dots, w_{2\ell}$ for $\mathfrak{a}_{\bar{0}}$ and $v_1, \dots, v_k$ for $\mathfrak{a}_{\bar{1}}$ such that (2.19) and (2.20) hold. Then (4.24) is again a dual basis, and we obtain a conformal vector

$$\omega = \frac{1}{2}\sum_{i=1}^{2\ell} w^i_{(-2)}w_i + \frac{1}{2}\sum_{j=1}^{k} v^i_{(-2)}v_i \tag{4.27}$$

In the even case ($\mathfrak{a} = \mathfrak{a}_{\bar{0}}$), the free superfermions are also known as symplectic bosons or as the bosonic ghost system (*$\beta\gamma$-system*). In the odd case ($\mathfrak{a} = \mathfrak{a}_{\bar{1}}$), they are called *free fermions*.

4.3 Lattice Vertex Algebras

Lattice vertex algebras were originally introduced by Borcherds. He used them to give a rigorous algebraic description of the vertex operators appearing in physics and representation theory (see Chapter 1). We begin our review of lattice vertex algebras by recalling the definition of a module of a vertex algebra.

Definition 4.3.1. Let V be a vertex algebra with vacuum vector $\mathbf{1}$. A *V-module* is a vector (super)space M together with an even state-field correspondence map: $Y^M : v \to \operatorname{Fie} M$ such that $Y^M(\mathbf{1}, z) = \mathrm{Id}$ on M, and the Borcherds identity (4.1) holds for all $a, b \in V$ and $c \in M$.

Remark 4.3.2. Equivalently, instead of requiring that the Borcherds identity holds, one may require that $Y^M(V)$ is a local collection and the nth product identity (4.13) holds for fields $Y^M(a, z)$ and $Y^M(b, z)$, where $a, b \in V$.

Throughout the rest of this text, we will suppress the dependence on M in the fields corresponding to a V-module except when doing so would cause confusion.

The following example motivates the definition of lattice vertex algebras. Let $\mathfrak{h} = \operatorname{span}\{a\}$, and let $V = B^1(\mathfrak{h})$ be the corresponding free boson vertex algebra. We construct irreducible highest weight modules of V similarly to the construction of V itself. We note that (4.23) implies that $a_{(0)}$ is central in $\hat{\mathfrak{h}}$ and therefore acts diagonally on any representation of $\hat{\mathfrak{h}}$.

We consider the representation of $\hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^0$ on $\mathbb{C}$ where $\hat{\mathfrak{h}}^+\mathbb{C} = 0$, $K = I$ and $a_{(0)}$ acts with weight λ. Then we induce this representation to a highest-weight representation of $\hat{\mathfrak{h}}$:

$$\widetilde{V}^{1,\lambda} = \mathrm{Ind}_{\hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^0}^{\hat{\mathfrak{h}}}(\mathbb{C}).$$

Note that $\widetilde{V}^{1,0} = \widetilde{V}^1$ as $\hat{\mathfrak{h}}$-modules. Thus it is natural to ask whether $\widetilde{V}^{1,\lambda}$ can be included in some larger vertex algebra structure. If such a structure exists, it must contain the highest weight vector in $\widetilde{V}^{1,\lambda}$, which we denote by e^λ. The weight $\lambda\colon \mathfrak{h} \to \mathbb{C}$ can be identified with an element of $\mathfrak{h}$ using the bilinear form $(\cdot|\cdot)$. We will denote this element as $\lambda \in \mathfrak{h}$. Then we have

$$a_{(0)}e^\lambda = (a|\lambda)e^\lambda.$$

In the attempt to construct a vertex algebra including the highest weight vector e^λ, we must assume that $|\lambda|^2 = (\lambda|\lambda) \in \mathbb{Z}$ and include all the highest weight modules of $\hat{\mathfrak{h}}$ with highest weight vectors $e^{m\lambda},\ m \in \mathbb{Z}$. Then the weights form an integral lattice

$$Q = \mathrm{span}_{\mathbb{Z}}\{\lambda\}$$

satisfying $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$. In order to complete the construction, one must find the fields corresponding to the highest weight vectors: $Y(e^{m\lambda}, z),\ m \in \mathbb{Z}$. Such fields are known as *vertex operators*, and their derivation can be found in, e.g, [60]. We define them explicitly in the general case below.

Armed with the intuition afforded by this example of a single free boson, we proceed to the formal definition of a lattice vertex algebra. We start with a *lattice* Q of rank d, i.e., a free abelian group on d generators:

$$Q = \mathrm{span}_{\mathbb{Z}}\{\lambda_1, \ldots, \lambda_d\}.$$

We give Q the structure of an *integral lattice*, meaning we take Q together with a bilinear form $(\cdot|\cdot)\colon Q \times Q \to \mathbb{Z}$.

We let $\varepsilon : Q \times Q \to \{\pm 1\}$ be a bimultiplicative function which satisfies

$$\varepsilon(\lambda, \lambda) = (-1)^{|\lambda|^2(|\lambda|^2+1)/2}, \qquad \lambda \in Q. \tag{4.28}$$

A unique such function exists up to equivalence and satisfies

$$\varepsilon(\lambda,\mu)\varepsilon(\mu,\lambda)=(-1)^{(\lambda|\mu)+|\lambda|^2|\mu|^2}, \qquad \lambda,\mu\in Q. \tag{4.29}$$

Then the *twisted group algebra* $\mathbb{C}_\varepsilon[Q]$ has basis $\{e^\lambda\}_{\lambda\in Q}$, with multiplication given by

$$e^\lambda e^\mu=\varepsilon(\lambda,\mu)e^{\lambda+\mu}, \qquad \lambda,\mu\in Q. \tag{4.30}$$

Using bilinearity, we extend $(\cdot|\cdot)$ to $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}Q$. Let $\hat{\mathfrak{h}}=\mathfrak{h}[t,t^{-1}]\oplus\mathbb{C}K$ be the corresponding Heisenberg (free boson) current algebra with brackets (2.16), and let $S=\widetilde{V}^1$ be its Fock space representation as in Section 4.2. Then we extend this representation to the space $V_Q=S\otimes\mathbb{C}_\varepsilon[Q]$ via

$$(at^m)(s\otimes e^\lambda)=(at^m+\delta_{m,0}(a|\lambda))s\otimes e^\lambda, \qquad (m\geq 0). \tag{4.31}$$

In particular, we have

$$a_{(0)}(s\otimes e^\lambda)=(a|\lambda)s\otimes e^\lambda, \qquad a\in\mathfrak{h},\ \lambda\in Q. \tag{4.32}$$

The twisted group algebra $\mathbb{C}_\varepsilon[Q]$ can also be represented on V_Q by the following action:

$$e^\lambda(s\otimes e^\mu)=\varepsilon(\lambda,\mu)s\otimes e^{\lambda+\mu}. \tag{4.33}$$

Following the notation of [10], we write e^λ (resp. at^m) for $1\otimes e^\lambda$ (resp. $ht^m\otimes 1$). Then equations (4.31) and (4.33) together with the relation

$$e^\lambda(at^m)=(at^m-\delta_{m,0}(\lambda|a))e^\lambda$$

define a representation of the associative algebra $\mathcal{A}_Q=\mathcal{U}(\hat{\mathfrak{h}})\otimes\mathbb{C}_\varepsilon[Q]$ in V_Q. The fields

$$Y(a,z)=\sum_{m\in\mathbb{Z}}a_{(m)}z^{-m-1}, \qquad a\in\mathfrak{h}, \tag{4.34}$$

on V_Q are called *currents*.

The lattice vectors $\lambda\in Q$ act semisimply with integral eigenvalues on V_Q via the action of the zero modes given by (4.32). This allows us to define operators z^λ by

$$z^\lambda(s \otimes e^\mu) = z^{(\lambda|\mu)}(s \otimes e^\mu). \tag{4.35}$$

Let $\mathbb{Z}^\pm = \pm\mathbb{N}$. The fields

$$\begin{aligned} Y(e^\lambda, z) &= e^\lambda \colon \exp \int Y(\lambda, z) \colon \\ &= e^\lambda z^\lambda \exp\Big(\sum_{n\in\mathbb{Z}^-} \lambda_{(n)} \frac{z^{-n}}{-n}\Big)\Big(\sum_{n\in\mathbb{Z}^+} \lambda_{(n)} \frac{z^{-n}}{-n}\Big) \end{aligned} \tag{4.36}$$

are called *vertex operators*. Then the currents (4.34) and the vertex operators (4.36) generate a vertex algebra structure on V_Q, called a *lattice vertex algebra*, where the vacuum vector is given by $\mathbf{1} \otimes e^0$, the infinitesimal translation operator acts as

$$Te^\lambda = \lambda_{(-1)} e^\lambda, \tag{4.37}$$

and the parity in V_Q is given by $p(a \otimes e^\lambda) = (\lambda|\lambda)$. Reminiscent of (4.30), in the vertex algebra the following formula holds:

$$e^\lambda_{(-1-(\lambda|\mu))} e^\mu = \varepsilon(\lambda, \mu) e^{\lambda+\mu}. \tag{4.38}$$

The locality of the vertex operators $Y(e^\lambda, z)$ and $Y(e^\mu, z)$ is established by showing that

$$(z_1 - z_2)^{-(\lambda|\mu)} Y(e^\lambda, z_1) Y(e^\mu, z_2) = (-1)^{|\lambda|^2|\mu|^2} (z_1 - z_2)^{-(\lambda|\mu)} Y(e^\mu, z_2) Y(e^\lambda, z_1). \tag{4.39}$$

The currents (4.34) generate the Heisenberg algebra $B^1(\mathfrak{h})$. Let $d = \dim \mathfrak{h}$ and $\{v_1, \ldots, v_d\}$ be a basis of $\mathfrak{h}$. Also let $\{v^1, \ldots, v^d\}$ its dual basis with respect to $(\cdot|\cdot)$. Then we can construct a conformal vector (cf. (4.25))

$$\omega = \frac{1}{2} \sum_{i=1}^{d} v^i_{(-1)} v_i \tag{4.40}$$

for $B^1(\mathfrak{h})$. In fact (4.40) is a conformal vector for the lattice vertex algebra V_Q (see, e.g., [60]) The corresponding Virasoro field

$$L(z) = Y(\omega, z) = \sum_{m\in\mathbb{Z}} L_m z^{-m-2} \tag{4.41}$$

has central charge d.

It will be important to a later discussion that we know the explicit action of L_0 on the vectors e^λ. From (4.40) and the fact that

$$Y(a_{(-1)}b, z) = {:}Y(a,z)Y(b,z){:}$$

we have

$$L_0 = \frac{1}{2}\sum_{i=1}^{d}\sum_{m=1}^{\infty} v^i_{(-m)}v_{i(m)} + \frac{1}{2}\sum_{i=1}^{d}\sum_{m=0}^{\infty} v_{i(-m)}v^i_{(m)}.$$

We recall that any vector $h \in \mathfrak{h}$ can be written as

$$h = \sum_{i=1}^{d}(v^i|h)v_i.$$

Then from (4.31) we have

$$L_0 e^\lambda = \frac{1}{2}\sum_{i=1}^{d} v_{i(0)}v^i_{(0)}e^\lambda = \frac{1}{2}\sum_{i=1}^{d}(v^i|\lambda)v_{i(0)}e^\lambda = \frac{1}{2}\lambda_{(0)}e^\lambda = \frac{1}{2}|\lambda|^2 e^\lambda. \tag{4.42}$$

4.4 Twisted Logarithmic Modules

Vertex operators often appear in applications in a "twisted" form. For example, the monster vertex algebra was constructed using twisted vertex operators [42,43]. Twisted vertex operators are formalized by twisted modules of vertex algebras, which are constructed using an automorphism of the vertex algebra. We recall that an automorphism of a vertex algebra V is a vector space isomorphism $\sigma\colon V \to V$ on the space of states which respects all nth products:

$$(\sigma a)_{(n)}(\sigma b) = \sigma(a_{(n)}b), \qquad a, b \in V,\ n \in \mathbb{Z}.$$

Until Huang's article [51] entitled *Generalized twisted modules associated to general automorphisms of a vertex operator algebra* published in 2010, σ-twisted modules were only defined for automorphisms σ of finite order. However, there are instances of vertex operators which cannot be described by a σ-twisted module for an automorphism σ of finite order. Such is the case with logarithmic conformal field theory (LCFT) [20], which has applications to percolation [81] and the quantum Hall effect [93] (see Section 7.4).

Huang was hindered in the full development of the theory because of the lack of a usable definition for the nth products of twisted vertex operators. In 2015, also motivated by LCFT along with certain vertex operators arising in Gromov-Witten theory [11], Bakalov introduced a more general notion of a twisted logarithmic module (TLM) of a vertex algebra and defined the nth product of twisted vertex operators, allowing him to further develop the theory [6].

In this section, we review the definition of twisted logarithmic modules of vertex algebras following [6]. We also review several properties of such modules. The main distinguishing feature of twisted logarithmic modules is the presence of a second formal variable in the fields. This new formal variable ζ essentially plays the role of $\log z$. However, sometimes it is convenient to specialize ζ to values independently of z. So the formalism is developed as follows.

Let z and ζ be two independent formal variables. If we think of ζ as $\log z$, then the derivatives with respect to z and ζ become

$$D_z = \partial_z + z^{-1}\partial_\zeta, \qquad D_\zeta = z\partial_z + \partial_\zeta. \tag{4.43}$$

For a vector space W over $\mathbb{C}$, a *logarithmic (quantum) field* on W is a formal series

$$a(z) = a(z,\zeta) = \sum_{\alpha\in\mathcal{A}} \sum_{m\in\alpha} a_m(\zeta) z^{-m},$$

where $\mathcal{A}$ is a finite subset of $\mathbb{C}/\mathbb{Z}$, $a_m(\zeta)\colon W \to W[[\zeta]]$ is a power series in ζ with coefficients in $\operatorname{End} W$, and for any $w \in W$ we have $a_m(\zeta)w = 0$ for $\operatorname{Re} m \gg 0$. If all coefficients of each power series $a_m(\zeta)$ are even (resp. odd) elements of $\operatorname{End} W$, then $a(z)$ is said to be an *even* (resp. *odd*) logarithmic field. The space of all logarithmic fields on W is denoted $\operatorname{LFie}(W)$.

We use the notation

$$a(z)_+ = \sum_{\alpha\in\mathcal{A}} \sum_{\substack{m\in\alpha \\ \operatorname{Re} m\le 0}} a_m(\zeta) z^{-m},$$
$$a(z)_- = \sum_{\alpha\in\mathcal{A}} \sum_{\substack{m\in\alpha \\ \operatorname{Re} m> 0}} a_m(\zeta) z^{-m}.$$

Let $a(z), b(z) \in \mathrm{LFie}(W)$ be homogeneous fields with parities $p(a)$ and $p(b)$ respectively. The *normally ordered product* of $a(z)$ with $b(z)$ is given by

$$:a(z_1)b(z_2): = a(z_1)_+ b(z_2) + (-1)^{p(a)p(b)} b(z_2) a(z_1)_- . \tag{4.44}$$

The normally ordered product is then extended linearly to include all $a(z) \in \mathrm{LFie}(W)$. One can check that the normally ordered product is well-defined for $z_1 = z_2$ and that $:a(z)b(z):$ is again a logarithmic field of parity $p(a) + p(b)$.

Definition 4.4.1. Two logarithmic fields $a(z)$ and $b(z)$ are *local* if there exists an integer $N \geq 0$ such that

$$(z_1 - z_2)^N a(z_1)b(z_2) = (-1)^{p(a)p(b)} (z_1 - z_2)^N b(z_2)a(z_1). \tag{4.45}$$

The key to the full development of the theory of twisted logarithmic modules of vertex algebras is the following *nth product formula* of local logarithmic fields:

$$(a(z)_{(n)} b(z))w = D_{z_1}^{(N-1-n)} \Big((z_1 - z_2)^N a(z_1)b(z_2)w \Big) \Big|_{\substack{z_1=z_2=z \\ \zeta_1=\zeta_2=\zeta}}, \tag{4.46}$$

for $w \in W$ and $n \leq N - 1$ an integer. For $n \geq N$, we set the nth product equal to 0.

Definition 4.4.2 ([6])**.** Given a vertex algebra V and an even automorphism φ of V, a *φ-twisted V-module* is a vector superspace W equipped with an even linear map $Y : V \to \mathrm{LFie}(W)$ such that $Y(\mathbf{1}, z) = \mathrm{Id}$ is the identity on W, and $Y(V)$ is a local collection of logarithmic fields satisfying:

1. φ-equivariance:
$$Y(\varphi a, z) = e^{2\pi \mathrm{i} D_\zeta} Y(a, z) \tag{4.47}$$

2. n-th product identity:

$$Y(a_{(n)}b, z) = Y(a, z)_{(n)}Y(b, z) \tag{4.48}$$

for all $a, b \in V$ and $n \in \mathbb{Z}$.

We note that $Y(a, z)$ depends on ζ as well as z, though we do not indicate this in the notation. From (4.48) and (4.2) we obtain (cf. [51])

$$Y(Ta, z) = D_z Y(a, z), \qquad a \in V. \tag{4.49}$$

Let $\bar{V} \subseteq V$ be the subspace of V on which φ is *locally-finite* i.e., the subspace of vectors $v \in V$ such that $\operatorname{span}\{\varphi^i v \colon i \geq 0\}$ is finite-dimensional. Then we can write

$$\bar{\varphi} = \varphi\big|_{\bar{V}} = \sigma e^{-2\pi \mathrm{i} \mathcal{N}}, \tag{4.50}$$

where $\sigma \in \operatorname{Aut}(\bar{V})$, $\mathcal{N} \in \operatorname{Der}(\bar{V})$, σ and $\mathcal{N}$ commute, σ is semisimple, and $\mathcal{N}$ is locally nilpotent on $\bar{V}$.

We denote the eigenspaces of σ by

$$V_\alpha = \{a \in \bar{V} \mid \sigma a = e^{-2\pi \mathrm{i}\alpha} a\}, \qquad \alpha \in \mathbb{C}/\mathbb{Z}.$$

Then φ-equivariance implies that

$$X(a, z) = Y(e^{\zeta \mathcal{N}} a, z) = Y(a, z)\big|_{\zeta = 0} \tag{4.51}$$

is independent of ζ, and the exponents of z in $X(a, z)$ belong to $-\alpha$ for $a \in V_\alpha$. For $m \in \alpha$, the $(m + \mathcal{N})$*-th mode* of $a \in V_\alpha$ is defined as

$$a_{(m+\mathcal{N})} = \operatorname{Res}_z z^m X(a, z), \tag{4.52}$$

where, as usual, Res_z denotes the coefficient of z^{-1}. Then

$$Y(a, z) = X(e^{-\zeta \mathcal{N}} a, z) = \sum_{m \in \alpha} (z^{-m-\mathcal{N}-1} a)_{(m+\mathcal{N})}, \tag{4.53}$$

where we use the notation $z^{-\mathcal{N}} = e^{-\zeta \mathcal{N}}$. We note that $Y(a, z)$ is a polynomial of ζ for

$a \in \bar{V}$, since $\mathcal{N}$ is nilpotent on a.

One of the main results of [6] is the Borcherds identity for twisted logarithmic modules. In terms of fields, this identity is written using the *shifted delta-function*

$$\delta_{\alpha+\mathcal{N}}(z_1, z_2) = \sum_{m\in\alpha} z_1^{-m-1-\mathcal{N}} z_2^{m+\mathcal{N}}$$

as follows:

Proposition 4.4.3 (Borcherds Identity [6])**.** *Let φ be an even automorphism of a vertex algebra V. Let σ and $\mathcal{N}$ be defined by* (4.50)*. Then in any φ-twisted V-module the following identity holds:*

$$\begin{aligned} i_{z_1,z_2}(z_1-z_2)^n\ Y(a,z_1)Y(b,z_2)w - (-1)^{p(a)p(b)} i_{z_2,z_1}(z_1-z_2)^n\ Y(b,z_2)Y(a,z_1)w \\ = \sum_{j=0}^{\infty} Y\Big(\big(D_{z_2}^{(j)}\delta_{\alpha+\mathcal{N}}(z_1,z_2)a\big)_{(n+j)}b, z_2\Big)w \end{aligned}$$

for $a \in V_\alpha \subseteq \bar{V}$, $b \in V$, $w \in W$ and $n \in \mathbb{Z}$.

The fact that b is allowed to be an arbitrary element of V will be especially important in Chapter 6 when we study twisted logarithmic modules of lattice vertex algebras. From the Borcherds identity we have the following commutator formulas for modes:

$$\big[a_{(m+\mathcal{N})}, b_{(k+\mathcal{N})}\big] = \sum_{j=0}^{\infty}\Big(\Big(\binom{m+\mathcal{N}}{j}a\Big)_{(j)}b\Big)_{(m+k-j+\mathcal{N})} \tag{4.54}$$

for $a \in V_\alpha$, $b \in V_\beta$, and $m, n \in \mathbb{Z}$. We also have

$$\big[a_{(m+\mathcal{N})}, Y(b,z)\big] = \sum_{j=0}^{\infty} Y\Big(\Big(\binom{m+\mathcal{N}}{j} z^{m-j+\mathcal{N}}a\Big)_{(j)}b, z\Big) \tag{4.55}$$

for $a \in V_\alpha$, $b \in V$, and $m, n \in \mathbb{Z}$.

The (-1)-st product in a twisted logarithmic module is related to the normally ordered product (4.44) by the formula

$$:Y(a,z)Y(b,z): = \sum_{j=-1}^{N-1} z^{-j-1} Y\Big(\Big(\binom{\alpha_0+\mathcal{N}}{j+1}a\Big)_{(j)}b, z\Big) \tag{4.56}$$

for $a \in V_\alpha$ and $b \in V$.

Chapter 5

Twisted Logarithmic Modules of Free Fields

In this chapter, we investigate the twisted logarithmic modules of free field vertex algebras, extending the results on free bosons obtained in [6] to the cases of symplectic fermions, free fermions, and the bosonic ghost system (see Table 5.1). We have previously published the results included in this chapter in [13]. We begin by answering a question from linear algebra regarding canonical forms of an automorphism preserving a bilinear form. Then we consider the case of twisted logarithmic modules of free superbosons, including the odd case of symplectic fermions SF and the even case of free bosons which was previously done in [6]. We also consider the free superfermions, specifically the odd case of free fermions and the even case of the bosonic ghost system. In each of these cases, we realize the twisted logarithmic modules explicitly as highest-weight representations of certain Fock spaces, and we explicitly write the action of the Virasoro algebra on each. We show in each case that when $\mathcal{N} \neq 0$ the action of L_0 is not semisimple (cf. [4, 20, 72]).

5.1 Automorphisms Preserving a Bilinear Form

We recall from Chapter 4 that a superboson (resp. superfermion) vertex algebra is constructed using a finite-dimensional vector superspace $\mathfrak{h}$ (resp. $\mathfrak{a}$) together with a non-degenerate even supersymmetric (resp. skew-supersymmetric) bilinear form $(\cdot|\cdot)$ on the superspace. Thus in order to classify the twisted logarithmic modules of the superboson and superfermion vertex algebras, we must first obtain canonical forms for $(\cdot|\cdot)$ and the automorphisms φ preserving it.

A supersymmetric bilinear form is symmetric on $\mathfrak{h}_{\bar{0}}$ and skew-symmetric on $\mathfrak{h}_{\bar{1}}$. Similarly, a skew-supersymmetric bilinear form is skew-symmetric on $\mathfrak{h}_{\bar{0}}$ and symmetric (resp. symmetric) on $\mathfrak{h}_{\bar{1}}$. Thus it is enough to find canonical forms for an automorphism φ of a vector space V which preserves a nondegenerate bilinear form that is either symmetric or skew-symmetric.

Let V be a finite-dimensional vector space and $(\cdot|\cdot)\colon V \times V \to \mathbb{C}$ be a nondegenerate bilinear form. Let φ be an invertible linear operator on V which preserves the bilinear form, i.e., which satisfies

$$(\varphi a|\varphi b) = (a|b). \tag{5.1}$$

Via the multiplicative Jordan-Chevalley decomposition, we may uniquely write $\varphi = \sigma e^{-2\pi \mathrm{i} \mathcal{N}}$ such that σ is a semisimple invertible operator, $\mathcal{N}$ is nilpotent, and $\sigma\mathcal{N} = \mathcal{N}\sigma$. Then the φ-invariance (5.1) is equivalent to

$$(\sigma a|\sigma b) = (a|b), \qquad (\mathcal{N} a|b) + (a|\mathcal{N} b) = 0, \tag{5.2}$$

for all $a, b \in V$.

5.1.1 The symmetric case

The case when $(\cdot|\cdot)$ is symmetric was investigated previously in [6, Section 6]. The classification of all σ and $\mathcal{N}$ satisfying (5.2) can be deduced from the well-known description of the canonical Jordan forms of orthogonal and skew-symmetric matrices over $\mathbb{C}$ (see [45, 50]). We include it here for completeness.

In the following examples, V is a vector space with a basis $\{v_1, \dots, v_d\}$ such that $(v_i|v_j) = \delta_{i+j,d+1}$ for all i, j, and $\lambda = e^{-2\pi \mathrm{i}\alpha_0}$ for some $\alpha_0 \in \mathbb{C}$ such that $-1 < \operatorname{Re}\alpha_0 \leq 0$.

Example 5.1.1 ($d = 2\ell$)**.**

$$\sigma v_i = \begin{cases} \lambda v_i, & 1 \leq i \leq \ell, \\ \lambda^{-1} v_i, & \ell + 1 \leq i \leq 2\ell, \end{cases} \qquad \mathcal{N} v_i = \begin{cases} v_{i+1}, & 1 \leq i \leq \ell - 1, \\ -v_{i+1}, & \ell + 1 \leq i \leq 2\ell - 1, \\ 0, & i = \ell, 2\ell. \end{cases}$$

Let us write $\lambda^{-1} = e^{-2\pi \mathrm{i}\beta_0}$ where $-1 < \operatorname{Re}\beta_0 \leq 0$. Let

$$\mathbb{C}^+ = \{\gamma \in \mathbb{C}\colon\ \operatorname{Re}\gamma > 0\} \cup \{\gamma \in \mathbb{C}\colon\ \operatorname{Re}\gamma = 0,\ \operatorname{Im}\gamma > 0\}, \tag{5.3}$$

and let $\mathbb{C}^- = -\mathbb{C}^+$. The symmetry

$$v_i \mapsto (-1)^i v_{\ell+i}, \qquad v_{\ell+i} \mapsto (-1)^{i+\ell+1} v_i, \qquad (1 \leq i \leq \ell) \tag{5.4}$$

allows us to switch λ with λ^{-1} and assume that $\alpha_0 \in \mathbb{C}^- \cup \{0\}$. For example, if $\operatorname{Re}\alpha_0 = 0$, then we may switch the roles of α_0 and β_0 if necessary to ensure that $\operatorname{Im}\alpha_0 \leq 0$ as illustrated in Figure 5.1.

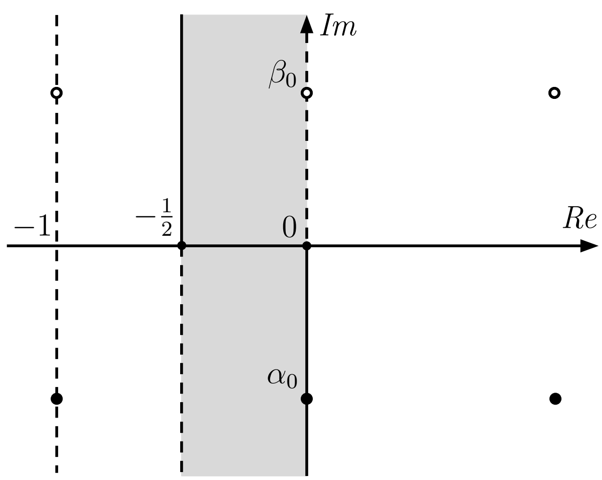

Figure 5.1: The case when $\operatorname{Re}\alpha_0 = 0$.

If $-1 < \operatorname{Re}\alpha_0 < 0$, then again employing the symmetry (5.4), we may assume $-1/2 \leq \operatorname{Re}\alpha_0 < 0$ as shown in Figure 5.2. Finally, if $\operatorname{Re}\alpha_0 = -1/2$, then (5.4) allows us to assume $\operatorname{Im}\alpha_0 \geq 0$ as shown in Figure 5.3. In summary, we may always assume α_0 is located in the gray region of Figures 5.1 through 5.3, where only the solid portions of the boundary are included.

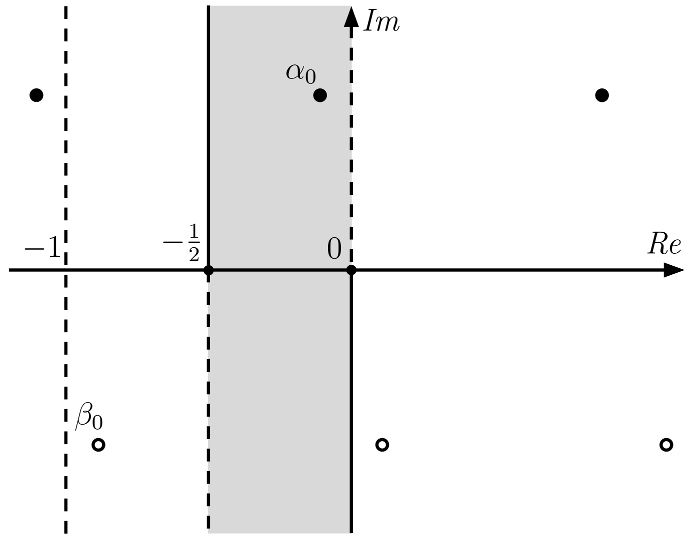

Figure 5.2: The case when $-1/2 < \operatorname{Re}\alpha_0 < 0$ and $\beta_0 = -\alpha_0 - 1$.

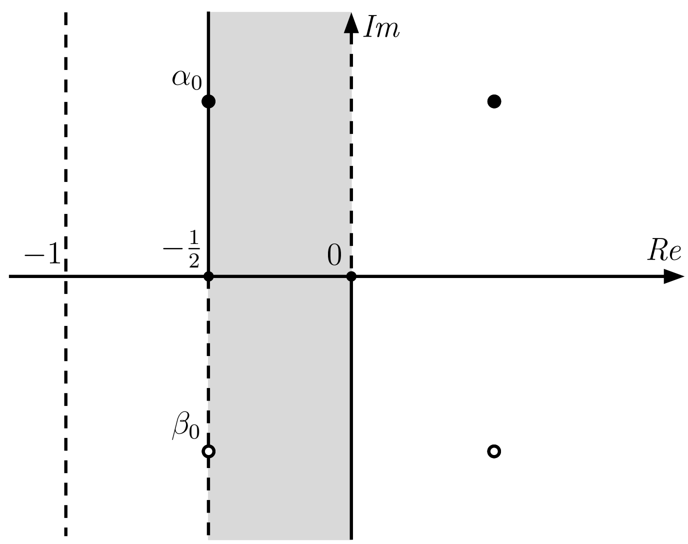

Figure 5.3: The case when $\operatorname{Re}\alpha_0 = -1/2$.

Example 5.1.2 ($d = 2\ell - 1$ and $\lambda = \pm 1$)**.**

$$\sigma v_i = \lambda v_i, \quad 1 \leq i \leq 2\ell - 1, \qquad \mathcal{N} v_i = \begin{cases} (-1)^{i+1} v_{i+1}, & 1 \leq i \leq 2\ell - 2, \\ 0, & i = 2\ell - 1. \end{cases}$$

Since $\lambda = \pm 1$, it follows that $\alpha_0 = 0$ or $-1/2$.

Remark 5.1.3. After rescaling the basis vectors in Example 5.1.2, the operator $\mathcal{N}$ can be written alternatively in the form

$$\mathcal{N} v_i = \begin{cases} v_{i+1}, & 1 \leq i \leq \ell - 1, \\ -v_{i+1}, & \ell \leq i \leq 2\ell - 2, \\ 0, & i = 2\ell - 1, \end{cases}$$

which more clearly shows the strong relationship between the symmetric and skew-symmetric cases (cf. Example 5.1.6 below).

Proposition 5.1.4 ([6])**.** *Let V be a finite-dimensional vector space, equipped with a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ and with commuting linear operators σ, $\mathcal{N}$ satisfying* (5.2)*, such that σ is invertible and semisimple, and $\mathcal{N}$ is nilpotent. Then V is an orthogonal direct sum of subspaces that are as in Examples 5.1.1 and 5.1.2.*

5.1.2 The skew-symmetric case

In the case when $(\cdot|\cdot)$ is a nondegenerate skew-symmetric bilinear form, we were unable to locate in the literature an analogous classification of symplectic matrices. Below we present two explicit examples of linear operators σ and $\mathcal{N}$ satisfying (5.2). Pick a basis $\{v_1, \ldots, v_{2\ell}\}$ for V such that

$$(v_i|v_j) = \delta_{i+j,2\ell+1} = -(v_j|v_i), \qquad 1 \leq i \leq j \leq 2\ell, \tag{5.5}$$

and let $\lambda = e^{-2\pi i \alpha_0}$ as before.

Example 5.1.5 (ℓ is odd, or ℓ is even and $\lambda \neq \pm 1$)**.**

$$\sigma v_i = \begin{cases} \lambda v_i, & 1 \leq i \leq \ell, \\ \lambda^{-1} v_i, & \ell+1 \leq i \leq 2\ell, \end{cases} \qquad \mathcal{N} v_i = \begin{cases} v_{i+1}, & 1 \leq i \leq \ell-1, \\ -v_{i+1}, & \ell+1 \leq i \leq 2\ell-1, \\ 0, & i = \ell, 2\ell. \end{cases}$$

As in Example 5.1.1, the symmetry $v_i \mapsto (-1)^i v_{\ell+i}$, $v_{\ell+i} \mapsto (-1)^i v_i$ ($1 \leq i \leq \ell$) allows us to assume that $\alpha_0 \in \mathbb{C}^- \cup \{0\}$, $-\frac{1}{2} \leq \operatorname{Re} \alpha_0 \leq 0$, and $\operatorname{Im} \alpha_0 \geq 0$ when $\operatorname{Re} \alpha_0 = -\frac{1}{2}$ (see Figures 5.1 through 5.3).

We have omitted the case when ℓ is even and $\lambda = \pm 1$ in Example 5.1.5, because it can be rewritten as an orthogonal direct sum of two copies of the following example.

Example 5.1.6 ($\lambda = \pm 1$)**.**

$$\sigma v_i = \lambda v_i, \quad 1 \leq i \leq 2\ell, \qquad \mathcal{N} v_i = \begin{cases} v_{i+1}, & 1 \leq i \leq \ell, \\ -v_{i+1}, & \ell+1 \leq i \leq 2\ell-1, \\ 0, & i = 2\ell. \end{cases}$$

Since $\lambda = \pm 1$, we have $\alpha_0 = 0$ or $-1/2$.

Theorem 5.1.7. *Consider a finite-dimensional vector space V with a nondegenerate skew-symmetric bilinear form $(\cdot|\cdot)$. Let σ and $\mathcal{N}$ be commuting linear operators satisfying* (5.2)*, such that σ is invertible and semisimple and $\mathcal{N}$ is nilpotent. Then V is an orthogonal direct sum of subspaces as in Examples 5.1.5 and 5.1.6.*

Proof. Denote by V_λ the eigenspaces of σ. Since the form $(\cdot|\cdot)$ is nondegenerate and σ-invariant, it gives isomorphisms $V_\lambda \cong (V_{\lambda^{-1}})^*$, while $V_\lambda \perp V_\mu$ for $\lambda \neq \mu^{-1}$. Hence, we can assume that $V = V_\lambda \oplus V_{\lambda^{-1}}$ ($\lambda \neq \pm 1$) or $V = V_{\pm 1}$.

In the first case, pick a basis $\{w_1, \dots, w_d\}$ for V_λ in which $\mathcal{N}$ is in lower Jordan form. Then $V_{\lambda^{-1}}$ has a basis $\{w_{d+1}, \dots, w_{2d}\}$ such that $(w_i | w_j) = \delta_{i+j, 2d+1}$ for all $i < j$, and V becomes an orthogonal direct sum of subspaces as in Example 5.1.5.

Now assume $V = V_{\pm 1}$. By a similar argument as above, we can assume that V has the form $V' + V''$ where V' has a basis $\{w_1, \dots, w_d\}$, and V'' has a basis $\{w_{d+1}, \dots, w_{2d}\}$ such that

$$\mathcal{N} \colon w_1 \mapsto w_2 \mapsto \cdots \mapsto w_d \mapsto 0, \tag{5.6}$$

and $(w_i|w_{d+j}) = \delta_{i+j,d+1}$ for all $1 \le i,j \le d$. However, V' and V'' may not be distinct, and we must consider two cases.

First we suppose V' and V'' are distinct, and hence $V = V' \oplus V''$. When d is odd, φ acts on $V' \oplus V''$ as in Example 5.1.5. When d is even, we make the following change of basis:

$$u_i' = \begin{cases} \frac{1}{\sqrt{2}}(w_i + (-1)^{i+1} w_{d+i}), & 1 \le i \le \frac{d}{2}, \\ \frac{1}{\sqrt{2}}((-1)^{i+1} w_i + w_{d+i}), & \frac{d}{2}+1 \le i \le d, \end{cases}$$
$$u_i'' = \begin{cases} \frac{1}{\sqrt{2}}(w_i + (-1)^{i} w_{d+i}), & 1 \le i \le \frac{d}{2}, \\ \frac{1}{\sqrt{2}}((-1)^{i} w_i + w_{d+i}), & \frac{d}{2}+1 \le i \le d. \end{cases}$$

Then $U' = \operatorname{span}\{u_1', \dots, u_d'\}$ and $U'' = \operatorname{span}\{u_1'', \dots, u_d''\}$ are as in Example 5.1.6, and $V = U' \oplus U''$ is an orthogonal direct sum.

Finally, consider the case when $V = V' = V''$. Then $d = 2\ell$ is even, and V has a basis $\{w_1, \dots, w_d\}$ such that (5.6) holds. Note that (5.2) and (5.6) imply that $(w_i|w_j) = 0$ whenever $i + j > 2\ell + 1$. With the appropriate rescaling we may assume $(w_1|w_{2\ell}) = 1$. Then

$$(w_i|w_{2\ell-i+1}) = (-1)^{i+1}, \qquad 1 \le i \le 2\ell. \tag{5.7}$$

A Gram–Schmidt process allows us to construct a new basis $w_1', \dots, w_{2\ell}'$ such that (5.6) and (5.7) still hold and $(w_i'|w_j') = 0$ when $i + j < 2\ell + 1$. Thus $(w_i'|w_j') = (-1)^{i+1}\delta_{i+j,2\ell+1}$ for all i,j. Rescaling the basis vectors, we see that V is as in Example 5.1.6. $\square$

Note that Proposition 5.1.4 can be proved similarly to Theorem 5.1.7.

5.2 Twisted Logarithmic Modules of $B^1(\mathfrak{h})$

Let φ be an even automorphism of $\mathfrak{h}$ such that $(\cdot|\cdot)$ is φ-invariant. As in Section 5.1, we write $\varphi = \sigma e^{-2\pi \mathrm{i} \mathcal{N}}$ and denote the eigenspaces of σ by

$$\mathfrak{h}_\alpha = \{a \in \mathfrak{h} \,|\, \sigma a = e^{-2\pi \mathrm{i}\alpha} a\}, \qquad \alpha \in \mathbb{C}/\mathbb{Z}.$$

Definition 5.2.1 (cf. [6])**.** The *φ-twisted affinization* $\hat{\mathfrak{h}}_\varphi$ is the Lie superalgebra spanned by an even central element K and elements $a_{(m+\mathcal{N})} = at^m$ ($a \in \mathfrak{h}_\alpha$, $m \in \alpha$) with parity $p(a_{(m+\mathcal{N})}) = p(a)$, and the Lie superbracket

$$[a_{(m+\mathcal{N})}, b_{(n+\mathcal{N})}] = \delta_{m,-n}((m+\mathcal{N})a|b)K \tag{5.8}$$

for $a \in \mathfrak{h}_\alpha$, $b \in \mathfrak{h}_\beta$, $m \in \alpha$, $n \in \beta$.

An $\hat{\mathfrak{h}}_\varphi$-module W is called *restricted* if for every $a \in \mathfrak{h}_\alpha$, $m \in \alpha$, $v \in W$, there is an integer L such that $(at^{m+i})v = 0$ for all $i \in \mathbb{Z}$, $i \geq L$. We note that every highest weight $\hat{\mathfrak{h}}_\varphi$-module is restricted (see [59]). The automorphism φ naturally induces automorphisms of $\hat{\mathfrak{h}}$ and $B^1(\mathfrak{h})$, which we denote again by φ.

Theorem 5.2.2 (See [6], Theorem 6.3)**.** *Every φ-twisted $B^1(\mathfrak{h})$-module W has the structure of a restricted $\hat{\mathfrak{h}}_\varphi$-module with $(at^m)v = a_{(m+\mathcal{N})}v$ for $a \in \mathfrak{h}_\alpha$, $m \in \alpha$, $v \in W$. Conversely, every restricted $\hat{\mathfrak{h}}_\varphi$-module uniquely extends to a φ-twisted $B^1(\mathfrak{h})$-module.*

As before, we split $\mathbb{C}$ as a disjoint union of subsets $\mathbb{C}^+$, $\mathbb{C}^- = -\mathbb{C}^+$ and $\{0\}$ (see (5.3)). Then the φ-twisted affinization $\hat{\mathfrak{h}}_\varphi$ has a triangular decomposition

$$\hat{\mathfrak{h}}_\varphi = \hat{\mathfrak{h}}_\varphi^- \oplus \hat{\mathfrak{h}}_\varphi^0 \oplus \hat{\mathfrak{h}}_\varphi^+, \tag{5.9}$$

where

$$\hat{\mathfrak{h}}_\varphi^\pm = \operatorname{span}\{at^m \,|\, a \in \mathfrak{h}_\alpha,\, \alpha \in \mathbb{C}/\mathbb{Z},\, m \in \alpha \cap \mathbb{C}^\pm\}$$

and

$$\hat{\mathfrak{h}}_\varphi^0 = \operatorname{span}\{at^0 \,|\, a \in \mathfrak{h}_0\} \oplus \mathbb{C}K.$$

Starting from an $\hat{\mathfrak{h}}_\varphi^0$-module R with $K = I$, the (generalized) *Verma module* is defined by

$$M_\varphi(R) = \operatorname{Ind}_{\hat{\mathfrak{h}}_\varphi^+ \oplus \hat{\mathfrak{h}}_\varphi^0}^{\hat{\mathfrak{h}}_\varphi} R,$$

where $\hat{\mathfrak{h}}_\varphi^+$ acts trivially on R. These are φ-twisted $B^1(\mathfrak{h})$-modules.

5.2.1 Virasoro Fields

Choose dual bases $\{v_i\}$ and $\{v^i\}$ of $\mathfrak{h}$ of homogeneous elements, i.e., satisfying

$$(v_i|v^j) = \delta_{i,j}, \qquad p(v_i) = p(v^i), \quad 1 \leq i, j \leq d. \tag{5.10}$$

Let ω be the conformal vector (4.25), and let $\mathcal{S}\colon \mathfrak{h} \to \mathfrak{h}$ be the linear operator given by $\mathcal{S}a = \alpha_0 a$ for $a \in \mathfrak{h}_\alpha$, where, as before, $\alpha_0 \in \alpha$ is such that $-1 < \operatorname{Re}\alpha_0 \leq 0$. The following result is the super analogue of [6, Lemma 6.4] in the special case when $\mathfrak{g} = \mathfrak{h}$ is abelian. We use the normally ordered product (4.44).

Proposition 5.2.3. *In every φ-twisted $B^1(\mathfrak{h})$ module, we have*

$$2Y(\omega, z) = \sum_{i=1}^{d} {:}X(v^i, z)X(v_i, z){:} - z^{-2} \operatorname{str} \binom{\mathcal{S}}{2} I. \tag{5.11}$$

Proof. Equation (4.56) together with the nth products (4.23) give the equation:

$$Y(a_{(-1)}b, z) = {:}Y(a, z)Y(b, z){:} - z^{-2} \Big(\binom{\mathcal{S}+\mathcal{N}}{2} a \Big| b \Big) I \tag{5.12}$$

for $a, b \in \mathfrak{h}$. Note that $(\mathcal{N}v^i|v_i) = 0$. Using this fact together with (5.12) and (4.25), we obtain

$$2Y(\omega, z) = \sum_{i=1}^{d} {:}Y(v^i, z)Y(v_i, z){:} - z^{-2} \sum_{i=1}^{d} \Big(\binom{\mathcal{S}}{2} v^i \Big| v_i \Big) I. \tag{5.13}$$

We also observe that $\varphi\omega = \omega$, which implies $Y(\omega, z)$ is independent of ζ. Thus we obtain the desired result by setting $\zeta = 0$ in (5.13) and noting that

$$\sum_{i=1}^{d} \Big(\binom{\mathcal{S}}{2} v^i \Big| v_i \Big) = \sum_{i=1}^{d} (-1)^{p(v^i)} \Big(v_i \Big| \binom{\mathcal{S}}{2} v^i \Big) = \operatorname{str} \binom{\mathcal{S}}{2} I.$$

$\square$

Throughout the next several sections, we will study the four types of free field algebras arising via the Weyl or Clifford affinization (cf. Section 2.4) of an even or odd abelian Lie superalgebra. These four types of free fields are listed in Table 5.1.

5.3 Symplectic fermions

Historically, the symplectic fermions provided the first example of a logarithmic conformal field theory, due to Kausch [72, 73]. An orbifold of the symplectic fermion algebra SF

Table 5.1: Even and odd free field algebras arising via the Clifford affinization $C_{\mathfrak{a}}$ or the Weyl affinization $\hat{\mathfrak{h}}$.

Vertex Algebra	$\mathfrak{g}$	dim	$(\cdot\|\cdot)$	Affinization
Free Bosons	$\mathfrak{h}=\mathfrak{h}_{\bar{0}}$	n	symmetric	$\hat{\mathfrak{h}}$
Symplectic Fermions	$\mathfrak{h}=\mathfrak{h}_{\bar{1}}$	2ℓ	skew-symmetric	$\hat{\mathfrak{h}}$
Free Fermions	$\mathfrak{a}=\mathfrak{a}_{\bar{1}}$	n	symmetric	$C_{\mathfrak{a}}$
Bosonic Ghost System	$\mathfrak{a}=\mathfrak{a}_{\bar{0}}$	2ℓ	skew-symmetric	$C_{\mathfrak{a}}$

under an automorphism of order 2 has the important properties of being C_2-cofinite but not rational [1]. More recently, the subalgebras of SF known as the triplet and singlet algebras have generated considerable interest (see [5, 74]). Other orbifolds of SF give rise to interesting $\mathcal{W}$-algebras [18].

In this section, we will continue to use the notation from Section 5.2. We will assume that $\mathfrak{h}$ is odd, i.e., $\mathfrak{h}=\mathfrak{h}_{\bar{1}}$. In this case, $B^1(\mathfrak{h})$ is called the *symplectic fermion algebra* and denoted SF (see [1,72,73]). Then the bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ is skew-symmetric. For $\mathfrak{h}$ and φ as in Examples 5.1.5 and 5.1.6, we will describe explicitly the φ-twisted affinization $\hat{\mathfrak{h}}_\varphi$ and its irreducible highest-weight modules $M_\varphi(R)$, together with the action of the Virasoro algebra on them.

5.3.1 Action of the Virasoro algebra

Choose a basis $\{v_i\}$ for $\mathfrak{h}$ satisfying (5.5), where φ acts either as in Example 5.1.5 or 5.1.6. Let $v^i=v_{2\ell-i+1}$ and $v^{\ell+i}=-v_{\ell-i+1}$ $(1\leq i\leq \ell)$. The basis $\{v^i\}$ is dual to $\{v_i\}$ with respect to $(\cdot|\cdot)$, so that $(v_i|v^j)=\delta_{i,j}$ (cf. (5.10)). Then

$$\omega=\frac{1}{2}\sum_{i=1}^{2\ell}v^i_{(-1)}v_i=\sum_{i=1}^{\ell}v^i_{(-1)}v_i\in B^1(\mathfrak{h})$$

is a conformal vector with central charge $c=\operatorname{sdim}\mathfrak{h}=-2\ell$. Since $\varphi\omega=\omega$, the modes of $Y(\omega,z)$ give a (untwisted) representation of the Virasoro Lie algebra on every φ-twisted $B^1(\mathfrak{h})$-module (cf. [6, Lemma 6.8]).

The triangular decomposition (5.9) induces the following normal ordering on the modes of $\hat{\mathfrak{h}}_\varphi$:

$$
{}^\circ_\circ(at^m)(bt^n){}^\circ_\circ = \begin{cases} (at^m)(bt^n), & m \in \mathbb{C}^-, \\ (-1)^{p(a)p(b)}(bt^n)(at^m), & m \in \mathbb{C}^+ \cup \{0\}. \end{cases} \tag{5.14}
$$

On the other hand, the normally ordered product $:Y(a,z)Y(b,z):$ of two logarithmic fields is defined by placing the part of $Y(a,z)$ corresponding to powers z^γ with $\operatorname{Re}\gamma < 0$ to the right of $Y(b,z)$ (see (4.44)). This also induces a normal ordering on the modes:

$$
:(at^m)(bt^n): = \begin{cases} (at^m)(bt^n), & \operatorname{Re} m < -1, \\ (-1)^{p(a)p(b)}(bt^n)(at^m), & \operatorname{Re} m \geq -1. \end{cases} \tag{5.15}
$$

Note that the two normal orderings (5.14) and (5.15) of the modes differ when $-1 < \operatorname{Re} m \leq 0$ and $\operatorname{Im} m < 0$ (see Figure 6.1.7).

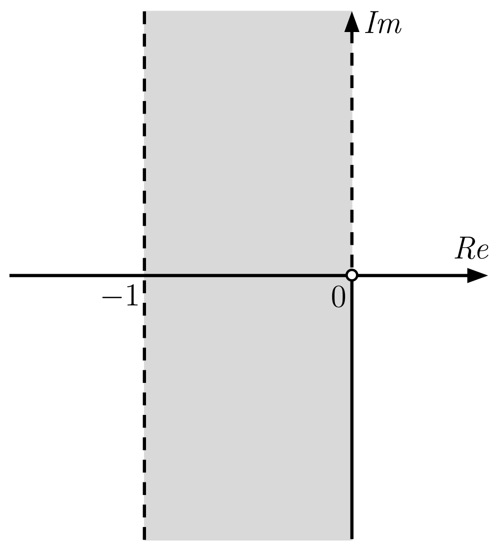

Figure 5.4: Choices of $m \in \mathbb{C}$ for which $:(at^m)(bt^n):$ and ${}^\circ_\circ(at^m)(bt^n){}^\circ_\circ$ differ.

The normal ordering (5.14) consistently moves all annihilation operators to the right of creation operators and hence is more convenient than (5.15) in explicitly calculating the action of Virasoro on $M_\varphi(R)$. We should note, however, that (5.14) moves all zero modes to the right, even those that act as creation operators on $M_\varphi(R)$. We correct this on a case-by-case basis depending on the choice of representation of the zero modes.

Proposition 5.3.1. *Assume $\mathfrak{h}$ and φ are as in Example 5.1.5 or 5.1.6. Then in every φ-twisted module of $SF = B^1(\mathfrak{h})$, we have*

$$L_k = \sum_{i=1}^{\ell} \sum_{m\in\alpha_0+\mathbb{Z}} {}^\circ_\circ(v^i t^{-m})(v_i t^{k+m})^\circ_\circ + \delta_{k,0}\frac{\ell}{2}\alpha_0(\alpha_0+1)I. \tag{5.16}$$

Proof. Assume $v_1,\dots,v_\ell \in \mathfrak{h}_\alpha$ and $v_{\ell+1},\dots,v_{2\ell}\in\mathfrak{h}_\beta$, where $\beta=-\alpha$. Using (4.56), (4.23), the skew-symmetry of $(\cdot|\cdot)$, and the fact that $(\mathcal{N}v^i|v_i)=0$ for $1\le i\le \ell$, we obtain

$$Y(\omega,z) = \sum_{i=1}^{\ell} {:}X(v^i,z)X(v_i,z){:} + z^{-2}\ell\binom{\beta_0}{2}I. \tag{5.17}$$

If $\operatorname{Re}\alpha_0=0$, then $\beta_0=-\alpha_0\in\mathbb{C}^+\cup\{0\}$. So the normal ordering in (5.17) coincides with (5.14), and $\binom{\beta_0}{2}=\frac{1}{2}\alpha_0(\alpha_0+1)$. Thus L_k is given by (5.16).

Now assume $\operatorname{Re}\alpha_0<0$. Then $\beta_0=-\alpha_0-1\in\mathbb{C}^-$, and the ordering of the modes in (5.17) differs from (5.14) when $k=0$ for

$$-(v_i t^{-\beta_0})(v^i t^{\beta_0}) = {}^\circ_\circ(v^i t^{\beta_0})(v_i t^{-\beta_0})^\circ_\circ + \beta_0 I.$$

Finally, we note that $\beta_0+\binom{\beta_0}{2}=\frac{1}{2}\alpha_0(\alpha_0+1)$. Thus after reordering to match (5.14), L_k is given by (5.16). $\square$

Note that the normal orderings in (5.16) can be omitted for $k\neq 0$. In the following subsections, we will compute explicitly the actions of $\hat{\mathfrak{h}}_\varphi$ and L_0 on $M_\varphi(R)$ when $\mathfrak{h}$ is odd as in Examples 5.1.5, 5.1.6.

5.3.2 The case of Example 5.1.5

Recall that for any φ-twisted SF-module, the logarithmic fields are given by (4.53). Assume that $\mathcal{N}$ acts on $\mathfrak{h}$ as in Example 5.1.5. Since this action is the same as in Example

5.1.1, the logarithmic fields $Y(v_j, z)$ are the same as in [6, Section 6.4]:

$$\begin{aligned} Y(v_j, z) &= \sum_{i=j}^{\ell} \sum_{m\in\alpha_0+\mathbb{Z}} \frac{(-1)^{i-j}}{(i-j)!} \zeta^{i-j}(v_i t^m) z^{-m-1}, \\ Y(v_{\ell+j}, z) &= \sum_{i=j}^{\ell} \sum_{m\in-\alpha_0+\mathbb{Z}} \frac{1}{(i-j)!} \zeta^{i-j}(v_{\ell+i} t^m) z^{-m-1}, \end{aligned} \tag{5.18}$$

for $1 \leq j \leq \ell$.

The Lie superalgebra $\hat{\mathfrak{h}}_\varphi$ is spanned by an even central element K and odd elements $v_i t^{m+\alpha_0}$, $v_{\ell+i} t^{m-\alpha_0}$ ($1 \leq i \leq \ell$, $m \in \mathbb{Z}$), where $\alpha_0 \in \mathbb{C}^- \cup \{0\}$ and $-1 < \operatorname{Re}\alpha_0 \leq 0$. By (5.8), the only nonzero brackets in $\hat{\mathfrak{h}}_\varphi$ are given by:

$$[v_i t^{m+\alpha_0}, v_j t^{n-\alpha_0}] = (m+\alpha_0)\delta_{m,-n}\delta_{i+j,2\ell+1} K + \delta_{m,-n}\delta_{i+j,2\ell} K, \tag{5.19}$$

for $1 \leq i \leq \ell$, $\ell+1 \leq j \leq 2\ell$, $m, n \in \mathbb{Z}$. Notice that the elements of $\hat{\mathfrak{h}}_\varphi^-$ act as creation operators on $M_\varphi(R)$. Throughout the rest of the section, we will represent them as anti-commuting variables as follows:

$$v_i t^{-m+\alpha_0} = \xi_{i,m}, \qquad v_j t^{-n-\alpha_0} = \xi_{j,n}, \tag{5.20}$$

for $1 \leq i \leq \ell$, $\ell+1 \leq j \leq 2\ell$, and $m \geq 0$, $n \geq 1$.

The precise triangular decomposition (5.9) depends on whether $\alpha_0 \in \mathbb{C}^-$ or $\alpha_0 = 0$. Suppose first that $\alpha_0 \in \mathbb{C}^-$. Then $\hat{\mathfrak{h}}_\varphi^0 = \mathbb{C}K$ and $R = \mathbb{C}$. Equations (5.19) and (5.20) imply that

$$M_\varphi(R) \cong \bigwedge(\xi_{i,m}, \xi_{\ell+i,m+1})_{1\leq i\leq \ell,\, m=0,1,2,\ldots}. \tag{5.21}$$

Using the commutation relations (5.19) and the fact that $\hat{\mathfrak{h}}_\varphi^+ R = 0$, we obtain the action of $\hat{\mathfrak{h}}_\varphi^+$ on $M_\varphi(R)$:

$$\begin{aligned} v_i t^{m+\alpha_0} &= (m+\alpha_0)\partial_{\xi_{2\ell-i+1,m}} + (1-\delta_{i,\ell})\partial_{\xi_{2\ell-i,m}}, \\ v_{\ell+i} t^{n-\alpha_0} &= -(n-\alpha_0)\partial_{\xi_{\ell-i+1,n}} + (1-\delta_{i,\ell})\partial_{\xi_{\ell-i,n}}, \end{aligned}$$

where $1 \le i \le \ell$, $m \ge 1$, $n \ge 0$. By Proposition 5.3.1, the action of L_0 is

$$\begin{aligned} L_0 = &\sum_{i=1}^{\ell}\sum_{m=0}^{\infty} \xi_{i,m}\Big((m-\alpha_0)\partial_{\xi_{i,m}} - (1-\delta_{i,1})\partial_{\xi_{i-1,m}}\Big) \\ &+ \sum_{i=1}^{\ell}\sum_{m=1}^{\infty} \xi_{\ell+i,m}\Big((m+\alpha_0)\partial_{\xi_{\ell+i,m}} + (1-\delta_{i,1})\partial_{\xi_{\ell+i-1,m}}\Big) \\ &+ \frac{\ell}{2}\alpha_0(\alpha_0+1)I. \end{aligned} \tag{5.22}$$

Now we consider the case when $\alpha_0 = 0$. Then $\hat{\mathfrak{h}}_\varphi^0 = \operatorname{span}\{v_i t^0\}_{1\le i\le 2\ell} \oplus \mathbb{C}K$. We let

$$R = \bigwedge(\xi_{i,0}, \xi_{2\ell,0})_{1\le i\le \ell},$$

where the action of $\hat{\mathfrak{h}}_\varphi^0$ on R is given by

$$\begin{aligned} v_i t^0 &= \xi_{i,0}, &&1 \le i \le \ell \ \text{ or } \ i = 2\ell, \\ v_j t^0 &= \partial_{\xi_{2\ell-j,0}}, &&\ell+1 \le j \le 2\ell-1. \end{aligned} \tag{5.23}$$

Therefore, by (5.20),

$$M_\varphi(R) \cong \bigwedge(\xi_{i,m}, \xi_{2\ell,m}, \xi_{j,m+1})_{1\le i\le \ell,\, \ell+1\le j\le 2\ell-1,\, m=0,1,2,\ldots}, \tag{5.24}$$

where the action of $\hat{\mathfrak{h}}_\varphi^+$ is given by

$$\begin{aligned} v_i t^m &= m\partial_{\xi_{2\ell-i+1,m}} + (1-\delta_{i,\ell})\partial_{\xi_{2\ell-i,m}}, \\ v_{\ell+i} t^m &= -m\partial_{\xi_{\ell-i+i,m}} + (1-\delta_{i,\ell})\partial_{\xi_{\ell-i,m}}, \end{aligned}$$

for $1 \le i \le \ell$ and $m \ge 1$. The action of L_0 is

$$\begin{aligned} L_0 = &\sum_{i=1}^{\ell}\sum_{m=0}^{\infty} \xi_{i,m}\Big(m\partial_{\xi_{i,m}} - (1-\delta_{i,1})\partial_{\xi_{i-1,m}}\Big) \\ &+ \sum_{i=1}^{\ell}\sum_{m=1}^{\infty} \xi_{\ell+i,m}\Big(m\partial_{\xi_{\ell+i,m}} + (1-\delta_{i,1})\partial_{\xi_{\ell+i-1,m}}\Big) - \xi_{1,0}\xi_{2\ell,0}. \end{aligned}$$

5.3.3 The case of Example 5.1.6

Let $\mathfrak{h}$ be as in Example 5.1.6. Then, by (4.53), in any φ-twisted SF-module,

$$\begin{aligned} Y(v_j, z) = & \sum_{i=j}^{\ell} \sum_{m \in \alpha_0 + \mathbb{Z}} \frac{(-1)^{i-j}}{(i-j)!} \zeta^{i-j} (v_i t^m) z^{-m-1} \\ & + (-1)^{j-\ell+1} \sum_{i=\ell+1}^{2\ell} \sum_{m \in \alpha_0 + \mathbb{Z}} \frac{1}{(i-j)!} \zeta^{i-j} (v_i t^m) z^{-m-1}, \\ Y(v_{\ell+j}, z) = & \sum_{i=j}^{\ell} \sum_{m \in \alpha_0 + \mathbb{Z}} \frac{1}{(i-j)!} \zeta^{i-j} (v_{\ell+i} t^m) z^{-m-1}, \end{aligned} \tag{5.25}$$

for $1 \leq j \leq \ell$ and $\alpha_0 = 0$ or $-1/2$.

The Lie superalgebra $\hat{\mathfrak{h}}_\varphi$ is spanned by an even central element K and odd elements $v_i t^m$ $(1 \leq i \leq 2\ell, m \in \alpha_0 + \mathbb{Z})$. The brackets in $\hat{\mathfrak{h}}_\varphi$ are:

$$\begin{aligned} [v_i t^m, v_j t^n] &= m \delta_{m,-n} \delta_{i+j, 2\ell+1} K + \delta_{m,-n} (1 - 2\delta_{i,\ell}) \delta_{i+j, 2\ell} K, \\ [v_{\ell+i} t^m, v_j t^n] &= -m \delta_{m,-n} \delta_{i+j, \ell+1} K + \delta_{m,-n} \delta_{i+j,\ell} K, \end{aligned}$$

for $1 \leq i \leq \ell$, $1 \leq j \leq 2\ell$, $m, n \in \alpha_0 + \mathbb{Z}$. Again, we will let the creation operators from $\hat{\mathfrak{h}}_\varphi^-$ act by (5.20). To determine explicitly $\hat{\mathfrak{h}}_\varphi^0$, we need to consider separately the cases $\alpha_0 = 0$ or $-1/2$.

First, we assume $\alpha_0 = 0$. Then $\hat{\mathfrak{h}}_\varphi^0 = \mathrm{span}\{v_i t^0\}_{1 \leq i \leq 2\ell} \oplus \mathbb{C}K$. We let

$$R = \bigwedge (\xi_{i,0}, \xi_{2\ell,0})_{1 \leq i \leq \ell} \qquad \Big(\text{where } \xi_{\ell,0}^2 = -\frac{1}{2}\Big),$$

with the action of $\hat{\mathfrak{h}}_\varphi^0$ given by (5.23). Thus

$$M_\varphi(R) \cong \bigwedge (\xi_{i,m}, \xi_{2\ell,m}, \xi_{j,m+1})_{1 \leq i \leq \ell, \ell+1 \leq j \leq 2\ell-1, m=0,1,2,\ldots}, \tag{5.26}$$

where $\xi_{\ell,0}^2 = -1/2$. The action of $\hat{\mathfrak{h}}_\varphi^+$ on $M_\varphi(R)$ is given by

$$\begin{aligned} v_i t^m &= m \partial_{\xi_{2\ell-i+1,m}} + (1 - 2\delta_{i,\ell}) \partial_{\xi_{2\ell-i,m}}, \\ v_{\ell+i} t^m &= -m \partial_{\xi_{\ell-i+1,m}} + (1 - \delta_{i,\ell}) \partial_{\xi_{\ell-i,m}}, \end{aligned}$$

for $1 \le i \le \ell$, $m \ge 1$. The action of L_0 is

$$L_0 = \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} \xi_{i,m}\Big(m\partial_{\xi_{i,m}} - (1-\delta_{i,1})\partial_{\xi_{i-1,m}}\Big)$$
$$+ \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} \xi_{\ell+i,m}\Big(m\partial_{\xi_{\ell+i,m}} + (1-2\delta_{i,1})\partial_{\xi_{\ell+i-1,m}}\Big) - \xi_{1,0}\xi_{2\ell,0}.$$

Second, we consider the case when $\alpha_0 = -1/2$. Then $\hat{\mathfrak{h}}^0_\varphi = \mathbb{C}K$ and the Verma module $M_\varphi(R)$ is given by (5.21). The action of $\hat{\mathfrak{h}}^+_\varphi$ on $M_\varphi(R)$ is

$$v_i t^{m-1/2} = \Big(m - \frac{1}{2}\Big)\partial_{\xi_{2\ell-i+1,m}} + (1-2\delta_{i,\ell})\partial_{\xi_{2\ell-i,m}},$$
$$v_{\ell+i} t^{n+1/2} = -\Big(n + \frac{1}{2}\Big)\partial_{\xi_{\ell-i+1,n}} + (1-\delta_{i,\ell})\partial_{\xi_{\ell-i,n}},$$

for $1 \le i \le \ell$, $m \ge 1$, $n \ge 0$. The action of L_0 is

$$L_0 = \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} \xi_{i,m}\Big(\Big(m + \frac{1}{2}\Big)\partial_{\xi_{i,m}} - (1-\delta_{i,1})\partial_{\xi_{i-1,m}}\Big)$$
$$+ \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} \xi_{\ell+i,m}\Big(\Big(m - \frac{1}{2}\Big)\partial_{\xi_{\ell+i,m}} + (1-2\delta_{i,1})\partial_{\xi_{\ell+i-1,m}}\Big) - \frac{\ell}{8}I.$$

5.4 Free Bosons

In the case when $\mathfrak{h}$ is even ($\mathfrak{h} = \mathfrak{h}_{\bar{0}}$), the free superbosons are known simply as free bosons and $B^1(\mathfrak{h})$ is called the Heisenberg vertex algebra. Its twisted logarithmic modules were described in [6, Section 6]. We include the results here for completeness.

Proposition 5.4.1. *Assume $\mathfrak{h}$ is even and φ is as in Example 5.1.1, we have*

$$L_k = \sum_{i=1}^{\ell} \sum_{m \in \alpha_0 + \mathbb{Z}} {}^{\circ}_{\circ}(v^i t^{-m})(v_i t^{k+m}){}^{\circ}_{\circ} - \delta_{k,0}\frac{\ell}{2}\alpha_0(\alpha_0+1)I \tag{5.27}$$

in any φ-twisted $B^1(\mathfrak{h})$-module. In the case of Example 5.1.2, we have

$$L_k = \frac{1}{2}\sum_{i=1}^{d}\sum_{m\in\alpha_0+\mathbb{Z}} {}^{\circ}_{\circ}(v^i t^{-m})(v_i t^{k+m}){}^{\circ}_{\circ} - \delta_{k,0}\frac{d}{4}\alpha_0(\alpha_0+1)I. \tag{5.28}$$

The proof is similar to that of Proposition 5.3.1 and is omitted. These results agree with [6, Section 6]. Again the normal orderings in (5.27), (5.28) can be omitted for $k \neq 0$. In the following subsections, we will list the actions of $\hat{\mathfrak{h}}_\varphi$ and L_0 on $M_\varphi(R)$ when $\mathfrak{h}$ is even as in Examples 5.1.1, 5.1.2. These calculations were originally computed in [6].

5.4.1 The case of Example 5.1.1

Let $\mathfrak{h}$ and φ be as in Example 5.1.1. By (4.53), since the action of $\mathcal{N}$ is the same as it was in Example 5.1.5, the logarithmic fields $Y(v_j, z)$ are again given by (5.18). The Lie algebra $\hat{\mathfrak{h}}_\varphi$ is spanned by a central element K and elements $v_i t^{m+\alpha_0}$, $v_{\ell+i}t^{m-\alpha_0}$ $(1 \le i \le \ell, m \in \mathbb{Z})$. By (5.8), the nonzero brackets in $\hat{\mathfrak{h}}_\varphi$ are:

$$[v_i t^m, v_j t^k] = m\delta_{m+k,0}\delta_{i+j,2\ell+1}K + \delta_{m+k,0}(1-\delta_{i,\ell})\delta_{i+j,2\ell}K\,, \tag{5.29}$$

$$[v_{\ell+i} t^m, v_j t^k] = m\delta_{m+k,0}\delta_{i+j,\ell+1}K - \delta_{m+k,0}(1-\delta_{i,\ell})\delta_{i+j,\ell}K \tag{5.30}$$

for $1 \le i \le \ell$, $1 \le j \le 2\ell$. We note that the elements of $\hat{\mathfrak{h}}_\varphi$ act as commuting creation operators on $M_\varphi(R)$. Throughout this section, we will represent them as follows:

$$v_i t^{\alpha_0 - m} = x_{i,m}\,, \qquad v_{\ell+i}t^{-\alpha_0-n} = x_{\ell+i,n}\,, \tag{5.31}$$

where $1 \le i \le \ell$, and $m \ge 0$, $n \ge 1$.

As in the case of the symplectic fermions, the precise triangular decomposition (5.9) of $\hat{\mathfrak{h}}_\varphi$ depends on whether $\alpha_0 \in \mathbb{C}^-$ or $\alpha_0 = 0$. Suppose first that $\alpha_0 \in \mathbb{C}^-$. Then we have $\hat{\mathfrak{h}}_\varphi^0 = \mathbb{C}K$. We let $\hat{\mathfrak{h}}_\varphi^+$ act as zero on $R = \mathbb{C}$ and K act as the identity operator. Then (5.29) and (5.31) imply that

$$M_\varphi(R) \cong \mathbb{C}[x_{i,0}, x_{j,n}]_{1\le i\le \ell,\, 1\le j\le 2\ell,\, n=1,2,3,\ldots}\,. \tag{5.32}$$

The action of $\hat{\mathfrak{h}}_\varphi^+$ on $M_\varphi(R)$ is given by

$$\begin{aligned} v_i t^{\alpha_0+n+1} &= (\alpha_0+n+1)\partial_{x_{2\ell+1-i,n+1}} + (1-\delta_{i,\ell})\partial_{x_{2\ell-i,n+1}}\,, \\ v_{\ell+i} t^{-\alpha_0+n} &= (-\alpha_0+n)\partial_{x_{\ell+1-i,n}} - (1-\delta_{i,\ell})\partial_{x_{\ell-i,n}}\,, \end{aligned}$$

where $1 \le i \le \ell$, $n \ge 0$. The action of L_0 is

$$\begin{aligned} L_0 = \sum_{i=1}^{\ell}\sum_{n=0}^{\infty} & x_{i,n}\big((-\alpha_0+n)\partial_{x_{i,n}} - (1-\delta_{i,1})\partial_{x_{i-1,n}}\big) \\ & + \sum_{i=1}^{\ell}\sum_{n=1}^{\infty} x_{\ell+i,n}\big((\alpha_0+n)\partial_{x_{\ell+i,n}} + (1-\delta_{i,1})\partial_{x_{\ell+i-1,n}}\big) \\ & - \frac{\ell}{2}(\alpha_0^2+\alpha_0)I\,. \end{aligned} \tag{5.33}$$

Now we consider the case when $\alpha_0 = 0$. Then $\hat{\mathfrak{h}}_\varphi^0 = \mathfrak{h}t^0 \oplus \mathbb{C}K$. We let

$$R_{a_1,a_2} = \mathbb{C}[x_{i,0}]_{1\le i\le \ell-1} \qquad (a_1, a_2 \in \mathbb{C})\,,$$

where the action of $\hat{\mathfrak{h}}_\varphi^0$ on R is given by

$$v_i t^0 = x_{i,0}, \quad v_{\ell+i}t^0 = -\partial_{x_{\ell-i,0}}, \quad v_\ell t^0 = a_1 I, \quad v_{2\ell}t^0 = a_2 I.$$

Therefore, by (5.31),

$$M_\varphi(R_{a_1,a_2}) \cong \mathbb{C}[x_{i,0}, x_{j,n}]_{1\le i\le \ell-1,\, 1\le j\le 2\ell,\, n=1,2,3,\ldots}\,, \tag{5.34}$$

where the action of $\hat{\mathfrak{h}}_\varphi^+$ is given by

$$\begin{aligned} v_i t^n &= n\partial_{x_{2\ell+1-i,n}} + (1-\delta_{i,\ell})\partial_{x_{2\ell-i,n}}\,, \\ v_{\ell+i} t^n &= n\partial_{x_{\ell+1-i,n}} - (1-\delta_{i,\ell})\partial_{x_{\ell-i,n}}\,, \end{aligned}$$

for $1 \leq i \leq \ell$ and $n \geq 1$. The action of L_0 is

$$
\begin{aligned}
L_0 = \sum_{i=1}^{\ell} \sum_{n=1}^{\infty} & x_{i,n}\big(n\partial_{x_{i,n}} - (1-\delta_{i,1})\partial_{x_{i-1,n}}\big) \\
& + \sum_{i=1}^{\ell} \sum_{n=1}^{\infty} x_{\ell+i,n}\big(n\partial_{x_{\ell+i,n}} + (1-\delta_{i,1})\partial_{x_{\ell+i-1,n}}\big) \\
& - \sum_{i=2}^{\ell-1} x_{i,0}\partial_{x_{i-1,0}} + a_2 x_{1,0} - a_1 \partial_{x_{\ell-1,0}} \,.
\end{aligned}
$$

5.4.2 The case of Example 5.1.2

Let $\mathfrak{h}$ be as in Example 5.1.2. Then by (4.53), the logarithmic fields are given by (cf. [6, Section 6.5]):

$$
Y(v_j, z) = \sum_{i=j}^{2\ell-1} \sum_{m\in\alpha_0+\mathbb{Z}} (-1)^{(i-j)(i+j-1)/2} \zeta^{(i-j)}(v_i t^m) z^{-m-1} \,,
$$

for $1 \leq j \leq 2\ell - 1$.

The Lie algebra $\hat{\mathfrak{h}}_\varphi$ is spanned by a central element K and elements $v_i t^{\alpha_0+n}$ ($1 \leq i \leq 2\ell-1$, $n \in \mathbb{Z}$), where $\alpha_0 = 0$ or $-1/2$. By (5.8), the nonzero brackets in $\hat{\mathfrak{h}}_\varphi$ are given by:

$$
[v_i t^m, v_j t^k] = m\delta_{m+k,0}\delta_{i+j,2\ell}K + (-1)^{i+1}\delta_{m+k,0}\delta_{i+j,2\ell-1}K
$$

for $1 \leq j \leq 2\ell - 1$, $m, k \in \mathbb{Z}$. Notice that the elements of $\hat{\mathfrak{h}}_\varphi^-$ act as creation operators $v_i t^{\alpha_0 - m} = x_{i,m}$ for $1 \leq i \leq 2\ell - 1$, $m \geq 0$.

First, we assume $\alpha_0 = -1/2$. Then $\hat{\mathfrak{h}}_\varphi^0 = \mathbb{C}K$ and $R = \mathbb{C}$, where K acts as the identity. Thus

$$
M_\varphi(R) \cong \mathbb{C}[x_{j,n}]_{1\leq j\leq 2\ell-1,\, n=0,1,2,\ldots} \,. \tag{5.35}
$$

The action of $\hat{\mathfrak{h}}_\varphi^+$ is given by

$$
v_i t^{\frac{1}{2}+n} = \Big(\frac{1}{2} + n\Big)\partial_{x_{2\ell-i,n}} + (-1)^{i+1}(1 - \delta_{i,2\ell-1})\partial_{x_{2\ell-1-i,n}} \,,
$$

where $1 \le i \le 2\ell - 1$ and $n \ge 0$. The action of L_0 is

$$L_0 = \sum_{i=1}^{2\ell-1} \sum_{n=0}^{\infty} x_{i,n} \Big(\Big(\frac{1}{2} + n \Big) \partial_{x_{i,n}} + (-1)^{i+1} (1 - \delta_{i,1}) \partial_{x_{i-1,n}} \Big) + \frac{1}{16}(2\ell - 1) I \,.$$

Now we assume $\alpha_0 = 0$. Then $\hat{\mathfrak{h}}^0_\varphi = \mathfrak{h} t^0 \oplus \mathbb{C} K$. We let

$$R_a = \mathbb{C}[x_{i,0}]_{1 \le i \le \ell - 1} \qquad (a \in \mathbb{C}) \,,$$

where the action of $\hat{\mathfrak{h}}^0_\varphi$ is

$$v_i t^0 = x_{i,0} \,, \quad v_{\ell-1+i} t^0 = (-1)^{\ell - i} \partial_{x_{\ell-i,0}} \,, \quad v_{2\ell-1} t^0 = aI \,,$$

for $1 \le i \le \ell - 1$. Thus

$$M_\varphi(R_a) \cong \mathbb{C}[x_{i,0}, x_{j,n}]_{1 \le i \le \ell-1,\, 1 \le j \le 2\ell-1,\, n=1,2,3,\dots} \,. \tag{5.36}$$

The action of $\hat{\mathfrak{h}}^+_\varphi$ is given by

$$v_i t^n = n \partial_{x_{2\ell-i,n}} + (-1)^{i+1} (1 - \delta_{i,2\ell-1}) \partial_{x_{2\ell-1-i,n}} \,.$$

where $1 \le i \le 2\ell - 1$ and $n \ge 1$. The action of L_0 is

$$\begin{aligned} L_0 = \sum_{i=1}^{2\ell-1} \sum_{n=1}^{\infty} x_{i,n} \big(n \partial_{x_{i,n}} + (-1)^{i+1} (1 - \delta_{i,1}) \partial_{x_{i-1,n}} \big) \\ + \sum_{i=2}^{\ell-1} (-1)^{i+1} x_{i,0} \partial_{x_{i-1,0}} + \frac{1}{2} \partial^2_{x_{\ell-1,0}} + a x_{1,0} \,. \end{aligned}$$

5.5 Twisted logarithmic modules of $F^1(\mathfrak{a})$

In this section, we study the twisted logarithmic modules of *free superfermions* (cf. [60]), which include in particular the *free fermions* and the symplectic bosons (also known as the bosonic ghost system or *$\beta\gamma$-system*). The $\beta\gamma$-system provides another interesting model of logarithmic conformal field theory [87].

Let $\mathfrak{a}$ be as in Section 4.2, and let $\varphi\colon \mathfrak{a} \to \mathfrak{a}$ be an automorphism such that $(\cdot|\cdot)$ is

φ-invariant, as before we write $\varphi = \sigma e^{-2\pi i \mathcal{N}}$, and denote the eigenspaces of σ by

$$\mathfrak{a}_\alpha = \{a \in \mathfrak{a} \,|\, \sigma a = e^{-2\pi i \alpha} a\}, \qquad \alpha \in \mathbb{C}/\mathbb{Z}.$$

Definition 5.5.1. The *φ-twisted Clifford affinization* $(C_\mathfrak{a})_\varphi$ is the Lie superalgebra spanned by elements at^m ($a \in \mathfrak{a}_\alpha, m \in \alpha$) with $p(at^m) = p(a)$ and an even central element K. The Lie superbracket in $(C_\mathfrak{a})_\varphi$ is given by

$$[at^m, bt^n] = \delta_{m,-n-1}(a|b)K, \qquad [K, at^m] = 0, \tag{5.37}$$

for $a \in \mathfrak{a}_\alpha$, $b \in \mathfrak{a}_\beta$, $m \in \alpha$, $b \in \beta$.

Remark 5.5.2. Since the brackets (5.37) do not depend on $\mathcal{N}$, we have $(C_\mathfrak{a})_\varphi = (C_\mathfrak{a})_\sigma$. In particular, $(C_\mathfrak{a})_\varphi = C_\mathfrak{a}$ if $\varphi = e^{-2\pi i \mathcal{N}}$.

As in the case of superbosons, φ naturally induces automorphisms of $C_\mathfrak{a}$ and $F^1(\mathfrak{a})$. As before, a $(C_\mathfrak{a})_\varphi$-module W will be called *restricted* if for every $a \in \mathfrak{a}_\alpha$, $m \in \alpha$, $v \in W$, there is an integer L such that $(at^{m+i})v = 0$ for all $i \in \mathbb{Z}$, $i \geq L$.

Theorem 5.5.3. *Every φ-twisted $F^1(\mathfrak{a})$-module W has the structure of a restricted $(C_\mathfrak{a})_\varphi$-module with $(at^m)v = a_{(m+\mathcal{N})}v$ for $a \in \mathfrak{a}_\alpha$, $m \in \alpha$, $v \in W$. Conversely, every restricted $(C_\mathfrak{a})_\varphi$-module uniquely extends to a φ-twisted $F^1(\mathfrak{a})$-module.*

The proof of the theorem is identical to that of [6, Theorem 6.3] and is omitted. It follows from $(C_\mathfrak{a})_\varphi = (C_\mathfrak{a})_\sigma$ that every φ-twisted $F^1(\mathfrak{a})$-module W has the structure of a σ-twisted $F^1(\mathfrak{a})$-module, and vice versa. More precisely, if $Y \colon F^1(\mathfrak{a}) \to \mathrm{LFie}(W)$ is the state-field correspondence as a φ-twisted module, then the state-field correspondence as a σ-twisted module is given by the map X from (4.51) for $a \in \mathfrak{a}$. Conversely, given X, we can determine Y from (4.53) for $a \in \mathfrak{a}$. However, the relationship is more complicated for elements $a \in F^1(\mathfrak{a})$ that are not in the generating set $\mathfrak{a}$. In particular, we will see below that the action of the Virasoro algebra is different, so that L_0 is not semisimple in a φ-twisted module while it is semisimple in a σ-twisted module.

We will split $\mathbb{C}$ as a disjoint union of subsets $\mathbb{C}^+_{-\frac{1}{2}}$, $\mathbb{C}^-_{-\frac{1}{2}}$ and $\{-\frac{1}{2}\}$ where

$$\mathbb{C}^+_{-\frac{1}{2}} = -\frac{1}{2} + \mathbb{C}^+, \qquad \mathbb{C}^-_{-\frac{1}{2}} = -\frac{1}{2} - \mathbb{C}^+, \tag{5.38}$$

and $\mathbb{C}^+$ is given by (5.3). The φ-twisted Clifford affinization $(C_\mathfrak{a})_\varphi$ has a triangular decomposition

$$(C_\mathfrak{a})_\varphi = (C_\mathfrak{a})_\varphi^- \oplus (C_\mathfrak{a})_\varphi^0 \oplus (C_\mathfrak{a})_\varphi^+, \tag{5.39}$$

where

$$(C_\mathfrak{a})_\varphi^\pm = \operatorname{span}\bigl\{at^m \bigm| a \in \mathfrak{a}_\alpha,\, \alpha \in \mathbb{C}/\mathbb{Z},\, m \in \alpha \cap \mathbb{C}^\pm_{-\frac{1}{2}}\bigr\}$$

and

$$(C_\mathfrak{a})_\varphi^0 = \operatorname{span}\bigl\{at^{-\frac{1}{2}} \bigm| a \in \mathfrak{a}_{-\frac{1}{2}}\bigr\} \oplus \mathbb{C}K.$$

Starting from a $(C_\mathfrak{a})_\varphi^0$-module R with $K = I$, the (generalized) *Verma module* is defined by

$$M_\varphi(R) = \operatorname{Ind}^{(C_\mathfrak{a})_\varphi}_{(C_\mathfrak{a})_\varphi^+ \oplus (C_\mathfrak{a})_\varphi^0} R,$$

where $(C_\mathfrak{a})_\varphi^+$ acts trivially on R. These are φ-twisted $F^1(\mathfrak{a})$-modules, and in the following sections we will realize them explicitly as Fock spaces and will determine the action of the Virasoro algebra on them.

5.5.1 Action of the Virasoro algebra

Pick dual bases $\{v_i\}$ and $\{v^i\}$ of $\mathfrak{a}$ satisfying (5.10). Then

$$\omega = \frac{1}{2}\sum_{i=1}^{d} v^i_{(-2)} v_i \in F^1(\mathfrak{a}), \qquad d = \dim \mathfrak{a}, \tag{5.40}$$

is a conformal vector with central charge $c = -\frac{1}{2}\operatorname{sdim}\mathfrak{a}$.

Let $\mathcal{S}\colon \mathfrak{a} \to \mathfrak{a}$ be the linear operator given by $\mathcal{S}a = \alpha_0 a$ for $a \in \mathfrak{a}_\alpha$, where, as before, $\alpha_0 \in \alpha$ is such that $-1 < \operatorname{Re}\alpha_0 \leq 0$. In the next result, we use the normally ordered product (4.44).

Proposition 5.5.4. *In every φ-twisted $F^1(\mathfrak{a})$-module, we have*

$$\begin{aligned} 2Y(\omega, z) = \sum_{i=1}^{d} {:}\bigl(\partial_z X(v^i, z)\bigr) X(v_i, z){:} - z^{-1} \sum_{i=1}^{d} {:}X(\mathcal{N}v^i, z) X(v_i, z){:} \\ - z^{-2} \operatorname{str}\binom{\mathcal{S}}{2} I. \end{aligned}$$

Proof. Recall that in any vertex algebra, $(Ta)_{(j)}b = -ja_{(j-1)}b$, where T is the translation operator (see e.g. [60]). By replacing a with Ta in [6, Lemma 5.8] and using [6, (4.3)], we obtain

$$:\!\big(D_z Y(a,z)\big)Y(b,z)\!: = -\sum_{j=-1}^{N-1} j z^{-j-1} Y\Big(\big(\binom{\mathcal{S}+\mathcal{N}}{j+1}a\big)_{(j-1)}b, z\Big)$$

for sufficiently large N (depending on a,b). Due to (4.26), when $a, b \in \mathfrak{a}$, the right-hand side reduces to

$$Y(a_{(-2)}b, z) - z^{-2}\Big(\binom{\mathcal{S}+\mathcal{N}}{2}a\Big|b\Big)I.$$

Now using (4.43) and (4.53), we observe that

$$D_z Y(a,z)\big|_{\zeta=0} = \partial_z X(a,z) - z^{-1}X(\mathcal{N}a, z). \tag{5.41}$$

Finally, we note that

$$\sum_{i=1}^{d}\Big(\binom{\mathcal{S}}{2}v^i\Big|v_i\Big) = -\sum_{i=1}^{d}(-1)^{p(v^i)}\Big(v_i\Big|\binom{\mathcal{S}}{2}v^i\Big) = -\operatorname{str}\binom{\mathcal{S}}{2}.$$

Then the rest of the proof is similar to that of Proposition 5.2.3. □

5.5.2 Other Virasoro fields

Now let us assume that $\mathfrak{a}$ can be written as the direct sum of two isotropic subspaces $\mathfrak{a}^- = \operatorname{span}\{v_i\}$ and $\mathfrak{a}^+ = \operatorname{span}\{v^i\}$ ($1 \le i \le \ell$), i.e., subspaces satisfying $(\mathfrak{a}^\pm|\mathfrak{a}^\pm) = 0$. Additionally, we will assume that as before, $(v_i|v^j) = \delta_{i,j}$ and $d = \dim\mathfrak{a} = 2\ell$. Following [60, Section 3.6], we let

$$\omega^\lambda = (1-\lambda)\omega^+ + \lambda\omega^- \qquad (\lambda \in \mathbb{C}), \tag{5.42}$$

where

$$\omega^+ = \sum_{i=1}^{\ell} v^i_{(-2)}v_i, \qquad \omega^- = -\sum_{i=1}^{\ell}(-1)^{p(v_i)}v_{i(-2)}v^i.$$

Then ω^λ is a conformal vector in $F^1(\mathfrak{a})$ with central charge

$$c_\lambda = (6\lambda^2 - 6\lambda + 1)\operatorname{sdim}\mathfrak{a}.$$

In particular, $\omega^{1/2}$ coincides with (5.40). We denote the corresponding family of Virasoro fields as

$$L^\lambda(z) = Y(\omega^\lambda, z) = (1-\lambda)L^+(z) + \lambda L^-(z).$$

Their action can be derived from the proof of Proposition 5.5.4 as follows.

Corollary 5.5.5. *If $\varphi(\omega^+) = \omega^+$, then in every φ-twisted $F^1(\mathfrak{a})$-module*

$$L^+(z) = \sum_{i=1}^{\ell} {:}\big(\partial_z X(v^i, z)\big) X(v_i, z){:} - z^{-1} \sum_{i=1}^{\ell} {:}X(\mathcal{N} v^i, z) X(v_i, z){:} \\ - z^{-2} \operatorname{str} \binom{\mathcal{S}^+}{2} I,$$

where $\mathcal{S}^+$ is the restriction of $\mathcal{S}$ to $\mathfrak{a}^+$.

If the automorphism φ is as in Example 5.1.6, then a short calculation gives $\mathcal{N}(\omega^\lambda) = (2\lambda - 1) v_{\ell+1(-2)} v_{\ell+1}$. This implies that only the modes of $Y(\omega^{1/2}, z) = L^{1/2}(z)$ yield an untwisted representation of the Virasoro algebra on a φ-twisted $F^1(\mathfrak{a})$-module. If φ is as in Examples 5.1.1 or 5.1.5, then $\mathcal{N}(\omega^\lambda) = 0$ and $L^\lambda(z)$ yields an untwisted representation of the Virasoro algebra for any $\lambda \in \mathbb{C}$.

Remark 5.5.6. In the special case when $\mathfrak{a}$ is even with $\dim \mathfrak{a} = 2$ (i.e., when we have a $\beta\gamma$-system of rank 1), the above Virasoro fields resemble but are different from those of [3, (3.15)].

5.6 Free fermions

In this section, we will compute explicitly the actions of $(C_{\mathfrak{a}})_\varphi$ and L_0 on $M_\varphi(R)$ when $\mathfrak{a}$ is odd as in Examples 5.1.1, 5.1.2. Let $\{v_i\}$ be a basis for $\mathfrak{a}$ such that $(v_i|v_j) = \delta_{i+j,d+1}$ $(1 \le i, j \le d)$, and φ acts as in Example 5.1.1 or 5.1.2. We obtain a dual basis satisfying (5.10) by letting $v^i = v_{d-i+1}$. Thus a conformal vector ω is given by (5.40).

5.6.1 The case of Example 5.1.1

Assume that $\dim \mathfrak{a} = 2\ell$, and σ and $\mathcal{N}$ act as in Example 5.1.1. The logarithmic fields $Y(v_j, z)$ are given by (5.18). The Lie superalgebra $(C_{\mathfrak{a}})_\varphi$ is spanned by an even central element K and odd elements $v_i t^{m+\alpha_0}$, $v_{\ell+i} t^{m-\alpha_0}$ ($1 \le i \le \ell$, $m \in \mathbb{Z}$). The nonzero brackets in $(C_{\mathfrak{a}})_\varphi$ are given by:

$$[v_i t^{m+\alpha_0}, v_j t^{n-\alpha_0}] = \delta_{m,-n-1}\delta_{i+j,2\ell+1} K, \tag{5.43}$$

for $1 \le i \le \ell$, $\ell+1 \le j \le 2\ell$, $m, n \in \mathbb{Z}$. The elements of $(C_{\mathfrak{a}})^-_\varphi$ act as creation operators on $M_\varphi(R)$. Throughout the rest of this section, we will represent them as anti-commuting variables as follows:

$$v_i t^{-m+\alpha_0} = \xi_{i,m}, \qquad v_j t^{-n-\alpha_0-1} = \xi_{j,n}, \tag{5.44}$$

for $v_i \in \mathfrak{a}_\alpha$, $v_j \in \mathfrak{a}_{-\alpha}$, and $m \ge 1$, $n \ge 0$. The precise triangular decomposition (5.39) depends on whether $\alpha_0 = -1/2$ or $\alpha_0 \in \mathbb{C}^+_{-\frac{1}{2}}$ (cf. (5.38)).

Suppose first that $\alpha_0 \in \mathbb{C}^+_{-\frac{1}{2}}$. Then $(C_{\mathfrak{a}})^0_\varphi = \mathbb{C}K$ and $R = \mathbb{C}$. Thus

$$M_\varphi(R) \cong \bigwedge(\xi_{i,m+1}, \xi_{\ell+i,m})_{1\le i\le \ell,\, m=0,1,2,\ldots}. \tag{5.45}$$

The action of $(C_{\mathfrak{a}})^+_\varphi$ on $M_\varphi(R)$ is given explicitly by

$$v_i t^{m+\alpha_0} = \partial_{\xi_{2\ell-i+1,m}}, \qquad v_{\ell+i} t^{n-\alpha_0-1} = \partial_{\xi_{\ell-i+1,n}},$$

where $1 \le i \le \ell$, $m \ge 0$, $n \ge 1$. By Proposition 5.5.4, the action of L_0 is

$$\begin{aligned} L_0 = &\sum_{i=1}^{\ell}\sum_{m=1}^{\infty} \xi_{i,m}\Big(\Big(m - \alpha_0 - \frac{1}{2}\Big)\partial_{\xi_{i,m}} - (1-\delta_{i,1})\partial_{\xi_{i-1,m}}\Big) \\ &+ \sum_{i=1}^{\ell}\sum_{m=0}^{\infty} \xi_{\ell+i,m}\Big(\Big(m + \alpha_0 + \frac{1}{2}\Big)\partial_{\xi_{\ell+i,m}} + (1-\delta_{i,1})\partial_{\xi_{\ell+i-1,m}}\Big) \\ &+ \frac{\ell}{2}\alpha_0^2 I. \end{aligned} \tag{5.46}$$

Using Corollary 5.5.5, we find that the action of L_0^+ is given by:

$$\begin{aligned} L_0^+ = &\sum_{i=1}^{\ell}\sum_{m=1}^{\infty} \xi_{i,m}\Big((m-\alpha_0-1)\partial_{\xi_{i,m}} - (1-\delta_{i,1})\partial_{\xi_{i-1,m}}\Big) \\ &+ \sum_{i=1}^{\ell}\sum_{m=0}^{\infty} \xi_{\ell+i,m}\Big((m+\alpha_0+1)\partial_{\xi_{\ell+i,m}} + (1-\delta_{i,1})\partial_{\xi_{\ell+i-1,m}}\Big) \\ &+ \frac{\ell}{2}\alpha_0(\alpha_0+1)I, \end{aligned} \tag{5.47}$$

which coincides with (5.22) up to relabeling of the variables.

Now we consider the case when $\alpha_0 = -1/2$. Then

$$(C_{\mathfrak{a}})^0_\varphi = \operatorname{span}\{v_i t^{-1/2}\}_{1\le i\le d} \oplus \mathbb{C}K, \tag{5.48}$$

where $d = 2\ell$. We let

$$R = \bigwedge(\xi_{\ell+i,0})_{1\le i\le \ell}$$

with

$$v_{\ell+i}t^{-1/2} = \xi_{\ell+i,0}, \qquad v_i t^{-1/2} = \partial_{\xi_{2\ell-i+1,0}} \qquad (1 \le i \le \ell).$$

Therefore, by (5.44), $M_\varphi(R)$ is again given by (5.45). The action of $(C_{\mathfrak{a}})^+_\varphi$ is given by

$$v_i t^{m-1/2} = \partial_{\xi_{2\ell-i+1,m}}, \qquad v_{\ell+i}t^{n-1/2} = \partial_{\xi_{\ell-i+1,n}},$$

for $1 \le i \le \ell$, $m \ge 0$, $n \ge 1$. The actions of L_0 and L_0^+ are given by (5.46) and (5.47) respectively, each with $\alpha_0 = -\frac{1}{2}$.

5.6.2 The case of Example 5.1.2

Let $\mathfrak{a}$ be as in Example 5.1.2. The logarithmic fields $Y(v_j, z)$ are the same as in [6, Section 6.5]:

$$Y(v_j, z) = \sum_{i=j}^{2\ell-1}\sum_{m\in\alpha_0+\mathbb{Z}} \frac{(-1)^{(i-j)(i+j-1)/2}}{(i-j)!}\,\zeta^{i-j}(v_i t^m)z^{-m-1}$$

for $1 \le j \le 2\ell-1$. The Lie superalgebra $(C_{\mathfrak{a}})_\varphi$ is spanned by an even central element K and odd elements $v_i t^{m+\alpha_0}$ ($1 \le i \le 2\ell-1$, $m \in \mathbb{Z}$) where $\alpha_0 = -\frac{1}{2}$ or 0. The brackets in

$(C_{\mathfrak{a}})_\varphi$ are given by

$$[v_i t^{m+\alpha_0}, v_j t^{n-\alpha_0}] = \delta_{m,-n-1}\delta_{i+j,2\ell}K,$$

for $1 \le i,j \le 2\ell-1$, $m,n \in \mathbb{Z}$. We let the creation operators from $(C_{\mathfrak{a}})_\varphi^-$ act by the first equation of (5.44). The triangular decomposition (5.39) depends on whether $\alpha_0 = -\frac{1}{2}$ or $\alpha_0 = 0$. We first consider the case when $\alpha_0 = 0$. Then $(C_{\mathfrak{a}})_\varphi^0 = \mathbb{C}K$ and

$$M_\varphi(R) \cong \bigwedge(\xi_{i,m})_{1\le i\le 2\ell-1,\, m=1,2,3,\ldots},$$

where the action of $(C_{\mathfrak{a}})_\varphi^+$ on $M_\varphi(R)$ is

$$v_i t^m = \partial_{\xi_{2\ell-i,m+1}}, \qquad 1 \le i \le 2\ell-1,\ m \ge 0.$$

The action of L_0 is

$$L_0 = \sum_{i=1}^{2\ell-1}\sum_{m=1}^{\infty} \xi_{i,m}\Big(\Big(m-\frac{1}{2}\Big)\partial_{\xi_{i,m}} + (-1)^{i+1}(1-\delta_{i,1})\partial_{\xi_{i-1,m}}\Big).$$

Now consider the case when $\alpha_0 = -1/2$. Then $(C_{\mathfrak{a}})_\varphi^0$ is given by (5.48) with $d = 2\ell-1$. We let

$$R = \bigwedge(\xi_{j,0})_{\ell\le j\le 2\ell-1} \qquad \Big(\text{where } \xi_{\ell,0}^2 = \frac{1}{2}\Big),$$

with

$$v_j t^{-1/2} = \xi_{j,0}, \qquad v_i t^{-1/2} = \partial_{\xi_{2\ell-i,0}}$$

for $1 \le i \le \ell-1$ and $\ell \le j \le 2\ell-1$. Therefore,

$$M_\varphi(R) \cong \bigwedge(\xi_{i,m+1}, \xi_{j,m})_{1\le i\le \ell-1,\, \ell\le j\le 2\ell-1,\, m=0,1,2,\ldots},$$

where the action of $(C_{\mathfrak{a}})_\varphi^+$ on $M_\varphi(R)$ is given by

$$v_i t^{m-1/2} = \partial_{\xi_{2\ell-i,m}}, \qquad 1 \le i \le 2\ell-1,\ m \ge 1.$$

The action of L_0 is

$$L_0 = \sum_{i=1}^{2\ell-1} \sum_{m=1}^{\infty} \xi_{i,m} \Big(m\partial_{\xi_{i,m}} - (-1)^i (1-\delta_{i,1}) \partial_{\xi_{i-1,m}} \Big) + \sum_{i=1}^{\ell-1} (-1)^i \xi_{\ell+i,0} \partial_{\xi_{\ell+i-1,0}} + \frac{2\ell-1}{16} I.$$

5.7 Bosonic ghost system

Now we will compute explicitly the actions of $(C_{\mathfrak{a}})_\varphi$ and L_0 on $M_\varphi(R)$ when $\mathfrak{a}$ is even as in Examples 5.1.5 and 5.1.6. Let $\{v_i\}_{1\le i\le 2\ell}$ be a basis for $\mathfrak{a}$ such that $(v_i|v_j) = \delta_{i+j,2\ell+1}$ $(1 \le i \le j \le 2\ell)$, and φ acts as in Example 5.1.5 or 5.1.6. Then the basis defined by $v^i = v_{2\ell-i+1}$, $v^{\ell+i} = -v_{\ell-i+1}$ $(1 \le i \le \ell)$ is dual to $\{v_i\}$ with respect to $(\cdot|\cdot)$, and a conformal vector is given by (5.40).

5.7.1 The case of Example 5.1.5

Assume that σ and $\mathcal{N}$ act as in Example 5.1.5. The logarithmic fields $Y(v_j, z)$ are given by (5.18). The Lie algebra $(C_{\mathfrak{a}})_\varphi$ is spanned by a central element K and elements $v_i t^{m+\alpha_0}$, $v_{\ell+i} t^{m-\alpha_0}$ $(1 \le i \le \ell,\ m \in \mathbb{Z})$. The nonzero brackets in $(C_{\mathfrak{a}})_\varphi$ are given by

$$[v_i t^{m+\alpha_0}, v_j t^{n-\alpha_0}] = \delta_{m,-n-1} \delta_{i+j,2\ell+1} K, \tag{5.49}$$

for $1 \le i \le \ell$, $\ell+1 \le j \le 2\ell$, $m,n \in \mathbb{Z}$. The elements of $(C_{\mathfrak{a}})_\varphi^-$ act as creation operators on $M_\varphi(R)$. Throughout the rest of this section, we will represent them as commuting variables as follows

$$v_i t^{-m+\alpha_0} = x_{i,m}, \qquad v_j t^{-n-\alpha_0-1} = x_{j,n}, \tag{5.50}$$

for $1 \le i \le \ell$, $\ell+1 \le j \le 2\ell$ and $m \ge 1$, $n \ge 0$.

Again, the precise triangular decomposition (5.39) depends on whether $\alpha_0 \in \mathbb{C}^+_{-\frac{1}{2}}$ or $\alpha_0 = -\frac{1}{2}$.

Consider first the case when $\alpha_0 \in \mathbb{C}^+_{-\frac{1}{2}}$. Then $(C_{\mathfrak{a}})_\varphi^0 = \mathbb{C}K$, $R = \mathbb{C}$, and

$$M_\varphi(R) = \mathbb{C}[x_{i,m+1}, x_{\ell+i,m}]_{1\le i\le \ell,\, m=0,1,2,\dots}. \tag{5.51}$$

The action of $(C_{\mathfrak{a}})_\varphi^+$ on $M_\varphi(R)$ is given explicitly by

$$v_i t^{m+\alpha_0} = \partial_{x_{2\ell-i+1,m}}, \qquad v_{\ell+i} t^{n-\alpha_0-1} = -\partial_{x_{\ell-i+1,n}}, \tag{5.52}$$

for $1 \le i \le \ell$, $m \ge 0$, $n \ge 1$. The action of L_0 is

$$\begin{aligned} L_0 = & \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} x_{i,m} \Big(\Big(m - \alpha_0 - \frac{1}{2} \Big) \partial_{x_{i,m}} - (1-\delta_{i,1}) \partial_{x_{i-1,m}} \Big) \\ & + \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} x_{\ell+i,m} \Big(\Big(m + \alpha_0 + \frac{1}{2} \Big) \partial_{x_{\ell+i,m}} + (1-\delta_{i,1}) \partial_{x_{\ell+i-1,m}} \Big) \\ & - \frac{\ell}{2} \alpha_0^2 I. \end{aligned} \tag{5.53}$$

By Corollary 5.5.5, the action of L_0^+ is given by:

$$\begin{aligned} L_0^+ = & \sum_{i=1}^{\ell} \sum_{m=1}^{\infty} x_{i,m} \Big(\Big(m - \alpha_0 - 1 \Big) \partial_{x_{i,m}} - (1-\delta_{i,1}) \partial_{x_{i-1,m}} \Big) \\ & + \sum_{i=1}^{\ell} \sum_{m=0}^{\infty} x_{\ell+i,m} \Big(\Big(m + \alpha_0 + 1 \Big) \partial_{x_{\ell+i,m}} + (1-\delta_{i,1}) \partial_{x_{\ell+i-1,m}} \Big) \\ & - \frac{\ell}{2} \alpha_0 (\alpha_0 + 1) I, \end{aligned} \tag{5.54}$$

which agrees with (5.33) up to relabeling of the variables.

Now consider the case when $\alpha_0 = -\frac{1}{2}$. Then $(C_{\mathfrak{a}})_\varphi^0$ is given by (5.48) with $d = 2\ell$. We let

$$R = \mathbb{C}[x_{\ell+i,0}]_{1 \le i \le \ell}, \tag{5.55}$$

with

$$v_{\ell+i} t^{-1/2} = x_{\ell+i,0}, \qquad v_i t^{-1/2} = \partial_{x_{2\ell-i+1,0}} \qquad (1 \le i \le \ell).$$

Then $M_\varphi(R)$ is given by (5.51), where the action of $(C_{\mathfrak{a}})_\varphi^+$ is given by

$$v_i t^{m-1/2} = \partial_{x_{2\ell-i+1,m}}, \qquad v_{\ell+i} t^{m+1/2} = -\partial_{x_{\ell-i+1,m+1}}, \tag{5.56}$$

for $1 \le i \le \ell$ and $m \ge 1$. The actions of L_0 and L_0^+ are given by (5.53) and (5.54) respectively, each with $\alpha_0 = -\frac{1}{2}$.

5.7.2 The case of Example 5.1.6

Let $\mathfrak{a}$ be as in Example 5.1.6. The logarithmic fields are given by (5.25). The Lie algebra $(C_{\mathfrak{a}})_\varphi$ is spanned by a central element K and elements $v_i t^{m+\alpha_0}$ ($1 \le i \le 2\ell$) where $\alpha_0 = 0$ or $-1/2$. The brackets in $(C_{\mathfrak{a}})_\varphi$ are given by (5.49). We let the creation operators from $(C_{\mathfrak{a}})_\varphi^-$ act on $M_\varphi(R)$ by (5.50). As before, the triangular decomposition depends on whether $\alpha_0 = 0$ or $-1/2$.

We first consider the case when $\alpha_0 = 0$. Then $(C_{\mathfrak{a}})_\varphi^0 = \mathbb{C}K$, $R = \mathbb{C}$, $M_\varphi(R)$ is given by (5.51), and the action of $(C_{\mathfrak{a}})_\varphi^+$ is given by (5.52) with $\alpha_0 = 0$. The action of L_0 is

$$\begin{aligned}
L_0 = &\sum_{i=1}^{\ell}\sum_{m=1}^{\infty} x_{i,m}\left(\left(m-\frac{1}{2}\right)\partial_{x_{i,m}} - (1-\delta_{i,1})\partial_{x_{i-1,m}}\right)\\
&+\sum_{i=1}^{\ell}\sum_{m=0}^{\infty} x_{\ell+i,m}\left(\left(m+\frac{1}{2}\right)\partial_{x_{\ell+i,m}} + (1-2\delta_{i,1})\partial_{x_{\ell+i-1,m}}\right).
\end{aligned}$$

Now assume $\alpha_0 = -1/2$. Then $(C_{\mathfrak{a}})_\varphi^0$, R, and $M_\varphi(R)$ are the same as in the case when $\alpha_0 = -1/2$ in Section 5.7.1. The action of L_0 is

$$\begin{aligned}
L_0 = &\sum_{i=1}^{\ell}\sum_{m=1}^{\infty} x_{i,m}\left(m\partial_{x_{i,m}} - (1-\delta_{i,1})\partial_{x_{i-1,m}}\right)\\
&+\sum_{i=1}^{\ell}\sum_{m=1}^{\infty} x_{\ell+i,m}\left(m\partial_{x_{\ell+i,m}} + (1-2\delta_{i,1})\partial_{x_{\ell+i-1,m}}\right)\\
&+\sum_{i=2}^{\ell} x_{\ell+i,0}\partial_{x_{\ell+i-1,0}} + \frac{1}{2}x_{\ell+1,0}^2 - \frac{\ell}{8}I.
\end{aligned}$$

Chapter 6

Twisted Logarithmic Modules of Lattice Vertex Algebras

Twisted modules for lattice vertex algebras formalize the relations between twisted vertex operators [28]. In [10], such modules were classified under some natural assumptions in the case when the automorphism has finite order. In the current chapter, we use the theory of twisted logarithmic modules as presented in [6] to develop a framework for studying the case of a general automorphism. In particular, we show that twisted logarithmic vertex operators on certain φ-twisted V_Q-modules M can be written in terms of the action of $\hat{\mathfrak{h}}_\varphi$ on M together with the action of a group, and that the question of classifying such φ-twisted V_Q-modules reduces to the classification of modules of this group. To demonstrate the utility of our approach, we construct explicit examples from 3 and 4-dimensional lattices.

6.1 The φ-twisted Vertex Operators

Continuing the notation from Section 4.3, we let Q be an integral lattice with bilinear form $(\cdot|\cdot)$, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$ be its complexification, and $S = \widetilde{V}^1$ be the Fock space representation of $\hat{\mathfrak{h}}$. We let $\varepsilon\colon Q \times Q \to \{\pm 1\}$ be a bimultiplicative function such that (4.28) holds and $V_Q = S \otimes \mathbb{C}_\varepsilon[Q]$ the corresponding lattice vertex algebra. We let φ be an automorphism of the lattice Q which preserves the bilinear form, i.e., such that (5.1) holds. We extend φ linearly, thereby obtaining an automorphism of $\mathfrak{h}$, which we denote again by φ. As in

Chapter 5, we write

$$\varphi = \sigma e^{-2\pi \mathrm{i} \mathcal{N}}, \tag{6.1}$$

where σ is semisimple and $\mathcal{N}$ is nilpotent on $\mathfrak{h}$, and $\sigma\mathcal{N} = \mathcal{N}\sigma$.

The map $Q \times Q \to \{\pm 1\}$ given by $(\lambda, \mu) \mapsto \varepsilon(\varphi\lambda, \varphi\mu)$ is a 2-cocycle satisfying (4.28). Since ε is unique up to equivalence, there exists some $\eta\colon Q \to \{\pm 1\}$ satisfying

$$\eta(\lambda)\eta(\mu)\varepsilon(\lambda,\mu) = \eta(\lambda+\mu)\varepsilon(\varphi\lambda,\varphi\mu), \qquad \lambda,\mu \in Q. \tag{6.2}$$

Proposition 6.1.1 (See [10], Proposition 4.1)**.** *The automorphism $\varphi\colon Q \to Q$ can be lifted to an automorphism of the lattice vertex algebra V_Q so that*

$$\varphi(a_{(m)}) = (\varphi a)_{(m)}, \qquad \varphi(e^{\lambda}) = \eta(\lambda)^{-1} e^{\varphi\lambda}, \qquad a \in \mathfrak{h},\ \lambda \in Q,\ m \in \mathbb{Z}. \tag{6.3}$$

Remark 6.1.2. In general the action of φ on e^{λ} is not locally finite, and the decomposition (6.1) is not valid for such vectors. However, (6.3) guarantees that φ is locally finite on the Heisenberg subalgebra $B^1(\mathfrak{h}) \subseteq V_Q$.

We have already considered the φ-twisted modules of the Heisenberg subalgebra $B^1(\mathfrak{h})$ in Chapter 5. Now we aim to extend such modules to be φ-twisted modules of V_Q. In particular, we need to find logarithmic fields $Y(e^{\lambda}, z)$ on the module which taken together with the fields corresponding to the Heisenberg subalgebra satisfy the axioms of a φ-twisted module (see Definition 4.4.2).

We begin by assuming that M is a φ-twisted V_Q-module. Then the logarithmic fields $Y(a, z)$, $a \in \mathfrak{h}$ generate a φ-twisted $B^1(\mathfrak{h})$-module structure on M. In Chapter 5 we dealt exclusively with modes $a_{(m+\mathcal{N})}$ where a was an eigenvector for σ. Here we do not have that luxury, since there may be no generating set for Q consisting of eigenvectors for σ. Instead we let S_σ be the *spectrum* of σ, i.e., its set of eigenvalues. Then we let

$$\mathcal{A} = \{\alpha \in \mathbb{C}/\mathbb{Z}\colon e^{-2\pi \mathrm{i}\alpha} \in S_\sigma\}.$$

Given $\alpha \in \mathcal{A}$, we let $\pi_\alpha\colon \mathfrak{h} \to \mathfrak{h}_\alpha$ be the projection onto the corresponding eigenspace. Then we can write any $\lambda \in Q$ uniquely as a sum of eigenvectors for σ:

$$\lambda = \sum_{\alpha \in \mathcal{A}} \pi_\alpha \lambda.$$

We adopt the convention that given $\alpha \in \mathcal{A}$, $m \in \alpha$, and any $a \in \mathfrak{h}$,

$$a_{(m+\mathcal{N})} = (\pi_\alpha a)_{(m+\mathcal{N})}. \tag{6.4}$$

We set $a_{(m+\mathcal{N})} = 0$ for $m \in \alpha$, where $\alpha \in (\mathbb{C}/\mathbb{Z}) \setminus \mathcal{A}$. We also use the notation $\pi_0 = \pi_{\mathbb{Z}}$, $\mathfrak{h}_0 = \mathfrak{h}_{\mathbb{Z}}$, and $a_0 = \pi_0 a$.

We recall that $\mathfrak{h}_\alpha \perp \mathfrak{h}_\beta$ for $\beta \neq -\alpha$. Thus for any $\alpha \in \mathcal{A}$, $a, b \in \mathfrak{h}$, we have

$$(\pi_\alpha a|b) = (\pi_\alpha a|\pi_{-\alpha} b). \tag{6.5}$$

Then from (4.23), (4.54), (6.4), and (6.5) we have

$$\bigl[a_{(m+\mathcal{N})}, b_{(n+\mathcal{N})}\bigr] = \bigl((m+\mathcal{N})\pi_\alpha a\big|b\bigr)\delta_{m,-n}, \qquad m \in \alpha,\ n \in \mathbb{C}. \tag{6.6}$$

Let $\lambda \in Q$ and $\alpha \in \mathcal{A}$. Then (4.32) and (4.55) imply

$$\bigl[a_{(m+\mathcal{N})}, Y(e^\lambda, z)\bigr] = \bigl(z^{m+\mathcal{N}}\pi_\alpha a\big|\lambda\bigr)Y(e^\lambda, z), \qquad a \in \mathfrak{h},\ m \in \alpha. \tag{6.7}$$

Using (4.53), we can write

$$Y(\lambda, z) = \sum_{\alpha\in\mathcal{A}}\sum_{m\in\alpha}\bigl(z^{-m-1-\mathcal{N}}\lambda\bigr)_{(m+\mathcal{N})} = \sum_{m\in\mathbb{C}}\bigl(z^{-m-1-\mathcal{N}}\lambda\bigr)_{(m+\mathcal{N})}. \tag{6.8}$$

For $m \in \mathbb{C} \setminus \{0\}$, the operator $(m+\mathcal{N})\colon \mathfrak{h} \to \mathfrak{h}$ is invertible on $\mathfrak{h}$ with

$$(m+\mathcal{N})^{-1}a = \sum_{j=0}^{\infty}\frac{(-\mathcal{N})^j}{m^{j+1}}a. \tag{6.9}$$

We recall that the operator $z^{-\mathcal{N}}$ acts on $\mathfrak{h}$ as

$$z^{-\mathcal{N}}a = e^{-\zeta\mathcal{N}}a = \sum_{j=0}^{\infty}(-\zeta)^{(j)}\mathcal{N}^j a. \tag{6.10}$$

The sums in both (6.9) and (6.10) are finite, since $\mathcal{N}$ is a nilpotent operator on $\mathfrak{h}$.

We define linear operators $\mathcal{P}^{\pm} = \mathcal{P}^{\pm}_{\zeta} \colon \mathfrak{h} \to \mathfrak{h}[\zeta]$ by

$$\mathcal{P}^{\pm} a = \pm \frac{1}{\zeta \mathcal{N}} \big(z^{\pm \mathcal{N}} - 1 \big) a.$$

These operators commute with σ and hence preserve its eigenspaces $\mathfrak{h}_{\alpha}$. We note that (5.2) implies that

$$(\mathcal{P}^{+} a | b) = (a | \mathcal{P}^{-} b), \qquad a, b \in \mathfrak{h}. \tag{6.11}$$

It immediately follows that

$$(\mathcal{P}^{+} a | a) = (\mathcal{P}^{-} a | a). \tag{6.12}$$

We also define the operator $\mathcal{P} = \mathcal{P}_{\zeta} \colon \mathfrak{h} \to \mathfrak{h}[\zeta]$ by

$$\mathcal{P} = \frac{\mathcal{P}^{+} - \mathcal{P}^{-}}{2}.$$

As an immediate consequence of (6.11), we have

$$(\mathcal{P} a | b) = -(a | \mathcal{P} b). \tag{6.13}$$

We define the logarithmic field

$$\phi_{\lambda}(z) = - \sum_{m \in \mathbb{C} \setminus \{0\}} \big((m + \mathcal{N})^{-1} z^{-m-\mathcal{N}} \lambda \big)_{(m+\mathcal{N})} + (\zeta \mathcal{P}^{-} \lambda)_{(0+\mathcal{N})}.$$

Then $D_z \phi_{\lambda}(z) = Y(\lambda, z)$.

Now we consider the exponentials

$$E_{\lambda}(z)_{\pm} = \exp\Big(- \sum_{m \in \mathbb{C}^{\mp}} \big((m + \mathcal{N})^{-1} z^{-m-\mathcal{N}} \lambda \big)_{(m+\mathcal{N})} \Big),$$

and define

$$E_{\lambda}(z) = E_{\lambda}(z)_{+} E_{\lambda}(z)_{-}.$$

Proposition 6.1.3. *Let $a \in \mathfrak{h}$ and $m \in \alpha$. Then*

$$\big[a_{(m+\mathcal{N})}, E_{\lambda}(z) \big] = \begin{cases} \big(z^{\mathcal{N}} \pi_{\alpha} a | \lambda \big) z^{m} E_{\lambda}(z), & m \neq 0, \\ 0, & m = 0. \end{cases}$$

Proof. The case when $m = 0$ follows immediately from (6.6). If $m \neq 0$, we use (2.8), (6.6), and the fact that $\mathcal{N}$ satisfies (5.2) to compute

$$\begin{aligned}\big[a_{(m+\mathcal{N})}, E_\lambda(z)\big] &= -\big[a_{(m+\mathcal{N})}, (-m+\mathcal{N})^{-1}(z^{m-\mathcal{N}}\lambda)_{(-m+\mathcal{N})}\big]E_\lambda(z)\\ &= -\big((m+\mathcal{N})\pi_\alpha a|(-m+\mathcal{N})^{-1}z^{m-\mathcal{N}}\lambda\big)E_\lambda(z)\\ &= \big(z^{m+\mathcal{N}}\pi_\alpha a|\lambda\big)E_\lambda(z).\end{aligned}$$

□

At times we use the notation $e^{\zeta q}$, where q is a semisimple operator on M. This should be understood to mean the operator z^q. If instead q is nilpotent, the notation z^q means the operator $e^{\zeta q}$. With this notational convention in mind, we define the operators

$$\theta_h = \theta_h(\zeta) = e^{(\zeta\mathcal{P}^- h)_{(0+\mathcal{N})}}, \qquad h \in \mathfrak{h}.$$

Then from (2.9) and the commutator

$$[(\zeta\mathcal{P}^- h)_{(0+\mathcal{N})}, (\zeta\mathcal{P}^- h')_{(0+\mathcal{N})}] = 2(\zeta\mathcal{P}h_0|h'), \qquad h, h' \in \mathfrak{h},$$

we have

$$\theta_h\theta_{h'} = e^{(\zeta\mathcal{P}h_0|h')}\theta_{h+h'} = e^{2(\zeta\mathcal{P}h_0|h')}\theta_{h'}\theta_h. \tag{6.14}$$

Then it follows from (6.13) and (6.14) that θ_h is invertible with $\theta_h^{-1} = \theta_{-h}$. We define the operators

$$U_\lambda(z) = E_\lambda(z)_+^{-1}Y(e^\lambda, z)\theta_\lambda E_\lambda(z)_-^{-1}, \qquad \lambda \in Q. \tag{6.15}$$

Proposition 6.1.4. *Let $a \in \mathfrak{h}$ and $\lambda \in Q$. Then*

$$\big[a_{(m+\mathcal{N})}, U_\lambda(z)\big] = \delta_{m,0}\big(a_0\big|\lambda\big)U_\lambda(z).$$

Proof. First assume $m \neq 0$. Then from (6.6) we have

$$\begin{aligned}&\Big[a_{(m+\mathcal{N})}, \big((n+\mathcal{N})^{-1}z^{-n-\mathcal{N}}\lambda\big)_{(n+\mathcal{N})}\Big]\\ &\qquad = \delta_{m,-n}\Big((m+\mathcal{N})\pi_\alpha a\Big|(-m+\mathcal{N})^{-1}z^{m-\mathcal{N}}\lambda\Big)\\ &\qquad = -\delta_{m,-n}\big(z^{\mathcal{N}}\pi_\alpha a\big|\lambda\big)z^m.\end{aligned} \tag{6.16}$$

Then (2.8), (6.7), and (6.16) imply that $\left[a_{(m+\mathcal{N})}, U_\lambda(z)\right] = 0$. Now assume that $m = 0$. Then

$$\begin{aligned}\left[a_{(0+\mathcal{N})}, U_\lambda(z)\right] &= \left(z^{\mathcal{N}} a_0 \middle| \lambda\right) U_\lambda(z) + \left[a_{(0+\mathcal{N})}, \left(\frac{1}{\mathcal{N}}\left(z^{-\mathcal{N}} - 1\right)\lambda\right)_{(0+\mathcal{N})}\right] U_\lambda(z) \\ &= \left(z^{\mathcal{N}} a_0 \middle| \lambda\right) U_\lambda(z) + \left(\mathcal{N} a_0 \middle| \frac{1}{\mathcal{N}}\left(z^{-\mathcal{N}} - 1\right)\lambda\right) U_\lambda(z) \\ &= \left(z^{\mathcal{N}} a_0 \middle| \lambda\right) U_\lambda(z) - \left((z^{\mathcal{N}} - 1) a_0 | \lambda\right) U_\lambda(z) \\ &= (a_0|\lambda) U_\lambda(z).\end{aligned}$$

□

From (6.15) we obtain

$$Y(e^\lambda, z) = E_\lambda(z)_+ U_\lambda(z) E_\lambda(z)_- \, \theta_\lambda.$$

Proposition 6.1.4 implies that $[E_\lambda(z)_+, U_\lambda(z)] = 0$. Thus

$$Y(e^\lambda, z) = U_\lambda(z) \theta_\lambda E_\lambda(z). \tag{6.17}$$

From (4.37) and (4.49) we have

$$D_z Y(e^\lambda, z) = Y(\lambda_{(-1)} e^\lambda, z). \tag{6.18}$$

We will use this equality to produce a partial differential equation in z and ζ satisfied by $U_\lambda(z)$. In order to do so, we need several lemmas.

Lemma 6.1.5. *Let A be an associative algebra and $D\colon A \to A$ a derivation. Assume $X \in A$ satisfies $[X, D(X)] = C$, where C commutes with X. Then for any $n \in \mathbb{N}$ we have*

$$\begin{aligned}D(X^n) &= nX^{n-1} D(X) - \frac{n(n-1)}{2} C X^{n-2} \\ &= nD(X) X^{n-1} + \frac{n(n-1)}{2} C X^{n-2}.\end{aligned} \tag{6.19}$$

Furthermore, in any representation R of A on which X can be exponentiated, we have

$$\begin{aligned} D(e^X) &= e^X\Big(D(X) - \frac{C}{2}\Big) \\ &= \Big(D(X) + \frac{C}{2}\Big)e^X. \end{aligned} \tag{6.20}$$

Proof. We prove (6.19) by induction on n. The claim is clearly true for $n = 0$ and $n = 1$. Now assume it is true for $n - 1$. Then

$$\begin{aligned} D(X^n) &= D(X^{n-1})X + X^{n-1}D(X) \\ &= \Big((n-1)X^{n-2}D(X) - \frac{(n-1)(n-2)}{2}CX^{n-3}\Big)X + X^{n-1}D(X) \\ &= (n-1)X^{n-2}(XD(X) - C) - \frac{(n-1)(n-2)}{2}CX^{n-2} + X^{n-1}D(X) \\ &= nX^{n-1}D(X) - \frac{n(n-1)}{2}CX^{n-2}. \end{aligned}$$

Similarly by shifting $D(X)$ to the left instead of right we obtain the second line of (6.19). We prove (6.20) using (6.19):

$$\begin{aligned} D(e^X) &= \sum_{n=0}^{\infty} \frac{D(X^n)}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\Big(nX^{n-1}D(X) - \frac{n(n-1)}{2}CX^{n-2}\Big) \\ &= \sum_{n=1} \frac{X^{n-1}}{(n-1)!}D(x) - \frac{C}{2}\sum_{n=2}^{\infty} \frac{X^{n-2}}{(n-2)!} \\ &= e^X\Big(D(X) - \frac{C}{2}\Big). \end{aligned}$$

We obtain the second line of (6.20) similarly using the second line of (6.19). $\square$

Lemma 6.1.6. *As in Chapter 5, we let $\mathcal{S}\colon \mathfrak{h} \to \mathfrak{h}$ be the operator $\mathcal{S}a = \alpha_0 a$ for $a \in \mathfrak{h}_\alpha$, $\alpha \in S_\sigma$ and extend linearly to $\mathfrak{h}$. Then for any $\lambda \in Q$ we have*

$$Y(\lambda_{(-1)}e^\lambda, z) = {:}Y(\lambda, z)Y(e^\lambda, z){:} - z^{-1}\big(\mathcal{S}\lambda\big|\lambda\big)Y(e^\lambda, z).$$

Furthermore, we have

$$(\mathcal{S}\lambda|\lambda) = -\frac{1}{2}|\lambda|^2 - \frac{1}{2}|\lambda_0|^2 + \sum_{\alpha\in\mathcal{A}^+\cup\mathbb{Z}} (\pi_\alpha\lambda|\lambda),$$

where

$$\begin{aligned}
\mathcal{A}^+ &= \{\alpha \in \mathcal{A}\colon \alpha_0 \in \mathbb{C}^+\} \\
&= \{\alpha \in \mathcal{A}\colon \operatorname{Re}\alpha_0 = 0 \textit{ and } \operatorname{Im}\alpha_0 > 0\}, \\
\mathcal{A}^- &= \{\alpha \in \mathcal{A}\colon \alpha_0 \in \mathbb{C}^-\} \\
&= \{\alpha \in \mathcal{A}\colon \operatorname{Re}\alpha_0 < 0 \textit{ or } \operatorname{Re}\alpha_0 = 0 \textit{ and } \operatorname{Im}\alpha_0 < 0\}.
\end{aligned}$$

Proof. From (4.32) and (4.56) we obtain

$$\begin{aligned}
{:}Y(\lambda,z)Y(e^\lambda,z){:} &= \sum_{\alpha\in\mathcal{A}} {:}Y(\pi_\alpha\lambda,z)Y(e^\lambda,z){:} \\
&= \sum_{\alpha\in\mathcal{A}} Y\big((\pi_\alpha\lambda)_{(-1)}e^\lambda,z\big) + \sum_{\alpha\in\mathcal{A}} z^{-1} Y\Big(\big((\alpha_0+\mathcal{N})\pi_\alpha\lambda\big)_{(0)}e^\lambda,z\Big) \\
&= Y(\lambda_{(-1)}e^\lambda,z) + z^{-1}\big((\mathcal{S}+\mathcal{N})\lambda\big|\lambda\big)Y(e^\lambda,z) \\
&= Y(\lambda_{(-1)}e^\lambda,z) + z^{-1}\big(\mathcal{S}\lambda\big|\lambda\big)Y(e^\lambda,z).
\end{aligned} \tag{6.21}$$

To obtain the last line of (6.21), we note that the symmetry of $(\cdot|\cdot)$ and (5.2) imply

$$(\mathcal{N}a|a) = 0, \qquad a \in \mathfrak{h}.$$

From (6.4) and the symmetry of $(\cdot|\cdot)$ we have

$$(\pi_\alpha\lambda|\lambda) = (\pi_{-\alpha}\lambda|\lambda). \tag{6.22}$$

We also recall that

$$(-\alpha)_0 = \begin{cases} -\alpha_0, & \operatorname{Re}\alpha_0 = 0, \\ -\alpha_0 - 1, & \operatorname{Re}\alpha_0 < 0. \end{cases} \tag{6.23}$$

Thus for $\alpha \neq \mathbb{Z}, -\frac{1}{2} + \mathbb{Z}$, we have

$$(\alpha_0 \pi_\alpha \lambda|\lambda) + ((-\alpha)_0 \pi_{-\alpha} \lambda|\lambda) = \begin{cases} 0 & \operatorname{Re} \alpha_0 = 0, \\ -(\pi_\alpha \lambda|\lambda) & \operatorname{Re} \alpha_0 < 0. \end{cases} \tag{6.24}$$

In the case of $\alpha_0 = -1/2$, we have $(-\alpha)_0 = \alpha_0$. Thus from (6.22) and (6.24) we obtain

$$\begin{aligned} (\mathcal{S}\lambda|\lambda) &= \sum_{\alpha \in \mathcal{A}} (\alpha_0 \pi_\alpha \lambda|\lambda) \\ &= -\frac{1}{2}(\pi_{1/2+\mathbb{Z}} \lambda|\lambda) + \sum_{\alpha \in \mathcal{A} \setminus \{1/2+\mathbb{Z}\}} \frac{1}{2}\big((\alpha_0 \pi_\alpha \lambda|\lambda) + ((-\alpha)_0 \pi_{-\alpha} \lambda|\lambda)\big) \\ &= -\frac{1}{2} \sum_{\substack{\alpha \in \mathcal{A} \\ \operatorname{Re} \alpha_0 < 0}} (\pi_\alpha \lambda|\lambda) \\ &= -\frac{1}{2} \sum_{\alpha \in \mathcal{A}} (\pi_\alpha \lambda|\lambda) - \frac{1}{2}|\lambda_0|^2 + \sum_{\alpha \in \mathcal{A}^+ \cup \{\mathbb{Z}\}} (\pi_\alpha \lambda|\lambda) \\ &= -\frac{1}{2}|\lambda|^2 - \frac{1}{2}|\lambda_0|^2 + \sum_{\alpha \in \mathcal{A}^+ \cup \{\mathbb{Z}\}} (\pi_\alpha \lambda|\lambda). \end{aligned}$$

□

Lemma 6.1.7. *For any $\lambda \in Q$ we have*

$$:Y(\lambda, z) Y(e^\lambda, z): = {}^\circ_\circ Y(\lambda, z) Y(e^\lambda, z) {}^\circ_\circ - z^{-1} |\lambda|^2 Y(e^\lambda, z) + z^{-1} \sum_{\alpha \in \mathcal{A}^+ \cup \{\mathbb{Z}\}} (\pi_\alpha \lambda|\lambda)\, Y(e^\lambda, z).$$

Proof.

$$\begin{aligned} :Y(\lambda, z) Y(e^\lambda, z): &= \sum_{\substack{m \in \mathbb{C} \\ \operatorname{Re} m \leq -1}} \big(z^{-m-1-\mathcal{N}} \lambda\big)_{(m+\mathcal{N})} Y(e^\lambda, z) + Y(e^\lambda, z) \sum_{\substack{m \in \mathbb{C} \\ \operatorname{Re} m > -1}} \big(z^{-m-1-\mathcal{N}} \lambda\big)_{(m+\mathcal{N})} \\ &= {}^\circ_\circ Y(\lambda, z) Y(e^\lambda, z) {}^\circ_\circ + \sum_{\alpha \in \mathcal{A}^-} \Big[Y(e^\lambda, z), \big(z^{-\alpha_0 - 1 - \mathcal{N}} \lambda\big)_{(\alpha_0 + \mathcal{N})} \Big]. \end{aligned}$$

Thus it only remains to note that (6.7) implies

$$\sum_{\alpha\in\mathcal{A}^-}\Big[(z^{-\alpha_0-1-\mathcal{N}}\lambda)_{(\alpha_0+\mathcal{N})}, Y(e^\lambda,z)\Big] = z^{-1}\sum_{\alpha\in\mathcal{A}^-}(\pi_\alpha\lambda|\lambda)Y(e^\lambda,z)$$
$$= z^{-1}|\lambda|^2 - z^{-1}\sum_{\alpha\in\mathcal{A}^+\cup\{\mathbb{Z}\}}(\pi_\alpha\lambda|\lambda)\,Y(e^\lambda,z).$$

□

Theorem 6.1.8. *Assume that M is a φ-twisted V_Q-module. Then there exist operators U_λ, $(\lambda\in Q)$ on M which are independent of z and ζ such that*

$$Y(e^\lambda,z) = U_\lambda\theta_\lambda e^{\zeta a_\lambda}z^{b_\lambda}E_\lambda(z), \tag{6.25}$$

where

$$a_\lambda = a_\lambda(\zeta) = \frac{(\mathcal{P}^-\lambda_0|\lambda)-|\lambda_0|^2}{2},$$
$$b_\lambda = \frac{|\lambda_0|^2-|\lambda|^2}{2},$$

and the operators U_λ satisfy

$$[a_{(m+\mathcal{N})}, U_\lambda] = \delta_{m,0}(a_0|\lambda)U_\lambda, \tag{6.26}$$
$$U_{\varphi\lambda} = \eta(\lambda)e^{2\pi\mathrm{i}c_\lambda}U_\lambda\tau_\lambda, \tag{6.27}$$

for $a\in\mathfrak{h}$, $m\in\mathbb{C}$, where

$$c_\lambda = 2\pi\mathrm{i}a_\lambda(2\pi\mathrm{i}) + b_\lambda = \frac{(\mathcal{P}^-_{2\pi\mathrm{i}}\lambda_0|\lambda)-|\lambda|^2}{2},$$

and

$$\tau_\lambda = \theta_\lambda(2\pi\mathrm{i}).$$

Proof. Using Lemma 6.1.5 with $D=D_z$, $X=(\zeta\mathcal{P}^-\lambda)_{(0+\mathcal{N})}$,

$$DX = \big(D_z(\zeta\mathcal{P}^-(\lambda))\big)_{(0+\mathcal{N})} = (z^{-\mathcal{N}-1}\lambda)_{(0+\mathcal{N})},$$

and

$$C = [(\zeta\mathcal{P}^-\lambda)_{(0+\mathcal{N})}, (z^{-\mathcal{N}-1}\lambda)_{(0+\mathcal{N})}] = z^{-1}\big((z^{-\mathcal{N}}-1)\lambda_0|\lambda\big),$$

we obtain

$$D_z\theta_\lambda = \theta_\lambda\Big((z^{-\mathcal{N}-1}\lambda)_{(0+\mathcal{N})} - \frac{1}{2}z^{-1}\big((z^{-\mathcal{N}}-1)\lambda_0\big|\lambda\big)\Big). \tag{6.28}$$

Next we compute

$$\begin{aligned} D_z E_\lambda(z) &= \Big(\sum_{m\in\mathbb{C}^-}(z^{-m-\mathcal{N}-1}\lambda)_{(m+\mathcal{N})}\Big)E_\lambda(z) + E_\lambda(z)\Big(\sum_{m\in\mathbb{C}^+}(z^{-m-\mathcal{N}-1}\lambda)_{(m+\mathcal{N})}\Big) \\ &= {}^\circ_\circ\big(Y(\lambda,z) - (z^{-1-\mathcal{N}})\lambda_{(0+\mathcal{N})}\big)E_\lambda(z){}^\circ_\circ. \end{aligned} \tag{6.29}$$

Thus from (6.28) and (6.29) we have

$$\begin{aligned} D_z Y(e^\lambda, z) = {}^\circ_\circ Y(\lambda,z)Y(e^\lambda,z){}^\circ_\circ + (D_z U_\lambda(z))\theta_\lambda E_\lambda(z) \\ - \frac{1}{2}z^{-1}\big((z^{-\mathcal{N}}-1)\lambda_0|\lambda\big)Y(e^\lambda,z). \end{aligned} \tag{6.30}$$

On the other hand, from (6.18), Lemma 6.1.6, and Lemma 6.1.7 we obtain

$$D_z Y(e^\lambda, z) = {}^\circ_\circ Y(\lambda,z)Y(e^\lambda,z){}^\circ_\circ + z^{-1}b_\lambda Y(e^\lambda,z). \tag{6.31}$$

By comparing (6.30) and (6.31), we find that $U_\lambda(z)$ satisfies the following partial differential equation in z and ζ:

$$D_z U_\lambda(z) = \frac{1}{2}z^{-1}\big((z^{-\mathcal{N}}-1)\lambda_0\big|\lambda\big)U_\lambda(z) + z^{-1}b_\lambda U_\lambda(z).$$

It follows immediately that the operator

$$U_\lambda = z^{-b_\lambda}e^{-\zeta a_\lambda}U_\lambda(z) \tag{6.32}$$

satisfies $D_z U_\lambda = 0$. Thus we may conclude that U_λ is a function of $ze^{-\zeta}$. Next we will show that U_λ is independent of z and ζ by proving that $\partial_z U_\lambda = 0$.

From Proposition 5.1.4, we know $\mathfrak{h}$ is a direct sum of subspaces

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_t,$$

where each subspace $\mathfrak{h}_j$ admits dual bases $\{v_i\}$ and $\{v^i\}$ on which φ and $(\cdot|\cdot)$ are as in

Example 5.1.1 or Example 5.1.2. Let

$$\omega_j = \frac{1}{2}\sum_{i=1}^{d_j} v^i_{(-1)}v_i, \qquad d_j = \dim \mathfrak{h}_j,\ 1 \le j \le t.$$

Then $\omega = \omega_1 + \cdots + \omega_t$ is a conformal vector in V_Q whose corresponding field $L(z) = Y(\omega, z)$ on V_Q has central charge $d = d_1 + \cdots + d_t$ (see (4.40)). Let $L^M(z) = Y^M(\omega, z)$ be the field corresponding to ω on M. The following equation holds for all $a \in V_Q$ (cf. [6]):

$$[L_0^M, Y(a,z)] = Y(L_0 a, z) + D_\varsigma Y(a,z). \tag{6.33}$$

We know from Chapter 5 that the action of L_0^M may not be semisimple. We split the action of L_0^M on M into its semisimple and nilpotent parts

$$L_0^M = L_0^{(s)} + L_0^{(n)}. \tag{6.34}$$

Then (6.33) yields the following equation for $L_0^{(s)}$:

$$[L_0^{(s)}, Y(a,z)] = Y(L_0 a, z) + z\partial_z Y(a,z), \qquad a \in V_Q. \tag{6.35}$$

In particular, we apply this equation for $a = e^\lambda$, obtaining

$$[L_0^{(s)}, Y(e^\lambda,z)] = Y(L_0 e^\lambda, z) + z\partial_z Y(e^\lambda,z). \tag{6.36}$$

By linearity, we can write

$$Y^M(\omega, z) = Y^M(\omega_1, z) + \cdots + Y^M(\omega_t, z).$$

We fix a subspace $\mathfrak{h}_j$ with $\{v_i\}$ and $\{v^i\}$ the dual bases of $\mathfrak{h}$ guaranteed by Proposition 5.1.4. Then either $\mathfrak{h}_j = \mathfrak{h}_{\alpha_j} \oplus \mathfrak{h}_{-\alpha_j}$, or $\mathfrak{h}_j = \mathfrak{h}_{\alpha_j}$ with $\alpha_j = \mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$. We let

$$L^j(z) = Y^M(\omega_j, z),$$

and we let $p_j \colon \mathfrak{h} \to \mathfrak{h}_j$ be the projection. Then from Proposition 5.4.1 and (6.6) we obtain

$$[L_0^j, h_{(m+\mathcal{N})}] = -((m+\mathcal{N})p_j h)_{(m+\mathcal{N})}, \qquad h \in \mathfrak{h},\ m \in \mathbb{C}. \tag{6.37}$$

Since $L_0^M = L_0^1 + \cdots + L_0^t$, it follows immediately from (6.37) that

$$[L_0^M, h_{(m+\mathcal{N})}] = -((m+\mathcal{N})h)_{(m+\mathcal{N})}. \tag{6.38}$$

The Jordan decomposition (6.34) of L_0^M implies the following Jordan decomposition of the adjoint action of L_0^M:

$$\mathrm{ad}_{L_0^M} = \mathrm{ad}_{L_0^{(s)}} + \mathrm{ad}_{L_0^{(n)}}, \tag{6.39}$$

where $\mathrm{ad}_{L_0^{(s)}}$ is semisimple, and $\mathrm{ad}_{L_0^{(n)}}$ is nilpotent. Then from (6.38) and (6.39) we immediately obtain

$$[L_0^{(s)}, h_{(m+\mathcal{N})}] = -mh_{(m+\mathcal{N})}, \qquad h \in \mathfrak{h}.$$

Then it is straightforward to check that

$$[L_0^{(s)}, E_\lambda(z)] = z\partial_z E_\lambda(z),$$

and by a calculation similar to (4.42), we have

$$L_0^{(s)} U_\lambda = \frac{|\lambda_0|^2}{2} U_\lambda L_0^{(s)}.$$

It follows that

$$[L_0^{(s)}, Y(e^\lambda, z)] = \frac{|\lambda_0|^2}{2} Y(e^\lambda, z) + U_\lambda \theta_\lambda e^{\zeta a_\lambda} z^{b_\lambda} \big(z\partial_z E_\lambda(z)\big). \tag{6.40}$$

From (4.42) we have

$$\begin{aligned} Y(L_0 e^\lambda, z) &+ z\partial_z Y(e^\lambda, z) \\ &= \frac{|\lambda|^2}{2} Y(e^\lambda, z) + (z\partial_z U_\lambda)\theta_\lambda e^{\zeta a_\lambda} z^{b_\lambda} E_\lambda(z) \\ &\qquad + b_\lambda Y(e^\lambda, z) \\ &\qquad + U_\lambda \theta_\lambda e^{\zeta a_\lambda} z^{b_\lambda} \big(z\partial_z E_\lambda(z)\big). \end{aligned} \tag{6.41}$$

By equating (6.40) with (6.41) we obtain

$$(z\partial_z U_\lambda)\theta_\lambda e^{\zeta a_\lambda} z^{b_\lambda} E_\lambda(z) = 0.$$

Then $\partial_z U_\lambda = 0$, and hence U_λ is independent of z and ζ.

From (4.47) and (6.3) we have

$$e^{2\pi \mathrm{i} D_\zeta} Y(e^\lambda, z) = \eta(\lambda)^{-1} Y(e^{\varphi\lambda}, z). \tag{6.42}$$

We make the following computations:

$$\begin{aligned} e^{2\pi \mathrm{i} D_\zeta} \theta_\lambda &= e^{\frac{1}{2}(\zeta \mathcal{P}^-(1-\varphi)\lambda_0|\lambda)} \tau_\lambda \theta_{\varphi\lambda}, \\ e^{2\pi \mathrm{i} D_\zeta} e^{\zeta a_\lambda} &= e^{\frac{1}{2}(\zeta \mathcal{P}^-(\varphi-1)\lambda_0|\lambda) + 2\pi \mathrm{i} a_\lambda(2\pi \mathrm{i})} e^{\zeta a_\lambda}. \end{aligned}$$

It is easy to check that $e^{2\pi \mathrm{i} D_\zeta} E_\lambda(z) = E_{\varphi\lambda}(z)$. Thus we have

$$e^{2\pi \mathrm{i} D_\zeta} Y(e^\lambda, z) = e^{2\pi \mathrm{i} c_\lambda} U_\lambda \tau_\lambda \theta_{\varphi\lambda} e^{\zeta \alpha_\lambda} z^{b_\lambda} E_{\varphi\lambda}(z). \tag{6.43}$$

On the other hand we have

$$Y(e^{\varphi\lambda}, z) = U_{\varphi\lambda} \theta_{\varphi\lambda} e^{\zeta a_\lambda} z^{b_\lambda} E_{\varphi\lambda}(z). \tag{6.44}$$

Then we obtain (6.27) by substituting (6.43) and (6.44) into (6.42). □

6.2 Products of Twisted Vertex Operators

Our next goal is to see how the operators U_λ commute. For each $\lambda, \mu \in Q$ we will need to write

$$(z_1 - z_2)^{-(\lambda|\mu)} E_\lambda(z_1) E_\mu(z_2)\big|_{z_1 = z_2 = z}$$

explicitly in terms of the exponential $E_{\lambda+\mu}(z)$. In order to do this, we make use of several well-known functions (see, e.g., [7, 8]), the first of which is the *Lerch transcendent*, a complex-valued function of three complex variables defined by the infinite series

$$\Phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(m+a)^s}, \qquad |z| < 1, \quad s \in \mathbb{C}, \quad a \neq 0, -1, -2, \dots. \tag{6.45}$$

The integral formula

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - z e^{-t}} \, dt \tag{6.46}$$

analytically extends Φ to include the cases

1. $\operatorname{Re} a > 0$, $\operatorname{Re} s > 0$, $z \in \mathbb{C} \setminus [1, \infty)$,

2. $\operatorname{Re} a > 0$, $\operatorname{Re} s > 1$, and $z = 1$.

Throughout the rest of this chapter, when dealing with $\ln(z)$, we will assume $-\pi < \arg(z) < \pi$. For $|z| > 1$, $z \notin (-\infty, 1) \cup (1, \infty)$, and $a \notin \mathbb{Z}$, the Lerch's transcendent $\Phi(z, 1, a)$ admits the following convergent expansion (see [32]):

$$\Phi(z, 1, a) = \pi \mathrm{i} z^{-a} \operatorname{sgn}(\omega(z)) + z^{-a} \pi \cot(\pi a) + z^{-1} \sum_{m=0}^{\infty} \frac{z^{-m}}{m + 1 - a}, \tag{6.47}$$

where $\omega(z) = \arg(\ln z)$. We will also make use of the following well-known properties of the Lerch transcendent:

$$\Phi(z, s, a) = z^n \Phi(z, s, a + n) + \sum_{k=0}^{n-1} \frac{z^k}{(k + a)^s}, \tag{6.48}$$

and

$$\partial_a \Phi(z, s, a) = -s \Phi(z, s + 1, a). \tag{6.49}$$

Other functions that can be derived from Φ are relevant to this text, including the *polylogarithm*:

$$\operatorname{Li}_s(z) = z \Phi(z, s, 1),$$

and the *Riemann zeta-function*:

$$\zeta(s) = \Phi(1, s, 1).$$

We also need the closely related *digamma function*:

$$\Psi(a) = -\gamma + \sum_{m=0}^{\infty} \Big(\frac{1}{m + 1} - \frac{1}{m + a} \Big), \qquad a \neq 0, -1, -2, \dots,$$

where $\gamma \approx 0.577216$ is the Euler-Mascheroni constant. The digamma function satisfies the following *reflection formula*:

$$\Psi(-a) - \Psi(a + 1) = \pi \cot \pi a = 2\pi \mathrm{i} \operatorname{Li}_0(e^{-2\pi \mathrm{i} a}) + \pi \mathrm{i}, \qquad a \in \mathbb{C} \setminus \{\mathbb{Z}\}. \tag{6.50}$$

The derivatives of the digamma function are the *polygamma functions*:

$$\Psi^{(j)}(a) = \partial_a^j \Psi(a) = (-1)^{j+1} j! \sum_{m=0}^{\infty} \frac{1}{(m+a)^{j+1}}, \qquad j \in \mathbb{N}, \quad a \neq -1, -2, \ldots. \tag{6.51}$$

The polygamma functions also satisfy a reflection formula obtained by taking the jth divided derivative of (6.50) with respect to a:

$$(-1)^j \Psi^{(j)}(-a) - \Psi^{(j)}(a+1) = -(-2\pi\mathrm{i})^{j+1} \operatorname{Li}_{-j}(e^{-2\pi\mathrm{i}a}), \qquad a \in \mathbb{C} \setminus \{\mathbb{Z}\}. \tag{6.52}$$

The polygamma functions are related to the Riemann zeta-function by the following formula:

$$\zeta(j+1) = \frac{(-1)^{j+1}}{j!} \Psi^{(j)}(1), \qquad j = 1, 2, 3, \ldots. \tag{6.53}$$

The values of the Reimann zeta-function for nonnegative even integers can be obtained using the following generating function:

$$\sum_{j=0}^{\infty} \zeta(2j) x^{2j} = -\frac{1}{2} \pi x \cot(\pi x). \tag{6.54}$$

Proposition 6.2.1. *Let*

$$\alpha_0' = \begin{cases} \alpha_0 + 1 & \alpha_0 \in \mathbb{C}^- \cup \{0\}, \\ \alpha_0 & \alpha_0 \in \mathbb{C}^+, \end{cases} \tag{6.55}$$

and define the operator $\mathcal{S}' \colon \mathfrak{h} \to \mathfrak{h}$ *by letting* $\mathcal{S}' a = \alpha_0' a$ *for* $a \in \alpha$ *and* $\alpha \in \mathcal{A}$ *and extending linearly. Then for any* $\lambda, \mu \in Q$ *we have*

$$(z_1 - z_2)^{-(\lambda|\mu)} E_\lambda(z_1) E_\mu(z_2)\big|_{z_1 = z_2 = z} = z^{-(\lambda|\mu)} B_{\lambda,\mu} \, E_{\lambda+\mu}(z),$$

where

$$B_{\lambda,\mu} = \exp\Big(\big((\Psi(\mathcal{S}' + \mathcal{N}) + \gamma)\lambda\big|\mu\big)\Big), \tag{6.56}$$

and the operator $(\Psi(\mathcal{S}'+\mathcal{N})+\gamma)\colon Q \to \mathfrak{h}$ *is understood via the following series expansion:*

$$
\begin{aligned}
(\Psi(\mathcal{S}'+\mathcal{N})+\gamma)\lambda &= e^{\mathcal{N}\partial_z}(\Psi(z)+\gamma)\Big|_{z=\mathcal{S}'}\lambda \\
&= \sum_{j=0}^{\infty}\frac{1}{j!}\partial_z^j(\Psi(z)+\gamma)\Big|_{z=\mathcal{S}'}\mathcal{N}^j\lambda \\
&= \sum_{j=0}^{\infty}\frac{1}{j!}(\Psi^{(j)}(\mathcal{S}')+\delta_{j,0}\gamma)\mathcal{N}^j\lambda \\
&= \sum_{\alpha\in\mathcal{A}}\sum_{j=0}^{\infty}\frac{1}{j!}(\Psi^{(j)}(\alpha_0')+\delta_{j,0}\gamma)\mathcal{N}^j\pi_\alpha\lambda.
\end{aligned}
\tag{6.57}
$$

Proof. Using (6.51) and (6.57) we rewrite (6.56) as

$$
B_{\lambda,\mu} = \exp\Big(\sum_{\alpha\in\mathcal{A}}\sum_{j=0}^{\infty} c_{\alpha,j}\big(\mathcal{N}^j\pi_\alpha\lambda\big|\mu\big)\Big),
$$

where the $c_{\alpha,j}$ are constants given by

$$
c_{\alpha,j} = \frac{1}{j!}\Psi^{(j)}(\alpha_0') + \delta_{j,0}\gamma.
$$

Using (2.9) and writing

$$
C = \Big[\sum_{m\in\mathbb{C}^+}\big((m+\mathcal{N})^{-1}z_1^{-m-\mathcal{N}}\lambda\big)_{(m+\mathcal{N})}, \sum_{n\in\mathbb{C}^-}\big((n+\mathcal{N})^{-1}z_2^{-n-\mathcal{N}}\mu\big)_{(n+\mathcal{N})}\Big],
$$

we obtain

$$
\begin{aligned}
E_\lambda(z_1)E_\mu(z_2) &= E_\lambda(z_1)_+E_\lambda(z_1)_-E_\mu(z_2)_+E_\mu(z_2)_- \\
&= \exp(C)E_\lambda(z_1)_+E_\mu(z_2)_+E_\lambda(z_1)_-E_\mu(z_2)_-.
\end{aligned}
$$

Then using (6.6) and (5.2), we write the commutator C as

$$
C = \sum_{\alpha\in\mathcal{A}}\sum_{j=0}^{\infty}(-1)^{j+1}\Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{N}}\mathcal{N}^j\pi_\alpha\lambda\Big|\mu\Big)\sum_{m\in\alpha^+}\frac{1}{m^{j+1}}\Big(\frac{z_2}{z_1}\Big)^m, \tag{6.58}
$$

where $\alpha^+ = \alpha\cap\mathbb{C}^+$, and the sum over j is finite due to the nilpotency of $\mathcal{N}$. For each

$\alpha \in \mathcal{A}$ and $j \geq 1$, the sum

$$\sum_{m\in\alpha^+} \frac{1}{m^{j+1}}\Big(\frac{z_2}{z_1}\Big)^m = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha_0')^{j+1}}\Big(\frac{z_2}{z_1}\Big)^{m+\alpha_0'} = \Big(\frac{z_2}{z_1}\Big)^{\alpha_0'}\Phi\Big(\frac{z_2}{z_1}, j+1, \alpha_0'\Big)$$

converges for $|z_2| \leq |z_1|$, and from (6.51) we obtain

$$\Big(\frac{z_2}{z_1}\Big)^{\alpha_0'}\Phi\Big(\frac{z_2}{z_1}, j+1, \alpha_0'\Big)\Big|_{z_1=z_2=z} = \frac{(-1)^{j+1}}{j!}\Psi^{(j)}(\alpha_0'). \tag{6.59}$$

Now we consider the case when $j = 0$. The sum is

$$\sum_{m\in\alpha^+} \frac{1}{m^{j+1}}\Big(\frac{z_2}{z_1}\Big)^m = \Big(\frac{z_2}{z_1}\Big)^{\alpha_0'}\Phi\Big(\frac{z_2}{z_1}, 1, \alpha_0'\Big), \qquad |z_2| \leq |z_1|,\ z_1 \neq z_2. \tag{6.60}$$

We cannot set $z_2 = z_1$ in (6.60), because both the sum (6.45) and the integral formula (6.46) for Φ diverge. Let $t = (\pi_\alpha\lambda|\mu)$. For $|z_2| \leq |z_1|$ and $z_1 \neq z_2$, we have

$$(z_1 - z_2)^{-t} = z_1^{-t}\exp\Big(-t\ln\Big(1 - \frac{z_2}{z_1}\Big)\Big) = \exp\Big(\sum_{m=1}^{\infty} t\frac{1}{m}\Big(\frac{z_2}{z_1}\Big)^m\Big). \tag{6.61}$$

Since the sums in both (6.60) and (6.61) converge for $|z_2| \leq |z_1|$ and $z_1 \neq z_2$, we obtain

$$\begin{aligned}(z_1 - z_2)^{-t}&\exp\Big(-\Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{N}}\pi_\alpha\lambda|\mu\Big)\Big(\frac{z_2}{z_1}\Big)^{\alpha_0'}\Phi\Big(\frac{z_2}{z_1}, 1, \alpha_0'\Big)\Big)\\ &= z_1^{-t}\exp\Big(\sum_{m=0}^{\infty}\Big(t\frac{1}{m+1}\Big(\frac{z_2}{z_1}\Big)^{m+1} - \Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{N}}\pi_\alpha\lambda|\mu\Big)\frac{1}{m+\alpha_0'}\Big(\frac{z_2}{z_1}\Big)^{m+\alpha_0'}\Big)\Big).\end{aligned} \tag{6.62}$$

The sum in the right-hand side of (6.62) converges for $z_1 = z_2$. Thus we have

$$\begin{aligned}\sum_{m=0}^{\infty}&\Big(t\frac{1}{m+1}\Big(\frac{z_2}{z_1}\Big)^{m+1} - \Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{N}}\pi_\alpha\lambda|\mu\Big)\frac{1}{m+\alpha_0'}\Big(\frac{z_2}{z_1}\Big)^{m+\alpha_0'}\Big)\Big|_{z_1=z_2=z}\\ &= \sum_{m=0}^{\infty}\Big(t\frac{1}{m+1} - t\frac{1}{m+\alpha_0'}\Big)\\ &= t(\Psi(\alpha_0') + \gamma).\end{aligned} \tag{6.63}$$

Therefore

$$
\begin{aligned}
&(z_1 - z_2)^{-(\lambda|\mu)} \exp(C)\big|_{z_1=z_2=z} \\
&\quad = \prod_{\alpha\in\mathcal{A}} \Big[(z_1-z_2)^{-(\pi_\alpha\lambda|\mu)} \exp\Big(-\Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{N}} \pi_\alpha\lambda \Big| \mu \Big) \sum_{m\in\alpha^+} \frac{1}{m} \Big(\frac{z_2}{z_1}\Big)^m \Big) \\
&\qquad \cdot \exp\Big(\sum_{j=1}^{\infty} (-1)^{j-1} \Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{N}} \mathcal{N}^j \pi_\alpha\lambda \Big| \mu \Big) \sum_{m\in\alpha^+} \frac{1}{m^{j+1}} \Big(\frac{z_2}{z_1}\Big)^m \Big) \Big] \Big|_{z_1=z_2=z} \\
&\quad = \prod_{\alpha\in\mathcal{A}} \Big[z^{-(\pi_\alpha\lambda|\mu)} \exp\Big(c_{\alpha,0}(\pi_\alpha\lambda|\mu) \Big) \exp\Big(\sum_{j=1}^{\infty} c_{\alpha,j} \big(\mathcal{N}^j \pi_\alpha\lambda \big| \mu \big) \Big] \\
&\quad = z^{-(\lambda|\mu)} \exp\Big(\sum_{\alpha\in\mathcal{A}} \sum_{j=0}^{\infty} c_{\alpha,j} \big(\mathcal{N}^j \pi_\alpha \lambda \big| \mu \big) \Big).
\end{aligned}
$$

□

Corollary 6.2.2. *For $\lambda, \mu \in Q$ we have*

$$(z_1 - z_2)^{-(\lambda|\mu)} Y(e^\lambda, z_1) Y(e^\mu, z_2)\big|_{z_1=z_2=z} = B_{\lambda,\mu} U_\lambda U_\mu \theta_{\lambda+\mu} e^{\zeta a_{\lambda+\mu}} z^{b_{\lambda+\mu}} E_{\lambda+\mu}(z), \tag{6.64}$$

Proof. Using (6.26) we obtain

$$\theta_\lambda U_\mu = e^{(\zeta\mathcal{P}^-\lambda_0|\mu)} U_\mu \theta_\lambda. \tag{6.65}$$

We compute

$$a_{\lambda+\mu} = a_\lambda + a_\mu + \Big(\frac{\mathcal{P}^+ + \mathcal{P}^-}{2} \lambda_0 \Big| \mu \Big) - (\lambda_0|\mu)$$

and

$$b_{\lambda+\mu} = b_\lambda + b_\mu + (\lambda_0|\mu) - (\lambda|\mu).$$

From these three equations and Proposition 6.2.1, we obtain the desired result. □

We recall that $\mathfrak{h}_0$ is the subset of elements of $\mathfrak{h}$ which are fixed by σ. Then the operator $(1-\sigma)$ is invertible on $(\mathfrak{h}_0)^\perp$. Thus there exists a unique element $a_* \in (\mathfrak{h}_0)^\perp$ satisfying

$$a = a_0 + (1-\sigma) a_*.$$

Proposition 6.2.3. *We have*

$$U_\lambda U_\mu = \varepsilon(\lambda,\mu) B^{-1}_{\lambda,\mu} U_{\lambda+\mu} = C_{\lambda,\mu} U_\mu U_\lambda, \tag{6.66}$$

where

$$C_{\lambda,\mu} = (-1)^{|\lambda|^2|\mu|^2} e^{\pi \mathrm{i}(\lambda_0|\mu)} e^{2\pi\mathrm{i}(\frac{1-\sigma}{1-\varphi}\lambda_*|\mu)} e^{(\frac{1}{\mathcal{N}}(\pi\mathrm{i}\mathcal{N}\frac{1+\varphi}{1-\varphi}-1)\lambda_0|\mu)}. \tag{6.67}$$

Proof. Since the vertex operators on V_Q satisfy (4.39), so do the logarithmic vertex operators on any φ-twisted V_Q-module. Thus (4.46) holds for logarithmic operators $Y(e^\lambda, z)$ and $Y(e^\mu, z)$ with $N = -(\lambda|\mu)$ and $n = -1-(\lambda|\mu)$. Then from (6.64) we obtain

$$\begin{aligned} Y(e^\lambda_{(-1-(\lambda|\mu))} e^\mu, z) &= (z_1 - z_2)^{-(\lambda|\mu)} Y(e^\lambda, z_1) Y(e^\mu, z_2)\big|_{z_1=z_2=z} \\ &= B_{\lambda,\mu} U_\lambda U_\mu \theta_{\lambda+\mu} e^{\zeta a_{\lambda+\mu}} z^{b_{\lambda+\mu}} E_{\lambda+\mu}(z). \end{aligned} \tag{6.68}$$

On the other hand, from (4.38) we have

$$\begin{aligned} Y(e^\lambda_{(-1-(\lambda|\mu))} e^\mu, z) &= \varepsilon(\lambda,\mu) Y(e^{\lambda+\mu}, z) \\ &= \varepsilon(\lambda,\mu) U_{\lambda+\mu} \theta_{\lambda+\mu} e^{\zeta a_{\lambda+\mu}} z^{b_{\lambda+\mu}} E_{\lambda+\mu}(z). \end{aligned} \tag{6.69}$$

Comparing (6.68) and (6.69), we obtain the first equality of (6.66).

From (6.23) and (6.55) we have

$$(-\alpha)'_0 = -\alpha'_0 + 1, \qquad \alpha \in \mathcal{A} \setminus \{\mathbb{Z}\}.$$

Then it is straightforward to check that

$$\big(\Psi(\mathcal{S}' + \mathcal{N})\mu \big| (1-\sigma)\lambda_*\big) = \big(\Psi(-\mathcal{S}' - \mathcal{N} + 1)(1-\sigma)\lambda_* \big| \mu\big). \tag{6.70}$$

From the reflection formulas (6.52) we obtain

$$\begin{aligned} (\Psi(-\mathcal{S}' - \mathcal{N} + 1) - &\Psi(\mathcal{S}' + \mathcal{N}))(1-\sigma)\lambda_* \\ &= e^{\mathcal{N}\partial_z}(\Psi(-z) - \Psi(z+1))\big|_{z=\mathcal{S}'-1}(1-\sigma)\lambda_* \\ &= e^{\mathcal{N}\partial_z}(\pi\cot(\pi z))\big|_{z=\mathcal{S}'-1}(1-\sigma)\lambda_* \\ &= \pi\cot(\pi(\mathcal{S}' + \mathcal{N}))(1-\sigma)\lambda_*. \end{aligned}$$

We recall that

$$\cot(z) = \mathrm{i}\frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{e^{\mathrm{i}z} - e^{-\mathrm{i}z}} = \mathrm{i}\frac{1 + e^{-2\mathrm{i}z}}{1 - e^{-2\mathrm{i}z}}. \tag{6.71}$$

Thus

$$\begin{aligned}\big(\pi\cot(\pi(\mathcal{S}' + \mathcal{N}))(1-\sigma)\lambda_*\big|\mu\big) &= \pi\mathrm{i}\Big(\frac{1 + e^{-2\pi\mathrm{i}(\mathcal{S}'+\mathcal{N})}}{1 - e^{-2\pi\mathrm{i}(\mathcal{S}'+\mathcal{N})}}(1-\sigma)\lambda_*\Big|\mu\Big)\\ &= \pi\mathrm{i}\Big(\frac{1+\varphi}{1-\varphi}(1-\sigma)\lambda_*\Big|\mu\Big)\\ &= 2\pi\mathrm{i}\Big(\frac{1-\sigma}{1-\varphi}\lambda_*\Big|\mu\Big) - \pi\mathrm{i}((1-\sigma)\lambda_*|\mu)\\ &= 2\pi\mathrm{i}\Big(\frac{1-\sigma}{1-\varphi}\lambda_*\Big|\mu\Big) - \pi\mathrm{i}(\lambda - \lambda^{\varphi}|\mu).\end{aligned}$$

Next we note that $\mathcal{S}'$ fixes each element of $\mathfrak{h}_0$. Thus by (6.57), (6.53), (6.54), and (6.71) we have

$$\begin{aligned}&((\Psi(\mathcal{S}'+\mathcal{N})+\gamma)\mu|\lambda_0) - (\Psi(\mathcal{S}'+\mathcal{N})+\gamma)\lambda_0|\mu)\\ &\qquad= ((\Psi(1-\mathcal{N}) - \Psi(1+\mathcal{N}))\lambda_0|\mu)\\ &\qquad= \Big(\Big(e^{-\mathcal{N}\partial_z}\Psi(z) - e^{\mathcal{N}\partial_z}\Psi(z)\Big)\Big|_{z=1}\lambda_0\big|\mu\Big)\\ &\qquad= \sum_{j=0}^{\infty}\Big(\Big((-1)^j\frac{\Psi^{(j)}(1)}{j!} - \frac{\Psi^{(j)}(1)}{j!}\Big)\mathcal{N}^j\lambda_0\big|\mu\Big)\\ &\qquad= -2\sum_{j=0}^{\infty}\zeta(2j+2)\big(\mathcal{N}^{2j+1}\lambda_0\big|\mu\big)\\ &\qquad= \Big(\frac{1}{\mathcal{N}}(\pi\mathcal{N}\cot(\pi\mathcal{N}) - 1)\lambda_0\big|\mu\Big)\\ &\qquad= \Big(\frac{1}{\mathcal{N}}\Big(\pi\mathrm{i}\mathcal{N}\frac{1+\varphi}{1-\varphi} - 1\Big)\lambda_0|\mu\Big).\end{aligned}$$

Thus

$$\begin{aligned}C_{\lambda,\mu} &= (-1)^{|\lambda|^2|\mu|^2 + (\lambda|\mu)}\frac{B_{\mu,\lambda}}{B_{\lambda,\mu}}\\ &= (-1)^{|\lambda|^2|\mu|^2 + (\lambda|\mu)}\exp\Big(\big((\Psi(\mathcal{S}'+\mathcal{N})+\gamma)\lambda\big|\mu\big) - \big((\Psi(\mathcal{S}'+\mathcal{N})+\gamma)\mu\big|\lambda\big)\Big)\\ &= (-1)^{|\lambda|^2|\mu|^2 + (\lambda|\mu)}e^{2\pi\mathrm{i}(\frac{1-\sigma}{1-\varphi}\lambda_*|\mu)}e^{-\pi\mathrm{i}(\lambda-\lambda_0|\mu)}e^{(\frac{1}{\mathcal{N}}(\pi\mathrm{i}\mathcal{N}\frac{1+\varphi}{1-\varphi}-1)\lambda_0|\mu)}\\ &= (-1)^{|\lambda|^2|\mu|^2}e^{\pi\mathrm{i}(\lambda_0|\mu)}e^{2\pi\mathrm{i}(\frac{1-\sigma}{1-\varphi}\lambda_*|\mu)}e^{(\frac{1}{\mathcal{N}}(\pi\mathrm{i}\mathcal{N}\frac{1+\varphi}{1-\varphi}-1)\lambda_0|\mu)}.\end{aligned}$$

$\square$

6.3 Reduction to Group Theory

The discussion in the previous section leads us to make the following definition. Let

$$G = \mathbb{C}^\times \times Q \times \exp\bigl(\mathfrak{h}_0\bigr) \tag{6.72}$$

be the set of elements

$$c\,U_\lambda e^h, \qquad c \in \mathbb{C}^\times,\ \lambda \in Q,\ h \in \mathfrak{h}_0. \tag{6.73}$$

We define the following multiplication in G:

$$\begin{aligned} e^h e^{h'} &= e^{\frac{1}{2}(\mathcal{N}h|h')} e^{h+h'}, \\ e^h U_\lambda e^{-h} &= e^{(h|\lambda_0)} U_\lambda, \\ U_\lambda U_\mu &= \varepsilon(\lambda,\mu) B_{\lambda,\mu}^{-1} U_{\lambda+\mu}. \end{aligned} \tag{6.74}$$

Proposition 6.3.1. *The set G with the multiplication defined above is a group with identity element $\mathbb{1} = 1U_0 e^0$.*

Proof. We have

$$U_\lambda e^h U_\mu e^{h'} = e^{(h|\mu_0 - \frac{1}{2}\mathcal{N}h')} \varepsilon(\lambda,\mu) B_{\lambda,\mu}^{-1} U_{\lambda+\mu} e^{h+h'}. \tag{6.75}$$

Then from (6.75) we see that $cU_\lambda e^{h(\zeta)}$ is invertible with

$$(cU_\lambda e^h)^{-1} = c^{-1} \varepsilon(\lambda,\lambda) B_{\lambda,\lambda}^{-1} e^{(h|\lambda_0)} U_{-\lambda} e^{-h}. \tag{6.76}$$

Since $\mathcal{N}$ and π_α are linear maps for $\alpha \in A$, the map $Q \times Q \to \mathbb{C}$ defined by

$$(\lambda,\mu) \mapsto \sum_{\alpha\in\mathcal{A}} \sum_{j=0}^{\infty} c_{\alpha,j} (\mathcal{N}^j \pi_\alpha \lambda|\mu)$$

is bilinear. Thus the map $(\lambda,\mu) \mapsto B_{\lambda,\mu}$ is bimultiplicative. We know that ε is a 2-cocycle. So the map $(\lambda,\mu) \mapsto \varepsilon(\lambda,\mu)B_{\lambda,\mu}^{-1}$ is also a 2-cocycle, i.e.,

$$\varepsilon(\lambda,\mu)\varepsilon(\lambda+\mu,\tau) B_{\lambda,\mu}^{-1} B_{\lambda+\mu,\tau} = \varepsilon(\lambda,\mu+\tau)\varepsilon(\mu,\tau) B_{\lambda,\mu+\tau} B_{\mu,\tau}^{-1}. \tag{6.77}$$

The associativity of G follows immediately from (6.74) and (6.77) via direct computation. $\square$

We have the following action of G on $\hat{\mathfrak{h}}_\varphi$ by conjugation:

$$\begin{aligned}(cU_\lambda e^h)(h'_{(m+\mathcal{N})} + c'K)(cU_\lambda e^h)^{-1} \\ = h'_{(m+\mathcal{N})} + \delta_{m,0}\big(h'\big|\mathcal{N}h - \lambda\big)K + c'K.\end{aligned} \tag{6.78}$$

This action is compatible with the adjoint action of $\hat{\mathfrak{h}}_\varphi^0$ on $\hat{\mathfrak{h}}_\varphi$. We say that a $\hat{\mathfrak{h}}_\varphi$-module is a $(\hat{\mathfrak{h}}_\varphi, G)$-module if it is also a G-module such that the adjoint action of G on $\hat{\mathfrak{h}}_\varphi$ satisfies (6.78). A $\hat{\mathfrak{h}}_\varphi$-module or a $(\hat{\mathfrak{h}}_\varphi, G)$-module is called *restricted* if for every $v \in M$ and $h \in \mathfrak{h}$, we have $h_{(m+\mathcal{N})}v = 0$ for $\operatorname{Re} m \gg 0$. We say that a $(\hat{\mathfrak{h}}_\varphi, G)$-module is of *level 1* if both $K \in \hat{\mathfrak{h}}_\varphi$ and $\mathbb{1} \in G$ act as 1.

We note that as defined, the elements of the group G do not necessarily satisfy the condition (6.27) on the operators U_λ induced by φ-equivariance. We amend this issue by considering a quotient group in which (6.27) holds.

Proposition 6.3.2. *The set*

$$N_\varphi = \{\eta(\lambda)e^{2\pi i c_\lambda}U_{\varphi\lambda}^{-1}U_\lambda\tau_\lambda \colon \lambda \in Q\}$$

is a central subgroup in G.

Proof. For each $\lambda \in Q$, we let

$$g_\lambda = \eta(\lambda)e^{2\pi i c_\lambda}U_{\varphi\lambda}^{-1}U_\lambda\tau_\lambda.$$

Then for every $h \in \mathfrak{h}_0$ we have

$$\begin{aligned}e^hU_{(1-\varphi)\lambda} &= e^{(h|(1-\varphi)\lambda)}U_{(1-\varphi)\lambda}e^h, \\ e^h\tau_\lambda &= e^{-(\mathcal{N}h|\frac{1}{\mathcal{N}}(\varphi-1)\lambda)}\tau_\lambda e^h = e^{-(h|(1-\varphi)\lambda)}\tau_\lambda e^h.\end{aligned}$$

It follows that

$$e^hg_\lambda e^{-h} = g_\lambda. \tag{6.79}$$

We note that

$$|(1-\varphi)\lambda|^2 = 2(\lambda|\lambda) - 2(\varphi\lambda|\lambda) \in 2\mathbb{Z}. \tag{6.80}$$

Now we let $\mu \in Q$ and compute

$$
\begin{aligned}
e^{2\pi \mathrm{i}(\frac{1-\sigma}{1-\varphi}\mu_*|(1-\varphi)\lambda)} &= e^{2\pi \mathrm{i}((1-\sigma)\frac{1-\varphi^{-1}}{1-\varphi}\mu_*|\lambda)} \\
&= e^{-2\pi \mathrm{i}((1-\sigma)\mu_*|\varphi\lambda)} \\
&= e^{(\mu_0|\varphi\lambda)}.
\end{aligned}
\tag{6.81}
$$

By a similar albeit longer computation we obtain

$$
e^{(\frac{1}{\mathcal{N}}(\pi \mathrm{i}\mathcal{N}\frac{1+\varphi}{1-\varphi}-1)\mu_0|(1-\varphi)\lambda)} = e^{-\pi \mathrm{i}(\mu_0|(1+\varphi)\lambda)} e^{2\pi \mathrm{i}(\mathcal{P}^-_{2\pi \mathrm{i}}\lambda_0|\mu)}. \tag{6.82}
$$

From (6.67), (6.80), (6.81), and (6.82) we have

$$
C_{\mu,(1-\varphi)\lambda} = e^{2\pi \mathrm{i}(\mathcal{P}^-_{2\pi \mathrm{i}}\lambda_0|\mu)}.
$$

Thus

$$
U_\mu U_{(1-\varphi)\lambda}\tau_\lambda U_\mu^{-1} = C_{\mu,(1-\varphi)\lambda} e^{-2\pi \mathrm{i}(\mathcal{P}^-_{2\pi \mathrm{i}}\lambda_0|\mu)} U_{(1-\varphi)\lambda}\tau_\lambda = U_{(1-\varphi)\lambda}\tau_\lambda. \tag{6.83}
$$

It follows from (6.79) and (6.83) that g_λ is central in G.

It is straightforward to check that

$$
e^{2\pi \mathrm{i}(c_\lambda + c_\mu)} = e^{-\pi \mathrm{i}((\mathcal{P}^+_{2\pi \mathrm{i}} + \mathcal{P}^-_{2\pi \mathrm{i}})\lambda_0|\mu)} e^{2\pi \mathrm{i} c_{\lambda+\mu}} \tag{6.84}
$$

and

$$
\tau_\lambda U_\mu \tau_\mu = e^{\pi \mathrm{i}((\mathcal{P}^+_{2\pi \mathrm{i}} + \mathcal{P}^-_{2\pi \mathrm{i}})\lambda_0|\mu)} U_\mu \tau_{\lambda+\mu}. \tag{6.85}
$$

We also have

$$
(U_{\varphi\lambda}U_{\varphi\mu})^{-1}U_\lambda U_\mu = \frac{\eta(\lambda+\mu)}{\eta(\lambda)\eta(\mu)} U^{-1}_{\varphi(\lambda+\mu)} U_{\lambda+\mu}. \tag{6.86}
$$

Then from (6.84), (6.85), (6.86), and the fact that g_λ is central in G we have

$$
\begin{aligned}
g_\lambda g_\mu &= \eta(\mu) e^{2\pi \mathrm{i} c_\mu} U^{-1}_{\varphi\mu} g_\lambda U_\mu \tau_\mu \\
&= \eta(\lambda)\eta(\mu) e^{2\pi \mathrm{i}(c_\lambda + c_\mu)} U^{-1}_{\varphi\mu} U^{-1}_{\varphi\lambda} U_\lambda \tau_\lambda U_\mu \tau_\mu \\
&= \eta(\lambda+\mu) e^{2\pi \mathrm{i} c_{\lambda+\mu}} U^{-1}_{\varphi(\lambda+\mu)} U_{\lambda+\mu}\tau_{\lambda+\mu} \\
&= g_{\lambda+\mu}.
\end{aligned}
$$

Thus N_φ is a subgroup of G. $\square$

We define G_φ as the quotient group $G_\varphi = G/N_\varphi$. Then the elements of G_φ satisfy the group relations (6.74) along with φ-equivariance (6.27). The following theorem, which is the main result of this chapter, reduces the classification of φ-twisted V_Q-modules to the classification of restricted $(\hat{\mathfrak{h}}_\varphi, G_\varphi)$-modules.

Theorem 6.3.3 (Main Theorem)**.** *Every φ-twisted V_Q-module is naturally a restricted $(\hat{\mathfrak{h}}_\varphi, G_\varphi)$-module of level 1. Conversely, any restricted $(\hat{\mathfrak{h}}_\varphi, G_\varphi)$-module of level 1 extends to a φ-twisted V_Q-module.*

Proof. Let M be a φ-twisted V_Q-module. As we mentioned previously, the logarithmic fields $Y(a,z)$, $(a \in \mathfrak{h})$ define a $B^1(\mathfrak{h})$-module structure on M. By Theorem 5.2.2, M is a restricted $\hat{\mathfrak{h}}_\varphi$-module of level 1. The action of the group elements $e^h \in \exp\big(\mathfrak{h}_0\big)$ is defined by

$$e^h = e^{h_{(0+\mathcal{N})}}, \qquad h \in \mathfrak{h}_0,$$

and the action of U_λ, $(\lambda \in Q)$ is defined by the operators (6.32). Then the group relations of G and the adjoint action (6.78) are satisfied (see (6.26), (6.27), and (6.66)). The φ-equivariance (6.27) implies $g_\lambda = 1$ on M for all $g_\lambda \in N_\varphi$. Thus M is a restricted $(\hat{\mathfrak{h}}_\varphi, G_\varphi)$-module of level 1.

Conversely, we assume M is a restricted $(\hat{\mathfrak{h}}_\varphi, G_\varphi)$-module of level 1. We define the logarithmic fields

$$Y(a,z), \quad Y(e^\lambda, z), \qquad a \in \mathfrak{h},\ \lambda \in Q, \tag{6.87}$$

by (4.53) and (6.25) respectively. Since the fields $Y(a,z)$ generate a φ-twisted module structure on M (see Theorem 5.2.2), they are local and satisfy the φ-equivariance (6.27). To show that $Y(a,z)$ is local with $Y(e^\lambda, z)$, we use (4.6), (6.7), and (6.8) to compute

$$[Y(a,z_1), Y(e^\lambda, z_2)] = \delta(z_1,z_2)\Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{S}+\mathcal{N}} a\big|\lambda\Big) Y(e^\lambda, z_2).$$

Thus locality follows from the fact that $(z_1 - z_2)\delta(z_1,z_2) = 0$. Let us temporarily assume that the fields $Y(e^\lambda, z)$ are mutually local. We let $\mathcal{W} \in \mathrm{LFie}(M)$ be the subspace of all the fields (6.87). Then the smallest subspace $\bar{\mathcal{W}} \subset \mathrm{LFie}(M)$ containing $\mathcal{W} \cup \{I\}$ and closed under D_ζ and all n-th products is a local collection, and the nth products endow $\bar{\mathcal{W}}$ with the structure of a vertex algebra with vacuum vector I and translation operator D_z (see [6, Theorem 3.7]). A φ-twisted V_Q-module structure on M is equivalent to a homomorphism of vertex algebras $V_Q \to \bar{\mathcal{W}}$. The lattice vertex algebra V_Q can be generated by the Heisenberg $\hat{\mathfrak{h}}$ and the elements e^λ, $\lambda \in Q$ subject to the following relations:

1. $h_{(n)}h' = \delta_{n,1}(h|h')\mathbf{1}$, $n \geq 0$,
2. $h_{(n)}e^\lambda = \delta_{n,0}(h|\lambda)e^\lambda$, $n \geq 0$,
3. $Te^\lambda = \lambda_{(-1)}e^\lambda$,
4. $e^\lambda{}_{(-(\lambda|\mu)-1)}e^\mu = \varepsilon(\lambda,\mu)e^{\lambda+\mu}$.

Thus to show that M carries the structure of a φ-twisted V_Q-module, it suffices to show that the same relations are satisfied by the twisted fields in $\mathcal{W}$. Relations 1. and 2. for the twisted fields are equivalent to (6.6) and (6.7), 3. is equivalent to (6.18), and 4. is equivalent to (6.69). Each of these equations holds for the twisted fields by construction. Thus M is a φ-twisted V_Q-module.

All that remains is to prove the locality of the fields $Y(e^\lambda, z)$ and $Y(e^\mu, z)$. We do so by proving that

$$(z_1 - z_2)^{-(\lambda|\mu)}Y(e^\lambda, z_1)Y(e^\mu, z_2) = (-1)^{|\lambda|^2|\mu|^2}(z_1 - z_2)^{-(\lambda|\mu)}Y(e^\mu, z_2)Y(e^\lambda, z_1). \tag{6.88}$$

We let $R_{\lambda,\mu}(z_1, z_2) = \exp(C)$, where C is given by (6.58). Using (6.74) we obtain

$$U_\lambda\theta_\lambda(\zeta_1)U_\mu\theta_\mu(\zeta_2) = C_{\lambda,\mu}e^{((\frac{1}{\mathcal{N}}((\frac{z_1}{z_2})^{\mathcal{N}}-1))\lambda_0|\mu)}U_\mu\theta_\mu(\zeta_2)U_\lambda\theta_\lambda(\zeta_1).$$

Thus to show (6.88), we may equivalently show that

$$\begin{aligned}(z_1 - z_2)^{-(\lambda|\mu)}&R_{\lambda,\mu}(z_1, z_2)\\ &= (-1)^{|\lambda|^2|\mu|^2}C_{\mu,\lambda}e^{((\frac{1}{\mathcal{N}}((\frac{z_1}{z_2})^{\mathcal{N}}-1))\lambda_0|\mu)}(z_1 - z_2)^{-(\lambda|\mu)}R_{\mu,\lambda}(z_2, z_1).\end{aligned} \tag{6.89}$$

From (6.45), (6.49), and (6.58) we have

$$R_{\lambda,\mu}(z_1, z_2) = \exp\Big(-\Big(\Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{S}'+\mathcal{N}}\Phi\Big(\frac{z_2}{z_1}, 1, \mathcal{S}' + \mathcal{N}\Big)\Big)\lambda\Big|\mu\Big)\Big), \tag{6.90}$$

where the right-hand side of (6.90) is expanded in the domain $|z_2| < |z_1|$. In the same domain, we have

$$(z_1 - z_2)^{-(\lambda|\mu)} = z_1^{-(\lambda|\mu)}\exp\Big(-(\lambda|\mu)\ln\Big(1 - \frac{z_2}{z_1}\Big)\Big).$$

Thus the left-hand side of (6.89) is equal to

$$z_1^{-(\lambda|\mu)}\exp\Big(-\big(\Big(\ln\Big(1-\frac{z_2}{z_1}\Big)+\Big(\frac{z_2}{z_1}\Big)^{\mathcal{S}'+\mathcal{N}}\Phi\big(\frac{z_2}{z_1},1,\mathcal{S}'+\mathcal{N}\big)\Big)\lambda\big|\mu\big)\Big), \tag{6.91}$$

expanded in the domain $|z_2|<|z_1|$. We note that (6.91) is regular at $z_1=z_2$. Indeed (see (6.56)),

$$\begin{aligned}-\big(\Big(\ln\Big(1-\frac{z_2}{z_1}\Big)+\Big(\frac{z_2}{z_1}\Big)^{\mathcal{S}'+\mathcal{N}}\Phi\big(\frac{z_2}{z_1},1,\mathcal{S}'+\mathcal{N}\big)\Big)\lambda\big|\mu\big)\Big|_{z_1=z_2=z}\\ =\big((\Psi(\mathcal{S}'+\mathcal{N})+\gamma)\lambda\big|\mu\big).\end{aligned}$$

It follows that the function (6.91) is holomorphic in the domain $z_2/z_1\notin(-\infty,\infty)$. Thus to complete the proof of locality, it suffices to show that the right-hand side of (6.89) is equal to (6.91) expanded in the domain

$$|z_1|<|z_2|, \qquad \frac{z_1}{z_2}\notin(-\infty,1)\cup(1,\infty). \tag{6.92}$$

For $z_2/z_1\notin(-\infty,\infty)$, we have

$$z_1^{-(\lambda|\mu)}\exp\big(-\ln\Big(1-\frac{z_2}{z_1}\Big)\big)=(-1)^{(\lambda|\mu)}z_2^{-(\lambda|\mu)}\exp\big(-\ln\Big(1-\frac{z_1}{z_2}\Big)\big). \tag{6.93}$$

Using the expansion formula (6.47), we obtain

$$\begin{aligned}&\exp\Big(-\big(\Big(\Big(\frac{z_2}{z_1}\Big)^{\mathcal{S}'+\mathcal{N}}\Phi\big(\frac{z_2}{z_1},1,\mathcal{S}'+\mathcal{N}\big)\Big)(1-\sigma)\lambda_*\big|\mu\big)\Big)\\ &\quad=e^{-\pi\mathrm{i}\,\mathrm{sgn}(\omega(z_2/z_1))((1-\sigma)\lambda_*|\mu)}\exp\Big(-\big(\pi\cot\big(\pi(\mathcal{S}+\mathcal{N})\big)(1-\sigma)\lambda_*\big|\mu\big)\Big)\\ &\qquad\cdot\exp\Big(-\big(\Big(\Big(\frac{z_1}{z_2}\Big)^{\mathcal{S}'+\mathcal{N}}\Phi\big(\frac{z_1}{z_2},1,\mathcal{S}'+\mathcal{N}\big)\Big)(1-\sigma)\mu_*\big|\lambda\big)\Big).\end{aligned} \tag{6.94}$$

Using (6.47) and (6.48), we obtain

$$
\begin{aligned}
&\exp\Big(-\big(\Big(\frac{z_2}{z_1}\Big)^{1+\mathcal{N}}\Phi\big(\frac{z_2}{z_1},1,1+\mathcal{N}\big)\lambda_0\big|\mu\big)\Big)\\
&\quad=\exp\Big(-e^{\mathcal{N}\partial_a}\big(\Big(\frac{z_2}{z_1}\Big)^{1+a}\Phi\big(\frac{z_2}{z_1},1,1+a\big)\lambda_0\big|\mu\big)\Big)\Big|_{a=0}\\
&\quad=\exp\Big(-e^{\mathcal{N}\partial_a}\big(\big(\Big(\frac{z_2}{z_1}\Big)^{a}\Phi\big(\frac{z_2}{z_1},1,a\big)-\frac{1}{a}\Big(\frac{z_2}{z_1}\Big)^{a}\big)\lambda_0\big|\mu\big)\Big)\Big|_{a=0}\\
&\quad=e^{-\pi\mathrm{i}\,\mathrm{sgn}(\omega(z_2/z_1))(\lambda_0|\mu)}\\
&\qquad\cdot\exp\Big(-e^{\mathcal{N}\partial_a}\big(\Big(\pi\cot(\pi a)-\frac{1}{a}\Big(\frac{z_2}{z_1}\Big)^{a}\Big)\lambda_0\big|\mu\big)\Big)\\
&\qquad\cdot\exp\Big(-e^{\mathcal{N}\partial_a}\big(\Big(\frac{z_1}{z_2}\Big)^{1-a}\Phi\big(\frac{z_1}{z_2},1,1-a\big)\lambda_0\big|\mu\big)\Big)\Big|_{a=0}\\
&\quad=e^{-\pi\mathrm{i}\,\mathrm{sgn}(\omega(z_2/z_1))(\lambda_0|\mu)}e^{((\frac{1}{\mathcal{N}}((\frac{z_2}{z_1})^{\mathcal{N}}-1))\lambda_0|\mu)}\\
&\qquad\cdot\exp\Big(-\big(\frac{1}{\mathcal{N}}(\pi\mathcal{N}\cot(\pi\mathcal{N})-1)\lambda_0\big|\mu\big)\Big)\\
&\qquad\cdot\exp\Big(-\big(\Big(\frac{z_1}{z_2}\Big)^{1-\mathcal{N}}\Phi\big(\frac{z_1}{z_2},1,1-\mathcal{N}\big)\lambda_0\big|\mu\big)\Big)\\
&\quad=e^{-\pi\mathrm{i}\,\mathrm{sgn}(\omega(z_2/z_1))(\lambda_0|\mu)}e^{((\frac{1}{\mathcal{N}}((\frac{z_2}{z_1})^{\mathcal{N}}-1))\lambda_0|\mu)}\\
&\qquad\cdot\exp\Big(-\big(\frac{1}{\mathcal{N}}(\pi\mathcal{N}\cot(\pi\mathcal{N})-1)\lambda_0\big|\mu\big)\Big)\\
&\qquad\cdot\exp\Big(-\big(\Big(\frac{z_1}{z_2}\Big)^{1+\mathcal{N}}\Phi\big(\frac{z_1}{z_2},1,1+\mathcal{N}\big)\mu_0\big|\lambda\big)\Big).
\end{aligned}
\tag{6.95}
$$

Using (6.93), (6.94), and (6.95), and noting that $\mathrm{sgn}(\omega(z_2/z_1))=\pm1$, we see that an expansion of (6.90) in the domain (6.92) is given by

$$
\begin{aligned}
&(-1)^{(\lambda|\mu)}z_2^{-(\lambda|\mu)}e^{-\pi\mathrm{i}\,\mathrm{sgn}(\omega(z_2/z_1))(\lambda|\mu)}e^{((\frac{1}{\mathcal{N}}((\frac{z_2}{z_1})^{\mathcal{N}}-1))\lambda_0|\mu)}\exp\big(-(\lambda|\mu)\ln\Big(1-\frac{z_1}{z_2}\Big)\big)\\
&\qquad\cdot\exp\Big(-\big(\pi\cot(\pi(\mathcal{S}'+\mathcal{N}))(1-\sigma)\lambda_*\big|\mu\big)\Big)\\
&\qquad\cdot\exp\Big(-\big(\frac{1}{\mathcal{N}}(\pi\mathcal{N}\cot(\pi\mathcal{N})-1)\lambda_0\big|\mu\big)\Big)\\
&\qquad\cdot\exp\Big(-\big(\Big(\frac{z_1}{z_2}\Big)^{\mathcal{S}'+\mathcal{N}}\Phi\big(\frac{z_1}{z_2},1,\mathcal{S}'+\mathcal{N}\big)\mu\big|\lambda\big)\Big).\\
&\quad=(-1)^{|\lambda|^2|\mu|^2+(\lambda|\mu)}e^{((\frac{1}{\mathcal{N}}((\frac{z_2}{z_1})^{\mathcal{N}}-1))\lambda_0|\mu)}(z_2-z_1)^{-(\lambda|\mu)}C_{\mu,\lambda}R_{\mu,\lambda}(z_2,z_1),
\end{aligned}
$$

which is equal to the right-hand side of (6.89). □

6.4 Examples of φ-twisted V_Q-modules

As we have seen in Chapter 5, a major advantage of the theory of twisted logarithmic modules of vertex algebras as developed in [6] is the framework for working out explicit examples. In this section, we demonstrate the utility of the results of the current chapter by working out two small-dimensional examples in detail.

Example 6.4.1. We begin with the integral lattice

$$Q = \mathrm{span}_{\mathbb{Z}}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\},$$

with bilinear form $(\cdot|\cdot)\colon Q \times Q \to \mathbb{Z}$ defined by

$$(\lambda_1|\lambda_4) = (\lambda_4|\lambda_1) = (\lambda_2|\lambda_3) = (\lambda_3|\lambda_2) = 1,$$

and all other scalar products of basis elements are zero. We let $\varphi\colon Q \to Q$ be the automorphism defined by

$$\varphi\lambda_1 = \lambda_1 - \lambda_2, \qquad \varphi\lambda_2 = \alpha_2, \qquad \varphi\lambda_3 = \lambda_3 + \lambda_4, \qquad \varphi\lambda_4 = \lambda_4.$$

It is easy to check that $(\cdot|\cdot)$ is φ-invariant. We let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$. Then we can write $\varphi = e^{-2\pi \mathrm{i}\mathcal{N}}$, where

$$\mathcal{N}\lambda_1 = \frac{1}{2\pi\mathrm{i}}\lambda_2, \qquad \mathcal{N}\lambda_2 = 0, \qquad \mathcal{N}\lambda_3 = -\frac{1}{2\pi\mathrm{i}}\lambda_4, \qquad \mathcal{N}\lambda_4 = 0,$$

and $\mathcal{N}^2\lambda_1 = \mathcal{N}^2\lambda_3 = 0$. We use the 2-cocycle

$$\varepsilon(\lambda_4, \lambda_1) = \varepsilon(\lambda_3, \lambda_2) = -1, \tag{6.96}$$

and $\varepsilon = 1$ on all other pairs of basis vectors. This allows us to also choose $\eta = 1$.

We consider the $\hat{\mathfrak{h}}^0_\varphi$-module

$$R = \mathbb{C}[x_{1,0}, q_1^{\pm 1}, q_2^{\pm 1}],$$

with the action

$$\lambda_{1(0+\mathcal{N})} = x_{1,0}, \quad \lambda_{2(0+\mathcal{N})} = q_1\partial_{q_1}, \quad \lambda_{3(0+\mathcal{N})} = -\frac{1}{2\pi\mathrm{i}}\partial_{x_{1,0}}, \quad \lambda_{4(0+\mathcal{N})} = q_2\partial_{q_2},$$

and we let $M = M_\varphi(R)$ be the corresponding generalized Verma module for $\hat{\mathfrak{h}}_\varphi$. We label the action of $\hat{\mathfrak{h}}_\varphi^-$ by the commuting variables:

$$\lambda_{1(-m+\mathcal{N})} = x_{1,m}, \qquad \lambda_{2(-m+\mathcal{N})} = 2\pi \mathrm{i} x_{2,m},$$
$$\lambda_{3(-m+\mathcal{N})} = \frac{1}{2\pi\mathrm{i}} x_{3,m}, \qquad \lambda_{4(-m+\mathcal{N})} = x_{4,m},$$

for $m \in \mathbb{N}$. Then

$$M_\varphi(R) \cong \mathbb{C}[x_{1,0}, q_1^{\pm 1}, q_2^{\pm 1}, x_{i,m}]_{1 \le i \le 4; m=1,2,\ldots},$$

and the action of $\hat{\mathfrak{h}}_\varphi^+$ is given by

$$\lambda_{1(m+\mathcal{N})} = m\partial_{x_{4,m}} + \partial_{x_{3,m}}, \qquad \lambda_{2(m+\mathcal{N})} = 2\pi\mathrm{i} m \partial_{x_{3,m}},$$
$$\lambda_{3(m+\mathcal{N})} = \frac{1}{2\pi\mathrm{i}}\big(m\partial_{x_{2,m}} - \partial_{x_{1,m}}\big), \qquad \lambda_{4(m+\mathcal{N})} = m\partial_{x_{1,m}},$$

for $m \in \mathbb{N}$.

We define the following operators on $M_\varphi(R)$:

$$U_{\lambda_1} = q_2, \quad U_{\lambda_2} = (-1)^{q_1\partial_{q_1}} e^{-2\pi\mathrm{i} x_{1,0}}, \quad U_{\lambda_3} = q_1 e^{-\frac{\pi\mathrm{i}}{6} q_2 \partial_{q_2}}, \quad U_{\lambda_4} = (-1)^{q_2\partial_{q_2}} e^{-\partial_{x_{1,0}}},$$

and

$$e^h = e^{h_{(0+\mathcal{N})}}, \qquad h \in \mathfrak{h}. \tag{6.97}$$

It is straightforward to check that (6.26) holds, and that the operators U_λ satisfy the following commutation relations:

$$U_{\lambda_1}U_{\lambda_2} = U_{\lambda_2}U_{\lambda_1}, \qquad U_{\lambda_2}U_{\lambda_3} = -U_{\lambda_3}U_{\lambda_2},$$
$$U_{\lambda_1}U_{\lambda_3} = e^{\frac{\pi\mathrm{i}}{6}} U_{\lambda_3}U_{\lambda_1}, \qquad U_{\lambda_2}U_{\lambda_4} = U_{\lambda_4}U_{\lambda_2},$$
$$U_{\lambda_1}U_{\lambda_4} - U_{\lambda_4}U_{\lambda_1}, \qquad U_{\lambda_3}U_{\lambda_4} = U_{\lambda_4}U_{\lambda_3},$$

which agree with the relations (6.74). Thus $M_\varphi(R)$ is a representation of the group G defined by (6.72). This G-module satisfies the adjoint action of G on $\hat{\mathfrak{h}}_\varphi$ given by (6.78). It is also straightforward to check that all elements of the normal subgroup N_φ act as 1 on $M_\varphi(R)$. Thus $M_\varphi(R)$ is a restricted $(\hat{\mathfrak{h}}_\varphi, G_\varphi)$-module of level 1. Thus by Theorem 6.3.3, the logarithmic fields

$$Y(a,z), \qquad a \in \mathfrak{h}, \tag{6.98}$$

and

$$Y(e^{\lambda}, z) = U_\lambda \theta_\lambda e^{\zeta a_\lambda} z^{b_\lambda} E_\lambda(z), \qquad \lambda \in Q, \tag{6.99}$$

generate a φ-twisted V_Q-module structure on $M_\varphi(R)$. The logarithmic vertex operators corresponding to the generators λ_i are

$$\begin{aligned} Y(e^{\lambda_1}, z) = q_2 e^{\zeta x_{1,0}} e^{-\frac{\zeta^2}{4\pi\mathrm{i}} q_1 \partial_{q_1}} \exp\Big(\sum_{m=1}^{\infty} \Big(\frac{1}{m} x_{1,m} + \frac{1 - m\zeta}{m^2} x_{2,m}\Big) z^m\Big) \\ \cdot \exp\Big(-\sum_{m=1}^{\infty} (\partial_{x_{4,m}} - \zeta \partial_{x_{3,m}}) z^{-m}\Big), \end{aligned}$$

$$Y(e^{\lambda_2}, z) = (-1)^{q_1\partial_{q_1}} e^{-2\pi\mathrm{i} x_{1,0}} z^{q_1\partial_{q_1}} \exp\Big(2\pi\mathrm{i} \sum_{m=1}^{\infty} x_{2,m} \frac{z^m}{m}\Big) \exp\Big(-2\pi\mathrm{i} \sum_{m=1}^{\infty} \partial_{x_{3,m}} z^{-m}\Big),$$

$$\begin{aligned} Y(e^{\lambda_3}, z) = q_1 e^{(\frac{\zeta^2}{4\pi\mathrm{i}} - \frac{\pi\mathrm{i}}{6}) q_2\partial_{q_2}} e^{-\frac{\zeta}{2\pi\mathrm{i}} \partial_{x_{1,0}}} \exp\Big(\frac{1}{2\pi\mathrm{i}} \sum_{m=1}^{\infty} \Big(\frac{1}{m} x_{3,m} + \frac{m\zeta - 1}{m^2} x_{4,m}\Big) z^m\Big) \\ \cdot \exp\Big(-\frac{1}{2\pi\mathrm{i}} \sum_{m=1}^{\infty} \Big(\partial_{x_{2,m}} + \zeta \partial_{x_{1,m}}\Big) z^{-m}\Big), \end{aligned}$$

$$Y(e^{\lambda_4}, z) = (-1)^{q_2\partial_{q_2}} e^{-\partial_{x_{1,0}}} z^{q_2\partial_{q_2}} \exp\Big(\sum_{m=1}^{\infty} x_{4,m} \frac{z^m}{m}\Big) \exp\Big(-\sum_{m=1}^{\infty} \partial_{x_{1,m}} z^{-m}\Big).$$

Let

$$v_1 = \lambda_1, \qquad v_2 = \frac{\lambda_2}{2\pi\mathrm{i}}, \qquad v_3 = 2\pi\mathrm{i}\lambda_3, \qquad v_4 = \lambda_4.$$

Then $\{v_i\}$ is a basis for $\mathfrak{h}$ for which φ and $(\cdot|\cdot)$ are as in Example 5.1.1. Let $\omega \in \mathfrak{h}$ be the conformal vector given by (4.25). Then the action of the Virasoro field $Y(\omega, z)$ on $M_\varphi(R)$ is given by Proposition 5.2.3 with $\alpha_0 = 0$. The action of L_0 is given explicitly by

$$\begin{aligned} L_0 = \sum_{i=1}^{4} \sum_{m=1}^{\infty} m x_{i,m} \partial_{x_{i,m}} + \sum_{m=1}^{\infty} \big(x_{4,m}\partial_{x_{3,m}} - x_{2,m}\partial_{x_{1,m}}\big) \\ + x_{1,0} q_2 \partial_{q_2} + \frac{1}{2\pi\mathrm{i}} q_1 \partial_{q_1} \partial_{x_{1,0}}. \end{aligned} \tag{6.100}$$

Example 6.4.2. For an example on which $\mathcal{N}$ acts as a single Jordan block, we revisit

Example 6.10 from [6]. Let $\mathfrak{h}$ be the Cartan subalgebra of a type $A_1^{(1)}$ affine Kac-Moody algebra. The dual space $\mathfrak{h}^*$ has a basis $\{\alpha_1, \delta, \Lambda_0\}$ with a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ defined by:

$$(\alpha_1|\alpha_1) = 2, \qquad (\delta|\Lambda_0) = (\Lambda_0|\delta) = 1,$$

with all other products of the basis vectors equal to 0. Thus $Q = \mathrm{span}_{\mathbb{Z}}\{\alpha_1, \delta, \Lambda_0\}$ is an integral lattice. We let $\varphi = t_{\alpha_1}$ be the element of the affine Weyl group which acts on $\mathfrak{h}^*$ by

$$\varphi\alpha_1 = \alpha_1 - 2\delta, \qquad \varphi\delta = \delta, \qquad \varphi\Lambda_0 = \Lambda_0 + \alpha_1 - \delta.$$

Then φ is an automorphism of the lattice, and $(\cdot|\cdot)$ is φ-invariant. We can write $\varphi = e^{-2\pi \mathrm{i}\mathcal{N}}$, where

$$\mathcal{N}\alpha_1 = \frac{\delta}{\pi\mathrm{i}}, \qquad \mathcal{N}\delta = 0, \qquad \mathcal{N}\Lambda_0 = -\frac{\alpha_1}{2\pi\mathrm{i}} \qquad \mathcal{N}^2\Lambda_0 = \frac{\delta}{2\pi^2},$$

and $\mathcal{N}^2\alpha_1 = \mathcal{N}^3\Lambda_0 = 0$. We use the 2-cocycle ε with

$$\varepsilon(\alpha_1, \alpha_1) = \varepsilon(\delta, \Lambda_0) = -1,$$

and $\varepsilon = 1$ on all other pairs of generators. Again we can assume $\eta = 1$ on Q.

We identify $\mathfrak{h}$ with its dual space and write $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} Q$. We consider the $\hat{\mathfrak{h}}_{\varphi}^0$-module

$$R = \mathbb{C}[x_{1,0}, q^{\pm 1}],$$

with the action

$$\Lambda_{0(0+\mathcal{N})} = -\frac{\sqrt{2}}{2\pi\mathrm{i}}\, x_{1,0}, \qquad \alpha_{1(0+\mathcal{N})} = -\sqrt{2}\,\partial_{x_{1,0}}, \qquad \delta_{(0+\mathcal{N})} = q\partial_q.$$

We let $M = M_{\varphi}(R)$ be the corresponding generalized Verma module for $\hat{\mathfrak{h}}_{\varphi}$. We label the action of $\hat{\mathfrak{h}}_{\varphi}^-$ by the commuting variables:

$$\Lambda_{0(-m+\mathcal{N})} = -\frac{\sqrt{2}}{2\pi\mathrm{i}}\, x_{1,m}, \quad \alpha_{1(-m+\mathcal{N})} = \sqrt{2}\, x_{2,m}, \quad \delta_{(-m+\mathcal{N})} = -\frac{2\pi\mathrm{i}}{\sqrt{2}}\, x_{3,m}, \quad m = 1, 2, 3, \ldots.$$

Then

$$M_{\varphi}(R) \cong \mathbb{C}[x_{1,0}, q^{\pm 1}, x_{i,m}]_{1 \le i \le 3; m=1,2,\ldots},$$

and the action of $\hat{\mathfrak{h}}_\varphi^+$ is given by

$$\begin{aligned}
\Lambda_{0(m+\mathcal{N})} &= -\frac{\sqrt{2}}{2\pi\mathrm{i}}\big(m\partial_{x_{3,m}} + \partial_{x_{2,m}}\big),\\
\alpha_{1(m+\mathcal{N})} &= \sqrt{2}\big(m\partial_{x_{2,m}} - \partial_{\partial_{x_{1,m}}}\big),\\
\delta_{(m+\mathcal{N})} &= -\frac{2\pi\mathrm{i}}{\sqrt{2}}\, m\partial_{x_{1,m}}.
\end{aligned}$$

We define the operators e^h ($h \in \mathfrak{h}$) again by (6.97), and the operators U_λ by

$$U_{\Lambda_0} = q, \qquad U_{\alpha_1} = -\mathrm{i}e^{\frac{\pi\mathrm{i}}{3}q\partial_q}e^{-\sqrt{2}\,x_{1,0}}, \qquad U_\delta = (-1)^{q\partial_q}e^{\frac{2\pi\mathrm{i}}{\sqrt{2}}\partial_{x_{1,0}}}.$$

Then these operators agree with (6.74). In particular we have

$$\begin{aligned}
U_{\Lambda_0}U_{\alpha_1} &= e^{-\frac{\pi\mathrm{i}}{3}}U_{\alpha_1}U_{\Lambda_0},\\
U_{\Lambda_0}U_\delta &= -U_\delta U_{\Lambda_0},\\
U_{\alpha_1}U_\delta &= U_\delta U_{\alpha_1}.
\end{aligned}$$

As in the previous example, $M_\varphi(R)$ is a restricted $(\hat{\mathfrak{h}}_\varphi, G_\varphi)$-module of level 1 and hence can be extended to be a φ-twisted V_Q-module. The logarithmic vertex operators corresponding to the generators are

$$\begin{aligned}
Y(e^{\Lambda_0}, z) = q e^{-\frac{\sqrt{2}}{2\pi\mathrm{i}}\zeta x_{1,0}} e^{-\frac{\sqrt{2}}{4\pi\mathrm{i}}\zeta^2\partial_{x_{1,0}} + \frac{\zeta^3}{12\pi^2}q\partial_q} e^{\frac{\zeta^3}{24\pi^2}}&\\
\cdot \exp\Big(-\frac{\sqrt{2}}{2\pi\mathrm{i}}\sum_{m=1}^\infty\Big(\frac{1}{m}x_{1,m} + \frac{1-m\zeta}{m^2}x_{2,m} - \frac{m^2\zeta^2 - 2m\zeta + 2}{m^3}x_{3,m}\Big)z^m\Big)&\\
\cdot \exp\Big(\frac{\sqrt{2}}{2\pi\mathrm{i}}\sum_{m=1}^\infty\Big(\partial_{x_{3,m}} - \zeta\partial_{x_{2,m}} - \frac{\zeta^2}{2}\partial_{x_{1,m}}\Big)z^{-m}\Big)&,
\end{aligned}$$

$$\begin{aligned}
Y(e^{\alpha_1}, z) = -\mathrm{i}e^{\frac{\pi\mathrm{i}}{3}q\partial_q}e^{-\sqrt{2}\,x_{1,0}}e^{-\sqrt{2}\zeta\partial_{x_{1,0}} - \frac{\zeta^2}{2\pi\mathrm{i}}q\partial_q}\exp\Big(\sqrt{2}\sum_{m=1}^\infty\Big(\frac{1}{m}x_{2,m} + \frac{m\zeta-1}{m^2}x_{3,m}\Big)z^m\Big)&\\
\cdot \exp\Big(-\sqrt{2}\sum_{m=1}^\infty\Big(\partial_{x_{2,m}} + \zeta\partial_{x_{1,m}}\Big)z^m\Big)&,
\end{aligned}$$

$$Y(e^{\delta}, z) = (-1)^{q\partial_q} e^{\frac{2\pi \mathrm{i}}{\sqrt{2}}\partial_{x_{1,0}}} z^{q\partial_q} \exp\Big(-\frac{2\pi \mathrm{i}}{\sqrt{2}} \sum_{m=1}^{\infty} x_{3,m} \frac{z^m}{m}\Big) \exp\Big(-\frac{2\pi \mathrm{i}}{\sqrt{2}} \sum_{m=1}^{\infty} \partial_{x_{1,m}} z^{-m}\Big).$$

Let

$$v_1 = -\frac{2\pi \mathrm{i}}{\sqrt{2}}\Lambda_0, \qquad v_2 = \frac{\alpha_1}{\sqrt{2}}, \qquad v_3 = -\frac{\sqrt{2}}{2\pi \mathrm{i}}\delta.$$

Then $\{v_i\}$ is a basis for $\mathfrak{h}$ for which φ and $(\cdot|\cdot)$ are as in Example 5.1.2. Again we let $\omega \in \mathfrak{h}$ be the conformal vector given by (4.25). The action of the Virasoro field $Y(\omega, z)$ on $M_{\varphi}(R)$ is given by Proposition 5.2.3 with $\alpha_0 = 0$. The action of L_0 is given explicitly by

$$\begin{aligned} L_0 = &\sum_{i=1}^{3} \sum_{m=1}^{\infty} m x_{i,m} \partial_{x_{i,m}} + \sum_{m=1}^{\infty} \big(x_{3,m}\partial_{x_{2,m}} - x_{2,m}\partial_{x_{1,m}}\big) \\ &- \frac{\sqrt{2}}{2\pi \mathrm{i}} x_{1,0} q \partial_q + \frac{1}{2}\partial_{x_{1,0}}^2. \end{aligned}$$

Chapter 7

Conclusion and Future Work

7.1 Conclusion

Vertex algebras have been of use in abstract settings, such as string theory and the representation theory of infinite-dimensional Lie algebras, as well as areas with known concrete applications, such as conformal field theory and integrable systems. Twisted modules of vertex algebras have important applications in both the abstract and concrete settings. Twisted logarithmic modules of vertex algebras extend the theory of twisted modules in a natural way. Where twisted modules have been useful in applications, we expect that twisted logarithmic modules will also have use.

We have classified (up to the choice of representation of zero modes) the twisted logarithmic modules of free field vertex algebras, extending the results on free bosons obtained in [6]. We began by answering a question from linear algebra regarding canonical forms of an automorphism preserving a nondegenerate skew-symmetric bilinear form. Then we considered the case of twisted logarithmic modules of free superbosons, including the odd case of symplectic fermions SF and the even case of free bosons which was previously done in [6]. We also considered the free superfermions, specifically the odd case of free fermions and the even case of the bosonic ghost system. In each of these cases, we realized the twisted logarithmic modules explicitly as highest-weight representations of certain Fock spaces. We explicitly wrote the action of the Virasoro algebra on each. In particular, we showed in each case that when $\mathcal{N} \neq 0$ the action of L_0 is not semisimple. We have previously published these results [13].

Supersymmetric and skew-supersymmetric bilinear forms can be used to construct interesting and useful algebras. The bilinear form used in such constructions is generally assumed to be even. We have shown that inhomogeneous supersymmetric bilinear forms, i.e., forms that are neither even nor odd also lead to the construction of interesting oscillator-like algebras. We have classified such forms up to dimension seven in the case when the restriction to even and odd parts of the superspace is nondegenerate. We have also provided an explicit example of an oscillator-like algebra constructed from such a form. These results have been submitted for publication [14].

Twisted modules for lattice vertex algebras formalize the relations between twisted vertex operators [28]. In [10], such modules were classified under some natural assumptions in the case when the automorphism has finite order. We have developed a framework for studying the case of a general automorphism. In particular, we showed that under suitable assumptions twisted logarithmic vertex operators on a φ-twisted V_Q-module M can be written in terms of an action of the φ-twisted Heisenberg algebra $\hat{\mathfrak{h}}_\varphi$ on M together with an action of a certain group G_φ. We also showed that the question of classifying such φ-twisted V_Q-modules reduces to the classification of modules of this group. To demonstrate the utility of our approach, we constructed explicit examples from 3 and 4-dimensional lattices.

In the remainder of this chapter, we mention several future project areas, some of which relate directly to the results in this text.

7.2 Subalgebras and Orbifolds of Free Field Algebras

Many important vertex algebras can be obtained as subalgebras of free field algebras, such as affine Kac-Moody algebras [33, 35, 38, 63, 91], toroidal Lie algebras [57, 58], $\mathcal{W}$-algebras [36, 37, 66], the $\mathcal{W}_{1+\infty}$-algebra [64, 71, 76], superconformal algebras [66, 70], and affine Lie superalgebras [69].

An *orbifold* of a vertex algebra is a subalgebra fixed pointwise by an automorphism φ or group of automorphisms Γ. Orbifold subalgebras are of particular interest to physicists, since they correspond to important 2-dimensional conformal field theories (see, e.g., [26]). Any φ-twisted logarithmic module of V restricts to a usual (untwisted) module for the orbifold V^φ. Furthermore, for any group of automorphisms Γ containing φ, the orbifold V^Γ is contained in V^φ, and any φ-twisted module of V restricts to an untwisted module for V^Γ. Orbifolds of free field algebras have been studied in such papers as [1] and [18].

For example, in [18], Creutzig and Linshaw study the orbifolds of the rank n symplectic fermions. Among other results, they show that the orbifolds corresponding to the full automorphism group $\Gamma = Sp(2n)$ and the subgroup $\Gamma = GL(n)$ are $\mathcal{W}$-algebras. For any element $\varphi \in \Gamma$, using our construction of φ-twisted modules for the symplectic fermions, we can consider questions such as

1. What is the fixed subalgebra V^φ?
2. What is the representation of V^φ obtained via the restriction from the φ-twisted module for V?
3. The orbifold V^Γ is of course a subalgebra of V^φ, but how do they compare? Which modules for V^Γ can be obtained via restriction from V^φ for some φ?

If V is one of the free-field algebras, $\varphi \in \mathrm{Aut}(V)$, and $A \subset V$ is a subalgebra such that $\varphi(A) \subset A$, then any φ-twisted V-module gives rise to a φ-twisted A-module by restriction. Moreover, such A-modules are untwisted if φ acts as the identity on A (i.e., if $A \subset V^\varphi$). Thus, we expect the twisted modules constructed in this paper to be useful for studying modules over subalgebras of free-field algebras.

Example 7.2.1 (The triplet algebra)**.** Let $\dim \mathfrak{h} = 2$ in Example 5.1.6. The *triplet algebra* $\mathfrak{W}(1,2) \subset SF$ is generated by the elements (see [20, 74]):

$$W^+ = -v_{1(-2)}v_1, \quad W^0 = -v_{1(-2)}v_2 - v_{2(-2)}v_1, \quad W^- = -v_{2(-2)}v_2.$$

Then $\sigma = I$ on $\mathfrak{W}(1,2)$ and $\mathcal{N}\colon W^+ \mapsto W^0 \mapsto 2W^- \mapsto 0$. Hence, the restriction of any φ-twisted module of SF to $\mathfrak{W}(1,2)$ is a φ-twisted module of $\mathfrak{W}(1,2)$, in which $Y(W^-,z)$ is independent of ζ while the fields $Y(W^+,z)$ and $Y(W^0,z)$ are logarithmic. Explicitly,

$$\begin{aligned}
Y(W^-,z) &= X(W^-,z), \\
Y(W^0,z) &= X(W^0,z) - 2\zeta X(W^-,z), \\
Y(W^+,z) &= X(W^+,z) - \zeta X(W^0,z) + \zeta^2 X(W^-,z).
\end{aligned}$$

Example 7.2.2 (Superbosons as a subalgebra of superfermions)**.** Suppose that $\dim \mathfrak{a} = 2\ell$ as in Examples 5.1.1, 5.1.5. It is well-known that the elements

$$u_i = v_i \in F^1(\mathfrak{a}), \quad u_{\ell+i} = Tv_{\ell+i} \in F^1(\mathfrak{a}) \qquad (1 \le i \le \ell)$$

(where T is the translation operator) are generators of the *free superboson algebra* $B^1(\mathfrak{a}) \subset F^1(\mathfrak{a})$. This is the Heisenberg vertex algebra in the case of Example 5.1.1, and the symplectic fermion algebra SF in the case of Example 5.1.5 (see e.g. [72]). Here u_i plays the role of v_i from Sections 5.3 and 5.4.

When φ is the automorphism of $F^1(\mathfrak{a})$ from Examples 5.1.1, 5.1.5, then φ restricts to an automorphism of $B^1(\mathfrak{a})$ of the same type, since φ commutes with T. Thus, any φ-twisted $F^1(\mathfrak{a})$-module restricts to a φ-twisted $B^1(\mathfrak{a})$-module. In such a module, the logarithmic fields corresponding to the generators are given by (cf. (4.49)):

$$Y(u_i, z) = Y(v_i, z), \quad Y(u_{\ell+i}, z) = D_z Y(v_{\ell+i}, z) \qquad (1 \le i \le \ell).$$

Then $u_{i(m+\mathcal{N})} = v_{i(m+\mathcal{N})}$ for $m \in \alpha_0 + \mathbb{Z}$, and using (5.41) we obtain

$$u_{\ell+i(m+\mathcal{N})} = -m v_{\ell+i(m-1+\mathcal{N})} + (1-\delta_{i,\ell}) v_{\ell+i+1(m-1+\mathcal{N})}$$

for $m \in -\alpha_0 + \mathbb{Z}$. The action of these modes on $M_\varphi(R)$ is related to the φ-twisted modules constructed in Sections 5.3 and 5.4 by a linear change of variables.

The free superboson algebra $B^1(\mathfrak{a})$ has a conformal vector (cf. (4.25))

$$\omega' = \sum_{i=1}^{\ell} u_{2\ell-i+1(-1)} u_i.$$

Since

$$u_{2\ell+1-i(-1)} u_i = (T v_{2\ell+1-i})_{(-1)} v_i = v_{2\ell+1-i(-2)} v_i = v^i_{(-2)} v_i,$$

we have $\omega' = \omega^+ \in F^1(\mathfrak{a})$ (see (5.42)). It follows that the action of $Y(\omega', z)$ on $M_\varphi(R)$ is equivalent to the action of $L^+(z)$ (cf. Sections 5.6 and 5.7).

Example 7.2.3 (The $\mathcal{W}_{1+\infty}$-algebra)**.** Other important subalgebras of $F^1(\mathfrak{a})$ are the $\mathcal{W}_{1+\infty}$-*algebra* and its subalgebras [64]. The following elements similar to ω^+ generate the $\mathcal{W}_{1+\infty}$-algebra:

$$\nu^n = \sum_{i=1}^{\ell} v^i_{(-n)} v_i, \qquad n = 1, 2, 3, \ldots \tag{7.1}$$

so that $\nu^2 = \omega^+$ and ν^1 generates the Heisenberg algebra. The automorphism φ of $F^1(\mathfrak{a})$ from Examples 5.1.1, 5.1.5 satisfies $\varphi(\nu^n) = \nu^n$ for all n. Therefore, any φ-twisted $F^1(\mathfrak{a})$-module restricts to an (untwisted) module of $\mathcal{W}_{1+\infty}$. The fields $Y(\nu^n, z)$ in such a module can be computed as in Proposition 5.5.4 and Corollary 5.5.5:

$$\begin{aligned} Y(\nu^n, z) = \frac{1}{(n-1)!} \sum_{i=1}^{\ell} &:X\big((\partial_z - z^{-1}\mathcal{N})^{n-1} v^i, z\big) X(v_i, z): \\ &+ (-1)^{n-1} z^{-n} \operatorname{str} \binom{\mathcal{S}^+}{n} I. \end{aligned} \tag{7.2}$$

Example 7.2.4 (The general linear Lie superalgebra $\widehat{\mathfrak{gl}}(m|n)$)**.** Other important realizations by free superfermions are those of *classical affine Lie (super)algebras* [33,38,63,69]. Here we discuss just one example. Let us assume, as before, that $\mathfrak{a}$ can be written as the direct sum of two isotropic subspaces $\mathfrak{a}^- = \operatorname{span}\{v_i\}$ and $\mathfrak{a}^+ = \operatorname{span}\{v^i\}$ $(1 \le i \le \ell)$, where $(v_i|v^j) = \delta_{i,j}$. We label the basis vectors so that the odd part $(\mathfrak{a}^-)_{\bar{1}}$ is spanned by $\{v_i\}_{i=1,\dots,m}$, and the even part $(\mathfrak{a}^-)_{\bar{0}}$ is spanned by $\{v_i\}_{i=m+1,\dots,m+n}$ where $\ell = m+n$. Then, by [69, Proposition 3.1], the elements $e_{ij} = v_{i(-1)}v^j$ $(1 \le i, j \le \ell)$ provide a realization of the affine Lie superalgebra $\widehat{\mathfrak{gl}}(m|n)$ inside the free superfermion algebra $F^1(\mathfrak{a})$. It is easy to see that if φ is the automorphism of $F^1(\mathfrak{a})$ from Examples 5.1.1, 5.1.5, then φ preserves $\widehat{\mathfrak{gl}}(m|n)$. In fact, φ acts as an inner automorphism of $\mathfrak{gl}(m|n)$; hence, the corresponding φ-twisted modules are as in Section 5.2.

7.3 Boson-Fermion Correspondences and Integrable Systems

A Hamiltonian system with j degrees of freedom is given by the differential equations

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}, \qquad \frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}},$$

where $\mathbf{p}$ and $\mathbf{q}$ are j-dimensional vectors describing momentum and position respectively. Such a system is (completely) integrable if there exist j independent first integrals. Some differential equations can be equivalently written as a Hamiltonian system. If the corresponding Hamiltonian system is integrable, then given one solution to the original differential equation, others can be obtained using the high degree of symmetry in the system. Nonlinear differential equations generally do not correspond to integrable Hamiltonian systems. However, some exceptional nonlinear PDEs do if one allows systems with infinitely many degrees of freedom [24]. The Korteweg-de Vries (KdV) equation was the first known example [54, 94]. Certain solutions, called soliton solutions, exhibit behavior similar to the superposition property of linear differential equations, allowing the theory to be rewritten in the language of linear differential operators [77].

The KdV hierarchy is an infinite hierarchy of PDEs that are commuting symmetries for the KdV equation and each other. Noncommuting symmetries of this hierarchy are more easily obtained in the language of differential operators. Then using the classical boson-fermion correspondence one realizes these symmetries as vertex operators acting on the space of solutions to the hierarchy. The classical boson-fermion correspondence is a highly nontrivial isomorphism between fermionic and bosonic Fock spaces [80]. Other boson-fermion correspondences are written in terms of twisted modules of vertex algebras, and some in terms of twisted vertex algebras. Such constructions lead to other important hierarchies (see, e.g., [3, 9, 29, 68]).

The results in this text prepare the way for the exploration of the possibility of boson-fermion correspondences for twisted logarithmic modules. For example, a free field algebra of k bosons can be realized using k pairs of fermions. For $k = 1$, this realization leads to the classical boson-fermion correspondence. For $k \geq 3$, the fermions admit φ-twisted modules where φ is not semisimple. We plan to investigate whether twisted logarithmic modules of the bosonic subalgebra can be obtained by restricting such φ-twisted modules. Such constructions may lead to new boson-fermion correspondences which may in turn lead to new hierarchies of PDEs.

7.4 Logarithmic Vertex Algebras and LCFT

In quantum field theory, one studies quantum mechanical models by describing interactions between particles in terms of the interactions of corresponding quantum fields. A conformal field theory (CFT) is a quantum field theory that is invariant under conformal transformations [26]. Vertex algebras provide a rigorous mathematical definition for chiral CFT (see, e.g., [60]). CFTs have important physical applications including statistical mechanics and condensed matter physics [89]. There are also more theoretical applications such as to string theory [92].

In certain models of conformal field theory, logarithms of the formal variables are required in the vertex operators and their operator product expansions, something not allowed in vertex algebras. Thus a more general structure is needed. Bakalov recently defined such a notion, calling it a logarithmic vertex algebra. Currently only a few examples of logarithmic vertex algebras are known, but they coincide with important LCFTs such as the symplectic fermion model [49, 73] and the WZW model on $GL(1|1)$ [85, 86].

Logarithmic vertex algebras look promising as a way to greatly simplify the methods used in developing and analyzing LCFTs. We plan to investigate methods of constructing other logarithmic vertex algebras not obtained from free fields. Possible candidates are the fractional level WZW model $\hat{\mathfrak{sl}(2)}_{-1/2}$ and the staggered modules discussed in [20].

7.5 Cryptography and Error-correcting Codes

Recent progress toward the construction of real quantum computers has led to strong interest in cryptosystems that are resistant to both quantum computer and classical computer attacks. Lattice-based cryptosystems, such as the Ajtai-Dwork, GGH, and NTRU cryptosystems, were first developed in the 1990s [2, 53, 56]. They are based on difficult lattice problems such as the shortest vector and the closest vector problems. Such cryptosystems have received renewed attention recently, because there are no known algorithms either for quantum computers or classical computers with the potential to efficiently attack them. However, one major disadvantage to such systems is that their underlying mathematical theory is not nearly as well understood as for cryptosystems based on factorization or the discrete log problem. As a long-term goal, I intend to help develop the theory needed to better analyze the security of such systems, and to possibly even develop other lattice-based systems. As a first step in this direction, I plan to explore possible applications of lattice vertex algebras to lattice-based cryptosystems.

Another important application of integral lattices is to error-correcting codes. Any medium used to transmit information is imperfect. Noise in the signal invariably arises, and this causes mistakes in the transmission. Error-correcting codes were first introduced in the mid-twentieth century in order to detect and correct these mistakes [55, 90]. Connections have already been made between error-correcting codes and vertex algebras [23, 46, 52], and I intend to use connections as a springboard to making contributions to this theory.

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