ABSTRACT

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Free and moving boundary fluid-structure interactions, physical processes in which a deformable or movable solid interfaces with a fluid, are ubiquitous in nature and engineering. Due to their highly nonlinear nature (the domain and boundary equations are nonlinear, and the common boundary is one of the unknowns in the system and has to be determined as part of the solution), they continue to be one of the most challenging topics to date. We consider two variations on PDE-constrained optimization problems governed by a free or moving boundary fluid-elasticity interaction.

The challenge of applying optimization tools to free or moving boundary fluid-structure interactions is the proper derivation of the adjoint sensitivity information with correct balancing conditions on the common interface. As the interaction is a coupling of Eulerian (fluid state) and Lagrangian (motion of the solid) quantities, sensitivity analysis on the system falls into the framework of shape analysis. We build upon the sensitivity theory developed in [12, 11, 14] and provide descriptions of the associated adjoint linearized models.

The adjoint linear models are necessary in the derivation of optimality conditions for controls, which feed into derivative-based optimization algorithms in free and moving boundary fluid-structure interactions. We derive the optimality conditions along with explicit representations of the gradient of the quadratic cost-functional associated with each problem; the cost function gradient is relevant to subsequent numerical investigations, as its explicit representation provides directions for descent. In the free boundary, steady state case, we provide rigorous justification for the characterization of the optimal control via the cost function gradient, in the form of a comprehensive well-posedness analysis.

Optimal Control in Free and Moving Boundary Couplings of Navier-Stokes and Nonlinear Elasticity

by
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BIOGRAPHY

The author was born and raised in a small town in southern Colorado. After she received her BA in Psychology and Philosophy from University of Colorado, Boulder, in 2002, she worked in industry for several years before returning to her first academic love, Mathematics. She completed the Post-Baccalaureate in Mathematics program at Smith College in 2012, where she also gained her first original research experiences and the advising necessary to enter the Applied Math PhD program at North Carolina State University.

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TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

A fluid-structure interaction (FSI) describes any physical process in which a deformable or movable solid interfaces with a fluid. Free and moving boundary fluid-structure interactions are ubiquitous in nature, with most known examples coming from industrial processes, aero-elasticity, and biomechanics (e.g., flutter analysis of airplanes, parachute FSI, robo-bees [53], blood flow in the cardiovascular system, and heart valve dynamics). Due to their highly nonlinear nature (the domain and boundary equations are nonlinear, and the common boundary is one of the unknowns in the system and has to be determined as part of the solution), they continue to be one of the most challenging topics to date, even though many publications have appeared focusing on the mathematical solvability [5, 6, 7, 15, 18, 19, 27, 28, 29, 33, 34, 36, 37, 38, 44, 45, 43, 48, 54] and numerical approximation [21, 23, 24, 25, 32, 46, 47] for these coupled systems. However, in most of these applications, the ultimate goal is not only the mathematical modeling and numerical simulation of the complex system, but rather the optimization or optimal control of the considered process, as well as related sensitivity analysis (with respect to relevant physical parameters) investigations. Regarding control problems in fluid-structure interactions, much of the literature is focused on the assumption of small but rapid oscillations of the solid body, so that the common interface may be assumed static [9, 39, 40, 41]. Recently, PDE constrained optimization problems governed by free boundary interactions (in which one works with a steady-state problem, and the position of the boundary is a spatial unknown) have been considered, with most research studies mainly addressed in the context of the numerical analysis of the finite element methods [4, 49, 51, 52].

In this work, we consider two variations on optimization problems involving FSI. The first is a free boundary problem of optimizing the fluid velocity in a system at steady state; the second is a moving boundary problem of minimizing flow turbulence in a nonlinear fluid-structure interaction model in which the domains and interface depend on time. We rely on the known well-posedness theory [27, 54, 19]. Sensitivity equations associated are obtained using shape optimization and tangential calculus techniques [12, 11, 14]. In comparison to earlier approaches, like transporting the equations to a fixed reference configuration, or using transpiration techniques [23, 24], the shape optimization route is most suited to incorporating the geometry of the problem into the analysis. This refined description brings up new terms in the matching of the normal stresses and velocities on the interface (in particular, the matrix of the interface's curvatures), terms that are missing if one just considers the coupling of linear Stokes flow and linear elasticity [31].

Sensitivity analysis represents the first step towards an optimization problem. As the coupled fluid-structure state is the solution of a system of PDEs that are coupled through continuity relations defined on the interface, itself a geometric unknown, sensitivity of the fluid state, which is an Eulerian quantity, with respect to the motion of the solid, which is a Lagrangian quantity, falls into moving shape analysis framework [22, 42].

This work builds upon the theory developed in [12, 11, 14], and we perform adjoint sensitivity analysis for both the free and the moving boundary problems. A central challenge of applying optimization tools to free or moving boundary FSI is the proper derivation of the adjoint sensitivity information with correct adjoint balancing conditions on the common interface. We are directly motivated to derive the necessary optimality conditions that describe optimal controls, which paves the way for subsequent numerical analysis. We describe and study the appropriate adjoint linearized models in both the free and moving boundary FSI, including a treatment of the well-posedness of the system in the free boundary case.

We note here that most of this work is independent of the choice of cost function, which encodes the quantity or process to be optimized, associated with the free or moving boundary interaction. Therefore the results have wide applicability in the field.

1.1 Notation

Throughout we use the Einstein summation convention as well as the following notation:

- $(Df(a))_{ij} = \partial_j f_i(a) \in \mathbb{M}^3$ is the gradient matrix at $a \in X$ of any vector field $f = (f_i) : X \subset \mathbb{R}^3 \to \mathbb{R}^3$.
- $\operatorname{div} f(a) = \partial_i f_i(a) \in \mathbb{R}$ is the divergence of $f : X \subset \mathbb{R}^3 \to \mathbb{R}^3$ at $a \in X$.
- $\operatorname{Div} T(a) = \partial_j T_{ij}(a) e_i \in \mathbb{R}^3$ is the divergence of any second-order tensor field $T = (T_{ij}) : X \subset \mathbb{R}^3 \to \mathbb{M}^3$ at $a \in X$.
- We will identify the set of all second-order tensors with the set $\mathbb{M}^3$ of all square matrices of order three.
- $A^* =$ transpose of A, for any $A \in \mathbb{M}^3$.
- $A..B = \operatorname{Tr}(A^* B) \in \mathbb{R}$ is the Frobenius inner product in $\mathbb{M}^3$.

Chapter 2

PDEs of Fluids and Structures

2.1 Reference and Deformed Configurations

Consider a domain $\widehat{\Omega} \subset \mathbb{R}^d$ $(d = 2, 3)$ with smooth boundary. The domain $\widehat{\Omega}$ is filled with a continuum medium; for our purposes $\widehat{\Omega}$ will be occupied by either an elastic solid or an incompressible fluid. We refer to $\widehat{\Omega}$ as the reference configuration of the domain. The domain $\widehat{\Omega}$ can be acted on by a map called a deformation, which deforms the domain. Let $\widehat{\varphi}$ be an injective map such that

$$\widehat{\varphi} : \widehat{\Omega} \to \Omega,$$

where $\Omega \subset \mathbb{R}^d$ is the deformed configuration (also referred to as the physical or current configuration). The deformation mapping $\widehat{\varphi}$ maps coordinates $\widehat{x} \in \widehat{\Omega}$ to coordinates in the deformed configuration $x \in \Omega$, that is $\widehat{\varphi}(\widehat{x}) = x$. The deformation gradient of the mapping $\widehat{\varphi}$ is a second-order tensor field (identified in the space $\mathbb{M}^d$ of $d \times d$ matrices) and denoted by $\widehat{F}_{ij} := \dfrac{\partial x_i}{\partial \widehat{x}_j}$, $i, j = \overline{1, d}$. The notation $\widehat{J} = \det \widehat{F}$ gives the Jacobian of the deformation. The deformations we consider are orientation-preserving, and as such $\widehat{J} > 0$.

Depending on the particular continuum medium, a quantity of interest may be the displacement of a coordinate $\widehat{x}$, which will be denoted by $\widehat{\eta}(\widehat{x})$. The displacement is the difference between a material point and it's location under the deformation.

$$\widehat{\eta}(\widehat{x}) = \widehat{\varphi}(\widehat{x}) - \widehat{x} \tag{2.1.1}$$

In the (common) case where the situation is dynamic, as opposed to static, we add the definition of a motion, which is a one-parameter family of deformations. Consider the smooth map

$$\widehat{\varphi}(t) : \widehat{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^3$$

with $\widehat{\varphi}(t)(\widehat{x}) = x$. If for each time $t \geq 0$ $\widehat{\varphi}(t) = \widehat{\varphi}(\cdot, t)$ is a deformation then $\widehat{\varphi}(t)$ is a motion. For each time t the current configuration is denoted by $\Omega(t) = \widehat{\varphi}(\widehat{\Omega}, t)$. That is,

$$\varphi(t) : \widehat{\Omega} \to \Omega(t).$$

In this context the reference configuration is typically considered to be the initial configuration, that is $\widehat{\Omega} = \Omega(0)$.

When the situation is dynamic, the velocity of the motion is a significant kinematic quantity which will appear in many calculations. The variable u is used to describe the velocity of the motion, which is given by

$$\widehat{u}(\widehat{x}, t) := \frac{\partial}{\partial t}\widehat{\eta}(\widehat{x}, t). \tag{2.1.2}$$

Notice that due to the rules governing partial derivatives and the definition of the deformation, it is also true that for each $t \geq 0$, $\widehat{u} = \dfrac{\partial \widehat{\varphi}}{\partial t}$.

When a quantity is described in the reference configuration, this is akin to the classical Lagrangian framework. The Lagrangian frame of observation entails following each individual particle (or parcel) $\widehat{x}$ as it moves through space and time. In contrast, when a quantity is described in the current configuration this is analogous to the classical Eulerian framework. The Eulerian frame of observation entails watching the movement of particles through a particular location and at a particular time (or interval of time). A commonly used analogy is that of a swimmer in the river. In the Lagrangian frame of reference, the observer is floating alongside the swimmer and monitoring her position and velocity. In the Eulerian frame of reference, the observer is sitting on the shore in a fixed spot and watching the swimmer's position and velocity evolve in time. Using the motion $\widehat{\varphi}$ and its inverse $\widehat{\varphi}^{-1} := \varphi$ we can write the relationship between a quantity in the Lagrangian vs. the Eulerian frame by composition with the motion. For example, for each $t \geq 0$,

$$u(x,t) := \widehat{u}(\cdot,t) \circ \varphi(x) = \widehat{u}(\widehat{x},t),$$
$$\eta(x,t) := \widehat{\eta}(\cdot,t) \circ \varphi(x) = \widehat{\eta}(\widehat{x},t).$$

In order to further expand on the relationship between the referential frames and quantities described within each of them, observe that we can write the inverse of the motion explicitly. Since for each t, $\widehat{\varphi}(\widehat{x},t) = \widehat{x} + \widehat{\eta}(\widehat{x},t)$ is the map that takes a material point to its Eulerian counterpart, i.e. $\widehat{\varphi}(\widehat{x},t) = x$, then the map that takes an Eulerian coordinate x back to its Lagrangian counterpart entails just subtracting off the displacement. That is, for each t,

$$\varphi(x,t) := x - \eta(x,t) = \widehat{x}.$$

With this relation we can define $F := D\varphi = I - D\eta$, and since $\widehat{\varphi} \circ \varphi = I$ we have $F\widehat{F} = I$. In other words, $F = \widehat{F}^{-1}$. Consequently, for $J := \det F$, the Jacobian of the inverse of the deformation, we have $J = 1/\widehat{J}$. The relations between the motion and its inverse are critically useful when casting expressions and equations in their equivalent forms in both the Eulerian and Lagrangian frames.

Regarding notation, in what follows we will not use boldface to distinguish between scalar and vectorial quantities, but will in most cases use capital letters to refer to tensorial quantities. We will typically use the 'hat' notation to refer to Lagrangian frame quantities, and the 'hatless' notation to refer to Eulerian frame quantities. Any distinctions not clear from context will be made explicit when necessary.

2.2 Formulas

We state several classical formulas which will be useful in developing the PDEs that comprise fluid-structure interaction models, as well as in the subsequent analysis. These formulas and the corresponding notation are found in, e.g., [25].

2.2.1 Change of Variables

If $\widehat{\varphi}(t)$ is a motion such that $\Omega(t) = \widehat{\varphi}(t)(\widehat{\Omega})$ for each $t \geq 0$ we have,

$$\int_{\Omega(t)} q dx = \int_{\widehat{\Omega}} \widehat{J}\widehat{q} dx, \tag{2.2.1}$$

where $\widehat{J} := \det D\widehat{\varphi}$.

2.2.2 Material Derivative

For an Eulerian field ϕ (minor adjustments can be made depending on whether ϕ is a scalar, vectorial, or tensorial quantity), define the material time-derivative of ϕ by

$$\frac{D\phi}{Dt} := \frac{\partial \widehat{\phi}}{\partial t} \circ \varphi.$$

Using the multivariable chain-rule we can write a more physically meaningful representation of the material time derivative

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \nabla\phi \cdot u, \tag{2.2.2}$$

where u is the Eulerian counterpart of the velocity of the motion, as described in Section 2.1.

2.2.3 Derivative of Jacobian

If $J(\varphi)$ is the Jacobian of a deformation (in an Eulerian frame) then the material time-derivative of J is given by

$$\frac{DJ}{Dt} = J \operatorname{div} u. \tag{2.2.3}$$

2.2.4 Gradient of composition

For an Eulerian field ϕ:

If ϕ is a scalar,

$$\nabla_{\widehat{x}}\widehat{\phi} = F^{-*}\nabla\phi. \tag{2.2.4}$$

On the other hand if ϕ is vector-valued then

$$D_{\widehat{x}}\widehat{\phi} = D\phi F^{-1}. \tag{2.2.5}$$

2.2.5 Reynold's Transport Formula

Let $V(t)$ be a domain associated with velocity u (as described in Section 2.1). If f is a continuously differentiable scalar field, then

$$\frac{d}{dt}\int_{V(t)} f dx = \int_{V(t)}\left(\frac{Df}{Dt} + f\operatorname{div} u\right)dx = \int_{V(t)}\left(\frac{\partial f}{\partial t} + \operatorname{div}(fu)\right)dx. \tag{2.2.6}$$

If f is a continuously differentiable vector field then

$$\frac{d}{dt}\int_{V(t)} f dx = \int_{V(t)}\left(\frac{Df}{Dt} + f\operatorname{div} u\right)dx = \int_{V(t)}\left(\frac{\partial f}{\partial t} + \operatorname{Div}(f\otimes u)\right)dx. \tag{2.2.7}$$

2.3 Conservation Laws

2.3.1 Mass

Fundamental principle: the mass of a body does not change when subjected to a motion.

If ρ is the density of a continuum medium in an arbitrary domain $V(t)$, then the mass of the domain is given by

$$\text{Mass of } V(t) = \int_{V(t)} \rho\ dx.$$

According to the principle of conservation of mass we have $\frac{d}{dt}(\text{Mass of } V(t)) = 0$, which implies

$$0 = \frac{d}{dt}\left(\int_{V(t)} \rho \, dx\right) \tag{2.3.1}$$

2.3.2 Momentum

Fundamental principle: the rate of change of momentum in a body is equal to the resultant of the forces acting on it.

Represent the resultant force on a body by $\mathbf{V}$. The quantity of momentum is given by the product of density and velocity, so the fundamental principle is translated as

$$\frac{d}{dt}\int_{V(t)} \rho u \, dx = \mathbf{V},$$

for an arbitrary domain $V(t)$. In order to represent conservation of momentum classically we decompose the force $\mathbf{V}$ into the force that acts in a distributed fashion throughout the volume of the body, $\mathbf{V_v}$, as well as the force that acts on the surface of the body, $\mathbf{V_s}$. The volume force $\mathbf{V_v}$ can be written as the accumulation the product of the density and a specific force v, that is

$$\mathbf{V_v} = \int_{V(t)} \rho v \, dx.$$

The surface force, on the other hand, can be represented by the surface integral of the Cauchy stress. According to a principle of Cauchy, the Cauchy stress is computed by the contraction of the Cauchy stress tensor σ (a second-order symmetric tensor) and the outer normal to $\partial V(t)$. That is,

$$\mathbf{V_s} = \int_{\partial V(t)} \sigma n \, dx.$$

Altogether, the principle of conservation of momentum can be written

$$\frac{d}{dt}\int_{V(t)} \rho u \, dx = \int_{V(t)} \rho v \, dx + \int_{\partial V(t)} \sigma n \, dx = \int_{V(t)} \rho v + \operatorname{Div} \sigma \, dx, \tag{2.3.2}$$

where the last equality entails an application of the divergence theorem.

2.4 PDEs of Fluids and Structures

There are two different ways to proceed in order to use general physical and conservation principles (2.3.1) and (2.3.2) to recover PDEs describing the evolution/state of fluids or structures. In order to compute the time derivative of the integrals we can move the integral to a fixed/reference domain, bring the time derivative under the integral, and then directly recover a continuity equation (owing to the arbitrariness of the domain). The result is an equation in the Lagrangian frame. Alternatively, after moving the time derivative under the integral, we can move the integral back to the Eulerian frame and then recover a continuity equation. The result is a continuity equation in the Eulerian frame. The approach that is more appropriate for a particular setting depends on the continuum medium of interest.

2.4.1 Equations of Solid

In the case of a solid, we are primarily concerned with the displacement of particles (as opposed to the velocity). As a result, it stands to reason that working in the Lagrangian framework is more suitable for deriving the equations governing solids; we will move the integral formulations of mass and momentum conservation to reference, time-independent domains and then derive the corresponding continuity equations and PDEs. While we will make use of the abstract quantity $\widehat{u}$, the velocity of the continuum medium, the physical quantities of interest are typically $\widehat{\eta}$ and $\widehat{\varphi}$, the displacement and the deformation, respectively. Recall that these quantities are related by relations (2.1.1) and (2.1.2).

2.4.1.1 Time-dependent Setting

Using the change of variables $\widehat{\varphi}$ to move (2.3.1) to the reference domain we obtain

$$0 = \frac{d}{dt}\left(\int_{\widehat{V}} \widehat{J}\widehat{\rho} d\widehat{x}\right) = \int_{\widehat{V}} \frac{\partial(\widehat{J}\widehat{\rho})}{\partial t} d\widehat{x}.$$

Define $\widehat{\rho}_0 := \widehat{J}\widehat{\rho}$. Owing to fact that $\widehat{V}$ is arbitrary, we obtain the continuity equation for mass conservation in $\widehat{\Omega}$

$$\frac{\partial \widehat{\rho}_0}{\partial t} = 0. \tag{2.4.1}$$

Note that (2.4.1) implies that the material density is constant in time. In the case where density is constant in space as well, the definition of $\widehat{\rho}_0$ implies that $\widehat{J} = 1$ for all $t \geq 0$.

According to the same reasoning we turn our attention to (2.3.2). Moving the integral back to the reference domain we obtain

$$\frac{d}{dt}\left(\int_{\widehat{V}} \widehat{J}\widehat{\rho}\frac{\partial \widehat{\eta}}{\partial t} d\widehat{x}\right) - \int_{\widehat{V}} \widehat{J}\widehat{\mathrm{Div}\,\sigma} d\widehat{x} = \int_{\widehat{V}} \widehat{\rho} v d\widehat{x}.$$

The first term on the left-hand side can be simplified as follows:

$$\begin{aligned}
\frac{d}{dt}\left(\int_{\widehat{V}} \widehat{J}\widehat{\rho}\frac{\partial \widehat{\eta}}{\partial t} d\widehat{x}\right) &= \int_{\widehat{V}} \frac{\partial}{\partial t}\left(\widehat{J}\widehat{\rho}\frac{\partial \widehat{\eta}}{\partial t}\right) d\widehat{x}, \\
&= \int_{\widehat{V}} \frac{\partial}{\partial t}\left(\widehat{\rho}_0\frac{\partial \widehat{\eta}}{\partial t}\right) d\widehat{x} \text{ by definition of } \widehat{\rho}_0, \\
&= \int_{\widehat{V}} \widehat{\rho}_0 \frac{\partial^2 \widehat{\eta}}{\partial t^2} d\widehat{x} \text{ by (2.4.1).}
\end{aligned}$$

Substituting we obtain

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{\eta}}{\partial t^2} - \widehat{J}\widehat{\mathrm{Div}\,\sigma} = \widehat{\rho}_0 v.$$

It remains to discuss the term $\widehat{\mathrm{Div}\,\sigma}$. Recall that the 'hat' notation in this context is defined as $\widehat{\mathrm{Div}\,\sigma} = \mathrm{Div}\,\sigma \circ \widehat{\varphi}$. In other words, the derivative is still being taken with respect to the Eulerian frame variables. We can formally carry out the composition in order to represent the PDE entirely in the Lagrangian frame, such that all derivative operations are with respect to the Lagrangian variables. The composition is represented by the Piola transformation. When σ is a two-tensor defined on the Eulerian frame, the

Piola transform of σ associated to a particular deformation $\widehat{\varphi}$ is defined by

$$\widehat{\mathcal{P}}_{\widehat{\varphi}}(\sigma) := \widehat{J}(\widehat{x})\sigma\big(\widehat{\varphi}(\widehat{x})\big)\widehat{F}^{-T}(\widehat{x}) = \widehat{J}\widehat{\sigma}\widehat{F}^{-T}. \tag{2.4.2}$$

We will use the shorthand $\widehat{\mathcal{P}}$ and assume that the dependence on both $\widehat{\varphi}$ and σ are understood.

We have the following further important relationship between σ and $\mathcal{P}$:

$$\operatorname{Div}_{\widehat{x}}\widehat{\mathcal{P}} = \widehat{J}(\operatorname{Div}\sigma \circ \widehat{\varphi}). \tag{2.4.3}$$

Consequently we can write the continuity equation for displacement of a solid. It is now understood that the divergence is taken with respect to the variables in the Lagrangian frame.

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{\eta}}{\partial t^2} - \operatorname{Div}\widehat{\mathcal{P}} = \widehat{\rho}_0 \upsilon.$$

Note that $\widehat{\mathcal{P}}$ is not symmetric; because symmetric tensors are computationally convenient we will often write equations for solids instead in terms of the second Piola tensor $\widehat{\Sigma} := \widehat{F}^{-1}\widehat{\mathcal{P}}$, which is symmetric. Then the equations of elastodynamics in Lagrangian frame written

$$\widehat{\rho}_0 \frac{\partial^2 \widehat{\eta}}{\partial t^2} - \operatorname{Div}\{\widehat{F}\widehat{\Sigma}\} = \widehat{\rho}_0 \upsilon. \tag{2.4.4}$$

Further assumptions are on the nature of the specific elastic material are required in order to write $\widehat{\mathcal{P}}$ or $\widehat{\Sigma}$ explicitly in terms of $\widehat{\eta}$ or $\widehat{u}$.

2.4.1.2 Constitutive Equations for Isotropic Elasticity

The Cauchy stress tensor is an abstraction. In order for the PDE to describe a particular elastic material we require a discussion of the relationship between the abstract Cauchy stress tensor and the kinematic quantities of the material. That is, we require a constitutive equation. Elastic materials are memoryless; the description of stress on an elastic material is a only function of space (the deformation), and is independent of time. In the case of homogenous elastic materials, mechanical properties do not vary in space. In the case of isotropic elastic materials, the material response to deformation is the same in all directions.

For isotropic elastic materials, we write the dependency of the constitutive equation on the Green-Lagrange strain tensor

$$\widehat{E} := \frac{1}{2}\big(\widehat{F}^T\widehat{F} - I\big).$$

It is evident that $\widehat{E}$ can also be written in terms of the displacement $\widehat{\eta}$:

$$\widehat{E} = \frac{1}{2}(D\widehat{\eta} + (D\widehat{\eta})^* + (D\widehat{\eta})^* D\widehat{\eta}).$$

Writing $\widehat{E}$ in this way makes clear that the strain is not affected by rigid body motions, as $\widehat{E}$ is dependent only on the gradient of the displacement.

Constitutive equations for elastic materials whose stress-strain relationship may be non-linear can be written in terms of a density of elastic energy. In particular, given an energy functional $W : \mathbb{M}^3 \to \mathbb{R}^+$ we set

$$\Sigma(\widehat{E}) = \frac{\partial W}{\partial \widehat{E}}(\widehat{E}).$$

The natural state of an elastic material is a configuration where the Cauchy stress tensor is identically zero. A straightforward choice for the strain-energy density function for a homogenous, isotropic material whose reference configuration is the natural state is given by Saint-Venant Kirchoff model. The strain-energy density functional is

$$W(\widehat{E}) = \frac{\lambda}{2}[\mathrm{tr}\,(\widehat{E})]^2 + \mu \mathrm{tr}\,(E^2), \tag{2.4.5}$$

for the Lamé parameters $\lambda, \mu > 0$, so the second Piola stress tensor for the Saint-Venant

Kirchoff model is

$$\Sigma(\widehat{E}) = \frac{\partial}{\partial \widehat{E}} \left(\frac{\lambda}{2} [\mathrm{tr}\,(\widehat{E})]^2 + \mu \mathrm{tr}\,(E^2) \right) = \lambda (\mathrm{tr}\, \widehat{E}) I + 2\mu \widehat{E}. \tag{2.4.6}$$

Taken together, the equations of elastodynamics describing the deformation of a homogenous, isotropic elastic material are

$$\begin{cases} \widehat{\rho}_0 \dfrac{\partial^2 \widehat{\varphi}}{\partial t^2} - \mathrm{Div}\, \left\{ \widehat{F} \Sigma\big(\widehat{E}\big) \right\} = \widehat{\rho}_0 v, \\ \widehat{u} = \dfrac{\partial \widehat{\varphi}}{\partial t} \end{cases} \tag{2.4.7}$$

for

$$\begin{cases} \widehat{E}(\widehat{\varphi}) := \frac{1}{2}\big(\widehat{F}^T \widehat{F} - I\big), \\ \Sigma(\widehat{E}) := \lambda (\mathrm{tr}\, \widehat{E}) I + 2\mu \widehat{E}. \end{cases} \tag{2.4.8}$$

2.4.1.3 Elastodynamics in the Eulerian Frame

For fluid-structure interaction problems it can be analytically advantageous to represent the equations of fluid and structure in the same frame of reference. In order to represent the equations (2.4.7) in the Eulerian framework we work with the integral transformation and use (2.2.1) as well as (2.2.2) and (2.2.5). Using the inverse of the deformation, φ, we can write the Eulerian counterparts of the Green-Lagrange strain tensor as well as the second Piola stress tensor:

$$\begin{aligned} E &:= \frac{1}{2}(F^{-*} F^{-1} - I), \\ \Sigma(E) &:= \lambda (\mathrm{tr}\, E) I + 2\mu E. \end{aligned}$$

Due to (2.2.5), the stress tensor that appears in the equation is the Cauchy stress tensor

$$\sigma(\Sigma(E))) := J F^{-1} \Sigma(E) F^{-*}. \tag{2.4.9}$$

Altogether, the equations of elastodynamics in the Eulerian frame are

$$\begin{cases} \widehat{\rho}_0 J \left(\dfrac{\partial u}{\partial t} + Du \cdot u \right) - \text{Div} \, \{\sigma(\Sigma(E))\} = \widehat{\rho}_0 J v, \\ u = \dfrac{\partial \eta}{\partial t} + D\eta \cdot u. \end{cases} \tag{2.4.10}$$

In order to write the equations more simply we let $\rho = \widehat{\rho}_0 J$ represent the density of the solid (motivated by the fact that $\widehat{\rho}_0 = \widehat{J}\widehat{\rho}$ implies $\widehat{\rho} = \widehat{\rho}_0 J$). Then we have

$$\begin{cases} \rho \left(\dfrac{\partial u}{\partial t} + Du \cdot u \right) - \text{Div} \, \{\sigma(\Sigma(E))\} = \rho v, \\ u = \dfrac{\partial \eta}{\partial t} + D\eta \cdot u. \end{cases} \tag{2.4.11}$$

2.4.1.4 Steady Setting

The equations of elasticity in the stationary setting can be derived on their own merits, using the principles of force and moment balance and the theorem of Cauchy that the Cauchy stress vector field depends linearly on the normal vector. Alternatively, we can consider the stationary setting to describe the state of the elastic body at equilibrium, in which case we can extract the steady state elasticity equations by assuming $\dfrac{\partial \widehat{\eta}}{\partial t} = 0$. An elastic body at steady state satisfies the following equations in the Lagrangian frame:

$$-\text{Div} \left\{ \widehat{F} \Sigma\big(\widehat{E}\big) \right\} = \widehat{\rho}_0 v. \tag{2.4.12}$$

An elastic body at steady state satisfies the following equations in the Eulerian frame:

$$-\text{Div} \, \{\sigma(\Sigma(E))\} = \rho v. \tag{2.4.13}$$

2.4.2 Equations of Fluid

It is both conventional and natural to consider evolution of a fluid domain in an Eulerian framework. The quantities of interest in the evolution of a fluid domain are typically the fluid velocity and pressure. Considering the displacement of the fluid typically is not of interest; the displacements are large and not physically relevant. We are typically interested in snapshots of the velocity field throughout a domain, at particular times. This corresponds classically with the Eulerian formulation.

2.4.2.1 Time-dependent setting

Using (2.2.7) in (2.3.1), we obtain

$$0 = \frac{d}{dt}\left(\int_{V(t)} \rho \ dx\right) = \int_{V(t)} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) dx.$$

Since $V(t)$ is an arbitrary domain we recover

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \tag{2.4.14}$$

in $\Omega(t)$ for all $t > 0$. Observe that if the fluid density is constant then $\frac{\partial \rho}{\partial t} = 0$, which implies $\operatorname{div}(u) = 0$. Constant density fluids are called incompressible, and incompressibility is marked by this divergence-free condition. In the sequel we will work in the framework of a constant density fluid, which will be considered as synonymous with incompressible.

On the other hand, when we use (2.2.7) on the lefthand side of (2.3.2) we obtain

$$\frac{d}{dt}\int_{V(t)} \rho u \, dx = \int_{V(t)} \frac{\partial(\rho u)}{\partial t} + \operatorname{Div}(\rho u \otimes u) dx,$$

from which we obtain the integral equation

$$\int_{V(t)} \frac{\partial(\rho u)}{\partial t} + \operatorname{Div}(\rho u \otimes u) dx = \int_{V(t)} \rho v + \operatorname{Div} \sigma \, dx.$$

Since $V(t)$ is arbitrary we obtain

$$\frac{\partial(\rho u)}{\partial t} + \operatorname{Div}(\rho u \otimes u) - \operatorname{Div}\sigma = \rho v.$$

Under the assumption that the fluid density is constant we have

$$\rho\left(\frac{\partial u}{\partial t} + \operatorname{Div}(u \otimes u)\right) - \operatorname{Div}\sigma = \rho v. \tag{2.4.15}$$

The continuity equation (2.4.15) can be further simplified using the decomposition

$$\operatorname{Div}(u \otimes u) = Du \cdot u + (\operatorname{div} u)u.$$

Since we are working with divergence-free fluids we have that (2.4.15) simplifies to

$$\rho\left(\frac{\partial u}{\partial t} + Du \cdot u\right) - \operatorname{Div}\sigma = \rho v. \tag{2.4.16}$$

2.4.2.2 Consitutive Equations for Incompressible Navier-Stokes

The quantity of interest for the evolution of a fluid domain is the strain rate tensor

$$\varepsilon(u) := \frac{1}{2}\big(Du + (Du)^*\big),$$

as the behavior of interest is the rate at which a fluid adapts to a deformation. The Cauchy stress tensor for fluids will depend on $\varepsilon(u)$. In the case of incompressible, Newtonian fluids the dependence is linear. We have

$$\sigma(p, u) := -pI + 2v_1\varepsilon(u), \tag{2.4.17}$$

where p is the pressure and $v_1 > 0$ is the dynamic viscosity of the fluid. Suppose the viscosity v_1 is constant. Then in order to substitute (2.4.17) into (2.4.15) we observe that

$$\operatorname{Div}(pI) = \nabla p,$$

$$\mathrm{Div}\,(v_1\varepsilon(u)) = \mathrm{Div}\,(Du + (Du)^*) = v_1\Delta u,$$

where the last equality also utilizes the divergence-free condition on the fluid.

Then the Navier-Stokes equations for Newtonian, constant viscosity, incompressible fluids are

$$\begin{cases} \rho\left(\dfrac{\partial u}{\partial t} + (Du)u\right) - \mathrm{Div}\,\sigma(p,u) = \rho v, \\ \mathrm{div}\,u = 0. \end{cases} \tag{2.4.18}$$

Another common way that the Navier-Stokes equations are written (and what we will use more frequently) is to rescale p by p/ρ (without altering the notation) and let $\nu := v_1/\rho$, the kinematic viscosity of the fluid. We rewrite $\sigma(p,u) := -pI + 2\nu\varepsilon(u)$ and and then the Navier-Stokes equations are written

$$\begin{cases} \dfrac{\partial u}{\partial t} + (Du)u - \mathrm{Div}\,\sigma(p,u) = v, \\ \mathrm{div}\,u = 0. \end{cases} \tag{2.4.19}$$

2.4.2.3 Steady setting

When the fluid medium is at equilibrium, we assume that the fluid velocity is constant in time, but varies in space, as does the pressure. The velocity and pressure of the incompressible fluid in this state satisfies the following steady state Navier-Stokes equations.

$$\begin{cases} (Du)u - \mathrm{Div}\,\sigma(p,u) = v, \\ \mathrm{div}\,u = 0. \end{cases} \tag{2.4.20}$$

2.4.3 Configuration of Domain

Let Ω be a control volume containing both the elastic and fluid domains.

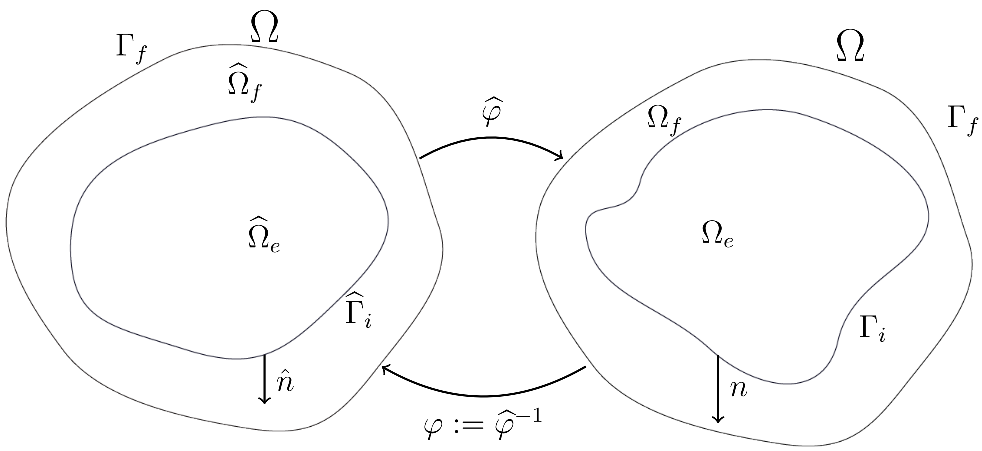

Figure 2.1 Steady state configuration of the control volume Ω on the reference and physcial domains.

In the stationary setting, shown in Figure 2.1, the elastic body occupies a domain $\Omega_e \subset \mathbb{R}^d$ ($d = 2, 3$), and is described according to a map acting in a fixed, reference domain $\widehat{\Omega}_e$. The fluid occupies domain $\Omega_f \subset \mathbb{R}^d$, and the interaction between the two media occurs via the shared interface, denoted by Γ_i. The external boundary of Ω is denoted by Γ_f. The material (or reference) configurations are denoted using the 'hat' notation, by $\widehat{\Omega}_e$, $\widehat{\Omega}_f$, $\widehat{\Gamma}_i$. Then the volume Ω is described by the deformation $\widehat{\varphi}$. For any $\widehat{x} \in \Omega_e$, $\widehat{\varphi}(\widehat{x}) = x$ represents the position of the material point $\widehat{x}$. On Ω_f, $\widehat{\varphi}$ is defined as an arbitrary extension of the restriction of $\widehat{\varphi}$ to Γ_i which preserves the boundary Γ_f. The quantity $\widehat{n}$ is the outer unit normal vector for $\widehat{\Omega}_e$ along $\widehat{\Gamma}_i$, and n is defined analogously.

In the fully dynamical setting, shown in Figure 2.2, the elastic body occupies a domain $\Omega_e(t) \subset \mathbb{R}^d$ ($d = 2, 3$), which depends on the parameter $t \geq 0$. The elastic domain is described according to a motion, acting in a fixed, reference domain. The fluid occupies domain $\Omega_f(t) \subset \mathbb{R}^d$, whose evolution is induced by the structural deformation through the shared interface, denoted by $\Gamma_i(t)$. The external boundary of Ω is denoted by Γ_f, and does not depend on time. The material (or reference) configurations are denoted using the 'hat' notation, and defined as the state of the domains at time $t = 0$, $\widehat{\Omega}_e := \Omega_e(0)$, $\widehat{\Omega}_f := \Omega_f(0)$, $\widehat{\Gamma}_i := \Gamma_i(0)$. In the dynamical setting the volume Ω is described by the motion $\widehat{\varphi}(t)$. For any $\widehat{x} \in \Omega_e$, $\widehat{\varphi}(\widehat{x}, t) = x$ represents the position of the material point $\widehat{x}$

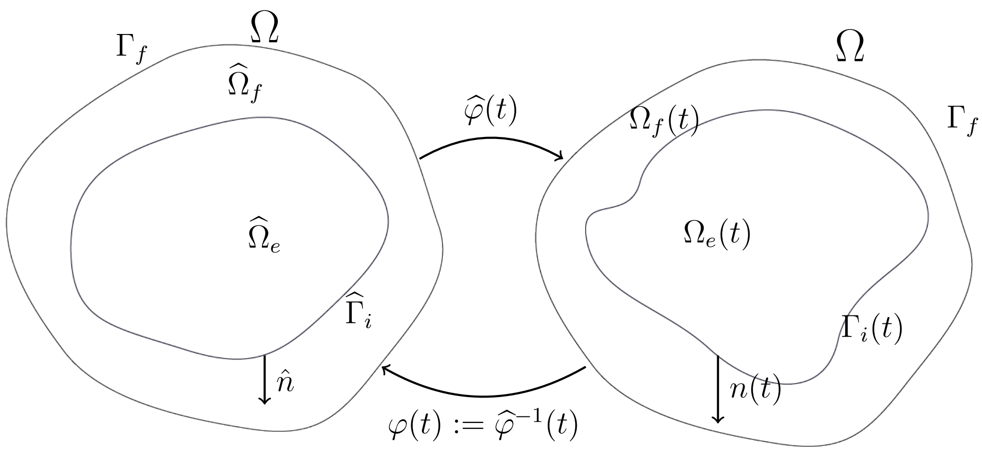

Figure 2.2 Dynamic state configuration of the control volume Ω on the reference and current domains.

at time t. On $\Omega_f(t)$, $\widehat{\varphi}(t)$ is defined as an arbitrary extension of the restriction of $\widehat{\varphi}(t)$ to $\Gamma_i(t)$ which preserves the boundary Γ_f.

In both settings the normal vectors on the reference and physical domains bear the following relationship to each other.

$$n \circ \widehat{\varphi} = \frac{(D\widehat{\varphi})^{-*}\widehat{n}}{|(D\widehat{\varphi})^{-*}\widehat{n}|} \tag{2.4.21}$$

Remark 2.4.1. *The interface $\Gamma_i(t)$ comprises a moving boundary that moves with the parameter t. Consequently the fully dynamical case entails a non-cylindrical configuration.*

2.5 Fluid-Structure Interaction Models

2.5.1 Steady State Fluid-Structure Interaction Model

The fluid-struction interaction problem entails a coupling of the nonlinear elasticity and incompressible steady state Navier-Stokes equations.

The subscripting notation is used to indicate in which domain the variable is acting, such that e.g. $u_f = u|_{\Omega_f}$. When the domain is clear from context the subscript may be omitted.

For the system in equilibrium we assume a no-slip condition for the fluid velocity on the entire fluid boundary $\partial\Omega_f = \Gamma_i \cup \Gamma_f$, as well as a matching of normal stresses on the interface.

Using these conditions to couple (2.4.13) and (2.4.20), we obtain the steady state PDE model on the deformed configuration.

$$\begin{cases} (Du_f)u_f - \operatorname{Div}\sigma_f(p, u_f) = v_f & \Omega_f \\ \operatorname{div} u_f = 0 & \Omega_f \\ -\operatorname{Div}\sigma_e(\varphi) = \rho_e v_e & \Omega_e \\ u_f = 0 & \Gamma_i \\ \sigma_e(\varphi)n = \sigma_f(p, u_f)n & \Gamma_i \\ u_f = 0 & \Gamma_f \end{cases} \tag{2.5.1}$$

where $\rho_e := \widehat{\rho}_0 J(\varphi)$ is the Eulerian density of the elastic body for $\widehat{\rho}_0$ the density in the reference configuration and the remaining notation is defined as follows:

$$\begin{aligned} \sigma_f(\psi, \phi) &:= -\psi I + 2\nu\varepsilon(\phi), \\ \varepsilon(\phi) &:= \frac{1}{2}(D\phi + (D\phi)^*), \\ \sigma_e(\phi) &:= J(\phi)(D\phi)^{-1}\Sigma(E(\phi))(D\phi)^{-*}, \\ \Sigma(\Phi) &:= \lambda \operatorname{tr}(\Phi) I + 2\mu\Phi, \\ E(\phi) &:= \frac{1}{2}((D\phi)^{-*}(D\phi)^{-1} - I), \end{aligned}$$

2.5.2 Dynamical Fluid-Structure Interaction Model

The fluid-struction interaction problem in the dynamical setting entails a coupling of the nonlinear elastodynamics and incompressible Navier-Stokes equations. For the coupling conditions we assume a matching of velocities and of normal stresses on the interface. We continue to assume a no-slip condition for the fluid velocity on the external boundary Γ_f.

Using these conditions to couple (2.4.11) and (2.4.19), we obtain the dynamic PDE model on the current configuration.

$$\begin{cases} \dfrac{\partial u_f}{\partial t} + (Du_f)u_f - \operatorname{Div} \sigma_f(p, u_f) = v_f & \Omega_f(t) \\ \operatorname{div} u_f = 0 & \Omega_f(t) \\ \rho_e \left(\dfrac{\partial u_e}{\partial t} + (Du_e)u_e \right) - \operatorname{Div} \sigma_e(\varphi) = \rho_e v_e & \Omega_e(t) \\ u_f = u_e & \Gamma_i(t) \\ \sigma_f(p, u_f)n = \sigma_e(\varphi)n & \Gamma_i(t) \\ u_f = 0 & \Gamma_f \\ \widehat{\varphi}(\cdot, 0) = \widehat{\varphi}^0, \widehat{\varphi}_t(\cdot, 0) = \widehat{\varphi}^1, u_f(\cdot, 0) = u^0, p(\cdot, 0) = p^0 & (\widehat{\Omega}_e)^2 \times (\Omega_f)^2 \end{cases} \tag{2.5.2}$$

Chapter 3

Existence of Optimal Control

3.1 Steady State Case

3.1.1 Objective

We consider a cost functional describing the difference between the fluid velocity and a given desired velocity, denoted by u_d. The objective is to reduce this difference; that is, for the fluid velocity to come as close as possible to matching a specified velocity.

The associated cost functional can be expressed as

$$J_S(v,u) = 1/2\|u - u_d\|^2_{L^2(\Omega_f)} + \frac{1}{2}\|v\|^2_{\mathcal{E}_S(\Omega)}. \tag{3.1.1}$$

The target velocity optimization problem can be formulated as

$$\min_{v \in Q^S_{\text{ad}}} J_S(v, u(v)), \tag{3.1.2}$$

for a space of admissable controls Q^S_{ad}. The control norm $\|\cdot\|_{\mathcal{E}_S(\Omega)}$ will be formulated from forthcoming known well-posedness results for the control-to-state map. Note that the inclusion in the cost functional of the term depending on the control, $\|v\|^2_{\mathcal{E}_S(\Omega)}$, is not physically required but is a mathematical necessity in the subsequent steps. Some physical justification is provided for including term by considering it as a cost of implementing the body force.

The goal is to prove the existence of an optimal control for (3.1.2); that is, we wish to show that the velocity of the fluid can be optimized by applying a force on the domain. We note that the proof relies on available existence results, which introduces two challenges. First, we require sufficiently high regularity of the control in order to utilize the existing theory. Second, the existing literature limits our consideration to the case of distributed control (control on the domain); while it would be of physical interest to consider boundary control, well-posedness analysis does not exist in the literature for that case, to the author's knowledge.

3.1.2 Lagrangian framework

The well-posedness analysis casts the coupled system (2.5.1) in Lagrangian coordinates by transporting the Navier-Stokes equation onto the reference domain.

Remark 3.1.1. *It is beneficial for the numerical solution of the optimization problem to express the system in Lagrangian coordinates. Solving the state equation on a reference domain bypasses the need to update the computational mesh at each step of the optimization algorithm.*

The transformation $\widehat{\varphi}$ is volume-preserving on $\widehat{\Omega}_f$, and so $\widehat{J}|_{\widehat{\Omega}_f} = 1$ and $[\mathrm{cof}(D\widehat{\varphi})]^*\big|_{\widehat{\Omega}_f} = (D\widehat{\varphi})^{-1}$, both facts which are used in the transport.

The PDE model can be written in the Lagrangian framework $(\widehat{\varphi}, \widehat{u}, \widehat{p})$ as in [27]:

$$\begin{cases} -\nu \mathrm{div}\,[(D\widehat{\varphi})^{-1}(D\widehat{\varphi})^{-*}D\widehat{u}_f] + (D\widehat{\varphi})^{-*}\nabla\widehat{p} + D\widehat{u}_f(D\widehat{\varphi})^{-*}\widehat{u}_f = \widehat{v}_f & \widehat{\Omega}_f \\ \mathrm{div}\,((D\widehat{\varphi})^{-1}\widehat{u}_f) = 0 & \widehat{\Omega}_f \\ -\mathrm{Div}\,\widehat{\mathcal{P}}(\widehat{\varphi}) = \widehat{\rho}_0\widehat{v}_e & \widehat{\Omega}_e \\ \widehat{u}_f = 0 & \widehat{\Gamma}_i \\ \widehat{\mathcal{P}}\widehat{n} = \nu(D\widehat{\varphi})^{-1}(D\widehat{\varphi})^{-*}D\widehat{u}_f\widehat{n} - \widehat{p}(D\widehat{\varphi})^{-*}\widehat{n} & \widehat{\Gamma}_i \\ \widehat{u}_f = 0 & \Gamma_f, \end{cases} \tag{3.1.3}$$

where recall that $\widehat{\mathcal{P}}(\widehat{\varphi})$ is the nonlinear Piola-Kirchoff stress tensor,

$$\widehat{\mathcal{P}}(\widehat{\varphi}) := D\widehat{\varphi}\Sigma\big(\widehat{E}\big) \tag{3.1.4}$$

for

$$\begin{cases} \widehat{E}(\widehat{\varphi}) := \frac{1}{2}\big((D\widehat{\varphi})^* D\widehat{\varphi} - I\big), \\ \Sigma(\widehat{E}) := \lambda(\operatorname{tr}\widehat{E})I + 2\mu\widehat{E}. \end{cases} \tag{3.1.5}$$

The first term in the cost functional can also be transported to the reference configuration:

$$J_S(v, \widehat{u}) := 1/2\|\widehat{u} - \widehat{u}_d\|^2_{L^2(\widehat{\Omega}_f)} + 1/2\|v\|^2_{\mathcal{E}_S(\Omega)}, \tag{3.1.6}$$

where $\widehat{u}_d = u_d \circ \widehat{\varphi}$.

3.1.3 Well-posedness results

To prove the existence of an optimal solution $(\bar{v}, \bar{\widehat{u}}, \bar{\widehat{\varphi}})$ we follow the strategy of [1], which requires existence and uniqueness of solutions to (3.1.3). The first major work analyzing the well-posedness of the steady state free boundary coupling of Navier-Stokes and nonlinear elasticity is found in [27], in the case of the configuration of a fluid inside an elastic structure. The work of [54] expands on the well-posedness result of [27] by generalizing the configuration (to include the configuration of Figure 2.1); additionally, the authors of [54] show that the solution is differentiable with respect to the control.

We require the following assumptions to be imposed on the control data.

Assumption 3.1.2. *Assume the existence of $v_f \in L^p(\mathbb{R}^3)$ and $\widehat{v}_e \in L^p(\widehat{\Omega}_e)$, with $3 < p < \infty$. Assume that v_f and $\widehat{v}_e$ are small enough, that is there exists a constant $M > 0$ such that*

$$C\|\widehat{v}_e\|_{L^p(\widehat{\Omega}_e)} + \mathcal{Q}(\|v_f\|_{L^p(\mathbb{R}^3)}) \leq M,$$

where $\mathcal{Q}$ is a is a polynomial function such that $\mathcal{Q}(0) = 0$ and whose non zero coefficients are positive. Assume moreover that $\|v_f\|_{H^{-1}(\mathbb{R}^3)} \leq C\nu^2$.

The following existence and uniqueness result is drawn from [27] and [54]:

Theorem 3.1.3 (Existence and uniqueness of solutions to (2.5.1))**.** *Let $\Omega \subset \mathbb{R}^3$ be an open smooth, bounded domain and let $\widehat{\Omega}_e \subset \Omega$ be an open set such that $\overline{\widehat{\Omega}_e} \subset \Omega$. Let*

$\widehat{\Omega}_f = \Omega \cap (\overline{\widehat{\Omega}_e})^c$, *and let* $\nu > 0$, $\lambda > 0$, $\mu > 0$ *be given. Under Assumption 3.1.2, there exists a unique solution* $(\widehat{u}_f, \widehat{p})$ *of* (2.5.1), *with the following regularity:*

$$\begin{aligned} \widehat{u}_f &\in W^{2,p}(\widehat{\Omega}_f) \cap W_0^{1,p}(\widehat{\Omega}_f), \\ \widehat{p} &\in W^{1,p}(\widehat{\Omega}_f). \end{aligned}$$

Furthermore, the displacement of the structure resides in the space

$$B_\rho = \{b \in W^{2,p}(\widehat{\Omega}_e) | \|b\|_{W^{2,p}(\widehat{\Omega}_e)} \leq M_1\}, \tag{3.1.7}$$

where M_1 *is chosen to justify the transport of the coordinates (specifically the invertibility of* $D\widehat{\varphi}$*). The* M *of Assumption 3.1.2, as applied here, depends on* M_1*. We have* $\widehat{\varphi} \in W^{2,p}(\widehat{\Omega}_e) \cap W^{2,p}(\widehat{\Omega}_f)$*.*

Proving existence of optimal control will rely on the underlying PDE system having solutions of sufficiently high regularity, in order to use Sobolev embedding properties to show that established weak subsequential limits are in fact strong limits in specific spaces. To that end we offer the following corollary to Theorem 3.1.3, describing how we can obtain increased regularity in the solution by increasing regularity on the control.

Corollary 3.1.4 (Additional regularity in solutions to (2.5.1))**.** *For integer* $m \geq 1$*, given the described domain configuration such that* $\widehat{\Omega}_e, \widehat{\Omega}_f$ *are* C^{m+2}*, if in addition to Assumption 3.1.2 we also have,* $\widehat{v}_f \in W^{m,p}(\widehat{\Omega}_f)$ *and* $\widehat{v}_e \in W^{m,p}(\widehat{\Omega}_e)$*, with* $3 < p < \infty$*, such that there exists a constant* $M_2 > 0$ *such that*

$$C\|\widehat{v}_e\|_{W^{m,p}(\widehat{\Omega}_e)} + \mathcal{Q}(\|v_f\|_{W^{m,p}(\Omega_f)}) \leq M_2, \tag{3.1.8}$$

then the corresponding solution of (2.5.1) *possesses the additional regularity*

$$\begin{aligned} \widehat{u}_f &\in W^{m+2,p}(\widehat{\Omega}_f), \\ \widehat{p} &\in W^{m+1,p}(\widehat{\Omega}_f), \\ \widehat{\varphi} &\in W^{m+2,p}(\widehat{\Omega}_e). \end{aligned}$$

Furthermore, the following estimates are satisfied (for $C_1 > 0$*):*

$$\|\widehat{u}_f\|_{W^{m+2,p}(\widehat{\Omega}_f)} + \|\widehat{p}\|_{W^{m+1,p}(\widehat{\Omega}_f)} \leq \mathcal{Q}(\|\widehat{v}_f\|_{W^{m,p}(\widehat{\Omega}_f)}), \tag{3.1.9}$$

$$\|\widehat{\varphi}\|_{W^{m+2,p}(\widehat{\Omega}_e)} \leq C_1 \|\widehat{v}_e\|_{W^{m,p}(\widehat{\Omega}_e)}. \tag{3.1.10}$$

Proof. The dependence of the regularity of solutions to (2.5.1) on the regularity of the control is governed by the regularity properties of the fluid and structure subproblems [27, pg. 86, 89]. The needed regularity result and estimate for the Stokes subproblem is given in [3, Proposition 1], and the analogous result for the nonlinear elasticity subproblem is given in [17, pg. 298]. □

3.1.4 Existence of optimal controls

In order to fit the problem within the framework of the well-posedness analysis, we define the control norm and the space of admissible controls. Define the control norm, in the context of Corollary 3.1.4, to be

$$\|v\|^2_{\mathcal{E}_S(\Omega)} := \|v_e\|^2_{H^m(\Omega_e)} + \|v_f\|^2_{H^m(\Omega_f)}, \tag{3.1.11}$$

and we set the set of admissible controls to be

$$\begin{aligned} Q^S_{\text{ad}} :=& \{\widehat{v}_f \in W^{m,p}(\widehat{\Omega}_f), \widehat{v}_e \in W^{m,p}(\widehat{\Omega}_e) \mid C\|\widehat{v}_e\|_{W^{m,p}(\widehat{\Omega}_e)} + k\|v_f\|_{W^{m,p}(\Omega_f)} \leq M, \\ & \|v_f\|_{H^{-1}(\mathbb{R}^3)} \leq C\nu^2, \text{ and } \|v_f\|_{W^{m,p}(\Omega_f)} \leq 1\}, \end{aligned} \tag{3.1.12}$$

where k depends on the polynomial $\mathcal{Q}$ of (3.1.8), $m \geq 1$ is an integer, and $3 < p < \infty$. Let $\mathcal{A}_S : Q^S_{\text{ad}} \to W^{m+2,p}(\widehat{\Omega}_f) \cap W^{1,p}_0(\widehat{\Omega}_f) \times W^{m+1,p}(\widehat{\Omega}_f) \times W^{m+2,p}(\widehat{\Omega}_e) \cap W^{2,p}(\widehat{\Omega}_f)$ be the control-to-state map, which maps an admissible control function to the solution of the (transported to Lagrangian coordinates) steady state problem (3.1.3). Then the optimization problem in Lagrangian framework reads: *find $\overline{v} \in Q^S_{\text{ad}}$ such that for the corresponding velocity $\overline{\widehat{u}} = \overline{\widehat{u}}(\overline{v})$ and deformation $\overline{\widehat{\varphi}} = \overline{\widehat{\varphi}}(\overline{v})$, the functional J_S in* (3.1.6) satisfies

$$J_S\left(\overline{v}, \overline{\widehat{u}}, \overline{\widehat{\varphi}}\right) = \min_{v \in Q^S_{\text{ad}}} J_S\left(v, \widehat{u}(v), \widehat{\varphi}(v)\right). \tag{3.1.13}$$

Theorem 3.1.5 (Optimal control)**.** *The optimization problem* (3.1.13) *has a solution, i.e. there is $\overline{v} \in Q^S_{ad}$ and a solution to* (3.1.3)*, $(\overline{\widehat{u}}, \overline{\widehat{p}}) \in W^{m+2,p}(\widehat{\Omega}_f) \cap W^{1,p}_0(\widehat{\Omega}_f) \times W^{m+1,p}(\widehat{\Omega}_f)$ with the associated deformation map $\overline{\widehat{\varphi}} \in W^{m+2,p}(\widehat{\Omega}_e) \cap W^{2,p}(\widehat{\Omega}_f)$, so that the functional $v \mapsto J_S(v, \widehat{u}(v), \widehat{\varphi}(v))$ attains its minimum on Q^S_{ad} at $\overline{v}$, and $(\overline{\widehat{u}}, \overline{\widehat{p}})$ is the solution of*

(3.1.3) *with Lagrangian field* $\overline{\widehat{\varphi}}$ *and forcing term* $\overline{v}$*.*

3.1.5 Proof of Theorem 3.1.5

3.1.5.1 The minimizing sequence

Let $\{v_n\} \in Q^S_{\mathrm{ad}}$ be a minimizing sequence for J_S, and set $(\widehat{u}_n, \widehat{p}_n, \widehat{\varphi}_n) = (\widehat{u}(\widehat{v}_n), \widehat{p}(\widehat{v}_n), \widehat{\varphi}(\widehat{v}_n))$ to be the associated solution of (3.1.3) with right hand side $\widehat{v}_n = v_n \circ \widehat{\varphi}_n$. By the coercivity of J_S, we know that $\{v_n\}$ is a bounded sequence in $\mathcal{E}_S(\Omega)$ with weak subsequential limits residing in the closed convex subset Q^S_{ad}. Using the estimate (3.1.9),

$$\|(\widehat{u}_n, \widehat{p}_n)\|_{W^{m+2,p}(\widehat{\Omega}_f)\times W^{m+1,p}(\widehat{\Omega}_f)} \leq \mathcal{Q}(\|\widehat{v}_f\|_{W^{m,p}(\widehat{\Omega}_f)}), \quad \text{for all} \quad n, \tag{3.1.14}$$

where $\mathcal{Q}$ is a polynomial function such that $\mathcal{Q}(0) = 0$. Hence

$$(\widehat{u}_n, \widehat{p}_n) \text{ is bounded in } W^{m+2,p}(\widehat{\Omega}_f) \times W^{m+1,p}(\widehat{\Omega}_f)\,.$$

3.1.5.2 Identifying limits

For $3 < p < \infty$, the space $W^{m+2,p}(\widehat{\Omega}_f)$ is a natural subspace of $H^2(\widehat{\Omega}_f)$ and the space $W^{m+1,p}(\widehat{\Omega}_f)$ is a natural subspace of $H^1(\widehat{\Omega}_f)$. Because $(\widehat{u}_n, \widehat{p}_n)$ is bounded in $W^{m+2,p}(\widehat{\Omega}_f) \times W^{m+1,p}(\widehat{\Omega}_f)$, then $(\widehat{u}_n, \widehat{p}_n)$ is bounded in $H^2(\widehat{\Omega}_f) \times H^1(\widehat{\Omega}_f)$, and there exists $(\overline{\widehat{u}}, \overline{\widehat{p}})$ such that on a reindexed subsequence

$$(\widehat{u}_n, \widehat{p}_n) \quad \text{converges weakly to} \quad (\overline{\widehat{u}}, \overline{\widehat{p}}) \quad \text{in} \quad H^2(\widehat{\Omega}_f) \times H^1(\widehat{\Omega}_f).$$

By (3.1.10), we have that $\widehat{\varphi}_n$ is bounded in $W^{m+2,p}(\widehat{\Omega}_e)$. Thus there exists $\overline{\widehat{\varphi}}$ such that on a reindexed subsequence

$$\widehat{\varphi}_n \quad \text{converges weakly to} \quad \overline{\widehat{\varphi}} \quad \text{in} \quad H^2(\widehat{\Omega}_e).$$

3.1.5.3 Strong convergence of $\{\widehat{p}_n\}$, $\{\widehat{u}_n\}$ and $\{\widehat{\varphi}_n\}$

The following observations will be useful to establish convergence of the minimizing sequence to the solution of the problem (3.1.3) which also minimizes the cost functional.

For the pressure term, according to the Rellich-Kondrachov Theorem, e.g. [2, pg. 168], for $m \geq 1$ the space $W^{m+1,p}(\widehat{\Omega}_f)$ in fact embeds compactly into $H^1(\widehat{\Omega}_f)$. The compact embedding implies that there exists again a reindexed subsequence such that

$$\widehat{p}_n \text{ converges strongly to } \overline{\widehat{p}} \text{ in } H^1(\widehat{\Omega}_f). \tag{3.1.15}$$

For the velocity term, according to the Rellich-Kondrachov Theorem, for $m \geq 1$ the space $W^{m+2,p}(\widehat{\Omega}_f)$, embeds compactly into the space $H^2(\widehat{\Omega}_f)$. The compact embedding implies that there exists again a reindexed subsequence such that

$$\widehat{u}_n \text{ converges strongly to } \overline{\widehat{u}} \text{ in } H^2(\widehat{\Omega}_f). \tag{3.1.16}$$

For the deformation term, the compact embedding for $m \geq 1$ of $W^{m+2,p}(\widehat{\Omega}_e)$ into $H^2(\widehat{\Omega}_e)$ gives that on an again reindexed subsequence we have,

$$\widehat{\varphi}_n \text{ converges strongly to } \overline{\widehat{\varphi}} \text{ in } H^2(\widehat{\Omega}_e). \tag{3.1.17}$$

3.1.5.4 Convergence in the system (3.1.3)

It suffices to focus on the convergence of nonlinear terms in the interior and on the boundary.

Fluid region. On the fluid region $\widehat{\Omega}_f$ we require the terms to converge in $L^2(\widehat{\Omega}_f)$. Consider the individual summands from the corresponding equation in (3.1.3):

- For the term $(D\widehat{\varphi}_n)^{-*}\nabla\widehat{p}_n$ it suffices to have the convergence of the product in $H^1(\widehat{\Omega}_f)$. The necessary convergence follows from (3.1.15) and (3.1.17).
- Next, consider $\operatorname{div}[(D\widehat{\varphi}_n)^{-1}(D\widehat{\varphi}_n)^{-*}D\widehat{u}_n]$ in $L^2(\widehat{\Omega}_f)$. The necessary convergence follows from (3.1.16) and (3.1.17).

Solid region. On $\widehat{\Omega}_e$ observe that the term $\operatorname{Div}\widehat{\mathcal{P}}(\widehat{\varphi}_n)$ converges in $L^2(\widehat{\Omega}_e)$ by (3.1.17).

Interface. Next, consider the trace identity on the reference interface $\widehat{\Gamma}_i$. For the solutions of considered regularity, the traces of the limiting problem can be identified in $L^2(\widehat{\Gamma}_i)$.

For the term $(D\widehat{\varphi}_n)^{-1}(D\widehat{\varphi}_n)^{-*}D\widehat{u}_n$ the desired convergence result follows from the strong convergence of these products in $H^1(\widehat{\Omega}_f)$ and the continuity of the trace map $H^1(\widehat{\Omega}_f) \hookrightarrow H^{1/2}(\widehat{\Gamma}_i) \subset L^2(\widehat{\Gamma}_i)$. The term $\widehat{\mathcal{P}}(\widehat{\varphi}_n)$ likewise features convergence in $H^1(\widehat{\Omega}_e)$ which suffices for the traces to converge in $L^2(\widehat{\Gamma}_i)$. The analogous conclusion holds for the boundary product $(D\widehat{\varphi})^{-*}\nabla\widehat{p}_n$ using the interior convergence in $H^1(\widehat{\Omega}_f)$.

Finally, since the reference configuration has $\widehat{\Gamma}_i$ of class C^1, then components $\widehat{n}_j$ of the normal vector do not affect the convergence in $L^2(\widehat{\Gamma}_i)$.

3.1.5.5 Attaining minimal cost

Some subsequence of controls $\{v_n\}$ goes weakly to $\bar{v}$ in $\mathcal{E}$, with the limit confined to the closed convex subset $Q^S_{\text{ad}} \subset \mathcal{E}$ (3.1.12). By the weak lower-semicontinuity of the norm

$$\|\bar{v}\|_{\mathcal{E}} \leq \liminf_{n\to\infty} \|v_n\|_{\mathcal{E}} \,.$$

In addition, from the strong convergence of $\widehat{u}_n$ and $\widehat{\varphi}_n$, we have,

$$\|\widehat{u}_n - \widehat{u}_d\|^2_{L^2(\widehat{\Omega}_f)} \to \|\overline{\widehat{u}} - \widehat{u}_d\|^2_{L^2(\widehat{\Omega}_f)}$$

Via the definition of the cost functional in (3.1.1) it follows

$$J_S(\overline{v}, \overline{\widehat{u}}, \overline{\widehat{\varphi}}) \leq J_S(v, \widehat{u}(v), \widehat{\varphi}(v)) \quad \text{for all} \quad v \in Q^S_{\text{ad}} \,.$$

completing the proof of Theorem 3.1.5.

3.2 Dynamical Case

3.2.1 Objective

In the dynamical case, the objective is to minimize the turbulence (mathematically, the vorticity) inside the fluid in the configuration described by (2.5.2). The associated cost functional is

$$J(v,u) := \frac{1}{2}\int_0^T \int_{\Omega_f(t)} |\operatorname{curl}(u)|^2 \, dx \, dt + \frac{1}{2}\|v\|^2_{\mathcal{E}(0,T;\Omega)}. \tag{3.2.1}$$

Denoting by $u = u(v)$ the flow-field corresponding to the control v, the constrained drag minimization problem can be formulated as

$$\min_{v\in Q_{\mathrm{ad}}} J(v,u(v)). \tag{3.2.2}$$

As in the steady state case, the control space Q_{ad} will be defined, and the control norm $\|\cdot\|_{\mathcal{E}(0,T;\Omega)}$ will be derived from the known well-posedness results for the control-to-state map. Furthermore, the analysis inherits the same challenges as the steady state case regarding the use of existing work on well-posedness.

3.2.2 Existence of optimal controls

The analogous result to the previous section for existence of optimal controls in the dynamical case is given in [10]. The details of the proof bear a great deal of similarity to the steady state case, so here we only summarize the results on well-posedness for the nonlinear time-dependent case from the literature and the main result from [10].

We will require the following assumptions to be imposed on the control term v and the initial data:

Assumption 3.2.1. *Suppose that control function $v(t)$ is defined for $t \in [0,\overline{T}]$, for some $\overline{T} > 0$. Assume*

$$v \in E(\overline{T}) := \{\phi \in L^2(0,\overline{T};H^3(\Omega)) \;\mid\; \partial_t^n \phi \in L^2(0,\overline{T};H^{3-n}(\Omega)) \quad \textit{for} \quad n = 1,2,3\}, \tag{3.2.3}$$

$$v(0) \in H^4(\Omega), \quad v_t(0) \in H^4(\Omega).$$

Then the following existence and uniqueness result is provided in [19]:

Theorem 3.2.2. *Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain of class H^4, and let $\widehat{\Omega}_e \subset \Omega$ be an open set of class H^4 such that $\overline{\widehat{\Omega}_e} \subset \Omega$. Suppose $u_0 \in H^6(\widehat{\Omega}_f) \cap H^6(\widehat{\Omega}_e) \cap H_0^1(\Omega) \cap L^2_{div,f}$. Let $\widehat{\Omega}_f = \Omega \cap (\overline{\widehat{\Omega}_e})^c$, and let $\nu > 0$, $\lambda > 0$, $\mu > 0$ be given. Under Assumption 3.2.1 and necessary compatibility conditions on the initial data (provided in [19, Thm. 1]), there exists $T \in (0, \overline{T})$ depending on u_0, v, and $\widehat{\Omega}_f$, such that there is a unique solution $(\widehat{u}, \widehat{p}) \in W_T \times Z_T$ of the (transported to Lagrangian coordinates) problem* (2.5.2)*. Furthermore,*

$$\widehat{\varphi} \in C([0,T]; H^4(\widehat{\Omega}_f) \cap H^4(\widehat{\Omega}_e) \cap H^1(\Omega)) \,. \tag{3.2.4}$$

The functional framework is defined as follows:

$$\begin{aligned}
V_f^4(T) &:= \left\{ \phi \in L^2(0,T; H^4(\widehat{\Omega}_f)) \mid \partial_t^n \phi \in L^2(0,T; H^{4-n}(\widehat{\Omega}_f)), n = 1,2,3 \right\}, \\
V_e^4(T) &:= \left\{ \psi \in L^2(0,T; H^4(\widehat{\Omega}_e)) \mid \partial_t^n \psi \in L^2(0,T; H^{4-n}(\widehat{\Omega}_e)), n = 1,2,3 \right\}, \\
L^2_{div,f}(\Omega) &:= \{ \psi \in L^2(\Omega) \mid div\, \psi = 0 \quad in \quad \widehat{\Omega}_f, \ \psi \cdot \widehat{n} = 0 \quad on \quad \partial\Omega \}, \\
X_T &:= \left\{ \widehat{u} \in L^2(0,T; H_0^1(\Omega)) \quad \mid \quad \left(\widehat{u}_f, \int_0^{(\cdot)} \widehat{u}_e \right) \in V_f^4(T) \times V_e^4(T) \right\}, \\
W_T &:= \left\{ \widehat{u} \in X_T \mid \widehat{u}_{ttt} \in L^\infty(0,T; L^2(\Omega)), \ \partial_t^n \int_0^{(\cdot)} \widehat{u}_e \in L^\infty(0,T; H^{4-n}(\widehat{\Omega}_e)), \right. \\
&\qquad \left. n = 0,1,2,3 \right\}, \\
Y_T &:= \left\{ \widehat{p} \in L^2\big(0,T; H^3(\widehat{\Omega}_f)\big) \mid \partial_t^n \widehat{p} \in L^2\big(0,T; H^{3-n}(\widehat{\Omega}_f)\big), \ n = 1,2 \right\}, \\
Z_T &:= \left\{ \widehat{p} \in Y_T \mid \widehat{p}_{tt} \in L^\infty\big(0,T; L^2(\widehat{\Omega}_f)\big) \right\} = Y_T \cap W^{2,\infty}(0,T; L^2(\widehat{\Omega}_f)),
\end{aligned}$$

each with its natural norm.

The authors of [10] then obtain the following main result regarding existence of optimal control for (3.2.1) subject to (2.5.2), following the strategy of [1].

Theorem 3.2.3 (Optimal control)**.** *Let $u_0 \in H^6(\widehat{\Omega}_f) \cap H^6(\widehat{\Omega}_e) \cap H_0^1(\Omega) \cap L^2_{div,f}$ satisfying the compatibility conditions as in Theorem 3.2.2. Then the minimization problem* (3.2.2) *has a solution; that is, there is* $\overline{v}$, *residing in a space of admissible controls* Q_{ad} *(defined explicitly in [10]) and a solution* $(\overline{\widehat{u}}, \overline{\widehat{p}}) \in W_T \times Z_T$ *with the associated deformation map* $\overline{\widehat{\varphi}}$ *as in* (3.2.4), *so that the functional* $v \mapsto J(v, \widehat{u}(v), \widehat{\varphi}(v))$ *attains its minimum on* Q_{ad} *at* $\overline{v}$, *and* $(\overline{\widehat{u}}, \overline{\widehat{p}})$ *minimizes* (3.2.1) *subject to* (2.5.2) *with Lagrangian deformation* $\overline{\widehat{\varphi}}$ *and forcing term* $\overline{v}$.

Chapter 4

Linearization

4.1 Introduction

With the existence of the optimal control established in both the steady and dynamical cases, we are interested in deriving the first-order necessary optimality conditions associated with problems (3.1.13) and (3.2.2), in order to characterize the optimal control. The typical way to derive the first order optimality conditions would be to apply min-max theory. That entails formulating the Lagrangian functional, which is the cost function minus weak form of the system. Then the cost function gradient reduces to the derivative of the Lagrangian with respect to the control at a saddle point. However, in this case due to the nonlinearity of the of the state equations, the Lagrangian functional associated with the coupled system is not convex-concave, and min-max theory does not apply.

Consequently, the optimality conditions must be derived from differentiability arguments on the cost functional with respect to the control. A challenge stems from the dependence of the cost integrals in the cost functional on an unknown domain and interface, which also depend on the control v.

In this section we will compute the directional derivative of the cost functional in a given but arbitrary direction. The computation of the directional derivatives of the cost functionals introduces new variables, about which sensitivity information must be derived. As the interaction is a coupling of Eulerian and Lagrangian quantities, the required sensitivity analysis on the system falls into the framework of shape analysis. We start with a summary of shape and tangential calculus provided in [20].

4.2 Shape and Tangential Calculus

4.2.1 Tangential Calculus

In [20], a differential calculus is developed avoiding the use of local bases and coordinates. The differential calculus relies on the oriented distance function, which is defined as follows:

Definition 4.2.1. *Given $\Omega \subset \mathbb{R}^N$, the distance function from a point x to Ω is*

$$d_\Omega(x) := \begin{cases} \inf_{y\in\Omega} |y - x| & \Omega \neq \emptyset, \\ +\infty & \Omega = \emptyset. \end{cases}$$

The *oriented distance function* from $x \in \mathbb{R}^N$ to $\Omega \subset \mathbb{R}^N$ is

$$b_\Omega(x) := d_\Omega(x) - d_{\Omega^c}(x),$$

where $\Omega^c = \mathbb{R}^N \setminus \Omega$ is the complement of Ω.

Observe that the oriented distance function is finite if and only if $\emptyset \neq \Omega \neq \mathbb{R}^N$. Equivalently, the oriented distance function is finite if and only if $\partial\Omega \neq \emptyset$. This is the framework in which the fluid structure interaction problems we analyze here are set, since we work with bounded subsets of $\mathbb{R}^N$. In this framework the oriented distance function is exactly the *algebraic distance function* to the boundary $\partial\Omega$,

$$b_\Omega(x) = \begin{cases} d_\Omega(x) = d_{\partial\Omega}(x) & x \in \text{int } \Omega' \\ 0 & x \in \partial\Omega \\ -d_{\Omega'}(x) = -d_{\partial\Omega'}(x) & x \in \text{int } \Omega. \end{cases}$$

Let Ω be an open domain of class C^2 in $\mathbb{R}^N$ with compact boundary Γ. Then there exists

$h > 0$ such that $b = b_\Omega \in C^2(S_{2h}(\Gamma))$. Define the *projection* of a point $x \in \mathbb{R}^N$ onto Γ by

$$p(x) := x - b(x)\nabla b(x),$$

and the *orthogonal projection operator* of a vector onto the tangent plane $T_{p(x)}\Gamma$ by

$$P(x) := I - \nabla b(x) \otimes \nabla b(x).$$

Remark 4.2.2. *The following relationships provide some insight into the connection between the development of a differential geometry through the use of the oriented distance function and classical differential geometry.*

- $P(x)$ *coincides with the first fundamental form of* Γ*;*
- $D^2b(x)$ *coincides with the second fundamental form of* Γ*;*
- $D^2b(x)^2$ *coincides with the third fundamental form of* Γ*;*
- $Dp(x) = I - \nabla b \otimes \nabla b - bD^2b.$

Definition 4.2.3. *The tangential gradient of* $f \in C^1(\Gamma)$ *is*

$$\nabla_\Gamma f := \nabla F|_\Gamma - \frac{\partial F}{\partial n} n = \nabla(f \circ p)|_\Gamma,$$

where $F \in C^1(S_{2h}(\Gamma))$ *is a* C^1*-extension of* f*.*

We can now state the relationship between the tangential gradient and the oriented distance function by way of the projection operator.

Theorem 4.2.4. *[20] Given* Γ *compact and* $h > 0$*, for a tubular neighborhood* $S_{2h} := \{x \in \mathbb{R}^N | |b(x)| < 2h\}$*, such that* $b_\Omega \in C^2(S_{2h}(\Gamma))$ *and* $f \in C^1(\Gamma)$*,*

(i) $\nabla_\Gamma f = (P\nabla F)|_\Gamma$ *and* $n \cdot \nabla_\Gamma f = \nabla b \cdot \nabla_\Gamma f = 0$*;*

(ii) $\nabla(f \circ p) = [I - bD^2b]\nabla_\Gamma f \circ p$ *and* $\nabla(f \circ p)|_\Gamma = \nabla_\Gamma f$*.*

With the tangential gradient matrix defined, we can define several other tangential derivative operators that will appear in the calculations in the sequel.

Definition 4.2.5. *The tangential gradient matrix of* $v \in (C^1(\Gamma))^N$ *is*

$$\begin{aligned} D_\Gamma v &:= D(v \circ p)|_\Gamma, \\ &= DV|_\Gamma - DVn \otimes n, \end{aligned}$$

where $V \in C^1(S_{2h}(\Gamma))$ *is an extension of* v.

The tangential divergence of v *is*

$$\begin{aligned} div_\Gamma v &:= div(v \circ p)|_\Gamma, \\ &= tr[DV|_\Gamma - DVn \otimes n], \\ &= divV|_\Gamma - DVn \cdot n. \end{aligned}$$

The following lemma lays out several facts about the oriented distance function and the tangential derivative operators, each of which plays a role in the linearization and derivation of the adjoint sensitivity information.

Lemma 4.2.6.

$$n = \nabla b|_\Gamma \tag{4.2.1}$$
$$D_\Gamma(n) = D^2 b|_\Gamma = D^*_\Gamma(n) \tag{4.2.2}$$
$$\nabla_\Gamma \langle v, n\rangle = (D_\Gamma v)^* n + (D^2 b)v \tag{4.2.3}$$
$$(D^2 b)n = 0 \tag{4.2.4}$$

4.2.2 Derivatives of Domain and Boundary Integrals

The construction of the differentiability arguments involved in deriving the first order optimality conditions will require us to take derivatives of domain and boundary integrals with respect to a parameter on which the domain and boundary also depend.

In order to do so we will utilize formulas presented in [20]. For an arbitrary domain and its boundary, Ω and $\Gamma = \partial\Omega$, consider a family of transformations $\{T_s : 0 \leq s \leq s_0\}$ such that $T_s : \Omega \to \Omega_s$ and the associated family of velocity fields $\left\{V(s) := \dfrac{\partial T_s}{\partial s} : 0 \leq s \leq s_0\right\}$. Under certain smoothness conditions on $V(s)$, for a smooth function $\phi(s) = \phi(s, x)$ we have

$$\frac{\partial}{\partial s}\left(\int_{\Omega(s)}\phi(s)\right)\Big|_{s=0} = \int_{\Omega}\phi'(0) + \int_{\Gamma}\phi(0)\langle V(0), n\rangle \tag{4.2.5}$$

$$\frac{\partial}{\partial s}\left(\int_{\Gamma(s)}\phi(s)\right)\Big|_{s=0} = \int_{\Gamma}\phi'(0) + \left(\frac{\partial\phi}{\partial n} + H\phi\right)\langle V(0), n\rangle, \tag{4.2.6}$$

$$= \int_{\Gamma}\phi'(0) + D\phi\cdot V(0) + \phi(\operatorname{div} V(0) - DV(0)n\cdot n), \tag{4.2.7}$$

where $\phi'(0) = \frac{\partial\phi}{\partial s}\Big|_{s=0}$, n is the outer normal to Ω along Γ, $\frac{\partial\phi}{\partial n} = D\phi(0,x)\cdot n$, and $H := \Delta b$ is the additive curvature of the boundary.

The following result from [20] is perhaps the most important for the analysis that follows; it will be used many times as a method for integrating by parts with tangential derivatives on the boundary.

Theorem 4.2.7 (Tangential Green's Formula)**.** *For $f \in C^1(\Gamma)$ and $v \in (C^1(\Gamma))^n$,*

$$\int_{\Gamma} f\, div_{\Gamma} v + \langle\nabla_{\Gamma} f, v\rangle d\Gamma = \int_{\Gamma} H f\langle v, n\rangle d\Gamma. \tag{4.2.8}$$

The tangential Green's formula is easily extended for a vector-valued function and a matrix.

Corollary 4.2.8. *For $f \in C^1(\Gamma)^n$ and $V \in (C^1(\Gamma))^{n\times n}$,*

$$\int_{\Gamma} f\, Div_{\Gamma} V + V..D_{\Gamma} f d\Gamma = \int_{\Gamma} H\langle Vn, f\rangle d\Gamma, \tag{4.2.9}$$

4.3 Steady State Case

4.3.1 Strategy

We model the approach after the techniques in [12] and perturb the system through the control v. Specifically, suppose that $v^s = v(s,x)$ depends linearly on a small parameter of variation $s \in [0, s_0]$. That is $v^s = v + sv'$ for given functions $v, v' \in \mathcal{E}(\Omega)$ such that

$\operatorname{div} v' = 0$ and $v'|_{\Gamma^f} = 0$. The perturbation can be thought of as a perturbation of the interface Γ_i^s determined by the family of transformations

$$
\begin{aligned}
\{T_s : 0 \leq s \leq s_0\} \\
T_s : \Omega_e \to \Omega_e^s \\
T_s = \widehat{\varphi}^s \circ \varphi,
\end{aligned}
\tag{4.3.1}
$$

associated to the family of velocity fields

$$
\{V(s) := \frac{\partial \widehat{\varphi}^s}{\partial s} \circ \varphi^s : 0 \leq s \leq s_0\}. \tag{4.3.2}
$$

Then we consider the perturbed functional $J_S(v + sv')$ and calculate the derivative at $s = 0$ of the function $s \to J_S(v + sv')$.

$$
\begin{aligned}
\partial J_S(v; v') &= \lim_{s \to 0} \frac{J_S(v + sv') - J_S(v)}{s} = \frac{\partial}{\partial s} J_S(v + sv') \bigg|_{s=0} \\
&= \frac{\partial}{\partial s} \left[\frac{1}{2} \int_{\Omega_f^s} |u_f^s - u_d|^2 + \frac{1}{2} \|v + sv'\|^2_{\mathcal{E}(\Omega)} \right] \Bigg|_{s=0} \\
&= \int_{\Omega_f} \left\langle u_f - u_d, \frac{\partial u_f}{\partial s} \bigg|_{s=0} \right\rangle - \frac{1}{2} \int_{\Gamma_i} \langle |u_f^s - u_d|^2 n, V(0) \rangle + (v, v')_{\mathcal{E}(\Omega)},
\end{aligned}
\tag{4.3.3}
$$

using the formula for domain integrals (4.2.5) in the last step.

The calculation of the directional derivative of the cost functional introduces the new variables $\dfrac{\partial u_f^s}{\partial s}\bigg|_{s=0}$ and $V(0) := \dfrac{\partial \widehat{\varphi}^s}{\partial s}\bigg|_{s=0} \circ \varphi$. The forthcoming linearization of (2.5.1) entails deriving the system of equations satisfied by these variables.

4.3.1.1 Perturbed System

The result of the perturbation by T_s as described in (4.3.1) is that all of the unknowns in (2.5.1) become functions of s, $(u^s, p^s, \widehat{\varphi}^s)$, as do the domains $(\Omega_f^s, \Omega_e^s, \Gamma_i^s)$.

The perturbed system is as follows:

$$\begin{cases} (Du_f^s)u_f^s - \operatorname{Div}\sigma_f(p^s, u_f^s) = v_f^s & \Omega_f^s \\ \operatorname{div} u_f^s = 0 & \Omega_f^s \\ -\operatorname{Div}\sigma_e(\varphi^s) = \rho_e^s v_e^s & \Omega_e^s \\ u_f^s = 0 & \Gamma_i^s \\ \sigma_e^s(\varphi^s)n^s = \sigma_f(p^s, u_f^s)n^s & \Gamma_i^s \\ u_f^s = 0 & \Gamma_f \end{cases} \tag{4.3.4}$$

where n^s is the unit outer normal vector along Γ_i^s with respect to Ω_e^s and $\rho_e^s = \widehat{\rho}_0 J(\varphi^s)$.

The strategy is to differentiate (4.3.4) with respect to s at $s = 0$ and obtain a linearized system for the s-derivatives of $(u^s, p^s, \widehat{\varphi}^s)$. Linearizing in this manner will represent a *total* linearization of (4.3.4) around an arbitrary regime.

In this framework, the s-derivatives of $(u^s, p^s, \widehat{\varphi}^s)$ are essentially shape derivatives with respect to the speed V. However, in this context V is a vector field which depends on $\widehat{\varphi}^s$ and is not given a priori, so the s-derivatives are in a sense *pseudo*-shape derivatives. Nonetheless, the standard theory on shape differentiation applies here.

We employ the prime notation to denote the linearized variables.

$$u' := \frac{\partial}{\partial s}u^s\bigg|_{s=0}, p' := \frac{\partial}{\partial s}p^s\bigg|_{s=0}, \widehat{\varphi}' := \frac{\partial}{\partial s}\widehat{\varphi}^s\bigg|_{s=0}.$$

When computing the shape derivatives, the speed associated with the flow T_s will appear only at $s = 0$. Thus we use the notation $V(0) := \widehat{\varphi}' \circ \varphi$.

We also associate the following differentiability assumptions with (4.3.4):

Assumption 4.3.1. *For all $s \in [0, s_0)$ the following weak derivatives with respect to s exist:*

1. $\forall \widehat{\phi} \in H^{-1}(\Omega) : \left(s \to \int_\Omega \langle \widehat{\varphi}_s, \widehat{\phi} \rangle\right) \in C^1([0, s_0))$

2. $\forall \phi \in H^{-1}(\Omega \setminus \overline{\Omega_e}) : \left(s \to \int_{\Omega \setminus \overline{\Omega_e}} \langle u_f^s \circ T_s, \phi \rangle\right) \in C^1([0, s_0))$

3. $\forall \pi \in L^2(\Omega \setminus \overline{\Omega_e}) : \left(s \to \int_{\Omega\setminus\overline{\Omega_e}} \langle p_s \circ T_s, \pi \rangle \right) \in C^1([0, s_0))$

Remark 4.3.2. *The authors of [54] expand on the work of [27] to show differentiability of the solution of* (2.5.1) *with respect the the control in the case of linear elasticity, providing some justification for Assumption 4.3.1.*

4.3.2 Linearized Steady State Model

The following result is obtained using the techniques demonstrated in [12]; in that work, the authors approach the linearization of a steady-state coupling of homogenous Navier-Stokes and homogeneous nonlinear elasticity in the case of a fluid flowing through a channel, with control on the normal velocity of the fluid flowing in.

Theorem 4.3.3 (Linearization of steady-state coupling of Navier-Stokes and Nonlinear Elasticity)**.** *For any* $s \in [0, s_0]$ *let* $(u^s, p^s, \widehat{\varphi}^s)$ *be a smooth solution of* (4.3.4) *that satisfies Assumption 4.3.1, and let the perturbed control,* v_s*, satisfy the criteria in Assumption 3.1.2. Then the following linear system is satisfied by* $u' := \frac{\partial}{\partial s} u^s\big|_{s=0}$, $p' := \frac{\partial}{\partial s} p^s\big|_{s=0}$, $\widehat{\varphi}' := \frac{\partial}{\partial s} \widehat{\varphi}^s\big|_{s=0}$:

$$\begin{cases} (Du'_f)u_f + (Du_f)u'_f - Div\, \sigma_f(p', u'_f) = v'_f & \Omega_f \\ div\, u'_f = 0 & \Omega_f \\ -Div \left\{ \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)} \right\} = \rho_e v'_e & \Omega_e \\ u'_f + (Du_f)(\widehat{\varphi}' \circ \varphi) = 0 & \Gamma_i \\ \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)} n = \sigma_f(p', u'_f)n + \mathcal{B}(\widehat{\varphi}' \circ \varphi) & \Gamma_i \\ u'_f = 0 & \Gamma_f \end{cases} \tag{4.3.5}$$

where

$$\begin{aligned} \overline{\Sigma}(\Phi, \phi) :=& (D\phi)^{-1} \Sigma\,(\Phi)\,(D\phi)^{-*}, \\ \widetilde{\varepsilon}(\psi, \phi) :=& (D\phi)^{-*} \varepsilon(\psi) (D\phi)^{-1}, \\ \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)} :=& D(\widehat{\varphi}' \circ \varphi)\sigma_e(\varphi) + J(\varphi)\overline{\Sigma}(\widetilde{\varepsilon}(\widehat{\varphi}' \circ \varphi, \varphi)), \\ \mathcal{B}(\widehat{\varphi}' \circ \varphi) :=& [\sigma_e(\varphi) - \sigma_f(p, u_f)] \cdot \nabla_{\Gamma_i} \langle \widehat{\varphi}' \circ \varphi, n \rangle + (D_{\Gamma_i} \sigma_e . (\widehat{\varphi}' \circ \varphi))n + (div_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n \end{aligned}$$

$$- \sigma_e(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n + \langle \widehat{\varphi}' \circ \varphi, n \rangle \left(H\sigma_f(p, u_f)n + \frac{\partial \sigma_f}{\partial n} n \right). \tag{4.3.6}$$

Observe that the boundary conditions of the linearized system are quite complicated. For one, there is a double coupling of $(u', \widehat{\varphi}')$ on Γ_i, unlike in the nonlinear system. Further, and more importantly, terms involving the curvatures of the boundary and the boundary acceleration are present in the linearization. This makes it clear the common interface can not be neglected while performing sensitivity analysis on the nonlinear coupled system (2.5.1).

4.3.3 Proof of Theorem 4.3.3

The general approach for proving the linearization in Theorem 4.3.3 is to write the $s-$perturbed system (4.3.4) in variational form and apply the formulas (4.2.5), (4.2.6), (4.2.7), for derivatives of domain and boundary integrals. The underlying mechanics of this approach utilize the material derivatives of $(u_s, p_s, \widehat{\varphi}_s)$, which are defined as

$$(\dot{u}, \dot{p}, \dot{\widehat{\varphi}}) := \frac{\partial}{\partial s}(u_s \circ T_s, p_s \circ T_s, \widehat{\varphi}_s).$$

The material derivatives bare the analogous relationship to (2.2.2) to the shape derivatives,

$$(u', p') = (\dot{u}, \dot{p}) - (Du \cdot (\widehat{\varphi}' \circ \varphi), \nabla p \cdot (\widehat{\varphi}' \circ \varphi)). \tag{4.3.7}$$

4.3.3.1 The linearized sticking condition on Γ_i

Take a test function $\phi \in C^1(\Omega)$. Then the sticking condition on Γ_i^s can be written in weak form as

$$0 = \int_{\Gamma_i^s} \langle u_f^s, \phi \rangle.$$

We take the derivative with respect to s at $s = 0$ by applying (4.2.7) and obtain

$$0 = \frac{\partial}{\partial s}\left(\int_{\Gamma_i^s} \langle u_f^s, \phi\rangle\right)\Bigg|_{s=0}$$
$$= \int_{\Gamma_i} \langle u_f' + (Du_f)(\widehat{\varphi}' \circ \varphi) + u_f(\operatorname{div}(\widehat{\varphi}' \circ \varphi) - (D(\widehat{\varphi}' \circ \varphi))n \cdot n), \phi\rangle,$$

which holds for all $\phi \in C^1(\Omega)$, so we extract the boundary condition

$$u_f' + (Du_f)(\widehat{\varphi}' \circ \varphi) + u_f(\operatorname{div}(\widehat{\varphi}' \circ \varphi) - (D(\widehat{\varphi}' \circ \varphi))n \cdot n) = 0.$$

Substituting the sticking condition from (2.5.2), we recover the final linearized boundary condition on Γ_i:

$$u_f' + (Du_f)(\widehat{\varphi}' \circ \varphi) = 0$$

4.3.3.2 Linearized fluid shape derivative on Ω_f

Take a test function $\phi \in C_0^1(\Omega_f)$, and we can obtain the weak formulation of the perturbed Navier-Stokes from (4.3.4):

$$\int_{\Omega_f^s} \langle (Du_f^s)u_f^s, \phi\rangle + \sigma_f(p^s, u_f^s)..D\phi = \int_{\Omega_f^s} \langle v_f^s, \phi\rangle. \tag{4.3.8}$$

We use (4.2.5) to differentiate with respect to s at $s = 0$.

$$\frac{\partial}{\partial s}\left(\int_{\Omega_f^s} \langle (Du_f^s)u_f^s, \phi\rangle + \sigma_f(p^s, u_f^s)..D\phi\right)\Bigg|_{s=0}$$

$$= \int_{\Omega_f} \langle (Du_f)u_f' + (Du_f')u_f, \phi\rangle + \sigma_f(p', u_f')..D\phi + \text{boundary terms}$$

Note that $\frac{\partial}{\partial s}(\sigma_f(p^s, u_f^s))\Big|_{s=0} = \sigma_f(p', u_f')$ because σ_f is linear.

$$\frac{\partial}{\partial s}\left(\int_{\Omega_f^s} \langle v_f^s, \phi\rangle\right)\Bigg|_{s=0} = \int_{\Omega_f} \langle v_f', \phi\rangle$$

We are unconcerned with the boundary terms since we have already recovered the linearized sticking condition on the interface Γ_i, which is why we take ϕ to have compact support on Γ_i; we recover the linearized Navier-Stokes on Ω_f:

$$(Du_f)u_f' + (Du_f')u_f - \operatorname{Div}\sigma_f(p', u_f') = v_f'.$$

4.3.3.3 Linearized Elasticity on Ω_e

Take a test function $\phi \in C^1(\Omega)$. Then we have the weak formulation of the s-perturbed elasticity equation:

$$\int_{\Omega_e^s} \sigma_e(\varphi^s)..D\phi - \int_{\Gamma_i^s} \langle \sigma_e(\varphi^s)n_s, \phi\rangle = \int_{\Omega_e^s} \langle v_e^s, \phi\rangle. \tag{4.3.9}$$

Recall that the s-derivative of the elastic deformation is defined as $\widehat{\varphi}' = \left.\dfrac{\partial}{\partial s}\widehat{\varphi}^s\right|_{s=0}$, on the reference configuration $\widehat{\Omega}_e$. So in order to take the $s-$derivative of the domain term on the left-hand side of (4.3.9) we will transport the integral to the reference configuration, take the derivative, and then transport back to the deformed configuration to recover the linearized elasticity equation. Let $\widehat{\phi}^s = \phi \circ \widehat{\varphi}^s$. According to Lemma 5.1 in [14] we have

$$\int_{\Omega_e^s} \sigma_e(\varphi^s)..D\phi = \int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}^s..D\widehat{\phi}^s,$$

where $\widehat{\mathcal{P}}$ is the $s-$perturbed Piola transform,

$$\begin{aligned} \widehat{\mathcal{P}}^s &:= D\widehat{\varphi}^s\Sigma(\widehat{E}(\widehat{\varphi}^s)), \\ \widehat{E}(\widehat{\phi}) &:= \frac{1}{2}\left(D\widehat{\phi}^* D\widehat{\phi} - I\right). \end{aligned}$$

Then we take the $s-$derivative of the domain term on the left hand side of (4.3.9). Notice that the reference configuration $\widehat{\Omega}_e$ is invariant to the perturbation, so the $s-$derivative can be brought directly under the integral.

$$\frac{\partial}{\partial s}\left(\int_{\Omega_e^s} \sigma_e(\varphi^s)..D\phi\right)\bigg|_{s=0} = \frac{\partial}{\partial s}\left(\int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}^s..D\widehat{\phi}^s\right)\bigg|_{s=0},$$

$$= \int_{\widehat{\Omega}_e} \frac{\partial}{\partial s} \left(\widehat{\mathcal{P}}^s .. D\widehat{\phi}^s \right) \Big|_{s=0},$$
$$= \int_{\widehat{\Omega}_e} \frac{\partial \widehat{\mathcal{P}}^s}{\partial s} \Big|_{s=0} .. D\widehat{\phi} + \widehat{\mathcal{P}} .. \frac{\partial}{\partial s} \left(D\widehat{\phi}^s \right) \Big|_{s=0},$$

where $\widehat{\phi} = \phi \circ \widehat{\varphi}$ is the Lagrangian counterpart of the test function ϕ. For the $s-$derivative of the Piola transform we have

$$\frac{\partial \widehat{\mathcal{P}}^s}{\partial s} \Big|_{s=0} = D\widehat{\varphi}' \Sigma(\widehat{E}(\widehat{\varphi})) + D\widehat{\varphi} \Sigma \left(\frac{1}{2} \left((D\widehat{\varphi}')^* D\widehat{\varphi} + (D\widehat{\varphi})^* D\widehat{\varphi}' \right) \right).$$

Define $\widehat{E}'(\widehat{\varphi}') := \frac{1}{2} \left((D\widehat{\varphi}')^* D\widehat{\varphi} + (D\widehat{\varphi})^* D\widehat{\varphi}' \right)$, then we can define

$$\widehat{\mathcal{P}}' := D\widehat{\varphi}' \Sigma(\widehat{E}(\widehat{\varphi})) + D\widehat{\varphi} \Sigma \left(\widehat{E}'(\widehat{\varphi}') \right). \tag{4.3.10}$$

Regarding the second term, we have

$$\frac{\partial \widehat{\phi}^s}{\partial s} \Big|_{s=0} = \frac{\partial}{\partial s} \left(\phi \circ \widehat{\varphi}^s \right) \Big|_{s=0} = (D\phi \circ \widehat{\varphi}) \widehat{\varphi}',$$

so

$$\frac{\partial}{\partial s} \left(\int_{\Omega_e^s} \sigma_e(\varphi^s) .. D\phi \right) \Big|_{s=0} = \frac{\partial}{\partial s} \left(\int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}^s .. D\widehat{\phi}^s \right) \Big|_{s=0},$$
$$= \int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}' .. D\widehat{\phi} + \widehat{\mathcal{P}} .. D \left[(D\phi \circ \widehat{\varphi}) \widehat{\varphi}' \right]. \tag{4.3.11}$$

Remark 4.3.4. *At this juncture we could integrate by parts in* (4.3.11) *and obtain the linearized elasticity equation on the reference configuration* $\widehat{\Omega}_e$. *However, it is our goal to express the linearized system fully on the deformed configuration, so we instead work to transport* (4.3.11) *back to the deformed configuration* Ω_e *in order to recover the linearized elasticity equation.*

We will work term-by-term to transport (4.3.11) to the deformed configuration.

$$\begin{aligned}\int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}'..D\widehat{\phi} &= \int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}'(D\widehat{\varphi})^*..D\widehat{\phi}(D\widehat{\varphi})^{-1} \\ &= \int_{\Omega_e} J(\varphi)\left(\widehat{\mathcal{P}}'(D\widehat{\varphi})^*\right)\circ\varphi..\left(D\widehat{\phi}(D\widehat{\varphi})^{-1}\right)\circ\varphi, \\ &= \int_{\Omega_e} J(\varphi)\left(\widehat{\mathcal{P}}'(D\widehat{\varphi})^*\right)\circ\varphi..D\phi.\end{aligned}$$

It remains to compute $\left(\widehat{\mathcal{P}}'(D\widehat{\varphi})^*\right)\circ\varphi$:

$$\begin{aligned}\left(\widehat{\mathcal{P}}'(D\widehat{\varphi})^*\right)\circ\varphi &= \left\{\left(D\widehat{\varphi}'\Sigma(\widehat{E}(\widehat{\varphi})) + D\widehat{\varphi}\Sigma\left(\widehat{E}'(\widehat{\varphi}')\right)\right)(D\widehat{\varphi})^*\right\}\circ\varphi, \\ &= \left(D(\widehat{\varphi}'\circ\varphi)(D\varphi)^{-1}\Sigma(E(\varphi)) + (D\varphi)^{-1}\Sigma\left(\widehat{E}'(\widehat{\varphi}')\circ\varphi\right)\right)(D\varphi)^{-*}, \\ &=\Big(D(\widehat{\varphi}'\circ\varphi)(D\varphi)^{-1}\Sigma(E(\varphi)) + \\ &\quad (D\varphi)^{-1}\Sigma\left(\frac{1}{2}\left((D\varphi)^{-*}\varepsilon(\widehat{\varphi}'\circ\varphi)(D\varphi)^{-1}\right)\right)\Big)(D\varphi)^{-*}\end{aligned}$$

We make the following definitions for the sake of exposition:

$$\overline{\Sigma}(\Phi,\phi) :=(D\phi)^{-1}\Sigma\left(\Phi\right)(D\phi)^{-*}, \tag{4.3.12}$$

$$\widetilde{\varepsilon}(\psi,\phi) :=(D\phi)^{-*}\varepsilon(\psi)(D\phi)^{-1}. \tag{4.3.13}$$

With this notation we have

$$J(\varphi)\left(\widehat{\mathcal{P}}'(D\widehat{\varphi})^*\right)\circ\varphi = D(\widehat{\varphi}'\circ\varphi)\sigma_e(\varphi) + J(\varphi)\overline{\Sigma}(\widetilde{\varepsilon}((\widehat{\varphi}'\circ\varphi),\varphi)).$$

We let

$$\begin{aligned}\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)} :=& J(\varphi)\left(\widehat{\mathcal{P}}'(D\widehat{\varphi})^*\right)\circ\varphi \\ =& D(\widehat{\varphi}'\circ\varphi)\sigma_e(\varphi) + J(\varphi)\overline{\Sigma}(\widetilde{\varepsilon}((\widehat{\varphi}'\circ\varphi),\varphi)),\end{aligned} \tag{4.3.14}$$

which is consistent with the notation used in [11]. Ultimately for the first term in (4.3.11) we obtain

$$\int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}'..D\widehat{\phi} = \int_{\Omega_e} \left\langle -\mathrm{Div}\, \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}, \phi \right\rangle + \int_{\Gamma_i} \left\langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n, \phi \right\rangle. \tag{4.3.15}$$

Next we work with the second term in (4.3.11). First we integrate by parts and change variables to move back to the deformed configuration:

$$\begin{aligned} \int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}..D\left[(D\phi \circ \widehat{\varphi})\widehat{\varphi}'\right] &= \int_{\widehat{\Omega}_e} \left\langle -\mathrm{Div}\, \widehat{\mathcal{P}}, (D\phi \circ \widehat{\varphi})\widehat{\varphi}' \right\rangle + \int_{\widehat{\Gamma}_i} \left\langle \widehat{\mathcal{P}}\hat{n}, (D\phi \circ \widehat{\varphi})\widehat{\varphi}' \right\rangle, \\ &= \int_{\Omega_e} \langle -\mathrm{Div}\, \sigma_e(\varphi), (D\phi)(\widehat{\varphi}' \circ \varphi) \rangle \\ &\quad + \int_{\Gamma_i} \langle \sigma_e(\varphi) n, (D\phi)(\widehat{\varphi}' \circ \varphi) \rangle. \end{aligned} \tag{4.3.16}$$

For convencience, we state a useful lemma from [14] for integration by parts:

Lemma 4.3.5. *Let $\theta, \xi, \psi \in C^1(\Omega)$ be vector-valued. Then the following identity is satisfied:*

$$\begin{aligned} \int_{\Omega} \langle \xi, (D\psi)\theta \rangle &= -\int_{\Omega} \langle (D\xi)\theta + (div\,\theta)\xi, \psi \rangle + \int_{\partial\Omega} \langle \langle \theta, n \rangle \xi, \psi \rangle \\ &= -\int_{\Omega} \langle Div\,(\xi \otimes \theta), \psi \rangle + \int_{\partial\Omega} \langle \langle \theta, n \rangle \xi, \psi \rangle \end{aligned} \tag{4.3.17}$$

Additionally, in the case where $D\psi \cdot n = 0$ on $\partial\Omega$ we have

$$\int_{\partial\Omega} \langle \xi, (D_\Gamma \psi)\theta \rangle = -\int_{\partial\Omega} \langle Div_\Gamma(\xi \otimes \theta), \psi \rangle, \tag{4.3.18}$$

which follows from (4.2.9) *with $V = \xi \otimes \theta$.*

First we apply (4.3.17) from Lemma 4.3.5 to the domain term in (4.3.16) and obtain

$$\begin{aligned} \int_{\Omega_e} \langle -\mathrm{Div}\, \sigma_e(\varphi), (D\phi)(\widehat{\varphi}' \circ \varphi) \rangle &= \int_{\Omega_e} \langle \mathrm{Div}\, \left(\mathrm{Div}\, \sigma_e(\varphi) \otimes (\widehat{\varphi}' \circ \varphi) \right), \phi \rangle \\ &\quad - \int_{\Gamma_i} \langle \langle (\widehat{\varphi}' \circ \varphi), n \rangle \mathrm{Div}\, \sigma_e(\varphi), \phi \rangle. \end{aligned}$$

For the purposes of the identities we are proving, it suffices to have ϕ such that $D\phi \cdot n = 0$.

Working in that framework we apply (4.3.18) from Lemma 4.3.5 to the boundary term in (4.3.16) and obtain

$$\int_{\Gamma_i} \langle \sigma_e(\varphi)n, (D\phi)(\widehat{\varphi}' \circ \varphi)\rangle = \int_{\Gamma_i} \langle -\mathrm{Div}_{\Gamma_i}(\sigma_e(\varphi)n \otimes (\widehat{\varphi}' \circ \varphi)), \phi\rangle \,.$$

Together, we obtain

$$\begin{aligned}\int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}..D\left[(D\phi \circ \widehat{\varphi})\widehat{\varphi}'\right] = &\int_{\Omega_e} \langle \mathrm{Div}\ (\mathrm{Div}\,\sigma_e(\varphi) \otimes (\widehat{\varphi}' \circ \varphi))\,, \phi\rangle \\ &- \int_{\Gamma_i} \langle \langle (\widehat{\varphi}' \circ \varphi), n\rangle \mathrm{Div}\,\sigma_e(\varphi) + \mathrm{Div}_{\Gamma_i}(\sigma_e(\varphi)n \otimes (\widehat{\varphi}' \circ \varphi)), \phi\rangle \,.\end{aligned}$$

All together we obtain the following expression for the shape derivative of the left hand side of the elasticity domain integral on the deformed configuration:

$$\begin{aligned}\frac{\partial}{\partial s}\left(\int_{\Omega_e^s} \sigma_e(\varphi^s)..D\phi\right)\Bigg|_{s=0} = &\int_{\Omega_e} \left\langle -\mathrm{Div}\ \left\{\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} - \mathrm{Div}\,\sigma_e(\varphi) \otimes (\widehat{\varphi}' \circ \varphi)\right\}, \phi\right\rangle \\ &+ \int_{\Gamma_i} \left\langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n - \langle (\widehat{\varphi}' \circ \varphi), n\rangle \mathrm{Div}\,\sigma_e(\varphi), \phi\right\rangle \\ &- \int_{\Gamma_i} \langle \mathrm{Div}_{\Gamma_i}(\sigma_e(\varphi)n \otimes (\widehat{\varphi}' \circ \varphi)), \phi\rangle \,. \qquad (4.3.19)\end{aligned}$$

Now we deal with the right hand side of (4.3.9). Recall that $\widehat{v}_e := v_e \circ \widehat{\varphi}$ and $\widehat{v}_e' := v_e' \circ \widehat{\varphi}$ are the Lagrangian counterparts of the perturbed control that $\widehat{\phi}^s = \phi \circ \widehat{\varphi}^s$. Then we obtain,

$$\begin{aligned}\frac{\partial}{\partial s}\left(\int_{\Omega_e^s} \langle \rho_e^s v_e^s, \phi\rangle\right)\Bigg|_{s=0} &= \frac{\partial}{\partial s}\left(\int_{\Omega_e^s} \langle \widehat{\rho}_0 J(\varphi^s)(v_e + s v_e'), \phi\rangle\right)\Bigg|_{s=0}, \\ &= \frac{\partial}{\partial s}\left(\int_{\widehat{\Omega}_e} \langle \widehat{\rho}_0(\widehat{v}_e + s\widehat{v}_e'), \widehat{\phi}^s\rangle\right)\Bigg|_{s=0}, \\ &= \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0\, \frac{\partial}{\partial s}(\widehat{v}_e + s\widehat{v}_e')\Big|_{s=0}, \widehat{\phi}\right\rangle + \left\langle \widehat{\rho}_0 \widehat{v}_e, \frac{\partial \widehat{\phi}^s}{\partial s}\Big|_{s=0}\right\rangle, \\ &= \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \widehat{v}_e', \widehat{\phi}\right\rangle + \langle \widehat{\rho}_0 \widehat{v}_e, (D\phi \circ \widehat{\varphi})\widehat{\varphi}'\rangle\,,\end{aligned}$$

$$= \int_{\Omega_e} \langle \widehat{\rho}_0 J(\varphi) v_e', \phi \rangle + \langle \widehat{\rho}_0 J(\varphi) v_e, (D\phi)(\widehat{\varphi}' \circ \varphi) \rangle \,,$$
$$= \int_{\Omega_e} \langle \rho_e v_e', \phi \rangle + \langle \rho_e v_e, (D\phi)(\widehat{\varphi}' \circ \varphi) \rangle \,.$$

We substitute the nonlinear elasticity domain equation and apply (4.3.17) to the second term in the sum.

$$\begin{aligned}
\int_{\Omega_e} \langle \rho_e v_e, (D\phi)(\widehat{\varphi}' \circ \varphi) \rangle &= \int_{\Omega_e} \langle -\mathrm{Div}\, \sigma_e(\varphi), (D\phi)(\widehat{\varphi}' \circ \varphi) \rangle \,, \\
&= \int_{\Omega_e} \langle \mathrm{Div}\, (\mathrm{Div}\, \sigma_e(\varphi) \otimes (\widehat{\varphi}' \circ \varphi)), \phi \rangle \\
&\quad - \int_{\Gamma_i} \langle \langle (\widehat{\varphi}' \circ \varphi), n \rangle \mathrm{Div}\, \sigma_e(\varphi), \phi \rangle \,.
\end{aligned}$$

All together we obtain,

$$\begin{aligned}
\frac{\partial}{\partial s} \left(\int_{\Omega_e^s} \langle \rho_e^s v_e^s, \phi \rangle \right) \bigg|_{s=0} &= \int_{\Omega_e} \langle \rho_e v_e' + \mathrm{Div}\, (\mathrm{Div}\, \sigma_e(\varphi) \otimes (\widehat{\varphi}' \circ \varphi)), \phi \rangle \\
&\quad - \int_{\Gamma_i} \langle \langle (\widehat{\varphi}' \circ \varphi), n \rangle \mathrm{Div}\, \sigma_e(\varphi), \phi \rangle
\end{aligned} \tag{4.3.20}$$

Setting (4.3.19) and (4.3.20) equal to each other and allowing ϕ to have compact support on Ω_e we obtain the linearized elasticity domain equation on Ω_e:

$$-\mathrm{Div}\, \left\{ \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} \right\} = \rho_e v_e'.$$

4.3.3.4 The stress matching condition on Γ_i

By substituting the linearized elasticity domain equation and (4.3.20) in to (4.3.9) we retain the boundary terms:

$$\int_{\Gamma_i} \left\langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n - \mathrm{Div}\,_{\Gamma_i} (\sigma_e(\varphi) n \otimes (\widehat{\varphi}' \circ \varphi)), \phi \right\rangle$$

$$= \frac{\partial}{\partial s}\left(\int_{\Gamma_i^s} \langle \sigma_e(\varphi^s) n^s, \phi\rangle\right)\bigg|_{s=0}. \tag{4.3.21}$$

Consequently the next step is to compute the $s-$derivative of the RHS of (4.3.21). The first step is to substitute the stress-matching boundary condition from (4.3.4),

$$\frac{\partial}{\partial s}\left(\int_{\Gamma_i^s} \langle \sigma_e(\varphi^s) n^s, \phi\rangle\right)\bigg|_{s=0} = \frac{\partial}{\partial s}\left(\int_{\Gamma_i^s} \langle \sigma_f(p^s, u_f^s) n^s, \phi\rangle\right)\bigg|_{s=0}.$$

Then using the divergence theorem (with $n^s = -n_f^s$) and (4.2.5) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(\int_{\Gamma_i^s} \langle \sigma_f(p^s, u_f^s) n^s, \phi\rangle\right)\bigg|_{s=0} \\
&= \frac{\partial}{\partial s}\left(\int_{\Omega_f^s} -\mathrm{div}\,(\sigma_f(p^s, u_f^s)\phi)\right)\bigg|_{s=0}, \\
&= \int_{\Omega_f} -\mathrm{div}\,(\sigma_f(p', u_f')\phi) + \int_{\Gamma_i} \mathrm{div}\,(\sigma_f(p, u_f)\phi)\langle(\widehat{\varphi}' \circ \varphi), n\rangle, \\
&= \int_{\Gamma_i} \langle \sigma_f(p', u_f') n, \phi\rangle + \mathrm{div}\,(\sigma_f(p, u_f)\phi)\langle(\widehat{\varphi}' \circ \varphi), n\rangle.
\end{aligned} \tag{4.3.22}$$

We apply (4.2.8) to (4.3.22) and obtain

$$\begin{aligned}
&\int_{\Gamma_i} \langle \sigma_f(p', u_f') n, \phi\rangle + \mathrm{div}\,(\sigma_f(p, u_f)\phi)\langle(\widehat{\varphi}' \circ \varphi), n\rangle \\
&= \int_{\Gamma_i} \langle \sigma_f(p', u_f') n, \phi\rangle + \left\{\mathrm{div}_{\Gamma_i}(\sigma_f(p, u_f)\phi) + \langle D[\sigma_f(p, u_f)\phi] n, n\rangle\right\}\langle(\widehat{\varphi}' \circ \varphi), n\rangle \\
&= \int_{\Gamma_i} \langle \sigma_f(p', u_f') n + \langle(\widehat{\varphi}' \circ \varphi), n\rangle H \sigma_f(p, u_f) n - \sigma_f(p, u_f)\cdot \nabla_{\Gamma_i}\langle(\widehat{\varphi}' \circ \varphi), n\rangle, \phi\rangle \\
&+ \int_{\Gamma_i} \langle D[\sigma_f(p, u_f)\phi] n, n\rangle\langle(\widehat{\varphi}' \circ \varphi), n\rangle.
\end{aligned} \tag{4.3.23}$$

We can simplify the last term in (4.3.23) by choosing a test function such that $D\phi \cdot n = 0$ on Γ_i. In that case we have $D[\sigma_f(p, u_f)\phi] n = \frac{\partial \sigma_f}{\partial n} n$. Substituting the simplification in to

(4.3.23) we obtain the final expression for the shape derivative of the boundary integral:

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(\int_{\Gamma_i^s}\langle\sigma_f(p^s,u_f^s)n^s,\phi\rangle\right)\Bigg|_{s=0}\\
&=\int_{\Gamma_i}\langle\sigma_f(p',u_f')n+\langle(\widehat{\varphi}'\circ\varphi),n\rangle H\sigma_f(p,u_f)n,\phi\rangle\\
&+\int_{\Gamma_i}\left\langle\langle(\widehat{\varphi}'\circ\varphi),n\rangle\frac{\partial\sigma_f}{\partial n}n-\sigma_f(p,u_f)\cdot\nabla_{\Gamma_i}\langle(\widehat{\varphi}'\circ\varphi),n\rangle,\phi\right\rangle. \qquad (4.3.24)
\end{aligned}$$

Substituting (4.3.24) back in to (4.3.21) we obtain

$$\begin{aligned}
&\int_{\Gamma_i}\left\langle\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}n-\operatorname{Div}_{\Gamma_i}(\sigma_e(\varphi)n\otimes(\widehat{\varphi}'\circ\varphi)),\phi\right\rangle\\
&=\int_{\Gamma_i}\langle\sigma_f(p',u_f')n+\langle(\widehat{\varphi}'\circ\varphi),n\rangle H\sigma_f(p,u_f)n,\phi\rangle\\
&+\int_{\Gamma_i}\left\langle\langle(\widehat{\varphi}'\circ\varphi),n\rangle\frac{\partial\sigma_f}{\partial n}n-\sigma_f(p,u_f)\cdot\nabla_{\Gamma_i}\langle(\widehat{\varphi}'\circ\varphi),n\rangle,\phi\right\rangle. \qquad (4.3.25)
\end{aligned}$$

Then we recover the boundary condition on Γ_i:

$$\begin{aligned}
\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}n&=\sigma_f(p',u_f')n+\langle(\widehat{\varphi}'\circ\varphi),n\rangle\left(H\sigma_f(p,u_f)n+\frac{\partial\sigma_f}{\partial n}n\right)\\
&\quad-\sigma_f(p,u_f)\cdot\nabla_{\Gamma_i}\langle(\widehat{\varphi}'\circ\varphi),n\rangle+\operatorname{Div}_{\Gamma_i}(\sigma_e(\varphi)n\otimes(\widehat{\varphi}'\circ\varphi)). \qquad (4.3.26)
\end{aligned}$$

We can make some substitutions so that the representation of the boundary condition (4.3.26) is more consistent with [11]. We have the identity, for a suitably regular matrix A and vectors c and d,

$$\operatorname{Div}(Ac\otimes d)=(DA.d)c+A(Dc)d+(\operatorname{div}d)Ac,$$

and its tangential counterpart,

$$\operatorname{Div}_\Gamma(Ac\otimes d)=(D_\Gamma A.d)c+A(D_\Gamma c)d+(\operatorname{div}_\Gamma d)Ac,$$

So we can expand the term $\mathrm{Div}_{\Gamma_i}(\sigma_e(\varphi)n \otimes (\widehat{\varphi}' \circ \varphi))$:

$$\begin{aligned}
&\mathrm{Div}_{\Gamma_i}(\sigma_e(\varphi)n \otimes (\widehat{\varphi}' \circ \varphi)) \\
&= (D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n + \sigma_e(D_{\Gamma_i}n)(\widehat{\varphi}' \circ \varphi) + (\mathrm{div}_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n, \\
&= (D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n + \sigma_e(D^2 b_{\Omega_e})(\widehat{\varphi}' \circ \varphi) + (\mathrm{div}_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n \\
&= (D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n + \sigma_e(D^2 b_{\Omega_e})(\widehat{\varphi}' \circ \varphi)_{\Gamma_i} + (\mathrm{div}_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n,
\end{aligned}$$

since $(\widehat{\varphi}' \circ \varphi) = (\widehat{\varphi}' \circ \varphi)_{\Gamma_i} + \langle(\widehat{\varphi}' \circ \varphi), n\rangle n$ and $D^2 b_{\Omega_e} n = D^2 b_{\Omega_e} \nabla b = 0$.

Substituting the expansion in to (4.3.26) and adding/subtracting the quantity $\sigma_e(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n$ we obtain the equivalent form of the boundary condition:

$$\begin{aligned}
\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n = {} & \sigma_f(p', u_f')n + [\sigma_e(\varphi) - \sigma_f(p, u_f)] \cdot \nabla_{\Gamma_i}\langle(\widehat{\varphi}' \circ \varphi), n\rangle \\
& + \langle(\widehat{\varphi}' \circ \varphi), n\rangle \left(H\sigma_f(p, u_f)n + \frac{\partial \sigma_f}{\partial n} n \right) \\
& + (D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n + (\mathrm{div}_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n - \sigma_e(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n, \qquad (4.3.27)
\end{aligned}$$

since $\nabla_{\Gamma_i}\langle(\widehat{\varphi}' \circ \varphi), n\rangle = (D^2 b_{\Omega_e})(\widehat{\varphi}' \circ \varphi)_{\Gamma_i} + (D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n$

In order to write the boundary condition concisely we define,

$$\begin{aligned}
\mathcal{B}(\widehat{\varphi}' \circ \varphi) := & [\sigma_e(\varphi) - \sigma_f(p, u_f)] \cdot \nabla_{\Gamma_i}\langle \widehat{\varphi}' \circ \varphi, n\rangle + (D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n \\
& + (\mathrm{div}_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n - \sigma_e(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n \\
& + \langle \widehat{\varphi}' \circ \varphi, n\rangle \left(H\sigma_f(p, u_f)n + \frac{\partial \sigma_f}{\partial n} n \right), \qquad (4.3.28)
\end{aligned}$$

in which case we can write

$$\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n = \sigma_f(p', u_f')n + \mathcal{B}(\widehat{\varphi}' \circ \varphi)$$

on Γ_i.

This completes the proof of Theorem 4.3.3.

4.4 Dynamical Case

In the linearization of time dependent case we will work with the following alternative formulation of (2.5.2):

$$\begin{cases} \dfrac{\partial u_f}{\partial t} + (Du_f)u_f - \operatorname{Div}\sigma_f(p,u_f) = v_f & \Omega_f(t) \\ \operatorname{div} u_f = 0 & \Omega_f(t) \\ \rho_e\left(\dfrac{\partial^2\widehat{\varphi}}{\partial t^2}\right)\circ\varphi - \operatorname{Div}\sigma_e(\varphi) = \rho_e v_e & \Omega_e(t) \\ u_f = u_e & \Gamma_i(t) \\ \sigma_f(p,u_f)n = \sigma_e(\varphi)n & \Gamma_i(t) \\ u_f = 0 & \Gamma_f \\ \widehat{\varphi}(\cdot,0) = \widehat{\varphi}^0, \widehat{\varphi}_t(\cdot,0) = \widehat{\varphi}^1, u_f(\cdot,0) = u^0, p(\cdot,0) = p^0 & (\widehat{\Omega}_e)^2\times(\Omega_f)^2 \end{cases} \tag{4.4.1}$$

Recall that $\rho_e = \left(\dfrac{\widehat{\rho}_0}{\widehat{J}}\right)\circ\varphi$, where $\widehat{\rho}_0$ is the reference elastic density, and $\sigma_e(\varphi) = \left(\dfrac{\widehat{\mathcal{P}}(D\widehat{\varphi})^*}{\widehat{J}_e}\right)\circ\varphi$ is the transformation of Piola tensor to Cauchy tensor.

The specific difference is that the time derivative in the elasticity equation is not transported to the current domain, but is left in terms of the deformation on the reference domain. As a result (4.4.1) can be considered an Arbitrary Lagrangian-Eulerian (ALE) system, although we take the ALE map to be exactly the deformation φ as opposed to introducing an auxiliary map. This ALE formulation is possibly less elegant than the Eulerian-Eulerian formulation of (2.5.2), obtained by transporting all terms on to the moving domain, but it is more convenient to compute the linearized elasticity domain equation in this framework.

4.4.1 Strategy

The strategy to linearize the time-dependent case, (4.4.1), is similar to that used for the steady-state case, with adaptations made for the dependence of both the variables and the domains on time. We model the approach after the techniques used in [14], in which the authors consider the same coupling of nonlinear elasticity and Navier-Stokes, but with control only on the fluid domain, and homogenous elastodynamics equations.

We take a small parameter of variation $s \in [0, s_0]$ and perturb the system through the body force on the fluid domain v^f. We assume linear dependence on s; that is

$$v_f^s = v_f + sv'$$

for given v_f, v' of suitable regularity for well-posedness purposes, and such that $\operatorname{div} v' = 0$ and $v'_{\Gamma_f} = 0$. Consequently, all of the unknowns in (4.4.1) become functions of s as well as the geometric domains. The result is a moving boundary that moves with the parameter s. More precisely, for each t the perturbation $\Gamma_i^s(t)$ of $\Gamma_i(t)$ is built by the family of transformations

$$T_s(t) := \widehat{\varphi}^s(t) \circ \varphi, \tag{4.4.2}$$

associated with the family of speeds

$$V(s) = V(s,t,x) := \frac{\partial T^s}{\partial s} \circ (T^s)^{-1} = \frac{\partial \widehat{\varphi}^s}{\partial s} \circ (\widehat{\varphi}^s)^{-1} = \frac{\partial \widehat{\varphi}^s}{\partial s} \circ \varphi^s. \tag{4.4.3}$$

For example, $T_e^s = T^s|_{\Omega_e^s}$ and

$$T_e^s(t) : \Omega_e(t) \xrightarrow{\varphi} \widehat{\Omega}_e \xrightarrow{\widehat{\varphi}^s} \Omega_e^s(t).$$

For each t we have $\Gamma_i^s(t) = T^s(\Gamma_i(t))$.

Then we consider the perturbed functional $J(v + sv')$, where J is defined in (3.2.1), and calculate the derivative at $s = 0$ of the function $s \to J(v + sv')$.

$$\begin{aligned}
\partial J(v; v') &= \lim_{s\to 0} \frac{J(v+sv') - J(v)}{s} = \frac{\partial}{\partial s} J(v+sv')\bigg|_{s=0} \\
&= \frac{\partial}{\partial s}\left[\frac{1}{2}\int_0^T \int_{\Omega_f^s(t)} |\operatorname{curl}(u_f^s)|^2 + \frac{1}{2}\|v+sv'\|^2_{\mathcal{E}(0,T;\Omega)}\right]\Bigg|_{s=0} \\
&= \frac{1}{2}\int_0^T \int_{\Omega_f(t)} \frac{\partial}{\partial s} |\operatorname{curl}(u_f^s)|^2\bigg|_{s=0}
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \int_{\Gamma_i(t)} \frac{1}{2} |\operatorname{curl}(u_f)|^2 \langle V(0), n_f \rangle + (v, v')_{\mathcal{E}(0,T;\Omega)}, \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle \operatorname{curl}(u_f), \operatorname{curl}\left(\left.\frac{\partial u_f^s}{\partial s}\right|_{s=0} \right) \right\rangle \\
&- \int_0^T \int_{\Gamma_i(t)} \frac{1}{2} |\operatorname{curl}(u_f)|^2 \langle V(0), n \rangle + (v, v')_{\mathcal{E}(0,T;\Omega)}, \qquad (4.4.4)
\end{aligned}
$$

which requires the use of the formula for shape derivatives of domain integrals (4.2.5).

As in the steady state case, the calculation of the directional derivative of the cost functional introduces the new variables $\left.\frac{\partial u_f}{\partial s}\right|_{s=0}$ and $V(0) := \left.\frac{\partial \widehat{\varphi}^s}{\partial s}\right|_{s=0} \circ \varphi$. As in the previous section, the linearization of (4.4.1) entails deriving the system of equations satisfied by these variables.

4.4.2 Perturbed PDE System

The result of the perturbation by $T_s(t)$ as described in (4.4.2) is that all of the unknowns (4.4.1) become functions of s, $(u_s, p_s, \widehat{\varphi}_s)$, as do the domains $(\Omega_f^s(t), \Omega_e^s(t), \Gamma_i^s(t))$. The resulting perturbed PDE system is:

$$
\begin{cases}
\frac{\partial u_f^s}{\partial t} + (Du_f^s)u_f^s - \operatorname{Div} \sigma_f(p^s, u_f^s) = v_f^s & \Omega_f^s(t) \\
\operatorname{div} u_f^s = 0 & \Omega_f^s(t) \\
\rho_e^s \left(\frac{\partial^2 \widehat{\varphi}^s}{\partial t^2} \right) \circ \varphi^s - \operatorname{Div} \sigma_e(\varphi^s) = \rho_e^s v_e^s & \Omega_e^s(t) \\
u_f^s = u_e^s & \Gamma_i^s(t) \\
\sigma_f^s(p^s, u_f^s) n^s = \sigma_e(\varphi^s) n^s & \Gamma_i^s(t) \\
u_f^s = 0 & \Gamma_f \\
\widehat{\varphi}^s(\cdot, 0) = \widehat{\varphi}^0, \widehat{\varphi}_t^s(\cdot, 0) = \widehat{\varphi}^1, u_f^s(\cdot, 0) = u^0, p^s(\cdot, 0) = p^0 & (\widehat{\Omega}_e)^2 \times (\Omega_f)^2
\end{cases}
\qquad (4.4.5)
$$

In the s-system n^s is the unit outer normal vector along $\Gamma_i^s(t)$ with respect to $\Omega_e^s(t)$.

As in the previous section, strategy is to differentiate (4.4.5) with respect to s at $s = 0$

and obtain a linearized system for the s-derivatives of $(u^s, p^s, \widehat{\varphi}^s)$, in order to compute a *total* linearization of (4.4.5) around an arbitrary regime.

We again employ the prime notation to denote the linearized variables.

$$u' := \frac{\partial}{\partial s} u^s \bigg|_{s=0}, p' := \frac{\partial}{\partial s} p^s \bigg|_{s=0}, \widehat{\varphi}' := \frac{\partial}{\partial s} \widehat{\varphi}^s \bigg|_{s=0}.$$

When computing the shape derivatives, the vector field associated with the flow $T_s(t)$ will appear only at $s = 0$. Thus we use the notation $V(0) := V(0, t, x) = \widehat{\varphi}' \circ \varphi$.

We also associate the following differentiability assumptions with the (4.4.5):

Assumption 4.4.1. *For all $s \in [0, s_0)$ the following weak derivatives with respect to s exist:*

1. $\forall \widehat{\phi} \in L^2(0, T, H^{-1}(\Omega)) : \left(s \to \int_0^T \int_\Omega \langle \widehat{\varphi}_s, \widehat{\phi} \rangle \right) \in C^1([0, s_0))$
2. $\forall \phi \in L^2(0, T, H^{-1}(\Omega \setminus \overline{\Omega_e})) : \left(s \to \int_0^T \int_{\Omega \setminus \overline{\Omega_e(t)}} \langle u_f^s \circ T_s, \phi \rangle \right) \in C^1([0, s_0))$
3. $\forall \pi \in L^2(0, T, L^2(\Omega \setminus \overline{\Omega_e})) : \left(s \to \int_0^T \int_{\Omega \setminus \overline{\Omega_e(t)}} \langle p_s \circ T_s, \pi \rangle \right) \in C^1([0, s_0))$

4.5 Linearized dynamical case model

Theorem 4.5.1 (Linearization of time-dependent coupling of Navier-Stokes and Non-linear Elasticity)**.** *For any $s \in [0, s_0]$, assume that the initial condition $(\widehat{\varphi}^0, \widehat{\varphi}^1, u_0, p^0)$ associated with* (4.4.5) *solves* (4.4.1)*. Further, let $(u^s, p^s, \widehat{\varphi}^s)$ be a smooth solution of* (4.4.5) *on $[0, T]$ satisfying Assumption 4.4.1. Finally let $v^s(x, t) \in \mathcal{E}(0, T; \Omega)$. Then*

$u' = \frac{\partial}{\partial s} u^s \Big|_{s=0}, p' = \frac{\partial}{\partial s} p^s \Big|_{s=0}, \widehat{\varphi}' := \frac{\partial}{\partial s} \widehat{\varphi}^s \Big|_{s=0}$ *satisfy the following linear system:*

$$\begin{cases} \dfrac{\partial u_f'}{\partial t} + (Du_f')u_f + (Du_f)u_f' - Div\,\sigma_f(p', u_f') = v_f' & \Omega_f(t) \\ div\, u_f' = 0 & \Omega_f(t) \\ \rho_e \left(\dfrac{\partial^2 \widehat{\varphi}'}{\partial t^2}\right) \circ \varphi - Div \left\{ \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} \right\} = \rho_e v_e' & \Omega_e(t) \\ u_f' = u_e' + (Du_e - Du_f)(\widehat{\varphi}' \circ \varphi) & \Gamma_i(t) \\ \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n = \sigma_f(p', u_f')n + \mathcal{B}(\widehat{\varphi}' \circ \varphi) & \Gamma_i(t) \\ u_f' = 0 & \Gamma_f \\ \widehat{\varphi}'(0) = 0; \dfrac{\partial \widehat{\varphi}'}{\partial t}(0) = 0; u_f'(0) = 0 & (\widehat{\Omega}_e)^2 \times (\Omega_f)^2 \end{cases} \tag{4.5.1}$$

where

$$\begin{aligned} \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} :=& D(\widehat{\varphi}' \circ \varphi)\sigma_e(\varphi) + J(\varphi)\overline{\Sigma}(\widetilde{\varepsilon}(\widehat{\varphi}' \circ \varphi, \varphi)) \text{ for} \\ \overline{\Sigma}(\Phi, \phi) :=& (D\phi)^{-1} \Sigma\,(\Phi)\,(D\phi)^{-*}, \\ \widetilde{\varepsilon}(\psi, \phi) :=& (D\phi)^{-*} \varepsilon(\psi)(D\phi)^{-1}, \text{ and} \end{aligned}$$

$$\begin{aligned} \mathcal{B}(\xi) := (\sigma_e(\varphi) - \sigma_f(p, u_f)) \cdot \nabla_{\Gamma_i(t)} \langle \xi, n \rangle + \langle \xi, n \rangle \left(H\sigma_f(p, u_f)n + \frac{\partial \sigma_f}{\partial n} n \right) \\ + (D_{\Gamma_i}\sigma_e.\xi)n + (div_{\Gamma_i}\xi)\sigma_e n - \sigma_e(\varphi)(D_{\Gamma_i}\xi)^* n, \end{aligned}$$

as in Theorem 4.3.3, and $(\widehat{\varphi}, u, p)$ *is a smooth solution of* (4.4.1).

As in the steady state case of the previous section, the boundary conditions are quite complicated, and contain terms referring to the acceleration and curvature of the common interface $\Gamma_i(t)$.

4.5.1 Proof of Linearization

In order to prove Theorem 4.5.1 we employ a similar strategy as in the steady state case and apply results obtained in that case when possible. The relationship between the shape and material derivatives, as stated in (4.3.7), remains the case in the time-dependent setting.

4.5.1.1 Linearized Sticking Condition on $\Gamma_i(t)$

In the s-perturbed model we have the velocity matching condition $u_f^s = u_e^s$ on $\Gamma_i^s(t)$. We take a test function $\phi = \phi(x,t) \in C^1(0,T,\Omega)$ and obtain the weak formulation of the boundary condition

$$0 = \int_0^T \int_{\Gamma_i^s(t)} \langle u_f^s - u_e^s, \phi \rangle. \tag{4.5.2}$$

We use (4.2.7) to obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial s}\left(\int_0^T \int_{\Gamma_i^s(t)} \langle u_f^s - u_e^s, \phi \rangle \right)\Bigg|_{s=0}, \\ &= \int_0^T \int_{\Gamma_i(t)} \langle u_f' - u_e' + (Du_f - Du_e)\cdot V(0) + (u_f - u_e)(\operatorname{div} V(0) - DV(0)n\cdot n), \phi \rangle. \end{aligned}$$

Substituting the velocity-matching boundary condition from (4.4.1) we obtain:

$$0 = \int_0^T \int_{\Gamma_i(t)} \langle u_f' - u_e' + (Du_f - Du_e)\cdot (\widehat{\varphi}' \circ \varphi), \phi \rangle$$

Owing to the arbitrariness of the test function ϕ we obtain the linearized boundary condition on $\Gamma_i(t)$

$$u_f' = u_e' + (Du_e - Du_f)(\widehat{\varphi}' \circ \varphi). \tag{4.5.3}$$

Remark 4.5.2. *Observe that due to* (4.3.7) *boundary condition* (4.5.3) *can equivalently be written*

$$u_f' + (Du_f)(\widehat{\varphi}' \circ \varphi) = \left(\frac{\partial \widehat{\varphi}'}{\partial t} \right) \circ \varphi. \tag{4.5.4}$$

This formulation proves more convenient when recovering the adjoint sensitivity information in the following section, while the formulation (4.5.3) *better highlights the role of the variable of interest,* $\widehat{\varphi}' \circ \varphi$.

4.5.1.2 Fluid shape derivative on $\Omega^f(t)$

The s-perturbed Navier-Stokes equation on $\Omega_f^s(t)$ is

$$\frac{\partial u_f^s}{\partial t} + (Du_f^s)u_f^s - \operatorname{Div}\sigma_f(p^s, u_f^s) = v_f^s. \tag{4.5.5}$$

We take inner product with a smooth test function $\phi = \phi(x,t) \in C_0^1(0,T;\Omega_f)$ and integrate in time and space to obtain a variational formulation:

$$\int_0^T \int_{\Omega_f^s(t)} \left\langle \frac{\partial u_f^s}{\partial t} + (Du_f^s)u_f^s - \operatorname{Div}\sigma_f(p^s, u_f^s), \phi \right\rangle = \int_0^T \int_{\Omega_f^s(t)} \langle v_f^s, \phi \rangle. \tag{4.5.6}$$

By applying formula (4.2.5) we obtain for the left hand side of (4.5.6)

$$\frac{\partial}{\partial s}\left(\int_0^T \int_{\Omega_f^s(t)} \left\langle \frac{\partial u_f^s}{\partial t} + (Du_f^s)u_f^s - \operatorname{Div}\sigma_f(p^s, u_f^s), \phi \right\rangle \right)\Bigg|_{s=0}$$
$$= \int_0^T \int_{\Omega_f(t)} \left\langle \frac{\partial u_f'}{\partial t} + (Du_f')u_f + (Du_f)u_f' - \operatorname{Div}\sigma_f(p', u_f'), \phi \right\rangle + \text{boundary terms}.$$

We are unconcerned with the boundary terms since we already recovered the linearized velocity matching boundary condition on $\Gamma_i(t)$. We likewise apply (4.2.5) to the right hand side of (4.5.6) and recover the linearized Navier-Stokes equation on $\Omega_f(t)$,

$$\frac{\partial u_f'}{\partial t} + (Du_f')u_f + (Du_f)u_f' - \operatorname{Div}\sigma_f(p', u_f') = v_f'. \tag{4.5.7}$$

4.5.1.3 Elasticity shape derivative on $\Omega^e(t)$

The s-perturbed elastodynamics equations on $\Omega_e^s(t)$ are

$$\rho_e^s \left(\frac{\partial^2 \widehat{\varphi}^s}{\partial t^2} \right) \circ \varphi^s - \operatorname{Div}\sigma_e(\varphi^s) = \rho_e^s v_e^s, \tag{4.5.8}$$

which for a smooth test function $\phi = \phi(x,t) \in C^1(0,T,\Omega)$ admits the variational formulation

$$\int_0^T \int_{\Omega_e^s(t)} \left\langle \rho_e^s \left(\frac{\partial^2 \widehat{\varphi}^s}{\partial t^2}\right) \circ \varphi^s, \phi \right\rangle + \sigma_e(\varphi^s)..D\phi$$
$$= \int_0^T \int_{\Omega_e^s(t)} \langle \rho_e^s v_e^s, \phi \rangle + \int_0^T \int_{\Gamma_i^s(t)} \langle \sigma_e(\varphi^s) n^s, \phi \rangle. \tag{4.5.9}$$

We will first take the s-derivative of the left hand side. From (4.3.19) we have

$$\frac{\partial}{\partial s} \left(\int_0^T \int_{\Omega_e^s(t)} \sigma_e(\varphi^s)..D\phi \right)\Bigg|_{s=0}$$
$$= \int_0^T \int_{\Omega_e(t)} \left\langle -\mathrm{Div} \left\{ \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} - \mathrm{Div}\, \sigma_e(\varphi) \otimes (\widehat{\varphi}' \circ \varphi) \right\}, \phi \right\rangle$$
$$+ \int_0^T \int_{\Gamma_i(t)} \left\langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n - \langle \widehat{\varphi}' \circ \varphi, n \rangle \mathrm{Div}\, \sigma_e(\varphi), \phi \right\rangle$$
$$- \int_0^T \int_{\Gamma_i(t)} \langle \mathrm{Div}_{\Gamma_i} (\sigma_e(\varphi) n \otimes (\widehat{\varphi}' \circ \varphi)), \phi \rangle \tag{4.5.10}$$

matching the notation in (4.3.12), (4.3.13), and (4.3.14).

It remains to compute

$$\frac{\partial}{\partial s} \left(\int_0^T \int_{\Omega_e^s(t)} \left\langle \rho_e^s \left(\frac{\partial^2 \widehat{\varphi}^s}{\partial t^2}\right) \circ \varphi^s, \phi \right\rangle \right)\Bigg|_{s=0}$$

(which of course is the term not present in the steady state problem). We will compute the derivative in the same fashion as in the steady state case, by transporting the integral to the fixed domain, taking the s-derivative, and then moving back to the current configuration.

$$\frac{\partial}{\partial s} \left(\int_0^T \int_{\Omega_e^s(t)} \left\langle \rho_e^s \left(\frac{\partial^2 \widehat{\varphi}^s}{\partial t^2}\right) \circ \varphi^s, \phi \right\rangle \right)\Bigg|_{s=0}$$
$$= \frac{\partial}{\partial s} \left(\int_0^T \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial^2 \widehat{\varphi}^s}{\partial t^2}, \widehat{\phi}^s \right\rangle \right)\Bigg|_{s=0},$$
$$= \int_0^T \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial^2 \widehat{\varphi}'}{\partial t^2}, \widehat{\phi} \right\rangle + \left\langle \widehat{\rho}_0 \frac{\partial^2 \widehat{\varphi}}{\partial t^2}, (D\phi \circ \widehat{\varphi}) \widehat{\varphi}' \right\rangle,$$

$$= \int_0^T \int_{\Omega_e(t)} \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}'}{\partial t^2} \right) \circ \varphi, \phi \right\rangle + \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2} \right) \circ \varphi, (D\phi)(\widehat{\varphi}' \circ \varphi) \right\rangle,$$

$$= \int_0^T \int_{\Omega_e(t)} \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}'}{\partial t^2} \right) \circ \varphi, \phi \right\rangle + \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2} \right) \circ \varphi, (D\phi)(\widehat{\varphi}' \circ \varphi) \right\rangle.$$

Next we apply (4.3.17) to the last line and obtain

$$\begin{aligned}
&\frac{\partial}{\partial s} \left(\int_0^T \int_{\Omega_e^s(t)} \left\langle \rho_e^s \left(\frac{\partial^2 \widehat{\varphi}^s}{\partial t^2} \right) \circ \varphi^s, \phi \right\rangle \right) \Bigg|_{s=0} \\
&= \int_0^T \int_{\Omega_e(t)} \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}'}{\partial t^2} \right) \circ \varphi - \operatorname{Div} \left\{ \rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2} \right) \circ \varphi \otimes (\widehat{\varphi}' \circ \varphi) \right\}, \phi \right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle \langle \widehat{\varphi}' \circ \varphi, n \rangle \rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2} \right) \circ \varphi, \phi \right\rangle
\end{aligned} \tag{4.5.11}$$

Taking (4.3.19) and (4.5.11) together we have an expression for the s-derivative of the left hand side of (4.5.9):

$$\begin{aligned}
&\frac{\partial}{\partial s} \left(\int_0^T \int_{\Omega_e^s(t)} \left\langle \rho_e^s \left(\frac{\partial^2 \widehat{\varphi}^s}{\partial t^2} \right) \circ \varphi^s, \phi \right\rangle + \sigma_e(\varphi^s)..D\phi \right) \Bigg|_{s=0} \\
&= \int_0^T \int_{\Omega_e(t)} \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}'}{\partial t^2} \right) \circ \varphi - \operatorname{Div} \left\{ \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} \right\}, \phi \right\rangle \\
&- \int_0^T \int_{\Omega_e(t)} \left\langle \operatorname{Div} \left\{ \left[\rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2} \right) \circ \varphi - \operatorname{Div} \sigma_e(\varphi) \right] \otimes (\widehat{\varphi}' \circ \varphi) \right\}, \phi \right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n - \operatorname{Div}_{\Gamma_i} (\sigma_e(\varphi) n \otimes (\widehat{\varphi}' \circ \varphi)), \phi \right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle \langle \widehat{\varphi}' \circ \varphi, n \rangle \left(\rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2} \right) \circ \varphi - \operatorname{Div} \sigma_e(\varphi) \right), \phi \right\rangle.
\end{aligned} \tag{4.5.12}$$

Some of the particularly complicated terms will cancel in the next step.

Next we will work with the domain term on the right hand side of (4.5.9). As in the steady state case, the density ρ_e implicitly depends on $\widehat{\varphi}$ so we will transport the integral back to the fixed (wrt s) elastic domain in order to compute the derivative. As a consequence of (4.3.20) we have,

$$\frac{\partial}{\partial s}\left(\int_0^T \int_{\Omega_e^s(t)} \langle \rho_e^s v_e^s, \phi\rangle\right)\Bigg|_{s=0} = \int_0^T \int_{\Omega_e(t)} \langle \rho_e v_e', \phi\rangle + \langle \rho_e v_e, (D\phi)(\widehat{\varphi}' \circ \varphi)\rangle,$$

and by substituting the elasticity domain equation from (4.4.1) we obtain,

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(\int_0^T \int_{\Omega_e^s(t)} \langle \rho_e^s v_e^s, \phi\rangle\right)\Bigg|_{s=0} \\
&= \int_0^T \int_{\Omega_e(t)} \langle \rho_e v_e', \phi\rangle \\
&\quad + \int_0^T \int_{\Omega_e(t)} \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2}\right) \circ \varphi - \operatorname{Div} \sigma_e(\varphi), (D\phi)(\widehat{\varphi}' \circ \varphi)\right\rangle.
\end{aligned}$$

An application of (4.3.17) yields

$$\begin{aligned}
&\frac{\partial}{\partial s}\left(\int_0^T \int_{\Omega_e^s(t)} \langle \rho_e^s v_e^s, \phi\rangle\right)\Bigg|_{s=0} \\
&= \int_0^T \int_{\Omega_e(t)} \langle \rho_e v_e', \phi\rangle\,, \\
&\quad - \int_0^T \int_{\Omega_e(t)} \left\langle -\operatorname{Div}\left\{\left[\rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2}\right) \circ \varphi - \operatorname{Div} \sigma_e(\varphi)\right] \otimes (\widehat{\varphi}' \circ \varphi)\right\}, \phi\right\rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \left\langle \langle \widehat{\varphi}' \circ \varphi, n\rangle \left(\rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2}\right) \circ \varphi - \operatorname{Div} \sigma_e(\varphi)\right), \phi\right\rangle. \qquad (4.5.13)
\end{aligned}$$

We take (4.5.12) and (4.5.13) together and substitute back in to (4.5.9) to obtain the following:

$$\begin{aligned}
&\int_0^T \int_{\Omega_e(t)} \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}'}{\partial t^2}\right) \circ \varphi - \operatorname{Div}\left\{\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}\right\}, \phi\right\rangle \\
&- \int_0^T \int_{\Omega_e(t)} \left\langle \operatorname{Div}\left\{\left[\rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2}\right) \circ \varphi - \operatorname{Div} \sigma_e(\varphi)\right] \otimes (\widehat{\varphi}' \circ \varphi)\right\}, \phi\right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n - \operatorname{Div}_{\Gamma_i}(\sigma_e(\varphi) n \otimes (\widehat{\varphi}' \circ \varphi)), \phi\right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle \langle \widehat{\varphi}' \circ \varphi, n\rangle \left(\rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2}\right) \circ \varphi - \operatorname{Div} \sigma_e(\varphi)\right), \phi\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_{\Omega_e(t)} \langle \rho_e v_e', \phi \rangle \\
&\quad - \int_0^T \int_{\Omega_e(t)} \left\langle \text{Div} \left\{ \left[\rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2} \right) \circ \varphi - \text{Div}\, \sigma_e(\varphi) \right] \otimes (\widehat{\varphi}' \circ \varphi) \right\}, \phi \right\rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \left\langle \langle \widehat{\varphi}' \circ \varphi, n \rangle \left(\rho_e \left(\frac{\partial^2 \widehat{\varphi}}{\partial t^2} \right) \circ \varphi - \text{Div}\, \sigma_e(\varphi) \right), \phi \right\rangle \\
&\quad + \frac{\partial}{\partial s} \left(\int_0^T \int_{\Gamma_i^s(t)} \langle \sigma_e(\varphi^s) n^s, \phi \rangle \right) \Bigg|_{s=0}.
\end{aligned}
$$

After cancellation we have,

$$
\begin{aligned}
&\int_0^T \int_{\Omega_e(t)} \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}'}{\partial t^2} \right) \circ \varphi - \text{Div} \left\{ \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} \right\}, \phi \right\rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \left\langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n - \text{Div}_{\Gamma_i} (\sigma_e(\varphi) n \otimes (\widehat{\varphi}' \circ \varphi)), \phi \right\rangle \\
&= \int_0^T \int_{\Omega_e(t)} \langle \rho_e v_e', \phi \rangle + \frac{\partial}{\partial s} \left(\int_0^T \int_{\Gamma_i^s(t)} \langle \sigma_e(\varphi^s) n^s, \phi \rangle \right) \Bigg|_{s=0}.
\end{aligned}
\tag{4.5.14}
$$

Letting ϕ have compact support we recover the linearized domain equation on $\Omega_e(t)$:

$$
\rho_e \left(\frac{\partial^2 \widehat{\varphi}'}{\partial t^2} \right) \circ \varphi - \text{Div} \left\{ \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} \right\} = \rho_e v_e' \tag{4.5.15}
$$

4.5.1.4 Linearized Stress Matching Condition on $\Gamma_i(t)$

Substituting (4.5.15) in to (4.5.14), we are left with the following equation on the interface $\Gamma_i(t)$:

$$
\begin{aligned}
&\int_0^T \int_{\Gamma_i(t)} \left\langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n - \text{Div}_{\Gamma_i} (\sigma_e(\varphi) n \otimes (\widehat{\varphi}' \circ \varphi)), \phi \right\rangle \\
&\qquad = \frac{\partial}{\partial s} \left(\int_0^T \int_{\Gamma_i^s(t)} \langle \sigma_e(\varphi^s) n^s, \phi \rangle \right) \Bigg|_{s=0}.
\end{aligned}
\tag{4.5.16}
$$

This is precisely the scenario found in the steady state case, (4.3.21), with the exception of the integral in time and that the domains depend on t. The same analysis carries through, however, and we have the boundary condition for the dynamical case:

$$\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n = \sigma_f(p', u_f')n + \mathcal{B}(\widehat{\varphi}' \circ \varphi), \tag{4.5.17}$$

where $\mathcal{B}$ is defined as in (4.3.6).

This completes the proof of Theorem 4.5.1.

Chapter 5

Linear Adjoint

5.1 Introduction

In the previous chapter we found the sensitivity equations for the steady state case, (4.3.5), and the time-dependent case, (4.5.1). These sensitivity equations provide the needed characterization for the s-derivatives $((u_f', p'), \widehat{\varphi}' \circ \varphi)$ that appear in the formulas (4.3.3) of the directional derivative of the cost functional $\partial J_S(v; v')$ and (4.4.4) for $\partial J(v; v')$. However, in both cases the directional derivative expression does not actually represent the gradient of the cost functional, since it is not explicitly linear in v'. In fact, in both cases v' does not even appear in the chain rule computation, since it is hidden in the sensitivity equations, (4.3.5) and (4.5.1), for the s-derivatives u_f', p', and $\widehat{\varphi}' \circ \varphi$.

This dependence can be fleshed out by the introduction of a suitable adjoint problem that eliminates the s-derivatives and provides an explicit representation of the gradient of the cost functional. In this chapter we will derive the linear adjoint equations for the steady state case, in which the gradient of J_S at v is denoted by $J_S'(v; v')$, and for the time-dependent case, in which the gradient of J at v is denoted by $J'(v; v')$. With the development of the appropriate linear adjoint systems we can also (formally) write the first-order necessary optimality conditions for the optimal control problems (3.1.13) and (3.2.2).

5.2 Steady State Case

5.2.1 Goal and strategy

The goal is to emulate the proof of Lemma 1.2.2 in [1] in order to make precise the dependence of the linearized variables $((u'_f, p'), \widehat{\varphi}' \circ \varphi)$ on the control v'. The strategy is to derive a bilinear form based on a variational form of (4.3.5), and to use the bilinear form to extract the adjoint equations.

5.2.2 Variational Form

The first step is to write (4.3.5) in a variational form. Using test functions Q, P, and R, we obtain a weak formulation for (4.3.5).

$$\begin{aligned} &\int_{\Omega_f} \langle (Du'_f)u_f + (Du_f)u'_f - \operatorname{Div} \sigma_f(u'_f, p'), Q\rangle - \int_{\Omega_f} P \operatorname{div} u'_f \\ &- \int_{\Omega_e} \langle \operatorname{Div} \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}, R\rangle = \int_{\Omega_f} \langle v'|_{\Omega_f}, Q\rangle + \int_{\Omega_e} \langle v'|_{\Omega_e}, R\rangle \end{aligned} \tag{5.2.1}$$

We integrate by parts in the first term of the lefthand side of (5.2.1).

By (4.3.17) we have

$$\int_{\Omega_f} \langle (Du'_f)u_f, Q\rangle = -\int_{\Omega_f} \langle (DQ)u_f + (\operatorname{div} u_f)Q, u'_f\rangle + \int_{\partial\Omega_f} \langle \langle u_f, n_{\partial\Omega_f}\rangle Q, u'_f\rangle,$$

and using the facts that $u_f = 0$ on $\partial\Omega_f$ and $\operatorname{div} u'_f = 0$ on Ω_f we obtain

$$\int_{\Omega_f} \langle (Du'_f)u_f, Q\rangle = -\int_{\Omega_f} \langle (DQ)u_f, u'_f\rangle \tag{5.2.2}$$

In order to integrate the fluid stress tensor we require an adaptation of Green's first identity for a suitably regular matrix-valued function B and vector-valued function a

(which will be useful elsewhere as well):

$$\int_\Omega B..Da + \langle \operatorname{Div} B, a\rangle = \int_\Gamma \langle Bn, a\rangle \tag{5.2.3}$$

An application of (5.2.3), as well as the definition of matrix inner product, defined in Section 1.1, and the property of the trace that it is invariant under cyclic permutations, yields

$$\begin{aligned}
\int_{\Omega_f} \langle -\operatorname{Div}\sigma_f(u_f', p'), Q\rangle &= \int_{\Omega_f} \sigma_f(u_f', p')..DQ - \int_{\partial\Omega_f} \langle \sigma_f(u_f', p') n_{\partial\Omega_f}, Q\rangle, \\
&= \int_{\Omega_f} 2\nu\varepsilon(u_f')..DQ - p'\operatorname{div}(Q) \\
&\quad - \int_{\partial\Omega_f} \langle \sigma_f(u_f', p') n_{\partial\Omega_f}, Q\rangle, \\
&= \int_{\Omega_f} Du_f'..2\nu\varepsilon(Q) - p'\operatorname{div}(Q) \\
&\quad - \int_{\partial\Omega_f} \langle \sigma_f(u_f', p') n_{\partial\Omega_f}, Q\rangle, \\
&= \int_{\Omega_f} \langle u_f', -\operatorname{Div}(2\nu\varepsilon(Q)) - p'\operatorname{div}(Q) \\
&\quad + \int_{\partial\Omega_f} \langle u_f', 2\nu\varepsilon(Q) n_{\partial\Omega_f}\rangle - \langle \sigma_f(u_f', p') n_{\partial\Omega_f}, Q\rangle, \\
&= \int_{\Omega_f} \langle u_f', -\operatorname{Div}(2\nu\varepsilon(Q)) - p'\operatorname{div}(Q) \\
&\quad + \int_{\Gamma_i} \langle \sigma_f(u_f', p') n, Q\rangle - \langle u_f', 2\nu\varepsilon(Q) n\rangle \\
&\quad - \int_{\Gamma_f} \langle \sigma_f(u_f', p') n_{\Gamma_f}, Q\rangle.
\end{aligned}$$

where in the last step we used the fact that $u_f' = 0$ on Γ_f and that $n = -n_f$ on Γ_i. Together we have

$$\begin{aligned}
&\int_{\Omega_f} \langle (Du_f')u_f + (Du_f)u_f' - \operatorname{Div}\sigma_f(u_f', p'), Q\rangle = \\
&= \int_{\Omega_f} \langle u_f', (Du_f)^* Q - (DQ)u_f - \operatorname{Div}(2\nu\varepsilon(Q))\rangle - p'(\operatorname{div} Q)
\end{aligned}$$

$$+\int_{\Gamma_i} \langle \sigma_f(u'_f, p')n, Q\rangle - \langle u'_f, 2\nu\varepsilon(Q)n\rangle$$
$$-\int_{\Gamma_f} \langle \sigma_f(u'_f, p')n_{\Gamma_f}, Q\rangle. \tag{5.2.4}$$

We integrate by parts in the second term of the lefthand side of (5.2.1) and use the fact that $u'_f = 0$ on Γ_f, We obtain

$$\begin{aligned}
-\int_{\Omega_f} P \operatorname{div} u'_f &= -\int_{\partial\Omega_f} P\langle u'_f, n_{\partial\Omega_f}\rangle + \int_{\Omega_f} \langle u'_f, \nabla P\rangle \\
&= \int_{\Gamma_i} P\langle u'_f, n\rangle + \int_{\Omega_f} \langle u'_f, \nabla P\rangle.
\end{aligned} \tag{5.2.5}$$

Next, the following lemma for integration by parts on the linearized elasticity stress tensor will be useful:

Lemma 5.2.1. *Consider a smooth test function R. Then the following integration by parts formula is valid for $\overline{\sigma'_e}$ (as defined in* (4.3.14)*):*

$$-\int_{\Omega_e} \left\langle Div\, \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}, R\right\rangle = -\int_{\Omega_e} \left\langle \widehat{\varphi}' \circ \varphi, Div\, \overline{\sigma'_e(R)}\right\rangle$$
$$+\int_{\Gamma_i} \left\langle \widehat{\varphi}' \circ \varphi, \overline{\sigma'_e(R)}n\right\rangle - \left\langle \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}n, R\right\rangle. \tag{5.2.6}$$

Proof. We apply (5.2.3) to the righthand side of (5.2.6) and then substitute the definition of the linearized stress tensor $\overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}$:

$$\begin{aligned}
&-\int_{\Omega_e} \langle \operatorname{Div} \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}, R\rangle = \int_{\Omega_e} \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}..DR - \int_{\Gamma_i} \langle \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}n, R\rangle \\
&= \int_{\Omega_e} [(D\widehat{\varphi}' \circ \varphi)\sigma_e]..DR + \int_{\Omega_e} J(\varphi)\overline{\Sigma}(\widetilde{\varepsilon}((\widehat{\varphi}' \circ \varphi), \varphi))..DR \\
&-\int_{\Gamma_i} \langle \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}n, R\rangle
\end{aligned} \tag{5.2.7}$$

In the subsequent steps we again use the definition of matrix inner product and the invariance under cyclic permutations of the trace. For the first term on the righthand side of (5.2.7) we recall that σ_e is symmetric and use (5.2.3) to obtain

$$\begin{aligned}\int_{\Omega_e} [(D\widehat{\varphi}' \circ \varphi)\sigma_e]..DR &= \int_{\Omega_e} D\widehat{\varphi}' \circ \varphi..[(DR)\sigma_e] \\ &= -\int_{\Omega_e} \langle \widehat{\varphi}' \circ \varphi, \mathrm{Div}\,[(DR)\sigma_e]\rangle + \int_{\Gamma_i} \langle \widehat{\varphi}' \circ \varphi, [(DR)\sigma_e]n\rangle \end{aligned} \tag{5.2.8}$$

For the second domain term on the righthand side of (5.2.7), we first note that

$$J(\varphi)\overline{\Sigma}(\widetilde{\varepsilon}(\widehat{\varphi}' \circ \varphi, \varphi))..DR = \Sigma(\widetilde{\varepsilon}(\widehat{\varphi}' \circ \varphi, \varphi))..\left[J\overline{DR}\right]$$

where $\overline{DR} := (D\varphi)^{-*}DR(D\varphi)^{-1}$ and $\Sigma(\widehat{\varphi}' \circ \varphi)$ is defined according to (2.4.6). Using properties of the trace we obtain

$$\begin{aligned} & \int_{\Omega_e} [\lambda \mathrm{Tr}\,(\overline{D\widehat{\varphi}' \circ \varphi})I]..\left[J\overline{DR}\right] = \int_{\Omega_e} D\widehat{\varphi}' \circ \varphi..\left[J\lambda(D\varphi)^{-1}\{\mathrm{Tr}\,(\overline{DR})I\}(D\varphi)^{-*}\right] \\ = \quad & -\int_{\Omega_e} \left\langle \widehat{\varphi}' \circ \varphi, \mathrm{Div}\,\left[J\lambda(D\varphi)^{-1}\{\mathrm{Tr}\,(\overline{DR})I\}(D\varphi)^{-*}\right]\right\rangle \\ & + \int_{\Gamma_i} \left\langle \widehat{\varphi}' \circ \varphi, \left[J\lambda(D\varphi)^{-1}\{\mathrm{Tr}\,(\overline{DR})I\}(D\varphi)^{-*}\right] n\right\rangle \end{aligned} \tag{5.2.9}$$

Similarly, we have

$$\begin{aligned} & \int_{\Omega_e} \mu[\overline{D\widehat{\varphi}' \circ \varphi} + (\overline{D\widehat{\varphi}' \circ \varphi})^*]..\left[J\overline{DR}\right] \\ = \quad & -\int_{\Omega_e} \left\langle \widehat{\varphi}' \circ \varphi, \mathrm{Div}\,\left\{J\mu(D\varphi)^{-1}[\overline{DR} + (\overline{DR})^*](D\varphi)^{-*}\right\}\right\rangle \\ & + \int_{\Gamma_i} \left\langle \widehat{\varphi}' \circ \varphi, \left\{J\mu(D\varphi)^{-1}[\overline{DR} + (\overline{DR})^*](D\varphi)^{-*}\right\} n\right\rangle \end{aligned} \tag{5.2.10}$$

Combining (5.2.9) with (5.2.10) we obtain

$$\begin{aligned}\int_{\Omega_e} J(\varphi)\overline{\Sigma}(\widetilde{\varepsilon}(\widehat{\varphi}' \circ \varphi, \varphi))..DR = \quad & -\int_{\Omega_e} \left\langle \widehat{\varphi}' \circ \varphi, \mathrm{Div}\,\left[J\overline{\Sigma}(\tilde{\varepsilon}(R, \varphi))\right]\right\rangle \\ & + \int_{\Gamma_i} \left\langle \widehat{\varphi}' \circ \varphi, \left[J\overline{\Sigma}(\tilde{\varepsilon}(R, \varphi))\right] n\right\rangle \end{aligned}$$

Using (5.2.8) and (5.2.11) back into (5.2.7), we obtain the integration by parts formula for the linearized elastic stress tensor:

$$
\begin{aligned}
-\int_{\Omega_e}\left\langle \operatorname{Div}\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}, R\right\rangle = &-\int_{\Omega_e}\left\langle \widehat{\varphi}'\circ\varphi, \operatorname{Div}\overline{\sigma_e'(R)}\right\rangle \\
&+\int_{\Gamma_i}\left\langle \widehat{\varphi}'\circ\varphi, \overline{\sigma_e'(R)}n\right\rangle - \left\langle \overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}n, R\right\rangle . \qquad (5.2.11)
\end{aligned}
$$

□

Taking (5.2.4), (5.2.5), together, applying Lemma 5.2.1, and substituting the boundary condition from (4.3.5), we obtain that (5.2.1) is equivalent to

$$
\begin{aligned}
&\int_{\Omega_f}\langle u_f', (Du_f)^*Q-(DQ)u_f-\operatorname{Div}\sigma_f(Q,P)\rangle - p'(\operatorname{div}Q)\\
&+\int_{\Gamma_i}\langle \sigma_f(u_f',p')n, Q\rangle\\
&-\int_{\Omega_e}\left\langle \widehat{\varphi}'\circ\varphi, \operatorname{Div}\overline{\sigma_e'(R)}\right\rangle + \int_{\Gamma_i}\left\langle \widehat{\varphi}'\circ\varphi, \overline{\sigma_e'(R)}n + (Du_f)^*\sigma_f(P,Q)n\right\rangle\\
&-\int_{\Gamma_i}\langle \sigma_f(u_f',p')n+\mathcal{B}(\widehat{\varphi}'\circ\varphi), R\rangle - \int_{\Gamma_f}\langle \sigma_f(u_f',p')n_{\Gamma_f}, Q\rangle\\
&=\int_{\Omega_f}\langle v'|_{\Omega_f}, Q\rangle + \int_{\Omega_e}\langle v'|_{\Omega_e}, R\rangle \qquad (5.2.12)
\end{aligned}
$$

This weak formulation of the state equations, (5.2.12), motivates the introduction of the bilinear form from which we will recover the adjoint system for (Q,P,R).

Definition 5.2.2 (Steady State Linear Adjoint Bilinear Form)**.**

$$
\begin{aligned}
&a((\alpha,\beta),\gamma;(Q,P),R) :=\\
&\int_{\Omega_f}\langle \alpha, (Du_f)^*Q-(DQ)u_f-Div\,\sigma_f(Q,P)\rangle - \beta(div\,Q)\\
&+\int_{\Gamma_i}\langle \sigma_f(\alpha,\beta)n, Q\rangle\\
&-\int_{\Omega_e}\left\langle \gamma, Div\,\overline{\sigma_e'(R)}\right\rangle + \int_{\Gamma_i}\left\langle \gamma, \overline{\sigma_e'(R)}n + (Du_f)^*\sigma_f(P,Q)n\right\rangle\\
&-\int_{\Gamma_i}\langle \sigma_f(\alpha,\beta)n+\mathcal{B}(\gamma), R\rangle - \int_{\Gamma_f}\langle \sigma_f(\alpha,\beta)n_{\Gamma_f}, Q\rangle
\end{aligned}
$$

$$= \int_{\Omega_f} \langle v'|_{\Omega_f}, Q\rangle + \int_{\Omega_e} \langle v'|_{\Omega_e}, R\rangle \tag{5.2.13}$$

5.2.3 Adjoint Derivation

For smooth test functions α, β, γ, we set,

$$a((\alpha,\beta),\gamma;(Q,P),R) = \int_{\Omega_f} \langle u_f - u_d, \alpha\rangle - \frac{1}{2}\int_{\Gamma_i} \langle |u_f - u_d|^2 n, \gamma\rangle + (v, v')_{\mathcal{E}(\Omega)}, \tag{5.2.14}$$

where the righthand side comes from (4.3.3). We proceed to consider cases in order to recover the system for the adjoint variables $((Q,P),R)$.

5.2.3.1 Case 1: $\alpha = \gamma = 0$

Consider $\alpha = \gamma = 0$. Then for all β we have

$$0 = -\int_{\Omega_f} \beta(\operatorname{div} Q) + \int_{\Gamma_i} \beta\langle n, R - Q\rangle - \int_{\Gamma_f} \beta\langle n_{\Gamma_f}, Q\rangle. \tag{5.2.15}$$

Take β to have compact support on $\partial\Omega_f$. Then (5.2.15) reduces to

$$0 = -\int_{\Omega_f} \beta(\operatorname{div} Q),$$

from which we recover $\operatorname{div} Q = 0$ in Ω_f.

Next, we take β such that $\beta|_{\Gamma_f} = 0$. For all such β we have

$$0 = \int_{\Gamma_i} \beta \left\langle n, R - Q\right\rangle,$$

Which implies that $\langle R, n\rangle = \langle Q, n\rangle$ on Γ_i. Substituting this condition back in to (5.2.15) we recover $\langle Q, n_{\Gamma_f}\rangle = 0$ on Γ_f.

In the sequel we will enforce the following stronger boundary conditions, in order to recover the remaining equations in the adjoint sensitivity system:

$$Q = R \text{ on } \Gamma_i, \tag{5.2.16}$$

$$Q = 0 \text{ on } \Gamma_f. \tag{5.2.17}$$

5.2.3.2 Case 2: $\beta = \gamma = 0$

Next, we consider the case that $\beta = \gamma = 0$. Then, from (5.2.13) and (5.2.14), for all α we have,

$$\begin{aligned}
\int_{\Omega_f} \langle u_f - u_d, \alpha \rangle = & \int_{\Omega_f} \langle \alpha, (Du_f)^* Q - (DQ) u_f - \operatorname{Div} \sigma_f(Q, P) \rangle \\
& + \int_{\Gamma_i} \langle 2\nu\varepsilon(\alpha) n, Q - R \rangle - \int_{\Gamma_f} \langle 2\nu\varepsilon(\alpha) n_{\Gamma_f}, Q \rangle, \\
= & \int_{\Omega_f} \langle \alpha, (Du_f)^* Q - (DQ) u_f - \operatorname{Div} \sigma_f(Q, P) \rangle,
\end{aligned} \tag{5.2.18}$$

after substituting (5.2.16) and (5.2.17). Since (5.2.18) is true for all α, we recover the domain equation for Q and P,

$$(Du_f)^* Q - (DQ) u_f - \operatorname{Div} \sigma_f(Q, P) = u_f - u_d \text{ in } \Omega_f. \tag{5.2.19}$$

5.2.3.3 Case 3: $\alpha = \beta = 0$

Finally, we consider the case that $\alpha = \beta = 0$. Then, from (5.2.13) and (5.2.14), we have,

$$\begin{aligned}
\frac{1}{2} \int_{\Gamma_i} \langle |u_f - u_d|^2 n_f, \gamma \rangle = & - \int_{\Omega_e} \left\langle \gamma, \operatorname{Div} \overline{\sigma_e'(R)} \right\rangle \\
& + \int_{\Gamma_i} \left\langle \gamma, \overline{\sigma_e'(R)} n + (Du_f)^* \sigma_f(P, Q) n \right\rangle \\
& - \int_{\Gamma_i} \langle \mathcal{B}(\widehat{\varphi}' \circ \varphi), R \rangle .
\end{aligned} \tag{5.2.20}$$

As previously, we first consider the case where γ has compact support on Ω_e. Then $\gamma|_{\Gamma_i} = 0$. Moreover, $\mathcal{B}(\gamma)|_{\Gamma_i} = 0$ because $\mathcal{B} : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is tangential, according to

Proposition 6.2 in [11]. Then for all γ we have

$$0 = -\int_{\Omega_e} \left\langle \gamma, \operatorname{Div} \overline{\sigma_e'(R)} \right\rangle,$$

from which we recover the domain equation

$$\operatorname{Div} \overline{\sigma_e'(R)} = 0 \text{ on } \Omega_e. \tag{5.2.21}$$

Substituting (5.2.21) back in to (5.2.18) we have,

$$\begin{aligned} \frac{1}{2}\int_{\Gamma_i} \langle |u_f - u_d|^2 n_f, \gamma\rangle = \int_{\Gamma_i} &\left\langle \gamma, \overline{\sigma_e'(R)} n + (Du_f)^* \sigma_f(P,Q) n \right\rangle \\ &- \int_{\Gamma_i} \langle \mathcal{B}(\gamma), R\rangle\,. \end{aligned} \tag{5.2.22}$$

In order to recover the corresponding adjoint boundary equation the following lemma will be useful (and will be helpful in the following section as well).

Lemma 5.2.3 (Integration by parts of linearized boundary terms.)**.** *For $R, \gamma \in H^{1/2}(\Gamma_i)$ we have the following integration by parts formula:*

$$-\int_{\Gamma_i} \langle \mathcal{B}(\gamma), R\rangle = \int_{\Gamma_i} \langle \gamma, \mathcal{B}_A(R)\rangle. \tag{5.2.23}$$

where $\mathcal{B}(\gamma)$ is defined in (4.3.6) *and*

$$\begin{aligned} \mathcal{B}_A(R) :=& div_{\Gamma_i}[(\sigma_e(\varphi) - \sigma_f(p,u_f))R]n - \left\langle H\sigma_f(p,u_f)n + \frac{\partial \sigma_f}{\partial n} n, R \right\rangle n \\ &- (D\sigma_e^{\Delta}.n)^* R - Div_{\Gamma_i}(n \otimes \sigma_e R) + \nabla_{\Gamma_i}\langle \sigma_e n, R\rangle. \end{aligned} \tag{5.2.24}$$

Proof. We recall the definition of $\mathcal{B}(\gamma)$:

$$\begin{aligned} \mathcal{B}(\gamma) :=& [\sigma_e(\varphi) - \sigma_f(p,u_f)] \cdot \nabla_{\Gamma_i}\langle \widehat{\varphi}' \circ \varphi, n\rangle + \langle \gamma, n\rangle \left(H\sigma_f(p,u_f)n + \frac{\partial \sigma_f}{\partial n} n \right) \\ &- \sigma_e (D_{\Gamma_i}\gamma)^* n + (\operatorname{div}_{\Gamma_i}\gamma)\sigma_e n + (D_{\Gamma_i}\sigma_e.\gamma)n. \end{aligned} \tag{5.2.25}$$

For simplicity of exposition, we define

$$B_1 := \sigma_e(\varphi) - \sigma_f(p, u_f), \tag{5.2.26}$$

$$B_2 = H\sigma_f(p, u_f)n + \frac{\partial \sigma_f}{\partial n} n. \tag{5.2.27}$$

Observe that B_1 is a symmetric matrix such that $B_1 n = 0$ (from (2.5.1)), and B_2 is a vector. Both depend only on the solution to the nonlinear coupled system (2.5.1) and not on γ.

We substitute the definition of $\mathcal{B}$, defined in terms of B_1 and B_2, and carry out the proof term-by-term. We have,

$$\begin{aligned}
&\int_{\Gamma_i} \langle -\mathcal{B}(\gamma), R\rangle \\
&= \int_{\Gamma_i} \langle -B_1 \cdot \nabla_{\Gamma_i} \langle \gamma, n\rangle - \langle \gamma, n\rangle B_2 + \sigma_e (D_{\Gamma_i}\gamma)^* n, R\rangle \\
&- \int_{\Gamma_i} \langle (\operatorname{div}_{\Gamma_i}\gamma)\sigma_e n + (D_{\Gamma_i}\sigma_e . \gamma) n, R\rangle,
\end{aligned} \tag{5.2.28}$$

By arithmetic with B_2 we obtain,

$$\int_{\Gamma_i} \langle -\langle \gamma, n\rangle B_2, R\rangle = \int_{\Gamma_i} -\langle \gamma, \langle B_2, R\rangle n\rangle. \tag{5.2.29}$$

Next we use the symmetry of B_1 and (4.2.8) to obtain,

$$\begin{aligned}
\int_{\Gamma_i} \langle -B_1 \cdot \nabla_{\Gamma_i}\langle \gamma, n\rangle, R\rangle &= \int_{\Gamma_i} -\langle \nabla_{\Gamma_i}\langle \gamma, n\rangle, B_1 \cdot R\rangle, \\
&= \int_{\Gamma_i} \langle \gamma, \operatorname{div}_{\Gamma_i}(B_1 R) n - H\langle B_1 R, n\rangle n\rangle, \\
&= \int_{\Gamma_i} \langle \gamma, \operatorname{div}_{\Gamma_i}(B_1 R) n\rangle,
\end{aligned} \tag{5.2.30}$$

where in the last step we use the aforementioned fact for the symmetric B_1 that $B_1 n = 0$.

For the next term in (5.2.28) we use the symmetry of σ_e and the identity for a matrix A

and vectors b, c: $\langle A\cdot b, c\rangle = c\otimes b..A$. We can then apply (4.2.9) and obtain,

$$\begin{aligned}\int_{\Gamma_i}\langle\sigma_e(D_{\Gamma_i}\gamma)^* n, R\rangle &= \int_{\Gamma_i}\langle n, (D_{\Gamma_i}\gamma)\sigma_e R\rangle,\\ &= \int_{\Gamma_i} n\otimes\sigma_e R..D_{\Gamma_i}\gamma,\\ &= \int_{\Gamma_i}\langle\gamma, H(n\otimes\sigma_e R)n - \mathrm{Div}_{\Gamma_i}(n\otimes\sigma_e R)\rangle. \qquad (5.2.31)\end{aligned}$$

We apply (4.2.8) to the next term in (5.2.28) to obtain,

$$-\int_{\Gamma_i}\langle(\mathrm{div}_{\Gamma_i}\gamma)\sigma_e n, R\rangle = \int_{\Gamma_i}\langle\gamma, \nabla_{\Gamma_i}\langle\sigma_e n, R\rangle - H\langle\sigma_e n, R\rangle n\rangle. \qquad (5.2.32)$$

Finally, we consider the term $-\int_{\Gamma_i}\langle(D_{\Gamma_i}\sigma_e.\gamma)n, R\rangle$. Recall that the term $D\sigma_e$ is a three entries tensor representing the gradient matrix of σ_e, which arose in the derivation of the s-derivative of the Cauchy stress tensor σ_e^s. In particular, since

$$\forall i, j, ((\sigma_e^s)_{ij}\circ\varphi_s)' = (\sigma_e')_{ij}\circ\varphi + \langle(\nabla(\sigma_e)_{ij})\circ\varphi, \varphi'\rangle = (\sigma_e')_{ij}\circ\varphi + [(\partial_k(\sigma_e)_{ij})\circ\varphi]\varphi_k',$$

$D\sigma_e$ is defined as $(D\sigma_e.f)_{ij} = (\partial_k(\sigma_e)_{ij})f_k$. We introduce the following notation,

$$(D\sigma_e^{\Delta}.f)_{ik} := \partial_k(\sigma_e)_{ij}f_j,$$

and then we can integrate by parts in components.

$$\begin{aligned}\int_{\Gamma_i}\langle(D\sigma_e.\gamma)\cdot n, R\rangle &= \int_{\Gamma_i}(\partial_k(\sigma_e)_{ij}\gamma_k)(n)_j R_i\\ = \int_{\Gamma_i}\gamma_k(\partial_k(\sigma_e)_{ij}(n)_j R_i) &= \int_{\Gamma_i}\langle\gamma, (D\sigma_e^{\Delta}.n)^* R\rangle. \qquad (5.2.33)\end{aligned}$$

Substituting (5.2.29), (5.2.30), (5.2.31), (5.2.32), and (5.2.33) back in to (5.2.28) and cancelling where possible, we obtain,

$$\int_{\Gamma_i}\langle-\mathcal{B}(\gamma), R\rangle = \int_{\Gamma_i}\langle\gamma, \mathrm{div}_{\Gamma_i}(B_1 R)n - \langle B_2, R\rangle n - (D\sigma_e^{\Delta}.n)^* R\rangle,$$

$$+\int_{\Gamma_i}\langle\gamma,-\mathrm{Div}_{\Gamma_i}(n\otimes\sigma_e R)+\nabla_{\Gamma_i}\langle\sigma_e n,R\rangle\rangle. \tag{5.2.34}$$

The definition

$$\mathcal{B}_A(R):=\mathrm{div}_{\Gamma_i}(B_1R)n-\langle B_2,R\rangle n-(D\sigma_e^{\Delta}.n)^*R-\mathrm{Div}_{\Gamma_i}(n\otimes\sigma_e R)+\nabla_{\Gamma_i}\langle\sigma_e n,R\rangle, \tag{5.2.35}$$

and this completes the proof.

□

Substituting the result of Lemma 5.2.3 in to (5.2.22) we obtain,

$$\begin{aligned}\int_{\Gamma_i}\left\langle\frac{1}{2}|u_f-u_d|^2n_f,\gamma\right\rangle=&\int_{\Gamma_i}\left\langle\gamma,\overline{\sigma_e'(R)}n+(Du_f)^*\sigma_f(P,Q)n\right\rangle\\&+\int_{\Gamma_i}\langle\gamma,\mathcal{B}_A(R)\rangle,\end{aligned} \tag{5.2.36}$$

which is true for all test functions γ and thus implies the boundary condition

$$-\frac{1}{2}|u_f-u_d|^2n=\overline{\sigma_e'(R)}n+(Du_f)^*\sigma_f(P,Q)n+\mathcal{B}_A(R)\text{ on }\Gamma_i. \tag{5.2.37}$$

5.2.4 First Order Optimality Conditions and Gradient Recovery

With the adjoint equations derived we state the lemma which will allow us to state our main result:

Lemma 5.2.4. *Let v_1 be given in $\mathcal{E}_S(\Omega)$ and let*

$$((u_f'(v_1),p'(v_1)),\widehat{\varphi}'(v_1)):=(((\mathscr{D}u_f/\mathscr{D}v,\mathscr{D}p/\mathscr{D}v),\mathscr{D}\widehat{\varphi}/\mathscr{D}v)\cdot v_1)$$

be the Gâteaux derivative of the mapping $v\mapsto((u_f,p),\widehat{\varphi})$ in the direction v_1, which solves (4.3.5)*; then for every admissable v_2, we have*

$$\int_{\Omega_f}\left\langle v_2|_{\Omega_f},(u_f'(v_1),p'(v_1))\right\rangle+\int_{\Gamma_i}\langle v_2|_{\Gamma_i},(\widehat{\varphi}'\circ\varphi)(v_1)\rangle$$

$$= \int_{\Omega_f} \left\langle (Q(v_2), P(v_2)), v_1|_{\Omega_f} \right\rangle + \int_{\Omega_e} \langle R(v_2), v_1|_{\Omega_e} \rangle \tag{5.2.38}$$

where $((Q(v_2), P(v_2)), R(v_2))$ *is the solution of the linearized adjoint problem*

$$\begin{cases} (Du_f)^* Q - (DQ)u_f - Div\, \sigma_f(P, Q) = v_2|_{\Omega_f} & \Omega_f \\ div\,(Q) = 0 & \Omega_f \\ -Div\, \overline{\sigma_e'(R)} = 0 & \Omega_e \\ Q = R & \Gamma_i \\ \overline{\sigma_e'(R)} n + (Du_f)^* \sigma_f(P, Q) n + \mathcal{B}_A(R) = v_2|_{\Gamma_i} & \Gamma_i \\ Q = 0 & \Gamma_f \end{cases} \tag{5.2.39}$$

with

$$\mathcal{B}_A(R) := div_{\Gamma_i}(B_1 R)n - \langle B_2, R\rangle n - (D\sigma_e^{\Delta}.n)^* R - Div_{\Gamma_i}(n \otimes \sigma_e R) + \nabla_{\Gamma_i} \langle \sigma_e n, R\rangle.$$

Proof. We start with the first term on the lefthand side of the (5.3.11) (suppressing the dependence on v_1 and v_2 for the moment).

$$\int_{\Omega_f} \left\langle v_2|_{\Omega_f}, (u_f', p') \right\rangle = \int_{\Omega_f} \left\langle (Du_f)^* Q - (DQ)u_f - \mathrm{Div}\, \sigma_f(P, Q), u_f' \right\rangle - p' \mathrm{div}\, Q \tag{5.2.40}$$

Working term-by-term, or the first term we use a matrix transpose property and obtain

$$\int_{\Omega_f} \left\langle (Du_f)^* Q, u_f' \right\rangle = \int_{\Omega_f} \left\langle Q, (Du_f) u_f' \right\rangle \tag{5.2.41}$$

In the next term we use (4.3.17) and the facts that $u_f = 0$ on $\partial\Omega_f$ and $\mathrm{div}\, u_f = 0$ in Ω_f from the original nonlinear coupled system (2.5.1) to obtain,

$$\begin{aligned} &\int_{\Omega_f} - \left\langle (DQ)u_f, u_f' \right\rangle \\ &= \int_{\Omega_f} \langle Q, (Du_f')u_f + (\mathrm{div}\, u_f) u_f' \rangle - \int_{\partial\Omega_f} \langle Q, (u_f \cdot n_{\partial\Omega_f}) u_f' \rangle, \end{aligned}$$

$$= \int_{\Omega_f} \langle Q, (Du'_f)u_f \rangle. \tag{5.2.42}$$

Next, we apply the matrix formulation of Green's identity, (5.2.3), as well as $\operatorname{div} u_f = 0$ in Ω_f, and simplify to obtain,

$$\begin{aligned}
&\int_{\Omega_f} \left\langle -\operatorname{Div} \sigma_f(P, Q), u'_f \right\rangle \\
&= \int_{\Omega_f} \sigma_f(P, Q)..Du_f - \int_{\Gamma_i} \langle \sigma_f(P, Q)n_f, u'_f \rangle, \\
&= \int_{\Omega_f} -P\operatorname{div} u'_f + 2\nu\varepsilon(Q)..Du'_f + \int_{\Gamma_i} \langle \sigma_f(P, Q)n, u'_f \rangle \\
&= \int_{\Omega_f} -P\operatorname{div} u'_f + 2\nu\varepsilon(u_f)..DQ + \int_{\Gamma_i} \langle \sigma_f(P, Q)n, u'_f \rangle \\
&= \int_{\Omega_f} -P\operatorname{div} u'_f - \langle \operatorname{Div} 2\nu\varepsilon(u_f), Q \rangle + \int_{\Gamma_i} \langle \sigma_f(P, Q)n, u'_f \rangle \\
&+ \int_{\Gamma_i} \langle 2\nu\varepsilon(u'_f)n_f, Q \rangle, \\
&= \int_{\Omega_f} -\langle \operatorname{Div} 2\nu\varepsilon(u_f), Q \rangle + \int_{\Gamma_i} \langle \sigma_f(P, Q)n, u'_f \rangle - \langle 2\nu\varepsilon(u'_f)n, Q \rangle.
\end{aligned} \tag{5.2.43}$$

Finally, we use Green's identity and the boundary condition in (5.2.39), $Q = 0$ on Γ_f, to obtain,

$$\begin{aligned}
-\int_{\Omega_f} p'\operatorname{div} Q &= -\int_{\partial\Omega_f} p'(Q \cdot n_{\partial\Omega_f}) + \int_{\Omega_f} \langle Q, \nabla p' \rangle \\
&= -\int_{\Gamma_i} p'(Q \cdot n_f) + \int_{\Omega_f} \langle Q, \nabla p' \rangle \\
&= \int_{\Gamma_i} \langle Q, pn \rangle + \int_{\Omega_f} \langle Q, \nabla p' \rangle.
\end{aligned} \tag{5.2.44}$$

Taking (5.2.41), (5.2.42), (5.2.43), and (5.2.44), together and substituting back in to (5.2.40), we have,

$$\int_{\Omega_f} \left\langle v_2|_{\Omega_f}, (u_f', p')\right\rangle = \int_{\Omega_f} \left\langle Q, (Du_f')u_f + (Du_f)u_f' - \operatorname{Div}\sigma_f(p', u_f')\right\rangle$$
$$+ \int_{\Gamma_i} \langle \sigma_f(P,Q)n, u_f'\rangle - \langle \sigma_f(p', u_f')n, Q\rangle,$$

and by substituting the linearized Navier-Stokes domain equation from (4.3.5) we obtain,

$$\int_{\Omega_f} \left\langle v_2|_{\Omega_f}, (u_f', p')\right\rangle = \int_{\Omega_f} \left\langle Q, v_1|_{\Omega_f}\right\rangle$$
$$+ \int_{\Gamma_i} \langle \sigma_f(P,Q)n, u_f'\rangle - \langle \sigma_f(p', u_f')n, Q\rangle. \tag{5.2.45}$$

The remaining extra terms will cancel in the last step.

We then work with the second term on the lefthand side of the integral in the (5.3.11):

$$\int_{\Gamma_i} \langle v_2|_{\Gamma_i}, \widehat{\varphi}' \circ \varphi\rangle = \int_{\Gamma_i} \left\langle \left[\overline{\sigma_e'(R)} + (Du_f)^* \sigma_f(P,Q)\right] n + \mathcal{B}_A(R), \widehat{\varphi}' \circ \varphi \right\rangle, \tag{5.2.46}$$

and consider each term in turn. For the first term, we use Lemma 5.2.1 and obtain,

$$\int_{\Gamma_i} \left\langle \overline{\sigma_e'(R)} n, \widehat{\varphi}' \circ \varphi \right\rangle = \int_{\Gamma_i} \left\langle R, \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n \right\rangle$$
$$+ \int_{\Omega_e} \left\langle R, \operatorname{Div} \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} \right\rangle - \left\langle \operatorname{Div} \overline{\sigma_e'(R)}, \widehat{\varphi}' \circ \varphi \right\rangle. \tag{5.2.47}$$

We then substitute the elasticity domain equations from the linear coupled system (4.3.5) and the adjoint system (5.2.39) to obtain,

$$\int_{\Gamma_i} \left\langle \overline{\sigma_e'(R)} n, \widehat{\varphi}' \circ \varphi \right\rangle = \int_{\Gamma_i} \left\langle R, \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n \right\rangle + \int_{\Omega_e} \langle R, v_1|_{\Omega_e}\rangle. \tag{5.2.48}$$

For the next term substitute the linearized velocity matching boundary condition from (4.3.5) to obtain,

$$\int_{\Gamma_i} \langle (Du_f)^* \sigma_f(P,Q)n, \widehat{\varphi}' \circ \varphi\rangle = \int_{\Gamma_i} \langle \sigma_f(P,Q), (Du_f)(\widehat{\varphi}' \circ \varphi)\rangle,$$

$$= \int_{\Gamma_i} \left\langle \sigma_f(P,Q)n, -u_f' \right\rangle . \tag{5.2.49}$$

For the final term in (5.2.46) we apply Lemma 5.2.3.

$$\int_{\Gamma_i} \langle \widehat{\varphi}' \circ \varphi, \mathcal{B}_A(R) \rangle = \int_{\Gamma_i} \langle -\mathcal{B}(\widehat{\varphi}' \circ \varphi), R \rangle . \tag{5.2.50}$$

Substituting (5.2.48), (5.2.49), and (5.2.50) in to (5.2.46) we obtain,

$$\begin{aligned}\int_{\Gamma_i} \langle v_2|_{\Gamma_i}, \widehat{\varphi}' \circ \varphi \rangle = \int_{\Omega_e} \langle R, v_1|_{\Omega_e} \rangle + \int_{\Gamma_i} \left\langle R, \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n - \mathcal{B}(\widehat{\varphi}' \circ \varphi) \right\rangle \\ - \int_{\Gamma_i} \left\langle \sigma_f(P,Q)n, u_f' \right\rangle \end{aligned} \tag{5.2.51}$$

Finally we add (5.2.45) and (5.2.51).

$$\begin{aligned} &\int_{\Omega_f} \left\langle v_2|_{\Omega_f}, (u_f', p') \right\rangle + \int_{\Gamma_i} \langle v_2|_{\Gamma_i}, \widehat{\varphi}' \circ \varphi \rangle \\ = &\int_{\Omega_e} \langle R, v_1|_{\Omega_e} \rangle + \int_{\Omega_f} \left\langle (Q,P), v_1|_{\Omega_f} \right\rangle \\ &+ \int_{\Gamma_i} \left\langle R, \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n - \mathcal{B}(\widehat{\varphi}' \circ \varphi) \right\rangle - \langle \sigma_f(p', u_f')n, Q \rangle \\ &- \int_{\Gamma_i} \left\langle \sigma_f(P,Q)n, u_f' \right\rangle + \int_{\Gamma_i} \langle \sigma_f(P,Q)n, u_f' \rangle. \end{aligned}$$

We cancel terms and substitute the linearized stress matching boundary condition from (4.3.5) and the boundary condition $Q = R$ from (5.2.39) to complete the proof.

□

Lemma 5.2.4 allows us to state our main result.

Theorem 5.2.5. *[First Order Optimality Conditions for Steady State Optimal Control Problem] Let $(\overline{v}; (\overline{u}, \overline{p}), \overline{\widehat{\varphi}})$ be an optimal pair for the optimal control problem:*

$$\min J_S(u_f, v) = 1/2\|u - u_f\|^2_{L^2(\Omega_f)} + 1/2\|v\|^2_{\mathcal{E}_S(\Omega)},$$

subject to (2.5.1). *Then* $\overline{v}$ *satisfies the following system for* $m = 1$ *in Definition 3.1.11:*

$$\begin{cases} \overline{v}_f - \Delta \overline{v}_f + (Q,P)(\overline{u}_f - u_d) = 0 & \Omega_f \\ \overline{v}_e - \Delta \overline{v}_e + R\left(-\frac{1}{2}|\overline{u}_f - u_d|^2 n\right) = 0 & \Omega_e \\ (D\overline{v}_e)n = (D\overline{v}_f)n & \Gamma_i \\ (D\overline{v}_f)n = 0 & \Gamma_f, \end{cases} \tag{5.2.52}$$

where $((Q,P)(\overline{u}_f - u_d), R(-1/2|\overline{u}_f - u_d|^2 n))$ *is the adjoint state that is the solution of the linearized adjoint*

$$\begin{cases} (Du_f)^* Q - (DQ)u_f - Div\, \sigma_f(P,Q) = \overline{u}_f - u_d & \Omega_f \\ div\,(Q) = 0 & \Omega_f \\ -Div\, \overline{\sigma_e'(R)} = 0 & \Omega_e \\ Q = R & \Gamma_i \\ \overline{\sigma_e'(R)} n + (Du_f)^* \sigma_f(P,Q)n + \mathcal{B}_A(R) = -\frac{1}{2}|\overline{u}_f - u_d|^2 n & \Gamma_i \\ Q = 0 & \Gamma_f. \end{cases} \tag{5.2.53}$$

Proof. Let Let $(\overline{v}; (\overline{u}, \overline{p}), \widehat{\overline{\varphi}})$ be an optimal pair. Recall the computation of the directional derivative of J_S in the direction v', (4.3.3). We have,

$$\begin{aligned} \partial J_S(v; v') &= \int_{\Omega_f} \left\langle \overline{u} - u_d, u_f' \right\rangle - \frac{1}{2} \int_{\Gamma_i} \langle |u_f - u_d|^2 n, \widehat{\varphi}' \circ \varphi \rangle + (v, v')_{\mathcal{E}_S(\Omega)}, \\ &= \int_{\Omega_f} \left\langle (Q,P)(\overline{u} - u_d)), v'|_{\Omega_f} \right\rangle + \int_{\Omega_e} \left\langle R\left(-\frac{1}{2}|\overline{u}_f - u_d|^2 n\right), v'|_{\Omega_e} \right\rangle \\ &+ (v, v')_{\mathcal{E}_S(\Omega)}, \end{aligned}$$

by Lemma 5.2.4. We expand the term $(v, v')_{\mathcal{E}_S(\Omega)}$ (by definition for $m = 1$) and use (5.2.3):

$$\begin{aligned} (v, v')_{\mathcal{E}_S(\Omega)} &= \int_{\Omega_f} \langle v_f, v' \rangle + Dv_f .. Dv' + \int_{\Omega_e} \langle v_e, v' \rangle + Dv_e .. Dv', \\ &= \int_{\Omega_f} \langle v_f - \Delta v_f, v' \rangle + \int_{\Omega_e} \langle v_e - \Delta v_e, v' \rangle \end{aligned}$$

$$+ \int_{\partial\Omega_f} \langle (Dv_f) n_{\partial\Omega_f}, v' \rangle + \int_{\Gamma_i} \langle (Dv_e) n, v' \rangle.$$

Since J_S attains it's minimum at $\overline{v}$, we have that $\partial J_S(\overline{v}; v') = 0$ for all $v' \in \mathcal{E}_S(\Omega)$, so by considering cases for v' we recover the system (5.2.52).

□

The derivation of the adjoint system and Theorem 5.2.5 also yields the following corollary which is relevant to subsequent numerical investigations, as the explicit representation of the gradient of the cost functional provides directions for descent.

Corollary 5.2.6. *For the target velocity optimization problem, the gradient of cost functional J_S is given by*

$$J'_S(v; v') = (v', v)_{\mathcal{E}_S(\Omega)} + (v'|_{\Omega_f}, (Q, P)) + (v'|_{\Omega_e}, R), \tag{5.2.54}$$

where $((Q, P), R)$ solve the linear adjoint problem (5.2.53).

5.3 Dynamical Case

5.3.1 Goal and strategy

The goal is again to emulate the proof of Lemma 1.2.2 in [1] in order to make precise the dependence of the linearized variables $((u'_f, p'), \widehat{\varphi}' \circ \varphi)$ on the control v', this time in the context of the time-dependent case where $((u'_f, p'), \widehat{\varphi}' \circ \varphi)$ solve (4.5.1). The strategy is again to derive a bilinear form based on a variational from of (4.5.1), and to use the bilinear form to extract the adjoint equations, while attempting to make use of the results from the steady state case wherever possible.

5.3.2 Variational Form

The first step is to write (4.5.1) in a variational form. Using test functions Q, P, and R, we obtain a weak formulation for (4.5.1).

$$
\begin{aligned}
& \int_0^T \int_{\Omega_f(t)} \left\langle \underbrace{\frac{\partial u_f'}{\partial t}}_{\mathrm{I}} + (Du_f')u_f + (Du_f)u_f' - \operatorname{Div} \sigma_f(u_f', p'), Q \right\rangle \\
& - \int_0^T \int_{\Omega_f(t)} P \operatorname{div} u_f' + \int_0^T \int_{\Omega_e(t)} \left\langle \underbrace{\rho_e \left(\frac{\partial^2 \widehat{\varphi}_e'}{\partial t^2} \right) \circ \varphi}_{\mathrm{II}} - \operatorname{Div} \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}, R \right\rangle \\
= \quad & \int_0^T \int_{\Omega_f} \langle v'|_{\Omega_f}, Q \rangle + \int_0^T \int_{\Omega_e} \langle v'|_{\Omega_e}, R \rangle
\end{aligned}
\tag{5.3.1}
$$

Highlighted in the variational form (5.3.1) are the terms present in the dynamical case which were absent in the steady state case, so we consider those terms in the development of the bilinear form from which we will obtain the adjoint system.

For term I, it is helpful to adapt the formula for derivatives of domain integrals in [20]:
Lemma 5.3.1. *[Integration by Parts in Time over a Time-Dependent Domain] For $\phi, \xi \in L^2(0, T; L^2(\Omega))$, the following integration by parts formula holds:*

$$
\begin{aligned}
\int_0^T \int_{\Omega(t)} \left\langle \frac{\partial \phi}{\partial t}, \xi \right\rangle &= \int_{\Omega(T)} \langle \phi, \xi \rangle - \int_{\Omega(0)} \langle \phi, \xi \rangle \\
&\quad - \int_0^T \int_{\Omega(t)} \left\langle \phi, \frac{\partial \xi}{\partial t} \right\rangle - \int_0^T \int_{\partial\Omega(t)} \langle \phi, \xi \rangle \langle V(\Omega(t)), n_\Omega \rangle,
\end{aligned}
$$

where $V(\Omega(t))$ is the velocity which builds the boundary $\partial\Omega(t)$ and n_Ω is the unit outer normal with respect to $\partial\Omega$.

Using Lemma 5.3.1 we obtain,

$$
\begin{aligned}
\int_0^T \int_{\Omega_f(t)} \left\langle \frac{\partial u_f'}{\partial t}, Q \right\rangle &= \int_{\Omega_f(T)} \langle u_f', Q \rangle - \int_{\Omega_f(0)} \langle u_f', Q \rangle - \int_0^T \int_{\Omega_f(t)} \left\langle u_f', \frac{\partial Q}{\partial t} \right\rangle \\
&\quad - \int_0^T \int_{\partial\Omega_f(t)} \langle u_f', Q \rangle \left\langle V(\Omega_f(t)), n_{\partial\Omega_f(t)} \right\rangle .
\end{aligned}
$$

We recall that $u'_f = 0$ on Γ_f, and substitute $u'_f(0) = 0$ and $n = -n_f$. Further,we have that since $\widehat{\varphi} : \widehat{\Omega}_f \to \Omega_f(t)$, the speed which builds the boundary $\Gamma_i(t)$ is $V(\Omega_f(t)) = \dfrac{\partial \widehat{\varphi}}{\partial t} \circ \varphi$. So we obtain,

$$\begin{aligned}\int_0^T \int_{\Omega_f(t)} \left\langle \frac{\partial u'_f}{\partial t}, Q \right\rangle = \int_{\Omega_f(T)} \langle u'_f, Q\rangle - \int_0^T \int_{\Omega_f(t)} \left\langle u'_f, \frac{\partial Q}{\partial t} \right\rangle \\ + \int_0^T \int_{\Gamma_i(t)} \langle u'_f, Q\rangle \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle . \end{aligned} \tag{5.3.2}$$

In the case of term II in (5.3.1), there is a time derivative of $\widehat{\varphi}'$, which is defined on the Lagrangian frame. Consequently we take the approach of transporting the term to the reference configuration, integrating by parts, and transporting back to the current configuration. Define $\widehat{R} := R \circ \widehat{\varphi}$ and we obtain,

$$\begin{aligned}
&\int_0^T \int_{\Omega_e(t)} \left\langle \rho_e \left(\frac{\partial^2 \widehat{\varphi}'_e}{\partial t^2} \right) \circ \varphi, R \right\rangle \\
&= \int_0^T \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial^2 \widehat{\varphi}'_e}{\partial t^2}, R \circ \widehat{\varphi} \right\rangle \\
&= \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial \widehat{\varphi}'_e}{\partial t}, \widehat{R} \right\rangle \Bigg|_0^T - \int_0^T \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial \widehat{\varphi}'_e}{\partial t}, \frac{\partial \widehat{R}}{\partial t} \right\rangle, \\
&= \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial \widehat{\varphi}'_e}{\partial t}, \widehat{R} \right\rangle \Bigg|_0^T - \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \widehat{\varphi}', \frac{\partial \widehat{R}}{\partial t} \right\rangle \Bigg|_0^T + \int_0^T \int_{\widehat{\Omega}_e} \left\langle \widehat{\varphi}'_e, \widehat{\rho}_0 \frac{\partial^2 \widehat{R}}{\partial t^2} \right\rangle, \\
&= \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial \widehat{\varphi}'_e}{\partial t}(T), \widehat{R}(T) \right\rangle - \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \widehat{\varphi}'(T), \frac{\partial \widehat{R}}{\partial t}(T) \right\rangle \\
&+ \int_0^T \int_{\Omega_e(t)} \left\langle \widehat{\varphi}'_e \circ \varphi, \rho_e \left(\frac{\partial^2 \widehat{R}}{\partial t^2} \right) \circ \varphi \right\rangle ,
\end{aligned} \tag{5.3.3}$$

where we have applied that $\widehat{\varphi}'_e(0) = \dfrac{\partial \widehat{\varphi}'}{\partial t}(0) = 0$.

For the remaining terms in (5.3.1), the following relations on domain integrals follow readily from the steady state derivation:

$$\int_0^T \int_{\Omega_f(t)} \langle (Du'_f)u_f + (Du_f)u'_f, Q\rangle + \sigma_f(p', u'_f)..DQ - (\operatorname{div} u'_f)P$$

$$= \int_0^T \int_{\Omega_f(t)} \langle u_f', -(DQ)u_f + (Du_f)^* Q - \mathrm{Div}\,(\sigma_f(P,Q))\rangle - p'\mathrm{div}\,(Q)$$

$$+ \int_0^T \int_{\Gamma_i(t)} \langle u_f', -\sigma_f(P,Q)n\rangle, \tag{5.3.4}$$

which uses the facts that $u_f'|_{\Gamma_f} = 0$ and $n = -n_f$, and

$$\int_0^T \int_{\Omega_e(t)} \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} .. DR$$

$$= \int_0^T \int_{\Omega_e(t)} \langle \widehat{\varphi}' \circ \varphi, -\mathrm{Div}\, \overline{\sigma_e'(R)}\rangle + \int_0^T \int_{\Gamma_i(t)} \langle \widehat{\varphi}' \circ \varphi, \overline{\sigma_e'(R)} n\rangle, \tag{5.3.5}$$

from Lemma 5.2.1.

We substitute (5.3.2), (5.3.3), (5.3.4), and (5.3.5), in to (5.3.1):

$$\int_0^T \int_{\Omega_f(t)} \big\langle v_f', (Q,P)\big\rangle + \int_0^T \int_{\Omega_e(t)} \langle \rho_e v_e', R\rangle$$

$$= \int_0^T \int_{\Omega_f(t)} \left\langle u_f', -\frac{\partial Q}{\partial t} - (DQ)u_f + (Du_f)^* Q - \mathrm{Div}\,(\sigma_f(P,Q)) \right\rangle - p'\mathrm{div}\,(Q)$$

$$+ \int_0^T \int_{\Omega_e(t)} \left\langle \widehat{\varphi} \circ \varphi, \rho_e \left(\frac{\partial^2 \widehat{R}}{\partial t^2} \right) \circ \varphi - \mathrm{Div}\, \overline{\sigma_e'(R)} \right\rangle$$

$$+ \int_0^T \int_{\Gamma_i(t)} \langle \sigma_f(u_f', p')n, Q\rangle - \langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n, R\rangle + \langle \widehat{\varphi}' \circ \varphi, \overline{\sigma_e'(R)} n\rangle$$

$$+ \int_0^T \int_{\Gamma_i(t)} \left\langle u_f', \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q - \sigma_f(P,Q)n \right\rangle$$

$$+ \int_{\Omega_f(T)} \langle u_f', Q\rangle + \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial \widehat{\varphi}_e'}{\partial t}(T), \widehat{R}(T) \right\rangle - \left\langle \widehat{\rho}_0 \widehat{\varphi}'(T), \frac{\partial \widehat{R}}{\partial t}(T) \right\rangle. \tag{5.3.6}$$

Next we substitute the boundary conditions from (4.5.1) into (5.3.6) and obtain the following weak formulation of (4.5.1):

$$\int_0^T \int_{\Omega_f(t)} \big\langle v_f', (Q,P)\big\rangle + \int_0^T \int_{\Omega_e(t)} \langle \rho_e v_e', R\rangle$$

$$= \int_0^T \int_{\Omega_f(t)} \left\langle u_f', -\frac{\partial Q}{\partial t} - (DQ)u_f + (Du_f)^* Q - \mathrm{Div}\,(\sigma_f(P,Q)) \right\rangle - p'\mathrm{div}\,(Q)$$

$$
\begin{aligned}
&+\int_0^T\int_{\Omega_e(t)}\left\langle \widehat{\varphi}\circ\varphi, \rho_e\left(\frac{\partial^2\widehat{R}}{\partial t^2}\right)\circ\varphi - \mathrm{Div}\,\overline{\sigma_e'(R)}\right\rangle \\
&+\int_0^T\int_{\Gamma_i(t)}\langle\sigma_f(u_f',p')n,Q\rangle - \langle\sigma_f(u_f',p')n - \mathcal{B}(\widehat{\varphi}'\circ\varphi),R\rangle \\
&+\int_0^T\int_{\Gamma_i(t)}\langle\widehat{\varphi}'\circ\varphi,\overline{\sigma_e'(R)}n\rangle \\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle\left(\frac{\partial\widehat{\varphi}'}{\partial t}\right)\circ\varphi-(Du_f)(\widehat{\varphi}'\circ\varphi),\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle Q-\sigma_f(P,Q)n\right\rangle \\
&+\int_{\Omega_f(T)}\langle u_f',Q\rangle+\int_{\widehat{\Omega}_e}\left\langle\widehat{\rho}_0\frac{\partial\widehat{\varphi}_e'}{\partial t}(T),\widehat{R}(T)\right\rangle-\left\langle\widehat{\rho}_0\widehat{\varphi}'(T),\frac{\partial\widehat{R}}{\partial t}(T)\right\rangle, \\
&=\int_0^T\int_{\Omega_f(t)}\left\langle u_f',-\frac{\partial Q}{\partial t}-(DQ)u_f+(Du_f)^*Q-\mathrm{Div}\,(\sigma_f(P,Q))\right\rangle-p'\mathrm{div}\,(Q) \\
&+\int_0^T\int_{\Omega_e(t)}\left\langle \widehat{\varphi}\circ\varphi, \rho_e\left(\frac{\partial^2\widehat{R}}{\partial t^2}\right)\circ\varphi - \mathrm{Div}\,\overline{\sigma_e'(R)}\right\rangle \\
&+\int_0^T\int_{\Gamma_i(t)}\langle\sigma_f(u_f',p')n,Q\rangle - \langle\sigma_f(u_f',p')n + \mathcal{B}(\widehat{\varphi}'\circ\varphi),R\rangle \\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle\widehat{\varphi}'\circ\varphi,\overline{\sigma_e'(R)}n+(Du_f)^*\sigma_f(P,Q)n-\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle(Du_f)^*Q\right\rangle \\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle\left(\frac{\partial\widehat{\varphi}'}{\partial t}\right)\circ\varphi,\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle Q-\sigma_f(P,Q)n\right\rangle \\
&+\int_{\Omega_f(T)}\langle u_f',Q\rangle+\int_{\widehat{\Omega}_e}\left\langle\widehat{\rho}_0\frac{\partial\widehat{\varphi}_e'}{\partial t}(T),\widehat{R}(T)\right\rangle-\left\langle\widehat{\rho}_0\widehat{\varphi}'(T),\frac{\partial\widehat{R}}{\partial t}(T)\right\rangle. \qquad (5.3.7)
\end{aligned}
$$

To complete the weak formulation and derive the adjoint equations, we must consider the term $\int_0^T\int_{\Gamma_i(t)}\left\langle\left(\frac{\partial\widehat{\varphi}'}{\partial t}\right)\circ\varphi,\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle Q-\sigma_f(P,Q)n\right\rangle$. We require a formula for integration by parts in time on the boundary. First, for simplicity of exposition, define the operator

$$\mathcal{K}_{\Gamma(t)}[\psi] := \frac{\partial\psi}{\partial t}+(D\psi)V+\psi\mathrm{div}_{\Gamma(t)}V, \qquad (5.3.8)$$

where V is the speed that builds the boundary $\Gamma(t)$.

Lemma 5.3.2. *[Integration by Parts in Time on a Time-Dependent Boundary] For*

$\xi, \psi \in L^2(0,T;L^2(\partial\Omega))$, *the following integration by parts formula holds:*

$$\int_0^T \int_{\Gamma(t)} \left\langle \frac{\partial \widehat{\xi}}{\partial t} \circ \varphi, \psi \right\rangle$$
$$= \int_{\Gamma(T)} - \int_{\Gamma(0)} \left\langle \widehat{\xi} \circ \varphi, \psi \right\rangle - \int_0^T \int_{\Gamma(t)} \left\langle \widehat{\xi} \circ \varphi, \mathcal{K}_{\Gamma(t)}[\psi] \right\rangle .$$

The formula is obtained by transporting the boundary integral to the fixed (in time) reference configuration, performing the IBP, and transporting back to the moving boundary and carrying out the composition with ϕ.

Applying Lemma 5.3.2 to (5.3.7), we obtain the following equivalent weak formulation of (4.5.1):

$$\begin{aligned}
&\int_0^T \int_{\Omega_f(t)} \left\langle v'_f, (Q,P) \right\rangle + \int_0^T \int_{\Omega_e(t)} \langle \rho_e v'_e, R \rangle \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle u'_f, -\frac{\partial Q}{\partial t} - (DQ)u_f + (Du_f)^* Q - \operatorname{Div}(\sigma_f(P,Q)) \right\rangle - p' \operatorname{div}(Q) \\
&\quad + \int_0^T \int_{\Omega_e(t)} \left\langle \widehat{\varphi} \circ \varphi, \rho_e \left(\frac{\partial^2 \widehat{R}}{\partial t^2} \right) \circ \varphi - \operatorname{Div} \overline{\sigma'_e(R)} \right\rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \langle \sigma_f(u'_f, p')n, Q \rangle - \langle \sigma_f(u'_f,p')n + \mathcal{B}(\widehat{\varphi}' \circ \varphi), R \rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \left\langle \widehat{\varphi}' \circ \varphi, \overline{\sigma'_e(R)} n + (Du_f)^* \sigma_f(P,Q) n - \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle (Du_f)^* Q \right\rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \left\langle \widehat{\varphi}' \circ \varphi, -\mathcal{K}_{\Gamma_i(t)} \left[\left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q - \sigma_f(P,Q)n \right] \right\rangle \\
&\quad + \int_{\Omega_f(T)} \langle u'_f, Q \rangle + \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial \widehat{\varphi}'_e}{\partial t}(T), \widehat{R}(T) \right\rangle - \left\langle \widehat{\rho}_0 \widehat{\varphi}'(T), \frac{\partial \widehat{R}}{\partial t}(T) \right\rangle \\
&\quad - \int_{\Gamma_i(T)} \left\langle \widehat{\varphi}' \circ \varphi, \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q - \sigma_f(P,Q)n \right\rangle .
\end{aligned} \tag{5.3.9}$$

The weak formulation of the state equations (5.3.9) motivates the introduction of the bilinear form from which we will recover the adjoint system for (Q,P,R).

Definition 5.3.3 (Time Dependent Case Linear Adjoint Variational Form).

$$
\begin{aligned}
&a((\alpha,\beta),\gamma;(Q,P),R) := \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle u_f', -\frac{\partial Q}{\partial t} - (DQ)u_f + (Du_f)^* Q - Div(\sigma_f(P,Q)) \right\rangle - \beta\, div(Q) \\
&\quad + \int_0^T \int_{\Omega_e(t)} \left\langle \widehat{\varphi} \circ \varphi, \rho_e \left(\frac{\partial^2 \widehat{R}}{\partial t^2} \right) \circ \varphi - Div\, \overline{\sigma_e'(R)} \right\rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \langle \sigma_f(\alpha,\beta)n, Q \rangle - \langle \sigma_f(\alpha,\beta)n + \mathcal{B}(\gamma), R \rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \left\langle \gamma, \overline{\sigma_e'(R)}n + (Du_f)^*\sigma_f(P,Q)n - \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle (Du_f)^* Q \right\rangle \\
&\quad + \int_0^T \int_{\Gamma_i(t)} \left\langle \gamma, -\mathcal{K}_{\Gamma_i(t)} \left[\left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q - \sigma_f(P,Q)n \right] \right\rangle \\
&\quad + \int_{\Omega_f(T)} \langle \alpha, Q \rangle + \int_{\widehat{\Omega}_e} \left\langle \widehat{\rho}_0 \frac{\partial \widehat{\varphi}_e'}{\partial t}(T), \widehat{R}(T) \right\rangle - \left\langle \widehat{\rho}_0 \widehat{\varphi}'(T), \frac{\partial \widehat{R}}{\partial t}(T) \right\rangle \\
&\quad - \int_{\Gamma_i(T)} \left\langle \gamma, \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q - \sigma_f(P,Q)n \right\rangle .
\end{aligned}
\tag{5.3.10}
$$

Subsequently, the adjoint equations can be found according to a similar process as was carried out for the steady state case.

5.3.3 Gradient Recovery and First Order Optimality Conditions

We can now state and prove the following result, analogous to Lemma 5.2.4.

Lemma 5.3.4. *Let v_1 be given in $\mathcal{E}(\Omega)$ and let*

$$((u_f'(v_1), p'(v_1)), \widehat{\varphi}'(v_1)) := (((\mathscr{D}u_f/\mathscr{D}v, \mathscr{D}p/\mathscr{D}v), \mathscr{D}\widehat{\varphi}/\mathscr{D}v) \cdot v_1)$$

be the Gâteaux derivative of the mapping $v \mapsto ((u_f, p), \widehat{\varphi})$ in the direction v_1, which solves

(4.5.1); *then for every admissable* v_2, *we have*

$$\begin{aligned}
&\int_0^T \int_{\Omega_f(t)} \left\langle v_2|_{\Omega_f(t)}, (u_f'(v_1), p'(v_1))\right\rangle + \int_0^T \int_{\Gamma_i(t)} \langle v_2|_{\Gamma_i}, (\widehat{\varphi}' \circ \varphi)(v_1)\rangle \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle (Q(v_2), P(v_2)), v_1|_{\Omega_f(t)}\right\rangle + \int_0^T \int_{\Omega_e(t)} \left\langle R(v_2), v_1|_{\Omega_e(t)}\right\rangle
\end{aligned} \tag{5.3.11}$$

where $((Q(v_2), P(v_2)), R(v_2))$ *is the solution of the linearized adjoint problem*

$$\begin{cases}
-\dfrac{\partial Q}{\partial t} - (DQ)u_f + (Du_f)^* Q - Div(\sigma_f(P,Q)) = v_2|_{\Omega_f(t)} & \Omega_f(t) \\
div(Q) = 0 & \Omega_f(t) \\
\rho_e \left(\dfrac{\partial^2 \widehat{R}}{\partial t^2}\right) \circ \varphi - Div\, \overline{\sigma_e'(R)} = 0 & \Omega_e(t) \\
Q = R & \Gamma_i(t) \\
\overline{\sigma_e'(R)} n + (Du_f)^* \sigma_f(P,Q) n & \\
-\left\langle \dfrac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle (Du_f)^* Q + \mathcal{B}_A(R) & \\
-\mathcal{K}_{\Gamma_i(t)} \left[\left\langle \dfrac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q - \sigma_f(P,Q) n \right] = v_2|_{\Gamma_i} & \Gamma_i(t) \\
Q = 0 & \Gamma_f \\
\widehat{R}(T) = 0; \dfrac{\partial \widehat{R}}{\partial t}(T) = 0; Q(T) = 0; P(T) = 0 & (\widehat{\Omega}_e)^2 \times (\Omega_f)^2
\end{cases} \tag{5.3.12}$$

with

$$\begin{aligned}
\mathcal{B}_A(R) &:= div_{\Gamma_i}(B_1 R) n - \langle B_2, R\rangle n - (D\sigma_e^{\Delta}.n)^* R \\
&\quad - Div_{\Gamma_i}(n \otimes \sigma_e R) + \nabla_{\Gamma_i} \langle \sigma_e n, R\rangle, \\
\mathcal{K}_{\Gamma_i(t)}[W] &:= \frac{\partial W}{\partial t} + (DW)\left(\frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi\right) + W\, div_{\Gamma_i(t)} \left(\frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi\right).
\end{aligned}$$

Proof. We start with the first term on the LHS of (5.3.11).

$$\int_0^T \int_{\Omega_f(t)} \left\langle v_2|_{\Omega_f(t)}, (u_f'(v_1), p'(v_1))\right\rangle$$

$$= \int_0^T \int_{\Omega_f(t)} \left\langle -\frac{\partial Q}{\partial t}, u_f \right\rangle$$
$$+ \int_0^T \int_{\Omega_f(t)} \left\langle (Du_f)^* Q - (DQ)u_f - \operatorname{Div} \sigma_f(P,Q), u_f' \right\rangle + p' \operatorname{div} Q. \tag{5.3.13}$$

We apply Lemma 5.3.1 to the first term in (5.3.13).

$$\begin{aligned} \int_0^T \int_{\Omega_f(t)} \left\langle -\frac{\partial Q}{\partial t}, u_f \right\rangle = & \int_0^T \int_{\Omega_f(t)} \left\langle \frac{\partial u_f'}{\partial t}, Q \right\rangle \\ & + \int_{\Omega_f(T)} \langle u_f', Q \rangle - \int_{\Omega_f(0)} \langle u_f', Q \rangle \\ & + \int_0^T \int_{\partial\Omega_f(t)} \langle u_f', Q \rangle \left\langle V(\Omega_f(t)), n_{\partial\Omega_f(t)} \right\rangle . \end{aligned} \tag{5.3.14}$$

By substituting the initial conditions from (4.5.1) and (5.3.12), and the boundary condition $u_f'|_{\Gamma_f} = 0$ we obtain,

$$\begin{aligned} \int_0^T \int_{\Omega_f(t)} \left\langle -\frac{\partial Q}{\partial t}, u_f \right\rangle = & \int_0^T \int_{\Omega_f(t)} \left\langle \frac{\partial u_f'}{\partial t}, Q \right\rangle \\ & - \int_0^T \int_{\Gamma_i(t)} \langle u_f', Q \rangle \left\langle V(\Omega_f(t)), n \right\rangle . \end{aligned} \tag{5.3.15}$$

For the remaining terms in (5.3.13) we apply (5.3.4) and obtain,

$$\int_0^T \int_{\Omega_f(t)} \langle u_f', -(DQ)u_f + (Du_f)^* Q - \operatorname{Div}(\sigma_f(P,Q)) \rangle - p' \operatorname{div}(Q)$$
$$= \int_0^T \int_{\Omega_f(t)} \left\langle (Du_f')u_f + (Du_f)u_f', Q \right\rangle + \sigma_f(p', u_f')..DQ - (\operatorname{div} u_f')P$$
$$+ \int_0^T \int_{\Gamma_i(t)} \langle u_f', \sigma_f(P,Q)n \rangle. \tag{5.3.16}$$

Taking (5.3.15) and (5.3.16) together, and recalling that the speed which builds the boundary $\Gamma_i(t)$ is $V(\Omega_f(t)) = \dfrac{\partial \widehat{\varphi}}{\partial t} \circ \varphi$, we obtain,

$$\begin{aligned}
&\int_0^T \int_{\Omega_f(t)} \left\langle v_2|_{\Omega_f(t)}, (u_f'(v_1), p'(v_1))\right\rangle \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle \frac{\partial u_f'}{\partial t}, Q \right\rangle - (\operatorname{div} u_f')P \\
&+ \int_0^T \int_{\Omega_f(t)} \left\langle (Du_f')u_f + (Du_f)u_f', Q\right\rangle + \sigma_f(p', u_f')..DQ \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle u_f', \sigma_f(P,Q)n - \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q \right\rangle \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle \frac{\partial u_f'}{\partial t} - \operatorname{Div}(\sigma_f(p', u_f')), Q \right\rangle - (\operatorname{div} u_f')P \\
&+ \int_0^T \int_{\Omega_f(t)} \left\langle (Du_f')u_f + (Du_f)u_f', Q\right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle u_f', \sigma_f(P,Q)n - \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q \right\rangle \\
&- \int_0^T \int_{\Gamma_i(t)} \left\langle Q, \sigma_f(p', u_f')n\right\rangle .
\end{aligned} \tag{5.3.17}$$

Next we work with the second term on the lefthand side of (5.3.11).

$$\begin{aligned}
&\int_0^T \int_{\Gamma_i(t)} \left\langle v_2|_{\Gamma_i(t)}, (\widehat{\varphi}' \circ \varphi)(v_1)\right\rangle \\
&= \int_0^T \int_{\Gamma_i(t)} \left\langle \overline{\sigma_e'(R)}n + (Du_f)^*\sigma_f(P,Q)n - \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle (Du_f)^*Q, \widehat{\varphi}' \circ \varphi \right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle \mathcal{B}_A(R) - \mathcal{K}_{\Gamma_i(t)} \left[\left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q - \sigma_f(P,Q)n \right], \widehat{\varphi}' \circ \varphi \right\rangle .
\end{aligned} \tag{5.3.18}$$

We integrate by parts using (5.2.1) and obtain that (5.3.18) is equivalent to

$$\begin{aligned}
&\int_0^T \int_{\Omega_e(t)} \left\langle \operatorname{Div} \overline{\sigma_e'(R)}, \widehat{\varphi}' \circ \varphi \right\rangle - \left\langle R, \operatorname{Div} \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} \right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle R, \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n \right\rangle + \left\langle (Du_f)^*\sigma_f(P,Q)n - \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle (Du_f)^*Q, \widehat{\varphi}' \circ \varphi \right\rangle
\end{aligned}$$

$$+\int_0^T\int_{\Gamma_i(t)}\left\langle \mathcal{B}_A(R)-\mathcal{K}_{\Gamma_i(t)}\left[\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle Q-\sigma_f(P,Q)n\right],\widehat{\varphi}'\circ\varphi\right\rangle. \tag{5.3.19}$$

Next, we substitute the structure domain equation from (5.3.12). We also apply (5.3.3) and initial conditions from (4.5.1) and (5.3.12) to obtain that (5.3.19) is equivalent to

$$\begin{aligned}
&\int_0^T\int_{\Omega_e(t)}\left\langle \rho_e\left(\frac{\partial^2\widehat{R}}{\partial t^2}\right)\circ\varphi,\widehat{\varphi}'\circ\varphi\right\rangle-\left\langle R,\operatorname{Div}\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}\right\rangle\\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle R,\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}n\right\rangle+\left\langle (Du_f)^*\sigma_f(P,Q)n-\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle(Du_f)^*Q,\widehat{\varphi}'\circ\varphi\right\rangle\\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle \mathcal{B}_A(R)-\mathcal{K}_{\Gamma_i(t)}\left[\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle Q-\sigma_f(P,Q)n\right],\widehat{\varphi}'\circ\varphi\right\rangle,\\
&=\int_0^T\int_{\Omega_e(t)}\left\langle R,\rho_e\left(\frac{\partial^2\widehat{\varphi}_e'}{\partial t^2}\right)\circ\varphi-\operatorname{Div}\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}\right\rangle\\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle R,\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}n\right\rangle+\left\langle (Du_f)^*\sigma_f(P,Q)n-\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle(Du_f)^*Q,\widehat{\varphi}'\circ\varphi\right\rangle\\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle \mathcal{B}_A(R)-\mathcal{K}_{\Gamma_i(t)}\left[\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle Q-\sigma_f(P,Q)n\right],\widehat{\varphi}'\circ\varphi\right\rangle.
\end{aligned}\tag{5.3.20}$$

Next, we apply Lemma 5.2.3 and Lemma 5.3.2 and initial conditions from (4.5.1) and (5.3.12) to obtain that 5.3.20 is equivalent to,

$$\begin{aligned}
&\int_0^T\int_{\Omega_e(t)}\left\langle R,\rho_e\left(\frac{\partial^2\widehat{\varphi}_e'}{\partial t^2}\right)\circ\varphi-\operatorname{Div}\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}\right\rangle\\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle R,\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}n-\mathcal{B}(\widehat{\varphi}'\circ\varphi)\right\rangle\\
&+\int_0^T\int_{\Gamma_i(t)}\left\langle\left\langle\frac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle Q-\sigma_f(P,Q)n,\frac{\partial\widehat{\varphi}'}{\partial t}\circ\varphi-(Du_f)(\widehat{\varphi}'\circ\varphi)\right\rangle.
\end{aligned}\tag{5.3.21}$$

Finally, we substitute the linearized sticking condition from (4.5.1) and obtain,

$$\begin{aligned}
&\int_0^T\int_{\Gamma_i(t)}\left\langle v_2|_{\Gamma_i(t)},(\widehat{\varphi}'\circ\varphi)(v_1)\right\rangle\\
&=\int_0^T\int_{\Omega_e(t)}\left\langle R,\rho_e\left(\frac{\partial^2\widehat{\varphi}_e'}{\partial t^2}\right)\circ\varphi-\operatorname{Div}\overline{\sigma_e'(\widehat{\varphi}'\circ\varphi)}\right\rangle
\end{aligned}$$

$$+\int_0^T \int_{\Gamma_i(t)} \left\langle R, \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n - \mathcal{B}(\widehat{\varphi}' \circ \varphi) \right\rangle$$
$$+\int_0^T \int_{\Gamma_i(t)} \left\langle \left\langle \frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi, n \right\rangle Q - \sigma_f(P,Q)n, u_f' \right\rangle. \tag{5.3.22}$$

The final step is to add (5.3.17) and (5.3.22). We substitute the domain equations from (4.5.1) (including the divergence free condition) and we obtain

$$\int_0^T \int_{\Omega_f(t)} \langle v_2|_{\Omega_f(t)}, (u_f'(v_1), p'(v_1)) \rangle + \int_0^T \int_{\Gamma_i(t)} \langle v_2|_{\Gamma_i(t)}, (\widehat{\varphi}' \circ \varphi)(v_1) \rangle$$
$$= \int_0^T \int_{\Omega_f(t)} \langle Q, v_1|_{\Omega_f(t)} \rangle \int_0^T \int_{\Omega_e(t)} + \langle R, v_1|_{\Omega_e(t)} \rangle$$
$$+ \int_0^T \int_{\Gamma_i(t)} \left\langle R, \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n - \mathcal{B}(\widehat{\varphi}' \circ \varphi) \right\rangle - \langle \sigma_f(p', u_f')n, Q \rangle$$
$$= \int_0^T \int_{\Omega_f(t)} \langle Q, v_1|_{\Omega_f(t)} \rangle \int_0^T \int_{\Omega_e(t)} + \langle R, v_1|_{\Omega_e(t)} \rangle ,$$

where the last step entails substituting the linearized stress matching boundary condition from (4.5.1) and the matching boundary condition $R = Q$ on $\Gamma_i(t)$ from (5.3.12). This completes the proof.

□

With Lemma 5.3.4, we can derive a set of first order optimality conditions for the dynamical case. As in Theorem 5.2.5, deriving the set of first order optimality conditions will require integration by parts on the inner product $(v, v')_{\mathcal{E}(0,T;\Omega)}$, which comes from the control norm. For simplicity of computation, we consider a relaxation of the regularity requirements of Theorem 3.2.2, and derive first order optimality conditions using the following control norm:

$$\tilde{\mathcal{E}}(0,T;\Omega) := \sum_{n=0}^{3} \|\partial_t^n v\|^2_{L^2(0,T;H^1(\Omega))}.$$

Theorem 5.3.5. *[First Order Optimality Conditions for Dynamic Optimal Control Prob-*

lem] Let $(\overline{v};(\overline{u},\overline{p}),\overline{\widehat{\varphi}})$ *be an optimal pair for the optimal control problem:*

$$\min J(v,u_f)=\frac{1}{2}\int_0^T\int_{\Omega_f(t)}|\nabla\times u_f|^2+\frac{1}{2}\|v\|^2_{\tilde{\mathcal{E}}(0,T;\Omega)}. \tag{5.3.23}$$

subject to (2.5.2). *Then the following system is satisfied:*

$$\begin{cases}\sum_{k=0}^3(-1)^k\partial_t^{2k}(\overline{v}-\Delta\overline{v})\big|_{\Omega_f}+(Q,P)(\nabla\times\nabla\times\overline{u}_f)=0 & \Omega_f(t),\\ \sum_{k=0}^3(-1)^k\partial_t^{2k}(\overline{v}-\Delta\overline{v})\big|_{\Omega_e} & \\ +R\left(-\frac{1}{2}|\nabla\times\overline{u}_f|^2n+\{(D\overline{u}_f)^*+\mathcal{K}_{\Gamma_i(t)}\}[n\times\nabla\times\overline{u}_f]\right)=0 & \Omega_e(t),\\ [D\overline{v}-D(\partial_t^2\overline{v})+D(\partial_t^4\overline{v})-D(\partial_t^6\overline{v})]n_{\Gamma_f}=0 & \Gamma_f,\\ \partial_t^k\overline{v}(0)=\partial_t^k\overline{v}(T)=0,k=\overline{1,5}, & \end{cases} \tag{5.3.24}$$

where $\left((Q,P)(\nabla\times\nabla\times\overline{u}_f),R\left(-\frac{1}{2}|\nabla\times\overline{u}_f|^2n+\{(D\overline{u}_f)^*+\mathcal{K}_{\Gamma_i(t)}\}[n\times\nabla\times\overline{u}_f]\right)\right)$ *is the adjoint state that is the solution of the linearized adjoint*

$$\begin{cases}-\dfrac{\partial Q}{\partial t}-(DQ)u_f+(Du_f)^*Q-Div(\sigma_f(P,Q)) & \\ =\nabla\times\nabla\times\overline{u}_f & \Omega_f(t)\\ div(Q)=0 & \Omega_f(t)\\ \rho_e\left(\dfrac{\partial^2\widehat{R}}{\partial t^2}\right)\circ\varphi-Div\,\overline{\sigma_e'(R)}=0 & \Omega_e(t)\\ Q=R & \Gamma_i(t)\\ \overline{\sigma_e'(R)}n+(Du_f)^*\sigma_f(P,Q)n & \\ -\left\langle\dfrac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle(Du_f)^*Q+\mathcal{B}_A(R) & \\ -\mathcal{K}_{\Gamma_i(t)}\left[\left\langle\dfrac{\partial\widehat{\varphi}}{\partial t}\circ\varphi,n\right\rangle Q-\sigma_f(P,Q)n\right] & \\ =-\frac{1}{2}|\nabla\times\overline{u}_f|^2n+\{(D\overline{u}_f)^*+\mathcal{K}_{\Gamma_i(t)}\}[n\times\nabla\times\overline{u}_f] & \Gamma_i(t)\\ Q=0 & \Gamma_f\\ \widehat{R}(T)=0;\dfrac{\partial\widehat{R}}{\partial t}(T)=0;Q(T)=0;P(T)=0 & (\widehat{\Omega}_e)^2\times(\Omega_f)^2\end{cases} \tag{5.3.25}$$

with

$$
\begin{aligned}
\mathcal{B}_A(R) = &- div_{\Gamma_i}(B_1 R)n - \langle B_2, R\rangle n - (D\sigma_e^{\Delta}.n)^* R \\
&- Div_{\Gamma_i}(n \otimes \sigma_e R) + \nabla_{\Gamma_i}\langle \sigma_e n, R\rangle, \\
\mathcal{K}_{\Gamma_i(t)}[n \times \nabla \times \overline{u}_f] = &\frac{\partial (n \times \nabla \times \overline{u}_f)}{\partial t} + (D(n \times \nabla \times \overline{u}_f))\left(\frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi\right) \\
&+ (n \times \nabla \times \overline{u}_f) div_{\Gamma_i(t)}\left(\frac{\partial \widehat{\varphi}}{\partial t} \circ \varphi\right).
\end{aligned}
$$

Proof. Let Let $(\overline{v}; (\overline{u}, \overline{p}), \overline{\widehat{\varphi}})$ be an optimal pair. Recall the computation of Gâteaux derivative of J in the direction v', (4.4.4). We have,

$$
\begin{aligned}
\partial J(v; v') = &\int_0^T \int_{\Omega_f(t)} \langle \nabla \times u_f, \nabla \times u_f' \rangle \\
&- \int_0^T \int_{\Gamma_i(t)} \left\langle \widehat{\varphi}' \circ \varphi, \frac{1}{2}|\nabla \times u_f|^2 n \right\rangle + (v, v')_{\tilde{\mathcal{E}}(0,T;\Omega)}.
\end{aligned} \tag{5.3.26}
$$

The goal is to use Lemma (5.3.4) which requires further integration by parts in the first term of (5.3.26).

$$
\begin{aligned}
&\int_0^T \int_{\Omega_f(t)} \langle \nabla \times u_f, \nabla \times u_f' \rangle \\
&= \int_0^T \int_{\Omega_f(t)} \langle \nabla \times \nabla \times u_f, u_f' \rangle + \int_0^T \int_{\partial\Omega_f(t)} \langle \nabla \times u_f, u_f' \times n_{\partial\Omega_f} \rangle \\
&= \int_0^T \int_{\Omega_f(t)} \langle \nabla \times \nabla \times u_f, u_f' \rangle - \int_0^T \int_{\Gamma_i(t)} \langle \nabla \times u_f, u_f' \times n \rangle \\
&= \int_0^T \int_{\Omega_f(t)} \langle \nabla \times \nabla \times u_f, u_f' \rangle - \int_0^T \int_{\Gamma_i(t)} \langle n \times \nabla \times u_f, u_f' \rangle,
\end{aligned} \tag{5.3.27}
$$

where the last step involves taking advantage of the symmetries of the triple scalar product. Substituting the linearized sticking boundary condition from (4.5.1) and applying Lemma 5.3.1 we obtain,

$$
- \int_0^T \int_{\Gamma_i(t)} \langle n \times \nabla \times u_f, u_f' \rangle
$$

$$
\begin{aligned}
&= -\int_0^T \int_{\Gamma_i(t)} \left\langle n \times \nabla \times u_f, \left(\frac{\partial \widehat{\varphi}'}{\partial t}\right) \circ \varphi - (Du_f)^*(\widehat{\varphi}' \circ \varphi) \right\rangle, \\
&= \int_0^T \int_{\Gamma_i(t)} \langle (Du_f)^*[n \times \nabla \times u_f], \widehat{\varphi}' \circ \varphi \rangle - \left\langle n \times \nabla \times u_f, \left(\frac{\partial \widehat{\varphi}'}{\partial t}\right) \circ \varphi \right\rangle, \\
&= \int_0^T \int_{\Gamma_i(t)} \left\langle \{(Du_f)^* + \mathcal{K}_{\Gamma_i(t)}\}[n \times \nabla \times u_f], \widehat{\varphi}' \circ \varphi \right\rangle \\
&- \int_{\Gamma_i(T)} \langle n \times \nabla \times u_f, \widehat{\varphi}' \circ \varphi \rangle. \qquad (5.3.28)
\end{aligned}
$$

Now we can apply Lemma 5.3.4 and obtain,

$$
\begin{aligned}
\partial J(v; v') &= \int_0^T \int_{\Omega_f(t)} \left\langle \nabla \times u_f, \nabla \times u_f' \right\rangle \\
&- \int_0^T \int_{\Gamma_i(t)} \left\langle \widehat{\varphi}' \circ \varphi, \frac{1}{2}|\nabla \times u_f|^2 n \right\rangle + (v, v')_{\tilde{\mathcal{E}}(0,T;\Omega)}, \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle \nabla \times \nabla \times u_f, u_f' \right\rangle - \int_0^T \int_{\Gamma_i(t)} \langle n \times \nabla \times u_f, u_f' \rangle \\
&- \int_0^T \int_{\Gamma_i(t)} \left\langle \widehat{\varphi}' \circ \varphi, \frac{1}{2}|\nabla \times u_f|^2 n \right\rangle + (v, v')_{\tilde{\mathcal{E}}(0,T;\Omega)}, \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle \nabla \times \nabla \times u_f, u_f' \right\rangle \\
&+ \int_0^T \int_{\Gamma_i(t)} \left\langle \widehat{\varphi}' \circ \varphi, \{(Du_f)^* + \mathcal{K}_{\Gamma_i(t)}\}[n \times \nabla \times u_f] \right\rangle \\
&- \int_0^T \int_{\Gamma_i(t)} \left\langle \widehat{\varphi}' \circ \varphi, \frac{1}{2}|\nabla \times u_f|^2 n \right\rangle \\
&- \int_{\Gamma_i(T)} \langle n \times \nabla \times u_f, \widehat{\varphi}' \circ \varphi \rangle + (v, v')_{\tilde{\mathcal{E}}(0,T;\Omega)}, \\
&= \int_0^T \int_{\Omega_f(t)} \left\langle (Q, P)(\nabla \times \nabla \times u_f), v'|_{\Omega_f(t)} \right\rangle \\
&+ \int_0^T \int_{\Omega_e(t)} \left\langle R\left(-\frac{1}{2}|\nabla \times u_f|^2 n + \{(Du_f)^* + \mathcal{K}_{\Gamma_i(t)}\}[n \times \nabla \times u_f]\right), v'|_{\Omega_e(t)} \right\rangle \\
&+ (v, v')_{\tilde{\mathcal{E}}(0,T;\Omega)},
\end{aligned}
$$

where we have also used the initial condition $\widehat{R}(T) = R(T) = 0$.

Next, we expand the term $(v,v')_{\tilde{\mathcal{E}}(0,T;\Omega)}$. We have,

$$(v,v')_{\tilde{\mathcal{E}}(0,T;\Omega)} := \sum_{n=0}^{3}\int_0^T\int_\Omega \langle \partial_t^n v, \partial_t^n v'\rangle + D(\partial_t^n v)..D(\partial_t^n v'). \tag{5.3.29}$$

Integrating by parts in time we obtain,

$$\begin{aligned}(v,v')_{\tilde{\mathcal{E}}(0,T;\Omega)} = &\int_\Omega [\langle \partial_t^3 v, \partial_t^2 v'\rangle + D(\partial_t^3 v)..D(\partial_t^2 v') - \langle \partial_t^4 v, \partial_t v'\rangle - D(\partial_t^4 v)..D(\partial_t v')\\ &+ \langle \partial_t^5 v, v'\rangle + D(\partial_t^5 v)..Dv' + \langle \partial_t^2 v, \partial_t v'\rangle + D(\partial_t^2 v)..D(\partial_t v') - \langle \partial_t^3 v, v'\rangle\\ &- D(\partial_t^3 v)..Dv' + \langle \partial_t v, v'\rangle + D(\partial_t v)..Dv']_0^T\\ &+ \int_0^T\int_\Omega \langle v - \partial_t^2 v + \partial_t^4 v - \partial_t^6 v, v'\rangle + D(v - \partial_t^2 v + \partial_t^4 v - \partial_t^6 v)..Dv'.\end{aligned}$$

Then an application of (5.2.3) yields,

$$\begin{aligned}(v,v')_{\tilde{\mathcal{E}}(0,T;\Omega)} = &\int_\Omega [\langle \partial_t^3 v, \partial_t^2 v'\rangle + D(\partial_t^3 v)..D(\partial_t^2 v') - \langle \partial_t^4 v, \partial_t v'\rangle - D(\partial_t^4 v)..D(\partial_t v')\\ &+ \langle \partial_t^5 v, v'\rangle + D(\partial_t^5 v)..Dv' + \langle \partial_t^2 v, \partial_t v'\rangle + D(\partial_t^2 v)..D(\partial_t v') - \langle \partial_t^3 v, v'\rangle\\ &- D(\partial_t^3 v)..Dv' + \langle \partial_t v, v'\rangle + D(\partial_t v)..Dv']_0^T\\ &+ \int_0^T \sum_{k=0}^{3}(-1)^k\left[\int_\Omega \langle \partial_t^{2k}(v - \Delta v), v'\rangle + \int_{\partial\Omega}\langle D(\partial_t^{2k} v)n_\Omega, v'\rangle\right].\end{aligned} \tag{5.3.30}$$

Since J attains it's minimum at $\overline{v}$, we have that $\partial J(\overline{v}; v') = 0$ for all $v' \in \tilde{\mathcal{E}}(0,T;\Omega)$, and by varying spaces for v' we recover the system (5.3.24).

□

The derivation of the adjoint system and Theorem 5.3.5 also yields the following corollary which is relevant to subsequent numerical investigations, as the explicit representation of the gradient of the cost functional provides directions for descent.

Corollary 5.3.6. *For the vorticity minimization problem, the gradient of cost functional*

J is given by

$$J'(v; v') = (v', v)_{\mathcal{E}(0,T;\Omega)} + (v'|_{\Omega_f(t)}, (Q, P)) + (v'|_{\Omega_e(t)}, R), \tag{5.3.31}$$

where $((Q, P), R)$ solve the linear adjoint problem (5.3.25).

Corollary 5.3.6 is formal, in the sense that it is subject to a well-posedness investigation of the adjoint system (5.3.12).

Chapter 6

Well Posedness

6.1 Introduction

We turn our attention to the question of existence and uniqueness of solutions to the linearized and linear adjoint systems. The derivation of the linear adjoint equations is a formal process; we assumed that the functions under consideration possessed the necessary regularity to justify integration by parts and other operations. In this section we provide the technical justification in the form of well-posedness analysis. We focus here on the steady state framework, in which we have access to powerful results such as Lax-Milgram and Babuška-Brezzi Theorems for existence and uniqueness of solutions to linear PDE systems. We start by considering the steady state system 'at rest' and use the results obtained there to deal with with the more general steady state case.

6.2 Steady State Around 'Rest'

6.2.1 Linearization around 'rest'

We start with a consideration of the homogeneous, linearized steady state model 'around rest', which is to say let $u_f = 0$ in Ω_f. The resulting linearization is as follows:

$$
\begin{cases}
-\mathrm{Div}\,\sigma_f(p', u'_f) = 0 & \Omega_f \\
\mathrm{div}\, u'_f = 0 & \Omega_f \\
-\mathrm{Div}\,\left\{\overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}\right\} = 0 & \Omega_e \\
u'_f = 0 & \Gamma_i \\
\overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}n = \sigma_f(p', u'_f)n + \mathcal{B}^{\mathrm{rest}}(\widehat{\varphi}' \circ \varphi) & \Gamma_i \\
u'_f = 0 & \Gamma_f
\end{cases}
\tag{6.2.1}
$$

where

$$
\begin{aligned}
\mathcal{B}^{\mathrm{rest}}(\widehat{\varphi}' \circ \varphi) :=& [\sigma_e(\varphi) + pI] \cdot \nabla_{\Gamma_i}\langle \widehat{\varphi}' \circ \varphi, n\rangle + (D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n + (\mathrm{div}\,_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n \\
&- \sigma_e(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n + \langle(\widehat{\varphi}' \circ \varphi), n\rangle \left(Hpn + \langle \nabla p, n\rangle n\right).
\end{aligned}
$$

Remark 6.2.1. *Observe that the in the fluid domain the equations are exactly the incompressible Stokes equations $-\Delta u'_f + \nabla p' = v'_f$, $div u'_f = 0$ in Ω_f.*

The idea is to model the argumentation after that given in [13] to show existence and uniqueness of solution for (6.2.1), and then to carry that argumentation over to the more general case.

Before presenting our main result we observe that there is readily a simplification of the boundary operator $\mathcal{B}^{\mathrm{rest}}$. Recall the identity from [20, Ch. 9 (5.23)] : $\nabla_{\Gamma_i}\langle \widehat{\varphi}' \circ \varphi, n\rangle = (D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n + D^2 b_{\Omega_e}(\widehat{\varphi}' \circ \varphi)_{\Gamma_i}$, so

$$
\begin{aligned}
&[\sigma_e(\varphi) + pI] \cdot \nabla_{\Gamma_i}\langle \widehat{\varphi}' \circ \varphi, n\rangle - \sigma_e(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n \\
=&[\sigma_e(\varphi) + pI] D^2 b_{\Omega_e}(\widehat{\varphi}' \circ \varphi)_{\Gamma_i} + p(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n.
\end{aligned}
$$

Consequently we will write

$$
\begin{aligned}
\mathcal{B}^{\mathrm{rest}}(\widehat{\varphi}' \circ \varphi) :=& [\sigma_e(\varphi) + pI] D^2 b_{\Omega_e}(\widehat{\varphi}' \circ \varphi)_{\Gamma_i} + p(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n \\
&+ (D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n + (\mathrm{div}\,_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n + \langle \widehat{\varphi}' \circ \varphi, n\rangle \left(Hpn + \langle \nabla p, n\rangle n\right), \\
&= B_1^{\mathrm{rest}}(\varphi, p) D^2 b_{\Omega_e}(\widehat{\varphi}' \circ \varphi)_{\Gamma_i} + p(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n \\
&+ (D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n + (\mathrm{div}\,_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n + \langle \widehat{\varphi}' \circ \varphi, n\rangle B_2^{\mathrm{rest}}(p),
\end{aligned}
$$

where $B_1^{\text{rest}}(\varphi, p) := \sigma_e(\varphi) + pI$ is a symmetric matrix such that $B_1^{\text{rest}}(\varphi, p)n = 0$ (owing to the original nonlinear system taken "at rest"), and $B_2^{\text{rest}} := Hpn + \langle \nabla p, n \rangle n$.

We now state our main result.

Theorem 6.2.2. *There exists ρ^* such that $\forall \rho < \rho^*$ and $(\widehat{\varphi}, p) \in B_\rho = \{(\widehat{\varphi}, p) \in C^2(\overline{\widehat{\Omega}_e}) \times C^1(\overline{\Omega_f}) | \|\widehat{\varphi}\|_{C^2(\overline{\widehat{\Omega}_e})} \leq \rho, \|\widehat{p}\|_{C^1(\overline{\Omega_f})} \leq \rho\}$ the solution to (2.5.1) with $u_f = 0$, there exists a unique solution $(u_f', p', \widehat{\varphi}' \circ \varphi) \in H^1(\Omega_f) \times L^2(\Omega_f) \times H^1(\Omega_e)$ to (6.2.1).*

Proof of Theorem 6.2.2. The key observation for the proof is that the linearized system (6.2.1) is only weakly coupled. Therefore we will first consider the Stokes system, and pass that solution as boundary data to the elasticity system.

The Stokes System

We consider the component

$$\begin{cases} -\operatorname{Div} \sigma_f(p', u_f') = 0 & \Omega_f \\ \operatorname{div} u_f' = 0 & \Omega_f \\ u_f' = 0 & \partial\Omega_f. \end{cases} \tag{6.2.2}$$

The fluid subsystem is precisely the linear Stokes system with homogenous Dirichlet boundary conditions. Existence of a unique solution $(u_f', p') \in H^1(\Omega_f) \times L_2(\Omega_f)$ is well established, for example in [50, Theorem 2.1].

We require a discussion of the traces of components of $H^1(\Omega_f)$ to the interface Γ_i. We introduce the space

$$H_{\Gamma_f}(\Omega_f) = \{\theta \in H^1(\Omega_f) | \theta = 0 \text{ on } \Gamma_f\}. \tag{6.2.3}$$

Observe that the linearized fluid velocity $u_f' \in H_{\Gamma_f}(\Omega_f)$. This motivates the following lemma.

Lemma 6.2.3. *The traces of elements of $H^1_{\Gamma_f}(\Omega_f)$ describe the linear space $H^{1/2}(\Gamma_i)$, and there exists a linear mapping*

$$T_{\Gamma_i} \in \mathcal{L}(H^{1/2}(\Gamma_i), H^1_{\Gamma_f}(\Omega_f)) \text{ s.t. } \forall \theta \in H^{1/2}(\Gamma_i)\,, (T_{\Gamma_i}\theta)|_{\partial\Omega_f} = \theta.$$

Proof. Let $\theta \in H^{1/2}(\Gamma_i)$, and define $\zeta = T_{\Gamma_i}\theta$ as the solution of the Laplace problem with Dirichlet boundary condition:

$$\begin{cases} -\Delta\zeta = 0 & \Omega_f \\ \zeta = \tilde{\theta} & \partial\Omega_f, \end{cases} \tag{6.2.4}$$

where $\tilde{\theta} \in H^{1/2}(\partial\Omega_f)$ is the extension of θ outside of Γ_i by zero.

□

Lemma 6.2.3 then offers a weak characterization of this boundary element $-2\varepsilon(u'_f)n + p'n \in H^{1/2}(\Gamma_i)'$:

$$\langle -2\varepsilon(u'_f)n + p'n, \theta\rangle_{H^{1/2}(\Gamma_i)' \times H^{1/2}(\Gamma_i)} = \int_{\Omega_f} 2\varepsilon(u'_f)..\varepsilon(T_{\Gamma_i}\theta) - p'\mathrm{div}\,(T_{\Gamma_i}\theta). \tag{6.2.5}$$

The Elasticity System

We turn our attention to the elasticity subsystem. What remains is to show existence of a unique solution $\widehat{\varphi}' \circ \varphi \in H^1(\Omega_e)$ for the elastic component of (6.2.1). We have the following system:

$$\begin{cases} -\mathrm{Div}\,\left\{\overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}\right\} = 0 & \Omega_e \\ \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}n + \mathcal{B}^{\mathrm{rest}}(\widehat{\varphi}' \circ \varphi) = \sigma_f(p', u'_f)n & \Gamma_i \end{cases} \tag{6.2.6}$$

We observe that the system is linear-elliptic and has a Fourier-type boundary conditions. The goal is to cast the system in the framework of the Lax-Millgram Theorem in order to show existence and uniqueness of solution. There are four main steps in the proof.

1. Build a bilinear form $A_{\varphi,p}$ on $H^1(\Omega_e)$ in order to write (6.2.6) in variational form.
2. Transport the bilinear form $A_{\varphi,p}$ to the fixed-domain. I.e. find $a_{\widehat{\varphi},\widehat{p}} = A_{\varphi,p} \circ \widehat{\varphi}$, a nonsymmetric, bilinear form on $H^1(\widehat{\Omega}_e)$.
3. Show that the mapping $(\widehat{\varphi}, \widehat{p}) \to a_{\widehat{\varphi},\widehat{p}}(\cdot,\cdot)$ is equicontinuous.
4. Show $a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\varphi}')$ is coercive.

Step 1:

Define the following bilinear forms $\forall \widehat{\varphi}' \circ \varphi, \Theta \in H^1(\Omega_e)$:

$$A^0_{\varphi,p}(\widehat{\varphi}' \circ \varphi, \Theta) := \int_{\Omega_e} \left\langle -\mathrm{Div}\left\{\overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}\right\}, \Theta \right\rangle + \int_{\Gamma_i} \langle \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)} n, \Theta\rangle \tag{6.2.7}$$

and

$$\begin{aligned} A_{\varphi,p}(\widehat{\varphi}' \circ \varphi, \Theta) :=& A^0_{\varphi,p}(\widehat{\varphi}' \circ \varphi, \Theta) - \int_{\Gamma_i} \langle \mathcal{B}^{\mathrm{rest}}(\widehat{\varphi}' \circ \varphi), \Theta\rangle, \\ :=& A^0_{\varphi,p}(\widehat{\varphi}' \circ \varphi, \Theta) \\ & - \int_{\Gamma_i} \langle B^{\mathrm{rest}}_1(\varphi,p) D^2 b_{\Omega_e}(\widehat{\varphi}' \circ \varphi)_{\Gamma_i} + p(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n, \Theta\rangle \\ & + \int_{\Gamma_i} \langle -(D_{\Gamma_i}\sigma_e.(\widehat{\varphi}' \circ \varphi))n - (\mathrm{div}_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n, \Theta\rangle \\ & - \int_{\Gamma_i} \langle \langle \widehat{\varphi}' \circ \varphi, n\rangle B^{\mathrm{rest}}_2(p), \Theta\rangle. \end{aligned} \tag{6.2.8}$$

Then we can reframe the elasticity subproblem (6.2.6) as the variational problem to find $\widehat{\varphi}' \circ \varphi \in H^1(\Omega_e)$ that solves

$$A_{\varphi,p}(\widehat{\varphi}' \circ \varphi, \Theta) = \int_{\Gamma_i} \langle G, \Theta\rangle \tag{6.2.9}$$

for all $\Theta \in H^1(\Omega_e)$, where $G = \sigma_f(p', u'_f)n$ on Γ_i.

Using Lemma 6.2.3 and (6.2.5), we have that (6.2.9) is equivalent to the problem of finding $\widehat{\varphi}' \circ \varphi \in H^1(\Omega_e)$ such that for all $\Theta \in H^1(\Omega_e)$ we have

$$A_{\varphi,p}(\widehat{\varphi}' \circ \varphi, \Theta) = \int_{\Omega_f} 2\varepsilon(u'_f)..\varepsilon(T_{\Gamma_i}\Theta) - p'\mathrm{div}\,(T_{\Gamma_i}\Theta) \tag{6.2.10}$$

The spaces associated to the family of bilinear forms $A_{\varphi,p}$ depends on φ, so the next step is to transport $A_{\varphi,p}$ to the fixed elastic geometry, $\widehat{\Omega}_e$ with $\partial\widehat{\Omega}_e = \widehat{\Gamma}_i$.

Step 2:

First we will consider $a^0_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\Theta}) := A^0_{\varphi,p}(\widehat{\varphi}' \circ \varphi, \Theta) \circ \widehat{\varphi}$. Recall the identity (4.3.15) obtained in the linearization of the general steady state system (2.5.1):

$$\int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}'..D\widehat{\Theta} = \int_{\Omega_e} \left\langle -\text{Div}\left\{\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}\right\}, \Theta \right\rangle + \int_{\Gamma_i} \langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)} n, \Theta \rangle.$$

This directly implies that a representation of $A^0_{\varphi,p}$ on the fixed elastic geometry is

$$a^0_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\Theta}) = \int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}'..D\widehat{\Theta}, \tag{6.2.11}$$

where recall that $\widehat{\mathcal{P}}' = (D\widehat{\varphi}')\Sigma(\widehat{E}(\widehat{\varphi})) + D\widehat{\varphi}\Sigma(\widehat{E}'(\widehat{\varphi}))$ is the linearized Piola stress tensor.

The two remaining terms in $A_{\varphi,p}$ which require some attention when transporting to $\widehat{\Omega}_e$ are $\int_{\Gamma_i} \langle p(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^*, \Theta \rangle$ and $\int_{\Gamma_i} \langle -(\text{div}_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n, \Theta \rangle$ because they involve tangential derivatives of $\widehat{\varphi}' \circ \varphi$; we need formulas for how the composition with $\widehat{\varphi}$ acts on the tangential derivatives.

From [20, (5.25)] we have $D_\Gamma(v \circ g) = D_\Gamma v \circ g D_\Gamma g$ and $n \circ \widehat{\varphi} = \dfrac{(D\widehat{\varphi})^{-*}\widehat{n}}{|(D\widehat{\varphi})^{-*}\widehat{n}|}$, where $\widehat{n}$ is the outer normal to $\widehat{\Omega}_e$.

So by changing variables on the boundary we obtain,

$$\int_{\Gamma_i} \langle p(D_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))^* n, \Theta \rangle = \int_{\widehat{\Gamma}_i} \widehat{p}\widehat{J} \langle (D_{\widehat{\Gamma}_i}\widehat{\varphi})^{-*} D_{\widehat{\Gamma}_i}\widehat{\varphi}'(D\widehat{\varphi})^{-*}\widehat{n}, \widehat{\Theta} \rangle \tag{6.2.12}$$

For the tangential divergence we have the following lemma from [13, eq. (3.4)]:

Lemma 6.2.4.

$$div_\Gamma \phi \circ \widehat{\varphi} = \frac{1}{\widehat{J}} div_\Gamma \left(\widehat{J}(D_\Gamma \widehat{\varphi})^{-1} \widehat{\phi} \right)$$

So,

$$\int_{\Gamma_i} \langle -(\text{div}_{\Gamma_i}(\widehat{\varphi}' \circ \varphi))\sigma_e n, \Theta \rangle = \int_{\widehat{\Gamma}_i} \left\langle -(1/\widehat{J})\text{div}_{\widehat{\Gamma}_i}[\widehat{J}(D_{\widehat{\Gamma}_i}\widehat{\varphi})^{-1}\widehat{\varphi}']\widehat{\mathcal{P}}\widehat{n}, \widehat{\Theta} \right\rangle \tag{6.2.13}$$

Substituting these identities we have,

$$
\begin{aligned}
a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}',\widehat{\Theta}) = &\int_{\widehat{\Omega}_e} \widehat{\mathcal{P}}'..D\widehat{\Theta} \\
&- \int_{\Gamma_i} \omega(\widehat{\varphi})\langle B_1^{\text{rest}}(\varphi,p)\circ\widehat{\varphi}(D^2 b_{\Omega_e}\circ\widehat{\varphi})\widehat{\varphi}'_{\Gamma_i},\widehat{\Theta}\rangle \\
&+ \int_{\Gamma_i} \langle \widehat{p}\widehat{J}\langle (D_{\widehat{\Gamma}_i}\widehat{\varphi})^{-*}D_{\widehat{\Gamma}_i}\widehat{\varphi}'(D\widehat{\varphi})^{-*}\widehat{n} - \widehat{J}[(D_{\Gamma_i}\sigma_e\circ\widehat{\varphi}).\widehat{\varphi}'](D\widehat{\varphi})^{-*}\widehat{n},\widehat{\Theta}\rangle \\
&+ \int_{\Gamma_i} \langle -(1/\widehat{J})\mathrm{div}_{\widehat{\Gamma}_i}[\widehat{J}(D_{\widehat{\Gamma}_i}\widehat{\varphi})^{-1}\widehat{\varphi}']\widehat{\mathcal{P}}\widehat{n},\widehat{\Theta}\rangle \\
&- \int_{\Gamma_i} \widehat{J}\langle\langle\widehat{\varphi}',(D\widehat{\varphi})^{-*}\widehat{n}\rangle(B_2^{\text{rest}}(p)\circ\widehat{\varphi}),\widehat{\Theta}\rangle. \qquad (6.2.14)
\end{aligned}
$$

Step 3:

The next step is to show that the bilinear form $a_{\widehat{\varphi},\widehat{p}}(\cdot,\cdot)$ is continuous for all suitable $(\widehat{\varphi},p)$. We offer the following proposition:

Proposition 6.2.5. *The mapping* $(\widehat{\varphi},p)\to a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}',\widehat{\Theta})$ *is equicontinuous from* B_ρ *into* $\mathbb{R}$ *with respect to* $\widehat{\varphi}',\widehat{\Theta}\in S^1=\{u\in H^1(\Omega_e)|\|u\|_{H^1(\Omega_e)}=1\}$.

Proof. We state several estimates to establish the equicontinuity.

$$\forall\widehat{\varphi}\in B_\rho, \|\widehat{\varphi}\|_{C^2(\widehat{\Omega}_e)}\le\rho, \|\varphi\|_{C^2(\Omega_e)}\le\rho$$

We also have control on the norm of the trace operator via the continuity of the trace,

$$\exists K>0 \text{s.t. } \forall\widehat{\varphi}\in B_\rho, \|\gamma_{\widehat{\Gamma}_i}\|_{\mathcal{L}(H^1(\widehat{\Omega}_e),H^{1/2}(\widehat{\Gamma}_i))}\le K.$$

So for any $\widehat{\theta}\in S^1$ we have

$$\|\widehat{\theta}\|_{H^{1/2}(\widehat{\Gamma}_i)}\le K, \qquad (6.2.15)$$

and

$$\|D_{\widehat{\Gamma}_i}\widehat{\theta}\|_{H^{-1/2}(\widehat{\Gamma}_i)}\le cK. \qquad (6.2.16)$$

With these estimates established, we observe that the integrands in $a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}',\widehat{\Theta})$ depend on two types of terms. The terms which depend on $\widehat{\varphi}$ and $\widehat{p}$ are continuous given the definition of B_ρ. Then there are terms depending on $\widehat{\varphi}'$ and $\widehat{\Theta}$, which are continuous via the established bounds above. Therefore $(\widehat{\varphi}, \widehat{p}) \to a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\Theta})$ is uniformly equicontinuous with respect to $\widehat{\varphi}', \widehat{\Theta} \in S^1$. $\square$

Step 4:

The last step is to establish that the bilinear form $a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\varphi}')$ is coercive. First we consider $a_{0,0}(\widehat{\varphi}', \widehat{\varphi}')$, that is the bilinear form when $p = \widehat{\varphi} = 0$. The only terms in $a_{0,0}(\widehat{\varphi}', \widehat{\varphi}')$ which are nonzero come from $a^0_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\varphi}')$.

We have

$$\begin{aligned} a_{0,0}(\widehat{\varphi}', \widehat{\varphi}') =& a^0_{0,0}(\widehat{\varphi}', \widehat{\varphi}'), \\ =& \int_{\widehat{\Omega}_e} C(\lambda, \mu) D\widehat{\varphi}' .. D\widehat{\varphi}', \end{aligned}$$

where $C(\lambda, \mu) > 0$ depends on the Lamé constants. As a consequence of Korn's inequality [16], there exists $C_0 > 0$ such that $a_{0,0}(\widehat{\varphi}', \widehat{\varphi}') \geq C_0 \|\widehat{\varphi}'\|^2_{H^1(\widehat{\Omega}_e)}$. With the coercivity at $(\widehat{\varphi}, \widehat{p}) = (0, 0)$ established, we define the following mapping.

$$\pi_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}') := \frac{a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\varphi})}{\|\widehat{\varphi}'\|^2_{H^1(\widehat{\Omega}_e)}}$$

In the last step we showed that the mapping $(\widehat{\varphi}, \widehat{p}) \to a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\varphi}')$ is equicontinuous in B_ρ for $\widehat{\varphi}'$ in the $H^1(\Omega_e)$-unit sphere. It follows that the mapping $(\widehat{\varphi}, \widehat{p}) \to \pi_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}')$ is equicontinuous in B_ρ for $\widehat{\varphi} \in H^1(\widehat{\Omega})$. Thus $\inf\limits_{\widehat{\varphi}' \in S^1} \pi_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}')$ is continuous on B_ρ, which is to say there exists ρ^* such that for $(\widehat{\varphi}, \widehat{p}) \in B_\rho$ with $\rho \leq \rho^*$, we have that $\inf_{\widehat{\varphi}' \in S^1} \pi_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}') \geq C_0/2$, which is exactly coercivity of $a_{\widehat{\varphi},\widehat{p}}(\widehat{\varphi}', \widehat{\varphi}')$ for ρ^* small.

We have met the parameters of the Lax-Milgram theorem, so there exists a unique solution $\widehat{\varphi}'$ to (6.2.6), transported to the fixed geometry $\widehat{\Omega}_e$, $\widehat{\Gamma}_i$. The element $\widehat{\varphi}' \circ \varphi$ is the unique solution in $H^1(\Omega_e)$ to (6.2.6). $\square$

6.2.2 Linear adjoint 'around rest'

Next, we consider the homogenous linear adjoint 'at rest' associated to (6.2.1).

$$\begin{cases} -\mathrm{Div}\,\sigma_f(P,Q)=0 & \Omega_f \\ \mathrm{div}\,Q=0 & \Omega_f \\ -\mathrm{Div}\,\overline{\sigma_e'(R)}=0 & \Omega_e \\ Q=R & \Gamma_i \\ \overline{\sigma_e'(R)}n+\mathcal{B}_A^{\mathrm{rest}}(R)=0 & \Gamma_i \\ Q=0 & \Gamma_f \end{cases} \tag{6.2.17}$$

where

$$\begin{aligned}\mathcal{B}_A^{\mathrm{rest}}(R) := & -(D_{\Gamma_i}n)\sigma_e R+\langle\nabla_{\Gamma_i}p,R\rangle n+p(\mathrm{div}_{\Gamma_i}R)n \\ & -[D_{\Gamma_i}\sigma_e^{\Delta}.n]^*R+\nabla_{\Gamma_i}\langle\sigma_e n,R\rangle-\langle\langle\nabla p,n\rangle n,R\rangle n.\end{aligned}$$

We have an analogous result for Theorem 6.2.2 for (6.2.17):
Theorem 6.2.6. *There exists ρ^* such that $\forall\rho<\rho^*$ and $(\widehat{\varphi},p)\in B_\rho=\{(\widehat{\varphi},p)\in C^2(\overline{\widehat{\Omega}_e})\times C^1(\overline{\Omega_f})\,|\,\|\widehat{\varphi}\|_{C^2(\overline{\widehat{\Omega}_e})}\le\rho,\|\widehat{p}\|_{C^1(\overline{\Omega_f})}\le\rho\}$ the solution to (2.5.1) with $u_f=0$, there exists a unique solution $(Q,P,R)\in H^1(\Omega_f)\times L^2(\Omega_f)\times H^1(\Omega_e)$ to (6.2.17).*

The proof of Theorem 6.2.6 bears a great deal of similarity to its analogue in the previous section. The idea is to treat the system as weakly coupled since the second boundary condition depends only on R. So we decouple into the R- and (Q,P)-subsystems:

$$\begin{cases} -\mathrm{Div}\,\overline{\sigma_e'(R)}=0 & \Omega_e \\ Q=R & \Gamma_i \\ \overline{\sigma_e'(R)}n+\mathcal{B}_A^{\mathrm{rest}}(R)=0 & \Gamma_i \end{cases} \tag{6.2.18}$$

$$\begin{cases} -\operatorname{Div}\sigma_f(P,Q) = & \Omega_f \\ \operatorname{div} Q = 0 & \Omega_f \\ Q = R & \Gamma_i \\ Q = 0 & \Gamma_f \end{cases} \tag{6.2.19}$$

First we will solve the R-subsystem (6.2.18) using a similar argument as for the linearization (perturbation of Lax-Milgram). Next we will solve the (Q, P)-subsystem (6.2.19), passing in the solution for R as data.

The R-subsystem

We will carry out the following steps to solve the R-subsystem (where $\widehat{R} := R \circ \widehat{\varphi}$):

Steps:

1. Build a bilinear form $A^{\text{adj}}_{\varphi,p}$ on $H^1(\Omega_e)$ in order to write (6.2.18) in variational form.

2. Transport the bilinear form $A^{\text{adj}}_{\varphi,p}$ to the fixed-domain. That is, we will find $a^{\text{adj}}_{\widehat{\varphi},\widehat{p}} = A^{\text{adj}}_{\varphi,p} \circ \widehat{\varphi}$, a nonsymmetric, bilinear form on $H^1(\widehat{\Omega}_e)$.

3. Show $(\widehat{\varphi}, \widehat{p}) \to a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\cdot,\cdot)$ is equicontinuous.

4. Show $a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R}, \widehat{R})$ is coercive.

Step 1:

Define the following bilinear form $\forall (R, \Theta) \in H^2(\Omega_e)^2$:

$$\begin{aligned} A^{\text{adj}}_{\varphi,p}(R,\Theta) :=& A^0_{\varphi,p}(R,\Theta) + \int_{\Gamma_i} \langle \mathcal{B}^{\text{rest}}_A(R), \Theta\rangle \\ :=& A^0_{\varphi,p}(R,\Theta) + \int_{\Gamma_i} \langle -(D_{\Gamma_i} n)\sigma_e R + \langle \nabla_{\Gamma_i} p, R\rangle n + p(\operatorname{div}_{\Gamma_i} R) n \\ & - [D_{\Gamma_i}\sigma_e^{\Delta}.n]^* R + \nabla_{\Gamma_i}\langle \sigma_e n, R\rangle - \langle\langle \nabla p, n\rangle n, R\rangle n, \Theta\rangle, \end{aligned} \tag{6.2.20}$$

where $A^0_{\varphi,p}(R,\Theta)$ is given by (6.2.7).

Because the spaces on which $A^{\text{adj}}_{\varphi,p}(R,\Theta)$ depends themselves depend on the deformation, we will transport $A^{\text{adj}}_{\varphi,p}(R,\Theta)$ to the fixed domain and boundary, $\widehat{\Omega}_e$ and $\widehat{\Gamma}_i$.

Step 2:

Finding the composition $a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{\Theta}) := A^{\text{adj}}_{\varphi,p}(R,\Theta)\circ\widehat{\varphi}$ again involves Lemma 6.2.4. We can define the bilinear form $a_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{\Theta})$.

$$\begin{aligned}
a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{\Theta}) &:= A^{\text{adj}}_{\varphi,p}(R,\Theta)\circ\widehat{\varphi}\\
&= A^{0}_{\varphi,p}(R,\Theta)\circ\widehat{p} + \int_{\widehat{\Gamma}_i}\langle \mathcal{B}^{\text{rest}}_{A}(R)\circ\widehat{\varphi},\widehat{\Theta}\rangle\omega(\widehat{\varphi})\\
&= \int_{\widehat{\Omega}_e}\mathcal{P}'(\widehat{R})..D\widehat{\Theta} + \int_{\Gamma_i}\langle -(D_{\Gamma_i}n\circ\widehat{\varphi})(\sigma_e\circ\widehat{\varphi})\widehat{R} + \langle\nabla_{\Gamma_i}p\circ\widehat{\varphi},\widehat{R}\rangle(n\circ\widehat{\varphi})\\
&\quad + \int_{\Gamma_i}\widehat{p}(\operatorname{div}_{\Gamma_i}R)(n\circ\widehat{\varphi}) - ([D_{\Gamma_i}\sigma_e^{\Delta}.n]^*\circ\widehat{\varphi})\widehat{R} + (D_{\widehat{\Gamma}_i}\widehat{\varphi})^{-*}\nabla_{\widehat{\Gamma}_i}\langle\sigma_e n\circ\widehat{\varphi},\widehat{R}\rangle\\
&\quad + \int_{\Gamma_i} -\langle\langle\nabla p,n\rangle\circ\widehat{\varphi}(n\circ\widehat{\varphi}),\widehat{R}\rangle(n\widehat{\varphi}),\widehat{\Theta}\rangle\omega(\widehat{\varphi}).
\end{aligned}$$

Step 3:

Proposition 6.2.7. *The mapping $(\widehat{\varphi},p)\to a_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{\Theta})$ is equicontinuous from B_ρ into $\mathbb{R}$ with respect to $\widehat{R},\widehat{\Theta}\in S^1$, where S_1 is defined in Proposition* (6.2.5)*.*

Proof. We have the estimates on $\|\widehat{\varphi}\|_{C^2(\widehat{\Omega}_e)}$, $\|\varphi\|_{C^2(\Omega_e)}$, and the trace operator from the proof of Proposition (6.2.5).

Then, we observe that the integrands in $a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{\Theta})$ depend on two types of terms. The terms which depend on $\widehat{\varphi}$ and $\widehat{p}$ are continuous given the definition of B_ρ. Then there are terms depending on $\widehat{R}$, which are continuous via (6.2.15) and (6.2.16). Therefore $(\widehat{\varphi},\widehat{p})\to a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{\Theta})$ is uniformly continuous with respect to $\widehat{R},\widehat{\Theta}\in S^1$.

□

Step 4:

Proposition 6.2.8. *There exists ρ^* such that the bilinear form $a^{adj}_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{R})$ is coercive on $H^1(\widehat{\Omega}_e)$ for $(\widehat{\varphi},\widehat{p})\in B_\rho$ with $\rho\le\rho^*$.*

Proof. For the coercivity, observe that $a^{\text{adj}}_{0,0}(\widehat{R},\widehat{R}) = a_{0,0}(\widehat{R},\widehat{R})$. Thus, following a similar line of reasoning as for the linearization, we define the following mapping:

$$\pi^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R}) := \frac{a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{R})}{\|\widehat{R}\|^2_{H^1(\widehat{\Omega}_e)}}.$$

In the last step we showed that the mapping $a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{R})$ is equicontinuous in B_ρ for $\widehat{R}$ in the $H^1(\Omega_e)$-unit sphere. It follows that the mapping $(\widehat{\varphi},\widehat{p}) \to \pi^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R})$ is equicontinuous in B_ρ for $\widehat{R} \in H^1(\widehat{\Omega}_e)$. Thus $\inf\limits_{\widehat{R}\in S^1} \pi^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R})$ is continuous on B_ρ, which is to say there exists ρ^* such that for $(\widehat{\varphi},\widehat{p}) \in B_\rho$ with $\rho \leq \rho^*$, we have that $\inf\limits_{\widehat{R}\in S^1} \pi^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R}) \geq C_0/2$, which is exactly coercivity of $a^{\text{adj}}_{\widehat{\varphi},\widehat{p}}(\widehat{R},\widehat{R})$ for ρ^* small.

□

We have met the conditions of the Lax-Milgram theorem, so there exists a unique solution $\widehat{R}$ to (6.2.18), transported to the fixed geometry $\widehat{\Omega}_e$, $\widehat{\Gamma}_i$. The element $R := \widehat{R} \circ \varphi$ is the unique solution in $H^1(\Omega_e)$ to (6.2.18).

The (Q, P)-subsystem

We are left with the (Q, P)-subsystem, (6.2.19), which exactly the linear Stokes system in variables Q and P, with homogenous data on the domain equation and outer boundary Γ_f, and R on the boundary Γ_i. The known solution to the nonhomogenous Stokes system is given by, e.g., [50, Theorem 2.4], and is satisfied by $R \in H^{1/2}(\Gamma_i)$.

6.3 Steady State around Arbitrary Solution

We now generalize the well-posedness analysis to a consideration of the steady state linearization (4.3.5) and adjoint systems (5.2.39) around an arbitrary (albeit with forthcoming conditions imposed) solution to the original nonlinear system (2.5.1) . In both cases, the idea is to adapt the machinery developed in [11] to analyze the elastic component from the perspective of elliptic theory, and to subsequently cast the system in the Babuška-Brezzi framework in order to obtain existence and uniqueness of a weak solution. We will utilize the work from the 'at rest' cases, particularly in establishing the coercivity of the elastic component.

We define the state space:

$$\mathscr{H} := H^1(\Omega_e) \times L^2(\Omega_f) \times \mathcal{H}_f, \tag{6.3.1}$$

where $\mathcal{H}_f$ is the closure in $L^2(\Omega_f)$ of divergence-free functions vanishing on the outer boundary Γ_f:

$$\mathcal{H}_f := \left\{ V \in L^2(\Omega_f) | \operatorname{div} V = 0, \langle V, n\rangle|_{\Gamma_f} = 0 \right\}.$$

We will also again make use of the space $H^1_{\Gamma_f}(\Omega_f) := \{V \in H^1(\Omega_f) | V|_{\Gamma_f} = 0\}$.

6.3.1 Assumptions on the nonlinear steady regime

There are coefficients in (4.3.5) and (5.2.39) which depend on the solution (u_f, p, φ) to the nonlinear system (2.5.1). In order to develop the well-posedness, we need a short list of sufficient conditions on the regularity and size of (u_f, p, φ). The assertion of these conditions is justified by the smoothness of solutions to (2.5.1), as indicated in [27].

Assumption 6.3.1. *[Sufficient Conditions on the Nonlinear Steady Regime] Assume that* (u_f, p, φ)*, the solution to* (2.5.1)*, meets the following criteria.*

(a) Regularity conditions.

(i) u_f *has an extension to* $\overline{\Omega}$*:*

$$\underline{u_f} \in C^1(\overline{\Omega}) \text{ and } \underline{u_f} = u_f \text{ in } \Omega_f. \tag{6.3.2}$$

(ii) The coefficients of the first-order terms in the definition of $\mathcal{B}(V)$*,* (4.3.6)*, and* $\mathcal{B}_A(V)$*,* (5.2.24)*, are multipliers in* $H^{1/2}(\Gamma_i)$*:*

$$(\sigma_e)_{jk}, p, \text{ and } \partial_{x_i}(u_f)_k \text{ are multipliers on } H^{1/2}(\Gamma_i). \tag{6.3.3}$$

Additionally, the coefficients of the zero-order terms in $\mathcal{B}$ *and* $\mathcal{B}_A(V)$ *must be* *in* $L^\infty(\Gamma_i)$.

$$\partial_x p \text{ and } \partial_{x_i}\partial_{x_j}(u_f)_k \text{ belong to } L^\infty(\Gamma_i). \tag{6.3.4}$$

To elaborate on the preceding conditions, note that to be a multiplier on $H^{1/2}(\Gamma_i)$ *it suffices to be a member of* $W^{\frac{1}{2},4}(\Gamma_i) \cap L^\infty(\Gamma_i)$. *Also note that* $W^{\frac{1}{2}+\varepsilon,4}(\Gamma_i) \subset L^\infty(\Gamma_i)$ *for all* $\varepsilon > 0$. *The interior regularity associated to* $W^{\frac{1}{2}+\varepsilon,4}(\Gamma_i)$ *is* $W^{\frac{3}{4}+\varepsilon,4}(\Omega_f$ *or* $\Omega_e)$. *These facts motivate the following regularity assumptions:*

- u_f *is a* $W^{\frac{11}{4}+\varepsilon,4}$ *diffeomorphism,*
- $p \in W^{\frac{7}{4}+\varepsilon,4}(\Omega_f)$,
- φ *is a* $W^{\frac{11}{4}+\varepsilon,4}(\Omega_e)$ *diffeomorphism.*

(iii) Assume that $D^2 b_{\Omega_e} \in L^\infty(\Gamma_i)$ *and that* n *is a multiplier for* $H^{1/2}(\Gamma_i)$. *It suffices to assume that* $\partial\Omega_f = \Gamma_i \cup \Gamma_f$ *is of class* C^2, *and is locally on one side of the fluid domain.*

(b) Smallness conditions.

Let $\boldsymbol{R}$ *denote the regularity space* $W^{\frac{11}{4}+\varepsilon,4}(\Omega_f) \times W^{\frac{7}{4}+\varepsilon,4}(\Omega_f) \times W^{\frac{11}{4}+\varepsilon,4}(\Omega_e)$. *Take* $r := \|(u_f, p, \varphi)\|_{\boldsymbol{R}}$, *and assume that* r *is small enough that there exist constants* $0 < c_1, c_2$ *(depending on* r*) such that any* $\tilde{\tau} \in H^1_{\Gamma_f}(\Omega_f)$ *satisfies:*

(i) $$\left| \int_{\Omega_f} \langle (Du_f)^* \tilde{\tau} - (D\tilde{\tau})u_f, \tilde{\tau} \rangle \right| \leq c_1 \|\tilde{\tau}\|^2_{H^1_{\Gamma_f}(\Omega_f)},$$

(ii) $$\left| \int_{\Omega_f} \langle (D\tilde{\tau})u_f + (Du_f)\tilde{\tau}, \tilde{\tau} \rangle \right| \leq c_2 \|\tilde{\tau}\|^2_{H^1_{\Gamma_f}(\Omega_f)}.$$

Note that the smallness conditions of Assumption 6.3.1 are achieved by requiring small enough u_f and Du_f.

6.3.2 Elliptic theory for the elastic components

We set up a variational framework to analyze Neumann and Dirichlet elliptic boundary value problems associated with the elastic subsystems in (4.3.5) and (5.2.39). The systems described here are similar to those utilized in [11], with adaptations made to conform to the configuration presently of interest, as well as the steady state setting.

Define the operator

$$\boldsymbol{S} = -\mathrm{Div}\,\overline{\sigma_e}'(\cdot). \tag{6.3.5}$$

We consider the following systems for $\boldsymbol{S} : H^2(\Omega_e) \to L^2(\Omega_e)$:

$$\begin{cases} \boldsymbol{S}V = F & \Omega_e \\ \overline{\sigma_e}'(V)n + \mathcal{B}_{\mathrm{gen}}(V) = G \text{ or } V = G & \Gamma_i \end{cases} \tag{6.3.6}$$

Take $\mathcal{B}_{\mathrm{gen}}$ to be a first-order, tangential boundary operator with the property that the transformation $V \mapsto \mathcal{B}_{\mathrm{gen}}(V)$ continuously extends from $H^2(\Omega_e) \to L^2(\Gamma_i)$ to a bounded linear mapping $H^1(\Omega_e) \to H^{-1/2}(\Gamma_i)$.

We define the following bilinear form associated to $\boldsymbol{S}$:

$$a_{\boldsymbol{S}}^{\mathrm{gen}}(V, W) := \int_{\Omega_e} \overline{\sigma_e}'(V)..DW + \{\mathcal{B}_{\mathrm{gen}}(V), W\}_{\frac{1}{2},\Gamma_i}. \tag{6.3.7}$$

It is of interest to show that $a_{\boldsymbol{S}}^{\mathrm{gen}}$ is continuous and coercive on $H^1(\Omega_e)$. In that light, we have the following preliminary proposition.

Proposition 6.3.2. *Consider the linear transformation $H^1(\Omega_e) \to [H^1(\Omega_e)]'$:*

$$V \mapsto \left(W \mapsto a_{\boldsymbol{S}}^{gen} \textit{ for } W \in H^1(\Omega_e)\right).$$

If V belongs to the set

$$\mathcal{D}(\boldsymbol{S}) := \left\{V \in H^2(\Omega) : \overline{\sigma_e}'(V)n + \mathcal{B}_{gen} = 0\right\},$$

then for any $W \in H^1(\Omega_e)$ we have,

$$a_{\boldsymbol{S}}^{gen}(V, W) = (\boldsymbol{S}V, W)_{L^2(\Omega)}.$$

Furthermore, under the conditions of Assumption 6.3.1, the bilinear form $a_{\boldsymbol{S}}^{gen}$ is contin-

uous and elliptic on $H^1(\Omega_e)$*. That is, there exist* $m_{a_{\boldsymbol{S}}}$ *and* $M_{a_{\boldsymbol{S}}}$ *such that,*

$$m_{a_{\boldsymbol{S}}}\|V\|^2_{H^1(\Omega_e)} \leq a^{gen}_{\boldsymbol{S}}(V,V) \leq M_{a_{\boldsymbol{S}}}\|V\|^2,$$

for $V \in H^1(\Omega_e)$*.*

Proof. Observe that the bilinear operator $a^{\text{gen}}_{\boldsymbol{S}}$ is defined almost exactly as the bilinear operators for the elastic components in each 'at rest' case, (6.2.8) and (6.2.20), with the exception of some extra terms on the boundary. Consequently the result follows from Propositions 6.2.5 and 6.2.8, with the extra boundary terms being covered by the specification on $\mathcal{B}_{\text{gen}}$.

□

With Proposition 6.3.2 established, we can state existence and uniqueness results for variations on elliptic boundary value problems for $\boldsymbol{S}$. These results all follow from standard elliptic theory, but some detail on the proofs is stated here for the sake of a self-contained exposition.

Proposition 6.3.3 ($\boldsymbol{S}-$Neumann problem)**.** *For* $F \in [H^1(\Omega_e)]'$ *and* $G \in H^{-1/2}(\Gamma_i)$*, the Neumann boundary value problem*

$$\begin{cases} \boldsymbol{S}V = F & \Omega_e \\ \overline{\sigma_e'(V)}n + \mathcal{B}_{gen}(V) = G & \Gamma_i \end{cases}$$

has a unique weak solution $V \in H^1(\Omega_e)$*, where*

$$a^{gen}_{\boldsymbol{S}}(V,W) = (F,W)_{[H^1(\Omega_e)]';H^1(\Omega_e)} - \{G, W|_{\Gamma_i}\}_{\frac{1}{2},\Gamma_i}, \tag{6.3.8}$$

for all $W \in H^1(\Omega_e)$*, with* $a^{gen}_{\boldsymbol{S}}$ *given by* (6.3.7)*. Moreover,*

$$\|V\|_{H^1(\Omega_e)} \lesssim \max\{\|F\|_{H^1(\Omega_e)'}, \|W\|_{H^{-1/2}(\Gamma_i)}\}.$$

Proof. Recall that the trace map is surjective and continuous. Therefore, the right-hand side of (6.3.8) gives the image of W under a bounded linear functional on $H^1(\Omega_e)$. Then, along with the $H^1(\Omega_e)$ ellipticity of $a^{\text{gen}}_{\boldsymbol{S}}$, we meet the criteria of the Lax-Milgram theorem and have existence of a unique solution $V \in H^1(\Omega_e)$.

$\square$

Proposition 6.3.4 (Homogenous $\boldsymbol{S}-$Dirichlet Problem)**.** *For $G \in H^{-1}(\Omega_e)$, the homogeneous Dirichlet boundary value problem*

$$\begin{cases} \boldsymbol{S}V_0 = G & \Omega_e \\ V_0 = 0 & \Gamma_i \end{cases} \tag{6.3.9}$$

has a unique weak solution $V_0 \in H_0^1(\Omega_e)$, where

$$a_{\boldsymbol{S}}^{gen}(V_0, W) = (G, W)_{[H^1(\Omega_e)]';H^1(\Omega_e)}. \tag{6.3.10}$$

Proof. The proof has the same structure as that of Proposition 6.3.3, and depends on the $H^1(\Omega_e)$ ellipticity of $a_{\boldsymbol{S}}^{\mathrm{gen}}$. We make the observation that because $\mathcal{B}_{\mathrm{gen}}$ is a first-order tangential operator, we can associate the same bilinear form (6.3.7) with both the Neumann operator, (6.3.8), and the Dirichlet operator, (6.3.10). However, the optimization in the case of the Dirichlet operator takes place on the smaller subspace $H_0^1(\Omega_e) \subset H^1(\Omega_e)$.

$\square$

Proposition 6.3.5 ($\boldsymbol{S}-$Dirichlet Problem)**.** *For $F \in [H^1(\Omega_e)]'$ and $H \in H^{-1/2}(\Gamma_i)$, the Dirichlet boundary value problem*

$$\begin{cases} \boldsymbol{S}V = F & \Omega_e \\ V = H & \Gamma_i \end{cases} \tag{6.3.11}$$

has a unique weak solution $V \in H^1(\Omega_e)$ in the sense that $V = V_0 + \tilde{V}$, where $\tilde{V}$ has trace H in $H^{1/2}(\Gamma_i)$ and V_0 is the weak solution to (6.3.9)*, where $G(W) = F(W) - a_{\boldsymbol{S}}^{gen}(\tilde{V}, W)$ for all $W \in H_0^1(\Omega_e)$.*

Using (6.3.10)*, we have that V_0 and $\tilde{V}$ satisfy the variational identity,*

$$a_{\boldsymbol{S}}^{gen}(V_0, W) = (F, W)_{[H^1(\Omega_e)]';H^1(\Omega_e)} - a_{\boldsymbol{S}}^{gen}(\tilde{V}, W), \tag{6.3.12}$$

which is equivalent to

$$a_{\boldsymbol{S}}^{gen}(V,W) = (F,W)_{[H^1(\Omega_e)]';H^1(\Omega_e)}, \tag{6.3.13}$$

for all $W \in H_0^1(\Omega_e)$. *Additionally, we have the estimate*

$$\|V\|_{H^1(\Omega_e)} \leq C(\|F\|_{H^{-1}(\Omega_e)} + \|H\|_{H^{1/2}(\Gamma_i)}), \tag{6.3.14}$$

where C *depends on the ellipticity and continuity moduli of* $a_{\boldsymbol{S}}^{gen}$, $m_{a_{\boldsymbol{S}}}$ *and* $M_{a_{\boldsymbol{S}}}$.

6.3.2.1 Babuška-Brezzi Theorem

As indicated, the goal will be to express each of the systems in the Babuška-Brezzi framework, in order to show existence and uniqueness of solution.

For completeness of exposition, we restate the Babuška-Brezzi theorem (found in, e.g. [35, p. 116]) here.

Theorem 6.3.6 (Babuška-Brezzi)**.** *Let* X, V *be Hilbert spaces and* $a : X \times X \to R$, $b : X \times V \to R$, *bilinear forms which are continuous. Let*

$$Z = \{\xi \in X | b(\xi, v) = 0, \textit{ for every } v \in V\}.$$

Assume that $a(.,.)$ *is* $Z-$*elliptic, i.e. there exists a constant* $\alpha > 0$ *such that*

$$a(\xi,\xi) \geq \alpha \|\xi\|_X^2, \textit{ for every } \xi \in Z.$$

Assume further that there exists a constant $\beta > 0$ *such that*

$$\sup_{\tau \in X} \frac{b(\tau, v)}{\|\tau\|_X} \geq \beta \|v\|_V, \textit{ for every } v \in V.$$

Then there exists a unique pair $(\xi, \theta) \in X \times V$ *such that*

$$\begin{cases} a(\xi,\tau) + b(\tau,\theta) = F(\tau) & \textit{for every } \tau \in X \\ b(\xi, v) = 0 & \textit{for every } v \in V. \end{cases}$$

6.3.3 Existence and uniqueness for linearization around arbitrary solution

With the preliminaries in place, we can consider the well-posedness of the linearized steady state linearization.

$$\begin{cases} (Du'_f)u_f + (Du_f)u'_f - \operatorname{Div}\sigma_f(p', u'_f) = v'_f & \Omega_f \\ \operatorname{div} u'_f = 0 & \Omega_f \\ -\operatorname{Div}\left\{\overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}\right\} = \rho_e v'_e & \Omega_e \\ u'_f + (Du_f)(\widehat{\varphi}' \circ \varphi) = 0 & \Gamma_i \\ \overline{\sigma'_e(\widehat{\varphi}' \circ \varphi)}n = \sigma_f(p', u'_f)n + \mathcal{B}(\widehat{\varphi}' \circ \varphi) & \Gamma_i \\ u'_f = 0 & \Gamma_f \end{cases} \tag{6.3.15}$$

6.3.3.1 Main Result

Theorem 6.3.7. *Assume that Assumption 6.3.1 holds. For* $(v'_f, v'_e) \in \mathcal{H}_f \cap H^1_{\Gamma_f}(\Omega_f) \times H^1(\Omega_e)$*, there exists a unique solution* $(\widehat{\varphi}' \circ \varphi, p', u'_f) \in \mathscr{H}$ *for the steady state linear system,* (6.3.15).

The proof takes place in several steps.

6.3.3.2 Elliptic theory for the elastic component

We will use the elliptic theory established in Subsection 6.3.2 in order to recover the solution for the elastic subsystem from a linear map which depends on the extension of the fluid velocity as data.

We consider the following systems for $\boldsymbol{S} : H^2(\Omega_e) \to L^2(\Omega_e)$:

$$\begin{cases} \boldsymbol{S}(\widehat{\varphi}' \circ \varphi) = F & \Omega_e \\ \overline{\sigma_e}'(\widehat{\varphi}' \circ \varphi)n - \mathcal{B}(\widehat{\varphi}' \circ \varphi) = G \text{ or } \widehat{\varphi}' \circ \varphi = G & \Gamma_i \end{cases} \tag{6.3.16}$$

We define the following bilinear form associated to $\boldsymbol{S}$ and $\mathcal{B}$ for (6.3.16):

$$a_{\boldsymbol{S}}(V,W) := \int_{\Omega_e} \overline{\sigma_e}'(V)..DW + \{-\mathcal{B}(V), W\}_{\frac{1}{2},\Gamma_i}. \tag{6.3.17}$$

We have that $\mathcal{B}$ meets the criteria assumed in the elliptic theory:

Proposition 6.3.8 (Boundary operator $\mathcal{B}$)**.** *The transformation $V \mapsto \mathcal{B}(V)$ continuously extends from $H^2(\Omega_e) \to L^2(\Gamma_i)$ to a bounded linear mapping $H^1(\Omega_e) \to H^{-1/2}(\Gamma_i)$.*

Proof. By definition, $\mathcal{B}(V)$ is a linear operator. Furthermore, $\mathcal{B}(V)$ is defined only in terms of components of V and/or tangential derivatives of V. Consequently the result follows given the regularity assumptions on the coefficients, Assumption 6.3.1. □

Then from Proposition 6.3.2, we have that $a_{\boldsymbol{S}}$ is continuous and coercive on $H^1(\Omega_e)$.

Furthermore, we can write the following solution map.

Definition 6.3.9 ($\boldsymbol{S}-$Dirichlet extension for $\mathcal{B}$)**.** *Denote by $D_{\boldsymbol{S}}$ the bilinear solution map*

$$D_{\boldsymbol{S}} : (F, W) \mapsto \widehat{\varphi}' \circ \varphi$$

to the Dirichlet problem (6.3.11).

It follows from Proposition 6.3.5 that $D_{\boldsymbol{S}}$ is a continuous operator $[H^1(\Omega_e)]' \times H^{1/2}(\Gamma_i) \to H^1(\Omega_e)$.

Then, the elastic subsystem

$$\begin{cases} \boldsymbol{S}(\widehat{\varphi}' \circ \varphi) = \rho_e v_e' & \Omega_e \\ \widehat{\varphi}' \circ \varphi = -(Du_f)^{-1} u_f' & \Gamma_i \end{cases} \tag{6.3.18}$$

has a unique solution $\widehat{\varphi}' \circ \varphi = D_{\boldsymbol{S}}[\rho_e v_e', -(Du_f)^{-1} u_f']$.

Furthermore, the element $\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n - \mathcal{B}(\widehat{\varphi}' \circ \varphi) \in H^{-1/2}(\Gamma_i)$ is well-defined and satisfies,

$$\{\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n - \mathcal{B}(\widehat{\varphi}' \circ \varphi), W|_{\Gamma_i}\}_{\frac{1}{2},\Gamma_i} = a_{\boldsymbol{S}}(\widehat{\varphi}' \circ \varphi, W) - (\rho_e v_e', W)_{L^2(\Omega_e)}. \tag{6.3.19}$$

We can choose W such that in Ω_e, W is the $\boldsymbol{S}$-harmonic extension of $\tau \in H^{1/2}(\Gamma_i)$, in which case

$$W = D_{\boldsymbol{S}}[0, \tau].$$

Together, we have the identification

$$\begin{aligned}\{\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n - \mathcal{B}(\widehat{\varphi}' \circ \varphi), W|_{\Gamma_i}\}_{\frac{1}{2},\Gamma_i} =& a_{\boldsymbol{S}}(D_{\boldsymbol{S}}[0, -(Du_f)^{-1}u_f'], D_{\boldsymbol{S}}[0, \tau]) \\ &+ a_{\boldsymbol{S}}(D_{\boldsymbol{S}}[\rho_e v_e', 0], D_{\boldsymbol{S}}[0, \tau]) - (\rho_e v_e', D_{\boldsymbol{S}}[0, \tau])_{L^2(\Omega_e)},\end{aligned}$$

for all $\tau \in H^{1/2}(\Gamma_i)$.

6.3.3.3 Formulation as a Babuška-Brezzi System

In order to write the system in the Babuška-Brezzi framework, we need to write the variational equation satisfied by u_f' and p'. Integrating against a test function $\tau \in H^1_{\Gamma_f}(\Omega_f)$, we obtain,

$$\begin{aligned}\int_{\Omega_f} \langle v_f', \tau \rangle &= \int_{\Omega_f} \langle (Du_f')u_f + (Du_f)u_f' - \nu \Delta u_f' + \nabla p', \tau \rangle, \\ &= \int_{\Omega_f} 2\nu\varepsilon(u_f')..\varepsilon(\tau) + \langle (Du_f')u_f + (Du_f)u_f, \tau \rangle \\ &\quad + \int_{\Gamma_i} \langle -2\nu\varepsilon(u_f')n_f + p'n_f, \tau \rangle - \int_{\Omega_f} p' \mathrm{div}\,(\tau) \end{aligned} \tag{6.3.20}$$

Recall from (6.3.15) we have

$$\overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n - \mathcal{B}(\widehat{\varphi}' \circ \varphi) = 2\nu\varepsilon(u_f')n - p'n,$$

and substituting $n = -n_f$ into (6.3.20) we have,

$$\begin{aligned}\int_{\Omega_f} \langle v_f', \tau \rangle &= \int_{\Omega_f} \langle (Du_f')u_f + (Du_f)u_f' - \nu \Delta u_f' + \nabla p', \tau \rangle, \\ &= \int_{\Omega_f} 2\nu\varepsilon(u_f')..\varepsilon(\tau) + \langle (Du_f')u_f + (Du_f)u_f, \tau \rangle \\ &\quad + \int_{\Gamma_i} \langle \overline{\sigma_e'(\widehat{\varphi}' \circ \varphi)}n - \mathcal{B}(\widehat{\varphi}' \circ \varphi), \tau \rangle - \int_{\Omega_f} p' \mathrm{div}\,(\tau). \end{aligned} \tag{6.3.21}$$

We can now formulate (6.3.15) as a Babuška-Brezzi system. We have that u'_f and p' satisfy the following system:

$$\begin{cases} a_{BB}(u'_f, \tau) + b(\tau, p') = F(\tau) \text{ for all } \tau \in H^1_{\Gamma_f}(\Omega_f) \\ b(u'_f, v) = 0 \text{ for all } v \in L^2(\Omega_f) \end{cases} \tag{6.3.22}$$

where

$$\begin{aligned} a_{BB}(u'_f, \tau) :=& 2\nu \int_{\Omega_f} \varepsilon(u'_f)..\varepsilon(\tau) + \langle (Du'_f)u_f + (Du_f)u'_f, \tau\rangle \\ &+ a_{\boldsymbol{S}}(D_{\boldsymbol{S}}[0, -(Du_f)^{-1}u'_f], D_{\boldsymbol{S}}[0, \tau]), \end{aligned} \tag{6.3.23}$$

and

$$b(\tau, v) := -(v, \operatorname{div}(\tau))_{L^2(\Omega_f)}, \tag{6.3.24}$$

and

$$F(\tau) := (v'_f, \tau)_{L^2(\Omega_f)} - a_{\boldsymbol{S}}(D_{\boldsymbol{S}}[\rho_e v'_e, 0], D_{\boldsymbol{S}}[0, \tau]) + (\rho_e v'_e, D_{\boldsymbol{S}}[0, \tau])_{L^2(\Omega_e)}. \tag{6.3.25}$$

The following consequence of Korn's inequality [16, Thm. 6.15-4, p. 409] is useful for establishing both the continuity and the coercivity of a_{BB}.

Proposition 6.3.10. *The functional*

$$\tilde{\tau} \mapsto \left(\int_{\Omega_f} \varepsilon(\tilde{\tau})..\varepsilon(\tilde{\tau}) \right)^{1/2}$$

defines an equivalent norm on the $H^1_{\Gamma_f}(\Omega_f)$.

Regarding the continuity of a_{BB}, recall that $D_{\boldsymbol{S}}$ is continuous, and that $a_{\boldsymbol{S}}$ is continuous bilinear. The continuity of a_{BB} then follows from Proposition 6.3.10 and the smallness condition of Assumption 6.3.1.

The next step is to show the $H^1_{\Gamma_f}(\Omega_f)-$coercivity of a_{BB}. (Note that this is actually a stronger statement than what is required by Babuška-Brezzi).

By Proposition 6.3.10, we have $H^1_{\Gamma_f}(\Omega_f)-$coercivity of the term $2\nu \int_{\Omega_f} \varepsilon(\tau)..\varepsilon(\tau)$. Specifically, let m_ε be the ellipticity constant guaranteed by Proposition 6.3.10, and we have

$$2\nu \int_{\Omega_f} \varepsilon(\tau)..\varepsilon(\tau) \geq 2\nu m_\varepsilon \|\tau\|^2_{H^1_{\Gamma_f}(\Omega_f)}. \tag{6.3.26}$$

Next, we consider the term $\int_{\Omega_f} \langle (D\tau)u_f + (Du_f)\tau, \tau\rangle$. It suffices to consider $\int_{\Omega_f} \langle (D\tau)u_f + (Du_f)\tau, \tau\rangle < 0$ in which case we we have,

$$\begin{aligned} \langle (D\tau)u_f + (Du_f)\tau, \tau\rangle &= -\left|\langle (D\tau)u_f + (Du_f)\tau, \tau\rangle\right|, \\ &\geq -c_2\|\tau\|^2_{H^1_{\Gamma_f}(\Omega_f)}, \end{aligned}$$

from the smallness condition of Assumption 6.3.1.

Then, we consider the term $a_{\boldsymbol{S}}(D_{\boldsymbol{S}}[0, -(Du_f)^{-1}u_f'], D_{\boldsymbol{S}}[0,\tau])$. Recall that by the regularity condition of Assumption 6.3.1, we have $(Du_f)^{-1} \in \mathbb{M}^3(C(\overline{\Omega_f}))$. As a preliminary, we note that the topology on $\mathbb{M}^3(C(\overline{\Omega_f})$ defines multipliers on $H^{1/2}(\Gamma_i)$, according to the following estimate in [11]:

$$\|YV\|_{H^{1/2}(\Gamma_i)} \lesssim \|Y\|_{\mathbb{M}^3(C(\overline{\Omega_f}))}\|X\|_{H^{1/2}(\Gamma_i)}. \tag{6.3.27}$$

Using this estimate, as well as (6.3.14) (and recalling that the continuity constant of $a_{\boldsymbol{S}}$ is $M_{a_{\boldsymbol{S}}}$), we obtain,

$$\begin{aligned} &|a_{\boldsymbol{S}}(D_{\boldsymbol{S}}[0, -(Du_f)^{-1}\tau], D_{\boldsymbol{S}}[0,\tau])| \leq M_{a_{\boldsymbol{S}}}\|D_{\boldsymbol{S}}[0, -(Du_f)^{-1}\tau]\|_{H^1(\Omega_e)}\|D_{\boldsymbol{S}}[0,\tau]\|_{H^1(\Omega_e)} \\ &\leq M_{a_{\boldsymbol{S}}} C\|(Du_f)^{-1}\tau\|_{H^{1/2}(\Gamma_i)}\|\tau\|_{H^{1/2}(\Gamma_i)}, \\ &\leq M_{a_{\boldsymbol{S}}} C'\|(Du_f)^{-1}\|_{\mathbb{M}^3(C(\overline{\Omega_f}))}\|\tau\|^2_{H^{1/2}(\Gamma_i)} \\ &\leq M_{a_{\boldsymbol{S}}} C''\|\tau\|^2_{H^{1/2}(\Gamma_i)}, \end{aligned} \tag{6.3.28}$$

where in the last step we used that $\|(Du_f)^{-1}\|_{\mathbb{M}^3(C(\overline{\Omega_f}))}$ is small.

Substituting into (6.3.23) we have

$$a_{BB}(\tau,\tau) \geq 2\nu m_\varepsilon\|\tau\|^2_{H^1_{\Gamma_f}(\Omega_f)} - c_2\|\tau\|^2_{H^1_{\Gamma_f}(\Omega_f)} - M_{a_{\boldsymbol{S}}} C''\|\tau\|^2_{H^{1/2}(\Gamma_i)},$$

and the desired coercivity follows by assuming the viscosity is large enough to ensure $2\nu m_\varepsilon > c_2 + M_{a_S} C''$.

The last step in applying the Babuška-Brezzi theorem is to show that the bilinear form b meets the inf-sup condition, which is to say that there exists $\beta > 0$ such that

$$\sup_{\tau \in H^1_{\Gamma_f}(\Omega_f)} \frac{b(\tau, v)}{\|\tau\|_{H^1_{\Gamma_f}(\Omega_f)}} \geq \beta \|v\|_{L^2(\Omega_f)}, \text{ for every } v \in L^2(\Omega_f).$$

This argument is contained in [11] and elsewhere; we give the outline here in the interest of a self-contained exposition. Take $v \in L^2(\Omega_f)$ and consider the boundary value problem:

$$\begin{cases} \operatorname{div}(\omega) = -v & \Omega_f \\ \omega|_{\Gamma_f} = 0 & \Gamma_f \\ \omega_{\Gamma_i} = -\dfrac{\int_{\Omega_f} v}{|\Gamma_i|} n & \Gamma_i \end{cases} \tag{6.3.29}$$

From [26, (III.3.31), p. 176], (6.3.29) has a solution $\omega \in H^1_{\Gamma_f}(\Omega_f)$ and there exists $C > 0$ such that

$$\|\nabla \omega\|_{H^1_{\Gamma_f}(\Omega_f)} \leq C \|v\|_{L^2(\Omega_f)}.$$

Then we can consider the equivalent norm on $H^1_{\Gamma_f}$, $\|\nabla(\cdot)\|_{L^2(\Omega_f)}$, and for any $v \in L^2(\Omega_f)$ we obtain,

$$\begin{aligned} \sup_{\tau \in H^1_{\Gamma_f}(\Omega_f)} \frac{b(\tau, v)}{\|\tau\|_{H^1_{\Gamma_f}(\Omega_f)}} &= \sup_{\tau \in H^1_{\Gamma_f}(\Omega_f)} \frac{-(v, \operatorname{div}(\tau))_{L^2(\Omega_f)}}{\|\nabla \tau\|_{L^2(\Omega_f)}}, \\ &\geq \frac{-(v, \operatorname{div}(\omega))_{L^2(\Omega_f)}}{\|\nabla \omega\|_{L^2(\Omega_f)}}, \\ &= \frac{\|v\|^2_{L^2(\Omega_f)}}{\|\nabla \omega\|_{L^2(\Omega_f)}}, \\ &\geq \frac{1}{C} \|v\|_{L^2(\Omega_f)}. \end{aligned}$$

Thus the criteria of the Babuška-Brezzi theorem are met and we obtain the existence of the unique solution

$$(u_f', p') \in H^1_{\Gamma_f}(\Omega_f) \times L^2(\Omega_f)$$

of (6.3.15).

Observe that the second equation in (6.3.22) guarantees that u_f' is divergence-free, so in fact we have $u_f' \in \mathcal{H}_f$, and consequently can recover $\widehat{\varphi}' \circ \varphi \in H^1(\Omega_e)$ from the definition of $D_{\boldsymbol{S}}$. This completes the proof.

6.3.4 Existence and uniqueness for adjoint around arbitrary solution

Next, we turn our attention to the well-posedness of the linear, steady state adjoint system.

$$\begin{cases} (Du_f)^* Q - (DQ)u_f - \operatorname{Div} \sigma_f(P, Q) = v_1^A & \Omega_f \\ \operatorname{div}(Q) = 0 & \Omega_f \\ -\operatorname{Div} \overline{\sigma_e'(R)} = 0 & \Omega_e \\ Q = R & \Gamma_i \\ \overline{\sigma_e'(R)} n + (Du_f)^* \sigma_f(P, Q) n + \mathcal{B}_A(R) = v_2^A|_{\Gamma_i} & \Gamma_i \\ Q = 0 & \Gamma_f \end{cases} \tag{6.3.30}$$

6.3.5 Main Result

Theorem 6.3.11. *Assume that Assumption 6.3.1 holds. For $(v_1^A, v_2^A|_{\Gamma_i}) \in \mathcal{H}_f \cap H^1_{\Gamma_f}(\Omega_f) \times H^{1/2}(\Gamma_i)$, there exists a unique solution $(R, P, Q) \in \mathscr{H}$ for the steady state linear adjoint,* (6.3.30).

The proof of Theorem 6.3.11 takes place in several steps.

6.3.5.1 Analysis of the elastic component

Again, we will use the elliptic theory established in Subsection 6.3.2.

We consider the following systems for $\boldsymbol{S} : H^2(\Omega_e) \to L^2(\Omega_e)$:

$$\begin{cases} \boldsymbol{S}R = F & \Omega_e \\ \overline{\sigma_e}'(R)n + \mathcal{B}_A(R) = G \text{ or } R = G & \Gamma_i \end{cases} \tag{6.3.31}$$

Define the following bilinear form associated to $\boldsymbol{S}$ and $\mathcal{B}_A$:

$$a^A_{\boldsymbol{S}}(V, W) = \int_{\Omega_e} \overline{\sigma_e}'(V)..DW + \{\mathcal{B}_A(V), W\}_{\frac{1}{2},\Gamma_i}. \tag{6.3.32}$$

We have that $\mathcal{B}_A$ meets the criteria of the elliptic theory, analogous to Propostion 6.3.8.

Then we can define the following solution map.

Definition 6.3.12 ($\boldsymbol{S}-$Dirichlet extension)**.** *Denote by $D^A_{\boldsymbol{S}}$ the bilinear solution map*

$$D^A_{\boldsymbol{S}} : (F, W) \mapsto R$$

to the Dirichlet problem (6.3.11).

According to (6.3.14), $D^A_{\boldsymbol{S}}$ is a continuous operator $[H^1(\Omega_e)]' \times H^{1/2}(\Gamma_i) \to H^1(\Omega_e)$.

As in the previous case, we consider the elastic subsystem:

$$\begin{cases} \boldsymbol{S}R = 0 & \Omega_e \\ Q = R & \Gamma_i, \end{cases} \tag{6.3.33}$$

which has a unique solution $R = D_{\boldsymbol{S}}[0, Q]$. Furthermore, the element $\overline{\sigma'_e(R)}n - \mathcal{B}_A(R) \in H^{-1/2}(\Gamma_i)$ is well-defined and satisfies,

$$\{\overline{\sigma'_e(R)}n - \mathcal{B}_A(R), W|_{\Gamma_i}\}_{\frac{1}{2},\Gamma_i} = a_{\boldsymbol{S}}(R, W). \tag{6.3.34}$$

Thus, we have the identification

$$\{\overline{\sigma'_e(R)}n - \mathcal{B}_A(R), W|_{\Gamma_i}\}_{\frac{1}{2},\Gamma_i} = a_{\boldsymbol{S}}(D_{\boldsymbol{S}}[0, Q], D_{\boldsymbol{S}}[0, \tau]) \tag{6.3.35}$$

for all $\tau \in H^{1/2}(\Gamma_i)$.

6.3.5.2 Formulation as a Babuška-Brezzi System

We can now formulate (6.3.30) as a Babuška-Brezzi system. Here, we have that Q and P satisfy the following system:

$$\begin{cases} a^A_{BB}(Q,\tau) + b(\tau, P) = F_A(\tau) \text{ for all } \tau \in H^1_{\Gamma_f}(\Omega_f) \\ b(Q,v) = 0 \text{ for all } v \in L^2(\Omega_f) \end{cases} \tag{6.3.36}$$

where

$$\begin{aligned} a^A_{BB}(Q,\tau) :=& 2\nu \int_{\Omega_f} \varepsilon(Q)..\varepsilon(\tau) + \langle (Du_f)^* Q - (DQ)u_f, \tau\rangle \\ &+ a^A_{\boldsymbol{S}}(D^A_{\boldsymbol{S}}[0,Q], D^A_{\boldsymbol{S}}[0,\tau]), \\ F_A(\tau) :=& (v^A_1, \tau)_{L^2(\Omega_f)} - \{v^A_2|_{\Gamma_i}, \tau\}_{\frac{1}{2},\Gamma_i}, \end{aligned}$$

and b is defined as in (6.3.24).

The first step is to show the continuity of a^A_{BB}. Recall that $D^A_{\boldsymbol{S}}$ is continuous, and that $a^A_{\boldsymbol{S}}$ is continuous bilinear. The continuity of a^A_{BB} follows.

The next step is to show the $H^1_{\Gamma_f}(\Omega_f)-$coercivity of a^A_{BB}. The argument is analogous to that in the previous case. Along with an appeal to Proposition 6.3.10, we can show the coercivity of the term $\int_{\Omega_f} \langle (Du_f)^*\tau - (D\tau)u_f, \tau\rangle$ with the smallness condition of Assumption 6.3.1, and the coercivity of term $a^A_{\boldsymbol{S}}(D^A_{\boldsymbol{S}}[0,Q], D^A_{\boldsymbol{S}}[0,\tau])$ using the continuity of $a^A_{\boldsymbol{S}}$ and the estimates (6.3.14) and (6.3.27). We again require a sufficiently large viscosity, ν.

Finally, we have shown that b meets the inf-sup condition. Thus the criteria of the Babuška-Brezzi theorem are met and we obtain the existence of the unique solution

$$(Q,P) \in H^1_{\Gamma_f}(\Omega_f) \times L^2(\Omega_f)$$

of (6.3.30).

As in the previous case, the second equation in (6.3.36) guarantees that Q is divergence-free, so in fact we have $Q \in \mathcal{H}_f$; we recover $R \in H^1(\Omega_e)$ from the definition of $D^A_{\boldsymbol{S}}$. This completes the proof.

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