ABSTRACT

KARAN, CAGATAY. Some Problems on Black Litterman Model. (Under the direction of Tao Pang.)

The Black-Litterman Model (BLM) is a very important and useful extension of the classical mean-variance portfolio optimization model. In the BLM framework, the investor views can be integrated with the classical mean-variance portfolio optimization in a Bayesian manner to get the optimal portfolio. In the original setting, variance is used as a risk measure where the asset returns follow multivariate normal distribution. In our work we use the popular Conditional Value at Risk (CVaR) as the risk measure instead of variance proposed in the original model. In addition to that, elliptical uncertainty sets are used to model the uncertainty of asset returns. For constrained problem, driving the optimal solution analytically is extremely difficult. Hence, we propose an efficient approximation algorithm for the BLM type optimization problems under CVaR and we have established the convergence results. Based on the convergence we derived the closed-form solution of the portfolio optimization problems of Black-Litterman type with CVaR. However, if tail estimation is not good, CVaR does not give plausible results. Moreover, typically, the investor is not 100% sure about her view, so the confidence level of the view plays an important role in determining the optimal portfolio. We propose a new mapping technique for investor's confidence level. In addition to that, we propose a simple but meaningful method based on the investor confidence level on whether the market is a bull market. Full confidence level means a bull market and zero confidence level means a bear market. A confidence level between 0 and 100% means a mixed bull and bear market. Therefore, mixed Gaussian distributions are used to model the assets market returns under a mixed bull and bear market. Explicit solutions are derived under this mixed Gaussian distribution of the Black-Litterman framework, and the optimal portfolio can be obtained from the optimal portfolio weights under the bull market and the bear market.

Some Problems on Black Litterman Model

by
Cagatay Karan

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Operations Research

Raleigh, North Carolina

2017

APPROVED BY:

Jeffrey Scroggs	Negash Medhin
Peter Bloomfield	Tao Pang Chair of Advisory Committee

DEDICATION

To my wife Hayley Jones, mother Yildiz Karan, father Dr. Onur Karan, aunt Tomris Erkmen, grandmother Baise Erkmen and late grandmother Asuman Karan.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor Dr. Tao Pang for the continuous support of my Ph.D. study. I have been able to complete this work because of his constant support, encouragement and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I would like to mention my special thanks to Dr. Negash Medhin, Dr. Jeffrey Scroggs and Dr. Antje Berndt for their time, help and support throughout my Ph.D. program. Special thanks to Dr. Peter Bloomfield for his valuable time and willing to be on my dissertation committee. I am also grateful to Dr. Nesim Erkip, Dr. Murat Fadiloglu, Dr. Selim Akturk, Dr. Kagan Gokbayrak, Dr. Utku Gurler, Dr. Mustafa Pinar, Dr. Savas Dayanik and Dr. Ihsan Sabuncuoglu. I gained a lot of knowledge from them during my studies at Bilkent University. I would like to thank to my supervisors Tekinel Ozdemir, Seda Kocabalkan and Zeki Sen when I was working at Garanti Bank. I am also thankful to my family for their prayers and support. Moreover, I am thankful to all my friends especially to Azmat Hussain, Safak Yucel, Ismail Demir, Caglar Caglayan, Ramiz Boy, Osman Kocas, Ozgun Yanaz, Onur Uzunlar, Emre Uzun, Majed Baghdadi, Katherine Varga, Connie Yong, Yang Zhao, Aras Aslan, Yigit Harut, Deniz Cenk Demir, Altug Aslanoglu, Sarp Yildiz, Cagri Gungor, Engin Ozdemir, Cenk Atasaral, Akin Albayrak, Can Algan, Sahan Eken, Ahmet Eldazer, Burak Ozturk, Ahmet Boyaci, Cem Urulbay, Oguzcan Iren, Doruk Sargin, Chad Hall, Keisha Lancaster, Brittany Fisher, Sean Fisher, Ezgi Kaya, Emre Eyupoglu, Ismail Dogmusoz, Namik Temizer, Cenk Turkmen, Korhan Aras, Gokce Akin Aras, Aydin Beseli, Baris Kacar, and Thomas Smith for their encouragement and for always being there for any kind of help I needed. Special thanks to Ertugrul Alp, Kemal Tuncali, Oya Ercil, Nafia Altinok, Beyhan Yozgatli, Basak Ersen, Laki Vingas, and Arif Dulger.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

Introduction

Building a portfolio optimization model has three components; defining and estimating the parameters (see Stengel [46] and Lyuu [34]), specifying the objective function (utility function most of the case) and constraints (see for example Steinbach [45] and Kolm et. al. [25]).

First component is the parameter estimation. According to Lyuu [34], we can think about three major parameter estimation techniques; Least Square Method (minimizing the sum of squares of the deviations), Maximum Likelihood Estimator (maximizing the log-likelihood function for all moments) and Methods of Moments (maximizing the log-likelihood function for some moments). In addition to that some other techniques can be used such as; Bayesian Estimation (see for example Avramov and Zhou [3]) and Inverse Optimization (Ahuja and Orlin [1] and Heuberger [21]). In this thesis we use the Black-Litterman Model which is a combination of the Bayesian framework and Inverse Optimization.

Second component is the objective function. In this thesis, we assume that the purpose of the investors are to maximize the expected return on their portfolios. Therefore the objective function is taken as the expected return of the portfolio. These portfolios are composed of combination of stocks and a risk free asset.

Maximizing the expected return of the portfolio is challenging since return has two components; return with certainty and return with uncertainty. In this thesis we assume that the investors are risk averse which means that as the uncertainty on the return increases, investors require more reward. Hence, investors always want to access the uncertainty (i.e. risk). Accessing the uncertainty is twofold; first a probability distribution should either be assumed or fitted to asset returns, second a tool (risk measure) is needed to measure uncertainty.

Last component is the constraints. In this thesis, we have two constraints; one is related with the risk measure and the other one is related with the asset allocation. An upper bound for the risk measure is given to the model. In addition to that, we require that the sum of all the asset allocations (including the asset allocation to risk free asset) add up to one.

Investors have beliefs and/or information about what is going to happen in the market. However, we assume that it is not possible to use intuitions and/or inside information about the assets or market parameters to beat the market. In other words, we assume the Efficient Market Hypothesis. Moreover, we emphasize that it is impossible to come up with a model such that it predicts the outcome of any random event without an estimation error. The fundamental argument here is that the models are always wrong. This is the direct consequence of their definition and separates models from reality sharply.

All of these arguments stated above are vital atoms of this thesis. Because of the arguments that we have stated, the goal of this thesis is not to beat the market but is to establish a well diversified portfolio using; investor beliefs and historical data available in the market under different settings.

We continue with the Markowitz [29] model, also known as the mean-variance model (MVO).

1.1 Markowitz Portfolio Allocation Problem

According to Markowitz [29], process of selecting a portfolio has two stages. The first stage is about experience and observations, and beliefs about the future performance of the assets. In other words, selecting and estimating a probability distribution to asset returns. The second stage is finding a portfolio allocation vector where return is desirable and the risk is undesirable. Note that Markowitz [29] only consider the second stage in his novel paper.

Consider a market with n risky assets. Risky asset returns are denoted by the random vector $\mathbf{r} \in \mathbb{R}^n$ which is defined on the probability space $(\Omega, \mathcal{F}, P)$. The mean vector and the covariance matrix of risky asset returns are denoted by $\boldsymbol{\mu} = \mathbb{E}[\mathbf{r}]$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ respectively. The risk free rate of return is denoted as $r_f \in \mathbb{R}_+ \cup 0$. Moreover, $\mathbf{x} \in \mathbb{R}^n$ is the portfolio weight vector of risky assets and $(1 - \mathbf{e}'\mathbf{x})$ is the allocation on the risk-free asset, where $\mathbf{e} = (1, 1, \cdots, 1)'$ is a vector of ones in $\mathbb{R}^n$. Moreover, risk aversion coefficient is denoted as; λ (i.e. the amount of risk the investor is willing to take for a unit of return). The Markowitz Portfolio Allocation Problem is defined as:

Definition 1.1.1 (Markowitz's PAP)**.**

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \frac{\lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}{2} \right\}. \tag{1.1}$$

An investor can use Markowitz model [29] with the historical mean vector and covariance matrix of the risky assets and find the optimal portfolio allocation weights. If we were to assume that historical returns perfectly reflect the future returns, then an immediate question would follow that argument; if we could predict future asset returns using only historical data then every single player in the market would use the same strategy and that would cause a stable price structure in the market eventually. However, we do observe changes in the stock prices. Actually, that is the reason why we have volatility. Furthermore, mean variance optimization (i.e. Markowitz model [29] see Definition 1.1.1) considers only the first two moments of the return distribution. This restriction was consistent with reality if asset returns followed a normal distribution (Fabozzi [13] and Meucci [30]). But we know that asset return's coskewness and cokurtosis values differ from normality (for more information please see Harvey et. al. [18] and Jondeau and Rockinger [24]). In addition to that, optimal portfolio weights are highly sensitive to the parameters of the optimization problem (see Meucci [30] and Fabozzi et. al. [13]). Hence, Markowitz Model's optimal allocation vector is lack of diversification and/or has corner solutions. We have some techniques to cope with all the drawbacks listed above such as shrinkage estimates and bayesian approaches (see Fabozzi Frank J. [13]). Black and Litterman [9] (BLM) also proposes a bayesian approach. Their model, combines the intuitions about the selected assets or market parameters of the investor with the historical information of the market to update the mean vector and covariance matrix. In other words, BLM starts the portfolio allocation problem from the first stage denoted by Markowitz [29].

1.2 Black-Litterman Model (BLM)

We follow the same notation given as in Section 1.1 for the problem parameters and the portfolio allocation vector. Before we state the BLM, we first define the Forward and Inverse Optimization Problems, and give the relationship between inverse optimization and BLM.

1.2.1 Inverse Optimization Problems

Most of the optimization problems in finance are Forward Problems (i.e. Definition 1.2.1). In other words, we maximize (or minimize) an objective function (i.e. $f(\mathbf{x};\boldsymbol{\zeta})$) over a feasible set (i.e. $\Gamma(\boldsymbol{\zeta})$). This set is composed of the problem parameters and the constraints.

Definition 1.2.1 (Forward Problem)**.**

$$\max_{\mathbf{x}\in\Gamma(\boldsymbol{\zeta})} f(\mathbf{x};\boldsymbol{\zeta}), \tag{1.2}$$

where $\mathbf{x} \in \mathbf{R}^n$.

However, most of the time the optimal solution of the forward problem is supoptimal in reality since errors in parameter estimation cannot be avoided. A novel idea to overcome that issue is to define an Inverse Problem where for a given/observable candidate solution of the Forward Problem, we want to characterize the set of the problem parameters. In short, the idea of the Inverse Problem is that the parameters are perturbed so that the candidate solution will be optimal (i.e. Definition 1.2.2).

Definition 1.2.2 (Inverse Problem)**.** Find a value of $\boldsymbol{\zeta}$ such that

$$\hat{\mathbf{x}} = \operatorname*{argmin}_{\mathbf{x}\in\Gamma(\boldsymbol{\zeta})} f(\mathbf{x};\boldsymbol{\zeta}), \tag{1.3}$$

where $\mathbf{x} \in \mathbf{R}^n$.

Ahuja and Orlin [1] defines the Inverse Optimization as:"Let $\mathcal{S}$ represent a physical system. Assume that we are able to define a set of model parameters which completely define $\mathcal{S}$. All these parameters may not be directly measurable (such as the radius of Earth's Metallic core). We can operationally define some observable parameters whose actual values hopefully depend on the values of the model parameters. To solve the forward problem is to predict the values of the observable parameters given arbitrary values of the model parameters. To solve the inverse problem is to infer the values of the model parameters from given values of the observable parameters".

They continue: "A typical optimization problem is a forward problem because it identifies the values of the observable parameters (optimal decision variables), given the values of the model parameters (cost coefficients, right-hand side vector, and the constraint matrix). An inverse optimization problem consists of inferring the values of the model parameters (cost coefficients, right-hand side vector, and the constraint matrix), given the values of observable parameters (optimal decision variables)".

Moreover, according to Heuberger [21]: "The idea of the inverse optimization is to find values of the parameters which make the known solutions optimum and which differ from the given estimates as little as possible"

According to Ahuja and Orlin [1], Inverse Problems have lot of applications such as: Geophysical Sciences, Medical Imaging, Traffic Equilibrium and Portfolio Optimization. The Black-Litterman Model uses the inverse optimization to update the expected asset returns.

The BLM assumes that the market is in CAPM equilibrium. In other words, BLM assumes that every single investor in the market solves the problem given in Definition 1.1.1. The implied equilibrium return vector, $\mathbf{\Pi}$, is the customary equilibrium return vector used in the BLM and can be calculated using Inverse Optimization, and is equal to $2\lambda\mathbf{\Sigma}\mathbf{x}_{market}$ (details of the derivation is given in Chapter 2). We need estimations for risk aversion coefficient (λ), and covariance matrix ($\mathbf{\Sigma}$) to use the inverse optimization. In addition to that, a candidate optimal allocation vector is needed. BLM takes market capitalization weights (i.e. $\mathbf{x}_{market}$) as the candidate optimal allocation vector. Moreover, historical covariance matrix is used as the estimation for the covariance matrix. In addition to that, λ is fixed subjectively. Furthermore, BLM assumes that investors have views about possible outcomes of the market. This is captured by the view portfolio. This feature of the model is represented as the vector; $\mathbf{P}$. The investor provides the expected return of her view portfolio; $\mathbf{q}$. On the other hand, investor is not certain about the future. Therefore, she provides her uncertainty about the view portfolio (i.e. $\mathbf{\Omega}$). In addition to that she is not sure about the estimation of the covariance matrix. Hence, another uncertainty parameter (i.e. $\tau \in (0, \infty)$) is provided.

We give the Canonical Black-Litterman Model under unconstrainted setting next.

Definition 1.2.3 (Black Litterman Model)**.**

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}_{BL}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \frac{\lambda \mathbf{x}'\mathbf{\Sigma}_{BL}\mathbf{x}}{2} \right\}. \tag{1.4}$$

where the parameters of the model is given as:

$$\begin{aligned}
\boldsymbol{\mu}_{BL} &= \left((\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right)^{-1} \left((\tau\mathbf{\Sigma})^{-1}\mathbf{\Pi} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{q}\right), \\
\mathbf{\Sigma}_{BL}^{\boldsymbol{\mu}} &= \left((\tau\mathbf{\Sigma})^{-1} + \mathbf{P}'\mathbf{\Omega}^{-1}\mathbf{P}\right)^{-1}, \\
\mathbf{\Sigma}_{BL} &= \mathbf{\Sigma} + \mathbf{\Sigma}_{BL}^{\boldsymbol{\mu}}.
\end{aligned}$$

In this thesis, portfolio allocation problem (PAP) is examined in different settings and the rest of the thesis is organized as follows. In Chapter 2 we analyze the sensitivity of inputs in the Black-Litterman Model. In Chapter 3, we use CVaR as a risk measure instead of variance that is proposed in BLM as a risk measure. Moreover, elliptical uncertainty sets are used to model the uncertainty of asset returns. For constrained problem, driving the optimal solution analytically is extremely difficult. Hence, we proposed an efficient approximation algorithm for PAP under CVaR for the BLM type optimization problems under CVaR risk measure and derived the closed-form solutions. In Chapter 4 we extended the BLM in different ways; first of all we propose the BL-type estimations under multivariate mixed normal setting; second of all, since specifying the uncertainty of the investor views is difficult, we established the representation of the general allocation vector with the allocation vectors under full confidence and no confidence.

Chapter 2

Sensitivity of Inputs in the Black-Litterman Model

2.1 Introduction

The goal in modern portfolio optimization is to find the optimal portfolio allocation vector given problem parameters, objective function, and constraints. Setting the objective function can be done in many ways. For example, if the investor knows the true (or estimated) value of her risk reward trade off parameter, she can either maximize the unconstrained reward risk trade off optimization problem or minimize the unconstrained risk reward trade off optimization problem. However, estimating the risk reward trade off parameter is difficult. If the investor does not know the true value of her risk reward trade off parameter and cannot get a good estimation for it then, she solves the constrained problem by either setting a lower bound for the reward and minimizing risk or by setting an upper bound for the risk and maximizing reward.

In this chapter we consider two optimization problems: Markowitz Model [29] and Black-Litterman Model (BLM) [9] and for both models, expected return of the portfolio (reward) is maximized for a given upper bound for the variance of the portfolio (risk).

The traditional Markowitz Model uses historical mean returns ($\boldsymbol{\mu}$) and historical covariance matrix ($\boldsymbol{\Sigma}$) as inputs to find the optimal portfolio allocation vector. The resulting portfolio, is often not well diversified. Black-Litterman Model (BLM) attempts to fix this problem. BLM assumes that the market is at equilibrium under the Capital Asset Pricing Model (CAPM) and uses inverse optimization to get the implied equilibrium return vector (i.e. $\boldsymbol{\Pi}$). Investor views (i.e. $\mathbf{q}$) and $\boldsymbol{\Pi}$ are used in Bayesian framework to get the posterior distribution for the expected return vector. Hence a better estimate for expected asset returns can be found. The resulting posterior distribution has the updated expected return vector (i.e. $\boldsymbol{\mu}_{BL}$) and the updated covariance matrix (i.e. $\boldsymbol{\Sigma}_{BL}$). Then, these updated parameters are used in Markowitz Model to get the optimal portfolio allocation vector. Resulting optimal portfolios usually are more well diversified than the optimal portfolios generated by Markowitz Model (Bertsimas et. al. [7]).

The rest of the chapter is organized as follows. In Section 2.2 we analyze the Markowitz PAP with the risk free asset and without the risk free asset. In Section 2.3 we analyze the parameters of Black-Litterman Model and give the derivation of the implied equilibrium expected return. In Section 2.4 we conclude the chapter.

2.2 Markowitz Portfolio Allocation Problem

We consider a market with n risky assets. Risky asset returns are denoted by the random vector $\mathbf{r} \in \mathbb{R}^n$ which is defined on the probability space $(\Omega, \mathcal{F}, P)$. Vectors are defined as column vectors unless otherwise stated. The mean vector and the covariance matrix of risky asset returns are denoted by $\boldsymbol{\mu} = \mathbb{E}[\mathbf{r}] \in \mathbb{R}^n$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$ respectively. The risk free rate of return is denoted as $r_f \in \mathbb{R}_+ \cup 0$. Moreover, $\mathbf{x} \in \mathbb{R}^n$ is the portfolio allocation vector of risky assets and x_{r_f} is the allocation to the risk-free asset, where $\mathbf{e} = (1, 1, \cdots, 1)'$ is a vector of ones in $\mathbb{R}^n$.

If the investor knows the true value of her risk reward trade off (i.e. λ) then she can solve the mean-variance trade off PAP given in Definition 2.2.1

Definition 2.2.1 (Markowitz's PAP without risk free rate)**.**

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} - \frac{\lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}{2} : \mathbf{e}'\mathbf{x} = 1 \right\} \tag{2.1}$$

In addition to this model, if a risk-free asset present (i.e. $r_f \in \mathbb{R}_+ \cup 0$) in the market. Then one can solve the following PAP.

Definition 2.2.2 (Markowitz's PAP with risk free rate)**.**

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} + r_f x_{r_f} - \frac{\lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}{2} : \mathbf{e}'\mathbf{x} + x_{r_f} = 1 \right\} \tag{2.2}$$

We want our PAP to be as flexible as possible. Hence, if we write the constraint above as:

$$x_{r_f} = (1 - \mathbf{e}'\mathbf{x}).$$

Then we can incorporate this constraint with the objective function and get:

$$\boldsymbol{\mu}'\mathbf{x} + r_f(1 - \mathbf{e}'\mathbf{x}).$$

After this transformation we have the same PAP (see Boyd and Vandenberghe [10]) with no constraints:

Definition 2.2.3 (Markowitz's unconstrained PAP with risk free rate)**.**

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} + r_f(1 - \mathbf{e}'\mathbf{x}) - \frac{\lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}{2} \right\}, \tag{2.3}$$

However, estimating the risk reward trade off parameter is difficult. The easy way to handle this situation is to hold the minimum variance portfolio which is the optimal solution to the following problem (note that we assume risk-free rate is not present, since if a risk free rate is present then the minimum variance portfolio consist of only the risk free asset).

Definition 2.2.4 (Minimum variance PAP).

$$\min_{\mathbf{x}} \left\{ \mathbf{x}'\mathbf{\Sigma}\mathbf{x} : \mathbf{e}'\mathbf{x} = 1 \right\}. \tag{2.4}$$

On the other hand, most of the time investors use an upper bond for the portfolio risk in order to control the risk of their portfolios. Let us continue with the constrained Markowitz's portfolio allocation problem (PAP) (Markowitz [29]):

Definition 2.2.5 (Markowitz's PAP without risk free rate).

$$\begin{aligned} \max_{\mathbf{x}} \quad & \boldsymbol{\mu}'\mathbf{x} \\ \text{s. t.} \quad & \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \leq L^2 \\ & \mathbf{e}'\mathbf{x} = 1 \end{aligned}$$

where $L \in \mathbb{R}_+$ is a predefined risk tolerance level of the investor.

We can generate the efficient frontier by setting different values to L (see Figure 2.1). Note that, we assume that short sales are not allowed (i.e. $x_i \geq 0$ for all i) when we get the efficient frontier depicted in Figure 2.1). Moreover, if the risk free rate is present, then we get the following PAP:

Definition 2.2.6 (Markowitz's PAP).

$$\begin{aligned} \max_{\mathbf{x}} \quad & \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f \\ \text{s. t.} \quad & \mathbf{x}'\mathbf{\Sigma}\mathbf{x} \leq L^2 \end{aligned}$$

where $L \in \mathbb{R}_+$ is a predefined risk tolerance level of the investor.

The portfolios on the efficient frontier generated by PAP given in Definition 2.2.6 are the weighted average of the optimal risky portfolio and the risk free asset. Hence, the efficient frontier for this model is a straight line from the value of the risk-free asset on the y-axis that runs tangent to the frontier through the point with the highest Sharpe Ratio (i.e. optimal risky portfolio or market portfolio). The portion to the left of the optimal risky portfolio on our adjusted frontier relates to a portfolio where the investor will invest a portion of funds in the risk-free asset. This investment will result in the total risky allocation to be less than one, where the difference is invested in the risk-free asset. The points on the frontier to the right of the optimal risky portfolio are portfolios where the investor will borrow of the risk-free asset to provide additional capital to invest in the portfolio (we assume that risk free rate for borrowing and lending are same). This investment allows the weight of the optimal risky portfolio to be greater than one. Note that the total allocation of the risky and risk free investment always add up to one (also see Figure 2.1).

All of the problems given above can be represented as convex optimization problems. We use a *Matlab* package, 'cvx' by [16] to solve these optimization problems numerically.

2.2.1 Obtaining Data

We got historical stock prices for the 10 stocks. We chose the stocks from a variety of sectors over a span of 14 years (Jan. 1, 2000 - Dec. 31, 2013). Selecting stocks from different sectors is the best way to diversify the company based risk (unsystematic risk). The table below lists the stocks we use at this chapter along with their tickers, sectors, average annualized returns, and market capitalization weights. Note that we assume that the risk free rate is 1%.

The result of the Markowitz Model using the data in Table 2.1 is given in Figure 2.1.

Table 2.1: Data from Jan. 1, 2000 - Dec. 31, 2013

Stock	**Ticker**	**Sector**	**Avg. Return**	**Mkt. Weight**
Apple Inc.	AAPL	Consumer Goods	0.25	0.28
Coca-Cola Bottling Co.	COKE	Consumer Goods	0.08	0.0004
Walt Disney Co.	DIS	Services	0.12	0.071
Ford Motor Co.	F	Consumer Goods	0.03	0.03
Goldman Sachs Inc.	GS	Financial	0.13	0.04
Johnson & Johnson	JNJ	Healthcare	0.03	0.15
McDonald's Corp.	MCD	Services	0.1	0.05
Wal-Mart Stores Inc.	WMT	Services	0.04	0.12
Exxon Mobil Corp.	XOM	Basic Materials	0.06	0.23
Yahoo! Inc.	YHOO	Technology	0.0005	0.02

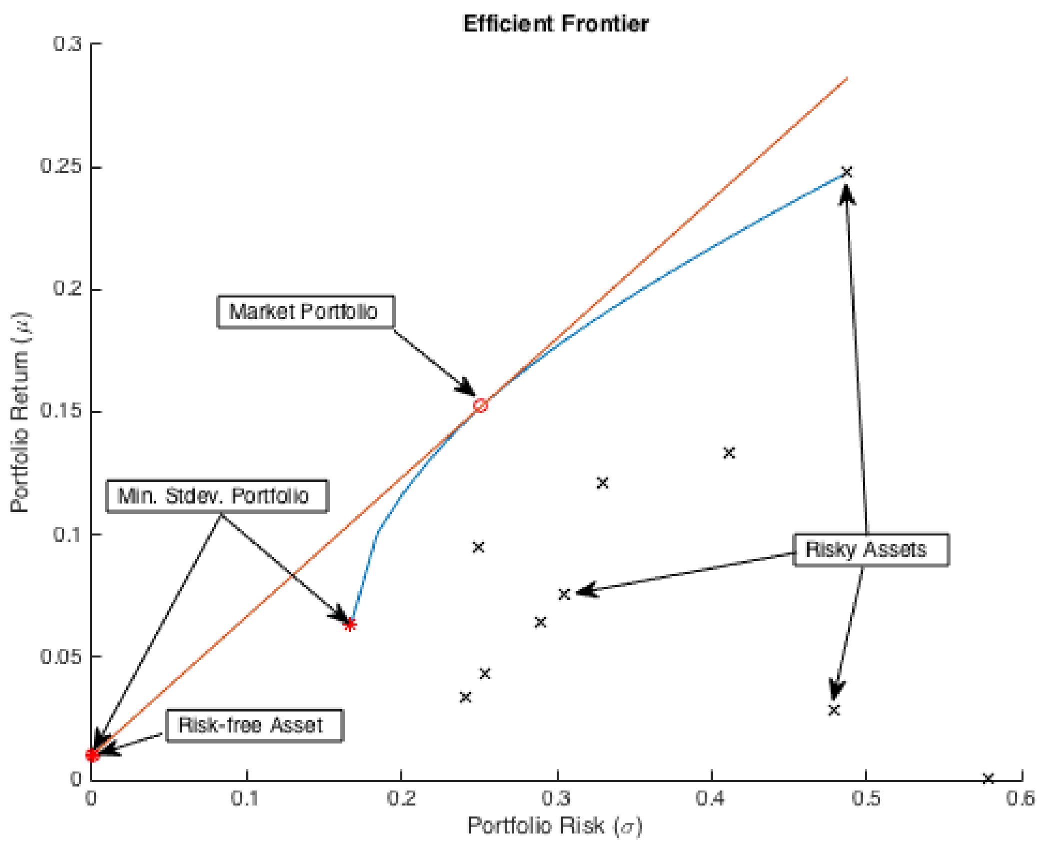

Figure 2.1: Efficient Frontier with and without risk free asset

2.3 Analyzing parameters of Black-Litterman Model

Black-Litterman Model combines investor views with market equilibrium to update the problem parameters. Using the Bayesian framework, BLM estimates the posterior distributions by taking market equilibrium data as the prior distribution and investor views as additional information. For both prior and posterior distributions multivariate normal distribution is assumed. The model assumes that the market is in CAPM equilibrium (see also Sharp [42] [43]). In other words, BLM assumes that every single investor in the market solves the problem given in Definition 2.2.1. The implied equilibrium return vector, $\mathbf{\Pi}$, is the customary equilibrium return vector used in the BLM and can be calculated using inverse optimization. We need to estimate the risk aversion coefficient (λ) and the covariance matrix ($\mathbf{\Sigma}$) to use them in the inverse optimization. In addition to that we need a candidate optimal allocation vector. In the BL framework this is given as the market capitalization weights (i.e. $\mathbf{x}_{market}$). We give two derivations of $\mathbf{\Pi}$ in Proposition 2.3.1 and Proposition 2.3.2.

Proposition 2.3.1. Suppose market is in CAPM equilibrium then if the value of the investor risk reward trade off coefficient, λ, is fixed as

$$\frac{\mu_{market} - r_f}{\sigma^2_{market}},$$

then the adjusted expected return is

$$\mathbf{\Pi} = \lambda \mathbf{\Sigma} \mathbf{x}_{market}, \tag{2.5}$$

where $\mathbf{\Sigma}$ is the historical covariance matrix and $\mathbf{x}_{market}$ is the market weights of the assets under consideration.

Proof. We assume that the market is CAPM equilibrium, therefore, for all risky assets ($i = 1, .., n$) we have:

$$\mu_i = r_f + \beta_i(\mu_{market} - r_f).$$

Since, this is true for all i from 1 to n we have,

$$\boldsymbol{\mu} = r_f\mathbf{e} + \boldsymbol{\beta}(\mu_{market} - r_f).$$

where, the value of $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \frac{Cov(\mathbf{r}, \mathbf{r}'\mathbf{x}_{market})}{\sigma^2_{market}}.$$

Plug the value of $\boldsymbol{\beta}$ to the above equation and get

$$\boldsymbol{\mu} = r_f\mathbf{e} + \frac{Cov(\mathbf{r}, \mathbf{r}'\mathbf{x}_{market})}{\sigma^2_{market}}(\mu_{market} - r_f).$$

This is nothing but

$$\boldsymbol{\mu} - r_f\mathbf{e} = Cov(\mathbf{r}, \mathbf{r}')\mathbf{x}_{market}\frac{(\mu_{market} - r_f)}{\sigma^2_{market}}.$$

Then we fix the value of λ as

$$\lambda = \frac{\mu_{market} - r_f}{\sigma^2_{market}}.$$

We get the desired result;

$$\boldsymbol{\Pi} = \lambda\boldsymbol{\Sigma}\mathbf{x}_{market}.$$

This completes the proof. □

Now, let us give another way to get the adjusted expected return.

Proposition 2.3.2. Suppose every single investor in the market solves the unconstrained PAP given in Definition 2.2.1 then the value of the adjusted expected return is:

$$\boldsymbol{\Pi} = \lambda\boldsymbol{\Sigma}\mathbf{x}_{market}, \tag{2.6}$$

where λ is the risk reward trade off coefficient, $\boldsymbol{\Sigma}$ is the historical covariance matrix, and $\mathbf{x}_{market}$ is the market weights of the assets under consideration.

Proof. The problem given in Definition 2.2.1 is an unconstrained optimization and can be represented as a convex optimization problem. Therefore, first order necessary condition is also sufficient. Hence, if we take the derivate with respect to the vector $\mathbf{x}$ and set the result equal to zero, we have

$$\boldsymbol{\mu} - r_f\mathbf{e} - \lambda\boldsymbol{\Sigma}\mathbf{x} = 0. \tag{2.7}$$

If we take $\mathbf{x}_{market}$ as the candidate optimal solution then by the virtue of the inverse optimization we get

$$\boldsymbol{\Pi} = \lambda\boldsymbol{\Sigma}\mathbf{x}_{market}. \tag{2.8}$$

This completes the proof. □

Note that as long as we fix the value of the risk reward trade-off as

$$\lambda = \frac{\mu_{market} - r_f}{\sigma^2_{market}},$$

then assuming CAPM or PAP given in Definition 2.2.1 does not matter. Estimating λ can also be subjective. For example, He and Litterman [19] fix λ to 1.25 (the reason for that can be found in their paper).

Suppose that our investor has no idea what is going in the market. However, using the inverse optimization she can update the expected return vector from $\boldsymbol{\mu}$ to $\boldsymbol{\Pi}$ and $\boldsymbol{\Sigma}$ to $(1+\tau)\boldsymbol{\Sigma}$. Then she can use the Markowitz model to get the optimal allocation vector. Note that after updating the return vector we get a new efficient frontier as well (see Figure 2.2, where $\tau = 1/14$). Moreover, the value of $\boldsymbol{\Pi}$ is based off of market equilibrium assumption and historical data. Since the efficient frontier derived from the historical data (red line) lies above the market equilibrium-based efficient frontier (blue line), it is reasonable to say that updating the expected return using the inverse optimization shrinks the efficient frontier.

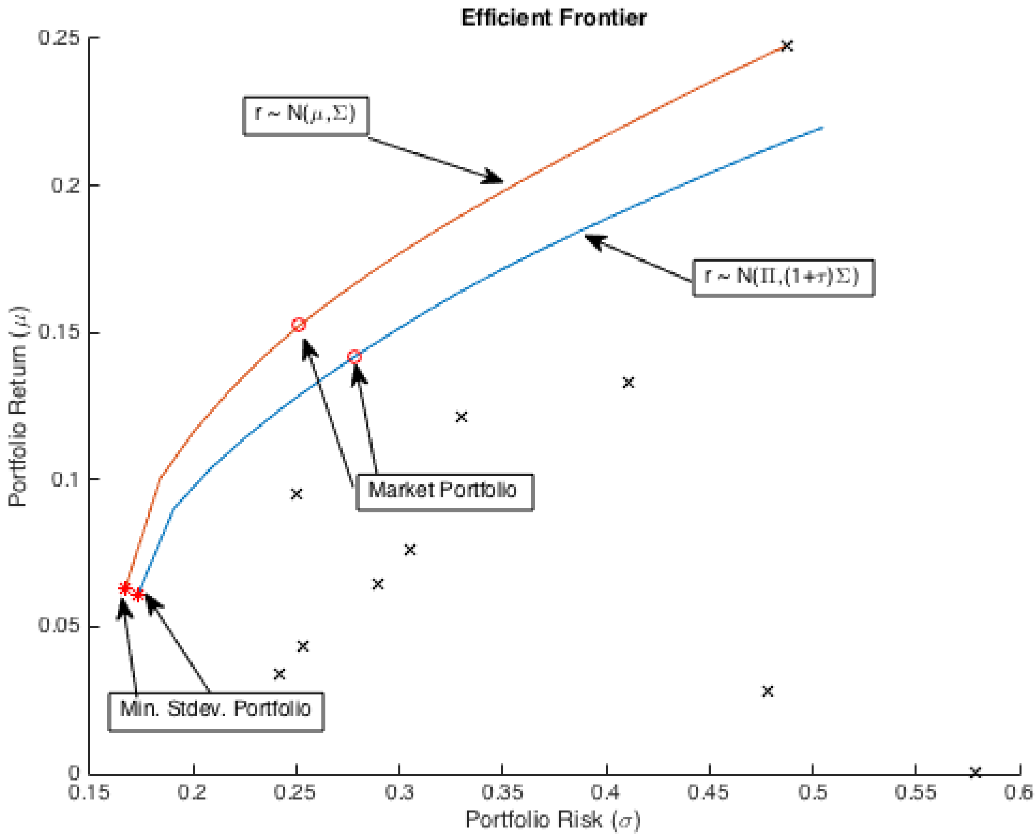

Figure 2.2: Markowitz, $N(\boldsymbol{\mu},\boldsymbol{\Sigma})$ vs. $N(\boldsymbol{\Pi},(1+\tau)\boldsymbol{\Sigma})$

Let us continue with the other inputs of BLM; the model takes historical data ($\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, $\mathbf{x}_{market}$), view portfolio ($\mathbf{P}$) and it's value ($\mathbf{q}$), and confidence in these views ($\boldsymbol{\Omega}$, τ) to derive an updated mean return vector (i.e. $\boldsymbol{\mu}_{BL}$), and covariance matrix (i.e. $\boldsymbol{\Sigma}_{BL}$). Using BLM investors can get a portfolio allocation vector based on their view(s). Following He and Litterman [19], the updated parameters can be represented as:

$$\begin{aligned}\boldsymbol{\mu}_{BL} &= \left((\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}\left((\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{q}\right), \\ \boldsymbol{\Sigma}_{BL} &= \boldsymbol{\Sigma} + \boldsymbol{\Sigma}^{\boldsymbol{\mu}}_{BL} \text{ (where } \boldsymbol{\Sigma}^{\boldsymbol{\mu}}_{BL} = \left((\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1})).\end{aligned}$$

2.3.1 Using Inverse Optimization to Get BLM

In this subsection, we follow the arguments given in Bertsimas et. al. (2012). They show that one can get Black-Litterman type estimations using only Inverse Optimization (see Proposition 2.3.3). They assume that all the investors in the market solve the unconstrained mean variance PAP. Moreover, the candidate optimal solution for the inverse problem is the market weights (i.e. $\mathbf{x}_{market}$). They combine the market equilibrium and private information to feed the inverse model. Hence, the nominal values for the inverse problem are the market weights and the investor view(s) respectively.

Proposition 2.3.3 (Proposition 2 of Bertsimas et. al. (2012))**.** BLM estimation for the mean vector can be directly found using inverse optimization and it is given as:

$$\boldsymbol{\mu}_{BL} = \boldsymbol{\Sigma}^{\mu}_{BL} \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix}' \boldsymbol{\Omega}_{\tau}^{-1} \begin{bmatrix} \boldsymbol{\Pi} \\ \mathbf{q} \end{bmatrix} \tag{2.9}$$

where,

$$\boldsymbol{\Sigma}^{\mu}_{BL} = \left[\begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix}' \boldsymbol{\Omega}_{\tau}^{-1} \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix} \right]^{-1}, \tag{2.10}$$

$$\boldsymbol{\Omega}_{\tau} = \begin{bmatrix} \tau\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega} \end{bmatrix}, \tag{2.11}$$

and $\mathbf{I}$ is an identity matrix.

Proof. We give sketch of the proof from Proposition 2 of Bertsimas et. al. (2012). Let us define the BLM in the sense of inverse optimization:

$$\min_{\boldsymbol{\mu},\boldsymbol{\Sigma},t} \left\{ t : \left\| \begin{bmatrix} \boldsymbol{\mu} - r_f - 2\delta\boldsymbol{\Sigma}\mathbf{x}_{mkt} \\ \mathbf{P}\boldsymbol{\mu} - \mathbf{q} \end{bmatrix} \right\| \leq t, \boldsymbol{\Sigma} \geq 0 \right\} \tag{2.12}$$

and, consider this problem under the weighted l_2 norm $\|z\|_2^{\mathbf{\Omega}_\tau} = \sqrt{z'\mathbf{\Omega}_\tau^{-1}z}$ where $\mathbf{\Sigma}$ is fixed. Then we get;

$$\min_{\boldsymbol{\mu}} \left[\begin{pmatrix} \mathbf{I} \\ \mathbf{P} \end{pmatrix} \boldsymbol{\mu} - \begin{pmatrix} \mathbf{\Pi} \\ \mathbf{q} \end{pmatrix} \right]' \mathbf{\Omega}_\tau^{-1} \left[\begin{pmatrix} \mathbf{I} \\ \mathbf{P} \end{pmatrix} \boldsymbol{\mu} - \begin{pmatrix} \mathbf{\Pi} \\ \mathbf{q} \end{pmatrix} \right]. \tag{2.13}$$

This problem can be represented as:

$$\min_{\boldsymbol{\mu}} \left\| \mathbf{\Omega}_\tau^{-1/2} \begin{pmatrix} \mathbf{I} \\ \mathbf{P} \end{pmatrix} \boldsymbol{\mu} - \mathbf{\Omega}_\tau^{-1/2} \begin{pmatrix} \mathbf{\Pi} \\ \mathbf{q} \end{pmatrix} \right\|_2^2 \tag{2.14}$$

Problem given in (2.14) is nothing but the least squares problem and can be represented as:

$$\min_{\mathbf{y}} \|\mathbf{A}\mathbf{y} - \mathbf{b}\|. \tag{2.15}$$

In Equation (2.15)

$$\mathbf{A} = \mathbf{\Omega}_\tau^{-1/2} \begin{pmatrix} \mathbf{I} \\ \mathbf{P} \end{pmatrix} \boldsymbol{\mu},$$

and

$$\mathbf{b} = \mathbf{\Omega}_\tau^{-1/2} \begin{pmatrix} \mathbf{\Pi} \\ \mathbf{q} \end{pmatrix}.$$

Closed form solution is given as:

$$\mathbf{y}^* = (\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'\mathbf{b}. \tag{2.16}$$

This is nothing but

$$\boldsymbol{\mu}_{BL} = \mathbf{\Sigma}_{BL}^{\mu} \begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix}' \mathbf{\Omega}_\tau^{-1} \begin{bmatrix} \mathbf{\Pi} \\ \mathbf{q} \end{bmatrix}$$

Hence, we get the mean vector of the Black-Litterman Model. □

Bertsimas et. al. (2012) does not provide how to get the estimation for the covariance matrix. Hence, we provide the Proposition below to give the result for the covariance matrix.

Proposition 2.3.4. Following the same arguments and the notation presented in Proposition 2.3.3, the covariance matrix for the Black-Litterman Model is given as:

$$\mathbf{\Sigma}_{BL} = \mathbf{\Sigma} + (\mathbf{A}'\mathbf{A})^{-1}.$$

Proof. Consider the problem

$$\min_{\mathbf{y}} \|\mathbf{A}\mathbf{y} - \mathbf{b}\|.$$

The error on the estimation of $\mathbf{y}$ is given by $(\mathbf{A}'\mathbf{A})^{-1}$ (by sections E.5 and E.6 of Bertsekas (2005)). Following the same notation presented in Proposition 2.3.3 and the arguments above, we can derive $\mathbf{\Sigma}_{BL}$:

$$\begin{aligned} \mathbf{\Sigma}_{BL} &= \mathbf{\Sigma} + \mathbb{E}[(\mathbf{y} - \mathbf{y}^*)(\mathbf{y} - \mathbf{y}^*)'] \\ &= \mathbf{\Sigma} + (\mathbf{A}'\mathbf{A})^{-1}. \end{aligned} \tag{2.17}$$

This is nothing but

$$\mathbf{\Sigma}_{BL} = \mathbf{\Sigma} + \mathbf{\Sigma}_{BL}^{\mu}.$$ □

In the following subsections, we analyze the sensitivity of the problem parameters: λ, $\mathbf{q}$, $\mathbf{\Omega}$, and τ.

2.3.2 Analyzing the sensitivity of λ

The parameter λ is the risk aversion coefficient. He and Litterman [19] fix it's value to 1.25. Note that λ is only used when we calculate the value of $\mathbf{\Pi}$ (i.e. the implied equilibrium return vector).

Consider the PAP given in Definition 2.2.6 and assume that short sales are not allowed (i.e. $x_i \geq 0$ for all i). In addition to that suppose the investor does not have any view. Then, an increase in λ creates a vertical shift in the mean variance efficient frontier as seen in Figure 2.3. It can be observed that as λ increases, the market portfolio shifts to the south east direction.

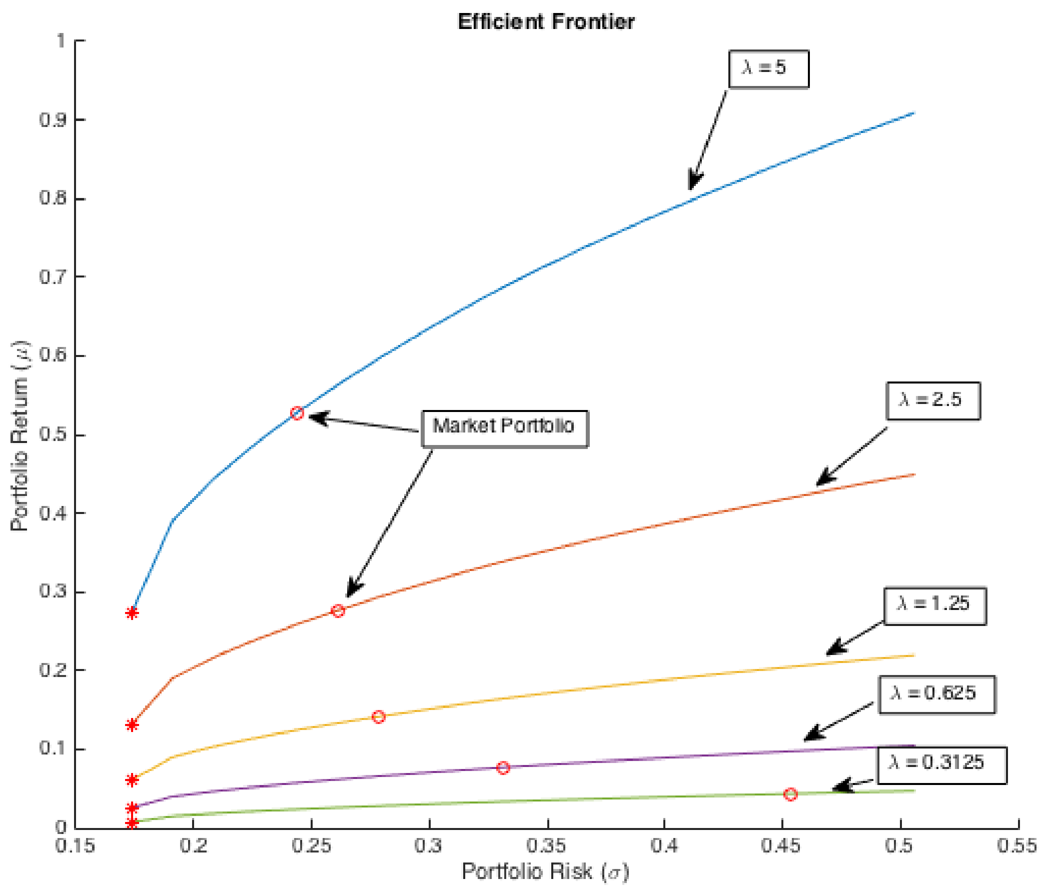

Figure 2.3: Effect of Varying λ

2.3.3 Analyzing the sensitivity of τ

The parameter τ represents the level of uncertainty an investor holds in the prior distribution. There is no convention in the literature about how to estimate the value of τ. For instance, Black and Litterman [9] says that τ should be close to 0 while Satchell and Scowcroft [40] states that τ should be closer to 1. Some authors, such as Meucci [30], have even proposed a model without the use τ to avoid this discrepancy (this model known as the Alternate Model).

Consider the case where the investor does not have any view. We can also assume that the investor has a view however her uncertainty about her view is ver large. Previously, we stated that as the investor uncertainty gets larger, the posterior covariance matrix goes to $(1+\tau)\boldsymbol{\Sigma}$. If that is case then, as τ gets closer to 0, then the posterior covariance becomes the historical covariance. Hence the BL efficient frontier converges to that of the mean variance model with updated $\boldsymbol{\mu}$ (i.e. $\boldsymbol{\Pi}$). Moreover, as τ increases from 0, uncertainty about our prior distribution increases, and thus, increases our risk due to the additional uncertainty. The effect of τ on the BL efficient frontier is a horizontal shift that moves efficient frontier right as τ increases. Our intuition is that the BL efficient frontier illustrates a horizontal shift of the constrained portfolio allocation problem with $r \sim N(\boldsymbol{\Pi}, \boldsymbol{\Sigma})$ efficient frontier by a scale of $\sqrt{1+\tau}$. This movement can be seen in Figure 2.4.

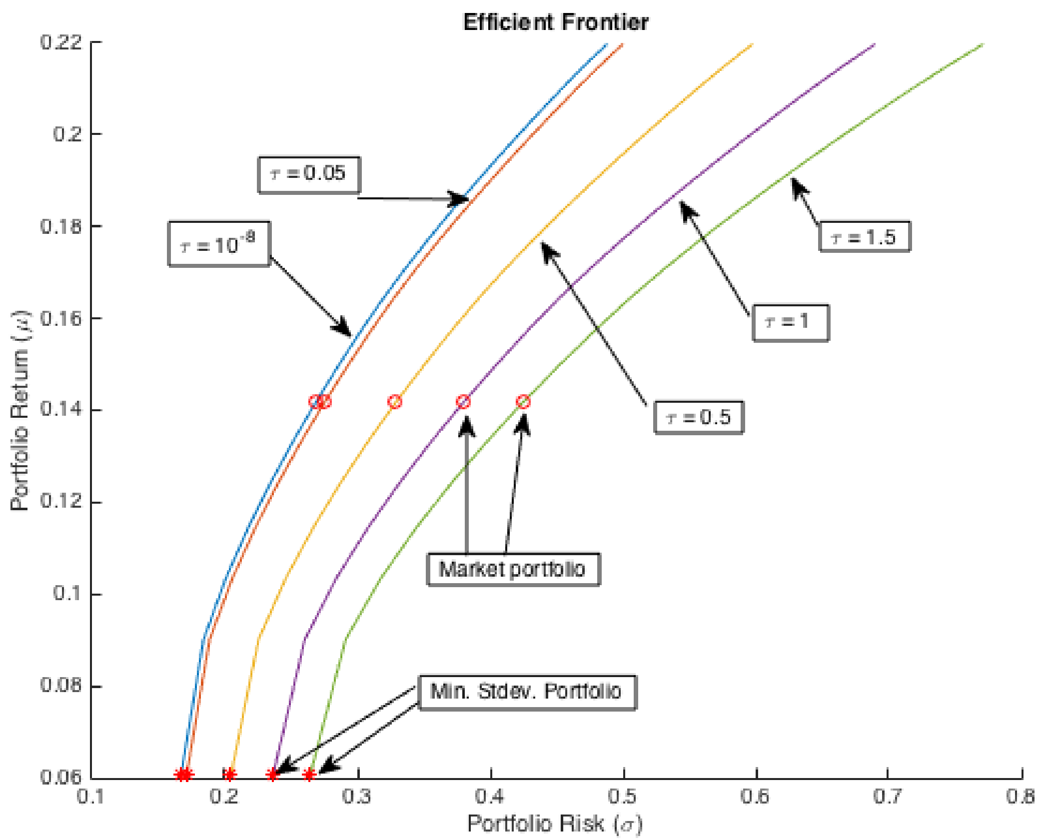

Figure 2.4: Effect of Varying τ

2.3.4 Analyzing the sensitivity of $\mathbf{\Omega}$

The $\mathbf{\Omega}$ enables investors to state their level of uncertainty about their views. If investor has k views then, $\mathbf{\Omega}$ is a $k \times k$ diagonal matrix where the level of uncertainty in each view is stated.

Estimating the value of $\mathbf{\Omega}$ is another challenge for the use of this model. Hence, He and Litterman [19] use the following idea to fix the value of the uncertainty matrix:

$$\mathbf{\Omega} = \omega \tau \mathbf{P}' \mathbf{\Sigma} \mathbf{P}. \tag{2.18}$$

Here, $\omega \in [0, \infty]$ represents the general uncertainty of the investors overall view portfolio(s). It can be observed that as ω approaches 0, the model will converge towards a frontier adjusted for complete certainty in the views. Conversely as the value of ω increase, the investor is less and less confident in the views, and the model shall move towards the MVO model adjusted for the given τ and $\mathbf{\Pi}$. If τ close to 0 and ω is increasing to infinity, the model converges to the MVO model with mean returns, $\mathbf{\Pi}$, and covariance matrix, $\mathbf{\Sigma}$ (see Figure 2.5). Here we give the following view to the model $P = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ and $q = 0.1$.

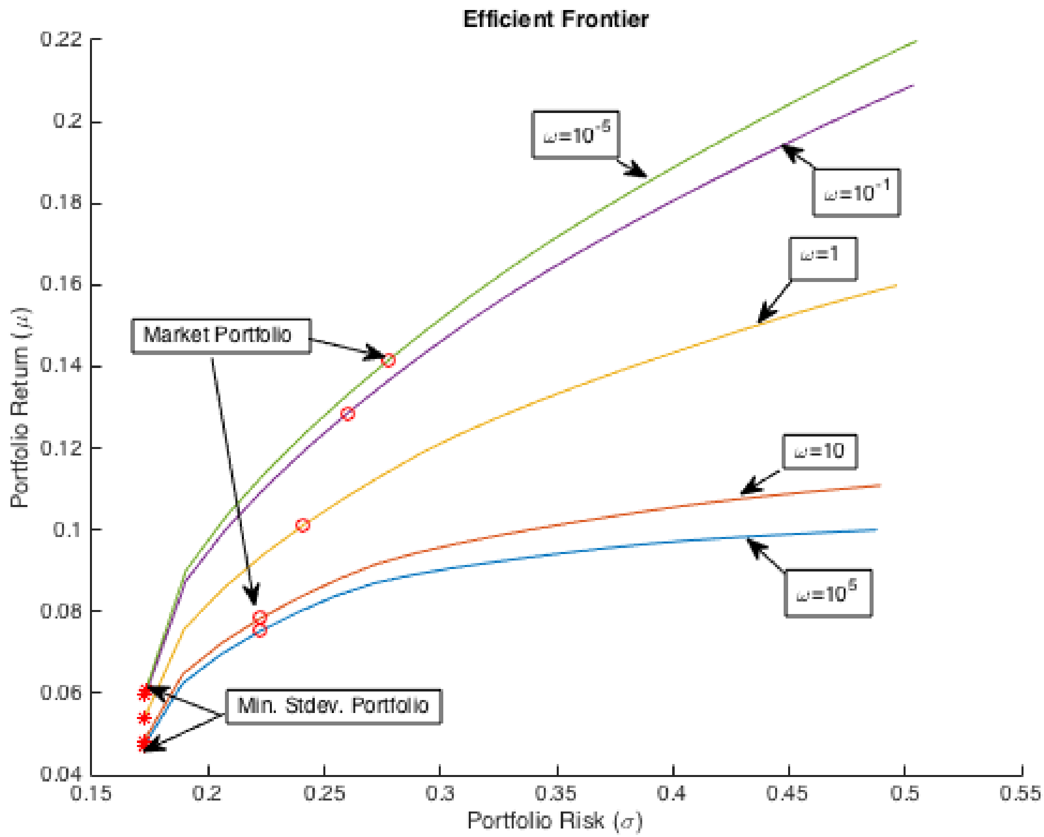

Figure 2.5: Testing Sensitivity of Ω

2.3.5 Analyzing the sensitivity of q

q is the investors expected return on her view portfolio. If investor has k views then, **q** is a $1 \times k$ vector. Estimating the value of **q** is also a challenge to the investor. If an investor is not certain about her view then she can use the method proposed by Fusai and Meucci [15] and adjust her view accordingly. It can be observed that as **q** increases, the efficient frontier gets larger. On the other hand, the movement of the market portfolio is interesting. As **q** increases market portfolio moves toward the north east direction. However, when **q** takes values between 0.025 and 0.05 it moves towards north (see Figure 2.6). Here we give the following view to the model $P = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$ and $\omega = 1$.

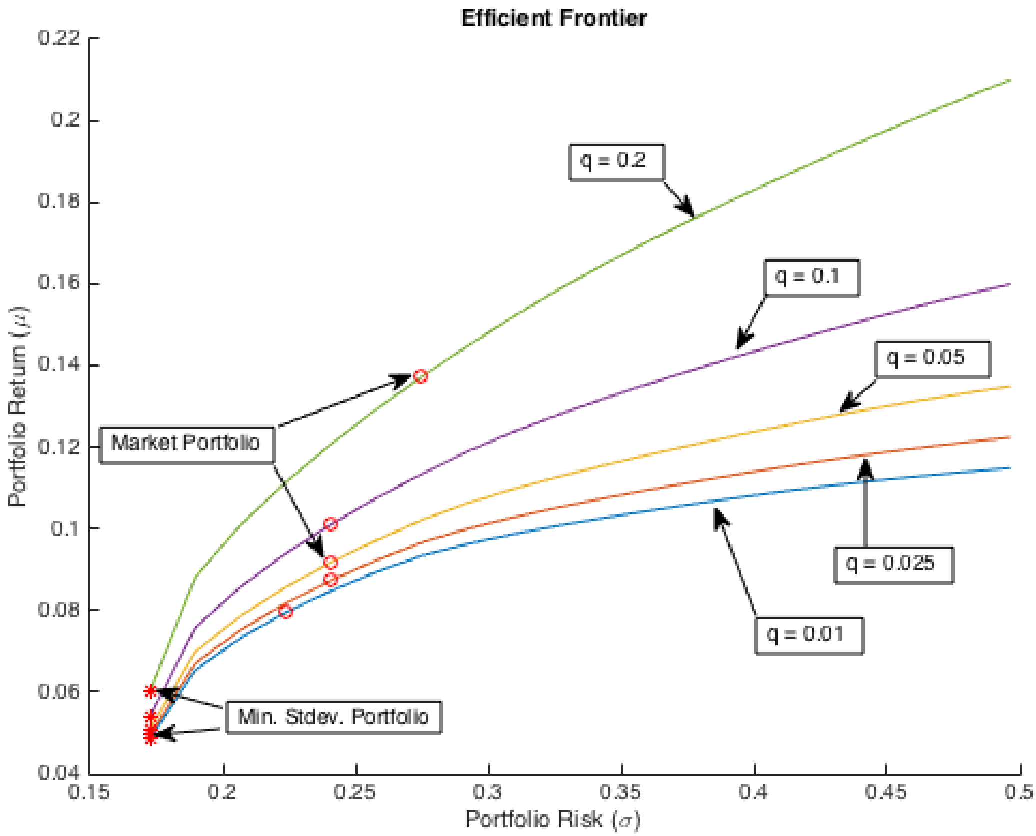

Figure 2.6: Testing Sensitivity of **q**

2.4 Conclusion

BLM demonstrates the elements necessary to construct a diversified portfolio that incorporates investor views with the CAPM equilibrium. The model allows for investors to quantitatively incorporate their views. In this chapter we tested the sensitivity of various inputs and observed the followings: An increase in λ results a vertical shift in the mean variance efficient frontier; as ω approaches 0, the model converges to a frontier adjusted for complete certainty in the views, conversely as the values in ω increase, the investor is less and less confident in the views, and the model move towards the Markowitz Model adjusted for the given τ; as τ increases BL efficient frontier horizontally shifts; as ω increases the investor is less and less confident in the views, and BL efficient frontier moves towards the MVO model efficient frontier adjusted for the given τ and $\mathbf{\Pi}$; as the values in $\mathbf{q}$ increases the efficient frontier gets bigger.

In this chapter, we only considered U.S. based stocks. Exploring other asset classes and expanding to foreign markets could be interesting directions to extend this research. Consideration of various asset classes should create more diverse portfolios and reduce volatility. In addition to that we assumed that the market is composed of the stocks in our data set. An investor can consider investing on the different sectors instead of stocks in order to get a well diversified portfolio and get a better estimation when updating the expected return of the asset returns.

Chapter 3

Closed-Form Solutions for Black-Litterman Models with Conditional Value at Risk

3.1 Introduction

In the classical Markowitz portfolio optimization model (also known as the mean-variance optimization model), the historical mean vector and covariance matrix of the risky assets are used to obtain the optimal portfolio allocation vector (Markowitz [29]). Mean-variance optimization (MVO) considers only the first two moments of the return distribution. This restriction was consistent with reality if asset returns followed a normal distribution (Fabozzi et. al.[13] and Meucci [30]). But we know that asset return's coskewness and cokurtosis values differ from normality (see more at Harvey et. al. [18] and Jondeau and Rockinger [24]). Moreover, historical data tells us that the covariance matrix is a random variable by itself (Hull [20]). Additionally, historical returns are not good estimates of the future returns and they are very difficult to estimate. Furthermore, the mapping between expected returns and portfolio weights are complicated. In addition to that, optimal portfolio weights are highly sensitive to the parameters of the optimization problem (see Meucci [30] and Fabozzi et. al. [13]). Because of the reasons stated above, Markowitz's optimal allocation vectors are lack of diversification and/or has corner solutions.

Black-Litterman Model (BLM) [9] proposes a parameter estimation technique in which the investor's view can be integrated with the historical performance to estimate the problem parameters. There are two different ways to approach this model in the literature; the original model (canonical model) and the later one (alternate model).

Canonical model combines the intuitions about the selected assets or asset classes of the investors with the historical information of the market in order to update the mean vector and covariance matrix. This is done by using the bayesian framework. BLM assumes that the expected returns are random variables themselves. These random variables are normally distributed and centered at the CAPM equilibrium returns with historical covariance matrix. BLM considers CAPM equilibrium returns as the prior distribution of the asset returns. Furthermore, investors have views on the assets (or asset classes) in the market and take the CAPM equilibrium as the reference point to specify their view(s). Those views are the additional information to the bayesian framework. On other hand, in the alternate model, market factors are considered instead of asset returns. In addition to that, the expected value of the market factors are not random variables (see also Walters [47] and Meucci [30]). Both models, use the updated parameters in the mean-variance framework to get the optimal portfolio allocation vector. Investor view(s) has two forms; relative view(s) (the sum of the weights of views add up to zero) and absolute view(s) (there is only one asset in this form of view, therefore, that asset's weight is always equal to one). Some of the authors use market capitalization weights when specifying the views (i.e. He and Litterman [19] and Idzorek T. [22]). On the other hand, some of them such as Meucci A. [31] and Satchell and Scowcroft [40] use an equal weighted scheme.

Most of the papers on the BLM use normal distributions with variance as the risk measure. In this chapter, we consider elliptical distributions for asset returns and use conditional value-at-risk as the risk measure. Note that CAPM holds as long as return distributions are elliptical (see Meucci [30] and references therein). Derivation of the posterior distribution for this case is given by Xiao and Valdez [48] (see also Proposition 3.5.1). Although, CVaR has become more and more popular as a coherent risk measure in the financial industry, derivation of the optimal solution analytically is extremely difficult under CVaR constraint. Hence, we propose an efficient approximation algorithm for optimization problems with CVaR constraint. Then, based on the approximation algorithm, we derive the closed-form solutions for the BLM with elliptical distributions and CVaR. To our best knowledge, no closed-form solutions for BLM with CVaR have been derived before.

3.2 Portfolio Allocation Problem (PAP)

We consider a market with n risky assets. Risky asset returns are denoted by the random vector $\mathbf{r} \in \mathbb{R}^n$ which is defined on the probability space $(\Omega, \mathcal{F}, P)$. Vectors are defined as column vectors unless otherwise stated. The mean vector and the covariance matrix of risky asset returns are denoted by $\boldsymbol{\mu} = \mathbb{E}[\mathbf{r}] \in \mathbb{R}^n$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ respectively. We assume that $\boldsymbol{\mu}$ is finite, and $\boldsymbol{\Sigma}$ is finite and positive definite. The risk free rate of return is denoted as $r_f \in \mathbb{R}_+ \cup 0$. Moreover, $\mathbf{x} \in \mathbb{R}^n$ is the portfolio weight vector of risky assets and $(1 - \mathbf{e}'\mathbf{x})$ is the allocation on the risk-free asset, where $\mathbf{e} = (1, 1, \cdots, 1)'$ is a vector of ones in $\mathbb{R}^n$.

First we define the space of portfolio returns for a given number of available risky assets and a risk-free asset.

Definition 3.2.1 (Space of Portfolio Returns)**.**

$$\mathcal{V} = \{\tilde{v} \in \mathbb{R} : \exists (r_f, \mathbf{x}) \text{ s.t. } \tilde{v} = v_0 + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}\} \tag{3.1}$$
$$\text{where } v_0 = \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f.$$

Note that in Definition 3.2.1 each portfolio return is represented as a combination of return with certainty (i.e. $\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f$) and return with uncertainty (i.e. $(\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}$).

Proposition 3.2.1. $\mathcal{V}$ is a convex set.

Proof. Let $\tilde{v}_1 \in \mathcal{V}$, $\tilde{v}_2 \in \mathcal{V}$ and $\gamma \in \mathbb{R}$ s.t. $0 \leq \gamma \leq 1$. Let $\tilde{v}_3 \equiv \gamma \tilde{v}_1 + (1-\gamma)\tilde{v}_2$. Then,

$$\begin{aligned}
\tilde{v}_3 &= \gamma \tilde{v}_1 + (1-\gamma)\tilde{v}_2 \\
&= \gamma(v_0^1 + \mathbf{z}'\mathbf{x}^1) + (1-\gamma)(v_0^2 + \mathbf{z}'\mathbf{x}^2) && (\text{where } \mathbf{z} = \mathbf{r} - \boldsymbol{\mu} \text{ and } \mathbf{x}^1, \mathbf{x}^2 \in \mathbb{R}) \\
&= \gamma v_0^1 + (1-\gamma)v_0^2 + \gamma(\mathbf{z}'\mathbf{x}^1) + (1-\gamma)(\mathbf{z}'\mathbf{x}^2) \\
&= v_0^3 + \gamma(\mathbf{z}'\mathbf{x}^1) + (1-\gamma)(\mathbf{z}'\mathbf{x}^2) && (\text{where } v_0^3 = \gamma v_0^1 + (1-\gamma)v_0^2) \\
&= v_0^3 + \mathbf{z}'\left(\gamma \mathbf{x}^1 + (1-\gamma)\mathbf{x}^2\right) \\
&= v_0^3 + \mathbf{z}'\mathbf{x}^3 && (\text{where } \mathbf{x}^3 = \gamma \mathbf{x}^1 + (1-\gamma)\mathbf{x}^2)
\end{aligned}$$

Hence, $\tilde{v}_3 \in \mathcal{V}$, therefore, $\mathcal{V}$ is a convex set. □

Definition 3.2.2 (Affine Function)**.** Let $\mathbf{z} = \mathbf{r} - \boldsymbol{\mu}$ and $\mathbf{x} \in \mathbb{R}^n$, $v_0 \in \mathbb{R}$. Then, every function of the form

$$f(\mathbf{x}) = v_0 + \langle \mathbf{z}, \mathbf{x} \rangle$$

is an affine function by Theorem 4.1 of Rockafellar [36].

Let us continue with the definition of the constrained Markowitz's portfolio allocation problem (PAP) (Markowitz [29]):

Definition 3.2.3 (Markowitz's PAP)**.**

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq L\}, \tag{3.2}$$

where $L \in \mathbb{R}_+$ is a predefined risk tolerance level of the investor.

In Markowitz's PAP, the variance is used as the risk measure. There are other risk measures that are widely used, such as value at risk (VaR) and conditional value at risk (CVaR).

Definition 3.2.4 (VaR)**.** Given $\alpha \in (0, 1)$ and a random variable (i.e. Y) the value at risk of the random variable with confidence level α is,

$$VaR_\alpha(\mathrm{Y}) = \inf\{\mathrm{y} \in \mathbb{R} : \mathbb{P}(\mathrm{y} + \mathrm{Y} \leq 0) \leq 1 - \alpha\}.$$

Definition 3.2.5 (PAP with VaR)**.**

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : VaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \leq L\} \tag{3.3}$$

VaR_α is not a coherent risk measure (see Artzner et. al. [2]). In particular, diversification benefit may not be present under VaR. On the other hand, CVaR is coherent (properties of CVaR can be found in Rockafellar and Uryasev [37] and [38]). This is the main reason that CVaR is a very popular risk measure. CVaR sometimes called as mean short fall or tail-VaR as well. Although, their definitions are different, they give the same result under continuous random variables (Krokhmal et. al. [26]).

Definition 3.2.6 (CVaR)**.** Given $\alpha \in (0, 1)$ and a random variable (i.e. Y) the conditional value at risk of the random variable with confidence level α is,

$$CVaR_\alpha(\mathrm{Y}) = -\mathbb{E}[\mathrm{Y}|\mathrm{Y} \leq -VaR_\alpha(\mathrm{Y})].$$

Here we consider the PAP with CVaR:

Definition 3.2.7 (PAP with CVaR)**.**

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \leq L\}. \tag{3.4}$$

More generally, we can use a generic coherent risk measure (denoted by $\rho(\tilde{v})$) instead of CVaR. We give the general portfolio allocation problem (GPAP) using generic coherent risk measure next.

Definition 3.2.8 (GPAP with generic coherent risk measure ρ).

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \rho((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \leq L\} \tag{3.5}$$

We want to solve the PAP given in Definition 3.2.7. CVaR is a coherent risk measure so we can use results from Natarajan et. al. [32] to find the robust counterpart of the CVaR constraint (with elliptical uncertainty sets to be specified below). In other words, we can convert the GPAP with generic coherent risk measure given in Definition 3.2.8 to the GPAP with robust optimization given in Definition 3.2.9 (see below) if a certain condition holds and vice versa (see Natarajan et. al. [32]). The closed form solution of the robust optimization with elliptical uncertainty sets are available as we mention below. Moreover, if we assume an elliptical distribution then we can get the closed form representation of CVaR (see Proposition 3.3.4 below).

We can use uncertainty sets in order to model the return with uncertainty (i.e. $(\mathbf{r} - \boldsymbol{\mu})$). In particular, we use elliptical uncertainty sets:

$$\mathcal{U}_\beta = \{\mathbf{r} - \boldsymbol{\mu} : (\mathbf{r} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{r} - \boldsymbol{\mu}) \leq \beta^2\}$$

where β is the scaling parameter which models the risk averseness of the investor from the deviation of the realized returns from their forecasted values.

The reasons of using elliptical uncertainty sets is twofold: First, when uncertainty set is elliptical then the robust programming can be converted into conic programming and the closed form solution exist (Ben Tal and Nemirovski [5]); second, elliptical uncertainty sets can be used for leptokurtic behavior of asset returns.

Once we define the uncertainty set then we can find the robust counterpart of the problem on hand. The robust counterpart is a deterministic problem. Now, we are ready to state the GPAP in the sense of robust optimization.

Definition 3.2.9 (GPAP with robust optimization)**.**

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f : (\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f \geq -L \ \forall \mathbf{r} - \boldsymbol{\mu} \in \mathcal{U}_\beta \ \}$$

The process of finding the robust counterpart is straight forward. We first start with defining the robust counterpart risk measure, $\eta_{\mathcal{U}_\beta}(\tilde{v})$, given by Natarajan et. al. [32] and it can be represented as:

$$\eta_{\mathcal{U}_\beta}(v_0 + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}) = -\min_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_\beta}(v_0 + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}) \tag{3.6}$$

Definition 3.2.10. The PAP with robust counterpart risk measure is given by

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f : \eta_{\mathcal{U}_\beta}\left((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f\right) \leq L\}$$

Let us show that problems given in Definitions 3.2.9 and 3.2.10 are the same problems. We are interested with the representation of the constraint, therefore, we consider the constraint given in Definition 3.2.9. In other words, we are interested in

$$(\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f \geq -L \ \forall \mathbf{r} - \boldsymbol{\mu} \in \mathcal{U}_\beta \ .$$

This can be written as:

$$(r_f\mathbf{e} - \mathbf{r})'\mathbf{x} - r_f \leq L \ \forall \mathbf{r} - \boldsymbol{\mu} \in \mathcal{U}_\beta$$

$$\Leftrightarrow -(\mathbf{r} - \boldsymbol{\mu})'\mathbf{x} - \boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f \leq L \ \forall \mathbf{r} - \boldsymbol{\mu} \in \mathcal{U}_\beta$$

$$\Leftrightarrow \max_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_\beta}\left(-(\mathbf{r} - \boldsymbol{\mu})'\mathbf{x} - \boldsymbol{\mu}'\mathbf{x} - (1-\mathbf{e}'\mathbf{x})r_f\right) \leq L$$

$$\Leftrightarrow \max_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_\beta} -\left((\mathbf{r} - \boldsymbol{\mu})'\mathbf{x} + \boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f\right) \leq L$$

$$\Leftrightarrow -\min_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_\beta}\left((\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}+\boldsymbol{\mu}'\mathbf{x}+(1-\mathbf{e}'\mathbf{x})r_f\right)\leq L$$

$$\Leftrightarrow \eta_{\mathcal{U}_\beta}\left(\boldsymbol{\mu}'\mathbf{x}+(1-\mathbf{e}'\mathbf{x})r_f+(\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}\right)\leq L$$

$$\Leftrightarrow \eta_{\mathcal{U}_\beta}\left((\mathbf{r}-r_f\mathbf{e})'\mathbf{x}+r_f\right)\leq L.$$

Proposition 3.2.2. If an elliptical uncertainty set is assumed in the constraint of the PAP given in Definition 3.2.9 then this PAP can be represented as:

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x}+(1-\mathbf{e}'\mathbf{x})r_f : -\left(\boldsymbol{\mu}'\mathbf{x}+(1-\mathbf{e}'\mathbf{x})r_f\right)+\beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}\leq L\}$$

Proof. Let us start with considering the constraint of the PAP given in Definition 3.2.9, in other words,

$$(\mathbf{r}-r_f\mathbf{e})'\mathbf{x}+r_f\geq -L\ \forall\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_\beta.$$

Note that this constraint is nothing but:

$$\eta_{\mathcal{U}_\beta}\left(\boldsymbol{\mu}'\mathbf{x}+(1-\mathbf{e}'\mathbf{x})r_f+(\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}\right)\leq L.$$

We can use Equation (3.6) to get

$$\eta_{\mathcal{U}_\beta}(v_0+(\mathbf{r}-\boldsymbol{\mu})'\mathbf{x})=-\min_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_\beta}(v_0+(\mathbf{r}-\boldsymbol{\mu})'\mathbf{x})$$

where $v_0=\boldsymbol{\mu}'\mathbf{x}+(1-\mathbf{e}'\mathbf{x})r_f$. We are minimizing an affine function,

$$\left((\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}+\boldsymbol{\mu}'\mathbf{x}+(1-\mathbf{e}'\mathbf{x})r_f\right),$$

over a single ellipsoidal constraint. This optimization problem has a closed form solution and the closed form solution is given by Ben Tal and Nemirovski[6]:

$$\min_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{E}_\beta}(v_0 + (\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}) = v_0 - \beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}. \tag{3.7}$$

Therefore,

$$-\left(\min_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{E}_\beta}(v_0 + (\mathbf{r}-\boldsymbol{\mu})'\mathbf{x})\right) = -v_0 + \beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}.$$

Hence,

$$\eta_{\mathcal{U}_\beta}(v_0 + (\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}) = -v_0 + \beta\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}$$

Use value of v_0 and get the desired result. □

Before we proceed, we would like to state some results when we assume multivariate normal distribution for return vector for PAP with VaR and CVaR. The proposition stated below gives a close form solution for VaR risk measure, defined on Definition 3.2.4, if a multivariate normal distribution is assumed for the asset returns.

Proposition 3.2.3. Let the asset return distribution, $\mathbf{r} \in \mathbb{R}^n$, follows a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ (i.e. $\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$) and $\mathbf{x}$ be the weight distribution of the risky assets. Then

$$VaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) = -(\mathbf{x}'\boldsymbol{\mu} + (1 - \mathbf{e}'\mathbf{x})r_f) + z_{1-\alpha}\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}.$$

Proof. Let us start with considering a standard normal random variable (i.e. $\tilde{v} \sim N(0,1)$). Note that $\tilde{v}$ is one dimensional. Now, suppose that the portfolio return distribution can be modeled as $\tilde{v}$. (i.e. portfolio return $\sim N(0,1)$). Then we have

$$VaR_\alpha(x) = z_{1-\alpha}$$

by definition of VaR. Next, suppose that $\tilde{v} \sim N(\mu, \sigma^2)$. Then, it is easy to show that

$$VaR_\alpha(x) = -\mu + z_{1-\alpha}\sigma$$

by using the definition of VaR and cumulative normal distribution. Next, let us consider the portfolio return distribution in a multidimensional space. If we replace μ and σ with the mean of the portfolio and standard deviation, respectively. Then we get:

$$VaR_\alpha(x) = -(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f) + z_{1-\alpha}\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \quad \square$$

Next, we give the closed form representations of CVaR when asset returns are multivariate normal.

Proposition 3.2.4. Let the asset return distribution, $\mathbf{r} \in \mathbb{R}^n$, follows a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ (i.e. $\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$). Then

$$CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) = -(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f) + \frac{f(z_{1-\alpha})}{1-\alpha}\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}.$$

Proof. Let $\tilde{v}$ be a standard normal random variable (i.e. $\tilde{v} \sim N(0, 1)$). Furthermore, suppose this is the distribution of the portfolio return. Then $CVaR_\alpha(\tilde{v})$ can be represented as follows,

$$CVaR_\alpha(\tilde{v}) = -\frac{\int_{-\infty}^{-z_{1-\alpha}} \tilde{v} f_{\tilde{v}}(\tilde{v}) d\tilde{v}}{1-\alpha}$$

$$\text{(where } z_{1-\alpha} = VaR_\alpha(\tilde{v}))$$

$$= -\frac{\int_{-\infty}^{-z_{1-\alpha}} \tilde{v} f_{\tilde{v}}(\tilde{v}) d\tilde{v}}{F(-z_{1-\alpha})}$$

$$= -\int_{-\infty}^{-z_{1-\alpha}} \tilde{v} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{\tilde{v}^2}{2}\right) d\tilde{v} \Big/ F(-z_{1-\alpha})$$

$$= \int_{z_{1-\alpha}^2}^{\infty} \frac{1}{2\sqrt{2\pi}} exp\left(-\frac{u}{2}\right) du \Big/ F(-z_{1-\alpha})$$

(make substitution $u = \tilde{v}^2$)

$$= f(z_{1-\alpha}) \Big/ F(-z_{1-\alpha}).$$

Now, suppose that $y \sim N(\mu_y, \sigma_y^2)$. Hence,

$$CVaR_\alpha(y) = -\mathbb{E}[y|y < \mu_y - z_{1-\alpha}\sigma_y]$$

(since $y \sim N(\mu_y, \sigma_y)$ and $z_{1-\alpha} = VaR_\alpha(y)$)

$$= -\mathbb{E}[\mu_y + \sigma_y x|\mu_y + \sigma_y x < \mu_y - z_{1-\alpha}\sigma_y] \qquad \text{(where } x = (y - \mu_y)/(\sigma_y) \text{)}$$

$$= -\mu_y - \sigma_y \mathbb{E}[x|\mu_y + \sigma_y x < \mu_y - z_{1-\alpha}\sigma_y]$$

$$= -\mu_y - \sigma_y \mathbb{E}[x|x < -z_{1-\alpha}]$$

$$= -\mu_y + \sigma_y \frac{f(-z_{1-\alpha})}{F(-z_{1-\alpha})} \qquad \text{(since } \tilde{v} \sim N(0,1)\text{)}$$

$$= -\mu_y + \sigma_y \frac{f(z_{1-\alpha})}{1-\alpha}. \qquad \text{(since } \tilde{v} \sim N(0,1)\text{)}$$

Replace μ_y and σ_y with the mean of the portfolio and standard deviation, respectively. Then we get;

$$CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) = -(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f) + \frac{f(z_{1-\alpha})}{1-\alpha}\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}. \quad \square$$

Now, we first state and give the proof of a proposition by Bertsimas D., Gupta V., Paschalidis Ioannis Ch.(2012) [7] to give an introduction to the relationship between uncertainty sets and risk measures.

Proposition 3.2.5. Consider the following uncertainty sets:

$$\mathcal{U}_1 = \{\mathbf{r} - r_f\mathbf{e} : (\mathbf{r} - r_f\mathbf{e})'\mathbf{\Sigma}^{-1}(\mathbf{r} - r_f\mathbf{e}) \leq 1\}, \tag{3.8}$$

$$\mathcal{U}_{z_{1-\alpha}} = \{\mathbf{r} - \boldsymbol{\mu} : (\mathbf{r} - \boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{r} - \boldsymbol{\mu}) \leq z_{1-\alpha}^2\} \tag{3.9}$$

(i) Problem given in Definition 3.2.9 with $\mathcal{U} = \mathcal{U}_1$ is equivalent to the Markowitz problem.

(ii) If $\mathbf{r}$ is distributed as a multivariate Gaussian, $\mathbf{r} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma})$ then Definition 3.2.9 with $\mathcal{U} = \mathcal{U}_{z_{1-\alpha}}$ is equivalent to the PAP with Value at Risk (3.3).

Proof. (i) Note that the objective functions of the problems are same, therefore, we shall focus only on the constraint. Let us define the robust counterpart risk measure of the general portfolio allocation problem (3.7).

For $\mathcal{U} = \mathcal{U}_1$ we have:

$$\eta_{\mathcal{U}_1}(v_0 + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}) = \eta_{\mathcal{U}_1}\left(r_f + (\mathbf{r} - r_f\mathbf{e})'\mathbf{x}\right)$$

$$= - \min_{\mathbf{r} - r_f\mathbf{e} \in \mathcal{U}_1} (r_f + (\mathbf{r} - r_f\mathbf{e})'\mathbf{x}) \qquad \text{(by equation (3.6))}$$

$$= -r_f + 1\sqrt{\mathbf{x}'\mathbf{\Sigma}\mathbf{x}} \qquad \text{(by equation (3.7))}.$$

Therefore the risk constraint turns into;

$$-r_f + \sqrt{\mathbf{x}'\mathbf{\Sigma}\mathbf{x}} \leq L,$$

this is nothing but

$$\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq L_1,$$

where $L_1 = L + r_f$.

(ii) Note that the objective functions of the problems are same, therefore, we shall focus only on the constraint on risk constraint. So that,

$$\begin{aligned}
\eta_{\mathcal{U}_{z_{1-\alpha}}}(v_0 + (\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}) &= \eta_{\mathcal{U}_{z_{1-\alpha}}}\left(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f + (\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}\right) \\
&= -\min_{\mathbf{r}\in\mathcal{U}_{z_{1-\alpha}}}\left(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f + (\mathbf{r}-\boldsymbol{\mu})'\mathbf{x}\right) && \text{by equation (3.6)} \\
&= -\left(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f\right) + z_{1-\alpha}\sqrt{\mathbf{x}\boldsymbol{\Sigma}\mathbf{x}} && \text{by equation (3.7)} \\
&= VaR_\alpha\left((\mathbf{r}-r_f\mathbf{e})'\mathbf{x} + r_f\right) && \text{by Proposition 3.2.3}
\end{aligned}$$

3.3 GPAP with Elliptical Distributions

In this section, we give the relationship between GPAP with robust optimization under some elliptical uncertainty sets and PAP with CVaR under some well known multivariate elliptical distributions (for details on elliptical distributions please see Fang et. al. [14]). Let us begin with the definition of elliptical distributions.

Definition 3.3.1 (Elliptical Distributions)**.** Let $\mathbf{r}$ be the n dimensional asset return vector with the following density

$$f_{\mathbf{r}}(\mathbf{r};\boldsymbol{\mu},\mathbf{D}) = |\mathbf{D}|^{-\frac{1}{2}} g_n\left((\mathbf{r}-\boldsymbol{\mu})'\,\mathbf{D}\,(\mathbf{r}-\boldsymbol{\mu})\right) \tag{3.10}$$

where $\boldsymbol{\mu}$ is a location vector, $\mathbf{D}$ is an positive definite dispersion matrix and $g_n(.)$ is the distribution specific density generator function. We use the notation $\mathbf{r} \sim ED_n(\boldsymbol{\mu}, \mathbf{D}, g_n)$. Moreover, if the density defined on equation (3.10) exist then it satisfies the following equation,

$$\int_0^\infty u^{\frac{n}{2}-1} g_n(u) du = \frac{\Gamma(n/2)}{\Pi^{n/2}}. \tag{3.11}$$

Interested readers can check the paper by Xiao and Valdez 2013 [48] to see how well known multivariate elliptical distributions are represented by (3.11).

Lemma 3.3.1. (Lemma 4.1 in Xiao and Valdez (2013) [48]) Let $\mathbf{v} \sim ED(\boldsymbol{\mu}, \mathbf{D}, g_n)$ and let $\mathbf{x} \in \mathbb{R}^n$ then,

$$CVaR_\alpha(\mathbf{r}'\mathbf{x}) = -\mathbf{x}'\boldsymbol{\mu} + CVaR_\alpha(Y)(\sqrt{\mathbf{x}'\mathbf{D}\mathbf{x}}) \tag{3.12}$$

where $Y \sim ED_1(0, 1, g_1)$ i.e. a spherical random variable. Moreover,

$$CVaR_\alpha(Y) = \frac{1}{1-\alpha} \bar{G}\left(\frac{y_{1-\alpha}^2}{2}\right) \tag{3.13}$$

where

$$\bar{G}(x) = G(\infty) - G(x) \text{ if } G(\infty) < \infty$$

such that

$$G(x) = \int_0^x g_1(2u) du,$$

and $y_{1-\alpha}$ satisfies

$$F_Y(y_{1-\alpha}) = \int_{-\infty}^{y_{1-\alpha}} g_1(u^2) du = 1 - \alpha \tag{3.14}$$

Proof. Please see Xiao and Valdez (2013) [48].

Driving the closed form representation of CVaR with multivariate normal distribution is given in Proposition 3.2.4. We give two important propositions on the closed form representation of CVaR with multivariate elliptical distributions next (also see Landsman [27]).

Proposition 3.3.2. Let $\mathbf{r} \sim t(\boldsymbol{\mu}, \mathbf{D}, m)$ and let $\mathbf{x} \in \mathbb{R}^n$ then equation (3.13) becomes

$$CVaR_\alpha(Y) = \frac{c_2 m}{(1-\alpha)(m-1)} \left(1 + \frac{y_{1-\alpha}^2}{m}\right)^{\frac{1-m}{2}},$$

where

$$c_2 = \frac{(\pi m)^{-1/2}\Gamma\left((m+1)/2\right)}{\Gamma\left(m/2\right)}.$$

Proof. We use the definitions of $G(\infty)$ and $G(y_{1-\alpha}^2/2)$ from Lemma 3.3.1. Let us start with calculating $G(\infty)$:

$$G(\infty) = \int_0^\infty g_1(2u)du$$

$$= \int_0^\infty c_2 \left(1 + \frac{2u}{m}\right)^{\frac{-(1+m)}{2}} du$$

$$= \frac{c_2 m}{2} \int_1^\infty v^{\frac{-(1+m)}{2}} dv \qquad \text{(make substitution } (1 + 2u/m = v))$$

$$= -\frac{c_2 m}{1-m}. \tag{3.15}$$

Next, we calculate $G(y_{1-\alpha}^2/2)$,

$$G(y_\alpha^2/2) = \int_0^{y_{1-\alpha}^2/2} g_1(2u)du$$

$$= \int_0^{y_{1-\alpha}^2/2} c_2 \left(1 + \frac{2u}{m}\right)^{\frac{-(1+m)}{2}} du$$

$$= \frac{c_2 m}{2} \int_1^{1+y_{1-\alpha}^2/m} v^{\frac{-(1+m)}{2}} dv \qquad \text{(make substitution } (1 + 2u/m = v))$$

$$= \frac{c_2 m}{1-m}\left(\left(1+\frac{y_{1-\alpha}^2}{m}\right)^{(1-m)/2} - 1\right). \tag{3.16}$$

Now, using equations (3.15) and (3.16) we get

$$CVaR_\alpha(Y) = \frac{1}{(1-\alpha)}\left(-\frac{c_2 m}{1-m} - \frac{c_2 m}{1-m}\left(\left(1+(y_{1-\alpha}^2/m)\right)^{(1-m)/2} - 1\right)\right)$$

$$= \frac{1}{(1-\alpha)}\left(\frac{c_2 m}{m-1}\left(1+\frac{y_{1-\alpha}^2}{m}\right)^{\frac{1-m}{2}}\right). \qquad \square$$

Proposition 3.3.3. Let $\mathbf{r} \sim ML(\boldsymbol{\mu}, \mathbf{D})$ and let $\mathbf{x} \in \mathbb{R}^n$ then equation (3.13) becomes

$$CVaR_\alpha(Y) = \frac{c_3}{2(1-\alpha)}\left(1 - \frac{1}{1+e^{-y_{1-\alpha}^2}}\right),$$

where

$$c_3 = \frac{\Gamma(n/2)}{\pi^{1/2}}\left(\int_0^\infty u^{n/2-1}\frac{e^{-u}}{(1+e^{-u})^2}du\right)^{-1}.$$

Proof. Note that definitions of $G(\infty)$ and $G(y_{1-\alpha}^2/2)$ are given in Lemma 3.3.1. We start with calculating $G(\infty)$:

$$G(\infty) = \int_0^\infty c_3\frac{e^{-2u}}{(1+e^{-2u})^2}du$$

$$= \int_2^1 \frac{c_3}{-2x^2}du \qquad \text{(make substitution } (1+e^{-2u} = x))$$

$$= \frac{c_3}{4}. \tag{3.17}$$

Next, we calculate $G(y^2_{1-\alpha}/2)$,

$$G(y^2_\alpha/2) = \int_0^{y^2_{1-\alpha}/2} c_3 \frac{u^{-2u}}{(1+e^{-2u})^2} du$$

$$= \int_2^{1+e^{-y^2_{1-\alpha}}} \frac{c_1}{-2x^2} du \qquad \text{(make substitution } (1+e^{-2u} = x))$$

$$= c_3 \left(\frac{-1}{2} + \frac{1}{1+e^{-y^2_{1-\alpha}}} \right). \tag{3.18}$$

Now, combine equations (3.17) and (3.18) we get

$$CVaR_\alpha(Y) = \frac{1}{(1-\alpha)} \left(\frac{c_1}{4} - \frac{c_1}{2} \left(\frac{-1}{2} + \frac{1}{1+e^{-y^2_{1-\alpha}}} \right) \right)$$

$$= \frac{c_3}{2(1-\alpha)} \left(1 - \frac{1}{1+e^{-y^2_{1-\alpha}}} \right) \qquad \square$$

We give the connection between GPAP with generic risk measure and GPAP in the sense of robust optimization in the following proposition. Note that we use CVaR as our coherent risk measure.

Proposition 3.3.4 (CVaR and Uncertainty Sets)**.** Consider the following uncertainty set:

$$\mathcal{U}_{\beta_\alpha} = \{\mathbf{r} - \boldsymbol{\mu} : (\mathbf{r} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{r} - \boldsymbol{\mu}) \leq \beta^2_\alpha\}, \tag{3.19}$$

(i) if $\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and

$$\beta_\alpha = f(z_{1-\alpha})/(1-\alpha),$$

where $f(\cdot)$ is the standard normal density function and $z_{1-\alpha}$ is the z-score,

(ii) if $\mathbf{r}$ follows a multivariate Student-t distribution i.e. $\mathbf{r} \sim t(\boldsymbol{\mu}, \boldsymbol{\Sigma}, m)$ where m stands for the degree of freedom, and

$$\beta_\alpha = \frac{c_2 m}{(1-\alpha)(m-1)}\left(1+\frac{y_{1-\alpha}^2}{m}\right)^{\frac{1-m}{2}},$$

where

$$c_2 = \frac{(\pi m)^{-1/2}\Gamma\left((m+1)/2\right)}{\Gamma\left(m/2\right)},$$

and $y_{1-\alpha}$ satisfying equation (3.14),

(iii) if $\mathbf{r}$ follows a multivariate Logistics distribution i.e. $\mathbf{r} \sim ML(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and

$$\beta_\alpha = \frac{c_3}{2(1-\alpha)}\left(1-\frac{1}{1+e^{-y_{1-\alpha}^2}}\right),$$

where

$$c_3 = \frac{\Gamma(n/2)}{\pi^{1/2}}\left(\int_0^\infty u^{n/2-1}\frac{e^{-u}}{(1+e^{-u})^2}du\right)^{-1},$$

and $y_{1-\alpha}$ satisfying equation (3.14),

then we get the PAP with CVaR for multivariate Normal, Student-t, and Logistics distributions, respectively.

Moreover, the closed-form solution of CVaR is:

$$CVaR_\alpha(\tilde{v}) = -(\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f) + \beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}. \tag{3.20}$$

Proof. For all three cases, we start with the using Theorem 4 of [32] to get the robust counterpart risk measure of a coherent risk measure and give the representation of CVaR (i.e. (3.20)).

(i)

$$\rho(v_0 + \mathbf{r}'\mathbf{x}) = \eta_{\mathcal{U}_{\beta_\alpha}}(v_0 + \mathbf{r}'\mathbf{x})$$

(by Theorem 4 of [32])

$$= \eta_{\mathcal{U}_{\beta_\alpha}}\left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}\right)$$

(by Definition 3.2.1)

$$= -\min_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_{\beta_\alpha}}\left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}\right)$$

(by Equation (3.6))

$$= -\left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f\right) + \beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}$$

(by equation (3.7))

$$= CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f).$$

(by Proposition 3.2.4)

(ii)

$$\rho(v_0 + \mathbf{r}'\mathbf{x}) = \eta_{\mathcal{U}_{\beta_\alpha}}(v_0 + \mathbf{r}'\mathbf{x})$$

(by Theorem 4 of [32])

$$= \eta_{\mathcal{U}_{\beta_\alpha}}\left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}\right)$$

(by Definition 3.2.1)

$$= -\min_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_{\beta_\alpha}}\left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}\right)$$

(by Equation (3.6))

$$= -\left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f\right) + \beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}$$

(by equation (3.7))

$$= CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x}) - r_f$$

(by Lemma 3.3.1 and Proposition 3.3.2)

$$= CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f).$$

(since CVaR is a coherent risk measure)

(iii)

$$\rho(v_0 + \mathbf{r}'\mathbf{x}) = \eta_{\mathcal{U}_{\beta_\alpha}}(v_0 + \mathbf{r}'\mathbf{x})$$

(by Theorem 4 of [32])

$$= \eta_{\mathcal{U}_{\beta_\alpha}}\left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}\right)$$

(by Definition 3.2.1)

$$= -\min_{\mathbf{r}-\boldsymbol{\mu}\in\mathcal{U}_{\beta_\alpha}} \left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f + (\mathbf{r} - \boldsymbol{\mu})'\mathbf{x}\right)$$

(by Equation (3.6))

$$= -\left(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f\right) + \beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}$$

(by equation (3.7))

$$= CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x}) - r_f.$$

(by Lemma 3.3.1 and Proposition 3.3.3)

$$= CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f).$$

(since CVaR is a coherent risk measure) □

Note that one can also use multivariate elliptical distributions directly and come up with the corresponding β_α values (see Landsman and Valdez [28]). For part (i) see also Bertsimas et. al. [7] and references there in.

3.4 CVaR Approximation

Consider the general constrained portfolio optimization problem (i.e. Definition 3.2.9) with uncertainty set given by (3.19). Then by using Proposition 3.3.4 we get the Lagrangian function of the optimization problem:

$$\begin{aligned}\mathcal{L}(\mathbf{x}, \delta) =& \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \\ & \left(CVaR_\alpha\left((\mathbf{r} - r_f\mathbf{e})'\,\mathbf{x} + r_f\right) - L\right)\delta \\ =& \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \\ & \left(-(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f) + \beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} - L\right)\delta \\ =& (1 + \delta)(-\boldsymbol{\mu}'\mathbf{x} - (1 - \mathbf{e}'\mathbf{x})r_f) - \delta\left(\beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} - L\right).\end{aligned}$$

Now, take the partial derivative with respect to $\mathbf{x}$ to get the first order necessary condition.

$$(1 + \delta)\,(\boldsymbol{\mu} - \mathbf{e}r_f) - \delta\left(\beta_\alpha(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})^{-1/2}\boldsymbol{\Sigma}\mathbf{x}\right) = 0.$$

Ladnsman [27] derives the closed-form solution of the problem of minimizing the root of a quadratic functional subject to some affine constraints. In addition to that, we can find the numerical solution using convex (or semidefinite) programming. But there is no explicit solution for constrained PAP with CVaR yet. Hence, we propose an algorithm to find the closed-form of the optimal solution for that problem.

The asset return distribution is assumed to be elliptical with the parameters defined in Proposition 3.5.1. Furthermore, historical mean vector and covariance matrix are taken as the mean of the asset returns and covariance matrix, respectively. We can rewrite the constraint as (using Proposition 3.3.4)

$$CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \leq L \tag{3.21}$$

$$\Leftrightarrow -(\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f) + \beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq L$$

$$\Leftrightarrow \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq \frac{(L + \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f)}{\beta_\alpha}.$$

Let

$$\tilde{L}(\mathbf{x}) \equiv \frac{(L + \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f)}{\beta_\alpha}. \tag{3.22}$$

Before we proceed to the main results we define the risk-adjusted return vector:

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\Sigma}^{-1/2}(\boldsymbol{\mu} - \mathbf{e}r_f).$$

Note that $\tilde{\boldsymbol{\mu}}$ is finite.

To solve the PAP with CVaR, we use an approximation approach. First, we define the feasible solution space $\mathcal{P}$ as

$$\mathcal{P} \equiv \{\mathbf{x} \in \mathbb{R}^n : CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \leq L\}. \tag{3.23}$$

Next, we define a sequence of vectors $\{\mathbf{x}_n\}_{n\geq 0}$ as follows:

$$\mathbf{x}_0 \in \mathcal{P}, \tag{3.24}$$

$$\mathbf{x}_{n+1} = \arg\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq \tilde{L}(\mathbf{x}_n)\}, \quad n \geq 0. \tag{3.25}$$

In other words, $\mathbf{x}_{n+1}$ is the solution of the following PAP

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq \tilde{L}(\mathbf{x}_n)\}. \tag{3.26}$$

We want to point out that the initial vector $\mathbf{x}_0$ can be any vector in $\mathcal{P}$. For example, we can choose $\mathbf{x}_0 = \mathbf{0}$. More details will be given in Proposition 3.4.2. As we can see, the above PAP is of Markowitz mean-variance type, and it is very easy to solve.

We have the following results.

Lemma 3.4.1. Let $\{\mathbf{x}_n\}_{n\geq 0}$ be given by (3.24)-(3.25). Then $\mathbf{x}_n \in \mathcal{P}$, $\forall n \geq 0$ and $\{\tilde{L}(\mathbf{x}_n)\}_{n\geq 0}$ is a non-decreasing sequence.

Proof. We prove the result by induction. First, from (3.24), we see that $\mathbf{x}_0 \in \mathcal{P}$. Further, by virtue of (3.23) and (3.21), it is easy to check that $\sqrt{\mathbf{x}_0'\boldsymbol{\Sigma}\mathbf{x}_0} \leq \tilde{L}(\mathbf{x}_0)$. By virtue of the definition of $\mathbf{x}_1$ (see (3.25)), we can get that

$$\boldsymbol{\mu}'\mathbf{x}_0 + (1 - \mathbf{e}'\mathbf{x}_0)r_f \leq \boldsymbol{\mu}'\mathbf{x}_1 + (1 - \mathbf{e}'\mathbf{x}_1)r_f.$$

Then, by the definition of $\tilde{L}(\mathbf{x})$ (3.22), we have

$$\tilde{L}(\mathbf{x}_0) \leq \tilde{L}(\mathbf{x}_1).$$

Using the definition of $\mathbf{x}_1$ (see (3.25)) and the above inequality, we have that

$$\sqrt{\mathbf{x}_1'\boldsymbol{\Sigma}\mathbf{x}_1} \leq \tilde{L}(\mathbf{x}_0) \leq \tilde{L}(\mathbf{x}_1).$$

Therefore, $\mathbf{x}_1 \in \mathcal{P}$. Now we can assume that $\mathbf{x}_n \in \mathcal{P}$ and $\tilde{L}(\mathbf{x}_{n-1}) \leq \tilde{L}(\mathbf{x}_n)$. Then using the same argument as we used for $\mathbf{x}_0$ and $\mathbf{x}_1$, we can show that

$$\mathbf{x}_{n+1} \in \mathcal{P}, \quad \tilde{L}(\mathbf{x}_n) \le \tilde{L}(\mathbf{x}_{n+1}).$$

This completes the proof. □

Unlike the classical mean-variance optimization problem which always have a solution, the PAP with CVaR may not have a bounded solution. For example, if $n = 1$, the CVaR constraint becomes $-(\mu - r_f)x - r_f + \beta_\alpha \sigma x \le L$, which might become redundant when $\mu - r_f > \sigma\beta_\alpha$, and the optimal solution is $x = \infty$. Therefore, some extra conditions are needed for the PAP with CVaR to be well-defined.

Define the risk-adjusted return vector $\tilde{\boldsymbol{\mu}}$ and a constant d as

$$\tilde{\boldsymbol{\mu}} \equiv \mathbf{\Sigma}^{-1/2}(\boldsymbol{\mu} - \mathbf{e}r_f), \tag{3.27}$$

$$d \equiv \sqrt{(\boldsymbol{\mu} - \mathbf{e}r_f)'\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f)}. \tag{3.28}$$

It is easy to see that $d = \|\tilde{\boldsymbol{\mu}}\|$. We assume the following condition holds:

$$\beta_\alpha - d > 0. \tag{3.29}$$

This condition is not a strong condition and it would usually hold. Note that as explained above, if it doesn't hold, the solution is trivial. Hence, condition (3.29) doesn't really constrain α. For example, for $n = 1$ with normal distributions, $\beta_\alpha = 2.0627$ for $\alpha = 95\%$ and d is nothing but the Sharp Ratio of the risky asset. Assume that $\mu = 20\%, r_f = 0\%$ and $\sigma = 16\%$, we can get $d = (\mu - r_f)/\sigma = 1.25$. Note that the risk-reward trade off is usually taken as $\delta = 1.25$ (see He and Litterman [19] and Bertsimas et. al. [7]). So condition (3.29) is easily satisfied.

Proposition 3.4.2 (Conv. of $\tilde{L}(\mathbf{x}_n)$)**.** Assume that (3.29) holds and define

$$\tilde{L}^* \equiv \frac{L + r_f}{\beta_\alpha - d}. \tag{3.30}$$

Let $\{\mathbf{x}_n\}_{n\geq 0}$ be given by (3.24)-(3.25). Then we have

$$\tilde{L}(\mathbf{x}_n) \leq \tilde{L}^*, \quad \forall n \geq 0, \tag{3.31}$$

and

$$\lim_{n\to\infty} \tilde{L}(\mathbf{x}_n) = \tilde{L}^*, \quad \forall \mathbf{x}_0 \in \mathcal{P}. \tag{3.32}$$

Further, we have

$$\tilde{L}(\mathbf{x}) \leq \tilde{L}^*, \quad \forall \mathbf{x} \in \mathcal{P}. \tag{3.33}$$

Proof. By virtue of the definition of $\mathbf{x}_{n+1}$ (see (3.25)), we know that $\mathbf{x}_{n+1}$ is the optimal solution of (3.26).The closed-form solution of this problem is well-known:

$$\mathbf{x}_{n+1} = \frac{\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f)}{2\delta}, \tag{3.34}$$

where δ is the Lagrange multiplier given by $\delta = \frac{d}{2\tilde{L}(\mathbf{x}_n)}$ and d is defined by (3.28). So we can get

$$\mathbf{x}_{n+1} = \frac{\tilde{L}(\mathbf{x}_n)}{d}\mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{e}). \tag{3.35}$$

Using the definition of $\tilde{L}(\cdot)$, we have

$$\tilde{L}(\mathbf{x}_{n+1}) = \frac{L + r_f}{\beta_\alpha} + \frac{(\boldsymbol{\mu} - r_f\mathbf{e})'\mathbf{x}_{n+1}}{\beta_\alpha}. \tag{3.36}$$

It is easy to verify that (3.35) and (3.36) imply

$$\tilde{L}(\mathbf{x}_{n+1}) = \frac{L + r_f}{\beta_\alpha} + \frac{d\tilde{L}(\mathbf{x}_n)}{\beta_\alpha}. \tag{3.37}$$

By virtue of Lemma 3.4.1, we have that $\tilde{L}(\mathbf{x}_{n+1}) \geq \tilde{L}(\mathbf{x}_n)$. So

$$\frac{L + r_f}{\beta_\alpha} + \frac{d\tilde{L}(\mathbf{x}_n)}{\beta_\alpha} \geq \tilde{L}(\mathbf{x}_n),$$

which is equivalent to

$$\tilde{L}(\mathbf{x}_n) \leq \frac{L + r_f}{\beta_\alpha - d}.$$

This holds for any $n \geq 0$. Therefore, (3.30) holds.

Further, by virtue of the recursive formula (3.37), we can get

$$\tilde{L}(\mathbf{x}_n) = \frac{L + r_f}{\beta_\alpha} \sum_{j=0}^{n-1} \left(\frac{d}{\beta_\alpha}\right)^j + \left(\frac{d}{\beta_\alpha}\right)^n \tilde{L}(\mathbf{x}_0). \tag{3.38}$$

By virtue of (3.29), we know that

$$\frac{d}{\beta_\alpha} < 1.$$

Therefore,

$$\lim_{n\to\infty} \tilde{L}(\mathbf{x}_n) = \frac{L + r_f}{\beta_\alpha} \frac{1}{1 - \frac{d}{\beta_\alpha}} = \frac{L + r_f}{\beta_\alpha - d} = \tilde{L}^*,$$

and it is true for any $\mathbf{x}_0 \in \mathcal{P}$. Therefore, (3.32) holds.

Finally, for any $\mathbf{x} \in \mathcal{P}$, we can take $\mathbf{x}_0 = \mathbf{x}$ in (3.24). Then by (3.31), we can get (3.33). This completes the proof. □

Now we can present the main result of this section.

Theorem 3.4.3. Assume that (3.29) holds. Then $\mathbf{x}^*$ is an optimal solution to the PAP with CVaR (i.e. problem given by Definition 3.2.7) if and only if $\tilde{L}(\mathbf{x}^*) = \tilde{L}^*$ and $\mathbf{x}^* \in \mathcal{P}$, where $\tilde{L}(\cdot), \tilde{L}^*$ and $\mathcal{P}$ are defined by (3.22), (3.30) and (3.23), respectively.

Proof. Define $\hat{\mathbf{x}}$ as a solution of

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \leq \tilde{L}^*\}. \tag{3.39}$$

Let $\{\mathbf{x}_n\}_{n\geq 0}$ be given by (3.24)-(3.25). By virtue of (3.31), we know that $\tilde{L}(\mathbf{x}_n) \leq \tilde{L}^*$. Then, by virtue of (3.39) and (3.25), we can get that

$$\boldsymbol{\mu}'\mathbf{x}_{n+1} + (1 - \mathbf{e}'\mathbf{x}_{n+1})r_f \leq \boldsymbol{\mu}'\hat{\mathbf{x}} + (1 - \mathbf{e}'\hat{\mathbf{x}})r_f.$$

By the definition of $\tilde{L}(\cdot)$ (see (3.22)), we can get $\tilde{L}(\mathbf{x}_{n+1}) \leq \tilde{L}(\hat{\mathbf{x}}),\ \forall n \geq 0$, which implies that

$$\tilde{L}^* = \lim_{n\to\infty} \tilde{L}(\mathbf{x}_n) \leq \tilde{L}(\hat{\mathbf{x}}). \tag{3.40}$$

Since $\hat{\mathbf{x}}$ is a solution of (3.39), the above equation implies that

$$\sqrt{\hat{\mathbf{x}}'\boldsymbol{\Sigma}\hat{\mathbf{x}}} \leq \tilde{L}(\hat{\mathbf{x}}). \tag{3.41}$$

Then, using (3.21), we can get that $\hat{\mathbf{x}} \in \mathcal{P}$. Now by virtue of (3.33), we know that

$$\tilde{L}(\hat{\mathbf{x}}) \leq \tilde{L}^*.$$

Together with (3.40), the above inequality implies that

$$\tilde{L}(\hat{\mathbf{x}}) = \tilde{L}^*. \tag{3.42}$$

Let $\mathbf{x}^*$ be an optimal solution of the PAP with CVaR:

$$\mathbf{x}^* \in \text{argmax}\{(\boldsymbol{\mu} - \mathbf{e})'\mathbf{x} + r_f : CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \leq L\}.$$

Then we have $\mathbf{x}^* \in \mathcal{P}$. Based on (3.41), we know that $\hat{\mathbf{x}} \in \mathcal{P}$. Since $\mathbf{x}^*$ is the optimal solution, we must have

$$(\boldsymbol{\mu} - \mathbf{e}r_f)'\hat{\mathbf{x}} + r_f \leq (\boldsymbol{\mu} - \mathbf{e}r_f)'\mathbf{x}^* + r_f \tag{3.43}$$

which implies

$$\tilde{L}(\hat{\mathbf{x}}) \le \tilde{L}(\mathbf{x}^*).$$

Now, taking $\mathbf{x}_0$ as $\mathbf{x}^*$ and using Lemma 3.4.1, we can get that

$$\tilde{L}(\mathbf{x}^*) \le \tilde{L}^* = \tilde{L}(\hat{\mathbf{x}}).$$

Therefore, we must have $\tilde{L}(\mathbf{x}^*) = \tilde{L}(\hat{\mathbf{x}}) = \tilde{L}^*$.

On the other hand, if there is an $\mathbf{x}^* \in \mathcal{P}$ such that $\tilde{L}(\mathbf{x}^*) = \tilde{L}^*$, then we can use (3.33) to derive that $\mathbf{x}^*$ maximizes $\tilde{L}(\cdot)$ over $\mathcal{P}$. By the definition of $\tilde{L}(\mathbf{x})$ (see (3.22)), we can see that maximizing $\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}r_f)\mathbf{x}$ is equivalent to maximizing $\tilde{L}(\mathbf{x})$. Therefore, $\mathbf{x}^*$ maximizes $\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f$ over $\mathcal{P}$ and it is an optimal solution of the PAP with CVaR given by Definition 3.2.7. This completes the proof. □

Another important result in this section is the following Theorem.

Theorem 3.4.4 (Conv. of $\tilde{L}(\mathbf{x}_n)$)**.** $\tilde{L}(\mathbf{x}_n) \xrightarrow{n\to\infty} \tilde{L}(\mathbf{x}^*)$ where $\mathbf{x}^*$ is the optimal solution to the PAP with CVaR (i.e. problem given by Definition 3.2.7).

Proof. From Proposition 3.4.2, we know that the sequence $\{\tilde{L}(\mathbf{x}_n)\}$ converges to a real number (i.e. $\tilde{L}^*$). We will show that $\tilde{L}^* = \tilde{L}(\mathbf{x}^*)$.

Define $\hat{\mathbf{x}}$ as the solution of

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \le \tilde{L}^*\}. \tag{3.44}$$

The closed form solution to this problem is well known:

$$\hat{\mathbf{x}} = \frac{\tilde{L}^*}{d}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{e}). \tag{3.45}$$

Using (3.22) we can get

$$\tilde{L}(\hat{\mathbf{x}}) = \frac{L + (\boldsymbol{\mu} - \mathbf{e}r_f)'\hat{\mathbf{x}} + r_f}{\beta_\alpha}. \tag{3.46}$$

Using equations (3.45) and (3.46) we get

$$\tilde{L}(\hat{\mathbf{x}}) = \frac{L + \tilde{L}^* d + r_f}{\beta_\alpha}$$

Since

$$\tilde{L}^* = \frac{L + r_f}{\beta_\alpha - d}$$

by Proposition 3.4.2, we get

$$\begin{aligned}
\tilde{L}(\hat{\mathbf{x}}) &= \frac{L + \frac{L+r_f}{\beta_\alpha - d} d + r_f}{\beta_\alpha} \\
&= \frac{\frac{(L+r_f)(\beta_\alpha - d + d)}{\beta_\alpha - d}}{\beta_\alpha} \\
&= \frac{L + r_f}{\beta_\alpha - d} \\
&= \tilde{L}^*.
\end{aligned}$$

Therefore,

$$\tilde{L}(\hat{\mathbf{x}}) = \tilde{L}^*. \tag{3.47}$$

On the other hand, by definition, we have

$$\mathbf{x}^* \in \operatorname{argmax}\{(\boldsymbol{\mu} - \mathbf{e})'\mathbf{x} + r_f : CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x} + r_f) \leq L\}.$$

Further, the constraint should be binding for the optimal solution, so we can get $CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x}^* + r_f) = L$. By the definition of $\tilde{L}$ (see (3.22)), we can get

$$CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\mathbf{x}^* + r_f) = L \Leftrightarrow \sqrt{(\mathbf{x}^*)'\boldsymbol{\Sigma}\mathbf{x}^*} = \tilde{L}(\mathbf{x}^*).$$

From (3.47) and (3.22), we can get

$$CVaR_\alpha((\mathbf{r} - r_f\mathbf{e})'\hat{\mathbf{x}} + r_f) \leq L.$$

Therefore, $\hat{\mathbf{x}}$ is a feasible solution to the PAP with CVaR. Because $\mathbf{x}^*$ is the optimal solution, we can get that

$$(\boldsymbol{\mu} - \mathbf{e}r_f)'\hat{\mathbf{x}} + r_f \leq (\boldsymbol{\mu} - \mathbf{e}r_f)'\mathbf{x}^* + r_f$$

$$\Rightarrow \frac{((\boldsymbol{\mu} - \mathbf{e}r_f)'\hat{\mathbf{x}} + r_f + L)}{\beta_\alpha} \leq \frac{((\boldsymbol{\mu} - \mathbf{e}r_f)'\mathbf{x}^* + r_f + L)}{\beta_\alpha}$$

$$\Rightarrow \tilde{L}(\hat{\mathbf{x}}) \leq \tilde{L}(\mathbf{x}^*). \tag{3.48}$$

Now, taking $\mathbf{x}_0$ as $\mathbf{x}^*$ and using Lemma 3.4.1, we can get that

$$\tilde{L}(\mathbf{x}^*) \leq \tilde{L}^* = \tilde{L}(\hat{\mathbf{x}}).$$

Therefore, we must have $\tilde{L}(\mathbf{x}^*) = \tilde{L}(\hat{\mathbf{x}}) = \tilde{L}^*$. This completes the proof. □

3.5 Elliptical Distributions and BLM

In this section, we state and examine the work of Xiao and Valdez (2013) [48] where they use multivariate elliptical distributions to model the asset returns and give optimal portfolio allocation vector for the alternate BLM under different type of risk measures such as VaR and CVaR in an unconstrained setting. Well known elliptical distributions other than multivariate normal are Student-t, multivariate Cauchy, multivariate Logistics and multivariate Stable (for details please see Fang et. al. [14]).

We continue with the BLM under elliptical distributions ($ED_n(\cdot)$) given in Xiao and Valdez

[48]. Let $\mathbf{r} \sim ED_n(\boldsymbol{\mu}, \mathbf{D}, g_n)$ be an n dimensional vector and denotes the market factors, where $\boldsymbol{\mu}$, $\mathbf{D}$ and g_n are the location parameter, dispersion matrix and the density generator function respectively. Furthermore, conditional random view vector is: $\mathbf{v}|\mathbf{r} \sim ED_k(\mathbf{P}\boldsymbol{\mu}, \boldsymbol{\Omega}, g_k(\cdot; p(\mathbf{r})))$ where $p(\mathbf{r}) = (\mathbf{r} - \boldsymbol{\Pi})'\mathbf{D}^{-1}(\mathbf{r} - \boldsymbol{\Pi})$ and $\boldsymbol{\Pi}$ is the CAPM equilibrium expected return and can be found via inverse optimization (for details see He and Litterman [19]). The posterior distribution is given by the following proposition.

Proposition 3.5.1 (Xiao and Valdez [48])**.** The posterior distribution is

$$\mathbf{r}|\mathbf{v} \sim ED_k(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}, g_n(\cdot; q(\mathbf{v})))$$

where

$$\boldsymbol{\mu}_{BL} = \boldsymbol{\Pi} + \mathbf{DP'}(\boldsymbol{\Omega} + \mathbf{P\Sigma P'})^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\Pi})$$

$$\mathbf{D}_{BL} = \boldsymbol{\Sigma} - \mathbf{DP'}(\boldsymbol{\Omega} + \mathbf{PDP'})^{-1}\mathbf{PD}$$

and

$$q(\mathbf{v}) = (\mathbf{v} - \mathbf{P}\boldsymbol{\Pi})'(\boldsymbol{\Omega} + \mathbf{P\Sigma P'})^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\Pi})$$

$$\boldsymbol{\Sigma}_{BL} = \mathbf{D}_{BL} C_k(q(\mathbf{v})/2),$$

where C_k is a distribution specific function from $\mathbb{R}$ to $\mathbb{R}$. (For details please see Xiao and Valdez [48]).

The key assumption of BLM is that every player in the market solves the Markowitz's problem. In other words, BLM takes CAPM equilibrium as prior for the excess return distribution. However, in our case, investors have views under the CVaR risk measure. There are other generalization for the BLM with CAPM equilibrium. For example, Silva et. al. [44] gives the BLM under active portfolio management, Giacometti et. al. [17] proposes a model where asset returns follow stable distributions with different types of risk measures. Unlike these models, here we consider elliptical distributions for asset returns and solve the constrained model.

Consider the unconstrained portfolio optimization problem with CVaR with uncertainty set given by (3.19). Then by using Proposition 3.3.4 we can represent the objective function as:

$$\begin{aligned}
\mathcal{G}(\mathbf{x}) =& \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \lambda CVaR_\alpha(\tilde{v}) \\
=& \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \left(-\mathbf{x}'(\boldsymbol{\mu} - \mathbf{e}r_f) - r_f + \beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}\right)\lambda \\
=& (1+\lambda)[\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f] - \lambda\beta_\alpha\sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}.
\end{aligned}$$

Now, take the partial derivative with respect to $\mathbf{x}$ to get the first order necessary condition and use inverse optimization to find an estimate of expected excess return vector:

$$\begin{aligned}
& \boldsymbol{\mu} - \mathbf{e}r_f - \left(-(\boldsymbol{\mu} - \mathbf{e}r_f) + \beta_\alpha(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})^{-1/2}\boldsymbol{\Sigma}\mathbf{x}\right)\lambda = 0 \\
\Leftrightarrow \quad & (1+\lambda)(\boldsymbol{\mu} - \mathbf{e}r_f) - \left(\beta_\alpha(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})^{-1/2}\boldsymbol{\Sigma}\mathbf{x}\right)\lambda = 0 \\
\Leftrightarrow \quad & (\boldsymbol{\mu} - \mathbf{e}r_f) = \frac{\lambda}{1+\lambda}\left(\beta_\alpha(\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x})^{-1/2}\boldsymbol{\Sigma}\mathbf{x}\right) \\
\Leftrightarrow \quad & \boldsymbol{\Pi} = \left(\frac{\lambda\beta_\alpha}{1+\lambda}(\mathbf{x}'_{mkt}\boldsymbol{\Sigma}\mathbf{x}_{mkt})^{-1/2}\right)\boldsymbol{\Sigma}\mathbf{x}_{mkt},
\end{aligned}$$

where we take $\mathbf{x} = \mathbf{x}_{mkt}$ as the market weights.

We get the return distribution for the updated excess return vector by using Proposition 3.5.1. We also need to determine the investor risk aversion parameter. In our problem, investor risk aversion is reflected in two parameters. The first one is the choice of the parameter β_α when the investor picks an elliptical distribution for the asset returns. The second parameter is λ, which is related with the risk reward trade off. Now, we can solve the portfolio optimization problem under CVaR (i.e. definition (3.2.7)) with the new (updated) excess return vector and dispersion matrix.

3.6 Closed-From Solutions of BLM with CVaR under Elliptical Distributions

In this section we give the closed-form solution for PAP with elliptical distributions under CVaR for the BLM. Under the BLM with CVaR, we have an updated mean vector $\boldsymbol{\mu}_{BL}$ and covariance matrix $\boldsymbol{\Sigma}_{BL}$ which are given by Proposition 3.5.1 using the new $\boldsymbol{\Pi}$ vector (i.e. equation (3.49)). Define

$$d_{BL} \equiv \sqrt{(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}, \tag{3.49}$$

$$\tilde{L}_{BL}^{*} \equiv \frac{L_{BL} + r_f}{\beta_\alpha - d_{BL}}. \tag{3.50}$$

Theorem 3.6.1. Let $\mathbf{x}^*$ be the optimal solution for PAP under CVaR then the closed-form solution is as follows:

$$\mathbf{x}^* = \frac{L + r_f}{(\beta_\alpha - d)d}\left(\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\right). \tag{3.51}$$

Proof. From Theorem 3.4.4 we know that the sequence $\tilde{L}(\mathbf{x}_n)$ converges. We now consider the sequence of $\{\mathbf{x}_n\}$ and will find the explicit optimal solution of the PAP with CVaR.

Let $\mathbf{x}_{n+1}$ be the optimal solution of the optimization problem defined by the problem parameters (i.e. $\boldsymbol{\mu}_{BL}$, r_f and $\boldsymbol{\Sigma}_{BL}$) and $\tilde{L}(\mathbf{x}_n)$. The closed-form solution of this problem is well-known:

$$\mathbf{x}_{n+1} = \frac{\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}{2\delta}, \tag{3.52}$$

where δ is the Lagrange multiplier. Since $\mathbf{x}_{n+1}$ is the optimal solution, the value of δ is given by:

$$\delta = \frac{d}{2\tilde{L}(\mathbf{x}_n)},$$

where d is defined by (3.49). If we plug in the value of δ to (3.52) and use the definition of $\tilde{L}(\cdot)$, we can get

$$\mathbf{x}_{n+1} = \frac{\mathbf{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\tilde{L}(\mathbf{x}_n)}{d}.$$

So we have

$$\tilde{L}(\mathbf{x}_{n+1}) = \tilde{L}\left(\frac{\mathbf{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\tilde{L}(\mathbf{x}_n)}{d}\right).$$

By virtue of Theorem 3.4.4, as $n \to \infty$, we can get

$$\tilde{L}(\mathbf{x}^*) = \tilde{L}\left(\frac{\mathbf{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\tilde{L}(\mathbf{x}^*)}{d}\right). \tag{3.53}$$

By the definition of $\tilde{L}$, the above equation is equivalent to

$$(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\mathbf{x}^* = \frac{(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\mathbf{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}{d}\tilde{L}(\mathbf{x}^*).$$

By the definition of $\tilde{L}$ again and canceling the d terms, it is equivalent to

$$(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\mathbf{x}^* = \frac{d}{\beta_\alpha}((\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\mathbf{x}^* + L + r_f).$$

After some algebra we get the following condition for optimal allocation vector:

$$(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\mathbf{x}^* = \frac{d}{\beta_\alpha - d}(L + r_f). \tag{3.54}$$

Now, let

$$\hat{\mathbf{x}} = \frac{L + r_f}{(\beta_\alpha - d)d}\left(\mathbf{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\right).$$

First we show that $\hat{\mathbf{x}}$ satisfies condition given by equation (3.54). Then we show that it is also a feasible solution. Equation (3.54) with $\hat{\mathbf{x}}$ yields,

$$(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\hat{\mathbf{x}} = \frac{(L + r_f)}{(\beta_\alpha - d)d}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\mathbf{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)$$

$$= \frac{d}{\beta_\alpha - d}(L + r_f).$$

Therefore, $\hat{\mathbf{x}}$ satisfies equation (3.54). Next we show that $\hat{\mathbf{x}} \in \mathcal{P}$. That is

$$-(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\hat{\mathbf{x}} + r_f + \beta_\alpha\sqrt{\hat{\mathbf{x}}'\boldsymbol{\Sigma}_{BL}\hat{\mathbf{x}}} \leq L.$$

When we plug the value of $\hat{\mathbf{x}}$ into the inequality above we get,

$$-(L + r_f)\frac{(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\boldsymbol{\Sigma}_{BL}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}{(\beta_\alpha - d)d}$$

$$+(L + r_f)\beta_\alpha\sqrt{\frac{(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)'\boldsymbol{\Sigma}_{BL}^{-1}\boldsymbol{\Sigma}_{BL}\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)}{(\beta_\alpha - d)^2 d^2}}$$

$$\leq L + r_f.$$

After some algebra we get that

$$\frac{-d}{\beta_\alpha - d} + \frac{\beta_\alpha}{\beta_\alpha - d} \leq 1$$

which is true and the constraint is binding. This completes the proof. □

We also have another proof.

Theorem 3.6.2. Let d_{BL}, $\tilde{L}_{BL}^*$ be given by (3.49), (3.50), respectively. Define

$$\hat{\mathbf{x}} \equiv \frac{\tilde{L}_{BL}^*}{d_{BL}}\left(\boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f)\right). \tag{3.55}$$

Assume that $\beta_\alpha > d_{BL}$. Then $\hat{\mathbf{x}}$ is an optimal solution of the PAP with CVaR.

Proof. By virtue of Theorem 3.4.4, it is sufficient to show that $\hat{\mathbf{x}} \in \mathcal{P}$ and $\tilde{L}(\hat{\mathbf{x}}) = \tilde{L}_{BL}^*$. Firstly, by virtue of (3.55) and (3.49), we can get that

$$\begin{aligned}
\sqrt{\hat{\mathbf{x}}'\mathbf{\Sigma}_{BL}\hat{\mathbf{x}}} &= \frac{\tilde{L}^*_{BL}}{d_{BL}}\sqrt{(\boldsymbol{\mu}_{BL}-\mathbf{e}r_f)'\mathbf{\Sigma}_{BL}^{-1}\mathbf{\Sigma}_{BL}\mathbf{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL}-\mathbf{e}r_f)} \\
&= \frac{\tilde{L}^*_{BL}}{d_{BL}}d_{BL} \\
&= \tilde{L}^*_{BL},
\end{aligned}$$

and

$$\begin{aligned}
(\boldsymbol{\mu}_{BL}-\mathbf{e}r_f)'\hat{\mathbf{x}} &= \frac{\tilde{L}^*_{BL}}{d_{BL}}(\boldsymbol{\mu}_{BL}-\mathbf{e}r_f)'\mathbf{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL}-\mathbf{e}r_f) \\
&= \frac{\tilde{L}^*_{BL}}{d_{BL}}d^2_{BL} \\
&= \tilde{L}^*_{BL}d.
\end{aligned}$$

Then, by virtue of (3.50), we have

$$\begin{aligned}
CVaR_\alpha((\mathbf{r}-r_f\mathbf{e})'\hat{\mathbf{x}}+r_f) &= -(\boldsymbol{\mu}_{BL}-\mathbf{e}r_f)'\hat{\mathbf{x}}-r_f+\beta_\alpha\sqrt{\hat{\mathbf{x}}'\mathbf{\Sigma}_{BL}\hat{\mathbf{x}}} \\
&= -\tilde{L}^*_{BL}d_{BL}-r_f+\beta_\alpha\tilde{L}^*_{BL} = (\beta_\alpha-d)\tilde{L}^*_{BL}+r_f \\
&= L.
\end{aligned}$$

Therefore, $\hat{\mathbf{x}} \in \mathcal{P}$. Secondly, using (3.22) and (3.49), we have

$$\tilde{L}(\hat{\mathbf{x}}) = \frac{L+r_f}{\beta_\alpha}+\frac{1}{\beta_\alpha}(\boldsymbol{\mu}_{BL}-\mathbf{e}r_f)'\hat{\mathbf{x}}$$

$$= \frac{\tilde{L}^*_{BL}(\beta_\alpha - d_{BL})}{\beta_\alpha} + \frac{1}{\beta_\alpha}\tilde{L}^*_{BL}d_{BL}$$

$$= \tilde{L}^*_{BL}.$$

This completes the proof. □

We continue with the relationship between the constrained models and unconstrained model of Portfolio Allocation Problem given above. First, let us start with minimum variance PAP:

$$\min_{\mathbf{x}} \left\{ \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} : \mathbf{e}'\mathbf{x} = 1 \right\}. \tag{3.56}$$

One can get the solution solving the dual of the minimum variance portfolio allocation problem. Optimum solution to this problem is: $\mathbf{x}^* = \frac{\lambda}{2}\boldsymbol{\Sigma}^{-1}\mathbf{e}$ where $\lambda = \frac{2}{\mathbf{e}'\boldsymbol{\Sigma}^{-1}\mathbf{e}}$. Therefore, we get:

$$x^* = (\mathbf{e}'\boldsymbol{\Sigma}^{-1}\mathbf{e})^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{e}.$$

Let us continue with the unconstrained model of PAP:

$$\max_{\mathbf{x}} \left\{ \boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f - \frac{\lambda\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}}{2} \right\}, \tag{3.57}$$

where λ denotes the investor's risk (variance in this case) reward trade-off. The optimal solution of this problem is:

$$\mathbf{x}^* = \lambda^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f).$$

We continue with the optimal allocation vector for the constrained model PAP as given by Definition 3.2.3. Using the same arguments given above, we can get the optimal solution

$$\mathbf{x}^* = \left(d\frac{1}{L}\right)^{-1}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f). \tag{3.58}$$

We can see that when the investor has an upper limit for her portfolio then she is doing nothing but changing her risk-reward coefficient as if she is solving an unconstrained model with an updated risk-reward trade-off coefficient (see also Steinbach [45]). Moreover, if we look at the problem with uncertainty sets. In particular, consider the problem given in Definition 3.2.9 with the uncertainty set given in (3.8). The optimal allocation vector to this problem is given as:

$$\mathbf{x}^* = \left(d \frac{1}{(L + r_f)} \right)^{-1} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f). \tag{3.59}$$

Here we see that using the uncertainty set given by (3.8) updates the optimal allocation vector. Lastly, we give the solution of the BLM with CVaR (see Theorem 3.6.2):

$$\mathbf{x}^* = \left(d \frac{(\beta_\alpha - d)}{(L + r_f)} \right)^{-1} \boldsymbol{\Sigma}_{BL}^{-1}(\boldsymbol{\mu}_{BL} - \mathbf{e}r_f). \tag{3.60}$$

Here, we can see that when the investor uses CVaR in an constrained setting then her risk-reward coefficient is updated with the risk free rate of return and $(\beta_\alpha - d)$. This is interesting and one can perturb the new risk-reward trade-off matrix in order to understand the behavior of the optimal portfolio vector under different problem settings.

Our model is a generalization of the multivariate case with variance as the risk measure. In this case, CAPM equilibrium is centered at $\boldsymbol{\Pi} = 2\delta\boldsymbol{\Sigma}\mathbf{x}_{mkt}$ where $\delta = 1.25$ is fixed. Note that if we assume multivariate normal distribution for asset returns and take λ in (3.49) such that

$$\frac{f(z_{1-\alpha})}{(1-\alpha)} \frac{\lambda}{(1+\lambda)} (\mathbf{x}'_{mkt}\boldsymbol{\Sigma}\mathbf{x}_{mkt})^{-1/2} = 2.5,$$

then we will get the same $\boldsymbol{\Pi}$ value. In addition, if we take the risk level L as $(L + r_f)/(\beta_\alpha - d)$, then we have the same setting as the multivariate normal models (see, for example, He and Litterman [19]).

3.7 Conclusion

Black and Litterman [9] proposed the BLM in order to overcome the Markowitz Model's drawbacks. Their model uses the Bayesian framework to combine the intuitions and/or inside information about the selected assets or market parameters with the historical information of the market to update the mean vector and covariance matrix. In our work, we use CVaR as a risk measure, instead of the variance risk measure proposed in the original model. In addition, elliptical uncertainty sets are used to model the uncertainty of asset returns in order to capture the non-normal behavior of the asset returns. For constrained problem, deriving the closed-form optimal solutions analytically is extremely difficult. Hence, we propose an efficient approximation algorithm for the BLM type optimization problems under CVaR and establish the convergence results. Based on the approximation, we derived the closed-form solution of the BLM with CVaR.

Chapter 4

Constructing Investor Views on Black-Litterman Model

4.1 Introduction

Markowitz model (Markowitz [29]) has some drawbacks as it only considers the first two moments of the return distribution (i.e. expected return vector and the variance-covariance matrix) when optimizing the portfolio allocation problem. If asset returns had normal distribution then Markowitz model (Markowitz [29]) would be plausible for portfolio allocation problem (Fabozzi [13] and Meucci [30]). One of the vital parts of the portfolio allocation problem is parameter estimation. Estimated parameters have errors (i.e. estimation risk). Furthermore, Chopra and Ziemba [11] reveals that portfolio allocation is highly sensitive to the expected return vector. Therefore, reducing the estimation error of the expected return is highly important. Black and Litterman [9] presents a framework to diminish the estimation error of the expected return vector. Their model blends the market data with the investor's views on expected asset returns using Bayesian framework. The proposed approach overcome some of the problems of Markowitz model such as; corner solutions and non-diversified solutions (for more discussion on these issues please see Black and Litterman [9]).

Suppose an investor has some idea/information/intuition about what is going on in the market. In other words, suppose she thinks certain portfolios (i.e. $\mathbf{P}$) might outperform their historical performances (i.e. $\mathbf{q}$). We call this as a view. Our investor can use her view(s) to update the parameters of the portfolio allocation problem using the Black-Litterman framework. Furthermore, the uncertainty/error of the view portfolio is captured by the uncertainty matrix (i.e. $\mathbf{\Omega}$). If the investor has only one view then, it is just a scalar, otherwise, it is a matrix. Specifying the values for this parameter is too abstract for non-quantitative investors. Hence, one of the most common extension of the Black-Litterman Model (BLM) is to replace this uncertainty matrix with the variance-covariance parameter of the view portfolio times a scalar, i.e. $\omega \in [0, \infty)$. ω, represents the overall uncertainty of the view portfolio. Using this extension of the model, investor is able to represent her uncertainty in terms of confidence (i.e. percentages). Providing the uncertainty as a confidence, is more intuitive (Idzorek [22]).

The original work of Black and Litterman assumes that the market is in CAPM equilibrium and they use market weights and inverse optimization to estimate the expected return vector. In this chapter we perturb that assumption about the market and assume that market is composed of two driving factors: bull and bear. Our investor estimates/guess the weight of the bull market i.e. $\alpha \in [0, 1]$. Furthermore, she provides a view for each of the market components; one for the bear component and the other for the bull component. We assume that the view portfolios are same for the bull and bear components. However, expectations of the view portfolios might be different for the components. We also assume that all the parameters are given or can be estimated.

The rest of the chapter is organized as follows. Definitions of the portfolio optimization problems are given in section 4.2. Representation of the allocation vectors of Black-Litterman Model (BLM) is given in detail in Section 4.3. The BL-type estimations under multivariate mixing models are given in Section 4.4. We conclude the chapter in Section 4.5.

4.2 Portfolio Allocation Problem

We give the notation of the chapter in this section. We use the abbreviation PAP for portfolio allocation problem. Let $\mathbf{x} \in \mathbb{R}^n$ be the portfolio allocation vector and $\mathbf{e}$ is a vector of ones in $\mathbb{R}^n$. Vectors are defined as column vectors unless otherwise stated. In addition to that risky asset returns and the expectation of risky asset returns are denoted by $\mathbf{r} \in \mathbb{R}^n$ and $\boldsymbol{\mu} = \mathbb{E}[\mathbf{r}] \in \mathbb{R}^n$ respectively. Moreover, the covariance matrix of the risky asset returns are denoted by $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$. $r_f \in \mathbb{R}_+ \cup 0$, is the risk free rate of return and $L \in \mathbb{R}_+$ is a predefined risk level of the investor. Moreover, $(1 - \mathbf{e}'\mathbf{x})$ is the allocation to the risk free asset.

If an investor knows her risk reward trade off parameter (i.e. δ) then she can use the unconstrained Markowitz model (see Definition 4.2.1) for PAP.

Definition 4.2.1 (Unconstraint Markowitz PAP)**.**

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x}) r_f - \delta \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}\} \tag{4.1}$$

Estimating the parameter δ is challenging. Therefore, investor might limit the risk of her portfolio. The constrained version of the Markowitz Model is given next.

Definition 4.2.2 (Markowitz PAP)**.**

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x}) r_f : \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \leq L^2\} \tag{4.2}$$

Variance is used as a risk measure in this problem. However, we are interested with other risk measures such as, VaR and CVaR. The definitions of these risk measures are given next.

Definition 4.2.3 (VaR)**.** Given $\alpha \in (0, 1)$ and a random variable Y, the VaR of the r.v. Y with confidence level α is;

$$VaR_\alpha(Y) = \inf\{y \in \mathbb{R} : \mathbb{P}(y + Y \leq 0) \leq 1 - \alpha\}.$$

VaR_α is not a coherent risk measure since diversification benefit might not be present (Artzner et. al. [2]). On the other hand, a commonly used risk measure CVaR is coherent and the properties of CVaR can be found in Rockafellar and Uryasev [37] and [38]. CVaR sometimes called as mean short fall or tail-VaR as well. Although, the definitions of them are different, they give the same results under continuous random variables (Krokhmal et. al. [26]).

Definition 4.2.4 (CVaR)**.** Given $\alpha \in (0,1)$ and a random variable (i.e. Y) the conditional value at risk of the random variable(i.e. Y) with confidence level α is,

$$CVaR_\alpha(Y) = -\mathbb{E}[Y|Y \leq -VaR_\alpha(Y)].$$

If the tail estimation of the model is not good then using CVaR causes problems (Sarykalin, Serraino & Uryasev [39], and Lim, Shanhikumar and Vahn [33]). Hence, practitioners sometimes prefer working with VaR instead of CVaR. We propose a model using mixing multivariate random variables to get a better tail estimation. Portfolio allocation problem using CVaR as a risk measure is given next.

Definition 4.2.5 (PAP with CVaR)**.**

$$\max_{\mathbf{x}}\{\boldsymbol{\mu}'\mathbf{x} + (1-\mathbf{e}'\mathbf{x})r_f : CVaR_\alpha((\mathbf{r}-r_f\mathbf{e})'\mathbf{x} + r_f) \leq L\} \tag{4.3}$$

4.3 Representation of the allocation vectors of BLM

In this section we give a well known model in modern portfolio optimization literature, i.e. the Black-Litterman Model (BLM), in detail. This model is a static (i.e one period) model and ignores the transaction costs. Before BLM there was no natural starting point for the expected excess return assumption.

Markowitz Model's (Defnition 4.2.2) optimal portfolio weights might be lack of diversification and/or might have corner solutions when short-sales are not allowed. Some reasons for that could be; historical returns are not a good estimate for the future returns and are very difficult to estimate. Furthermore, we know that asset returns have excess co-skewness and co-kurtosis (See also Harvey et. al. [18] and Jondeau and Rockinger [24]). Moreover, optimal portfolio weights are highly sensitive to the parameters of the optimization problem. Thus, small estimation error, especially the errors in expected excess return, might cause huge deviations on the resulting optimal portfolio weights (Chopra and Ziemba [11]). All of the drawbacks above can be overcome by using BLM as we explain below.

The detailed explanation of the model is given in He and Litterman [19] and an extensive survey for the model can be found at Walters [47]. There are two variations of this model in the literature; the canonical model and the alternate model. The fundamental difference between these two models is that the expected returns are not random variables in the alternate model.

BLM assumes CAPM equilibrium when estimating the expected excess returns as the prior distribution. In other words, CAPM equilibrium returns are the best guess for the expected excess return vector. CAPM equilibrium returns can be found using inverse optimization under the assumption that every single player in the market solves the unconstrained Markowitz Model (i.e. PAP in Definition 4.2.1). In other words, the expected excess return ($\mathbf{\Pi}$) can be estimated as the multiplying the risk-reward trade off parameter (δ), historical covariance matrix ($\mathbf{\Sigma}$) and the market weights ($\mathbf{x}_{market}$) (i.e. $\mathbf{\Pi} = 2\mathbf{\Sigma}\delta\mathbf{x}_{market}$).

Let us continue with the step where the parameters are updated using Bayesian framework. Note that, one can get the same estimations using Theil's Mixed Approximation Method (for details see Walters [47]) and using linear least square estimation (see Proposition 4.4.2). As we mentioned in the introduction, we suppose an investor have views on the assets (or asset classes) in the market. Furthermore, she takes the CAPM equilibrium as the reference point to specify their view(s). Those views are the additional information to the Bayesian framework.

The view(s) has two forms; relative view(s) (the sum of the weights add up to zero) and absolute view(s). In the absolute view, the weight is always one, since there is only one asset. Defining the weights on the view vector (this becomes a matrix if the investor has more than one view) is another matter of the model. Some of the authors use market capitalization weights when specifying the view portfolios (i.e. He and Litterman [19] and Idzorek [22]). On the other hand, some of them such as Meucci [31] and Satchell and Scowcroft [40] use an equal weighted scheme.

Papers that consider canonical model threat the expectation of the excess return as a multivariate normal variable itself. In addition to that the investor is not confidant in the estimation of the covariance matrix. Hence, a scaling parameter(τ) (which represents the uncertainty of the measurement of the covariance matrix) is used. This parameter is multiplied by the historical covariance matrix of the returns and used as the uncertainty on the CAPM equilibrium. Usually, $\tau = 1/d$ where d is the number of observations in the data set. In addition to that some of the authors (i.e. Satchell and Scowcroft [40]) take $\tau = 1$. The intuition behind that is the investors are fully confidant of their measure on the covariance matrix. The other models either eliminates τ (including Meucci [31]) or calibrates it (including Idzorek [22]).

On the other hand, the expected returns are not random variables in the alternate model. However, they are centered at the CAPM equilibrium (i.e. $\mathbf{\Pi}$). In addition to that, multivariate elliptical distributions (Xiao and Valdez [48], and Pang and Karan [35]) and stable distributions (Giacometti et. al. [17]) are also used to model the asset return distribution in this setting. Let us continue with the formal definition for the canonical model.

Specifying the $\mathbf{\Omega}$ matrix which is related with the confidence levels of the investors on their views is difficult. Therefore, we want to get rid of the $\mathbf{\Omega}$ matrix and directly calculate the portfolio weight vector. We consider the alternate version of the Black Litterman Model (ABLM) and consider two extreme cases where in the first case; investor has no confidence (i.e. $\omega \to \infty$), and in the second case; investor has full confidence on views (i.e. $\omega \to 0$). Let us continue with the Canonical Black Litterman Model (CBLM).

Let $\mathbf{r}$ be the asset return vector with a multivariate normal distribution, and defined as

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \mathbf{\Sigma}) \tag{4.4}$$

As we have stated earlier $\boldsymbol{\mu}$ is a random vector itself;

$$\boldsymbol{\mu} \sim N(\boldsymbol{\Pi}, \tau\boldsymbol{\Sigma}) \tag{4.5}$$

where $\boldsymbol{\Pi}$ and τ are defined above. Another way to denote the randomness in equation (4.5) is the following:

$$\boldsymbol{\mu} = \boldsymbol{\Pi} + \boldsymbol{\epsilon}^{eq} \tag{4.6}$$

where

$$\boldsymbol{\epsilon}^{eq} \sim N(\mathbf{0}, \tau\boldsymbol{\Sigma}).$$

The views of the investors are given in the following representation,

$$\mathbf{P}\boldsymbol{\mu} \sim N(\mathbf{q}, \boldsymbol{\Omega}) \tag{4.7}$$

where $\mathbf{P}$ is the view portfolio vector/matrix, $\mathbf{q}$ is the investor's expectation about her view portfolio, and $\boldsymbol{\Omega}$ is a diagonal positive semidefinite matrix stands for the uncertainty level of the investor's linear view(s). One can also use the following equation to model the views of the investors:

$$\mathbf{P}\boldsymbol{\mu} = \mathbf{q} + \boldsymbol{\epsilon}^{view} \tag{4.8}$$

where

$$\boldsymbol{\epsilon}^{view} \sim N(\mathbf{0}, \boldsymbol{\Omega}).$$

In the original model $\boldsymbol{\epsilon}^{eq}$ and $\boldsymbol{\epsilon}^{view}$ are independent. After using Bayesian framework we get the expectation and the covariance matrix for the expected excess return as:

$$\boldsymbol{\mu}_{BL} = \left((\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1} \left((\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{q}\right), \tag{4.9}$$

$$\boldsymbol{\Sigma}^{\mu}_{BL} = \left((\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right)^{-1}. \tag{4.10}$$

Now, since we have

$$\boldsymbol{\mu} \sim N(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}^{\mu}_{BL}) \tag{4.11}$$

the excess return vector will be the following because of equations (4.4), (4.6) and (4.11)

$$\mathbf{r} \sim N(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}) \tag{4.12}$$

where

$$\boldsymbol{\Sigma}_{BL} = \boldsymbol{\Sigma} + \boldsymbol{\Sigma}^{\mu}_{BL}.$$

Meucci [31] gives the compact version of the mean vector and the covariance matrix using Sherman-Morrison-Woodbury matrix formula:

$$\boldsymbol{\mu}_{BL} = \boldsymbol{\Pi} + \tau\boldsymbol{\Sigma}\mathbf{P}'(\boldsymbol{\Omega} + \tau\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi}), \tag{4.13}$$

$$\boldsymbol{\Sigma}_{BL} = (1+\tau)\boldsymbol{\Sigma} - \tau^2\boldsymbol{\Sigma}\mathbf{P}'(\boldsymbol{\Omega} + \tau\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{4.14}$$

He and Litterman [19] give the detailed explanation of the original model. Equation (17) in their model gives representation of the optimal weights under general view portfolio uncertainty matrix (i.e. no restrictions on $\boldsymbol{\Omega}$) for the weights under no view portfolios. They fix the view portfolio's uncertainty matrix (i.e. $\boldsymbol{\Omega}$) to $\omega\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'$ where $\omega \in [0, \infty)$ denotes the overall uncertainty of the view portfolio (Meucci [31] also uses this idea). Hence, we can analyze the BLM under two extremes. We give the connection between the weights, under no confidence (i.e. $\omega \to \infty$), full confidence (i.e. $\omega = 0$) and the general confidence (i.e. $\omega \in (0, \infty)$) below. Let us use Meucci's [31] nicer forms:

$$\boldsymbol{\mu}_{\omega} \;=\; \boldsymbol{\Pi} + \frac{\tau}{\tau+\omega}\boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi}), \tag{4.15}$$

$$\boldsymbol{\Sigma}_\omega = (1+\tau)\boldsymbol{\Sigma} - \frac{\tau^2}{\tau+\omega}\boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{4.16}$$

Next, we observe the updated model parameters in the two extremes (i.e. when $\omega \to 0$ and $\omega \to \infty$). In the case where $\omega \to 0$ (in other words, investor is 100% sure about her view portfolio) the parameters are updated to:

$$\boldsymbol{\mu}_{fc} = \boldsymbol{\Pi} + \boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi}), \tag{4.17}$$

$$\boldsymbol{\Sigma}_{fc} = (1+\tau)\boldsymbol{\Sigma} - \tau\boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{4.18}$$

On the other hand, when $\omega \to \infty$ (in other words, investor not sure at all, or we can say that she does not have any confidence about what is going on in the market) we have;

$$\boldsymbol{\mu}_{nc} = \boldsymbol{\Pi}, \tag{4.19}$$

$$\boldsymbol{\Sigma}_{nc} = (1+\tau)\boldsymbol{\Sigma}. \tag{4.20}$$

We want to represent the parameters under general uncertainty using parameters under full confidence and no confidence. After doing some algebra, we get,

$$\boldsymbol{\mu}_\omega = \boldsymbol{\mu}_{fc}\left(\frac{\tau}{\tau+\omega}\right) + \boldsymbol{\mu}_{nc}\left(1 - \frac{\tau}{\tau+\omega}\right) \tag{4.21}$$

$$\boldsymbol{\Sigma}_\omega = \boldsymbol{\Sigma}_{fc}\left(\frac{\tau}{\tau+\omega}\right) + \boldsymbol{\Sigma}_{nc}\left(1 - \frac{\tau}{\tau+\omega}\right) \tag{4.22}$$

Observe that the mean vector is the weighted average of the mean vector with full confidence and no confidence, and covariance matrix is also the weighted average of covariance matrix with full confidence and no confidence. We give the optimal portfolio weights for BLM with the uncertainty parameter, ω, in terms of optimal portfolio weights under no confidence and full confidence in Proposition 4.3.1.

Proposition 4.3.1. Consider CBLM unconstrained model and let $\mathbf{x}_{nc}$ and $\mathbf{x}_{fc}$ be the optimal allocation vectors with no confidence and full confidence respectively. Then, for any given $\omega \in [0, \infty)$ we have

$$\mathbf{x}_\omega = \left(\boldsymbol{\Sigma}_{fc}\left(\frac{\tau}{\tau+\omega}\right) + \boldsymbol{\Sigma}_{nc}\left(1 - \frac{\tau}{\tau+\omega}\right)\right)^{-1} \cdot \left[\boldsymbol{\Sigma}_{fc}\left(\frac{\tau}{\tau+\omega}\right)\mathbf{x}_{fc} + \boldsymbol{\Sigma}_{nc}\left(1 - \frac{\tau}{\tau+\omega}\right)\mathbf{x}_{nc}\right]. \tag{4.23}$$

Proof. Note that we have an unconstrained convex optimization problem, therefore, by using the first order necessity condition we get;

$$\mathbf{x}^*_\omega = (\delta \boldsymbol{\Sigma}_\omega)^{-1} \boldsymbol{\mu}_\omega. \tag{4.24}$$

Using this equation we get

$$\boldsymbol{\mu}_\omega = \delta \boldsymbol{\Sigma}_\omega \mathbf{x}^*_\omega. \tag{4.25}$$

Using the same argument we can get the corresponding equations for weights under full confidence and no confidence, $\boldsymbol{\mu}_{fc}$ and $\boldsymbol{\mu}_{nc}$ respectively. We use equation (4.21) to replace the left hand side of equation (4.25). The resulting equation is

$$\boldsymbol{\mu}_{fc}\left(\frac{\tau}{\tau+\omega}\right) + \boldsymbol{\mu}_{nc}\left(1 - \frac{\tau}{\tau+\omega}\right) = \delta \boldsymbol{\Sigma}_\omega \mathbf{x}^*_\omega.$$

Next we use the equations for $\boldsymbol{\mu}_{fc}$ and $\boldsymbol{\mu}_{nc}$ to replace the left hand side.

$$(\delta \boldsymbol{\Sigma}_{fc} \mathbf{x}^*_{fc})\left(\frac{\tau}{\tau+\omega}\right) + (\delta \boldsymbol{\Sigma}_{nc} \mathbf{x}^*_{nc})\left(1 - \frac{\tau}{\tau+\omega}\right) = \delta \boldsymbol{\Sigma}_\omega \mathbf{x}^*_\omega.$$

Then using equation (4.22) we get the desired result. □

Let us derive the parameter of the alternate model. Let $\mathbf{r}$ be the excess asset return vector.

Moreover, we have

$$\mathbf{r} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{4.26}$$

Now, $\boldsymbol{\mu}$ is a not random. It is centered at the CAPM equilibrium. Hence,

$$\boldsymbol{\mu} = \boldsymbol{\Pi}. \tag{4.27}$$

If we combine (4.26) with (4.27) we get,

$$\mathbf{r} \sim N(\boldsymbol{\Pi}, \boldsymbol{\Sigma}). \tag{4.28}$$

We use the following representation for the investor's view portfolio,

$$\mathbf{P}\boldsymbol{\mu} \sim N(\mathbf{q}, \boldsymbol{\Omega}). \tag{4.29}$$

We get the following distribution for the excess return vector by using Bayesian framework:

$$\boldsymbol{\mu}_{BL}^{alt} = \boldsymbol{\Pi} + \boldsymbol{\Sigma}\mathbf{P}' \left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1} (\mathbf{q} - \mathbf{P}\boldsymbol{\Pi}), \tag{4.30}$$

$$\boldsymbol{\Sigma}_{BL}^{alt} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}' \left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1} \mathbf{P}\boldsymbol{\Sigma}. \tag{4.31}$$

Hence,

$$\mathbf{r} \sim N(\boldsymbol{\mu}_{BL}^{alt}, \boldsymbol{\Sigma}_{BL}^{alt}). \tag{4.32}$$

Note that τ is missing in this representation.

We want to analyze the BLM under two extremes scenarios, full confidence (i.e. $\omega \to 0$), and no confidence (i.e. $\omega \to \infty$). Hence, let us again fix the view portfolio's uncertainty matrix (i.e. $\boldsymbol{\Omega}$) to $\omega\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}'$ where $\omega \in [0, \infty)$ denotes the overall uncertainty of the view portfolio. Meucci [31] gives the nicer forms of the expected return and covariance matrix using Sherman-

Morrison-Woodbury matrix formula;

$$\boldsymbol{\mu}_\omega = \boldsymbol{\Pi} + \frac{1}{1+\omega}\boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi}), \tag{4.33}$$

$$\boldsymbol{\Sigma}_\omega = \boldsymbol{\Sigma} - \frac{1}{1+\omega}\boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{4.34}$$

Next, we observe the updated model parameters in the two extremes (i.e. when $\omega \to 0$ and $\omega \to \infty$). In the case where $\omega \to 0$ our parameters are updated to:

$$\boldsymbol{\mu}_{fc} = \boldsymbol{\Pi} + \boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi}), \tag{4.35}$$

$$\boldsymbol{\Sigma}_{fc} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{4.36}$$

On the other hand, when $\omega \to \infty$ we have:

$$\boldsymbol{\mu}_{nc} = \boldsymbol{\Pi}, \tag{4.37}$$

$$\boldsymbol{\Sigma}_{nc} = \boldsymbol{\Sigma}. \tag{4.38}$$

We want to represent the parameters under general uncertainty using parameters under full confidence and no confidence. After doing some algebra, we get:

$$\boldsymbol{\mu}_\omega = \boldsymbol{\mu}_{fc}\left(\frac{1}{1+\omega}\right) + \boldsymbol{\mu}_{nc}\left(1 - \frac{1}{1+\omega}\right), \tag{4.39}$$

$$\boldsymbol{\Sigma}_\omega = \boldsymbol{\Sigma}_{fc}\left(\frac{1}{1+\omega}\right) + \boldsymbol{\Sigma}_{nc}\left(1 - \frac{1}{1+\omega}\right). \tag{4.40}$$

Observe that the mean vector is the weighted average of the mean vector with full confidence and no confidence, and covariance matrix is also the weighted average of covariance matrix with full confidence and no confidence.

Idzorek [22] states that it is more intuitive working with confidence levels (i.e. a number between 0 and 1) than working with uncertainties (i.e. a number between 0 and ∞). Idzorek [22], defines a function $f : [0, \infty) \to [0, 1]$ such that

$$f(\omega) = \frac{1}{1+\omega} \equiv c. \tag{4.41}$$

Using this f function, we have a new representation for the problem parameters:

$$\boldsymbol{\mu}_c = \boldsymbol{\mu}_{fc}\,(c) + \boldsymbol{\mu}_{nc}\,(1-c)\,, \tag{4.42}$$

$$\boldsymbol{\Sigma}_c = \boldsymbol{\Sigma}_{fc}\,(c) + \boldsymbol{\Sigma}_{nc}\,(1-c)\,. \tag{4.43}$$

Now, our investor can provide her confidence on her view portfolio (i.e. c). Moreover, Idzorek [22] proposes the following linear interpolation between the full confidence allocation vector, no confidence allocation vector and allocation vector with c confidence level:

$$\mathbf{x}_c = c\mathbf{x}_{fc} + (1-c)\mathbf{x}_{nc}. \tag{4.44}$$

We show in Proposition 4.3.2 that the optimal portfolio weights for BLM with c level of confidence in terms of optimal portfolio weights under no confidence and full confidence are in fact more complex than Izdorek's proposal (i.e. equation (4.44)).

Proposition 4.3.2. Let $\mathbf{x}_{nc}$ and $\mathbf{x}_{fc}$ be the weights of the assets with no confidence and full confidence on views under the ABLM. For any given $\omega \in [0, \infty)$ and under the function $f(\omega) = \frac{1}{1+\omega} = c$ we have:

$$\mathbf{x}_c = \left((1-c)\boldsymbol{\Sigma}_{nc} + c\boldsymbol{\Sigma}_{fc}\right)^{-1}\left(c\boldsymbol{\Sigma}_{fc}\mathbf{x}_{fc} + (1-c)\boldsymbol{\Sigma}_{nc}\mathbf{x}_{nc}\right). \tag{4.45}$$

Proof. Let us start with the optimal solutions of the unconstrained portfolio optimization problems (where δ is the risk-reward trade of coefficient of the investor) under full confidence, no confidence and c level of confidence;

$$\mathbf{x}_c^* = (\delta \boldsymbol{\Sigma}_c)^{-1} \boldsymbol{\mu}_c. \tag{4.46}$$

Using this equation we get

$$\boldsymbol{\mu}_c = \delta \boldsymbol{\Sigma}_c \mathbf{x}_c^*. \tag{4.47}$$

We can rewrite equation (4.47) using equation (4.42) and get

$$c\boldsymbol{\mu}_{fc} + (1-c)\boldsymbol{\mu}_{nc} = \delta \boldsymbol{\Sigma}_c \mathbf{x}_c^*.$$

Next we use the equations for $\boldsymbol{\mu}_{fc}$ and $\boldsymbol{\mu}_{nc}$ to replace the left hand side.:

$$c\left(\delta \boldsymbol{\Sigma}_{fc} \mathbf{x}_{fc}^*\right) + (1-c)\left(\delta \boldsymbol{\Sigma}_{nc} \mathbf{x}_{nc}^*\right) = \delta \boldsymbol{\Sigma}_c \mathbf{x}_c^*.$$

Now, note that

$$\boldsymbol{\Sigma}_{fc} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')\boldsymbol{\Sigma}\mathbf{P}, \tag{4.48}$$

$$\boldsymbol{\Sigma}_{nc} = \boldsymbol{\Sigma}, \text{ and} \tag{4.49}$$

$$\boldsymbol{\Sigma}_c = \boldsymbol{\Sigma} - c\boldsymbol{\Sigma}\mathbf{P}'(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')\boldsymbol{\Sigma}\mathbf{P}. \tag{4.50}$$

Using equations (4.48), (4.49), and (4.50) together to replace the $\boldsymbol{\Sigma}_c$ in the above equation gives the desired equation for the allocation vector under c level of confidence. □

So far we have been working with unconstrained models. Now, suppose that our investor cannot estimate her risk-reward trade-off. Therefore, she is going to use an upper bound to limit the risk of her portfolio. Note that problem parameters are updated using the Black-Litterman framework (either Canonical or Alternate). Once we have the updated parameters the rest is nothing but Markowitz Portfolio Allocation problem (i.e. problem given in Definition 4.2.2).

We know the explicit solution of Markowitz PAP. Finding the optimal value of the dual variable of the Markowitz PAP (i.e. λ) is given next.

Proposition 4.3.3 (Dual Variable of the Markowitz PAP at Optimality)**.** Consider the portfolio allocation problem (i.e. Definition 4.2.2). Then the Lagrangian variable λ (or investor risk reward parameter) is the following:

$$\lambda = \frac{\sqrt{(\boldsymbol{\mu} - \mathbf{e}r_f)' \, \boldsymbol{\Sigma}^{-1} \, (\boldsymbol{\mu} - \mathbf{e}r_f)}}{2L} \tag{4.51}$$

Proof. Note that we have the following problem on hand

$$\max_{\mathbf{x}} \{\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \leq L^2\}$$

Let us start by writing the Lagrangian Function of this optimization problem and continue by the Lagrangian approach to get the dual of that problem

$$\begin{aligned}
\min_{\lambda} \max_{\mathbf{x}} \left(\mathcal{L}(\mathbf{x}, \lambda)\right) &= \min_{\lambda} \max_{\mathbf{x}} \left[\boldsymbol{\mu}'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f + \lambda\left(L^2 - \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}\right)\right] \\
&= \min_{\lambda} \max_{\mathbf{x}} \left[\boldsymbol{\mu}'\mathbf{x} + r_f - \mathbf{e}'\mathbf{x}r_f + \lambda L^2 - \lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}\right] \\
&= \min_{\lambda} \max_{\mathbf{x}} \left[(\boldsymbol{\mu} - \mathbf{e}r_f)'\mathbf{x} + r_f + \lambda L^2 - \lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}\right] \\
&= \min_{\lambda} \left[\lambda L^2 + \max_{\mathbf{x}} (\boldsymbol{\mu} - \mathbf{e}r_f)'\mathbf{x} + r_f - \lambda \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}\right] \\
&= \min_{\lambda} \left[\lambda L^2 + \frac{(\boldsymbol{\mu} - \mathbf{e}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f)}{2\lambda} - \frac{(\boldsymbol{\mu} - \mathbf{e}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f)}{4\lambda} + r_f\right] \\
&\quad \left(\text{since } \mathbf{x}* = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f)}{2\lambda}\right)
\end{aligned}$$

$$= \min_{\lambda} \left[\lambda L^2 + \frac{(\boldsymbol{\mu} - \mathbf{e}r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}r_f)}{4\lambda} + r_f \right]$$

Therefore, the dual of the optimization problem is:

$$\min_{\lambda} \left\{ \lambda L^2 + \frac{(\boldsymbol{\mu} - \mathbf{e}'r_f)'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{e}'r_f)}{4\lambda} + r_f : \lambda \geq 0 \right\} \tag{4.52}$$

The first order necessary condition gives:

$$\lambda^* = \frac{\sqrt{(\boldsymbol{\mu} - \mathbf{e}r_f)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \mathbf{e}r_f)}}{2L}.$$

Note that L is nonnegative by definition and this implies that, λ^* is nonnegative. Therefore, λ^* is the optimal solution. □

Next, we use the optimal dual variables to find the optimal portfolio allocation vector with general uncertainty (confidence for alternate model) using the optimal allocation vectors with full confidence and no confidence.

Proposition 4.3.4. Let $\mathbf{x}_n$ and $\mathbf{x}_f$ be the weights of the assets with no confidence and full confidence levels under the Canonical Black-Litterman Model (CBLM). Then $\mathbf{x}_\omega$ can be represented as:

$$\mathbf{x}_\omega = {\lambda_\omega}^{-1} \left(\boldsymbol{\Sigma}_f \left(\frac{\tau}{\tau + \omega} \right) + \boldsymbol{\Sigma}_n \left(1 - \frac{\tau}{\tau + \omega} \right) \right)^{-1} \left(\left(\frac{\tau}{\tau + \omega} \right) \lambda_f \boldsymbol{\Sigma}_f \mathbf{x}_f + \left(1 - \frac{\tau}{\tau + \omega} \right) \lambda_n \boldsymbol{\Sigma}_n \mathbf{x}_n \right)$$

where, λ's have the following form:

$$\lambda = \frac{\sqrt{(\boldsymbol{\mu} - \mathbf{e}r_f)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \mathbf{e}r_f)}}{L}.$$

Proof. Note that we have a constrained convex optimization problem, therefore, by using the

Lagrangian Function we get;

$$\mathbf{x}^*_\omega = (\lambda_\omega \mathbf{\Sigma}_\omega)^{-1} \boldsymbol{\mu}_\omega \tag{4.53}$$

where λ_ω is the solution to Proposition 4.3.3 under specific problem parameters. Using this equation we get

$$\boldsymbol{\mu}_\omega = \lambda_\omega \mathbf{\Sigma}_\omega \mathbf{x}^*_\omega. \tag{4.54}$$

We use equation (4.21) to replace the left hand side of equation (4.54). The resulting equation is:

$$\boldsymbol{\mu}_{fc}\left(\frac{\tau}{\tau+\omega}\right) + \boldsymbol{\mu}_{nc}\left(1 - \frac{\tau}{\tau+\omega}\right) = \lambda_\omega \mathbf{\Sigma}_\omega \mathbf{x}^*_\omega.$$

Next follow the same logic when we get $\boldsymbol{\mu}_\omega$ to get the corresponding equations for expected return vectors under full confidence and no confidence, and we use these equations to replace $\boldsymbol{\mu}_{fc}$ and $\boldsymbol{\mu}_{nc}$ on the left hand side.

$$\left(\lambda_{fc}\mathbf{\Sigma}_{fc}\mathbf{x}^*_{fc}\right)\left(\frac{\tau}{\tau+\omega}\right) + \left(\lambda_{nc}\mathbf{\Sigma}_{nc}\mathbf{x}^*_{nc}\right)\left(1 - \frac{\tau}{\tau+\omega}\right) = \lambda_\omega \mathbf{\Sigma}_\omega \mathbf{x}^*_\omega.$$

Then using equation (4.22) we get the desired result. □

For the Alternate Model we have the representation given in Proposition 4.3.5.

Proposition 4.3.5. Let $\mathbf{x}_n$ and $\mathbf{x}_f$ be the weights of the assets with no confidence and full confidence levels under the Alternate Black-Litterman Model (ABLM). Then, for any given $\omega \in [0, \infty)$ and under the function $f(\omega) = \frac{1}{1+\omega} = c$ we have;

$$\mathbf{x}_c = \left(\lambda_c((1-c)\mathbf{\Sigma}_n + c\mathbf{\Sigma}_f)\right)^{-1}\left(c\lambda_f \mathbf{\Sigma}_f \mathbf{x}_f + (1-c)\lambda_n \mathbf{\Sigma}_n \mathbf{x}_n\right),$$

where λ's have the following form;

$$\lambda = \frac{\sqrt{(\boldsymbol{\mu} - \mathbf{e}r_f)' \mathbf{\Sigma}^{-1} (\boldsymbol{\mu} - \mathbf{e}r_f)}}{L}$$

Proof. We start with the optimal solution of the constrained PAP (where λ_c is the dual variable and can be explicitly found via using Proposition 4.3.3);

$$\mathbf{x}_c^* = (\lambda_c \boldsymbol{\Sigma}_c)^{-1} \boldsymbol{\mu}_c. \tag{4.55}$$

Using this equation we get

$$\boldsymbol{\mu}_c = \lambda_c \boldsymbol{\Sigma}_c \mathbf{x}_c^*. \tag{4.56}$$

We can rewrite equation (4.56) using equation (4.42) and get

$$c\boldsymbol{\mu}_{fc} + (1-c)\boldsymbol{\mu}_{nc} = \lambda_c \boldsymbol{\Sigma}_c \mathbf{x}_c^*.$$

Next we use the equations for $\boldsymbol{\mu}_{fc}$ and $\boldsymbol{\mu}_{nc}$ to replace the left hand side.

$$c\left(\lambda_{fc}\boldsymbol{\Sigma}_{fc}\mathbf{x}_{fc}^*\right) + (1-c)\left(\lambda_{nc}\boldsymbol{\Sigma}_{nc}\mathbf{x}_{nc}^*\right) = \lambda_c \boldsymbol{\Sigma}_c \mathbf{x}_c^*.$$

Using equations (4.48), (4.49), and (4.50), we can replace $\boldsymbol{\Sigma}_c$ and get the desired equation for the allocation vector under c level of confidence. □

We continue with the BLM under elliptical distributions ($ED_n(\cdot)$) given in Xiao and Valdez [48]. Let $\mathbf{r} \sim ED_n(\boldsymbol{\mu}, \mathbf{D}, g_n)$ be an n dimensional vector that denotes the market factors, where $\boldsymbol{\mu}$, $\mathbf{D}$ and g_n are the location parameter, dispersion matrix and the density generator function respectively (for more on elliptical distributions please see Fang et. al. [14]). Furthermore, conditional random view vector is: $\mathbf{v}|\mathbf{r} \sim ED_k(\mathbf{P}\boldsymbol{\mu}, \boldsymbol{\Omega}, g_k(\cdot; p(\mathbf{r})))$ where $p(\mathbf{r}) = (\mathbf{r}-\boldsymbol{\Pi})'\boldsymbol{\Sigma}'(\mathbf{r}-\boldsymbol{\Pi})$. The posterior distribution is given in the following proposition.

Definition 4.3.1 (Xiao and Valdez [48]). The posterior distribution is

$$\mathbf{r}|\mathbf{v} \sim ED_k(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}, g_n(.; q(\mathbf{v}))),$$

where

$$\boldsymbol{\mu}_{BL} = \boldsymbol{\Pi} + \mathbf{D}\mathbf{P}'(\boldsymbol{\Omega} + \mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{v} - \mathbf{P}\boldsymbol{\Pi}),$$
$$\mathbf{D}_{BL} = \boldsymbol{\Sigma} - \mathbf{D}\mathbf{P}'(\boldsymbol{\Omega} + \mathbf{P}\mathbf{D}\mathbf{P}')^{-1}\mathbf{P}\mathbf{D},$$

and

$$\boldsymbol{\Sigma}_{BL} = \mathbf{D}_{BL} C_k(q(\mathbf{v})/2),$$

where C_k is a distribution specific function from $\mathbb{R}$ to $\mathbb{R}$ and $q(\mathbf{v}) = (\mathbf{v}-\mathbf{P}\boldsymbol{\Pi})'(\boldsymbol{\Omega}+\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}')^{-1}(\mathbf{v}-\mathbf{P}\boldsymbol{\Pi})$.

We use the results from Chapter 3 (in particular Theorem 3.6.2) to construct the closed form solutions of the optimal asset allocation vector with CVaR and elliptically distributed asset returns for BLM. It is easy to see that Theorem 3.6.2 updates just the upper bound for the risk level (see the discussions on Chapter 3). We give result in Proposition 4.3.6.

Proposition 4.3.6. Let $\mathbf{x}_n$ and $\mathbf{x}_f$ be the weights of the assets with no confidence and full confidence levels under the Alternate Black-Litterman Model (ABLM). Then, for any given $\omega \in [0, \infty)$ and under the function $f(\omega) = \frac{1}{1+\omega} = c$ we have:

$$\mathbf{x}_c = (\lambda_c((1-c)\boldsymbol{\Sigma}_n + c\boldsymbol{\Sigma}_f))^{-1}\left(c\lambda_f\boldsymbol{\Sigma}_f\mathbf{x}_f + (1-c)\lambda_n\boldsymbol{\Sigma}_n\mathbf{x}_n\right),$$

where λ's have the following form:

$$\lambda = \frac{(\beta_\alpha - d)\, d}{L + r_f}$$

where

$$d = \sqrt{(\boldsymbol{\mu} - \mathbf{e}r_f)'\,\boldsymbol{\Sigma}^{-1}\,(\boldsymbol{\mu} - \mathbf{e}r_f)}.$$

Proof. We start with the optimal solution of the constrained PAP. Note that λ_c is the dual variable and can be explicitly found via using Theorem 3.6.2:

$$\mathbf{x}_c^* = (\lambda_c)^{-1}\,\boldsymbol{\Sigma}_c^{-1}\boldsymbol{\mu}_c. \tag{4.57}$$

Using this equation we get

$$\boldsymbol{\mu}_c = \lambda_c \boldsymbol{\Sigma}_c \mathbf{x}_c^*. \tag{4.58}$$

We can rewrite equation (4.58) using equation (4.42) and get

$$c\boldsymbol{\mu}_{fc} + (1-c)\boldsymbol{\mu}_{nc} = \lambda_c \boldsymbol{\Sigma}_c \mathbf{x}_c^*.$$

Next we use the equations for $\boldsymbol{\mu}_{fc}$ and $\boldsymbol{\mu}_{nc}$ to replace the left hand side.

$$c\left(\lambda_{fc}\boldsymbol{\Sigma}_{fc}\mathbf{x}_{fc}^*\right) + (1-c)\left(\lambda_{nc}\boldsymbol{\Sigma}_{nc}\mathbf{x}_{nc}^*\right) = \lambda_c \boldsymbol{\Sigma}_c \mathbf{x}_c^*.$$

We rewrite $\boldsymbol{\Sigma}_c$ using equations (4.48), (4.49), and (4.50) and get the desired equation for the allocation vector under c level of confidence. □

4.3.1 New mapping techniques for investors uncertainty

Fixing the uncertainty matrix has great benefits. In this section, we propose a method to generate the transformation functions such as $f(\omega) = \frac{1}{1+\omega} = c$. Note that $\omega \in [0, \infty)$ and $c \in [0, 1]$ is the general uncertainty and the confidence level, respectively. One can fix the uncertainty matrix using these transformation functions.

Let $\omega \in [0, \infty)$ be the general uncertainty and let $c \in [0, 1]$ be the confidence level. Then for given values of $\gamma \in \mathbb{R}$ and $i \in \mathbb{N}$, if we solve the equation below for c, then we get the new mapping from general uncertainty to confidence level.

$$\omega = \int_c^1 \gamma \left(\frac{1}{t}\right)^i dt. \tag{4.59}$$

Next, we give three examples for how to generate transformation functions using Equation (4.59).

Example 4.3.1. Let $\omega \in [0, \infty)$ be defined as before. Using (4.59), and letting $\gamma = 1$ and $i = 2$, we get

$$\omega = \int_c^1 \left(\frac{1}{y}\right)^2 dy$$

$$= \frac{1-c}{c}.$$

If we solve for c then we get our desired function $f : [0, \infty) \to [0, 1]$,

$$f(\omega) = \frac{1}{1+\omega} = c. \tag{4.60}$$

Note that, since $f'(\omega)$ is always positive in the domain of $f(\cdot)$, $f(\cdot)$ is a non increasing function. Hence, we need to fix the uncertainty matrix as;

$$\mathbf{\Omega} = (\omega)\, \mathbf{P\Sigma P'}. \tag{4.61}$$

Hence we get the Idzorek's setting.

Example 4.3.2. Let $\omega \in [0, \infty)$ be defined as before. Using (4.59), and letting $\gamma = 1/\beta$ and $i = 1$, we get

$$\omega = \int_c^1 \frac{1}{\beta}\left(\frac{1}{y}\right) dy$$

$$= -\frac{ln(c)}{\beta}$$

If we solve for c then we get our desired function $h : [0, \infty) \to [0, 1]$,

$$h(\omega) = e^{-\beta\omega} = c_{1,\beta}. \tag{4.62}$$

Note that, since $f'(\omega)$ is always positive in the domain of $f(\cdot)$, $f(\cdot)$ is a non increasing function.

Hence, we need to fix the uncertainty matrix as;

$$\mathbf{\Omega} = \left(\frac{1 - e^{-\beta\omega}}{e^{-\beta\omega}}\right) \mathbf{P\Sigma P}'. \tag{4.63}$$

All of the equations given for c level confidence levels can be safely updated with $c_{1,\beta}$.

Example 4.3.3. Let $\omega \in [0, \infty)$ be defined as before. Using (4.59), and letting $\gamma = (1/2\beta)$ and $i = 3$, we get

$$\begin{aligned}
\omega &= \int_c^1 \frac{1}{2\beta}\left(\frac{1}{y}\right)^3 dy \\
&= -\frac{1}{\beta}\left(1 - \frac{1}{c^2}\right) \\
&= \frac{1}{\beta}\left(\frac{1 - c^2}{c^2}\right).
\end{aligned}$$

If we solve for c then we get our desired function $l : [0, \infty) \to [0, 1]$,

$$l(\omega) = \frac{1}{\sqrt{1 + \omega\beta}} = c_{2,\beta}. \tag{4.64}$$

Note that, since $f'(\omega)$ is always positive in the domain of $f(\cdot)$, $f(\cdot)$ is a non increasing function. Hence, we need to fix the uncertainty matrix as;

$$\mathbf{\Omega} = \left(\sqrt{1 + \omega\beta} - 1\right) \mathbf{P\Sigma P}'. \tag{4.65}$$

All of the equations given for c level confidence levels can be safely updated with $c_{2,\beta}$.

4.4 Multivariate Mixing Model and BLM

The reason we want to use multivariate Mixed Normal is twofold. First of all, as we mentioned in Section 4.2, if the tail estimation of the return distribution is not good then using CVaR causes problems (Sarykalin, Serraino & Uryasev [39], and Lim, Shanhikumar and Vahn [33]). Hence, practitioners sometimes prefer working with VaR instead of CVaR. We propose a model using mixing multivariate random variables to get a better tail estimation. Hence, using CVaR will be more beneficial than using VaR. Second of all, we showed in Section 4.3 that $\mathbf{\Omega}$ can be eliminated from the model under some assumptions, and using the allocation vectors in two extreme cases: no confidence and full confidence. Therefore, in this section we use two components for multivariate Mixed Normal.

Before we proceed to the main results, let us have a look at some Stocks Indices under normal distribution and under Mixed Normal distribution with two components. We collected daily data from 11/16/14 to 11/16/15. For the Mixed Normal case, we used a build in Matlab function: "fitgmdist" with two components, the maximum iteration number is 3000 and a tolerance level of 1e-9 to estimate the Mixed Normal parameters. Then the estimated parameters are fed to another build in function ("random") and 20,000 random numbers are generated. Then the sample mean and sample standard deviation are found. Now, we present the quantile to quantile plots and their explanations.

Let us start with quantile to quantile plot of the S&P500 data (with expected value $-2.7087e-06$ and standard deviation 0.0096) v.s. Standard Normal, we observe that the dashed line does not have 45 degree angle this means that we have more skewness than normal distribution. In addition to that we have fatter tails than standard normal distribution (see Figure 4.1).

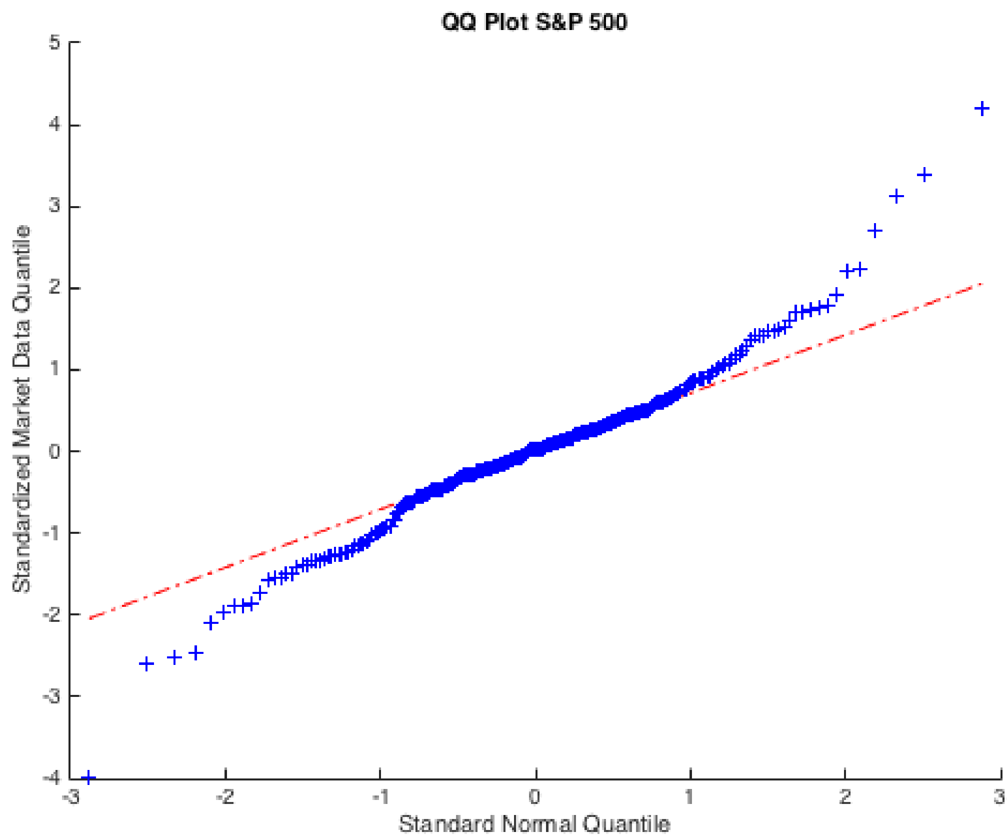

Figure 4.1: S&P 500 Normal Distribution

On the other hand, when we look at the quantile to quantile plot of the S&P500 v.s. Mixed Normal, we observe that the dashed line is almost 45 degree. Moreover, our tail estimation is relatively better (see Figure 4.2). Parameters for the Mixed Normal are; for component one: Mixing proportion: 0.645174, mean: -2.8835e-04, and sigma: 1.3480e-04; for the second component: Mixing proportion: 0.354826, mean: 5.1666e-04, and sigma: 1.2133e-05.

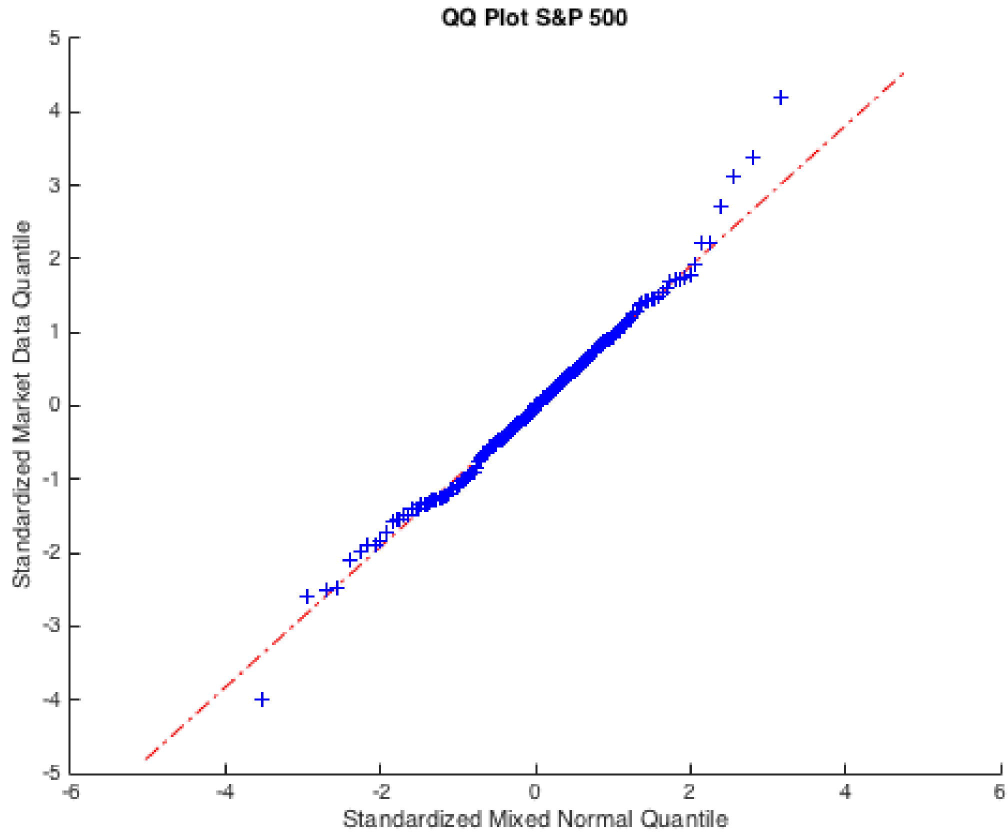

Figure 4.2: S&P 500 Mixed Normal

Let us continue with quantile to quantile plot of the BIST100 data (with expected value $-3.7759e-05$ and standard deviation 0.0134) and Standard Normal. Also in this case tail estimation is not good and we cannot model the skewness well (see Figure 4.3).

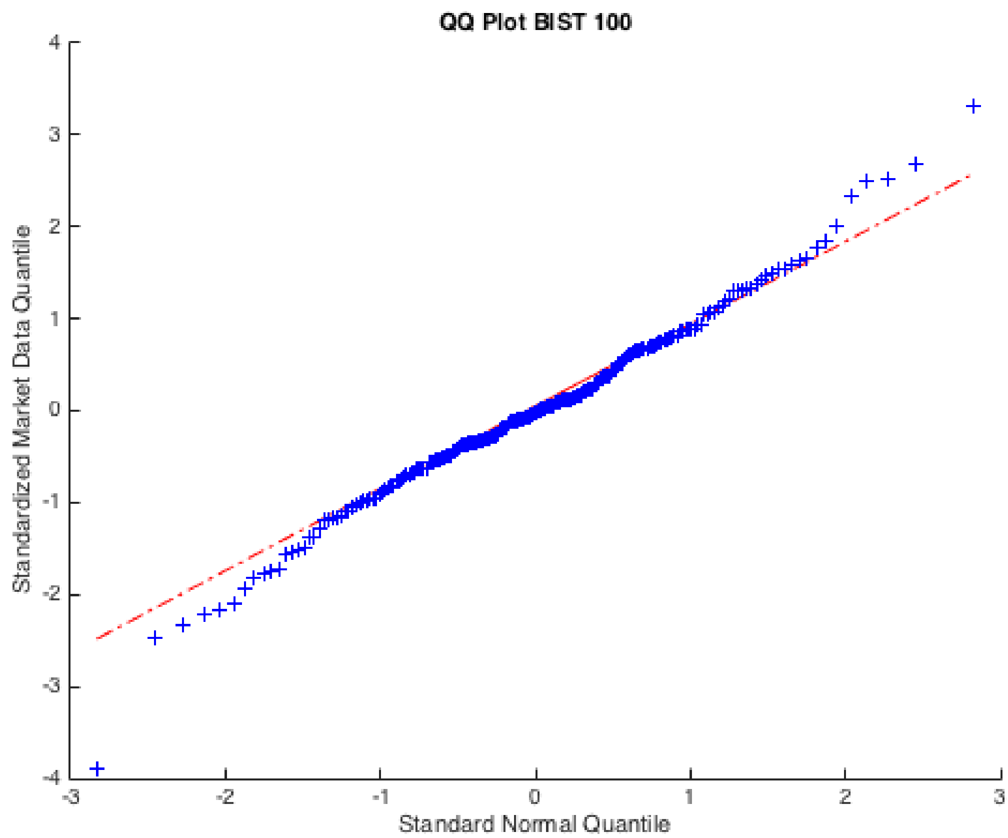

Figure 4.3: BIST 100 Normal Distribution

On the other hand, when we consider the Mixed Normal, then by looking at the quantile to quantile plot, we can say that we have a better fit in terms of skewness and tail estimation (see Figure 4.4). Parameters for the Mixed Normal are; for component one: Mixing proportion: 0.616407, mean: -2.6865e-04, and sigma: 9.2352e-05; for the second component: Mixing proportion: 0.383593, mean: 3.3326e-04, and sigma: 3.2044e-04.

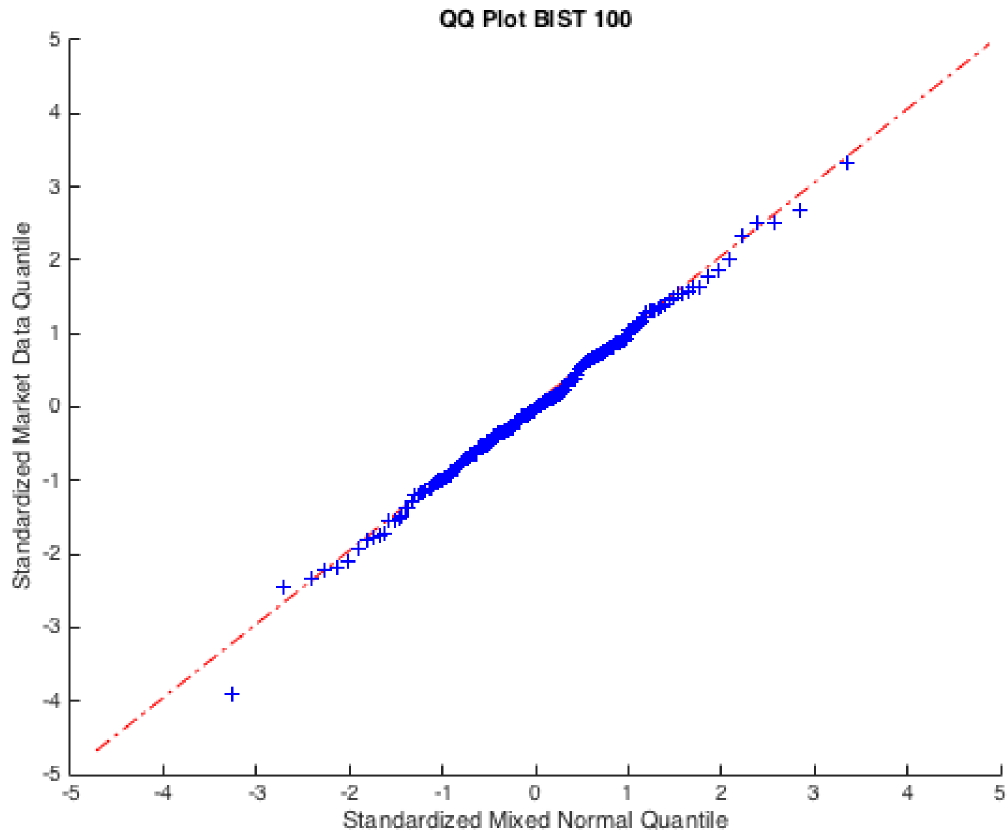

Figure 4.4: BIST 100 Mixed Normal

Our last index is Nikkei225 (expected value $-4.5039e-04$ and standard deviation 0.0131). Once more, when we compare the quantile to quantile plots under Standard Normal and Mixed Normal we can say that under Mixed Normal we have better estimations (see Figures 4.5 and 4.6). Parameters for the Mixed Normal are; for component one: Mixing proportion: 0.287528, mean: 0.0048, and sigma: 4.1075e-04; for the second component: Mixing proportion: 0.712472, mean: -0.0026, and sigma: 6.0364e-05.

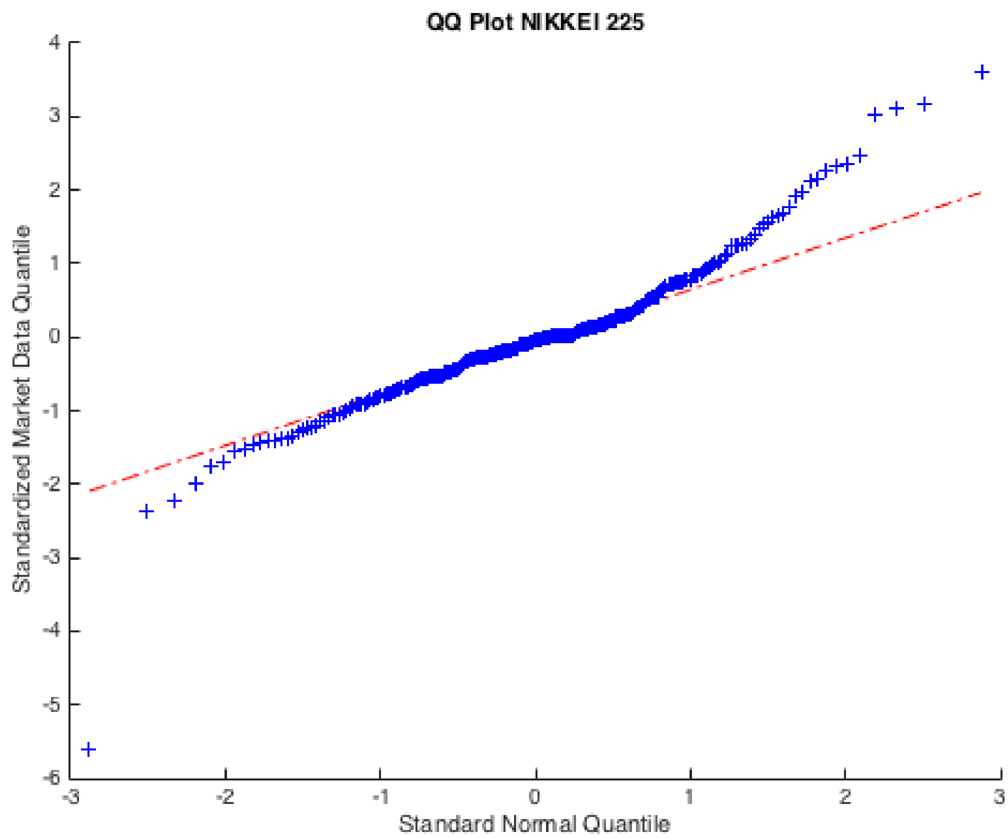

Figure 4.5: Nikkei 225 Normal Distribution

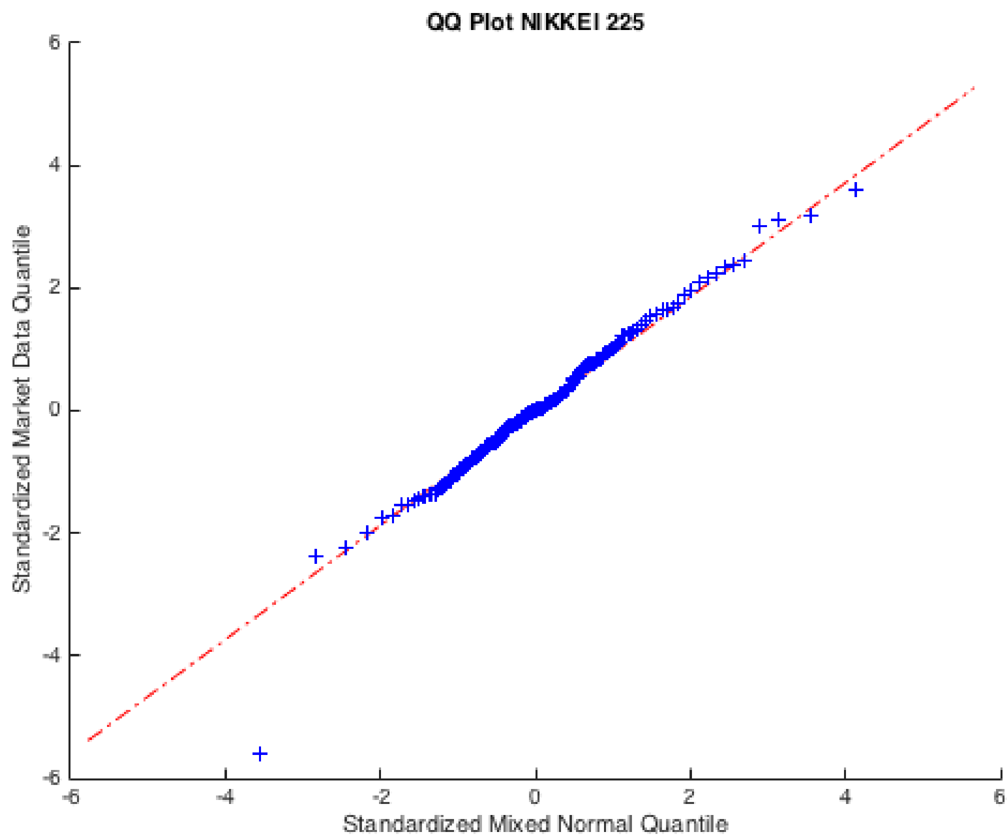

Figure 4.6: Nikkei 225 Mixed Normal

Let us continue with the definition of PAP with multivariate Mixed Normal random variable with two components (i.e. $\mathbf{r}_1$ and $\mathbf{r}_2$).

Definition 4.4.1 (Mixing Model)**.** Let $\mathbf{r}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $\mathbf{r}_2 \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$, and let $\mathbf{r}_\alpha$ be a random variable with the given moment generating function

$$\mathbb{E}[e^{\mathbf{t}\mathbf{r}_\alpha}] = \alpha \exp\left(\boldsymbol{\mu}_1'\mathbf{t} + \frac{\mathbf{t}'\boldsymbol{\Sigma}_1\mathbf{t}}{2}\right) + (1-\alpha)\exp\left(\boldsymbol{\mu}_2'\mathbf{t} + \frac{\mathbf{t}'\boldsymbol{\Sigma}_2\mathbf{t}}{2}\right). \tag{4.66}$$

Next, we give the PAP with the mixed return distribution, i.e., $\mathbf{r}_\alpha \sim MN(\alpha, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)$;

$$\max_{\mathbf{x}} \left\{ \mathbb{E}[\mathbf{r}_\alpha]'\mathbf{x} + (1 - \mathbf{e}'\mathbf{x})r_f : \sqrt{\mathbf{x}'\mathbb{E}[(\mathbf{r}_\alpha - \mathbb{E}[\mathbf{r}_\alpha])'(\mathbf{r}_\alpha - \mathbb{E}[\mathbf{r}_\alpha])]\mathbf{x}} \leq L \right\}. \tag{4.67}$$

We need to find $\mathbb{E}[\mathbf{r}_\alpha]$ and $\mathbb{E}[(\mathbf{r}_\alpha - \mathbb{E}[\mathbf{r}_\alpha])'(\mathbf{r}_\alpha - \mathbb{E}[\mathbf{r}_\alpha])]$. We accomplish this using the moment generating function of the multivariate normal distribution and it's components i.e. $\mathbf{r}_1$ and $\mathbf{r}_2$. Results given in the next proposition.

Proposition 4.4.1. Let $\mathbf{r}_1 \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ and $\mathbf{r}_2 \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$,and let $\mathbf{r}_\alpha$ be a random variable with the given moment generating function

$$\mathbb{E}[e^{\mathbf{t}\mathbf{r}_\alpha}] = \alpha \exp\left(\boldsymbol{\mu}_1'\mathbf{t} + \frac{\mathbf{t}'\boldsymbol{\Sigma}_1\mathbf{t}}{2}\right) + (1-\alpha)\exp\left(\boldsymbol{\mu}_2'\mathbf{t} + \frac{\mathbf{t}'\boldsymbol{\Sigma}_2\mathbf{t}}{2}\right). \tag{4.68}$$

Then the mean vector and the covariance matrix is given as follow respectively:

$$\boldsymbol{\mu}_\alpha = \alpha\boldsymbol{\mu}_1 + (1-\alpha)\boldsymbol{\mu}_2, \tag{4.69}$$

$$\boldsymbol{\Sigma}_\alpha = \alpha(1-\alpha)(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)' + \alpha\boldsymbol{\Sigma}_1 + (1-\alpha)\boldsymbol{\Sigma}_2. \tag{4.70}$$

Proof. It is easy to show that $\boldsymbol{\mu}_\alpha = \alpha\boldsymbol{\mu}_1 + (1-\alpha)\boldsymbol{\mu}_2$ using the moment generating function defined above. Now, let us find the covariance matrix of the mixed distribution.

$$\begin{aligned}
\mathbb{E}[\mathbf{x}'\mathbf{x}] &= \alpha \exp\left(\boldsymbol{\mu}_1'\mathbf{t} + \frac{\mathbf{t}'\boldsymbol{\Sigma}_1\mathbf{t}}{2}\right)(\boldsymbol{\mu}_1' + \boldsymbol{\Sigma}_1\mathbf{t})'(\boldsymbol{\mu}_1' + \boldsymbol{\Sigma}_1\mathbf{t}) \\
&+ \alpha \exp\left(\boldsymbol{\mu}_1'\mathbf{t} + \frac{\mathbf{t}'\boldsymbol{\Sigma}_1\mathbf{t}}{2}\right)\boldsymbol{\Sigma}_1 \\
&+ (1-\alpha)\exp\left(\boldsymbol{\mu}_2'\mathbf{t} + \frac{\mathbf{t}'\boldsymbol{\Sigma}_2\mathbf{t}}{2}\right)(\boldsymbol{\mu}_2' + \boldsymbol{\Sigma}_2\mathbf{t})'(\boldsymbol{\mu}_2' + \boldsymbol{\Sigma}_2\mathbf{t})
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (1-\alpha)\exp\left(\boldsymbol{\mu}_2'\mathbf{t} + \frac{\mathbf{t}'\boldsymbol{\Sigma}_2\mathbf{t}}{2}\right)\boldsymbol{\Sigma}_2|_{\mathbf{t}=\mathbf{0}} \\
&= \quad \alpha(\boldsymbol{\mu}_1'\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_1) + (1-\alpha)(\boldsymbol{\mu}_2'\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_2).
\end{aligned}
$$

Now, we are ready to derive the covariance matrix.

$$
\begin{aligned}
\boldsymbol{\Sigma}_\alpha &= \alpha(\boldsymbol{\mu}_1'\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_1) + (1-\alpha)(\boldsymbol{\mu}_2'\boldsymbol{\mu}_2 + \boldsymbol{\Sigma}_2) - (\alpha\boldsymbol{\mu}_1 + (1-\alpha)\boldsymbol{\mu}_2)'\left((1-\alpha)\boldsymbol{\mu}_1 + \alpha\boldsymbol{\mu}_2\right) \\
&= \alpha\boldsymbol{\Sigma}_1 + (1-\alpha)\boldsymbol{\Sigma}_2 + \alpha(\boldsymbol{\mu}_1'\boldsymbol{\mu}_1) + (1-\alpha)(\boldsymbol{\mu}_2'\boldsymbol{\mu}_2) \\
&- \alpha^2\boldsymbol{\mu}_1'\boldsymbol{\mu}_1 - (1-\alpha)^2\boldsymbol{\mu}_2'\boldsymbol{\mu}_2 - \alpha(1-\alpha)(\boldsymbol{\mu}_1'\boldsymbol{\mu}_2 + \boldsymbol{\mu}_2'\boldsymbol{\mu}_1) \\
&= \alpha\boldsymbol{\Sigma}_1 + (1-\alpha)\boldsymbol{\Sigma}_2 + \alpha(1-\alpha)(\boldsymbol{\mu}_1'\boldsymbol{\mu}_1) + \alpha(1-\alpha)(\boldsymbol{\mu}_2'\boldsymbol{\mu}_2) \\
&- \alpha(1-\alpha)(\boldsymbol{\mu}_1'\boldsymbol{\mu}_2 + \boldsymbol{\mu}_2'\boldsymbol{\mu}_1) \\
&= \alpha\boldsymbol{\Sigma}_1 + (1-\alpha)\boldsymbol{\Sigma}_2 + \alpha(1-\alpha)\boldsymbol{\mu}_1'(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) + \alpha(1-\alpha)\boldsymbol{\mu}_2'(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1) \\
&= \alpha\boldsymbol{\Sigma}_1 + (1-\alpha)\boldsymbol{\Sigma}_2 + \alpha(1-\alpha)(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)'(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2). \quad \square
\end{aligned}
$$

Next we state that the Alternate Black Litterman model can also be model as linear least square estimation.

Proposition 4.4.2 (ABLM with linear least square estimation)**.** Let $\mathbf{q} = \mathbf{P}\boldsymbol{\mu} + \boldsymbol{\epsilon}_1$ where $\boldsymbol{\epsilon}_1 \sim N(0, \boldsymbol{\Omega})$ and $\boldsymbol{\mu} = \boldsymbol{\Pi} + \boldsymbol{\epsilon}_2$ where $\boldsymbol{\epsilon}_2 \sim N(0, \boldsymbol{\Sigma})$. Then,

$$
\tilde{\boldsymbol{\mu}} = \boldsymbol{\Pi} + \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi}), \tag{4.71}
$$

$$\tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \tag{4.72}$$

Proof. Let $\mathbf{q} = \mathbf{P}\boldsymbol{\mu} + \boldsymbol{\epsilon}_1$ where $\boldsymbol{\epsilon}_1 \sim N(0, \boldsymbol{\Omega})$ and $\boldsymbol{\mu} = \boldsymbol{\Pi} + \boldsymbol{\epsilon}_2$ where $\boldsymbol{\epsilon}_2 \sim N(0, \boldsymbol{\Sigma})$. Hence,

$$\mathbb{E}[\boldsymbol{\mu}] = \boldsymbol{\Pi} \text{ and } \mathbb{E}[(\boldsymbol{\mu} - \boldsymbol{\Pi})(\boldsymbol{\mu} - \boldsymbol{\Pi})'] = \boldsymbol{\Sigma}.$$

And,

$$\mathbb{E}[\boldsymbol{\epsilon}_1] = 0 \text{ and } \mathbb{E}[\boldsymbol{\epsilon}_1'\boldsymbol{\epsilon}_1] = \boldsymbol{\Omega}.$$

Then by Corollary E.3.5 of Bertsekas [8] we have

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\Pi} + \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}(\mathbf{q} - \mathbf{P}\boldsymbol{\Pi}),$$

$$\tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}. \quad \square$$

We note that we do not need density functions anymore. Working with linear least square estimation gives us the flexibility to work with the first two moments. Next, we assume that asset returns follow a Mixed Normal distribution with two components.

4.4.1 Market Returns

"The definition of a market depends on the investor, who focuses on a specific pool of assets"(Meucci [30]). In this section we propose a new way to model the market returns using multivariate mixed gaussian distribution where the investors have views on the expected asset return (i.e. $\boldsymbol{\mu}$). In the BLM expected market returns follows a normal distribution centered at CAPM equilibrium. In this section we modify that assumption. In our problem expected asset returns have two components, bull and bear. The scalar α defines, the relationship (or weights) between them. We are going to work with the alternate BLM. Note that once we update the problem parameters then we can use them in the portfolio allocation problems.

Assumption 4.4.1 (market return assumption)**.** Expected market returns follows a mixed gaussian distribution:

$$\boldsymbol{\mu} \sim \alpha \boldsymbol{\mu}_{bull} + (1-\alpha)\boldsymbol{\mu}_{bear}. \tag{4.73}$$

Now, investor can have two different type of views one type of view can be related with α and the other type of view can be related with the expected returns on bull and bear market.

Next, we give a new BLM using Alternate Black Litterman Model under Assumption 4.4.1.

Proposition 4.4.3 (Mixing Model 1 for BLM)**.** Let $\mathbf{r}$ be the asset return vector;

$$\mathbf{r} \sim MN(\alpha, \boldsymbol{\mu}_{bull}, \boldsymbol{\mu}_{bear}, \boldsymbol{\Sigma}_{bull}, \boldsymbol{\Sigma}_{bear}), \tag{4.74}$$

and the investor's view be represented as:

$$\mathbf{P}\boldsymbol{\mu} \sim N(\mathbf{q}, \boldsymbol{\Omega}). \tag{4.75}$$

Then the BL-type estimates are:

$$\boldsymbol{\mu}_{BL}^{mn} = \boldsymbol{\mu}_{mn} + \boldsymbol{\Sigma}_{mn}\mathbf{P}' \left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1} \left(\mathbf{q} - \mathbf{P}\boldsymbol{\mu}_{mn}\right), \tag{4.76}$$

$$\boldsymbol{\Sigma}_{BL}^{mn} = \boldsymbol{\Sigma}_{mn} - \boldsymbol{\Sigma}_{mn}\mathbf{P}' \left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}' + \boldsymbol{\Omega}\right)^{-1} \mathbf{P}\boldsymbol{\Sigma}_{mn}. \tag{4.77}$$

Proof. Let $\mathbf{r}$ be the asset return vector:

$$\mathbf{r} \sim MN(\alpha, \boldsymbol{\mu}_{bull}, \boldsymbol{\mu}_{bear}, \boldsymbol{\Sigma}_{bull}, \boldsymbol{\Sigma}_{bear}). \tag{4.78}$$

Note that $\mathbf{r}$ follows a Mixed Normal distribution. Hence, using Proposition 4.4.1 we get it's mean vector ($\boldsymbol{\mu}_{mn}$) and covariance matrix ($\boldsymbol{\Sigma}_{mn}$):

$$\boldsymbol{\mu}_{mn} \quad = \quad \alpha \boldsymbol{\mu}_{bull} + (1-\alpha)\boldsymbol{\mu}_{bear}, \tag{4.79}$$

$$\boldsymbol{\Sigma}_{mn} = \alpha(1-\alpha)(\boldsymbol{\mu}_{bull}-\boldsymbol{\mu}_{bear})(\boldsymbol{\mu}_{bull}-\boldsymbol{\mu}_{bear})' + \alpha\boldsymbol{\Sigma}_{bull} + (1-\alpha)\boldsymbol{\Sigma}_{bear}. \tag{4.80}$$

We get the following using Proposition 4.4.2:

$$\boldsymbol{\mu}_{BL}^{mn} = \boldsymbol{\mu}_{mn} + \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}(\mathbf{q}-\mathbf{P}\boldsymbol{\mu}_{mn}),$$

$$\boldsymbol{\Sigma}_{BL}^{mn} = \boldsymbol{\Sigma}_{mn} - \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}'+\boldsymbol{\Omega}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}_{mn}. \quad \square$$

Now, we assume that the investor has different views for each component.

Proposition 4.4.4 (Mixing Model 2 for BLM)**.** Let $\mathbf{r}$ be the asset return vector;

$$\mathbf{r} \sim MN(\alpha, \boldsymbol{\mu}_{bull}, \boldsymbol{\mu}_{bear}, \boldsymbol{\Sigma}_{bull}, \boldsymbol{\Sigma}_{bear}), \tag{4.81}$$

and the investor's view be represented as;

$$\mathbf{P}\boldsymbol{\mu}_{bull} \sim N(\mathbf{q}_{bull}, \boldsymbol{\Omega}_{bull}), \tag{4.82}$$

$$\mathbf{P}\boldsymbol{\mu}_{bear} \sim N(\mathbf{q}_{bear}, \boldsymbol{\Omega}_{bear}). \tag{4.83}$$

Then the BL-type estimates are:

$$\begin{aligned}\boldsymbol{\mu}_{mn}^{BL} = \boldsymbol{\mu}_{mn} &+ \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}'+\boldsymbol{\Omega}_{bull}\right)^{-1}(\mathbf{q}_{bull}-\mathbf{P}\boldsymbol{\mu}_{bull}) \\ &+ \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}'+\boldsymbol{\Omega}_{bear}\right)^{-1}(\mathbf{q}_{bear}-\mathbf{P}\boldsymbol{\mu}_{bear}),\end{aligned} \tag{4.84}$$

$$\begin{aligned}\boldsymbol{\Sigma}_{mn}^{BL} = \boldsymbol{\Sigma}_{mn} &- \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}'+\boldsymbol{\Omega}_{bull}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}_{mn} \\ &- \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}'+\boldsymbol{\Omega}_{bear}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}_{mn}.\end{aligned} \tag{4.85}$$

Proof. Let $\mathbf{r}$ be the asset return vector;

$$\mathbf{r} \sim MN(\alpha, \boldsymbol{\mu}_{bull}, \boldsymbol{\mu}_{bear}, \boldsymbol{\Sigma}_{bull}, \boldsymbol{\Sigma}_{bear}). \tag{4.86}$$

Hence, using Proposition 4.4.1 we get it's mean vector ($\boldsymbol{\mu}_{mn}$) and covariance matrix ($\boldsymbol{\Sigma}_{mn}$);

$$\boldsymbol{\mu}_{mn} = \alpha\boldsymbol{\mu}_{bull} + (1-\alpha)\boldsymbol{\mu}_{bear}, \tag{4.87}$$

$$\boldsymbol{\Sigma}_{mn} = \alpha(1-\alpha)(\boldsymbol{\mu}_{bull} - \boldsymbol{\mu}_{bear})(\boldsymbol{\mu}_{bull} - \boldsymbol{\mu}_{bear})' + \alpha\boldsymbol{\Sigma}_{bull} + (1-\alpha)\boldsymbol{\Sigma}_{bear}. \tag{4.88}$$

Note that we have components for the linear least square estimation:

$$\mathbf{P}\boldsymbol{\mu}_{bull} \sim N(\mathbf{q}_{bull}, \boldsymbol{\Omega}_{bull}),$$

$$\mathbf{P}\boldsymbol{\mu}_{bear} \sim N(\mathbf{q}_{bear}, \boldsymbol{\Omega}_{bear}).$$

We get the updated parameters using linear least square method with two components. Hence, the expected return is updated as:

$$\begin{aligned}\boldsymbol{\mu}_{mn}^{BL} = \boldsymbol{\mu}_{mn} &+ \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}' + \boldsymbol{\Omega}_{bull}\right)^{-1}\left(\mathbf{q}_{bull} - \mathbf{P}\boldsymbol{\mu}_{bull}\right)\\ &+ \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}' + \boldsymbol{\Omega}_{bear}\right)^{-1}\left(\mathbf{q}_{bear} - \mathbf{P}\boldsymbol{\mu}_{bear}\right).\end{aligned}$$

In addition to that, the covariance matrix is updated as:

$$\begin{aligned}\boldsymbol{\Sigma}_{mn}^{BL} = \boldsymbol{\Sigma}_{mn} &- \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}' + \boldsymbol{\Omega}_{bull}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}_{mn}\\ &- \boldsymbol{\Sigma}_{mn}\mathbf{P}'\left(\mathbf{P}\boldsymbol{\Sigma}_{mn}\mathbf{P}' + \boldsymbol{\Omega}_{bear}\right)^{-1}\mathbf{P}\boldsymbol{\Sigma}_{mn}. \quad \square\end{aligned}$$

4.5 Conclusion

If the tail estimation of the return distribution is not good then using CVaR is nothing but error maximization. Moreover, specifying the $\mathbf{\Omega}$ is difficult for non-quantitative investors. Because of these two reasons, we propose a model using mixing multivariate random variables under BLM. We show that if $\mathbf{\Omega}$ is fixed as $\mathbf{\Omega} = \omega \mathbf{P}\mathbf{\Sigma}\mathbf{P}'$ then the closed form representation of the allocation vectors using allocation vectors under full confidence and no confidence can be found. In addition to that, new mapping techniques for investors uncertainty are presented. Moreover, we developed the BL-type estimations under multivariate Mixed Normal setting.

REFERENCES

[1] Ahuja RK, Orlin JB (2001) Inverse Optimization. Oper. Res. 49(5):771-783 HC04 Heuberger,

[2] Artzner P, Delbaen F, Eber JM, Heath D. (1999), Coherent measures of risk. Math. Finance 9(3):203-228

[3] Avramov, D., & Zhou, G. (2010). Bayesian portfolio analysis. Annu. Rev. Financ. Econ., 2(1), 25-47.

[4] Byrnes J., Clarke J., Dombrowski T., Kandi D., Pang T., Karan C. (Working Paper) The Black-Litterman Model: Exploring the Sensitivity of Inputs.

[5] Ben-Tal, A., Nemirovski, A. (1999). Robust solutions of uncertain linear programs. Operations Research Letters, 25(1), 1-13.

[6] Ben-Tal, A., A. Nemirovski. (2001), Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. MPR-SIAM Series on Optimization. SIAM, Philadelphia

[7] Bertsimas D., Gupta V., Paschalidis Ioannis Ch. (2012), Inverse Optimization: A new Perspective on the Black-Litterman Model. Oper. Res. 60(6):1389-1403

[8] Bertsekas D. (2005). Dynmic Programming and Optimal Control. Athena Scientific, Belmont, Massachusetts

[9] Black F, Litterman R (1992) Global portfolio optimization. Financial Analysts Journal. 48(5): 28-43

[10] Boyd, S., & Vandenberghe, L. (2004). Convex optimization. Cambridge university press.

[11] Chopra, V. K., & Ziemba, W. T. (1993). The effect of errors in means, variances, and covariances on optimal portfolio choice. Journal of Portfolio Management. Winter 1993 :6-11.

[12] Davis, M., & Lleo, S. (2013). Black-Litterman in continuous time: the case for filtering. Quantitative Finance Letters, 1(1), 30-35.

[13] Fabozzi Frank J., Kolm Petter N., Pachamanova Dessislava A., Focardi Sergio M. (2007), Robust Portfolio Optimization and Management, John Wiley & Sons, Inc., Hoboken, New Jersey.

[14] Fang, K. T., Kotz, S., & Ng, K. W. Symmetric Multivariate and Related Distributions. 1990. Chapman&Hall, London.

[15] Fusai A. and Meucci A., 2003. Assessing Views. Risk Magazine 16 (3): 18-21.

[16] (2014) CVX: Matlab software for disciplined convex programming, version 2.1. Accessed Jun 17, 2014 ,http://cvxr.com/cvx/

[17] Giacometti R., Bertocchi M., Rachev T. S. and Fabozzi F., 2007. Stable distributions in the Black-Litterman approach to asset allocation. Quantitative Finance 7(4):423-433

[18] Harvey R. C., Liechty C. J., Liechty W. M. and Muller P, 2010. Portfolio Selection with Higher Moments Quantitative Finance 10(5): 469-485

[19] He G., Litterman. R (1999) The intuition behind Black-Litterman model portfolios. Investment Management Research, Goldman Sachs & Company, New York

[20] Hull, John C. (2012) Risk Management and Financial Institutions. John Wiley & Sons, Inc. Hoboken, New Jersey

[21] Heuberger, C. (2004). Inverse combinatorial optimization: A survey on problems, methods, and results. Journal of combinatorial optimization, 8(3), 329-361.

[22] Idzorek T., (2005) A Step-By-Step guide to the Black-Litterman Model, Incorporating User-Specified Confidence Levels (Working Paper)

[23] Iyengar G, Kang W (2005) Inverse conic programming with applications. Oper. Res. Lett. 33(3):319-330

[24] Jondeau E. and Rockinger M., (2006). Optimal Portfolio Allocation under Higher Moments. European Financial Management 12(1): 29-55

[25] Kolm, P. N., Tutuncu, R., & Fabozzi, F. J. (2014). 60 Years of portfolio optimization: Practical challenges and current trends. European Journal of Operational Research, 234(2), 356-371.

[26] Krokhmal, Pavlo, Jonas Palmquist, and Stanislav Uryasev. "Portfolio optimization with conditional value-at-risk objective and constraints." Journal of risk 4 (2002): 43-68.

[27] Landsman, Z. (2008). Minimization of the root of a quadratic functional under a system of affine equality constraints with application to portfolio management. Journal of Computational and Applied Mathematics, 220(1), 739-748.

[28] Landsman, Z. M., & Valdez, E. A. (2003). Tail conditional expectations for elliptical distributions. North American Actuarial Journal, 7(4), 55-71.

[29] Markowitz, H. M. (1952). Portflio Selection. J. Finance 7 77-91

[30] Meucci A., (2005). Risk and Asset Allocation. Springer Finance, New York, NY.

[31] Meucci A., (2010). The Black Litterman Approach: Original Model and Extensions, The Encyclopedia of Quantitative Finance, Wiley. Available at http://ssrn.com/abstract=1117574.

[32] Natarajan K, Pachamanova D, Sim M (2009), Constructing risk measures from uncertainty sets. Oper. Res. 57(5):1129-1141

[33] Lim, A. E., Shanthikumar, J. G., & Vahn, G. Y. (2011). Conditional value-at-risk in portfolio optimization: Coherent but fragile. Operations Research Letters, 39(3), 163-171.

[34] Lyuu, Y. D. (2001). Financial engineering and computation: principles, mathematics, algorithms. Cambridge University Press.

[35] Pang T., Karan C. (2017) Closed-Form Solutions for Black-Litterman Model with Conditional Value at Risk(Working paper)

[36] Rockafellar, R. T. (1970). Convex Analysis. Princeton University Press, Princeton, NJ.

[37] Rockafellar RT, Uryasev S (2000) Optimization of conditional value-at-risk, Journal of Risk 2, 21-41

[38] Rockafellar RT, Uryasev S (2002) Conditional value-at-risk for general loss distributions. J. Banking Finance 26(7):1443-1471

[39] Sarykalin, S., Serraino, G., & Uryasev, S. (2008). Value-at-risk vs. conditional value-at-risk in risk management and optimization. Tutorials in Operations Research. INFORMS, Hanover, MD, 270-294.

[40] Satchell and Scowcroft (2000) A demystification of the Black-Litterman Model: Managing Quantitative and Traditional Portfolio Construction

[41] Schott J. R., (2005). Matrix Analysis for statistics. John Wiley & Sons, Inc., Hoboken, New Jersey.

[42] Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. The journal of finance, 19(3), 425-442.

[43] Sharpe, W. F. (1974). Imputing expected security returns from portfolio composition. Journal of Financial and Quantitative Analysis, 9(03), 463-472.

[44] Silva Da. S. A., Lee W. and Pornrojnangkool B. The Black-Litterman Model for Active Portfolio Management. The Journal of Portfolio Management Winter 2009: 61-70

[45] Steinbach, Marc C. "Markowitz revisited: Mean-variance models in financial portfolio analysis." SIAM review 43.1 (2001): 31-85.

[46] Stengel, R. F. (2012). Optimal control and estimation. Courier Corporation.

[47] Walters J (2014) The Black-Litterman model: A detailed exploration. Accessed June 15, 2014, http://www.blacklitterman.org

[48] Xiao Y. and Valdez E. A.(2015) A Black-Litterman asset allocation model under Elliptical distributions. Quantitative Finance 15(3):509-519