ABSTRACT

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Just as Lie algebras are generalizations of groups, Leibniz algebras are generalizations of Lie algebras. Many important results from Lie algebras have been extended to Leibniz algebras; for example, Engel's theorem and Lie's theorem. Ultimately from group theory, many other structures have more recently been examined in Lie algebras that give insight into the solvability or supersolvability of Lie algebras.

In this work, we introduce several generalizations of existing Lie and Leibniz algebra structures. David Towers introduced the idea of the ideal index of a maximal subalgebra of a Lie algebra, and Leila Goudarzi and Ali Reza Salemkar generalized this ideal index to all subalgebras of Lie algebras. Sara Chehrazi introduced two extensions of CAP-subalgebras of Lie algebras, CAP*- and SCAP-subalgebras. In this work, we define these concepts for Leibniz algebras and extend results on solvability and supersolvability.

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The Generalized Ideal Index and CAP*-subalgebras of Leibniz Algebras

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DEDICATION

To my mother for everything she has done for me.

BIOGRAPHY

The author loves finishing dissertations and does not love writing biographies.

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Thank you to my mother and sister. To Dr. Batten, without whom I would not be a mathematician. To DJ and Holly, without whom I would be very miserable. To Clover, Christina, and Elisabeth, without whom I would not have made it past the first year, and to Abby and Maggie, without whom I would not have any fun. To Jay, without whom I would have little support. To Dr. Stitzinger, without whom I would probably have dropped out. To Claire, without whom I would not make it past any particular Sunday night, and to Michael, without whom I would not have made it through February (and neither would this dissertation).

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1

Introduction

Group theory has a long history of study in mathematics, dating back to Cauchy and Galois in the 19th century. However, study in group theory was not commonplace until long after Galois' untimely death. In fact, much of what we know today about group theory was developed throughout the 20th century, and work in group theory continues even today.

In the mid-nineteenth century, another algebraic structure, the Lie algebra, was introduced from the study of Lie groups. In particular, Lie algebras found applications into applied mathematics and physics. Many results from group theory have been extended and generalized to Lie algebras over the past century, a work which continues today as well.

Several generalizations of the Lie algebra have been proposed and studied. In this work, we focus on the Leibniz algebra, which is a generalization of Lie algebras that is not antisymmetric. Leibniz algebras were defined in 1993 by Jean-Louis Loday [14]. Just as for groups and Lie algebras, many results from the study of Lie algebras have been generalized to Leibniz algebras, and we continue that process in this work.

Much of the work in this paper relies on other concepts introduced for Lie algebras by David Towers. Some of these concepts have been discussed and extended to Leibniz algebras by [**16**]. We generalize these concepts and provide more results to characterize solvable and supersolvable Leibniz algebras. Background concepts necessary for this work are discussed in Chapter 2.

Some links between solvable and supersolvable Leibniz algebras and the generalized ideal index of Chapter 3 are connected through the concept of c-ideals, introduced for Lie algebras by [**13**] and extended to Leibniz algebras by [**16**]. The concept of a c-ideal is analogous to c-normal subgroups in group theory, introduced in [**17**] by Wang. The necessary properties related to c-ideals are discussed in Chapter 2. Some theorems related to c-ideals were proved independently by [**16**] and this work. For these theorems, proofs are given in Chapter 2.

In Chapter 3, the primary object of study is the generalized ideal index. Ideal index for Lie algebras, discussed by Towers, are analogous to the concept of the index complex of a maximal subgroup of a finite group, discussed in [**7**] by Deskins. [**16**] studied the extension of this concept to Leibniz algebras. However, the ideal index is only defined in both these cases for maximal subalgebras. In [**8**], Goudarzi and Salemkar discuss a generalized ideal index for any subalgebra of a Lie algebra, not only those that are maximal. Chapter 3 extends this concept to Leibniz algebras and preserves properties that successfully extend to Leibniz algebras, such as solvability and supersolvability criteria.

David Towers introduced CAP-subalgebras of Lie algebras in [**15**]. CAP-subalgebras are analogous to ideas in finite group theory that provide some characterizations of finite solvable groups and their subgroups. Similarly, Towers showed that CAP-subalgebras of Lie algebras can characterize solvable finite-dimensional Lie algebras. Turner extended several of these results to Leibniz algebras in [**16**]. In Chapter 4, we study CAP*-subalgebras, a generalization of CAP-subalgebras discussed by Chehrazi in [**3**]. CAP* subalgebras are analogous to the CAP*-subgroup in finite group theory discussed by Li and Liu in [**9**]. Using these subalgebras, we obtain more characterizations for solvable and supersolvable Leibniz algebras.

Chapter 5 takes a look at a different generalization of CAP-subalgebras called SCAP-subalgebras, or semi-CAP subalbgebras, which cover or avoid a chief series of A rather than every chief factor. This concept is related to semi cover-avoiding subgroups in the theory of finite groups, studied by for example Li et al. in [**10**]. This concept was extended to Lie algebras by Chehrazi and Salemkar in [**4**], and in Chapter 5 we attempt to extend some of their properties to Leibniz algebras.

Chapter

2

Preliminaries

2.1 Leibniz Algebras

We begin with basic definitions and concepts that are used throughout. Much of the information about Leibniz algebras in this section comes from [**6**] unless specified otherwise.

A **(left) Leibniz algebra** is a vector space over a field $\mathbb{F}$ together with a bilinear map (multiplication)

$$[,]: A \times A \to A$$

that satisfies the left Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]]$$

for all $a, b, c \in A$.

For an element $a \in A$, the **left multiplication operator** $L_a : A \to A$ is defined by $L_a(b) = [a,b]$ for all $b \in A$. The left Leibniz identity is obtained from requiring all left multiplication operators to be derivations. Similarly, one may define a **right Leibniz algebra** by requiring the **right multiplication operator** $R_a : A \to A$ defined by $R_a(b) = [b, a]$ for all $b \in A$ to be a derivation, resulting in the **right Leibniz identity**

$$[[b,c],a] = [[b,a],c] + [b,[c,a]].$$

In general, right and left Leibniz algebras are different. Throughout this paper, we assume A is a finite dimensional left Leibniz algebra, and that the term 'Leibniz algebra' refers to a left Leibniz algebra.

For any Leibniz algebra, define Leib(A)=span$\{[a,a]|a \in A\}$. If Leib(A) is zero, then A is a Lie algebra. If B, C are subsets of the Leibniz algebra A, then the notation $[B,C] = \{[b,c]|b \in B$ and $c \in C\}$. We call $B \subseteq A$ a **subalgebra** of A if $[B,B] \subseteq B$. A subalgebra I of A is called a **left ideal** if $[A,I] \subseteq I$ or a **right ideal** if $[I,A] \subseteq I$. If both conditions hold, I is an **ideal** of A. For example, Leib(A) is an ideal of A, and it is **Abelian** as $[\text{Leib}(A),\text{Leib}(A)] = 0$. In fact, defining factor algebras in the usual way, Leib(A) is the smallest ideal such that $\frac{A}{\text{Leib}(A)}$ is a Lie algebra. The sum and intersection of two ideals is an ideal, but the product of two ideals may not be an ideal in a Leibniz algebra.

A Leibniz algebra is **simple** if it has no nonzero proper ideals. This agrees with the Lie algebra definition. However, there is another commonly used definition of a simple Leibniz algebra we will refer to when proving theorems about simple Leibniz algebras (see Chapter 3).

As with Lie algebras, we defined the center of A and the normalizer of a subalgebra B of A. The **left center** of A, denoted $Z^l(A) = \{x \in A|[x,a] = 0$ for all $a \in A\}$, and the **right center** of A, denoted $Z^r(A) = \{x \in A|[a,x] = 0$ for all $a \in A\}$. The **center** of A, denoted $Z(A) = Z^l(A) \cap Z^r(A)$. The **left normalizer** of B in A, denoted $N_A^l(B) = \{x \in A|[x,a] \in B$ for all $a \in B\}$, and the **right normalizer** of B in A, denoted $N_A^r(B) = \{x \in A|[a,x] \in B$ for all $a \in B\}$. The **normalizer** of B in A, denoted $N_A(B) = N_A^l(B) \cap N_A^r(B)$. The left normalizer and normalizer are subalgebras of A, but the right normalizer may not be.

A related concept is the **centralizer** of a subalgebra H of a Leibniz algebra, denoted $Z_A(H)$. It is defined as $Z_A^l(H) \cap Z_A^r(H)$ where $Z_A^l(H) = \{x \in A|[x,h] = 0$ for all $h \in H\}$ and $Z_A^r(H) = \{x \in A|[h,x] = 0$ for all $h \in H\}$. We use this concept in Chapter 4.

As in Lie algebras, solvable Leibniz algebras are of interest. We define the **derived series** of a Leibniz algebra A as the series $A \supseteq A^{(1)} \supseteq A^{(2)} \supseteq \cdots$ where $A^{(1)} = [A,A]$ and $A^{(k)} = [A^{(k-1)}, A^{(k-1)}]$. A Leibniz algebra A is said to be **solvable** if $A^{(k)} = 0$ for some positive integer k. The sum and intersection of any two solvable ideals of a Leibniz algebra is solvable, so every Leibniz algebra A contains a unique maximal solvable ideal, called the **radical** of A and denoted $rad(A)$, which contains all solvable ideals.

We will make frequent use of the following two lemmas throughout this work. They are results from Barnes in [**1**].

Lemma 2.1.1. *If N is a minimal ideal of A, then either $[N,A]=0$ or $[n,a]=-[a,n]$ for all $n \in N, a \in A$.*

Lemma 2.1.2. *Let M be an irreducible A-module. Let (T,S) be the associated representation. Then $A/Ker(T,S)$ is a Lie algebra and either $MA=0$ or $[a,m]=-[m,a]$.*

2.2 Ideals and c- Ideals

The following concepts have been studied by, for example, [**13**] for Lie algebras and [**16**] for Leibniz algebras. Future chapters will expand on these definitions and generalize them.

Definition 2.2.1. Let B be a subalgebra of A. The **core** of B with respect to A, B_A, is the largest ideal of A contained in B.

Definition 2.2.2. Let B be a subalgebra of A. B is a **c-ideal** of A if there is some ideal C such that $B + C = A$ and $B \cap C \subset B_A$.

Example 2.2.3. If B is itself an ideal of A, then it is a c-ideal of A, since $A = B + A$ and $B \cap A \subset B_A$ since $B \cap A = B = B_A$ since B is an ideal.

Lemma 2.2.4. 1. *If B is a c-deal of A and $B \subset K \subset A$, then B is a c-ideal of K.*

2. *If I is an ideal of A and $I \subset B$, then B is a c-ideal of A if and only if B/I is a c-ideal of A/I.*

Proof. 1. If B is a c-ideal of A, then there is an ideal C such that $A = B + C$ and $B \cap C \subset B_A$. Since $B \subset K, K = (B + C) \cap K = B + C \cap K$, and $C \cap K$ is an ideal of K. $B \cap (C \cap K) = (B \cap C) \cap K \subset B_A \cap K \subset B_K$., so B is a c-ideal of K complemented by $C \cap K$.

2. ($\Longrightarrow$) Suppose B is a c-ideal of A and I is an ideal of A contained in B. Then there is an ideal C such that $A = B + C$ and $B \cap C \subset B_A$. Suppose $a \in A$. Then, since $A = B + C$, $a = b + c$ for some $b \in B, c \in C$. Suppose $a + I \in A/I$. Then $a + I = (b + c) + I = (b + I) + (c + I)$. Since $b + I \in B/I, c + I \in C/I$, it follows that $A/I = B/I + C/I$, $\Longrightarrow$ $A/I = B/I + (C + I)/I$. Since C and I are ideals of A, $(C + I)/I$ is an ideal of A/I. Finally, $(B/I) \cap (C + I)/I = (B \cap (C + I))/I = (B \cap C + B \cap I)/I = (B \cap C + I)/I \subset B_A/I = (B/I)_{A/I}$. Thus B/I is a c-ideal of A/I complemented by $(C + I)/I$.

   Conversely, suppose B/I is a c-ideal of A/I. Then there is an ideal C/I such that $B/I + C/I = A/I$ and $(B/I) \cap (C/I) \subset (B/I)_{A/I} = B_A/I$. Similarly to the forward direction, this implies $A = B + C$. Since $(B/I) \cap (C/I) = (B \cap C)/I \subset (B/I)_{A/I} = B_A/I$, it follows that $B \cap C \subset B_A$

□

Theorems 2.0.6 and 2.0.7 have also been proved by [**16**]. The proofs are particularly nice, so we include them here.

Theorem 2.2.5. *All maximal subalgebras of a Leibniz algebra A are c-ideals of A if and only if A is solvable.*

Proof. Suppose A is a Leibniz algebra in which all maximal subalegbras are c-ideals, but A is not solvable. Choose the smallest such example of A. Consider $A/\text{Leib}(A)$. Note if $\text{Leib}(A)$ is $\{0\}$, then A is Lie and the theorem is known. If $\text{Leib}(A)$ is nonzero, then maximal subalgebras of $A/\text{Leib}(A)$ are of the form $M/\text{Leib}(A)$, where M is a maximal subalgebra of A. By Lemma 2.1 (ii), every maximal subalgebra of $A/\text{Leib}(A)$ is a c-ideal of $A/\text{Leib}(A)$.

On the other hand, suppose A is solvable. Let M be a maximal subalgebra of A. Let I be a minimal ideal of A. Suppose $I \not\subset M$. Then $M + I$ is larger than M, which is maximal, so $M + I = A$. Since $I \cap M = \{0\} \subset M_A$, M is a c-ideal of A.

Now, choose the smallest dimensional counterexample A (so that A is solvable, but it has a subalgebra M that is not a c-ideal of M). Suppose $I \subset M$. Since A/I is of smaller dimension than A, M/I is a c-ideal of A/I since it is maximal. However, Lemma 2.2.4 (ii) implies that M is a c-ideal of A, a contradiction. □

Theorem 2.2.6. *Let A be a Leibniz algebra that is either*

a) over a field F of characteristic zero, or

b) over an algebraically closed field F of characteristic greater than 5.

Then A has a solvable maximal subalgebra that is a c-ideal of A if and only if A is solvable.

Proof. Suppose A has a solvable maximal subalgebra that is a c-ideal of A, say M. Consider the radical of A, $\mathrm{rad}(A)$. First, suppose $\mathrm{rad}(A)$ is not contained in M. Then $M + \mathrm{rad}(A)$ is larger than M, a maximal subalgebra. So $M + \mathrm{rad}(A) = A$. But M and $\mathrm{rad}(A)$ are both solvable, so A is solvable.

Now suppose $\mathrm{rad}(A)$ is contained in M. Since $A/\mathrm{rad}(A)$ is semisimple, $A/\mathrm{rad}(A)$ is Lie since $\mathrm{Leib}(A)$=0. Since $\mathrm{rad}(A)$ is contained in M, by Lemma 2.1 $M/\mathrm{Leib}(A)$ is a c-ideal of the Lie algebra $A/\mathrm{rad}(A)$. Since $A/\mathrm{rad}(A)$ is a Lie algebra that has a maximal subalgebra which is a c-ideal, $A/\mathrm{rad}(A)$ is solvable by [Towers 3.2/3.3]. Since $\mathrm{rad}(A)$ is also solvable, we have that A is solvable.

Conversely, if A is solvable, Theorem 3.1 implies A has a maximal subalgebra that is a c-ideal. □

2.3 Index Complex

The index complex for maximal subalgebras was introduced by Towers in [**14**]

Definition 2.3.1. Let M be a maximal subalgebra of A. A subalgebra C of A is said to be a **completion** for M if C is not contained in M but every proper subalgebra of C that is an ideal of A is contained in M. The set of all completions of M, $I(M)$ is called the index complex of M in A.

Example 2.3.2. Below, in Lemma 2.0.12, it is shown that every maximal subalgebra of a Leibniz algebra A has at least one completion C, at least one of which is an ideal of A.

Definition 2.3.3. The **strict core** of a subalgebra $B \neq 0$, denoted $k(B)$, is the sum of all ideals of A that are proper subalgebras of B.

A simple example shows that the strict core and core can differ even when B is itself an ideal.

Example 2.3.4. Consider the Leibniz algebra $A = \text{span}\{x, a, b, c, d\}$ with $[a, b] = c, [b, a] = d$ $[x, a] = a = -[a, x], [x, c] = c, [x, d] = d, [c, x] = d, [d, x] = -d$, otherwise 0.

Consider the subalgebra $B = \text{span}\{d\}$. Note B is an ideal, so $B_A = B$. However, $k(B) = 0$.

Lemma 2.3.5. *If M is a maximal subalgebra of A then $I(M)$ is non-empty. In fact, $I(M)$ contains an ideal of A.*

Proof. The set of ideals of A which do not lie in M is a non-empty (e.g., A itself), partially ordered (by dimension) set. Choose C to be a minimal element of this set. Then $A = M + C$, and $k(C) \subset M$ since C is an ideal of minimal dimension not contained in M, so $C \in I(M)$. □

Definition 2.3.6. If C is a completion of M, and C is an ideal of A, then we call C an **ideal completion** of A.

The below theorem, which is proved for Lie algebra in [**14**] and for Leibniz algebras in [**16**], allows for the definition of ideal index.

Theorem 2.3.7. *Let C and D be ideal completions of the maximal subalgebra M of A. Then $C/k(C) \cong D/k(D)$.*

Proof. For contradiction, let A be the smallest dimensional Leibniz algebra that has a maximal subalgebra M with two ideal completions C and D such that $C/k(C) \ncong D/k(D)$. First suppose $k(C) \cap k(D) \neq 0$. Then, in $\frac{A}{k(C)\cap k(D)}$, $\frac{M}{k(C)\cap k(D)}$ is a maximal subalgebra with ideal completions $E = \frac{C}{k(C)\cap k(D)}$ and $F = \frac{D}{k(C)\cap k(D)}$. However, since $k(C) \cap k(D) \neq 0$ and A is the smallest counterexample, we know $E/k(E) \cong F/k(F)$ since they are ideal completions. This is a contradiction. Hence $k(C) \cap k(D) = 0$.

□

Definition 2.3.8. The ideal index of a maximal subalgebra M of A, denoted $\eta(A:M)$, is the dimension of $C/k(C)$, where C is an ideal completion of M in A.

Corollary 2.3.9. *If M is a maximal subalgebra of A, then $\eta(A:M)$ is well-defined.*

The following theorems on the ideal index of maximal subalgebras have been proved by [Turner] for Leibniz algebras. We will use these results in later chapters and attempt to expand on them, generalizing them to non maximal subalgebras as well as the concept of ideal index.

Proposition 2.3.10. *Let M be a maximal subalgebra of A and let B be an ideal of A with $B \subset M$. Let C/B be an ideal completion of M/B in A/B, suppose $k(C/B) = K/B$, and let D be an ideal completion of M in A. Then $C/K \cong D/k(D)$.*

Corollary 2.3.11. *Let M be a maximal subalgebra of A and let B be an ideal of A with $B \subset M$. Then $\eta(A/B : M/B) = \eta(A:M)$.*

Theorem 2.3.12. *Let M be a maximal subalgebra of A. Then M is a c-ideal of A if and only if $\eta(A:M) = dim(A/M)$.*

Corollary 2.3.13. *A is solvable if and only if $\eta(A:M) = dim(A/M)$ for all maximal subalgebras M of A.*

Corollary 2.3.14. *Let A be a Leibniz algebra over a field F, where F is of characteristic zero or F is algebraically closed with characteristic greater than 5. Then A has a solvable maximal subalgebra M with $\eta(A:M) = dim(A/M)$ if and only if A is solvable.*

Theorem 2.3.15. *If A has a supersolvable maximal subalgebra M with $\eta(A:M) = 1$ and $N(A)$ is not contained in M, then A is supersolvable.*

2.4 Nilpotency and Cartan Subalgebras

The topics of nilpotency, Engel subalgebras, and Cartan subalbgeras are of interest while studying CAP*-subalgebras in Chapter 4.

A Leibniz algebra A is **nilpotent** of class c if every product of $c+1$ elements is 0 and there is some product of c elements that is not zero. Equivalently, A is nilpotent if $A^{c+1}=0$ for some c. For example, A is solvable if and only if $[A,A]$ is nilpotent.

Suppose A is a Leibniz algebra. The Fitting null component of the left multiplication operator of an element a, L_a is denoted $E_A(a)$. It is called the **Engel** subalgebra corresponding to a. In a Lie algebra, a always belongs to such an Engel subalgebra, but this is not necessarily the case for Leibniz algebras.

A subalgebra H of A is called a Cartan subalgebra if it is nilpotent and self-normalizing ($N_A(H)=H$). The following theorems from [**2**] provides a realization and a useful result

Theorem 2.4.1. *A subalgebra of a Leibniz algebra A is a Cartan subalgebra if it is minimal in the set of all Engel subalgebras of A.*

Lemma 2.4.2. *Let H be a Cartan subalgebra of A and I an ideal of A. Then $(H+I)/I$ is a Cartan subalgebra of A/I.*

2.5 CAP-subalgebras

We also consider subalgebras that cover or avoid certain factor algebras called chief factors of a Leibniz algebra. These subalgebras are called CAP-subalgebras, and we offer generalizations of these subalgebras in later chapters.

Definition 2.5.1. Let A be a finite dimensional Leibniz algebra, C a subalgebra of A, and D an ideal of A. The factor algebra C/D is called a **chief factor** of A if C/D is a minimal ideal of A/D.

Definition 2.5.2. Let U be a subalgebra of the Leibniz algebra A. U is said to **avoid** the factor algebra C/D if $U\cap C=U\cap D$. Similary, U is said to **cover** the factor algebra C/D if $U+C=U+D$.

Definition 2.5.3. The subalgebra U has the **covering and avoidance property**, or is called a **CAP-subalgebra** if it covers or avoids every chief factor of A.

Of particular interest in later chapters are specifically Frattini and non-Frattini chief factors of A. The **Frattini subalgebra** of A, denoted $\mathrm{F}(A)$, is the intersection of all maximal subalgebras of A. The **Frattini ideal** of A, denoted $\phi(A)$, is the largest ideal which is contained in $\mathrm{F}(A)$.

The following lemma from [**1**] is useful in Chapter 4.

Lemma 2.5.4. *Let C be a Cartan subalgebra of A and K an ideal of A. Then $(C+K)/K$ is a Cartan subalgebra of A/K.*

Chapter

3

Generalized Ideal Index

3.1 Definitions and Properties

In [**14**], David Towers introduced the idea of the ideal index of a maximal subalgebra of a Lie algebra based on similar concepts from group theory. For Lie algebras, the ideal index provides useful results related to Lie algebra structure. Many of these useful properties were extended to Leibniz algebras by Turner in [**16**]. In [**8**], Goudarzi and Salemkar provide a generalization of the ideal index to all subalgebras of a Lie algebra. In this chapter, we extend this generalization to Leibniz algebras along with properties that still hold in the Leibniz case.

Definition 3.1.1. Let H be a subalgebra of a Leibniz algebra A and K an ideal of A with $A = H + K$. Then the dimension of the Leibniz algebra $\dfrac{K}{K \cap H_A}$ is said to be the **ideal index** of H in A, denoted by $\bar{\eta}(A : H)$, provided that $\dim \dfrac{T}{T \cap H_A} \geq \dim \dfrac{K}{K \cap H_A}$ for all ideals T of A such that $A = H + T$.

Example 3.1.2. Consider a nilpotent Lie algebra A of dimension 5 with basis $\{x, y, z, t, w\}$ and multiplication

$[x, w] = t, [x, t] = z, [x, z] = y, [t, w] = y$

Note $y \in Z(A)$. Consider the subalgebra H =span$\{x, y\}$. Then H_A =span$\{y\}$. Consider K spanned by $\{w, y, t, z\}$.

$A = H + K$ and $\dim \frac{K}{K \cap H_A} = 3$.

Any smaller dimensional choice for K would either not complete H or not be an ideal, and larger dimensional choices would yield larger values for $\dim \frac{K}{K \cap H_A}$. Thus $\bar{\eta}(A : H) = 3$.

Example 3.1.3. Consider the cyclic 2-dimensional Leibniz algebra spanned by x and $[x, x] = x^2$ with $[x, x^2] = x^2$.

Let H =span$\{x - x^2\}$. Note H is not an ideal since $[x - x^2, x] = x^2$. Since H is one dimensional, $H_A = 0$. The smallest ideal K such that $A = H + K$ must be one dimensional, e.g. K =span$\{x^2\}$. Then $\bar{\eta}(A : H)$ =$\dim \frac{K}{K \cap H_A} = 1$.

Lemma 3.1.4. *Let A be a Leibniz algebra, and H a proper subalgebra of A.*

1. $\bar{\eta}(A : H) \geq dim(A/H)$.
2. $\bar{\eta}(A : H) \geq max\{\bar{\eta}(T : H), \bar{\eta}(A : T)\}$ *for any subalgebra T of A containing H. Also, if $H \subseteq T \subset A$, then* $\bar{\eta}(A : H) > \bar{\eta}(T : H)$.
3. $\bar{\eta}(A : H) = dim(A/H)$ *if and only if H is a c-ideal of A.*
4. $\bar{\eta}(A : H) = \bar{\eta}(A/N : H/N)$ *for any ideal N of A that is contained in H.*
5. $\bar{\eta}(A : H) = \eta(A : H)$ *if H is a maximal subalgebra of A.*
6. *If* $\bar{\eta}(A : H) = dim(A/H)$*, then* $\bar{\eta}(T : H) = dim(T/H)$ *for all subalgebras T of A containing H.*

Proof. 1. For some ideal K of A, $\bar{\eta}(A : H) = \dim\left(\frac{K}{K \cap H_A}\right) \geq \dim\left(\frac{K}{K \cap H}\right) = \dim\left(\frac{A}{H}\right)$ since $A = H + K \implies \frac{A}{H} \cong \frac{H + K}{H} \cong \frac{K}{K \cap H}$.

2. Let T be a subalgebra of A containing H. Suppose K is an ideal of A such that $\bar{\eta}(A:H) = \dim\left(\frac{K}{K\cap H_A}\right)$, so that $A = H+K$. Then $K\cap T$ is an ideal in T and $T = H + K\cap T$. Thus

$$\bar{\eta}(A:H) = \dim\left(\frac{K}{K\cap H_A}\right) \geq \dim\left(\frac{K\cap T}{K\cap H_A}\right) = \dim\left(\frac{K\cap T}{K\cap T\cap H_A}\right) \geq \bar{\eta}(T:H).$$

In the above equation, we have if T is a proper subalgebra of A then $K\cap T$ is a proper subalgebra of K, giving a strict inequality $\bar{\eta}(A:H) > \bar{\eta}(T:H)$.

Also, since $H \subseteq T$, $H_A \subseteq T_A$. Additionally, $A = H+K \implies A = T+K$ so that

$$\bar{\eta}(A:H) = \dim\left(\frac{K}{K\cap H_A}\right) \geq \dim\left(\frac{K}{K\cap T_A}\right) \geq \bar{\eta}(A:T).$$

3. First suppose that $\bar{\eta}(A:H) = \dim(A/H)$. For some ideal K of A this means $\dim\left(\frac{K}{K\cap H_A}\right) = \dim\left(\frac{A}{H}\right) = \dim\left(\frac{H+K}{H}\right) = \dim\left(\frac{K}{K\cap H}\right) \implies H\cap K = H_A\cap K \subset H_A \implies$ H is a c-ideal of A.

   Now suppose H is a c-ideal of A. Then there is some ideal C of A such that $A = H+C$ and $C\cap H \subseteq H_A$. By definition,

$$\bar{\eta}(A:H) \leq \dim\left(\frac{C}{C\cap H_A}\right) \leq \dim\left(\frac{C}{C\cap H}\right) = \dim\left(\frac{C+H}{H}\right) = \dim\left(\frac{A}{H}\right).$$

   This along with part 1 implies $\bar{\eta}(A:H) = \dim(A/H)$.

4. Suppose C is an ideal of A contained in H. Then, for some ideal K of A,

$$\bar{\eta}(A:H) = \dim\left(\frac{K}{K\cap H_A}\right) = \dim\left(\frac{K+N}{(K\cap H_A)+N}\right) = \dim\left(\frac{K+N}{(K+N)\cap H_A}\right)$$

   Suppose now that C is an ideal of A that contains N with $A = H+C$. Then

$$\dim\left(\frac{C}{C\cap H_A}\right) = \dim\left(\frac{K}{K\cap H_A}\right)$$
$$\implies \dim\left(\frac{C/N}{(C/N)\cap(H/N)_{A/N}}\right) \geq \dim\left(\frac{(K+N)/N}{((K+N)/N)\cap(H/N)_{A/N}}\right)$$

$$\implies \bar{\eta}(A/N : H/N) = \dim\left(\frac{(K+N)/N}{((K+N)/N)\cap(H/N)_{A/N}}\right) = \dim\left(\frac{K+N}{(K+N)\cap H_A}\right) = \bar{\eta}(A:H).$$

5. Let H be a maximal subalgebra of A. We can asssume H_A is zero (if not, we may factor it out by part 4). This means that for some ideal K of A for which $A = H + K$, $\bar{\eta}(A:H) = \dim(K)$. K must be a minimal ideal; if it had any proper subalgebras that were ideals of A, then $\bar{\eta}(A:H)$ would be the dimension of that ideal. Thus $\eta(A:H) = \dim(K) = \bar{\eta}(A:H)$.

6. From part 3 we know that $\bar{\eta}(A:H) = \dim(A/H)$ if and only if H is a c-ideal of A. From [**16**] we know that H is a c-ideal of any subalgebra T containing it, thus $\bar{\eta}(T:H) = \dim(T/H)$.

□

For Lie algebras, the definition of a **simple** Lie algebra is one that has no nontrivial proper ideals. We may also use this definition for Leibniz algebra, so that the following corollary is true.

Corollary 3.1.5. *Let A be a Leibniz algebra. A is simple if and only if for any proper subalgebra H of A, $\bar{\eta}(A:H) = dim(A)$.*

However, consider the following. Suppose A is a Leibniz algebra and not a Lie algebra. This implies that Leib(A) is nonzero. Since Leib(A) is an ideal of A, this means that no non-Lie Leibniz algebra is simple by the Lie definition unless Leib(A)=A. One possible way to study theorems that are true for Lie algebras is to instead define a **Leibniz-simple** Leibniz algebra to be one that has no nontrivial proper ideals except Leib(A), and $A^2 \neq$ Leib(A). For this definition, we can modify the corollary but equivalence is lost.

Corollary 3.1.6. *Let A be a Leibniz algebra. If A is Leibniz-simple then for any proper subalgebra H of A, $\bar{\eta}(A:H) = dim(A)$, $\bar{\eta}(A:H) = dim(Leib(A))$, or $\bar{\eta}(A:H) = dim\dfrac{A}{Leib(A)}$.*

Proof. Let H be any proper subalgebra of A. The only possible ideals K of A such that $A = H + K$ are Leib(A) or A itself.

Suppose $A = H + \text{Leib}(A)$. Then Leib(A) is not contained in H, so the core of H is zero. Either $\bar{\eta}(A:H) = \dim\dfrac{A}{A\cap\{0\}} = \dim(A)$ or $\bar{\eta}(A:H) = \dim\dfrac{\text{Leib}(A)}{\text{Leib}(A)\cap\{0\}} = \dim(\text{Leib}(A))$.

Suppose now $A \neq H + \text{Leib}(A)$. Then the core of H is either zero or Leib(A), so either $\bar{\eta}(A : H) = \dim \dfrac{A}{A \cap \{0\}} = \dim(A)$ or $\bar{\eta}(A : H) = \dim \dfrac{A}{A \cap \text{Leib}(A)} = \dim \dfrac{A}{\text{Leib}(A)}$. □

3.2 Solvability

The following theorems give conditions on the ideal index of second maximal subalgebras for a Leibniz algbera to be solvable. For Lie algebras, the theorems and proofs may be found in [**8**]. For Leibniz algebras, many of the proofs utilize the fact that for any Leibniz algebra A, $A/\text{Leib}(A)$ is a Lie algebra. Leib(A) is also a solvable ideal, so if $A/\text{Leib}(A)$, we can conclude A is solvable as well.

Theorem 3.2.1. *A Leibniz algebra A is solvable if for any second maximal subalgebra H of A, there is an ideal K of A containing H_A such that $A = H + K$, $\bar{\eta}(A : H) = dim(K/H_A)$, and $C_{A/H_A}(K/H_A) \neq 0$.*

Proof. We will use induction. Assume A is the smallest dimensional Leibniz algebra that is as stated but not solvable. This theorem is known for Lie algebras, so assume that A is not Lie. Then there is a minimal ideal N that is solvable. Let N be a minimal ideal of A and H/N a second maximal subalegbra of A/N. By hypotheses, there is an ideal K of A containing H_A such that $A = H + K$, $\bar{\eta}(A : H) = \dim(K/H_A)$, and $C_{A/H_A}(K/H_A) \neq 0$. Since $N \subset H$ by assumption, $N \subset H_A$ since it is minimal and thus $N \subset K$ since $H_A \subset K$. Hence $N \subset H \cap K$ and

$$\bar{\eta}(A/N : H/N) = \dim\left(\frac{K/N}{K/N \cap H_{A/N}}\right) = \dim\left(\frac{K/N}{(H/N)_{A/N}}\right).$$

Also,

$$C_{\dfrac{A/N}{H_A/N}}\left(\frac{K/N}{(H/N)_{A/N}}\right) \cong C_{\dfrac{A}{H_A}}\left(\tfrac{K}{H_A}\right) \neq 0$$

This implies that A/N is solvable by the induction hypothesis. Since N is minimal, it is contained in Leib(A), which is Abelian. Hence N is solvable. Since A/N and N are solvable, we have that A is. □

Theorem 3.2.2. *A Leibniz algebra A is solvable if and only if for any second maximal subalgebra H of A, there is an ideal K of A containing H_A such that $A = H + K$, $\bar{\eta}(A : H) = dim(K/H_A)$, and K/H_A is solvable.*

Proof. ($\Longrightarrow$) First, suppose A is solvable and let H be a second maximal subalgebra of A.

Consider the factor algebra A/H_A. There is an ideal K/H_A such that $A/H_A = K/H_A + H/H_A$ (for example, A/H_A is itself a candidate) and

$\bar{\eta}(A/H_A : H/H_A) = \bar{\eta}(A : H) = \dim\left(\dfrac{K/H_A}{(K/H_A)\cap(H/H_A)_A}\right) = \dim\dfrac{K}{H_A}$ since H/H_A is core-free.

Clearly, K/H_A is solvable since A is.

In the opposite direction, we will use induction. Assume A is the smallest dimensional Leibniz algebra that is as stated but not solvable. This theorem is known for Lie algebras, so assume that A is not Lie. Hence Leib(A) is not zero. Let N be a minimal ideal of A and H/N a second maximal subalegbra of A/N. By hypotheses, there is an ideal K of A containing H_A such that $A = H + K$, $\bar{\eta}(A : H) = \dim(K/H_A)$, and K/H_A is solvable. Since $N \subset H$ by assumption, $N \subset H_A$ since it is minimal and thus $N \subset K$ since $H_A \subset K$. Hence $N \subset H \cap K$ and

$$\bar{\eta}(A/N : H/N) = \dim\left(\frac{K/N}{K/N \cap H_{A/N}}\right) = \dim\left(\frac{K/N}{(H/N)_{A/N}}\right).$$

Also,

$$\frac{K/N}{H_A/N} \cong \frac{K}{H_A} \text{ is solvable.}$$

This implies that A/N is solvable by the induction hypothesis. Since N is a minimal ideal, by Lemma 2.0.1 either $[N, N] = 0$ or $N \subseteq$Leib(A). In either case, N is solvable, so A is solvable. □

Theorem 3.2.3. *A Leibniz algebra A is solvable if and only if there exists a solvable second maximal subalgebra H of A and an ideal K of A containing H_A such that $A = H + K$, $\bar{\eta}(A : H) = dim(K/H_A)$, and K/H_A is solvable.*

Proof. First, suppose A is solvable and let H be a second maximal subalgebra of A.

Consider the factor algebra A/H_A. There is an ideal K/H_A such that $A/H_A = K/H_A + H/H_A$ (for example, A/H_A is itself a candidate) and

$\bar{\eta}(A/H_A : H/H_A) = \bar{\eta}(A : H) = \dim\left(\frac{K/H_A}{(K/H_A)\cap(H/H_A)_A}\right) = \dim\frac{K}{H_A}$ since H/H_A is core-free.

Clearly, K/H_A is solvable since A is.

Using induction, suppose that A is the smallest dimensional Leibniz algebra that is as above but not solvable. If $H_A \neq 0$, then we show that A/H_A is solvable. By Lemma 3.1.4(iv) $\bar{\eta}(A/H_A : H/H_A) = \bar{\eta}(A : H) = \dim(K/H_A) = \dim((K/H_A)/(H_A/H_A))$ and $(K/H_A)/(H_A/H_A) = K/H_A$ is solvable, which makes A/H_A solvable by the induction hypothesis. However, since H_A is solvable, A is.

Suppose instead that H_A is zero. Then K is solvable, and $\frac{A}{K} \cong \frac{H+K}{K} \cong \frac{H}{H\cap K}$, which are all solvable making A solvable. □

Corollary 3.2.4. *A Leibniz algebra A is solvable if it has a solvable second maximal subalgebra H with $\bar{\eta}(A : H) = 2$.*

Theorem 3.2.5. *Let A be a Leibniz algebra over a field of characteristic zero. If $\bar{\eta}(A : H) = dim(A/H)$ for all second maximal subalgebras of A, then A is solvable.*

Proof. If A is Lie, then the statement is known. Assume that A is not Lie, i.e. that Leib(A) is not zero. Hence A is not simple. We will use induction. Assume A is the smallest dimensional Leibniz algebra that is as stated but not solvable. Let N be a minimal ideal of A and H/N a second maximal subalegbra of A/N. By Lemma 3.1.4(iv), $\bar{\eta}(A/N : H/N) = \bar{\eta}(A : H) = \dim(A/H) = \dim\frac{A/N}{H/N}$.

Thus by the induction hypothesis A/N is solvable. N is solvable because either $[N, N] = 0$ or it is contained in Leib(A). Hence A is solvable. □

Theorem 3.2.6. *Let A be a Leibniz algebra over a field of characteristic zero. If*

$\bar{\eta}(A : H) = \bar{\eta}(A : M) + \bar{\eta}(M : H)$ *for any $H \subset M \subset A$, where H is maximal in M and M is maximal in A, then A is solvable.*

Proof. We will use induction. Assume A is the smallest dimensional Leibniz algebra that is as stated but not solvable. Also assume that A is not Lie, i.e. Leib(A) is not zero. In the Lie case, the statement is true.

Let N be a minimal ideal of A (thus solvable). Let $H/N \subset M/N \subset A/N$ be any maximal chain in A/N. By Lemma 3.1.4(iv),

$\bar{\eta}(A/N : H/N) = \bar{\eta}(A : H) = \bar{\eta}(A : M) + \bar{\eta}(M : H) = \bar{\eta}(A/N : M/N) + \bar{\eta}(M/N : H/N).$

Thus A/N is solvable. Since N is also solvable, A is solvable. □

Corollary 3.2.7. *If A is a Leibniz algebra of a field of characteristic zero, then*

$\bar{\eta}(A : H) =$dim(A/H) for any second maximal subalgebra H of A if and only if $\bar{\eta}(A : H) = \bar{\eta}(A : M) + \bar{\eta}(M : H)$ for any maximal chain $H \subseteq M \subseteq A$.

Proof. Suppose first that $\bar{\eta}(A : H) =\dim(A/H)$ for any second maximal subalgebra H of A. By Theorem 3.2.6, A is solvable. Let $H \subseteq M \subseteq A$ be a maximal chain. Then $\bar{\eta}(A : H) =\dim(A/H)$. Then by Lemma 3.1.4 part 6, $\bar{\eta}(M : H) =\dim(M/H)$. Also, by Lemma 3.1.4 part 3, $\bar{\eta}(A : M) =\dim(A/M)$ since M is maximal and A is solvable, making M a c-ideal of A. Then $\bar{\eta}(A : H) = \bar{\eta}(A : M) + \bar{\eta}(M : H)$ is equivalent to $\dim(A/H) =\dim(A/M)+\dim(M/H)$.

In the other direction, suppose $\bar{\eta}(A : H) = \bar{\eta}(A : M) + \bar{\eta}(M : H)$ for any maximal chain $H \subseteq M \subseteq A$. By Theorem 3.2.6, A is solvable. Let H be any second maximal subalgebra of A. Then we have $\bar{\eta}(A : H) = \bar{\eta}(A : M) + \bar{\eta}(M : H)$ where $\bar{\eta}(A : M) =\dim(A/M)$ since M is maximal in a solvable Leibniz algebra, making M a c-ideal. Then, by Lemma 3.1.4 part 6, $\bar{\eta}(M : H) =\dim(M/H)$ so that $\bar{\eta}(A : H) = \bar{\eta}(A : M) + \bar{\eta}(M : H) =\dim(A/M)+\dim(M/H) =\dim(A/H)$. □

Proposition 3.2.8. *Let A be a Leibniz algebra with maximal chain $H \subset M \subset A$, and K an ideal of A with $A = H + K$ and $\bar{\eta}(A : H) =$dim$\left(\frac{K}{K \cap H_A}\right)$. If $K/(K \cap H_A)$ is solvable, then $\bar{\eta}(A : M) < \bar{\eta}(A : H)$.*

Proof. Suppose that $K/(K \cap H_A)$ is a cheif factor of A. Then $K/(K \cap H_A)$ is Abelian since it is a solvable minimal ideal of $A/(K \cap H_A)$. Since $A = H + K = M + K$, then $K \cap H$ and $K \cap M$ are ideals of A. Since $H_A \subset H \subset M$, both these ideals contain $K \cap H_A$, implying that $K \cap H = K \cap M$ thus $\dim(H)=\dim(M)$, but this is a contradiction since H is maximal in M. Hence $\left(\frac{K}{K \cap H_A}\right)$ is not a chief factor of A and we can find a chief factor $\frac{N}{K \cap H_A}$ where N is an ideal of A with $N \subset K$.

Suppose N is a subalgebra of M. Then

$$\begin{aligned}\bar{\eta}(A : M) &\leq \dim\left(\frac{K}{K \cap M_A}\right) = \dim\left(\frac{K}{K \cap H_A}\right) - \dim\left(\frac{K \cap M_A}{K \cap H_A}\right)\\ &\leq \dim\left(\frac{K}{K \cap H_A}\right) - \dim\left(\frac{N}{K \cap H_A}\right) < \bar{\eta}(A : H).\end{aligned}$$

Now suppose N is not contained in M. Then since M is maximal and N is minimal, we have $A = N + M$ and so $\bar{\eta}(A : M) \leq \dim\left(\frac{N}{K \cap M_A}\right) < \bar{\eta}(A : H)$. □

3.3 Supersolvability

In the following, we establish criteria for a Leibniz algebra to be supersolvable (see Chapter 2) based on the generalized ideal index. These theorems and proofs for Lie algebras may be found in [**8**].

Corollary 3.3.1. *Let A be a Leibniz algebra and M a supersolvable maximal subalgebra of A such that $N(A)$ is not contained in M or $M_A = 0$. If $\bar{\eta}(A : H) = 2$ for some H maximal in M, then A is supersolvable.*

Proof. We know $H \subset M \subset A$ is a maximal chain. Suppose K is an ideal of A such that $\bar{\eta}(A : H) = \dim \frac{K}{K \cap H_A} = 2$. Thus, $\frac{K}{K \cap H_A}$ is solvable since it is two dimensional, and by Proposition 3.2.8, we have $\bar{\eta}(A : M) < \bar{\eta}(A : H) = 2$ so that $\bar{\eta}(A : M) = 1$.

Suppose $N(A)$ is not contained in M. Then [**16**] shows A is supersolvable. If instead $M_A = 0$, then $\bar{\eta}(A : M) = \dim \frac{K}{K \cap M_A} = 1$ for some ideal K of A with $A = M + K$ implies $\dim K = 1$, and since $M_A = 0$ and K is an ideal of A, $M \cap K = 0$. Thus $\frac{M}{M+K} = M \cong \frac{M+K}{K} = \frac{A}{K}$. Since M and K are supersolvable, so is A. □

Corollary 3.3.2. *Let A be a Leibniz algebra over a field of characteristic zero. Then A is supersolvable if $\bar{\eta}(A : H) = 2$ for any second maximal subalgebra H of A.*

Proof. Let M be any maximal subalgebra of A. Suppose $H \subset M \subset A$ a maximal chain. Suppose K is an ideal of A such that $\bar{\eta}(A : H) = \dim \frac{K}{K \cap H_A} = 2$. Thus, $\frac{K}{K \cap H_A}$ is solvable since it is two dimensional, and by Proposition 3.2.8, we have $\bar{\eta}(A : M) < \bar{\eta}(A : H) = 2$ so that $\bar{\eta}(A : M) = 1$.

Since M is maximal in A, $\bar{\eta}(A : M) = \eta(A : M) = 1$ and by [**16**], A is supersolvable. □

Chapter

4

CAP* Subalgebras

4.1 Definitions and Properties

CAP-subalgebras, or subalgebras with the cover-avoiding property discussed in Chapter 2, have many interesting properties in Lie and Leibniz algebras, some of which were also discussed in Chapter 2. We will now consider a generalization of CAP-subalgebras of Leibniz algebras which have been examined for Lie algebras by [**3**] called CAP*-subalgebras. The notion of a CAP*-subalgebra provides more results on solvability and supersolvability for Leibniz algebras.

Definition 4.1.1. Let A be a Leibniz algebra and C/D a chief factor of A. C/D is said to be a **Frattini chief factor** if $C/D \subseteq \phi(A/D)$. Similarly, C/D is a **non-Frattini chief factor** if $C/D \nsubseteq \phi(A/D)$.

Definition 4.1.2. A subalgebra H of a Leibniz algebra A is called a **CAP*-subalgebra** if H covers or avoids every non-Frattini chief factor of A.

A CAP*-subalgebra is a generalization of CAP-subalgebras. Clearly every CAP-subalgebra is a CAP*-subalgebra, but not every CAP*-subalgebra is a CAP-subalgebra. The following lemma is useful in supersolvable Leibniz algebras.

Lemma 4.1.3. *Let A be a Frattini-free Leibniz algebra and $K = N_1 \oplus \cdots \oplus N_r$, where N_i is a minimal ideal of A for every $1 \le i \le r$. Suppose every maximal subalgebra of K is a CAP*-subalgebra of A. Then dimN_i=1 for all $1 \le i \le r$.*

Proof. Let M be a maximal subalgebra of K. We claim M must contain all but one of the N_i. Suppose M did not contain two of the N_i, say N_1 and N_2. Then $M \subseteq M + N_1 \subset K$ and M is not maximal. Therefore M contains all N_i except one, say N_1.

Because A is Frattini-free, every chief factor of A is a non-Frattini chief factor. Since M is a CAP*-subalgebra of A, consider the chief factor $N_1/0$. M must cover or avoid this chief factor. Since $M + N_1 = M + 0$ would imply M contains N_1, we know it must avoid N_1 instead. Thus $M \cap N_1 = M \cap 0 = 0$. Since M is maximal, dimN_1=1. □

Lemma 4.1.4. *Let A be a Leibniz algebra. Let C be an ideal of A and C/D a non-nilpotent ideal of A/D. Then there exists a maximal subalgebra M of A containing an Engel subalgebra of A such that $A = C + M$ and D is a proper ideal of $C \cap M$.*

Proof. Since C/D is not nilpotent, there exists an element u in $C - D$ such that $C \nsubseteq E_A(u)+D$. Since $L_1(u) \subseteq C$ because C is an ideal, $A = C + E_A(u)$. Let M be any maximal subalgebra of A containing $E_A(u) + D$. Then $A = C + E_A(u)$, the result. □

The following lemma, which can be found in [**15**] for Lie algebras, contains many useful facts about the covering and avoiding properties and CAP-subalgebras. We prove them here for Leibniz algebras, though the details are the same as in [**15**].

Lemma 4.1.5. *Let H be a subalgebra of A and C/D a chief factor of A. Then*

1. *H covers C/D if and only if $H \cap C + D = C$.*

2. *H avoids C/D if and only if $(D+H)\cap C = D$.*

3. *If $H\cap C + D$ is an ideal of A, then H covers or avoids C/D. In particular, ideals are CAP-subalgebras.*

4. *The nontrivial Leibniz algebra A is simple if and only if it has no nontrivial proper CAP-subalgebras.*

5. *H covers or avoids C/D if and only if there exists an ideal N with $N \subseteq H\cap D$ and H/N covers or avoids $(C/N)/(D/N)$ respectively. Furthermore, H is a CAP-subalgebra of A if and only if there exists an ideal N of A such that $N \subseteq H$ and H/N is a CAP-subalgebra of A/N.*

6. *Let K be a subalgebra containing H. If C/D is covered (respectively, avoided) by H, then so is $(C\cap K)/(D\cap K)$.*

Proof. 1. Suppose H covers C/D. Then $H+C = H+D$. Let $x \in H\cap C+D$. Then x is in $H\cap C$ or x is in D. In either case, x is in C since $C \subseteq D$. Now suppose $x \in C$. Then $x \in H\cap C \subseteq H\cap C+D$.

Now suppose $H\cap C+D = C$. Let $x \in H+C$. Then x is in H or C. If x is in H, it is in $H+D$ as required. If it is in C and not H, then since $C = H\cap C+D$ and x is not in $H\cap C$, x is in D and thus $H+D$ as required. Then $H+C \subseteq H+D$. If x is in $H+D$, it is in $H+C$ since $D \subseteq C$. Thus $H+C = H+D$ and H covers C/D.

2. Suppose H avoids C/D. Then $H\cap C = H\cap D$. Then $(D+H)\cap C = D\cap C+H\cap C = D+H\cap D = D$.

Now suppose $(D+H)\cap C = D$. Then $D\cap C+H\cap C = D \implies D+H\cap C = D \implies H\cap D+H\cap C = H\cap D \implies H\cap C \subseteq H\cap D$. Since $D \subseteq C$, we have $H\cap D \subseteq H\cap C$ and so $H\cap C = H\cap D$ and H avoids C/D. Since $D \subseteq C$, $H\cap D \subseteq H\cap C$ and H avoids C/D as required.

3. Since C/D is a chief factor of A and $H\cap C, D \subseteq C$, we have either $H\cap C+D = D$ or $H\cap C+D = C$. If $H\cap C+D = C$, then by (i) it covers C/D. If $H\cap C+D = D$, then $(D+H)\cap C = D\cap C+H\cap C = D+H\cap C = D$ so that it avoids C/D by (ii).

4. Assume A is simple and it has a nontrivial proper CAP*-subalgebra. Since A is simple, the only chief factor of A is $A/0$. Let H be a nontrivial proper CAP-subalgebra. Then either $H \cap A = H \cap 0$ or $H + A = H + 0$. In either case, this is a contradiction.

   In the other direction, suppose A has no nontrivial proper CAP-subalgebras but it is nonsimple. Let I be a nontrivial proper ideal of A. Then I is a nontrivial proper CAP-subalgebra by (iii), a contradiction.

5. In particular, the ideal $N = (H \cap D)_A$ has the property that $H + C = H + D \iff H/N + C/N = H/N + D/N$ and $H \cap C = H \cap D \iff H/N \cap C/N = H/N \cap D/N$.

6. Suppose C/D is covered by $H \subset K$. Then $C + H = D + H$. Thus $C \cap K + H = C \cap K + H \cap K = (C + H) \cap K = (D + H) \cap K = D \cap K + H$. If C/D is avoided by $H \subset K$, then $C \cap H = D \cap H$ and thus $(C \cap K) \cap H = (C \cap H) \cap K = (D \cap H) \cap K = (D \cap K) \cap H$.

□

The following lemma is found for Lie algebras in [**3**]. However, we will also discuss the case when A is Leibniz-simple

Lemma 4.1.6. *A is simple if and only if it has no nontrivial proper CAP*-subalgebras.*

Proof. First, we prove the forward direction. Assume A is simple and it has a nontrivial proper CAP*-subalgebra. Since A is simple, the only chief factor of A is $A/0$. Let H be a nontrivial proper CAP*-subalgebra. Then either $H \cap A = H \cap 0$ or $H + A = H + 0$. In either case, this is a contradiction.

In the other direction, suppose A has no nontrivial proper CAP*-subalgebras but it is nonsimple. Let I be a nontrivial proper ideal of A.

Then I is a nontrivial proper CAP*-subalgebra, a contradiction. □

While it is true that a Leibniz algebra A being simple by the Lie definition is equivalent to having no nontrivial proper CAP*-subalgebras, it is interesting to consider the case where A is Leibniz-simple (i.e., $\mathrm{Leib}(A)$ is the only nontrivial proper ideal of A.) In this case, there is always a CAP*-subalgebra, in particular $\mathrm{Leib}(A)$, since all ideals are CAP-subalgebras and thus CAP*-subalgebras by Lemma

4.1.5(3). Conversely, if Leib(A) is the only CAP*-subalgebra of A, then it must be the only ideal since ideals are CAP*-subalgebras. However, if A is Leibniz-simple, it is not necessarily the case that Leib(A) is the only CAP*-subalgebra of A. For example, any subalgebra H that is core-free would cover Leib$(A)/0$, which is a non-Frattini chief factor assuming at least one maximal subalgbera is core-free. The following example and theorem summarize these points.

Example 4.1.7. Consider the set $sl(2,\mathbb{C})+\mathbb{C}^2$ with the following multiplication. If A, B are matrices from $sl(2,\mathbb{C})$ and v, w are vectors from $\mathbb{C}^2$, then $[A, v] = Av$ and $[A, B] = AB - BA$. In this example, $[v, A]$ and $[v, w]$ are zero.

Given $A+v$ and $B+w$ in this algebra, $[A+v, B+w] = [A, B]+[A, w]$, so $[sl(2,\mathbb{C})+\mathbb{C}^2, sl(2,\mathbb{C})+\mathbb{C}^2] = sl(2,\mathbb{C})+\mathbb{C}^2$. Since $[A+v, A+v] = [A, A]+[A, v] = Av$, Leib$(A) = \left\{[a, a] | a \in sl(2,\mathbb{C})+\mathbb{C}^2\right\} = \mathbb{C}^2$. This is the only ideal of $sl(2,\mathbb{C})+\mathbb{C}^2$.

Since $sl(2,\mathbb{C})$ is irreducible on $\mathbb{C}^2$, it is a maximal subalgebra which has a core of zero. Thus the Frattini ideal of this Leibniz algebra is zero. Hence $\mathbb{C}^2/0$ is a non-Frattini chief factor covered by $sl(2,\mathbb{C})$.

Lemma 4.1.8. *If $[A, A] \neq$Leib(A) is the only CAP*-subalgebra of A that is an ideal, then A is Leibniz-simple.*

The rest of the lemmas and properties in this section can be found for Lie algebras in [**3**].

Lemma 4.1.9. *Let C/D be a chief factor of a Leibniz algebra A. If I is an ideal of A contained in D, then C/D is a Frattini chief factor of A if and only if $(C/I)/(D/I)$ is a Frattini chief factor of A/I.*

Lemma 4.1.10. *Every non-Frattini chief factor of A is avoided by every subalgebra of $\phi(A)$.*

Proof. Let C/D be a non-Frattini chief factor of A and H a subalgebra of $\phi(A)$. Then C/D is not contained in $\phi(A/D)$. Let $I \subset \phi(A)$. Then $(I+D)/D \subset \phi(A/D)$. Since C/D is a chief factor of A, C/D is minimal in A/D and thus $C/D \cap \phi(A/D) = 0 \implies C/D \cap ((I+D)/D) = 0 \implies A \cap I = B \cap I$. $\square$

Lemma 4.1.11. *If I is an ideal of A and H is a CAP*-subalgebra of A, then we have the following:*

1. *$(H+I)/I$ is a CAP*-subalgebra of A/I.*

2. *$H \cap I$ is a CAP*-subalgebra of A.*

3. *If $I \subseteq \phi(A)$ then $H + I$ is a CAP*-subalgebra of A.*

4. *If $H + I$ is a maximal subalgebra of A, then $H + I$ is a CAP*-subalgebra of A.*

Proof. 1. Suppose $(C/I)/(D/I)$ is a non-Frattini chief factor of A/I. Lemma 4.1.9 shows that C/D is a non-Frattini chief factor of A. Since H is a CAP*-subalgebra of A, it covers or avoids C/D. If it covers C/D, then $H + C = H + D$. Then consider $x + I \in (H + I)/I + C/I$. Then $x+I \in (H+I)/I$ or $C/I \implies x \in H+I$ or $x \in C$. If $x \in H+I$ then clearly $x+I \in (H+I)/I+D/I$. If $x \in C$ then $x \in H + C = H + D \implies x + I \in (H + I)/I + D/I$. Similarly, we can show $(H + I)/I + C/I = (H + I)/I + D/I$ and we have that $(H + I)/I$ covers $(C/I)/(D/I)$.

Now suppose H avoids C/D. Then $H \cap C = H \cap D$. Then $(H+I) \cap C = (H \cap C)+I = (H \cap D)+I = (H + I) \cap B$ so that $(H + I)/I$ avoids $(C/I)/(D/I)$.

2. Suppose C/D is a non-Frattini chief factor of A. H or I avoids $C/D \implies H \cap I$ avoids C/D as well. Suppose both H and I cover C/D. If $I \subseteq \phi(A)$ then $H \cap I$ is a CAP*-subalgebra of A by Lemma 4.1.10. If I is minimal in A and $I \nsubseteq \phi(A)$ then $H \cap I$ is a CAP*-subalgebra of A. Now suppose I is not a minimal ideal of A. If $I \cap D$ is nonzero, then there is a minimal ideal J of A such that J is an ideal contained in $I \cap D$.

By (1) and induction, $(H \cap I + J)/J$ is a CAP*-subalgebra of A/J. If it covers $(C/J)/(D/J)$, then we have $H \cap I$+C $= H \cap I$ + D, and $H \cap I$ is a CAP*-subalgebra. If it instead avoids $(C/J)/(D/J)$, then $(H \cap I \cap C) + J = (H \cap I \cap D) + J$. However, J is minimal so by the dimension we can see that $H \cap I \cap C = H \cap I \cap D$.

Now, instead assume $I \cap D$ is zero. Let J be a minimal ideal of L contained in $I \cap C$, which is nonzero by assumption. Thus $J + C = J + D = C$, thus $H \cap I + C = H \cap I + J + B$. J is not contained in $\phi(A)$ because C/D is a non-Frattini chief factor. If H covers the non-Frattini chief factor $J/0$, then $H \cap I + C = H \cap I + J + D = H \cap I + D$. If H avoids $J/0$, then $H \cap J = 0$ so $H \cap I \cap J = 0$. Thus we can examine the dimension as follows:

dim$(H\cap I)$+dim$C$−dim$(H\cap I\cap C)$ =dim$((H\cap I)+C)$ =dim$((H\cap I)+J+D)$ =dim$(H\cap I)$+dim$C$−dim$(H\cap I\cap J)$−dim$(H\cap I\cap D)$+dim$(H\cap I\cap J\cap D)$.

Thus $H\cap I\cap C$ =0 and thus $H\cap I$ avoids C/D.

3. Let C/D be a non-Frattini chief factor of A. Since H is CAP*, if H covers C/D then so does $H+I$. The same is true if I covers C/D. So assume that H and I both avoid C/D. Then $(C+I)/(D+I)$ is a non-Frattini chief factor of A. Thus H covers or avoids it. If H covers this chief factor, then $H+I$ covers C/D. So instead assume H avoids $(C+I)/(D+I)$. Then $H\cap(C+I)=H\cap(D+I)$ and thus $(H\cap I)+C=(H\cap I)+D$.

4. Let C/D be a non-Frattini chief factor of A. If $H+C=H+D$, then $H+I+C=H+I+D$ and if $I+C=I+D$ then $H+I+C=H+I+D$ so it is proved if H or I covers C/D. Instead assume H and I both avoid C/D. Since $H+I$ is maximal, $(H+I)\cap C=H\cap C+I\cap C=H\cap D+I\cap D=(H+I)\cap D$, thus $H+I$ is a CAP*-subalgebra.

□

Theorem 4.1.12. *Let A be a Leibniz algebra and H a maximal subalgebra of A. Then H is a CAP*-subalgebra of A if and only if $\eta(A:H)$=dim(A/H).*

Proof. If A is Lie, then the theorem is true by Theorem 3.3 of [**3**]. Assume A is non-Lie, i.e. that Leib(A) is nonzero.

($\Longrightarrow$) Assume H is a CAP*-subalgebra of A. Then $\frac{H+H_A}{H_A}=\frac{H}{H_A}$ is a CAP*-subalgebra of $\frac{A}{H_A}$ by Lemma 4.0.9. Since H is maximal, we have the Frattini ideal of $\frac{A}{H_A}$ is 0. Thus since H/H_A is CAP*, every minimal ideal I/H_A of A/H_A has the property that $H/H_A+I/H_A=A/H_A$ and I/H_A is an ideal completion for H/H_A. Hence $\eta(A:H)$=dim(I/H_A)=dim(A/H_A).

($\Longleftarrow$) Suppose A is the smallest such Leibniz algebra such that every maximal subalgebra H of A has the property $\eta(A:H)$ =dim(A/H) but there exists such a maximal subalgebra that is not a CAP*-subalgebra.

Let H be a maximal subalgebra of A. Then we have

$$\eta\left(\frac{A}{H_A}:\frac{H}{H_A}\right)=\eta(A:H)=\dim(A/H)=\eta\left(\frac{A/H_A}{H/H_A}\right)$$

by Corollary 2.3.11. Since $\frac{A}{H_A}$ is a Lie algebra, Theorem 3.3 of [**3**] implies $\frac{H}{H_A}$ is a CAP*-subalgebra of $\frac{A}{H_A}$.

We will show H is a CAP*-subalgebra of A. Suppose C/D is a non-Frattini chief factor of A. If C and D are subalgebras of H, then $H+C=H+D$. If they are both not contained in H, then $H+C=H+D=A$ since H is maximal. This leaves the case where, $C\subseteq H$ and $D\not\subseteq H$.

Since H_A is an ideal, it is a CAP subalgebra, so that $H_A+C=H_A+D$ or $H_A\cap C=H_A\cap D$. Since $H_A\subseteq H$, $H_A+C=H_A+D$ implies $H+C=H+D$. Suppose then $H_A\cap C=H_A\cap D$. Then $(C+H_A)/H_A$ is a non-Frattini chief factor of A/H_A and so $C\cap H)+H_A=H_A$. Hence $C\cap H=D\cap H$ but then H is a CAP*-subalgbera, a contradiction. So we may assume H_A is zero.

Let I be a minimal ideal of A, which is non Lie and non simple. Then since H is maximal, $A=H+I$ and $\eta(A:H)=\dim(I)$. By hypothesis we have $\eta(A:H)=\dim(A/H)$ so $A\cap I=0$. Hence H is CAP* in A. □

4.2 Solvability

Theorem 4.2.1. *Let A be a Leibniz algebra and I an ideal of A. If all maximal subalgebras H of A such that $A=H+I$ are CAP*-subalgebras of A, then I is solvable.*

Proof. Suppose A is not solvable. Then there is a chief factor C/D of A that is non-nilpotent. By Lemma 4.1.4, there is a maximal subalgebra H of A containing an Engel subalgebra of A such that $A=C+H$ and D is a proper ideal of $C\cap H$. We will show that this H is not a CAP*-subalgebra. Clearly $H+C\neq H+D$ since $D\subset C$ and $A=C+H$. Also, $D\cap H=D$, but D is a proper ideal of $H\cap C$. □

Theorem 4.2.2. *Let A be a Leibniz algebra over a field of characteristic zero. A is solvable if and only if there exists a maximal subalgebra H of A such that H is a solvable CAP*-subalgebra of A.*

Proof. ($\Longrightarrow$) Let A be a non-Lie Leibniz algebra, i.e. Leib(A) is not zero. Suppose A is not solvable. Let H be a solvable maximal subalgebra of A. Clearly, $H + H_A$ is not A, or A would be solvable since Leib(A) is. Hence Leib(A) $\subset H$ and H/H_A is a maximal solvable subalgebra of A/H_A, which is a Lie algebra. If A/H_A is solvable, then A is, a contradiction. Thus assume A/H_A is not solvable. By Theorem 3.6 of [**3**], H/H_A is not a CAP*-subalgebra of A/H_A.

Assume H is a CAP*-subalgebra of A. Then, by Lemma 4.1.11(1), so is $(H + H_A)/H_A = H/H_A$ a CAP*-subalgebra of A/H_A, a contradiction.

($\Longleftarrow$) If A is solvable, then by Corollary 2.3.13, every maximal subalgebra H has the property $\eta(A : H)$=dim(A/H), and by Theorem 4.1.12, every such H is a CAP*-subalgebra. Since A is solvable, so is H. $\square$

The following lemma, similar to from Lemma 3.6.2 from [**5**] for Lie algebras, is used to get a condition on Cartan subalgebras.

Lemma 4.2.3. *Let A_1, A_2 be Leibniz algebras over a field of characteristic zero. If $\phi : A_1 \to A_2$ is a surjective homomorphism, and H is a Cartan subalgebra of A_2, then any Cartan subalgebra of $\phi^{-1}(H)$ is also a Cartan subalgebra of A_1.*

Proof. Let H be a Cartan subalgebra of A_2. Let J be a Cartan subalgebra of $\phi^{-1}(H)$. Then, using Lemma , $\phi(J)$ is a Cartan subalgebra of $\phi(\phi^{-1}(H)) = H$.

Let $x \in A_1$, $j \in J$ and suppose $[x, j] \in J$ and $[j, x] \in J$. Then $[\phi(x), \phi(j)]$ and $[\phi(j), \phi(x)]$ belong to $\phi(J)$, which is a Cartan subalgebra of H. Hence $\phi(x) \in \phi(J)$ and $x \in J$, so J is Cartan in A_1. $\square$

Theorem 4.2.4. *Let A be a Leibniz algebra over a field of characteristic zero. A is solvable if every Cartan subalgebra of A is a CAP*-subalgebra of A.*

Proof. Assume A is non-Lie. If it is Lie, it is true by Theorem 3.7 of [**3**]. Suppose by induction A is the smallest such Leibniz algebra that has all of its Cartan subalgebras being CAP*-subalgebras, but is not solvable.

Let I be a minimal ideal of A. Let C/I be a Cartan subalgebra of A/I. Applying Lemma 2.5.4 and Lemma 4.2.3, there is a Cartan subalgebra H of A such that $C = H + I$. Since H is Cartan, it is a

CAP*-subalgebra, and so by Lemma 4.1.11 (1), $(H+I)/I = C/I$ is a CAP*-subalgebra of A. Hence A/I is solvable. I is solvable since by Lemma 2.1.1 it is either Abelian or contained in Leib(A) which is Abelian. Thus we have A is solvable. □

4.3 Supersolvability

In this section we obtain some conditions on CAP*-subalgebras under which Leibniz algebras are supersolvable. Similar conditions for Lie algebras are proved in [**3**]. The following lemma, proved for Lie algebras in [**12**], is useful in this section.

Lemma 4.3.1. *Let A be a Leibniz algebra such that there is a series of ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_m = I$ where each I_i is an ideal of A and dim$(I_i/I_{i-1}) = 1$ for all $1 \le i \le m$. Then $A/Z_A(I)$ is supersolvable.*

Proof. Given the series of ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_m = I$, we may construct a basis for I, $\{i_1, i_2, \cdots, i_m\}$ such that $i_i + I_{i-1} \neq 0 \in I_i/I_{i-1}$. Let $a \in A$. Since each I_i is a (two-sided) ideal and has codimension one with each I_{i-1}, the matrix representation for left and right multiplication of a acting on I relative to this basis is upper triangular. Hence $A/Z^l_A(I)$ is isomorphic to a Lie subalgebra of the upper triangular matrices, and is thus supersolvable.

Consider $D = Z^l_A(I)/Z_A(I)$. Any element of D acts as 0 via left multiplication on I. Therefore it acts on each I_i as zero and hence by Lemma 2.1.2 right multiplication by D is zero on these I_i and thus K as well. Therefore by Engel's Theorem, D is nilipotent and $A/Z_A(I)$ is supersolvable. □

Theorem 4.3.2. *Let I be a solvable ideal of a Leibniz algebra A such that $N(I)$ is an ideal of A and A/I is supersolvable. If every maximal subalgebra of $N(I)$ is a CAP*-subalgebra of A, then A is supersolvable.*

Proof. First, we show that $\dfrac{N(I)+\phi(A)}{\phi(A)} = N\left(\dfrac{I+\phi(A)}{\phi(A)}\right)$.

Suppose $x+\phi(A) \in \dfrac{N(I)+\phi(A)}{\phi(A)}$, i.e. $x = n + p_1$ for some $n \in N(I), p_1 \in \phi(A)$. $x+\phi(A) \in N\left(\dfrac{I+\phi(A)}{\phi(A)}\right)$ if $[x+\phi(A), y+\phi(A)]$ and $[y+\phi(A), x+\phi(A)]$ belong to $\dfrac{I+\phi(A)}{\phi(A)}$ for all $y+\phi(A) \in \dfrac{I+\phi(A)}{\phi(A)}$, i.e. $y = i + p_2$ for some $i \in I, p_2 \in \phi(A)$.

$$[y+\phi(A), x+\phi(A)] = [i+p_2+\phi(A), n+p_1+\phi(A)] = [i+p_2, n+p_1]+\phi(A) = [i, n+p_1]+[p_2, n+p_1]+\phi(A).$$

The first term belongs to I since it is an ideal. The second belongs to $\phi(A)$ since it is also an ideal. The left multiplication is a similar calculation. Hence $\frac{N(I)+\phi(A)}{\phi(A)} \subseteq N\left(\frac{I+\phi(A)}{\phi(A)}\right)$. Similarly, $\frac{N(I)+\phi(A)}{\phi(A)} \supseteq N\left(\frac{I+\phi(A)}{\phi(A)}\right)$.

Due to this fact, using Lemma 4.1.11(1), we may assume that $\phi(A) = 0$. By Corollary 2.5 of [**2**], $N(A)$ is the direct sum of some number of minimal ideals of A. Since $N(I)$ is an ideal of A, $N(I) \subseteq N(A)$ and so $N(I)$ can be written as a direct sum of minimal ideals. By Lemma 4.1.3, the dimension of each of these minimal ideals is one.

By Lemma 4.3.1, this implies $A/Z_A(N(A))$ is supersolvable. A is solvable, so by [**2**] we have $Z_A(N(A)) = N(A)$. Hence A is supersolvable since $A/N(A)$ and $N(A)$ are. □

Corollary 4.3.3. *Let A be a solvable Leibniz algebra. If every maximal subalgebra of $N(A)$ is a CAP*-subalgbera of A, then A is supersolvable.*

Proof. This follows from Theorem 4.3.1 with $A = H$.

□

Chapter

5

SCAP Subalgebras

5.1 Definitions and Properties

In Chapter 4, we considered a generalization of CAP subalgebras which only cover or avoid those chief factors that are not contained in the Frattini ideal of A/D. In this chapter, we will now examine a generalization of CAP subalgebras which covers or avoids every chief factor in a chief series. This concept and related theorems for Lie algebras have been studied in [**4**] by Chehrazi and Salemkar.

Definition 5.1.1. Let A be a Leibniz algebra. A **chief series** of A is a series of ideals $0 = A_0 \subset A_1 \subset \cdots \subset A_t = A$ such that each A_i/A_{i-1} is a chief factor of A.

Definition 5.1.2. Let A be a Leibniz algebra. H is a **SCAP-subalgebra** of A if there is a chief series of A such that H covers or avoids each chief factor of the series.

Lemma 5.1.3. *Let H be a subalgebra of A and $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$ an ideal series. If H covers*

A_j/A_i, then H covers A_s/A_r for all $i \le r < s \le j$. If H avoids A_j/A_i, then H avoids A_s/A_r for all $i \le r < s \le j$.

Proof. Suppose H covers A_j/A_i. Then $H+A_j = H+A_i$, which implies A_j is an ideal of A_i+H. Since we have a chief series, A_s/A_i is an ideal of A_j/A_i, so A_s is an ideal of A_i+H. A_i is also an ideal of A_r, so therefore A_s is an ideal of A_r+H. Thus $A_s+H = A_r+H$ and so H covers A_s/A_r.

Similarly, if H avoids A_j/A_i, it avoids A_s/A_r. □

This lemma allows us to make a link between c-ideals, discussed previously, with SCAP subalgebras.

Lemma 5.1.4. *If H is a c-ideal of a Leibniz algebra A, then H is a SCAP-subalgebra of A.*

Proof. Let H be a c-ideal of A. Then there is an ideal K of A such that $A = K+H$ and $K \cap H \subseteq H_A$. Then $0 \subset H \cap K \subset K \subseteq A$ is an ideal series of A. Since $H = H+(H\cap K) = H+0$, $H+A = H+K$, and $H \cap K = H \cap (H \cap K)$, H covers or avoids each term of this series. By Lemma 5.1.3, any chief series is covered or avoided by H and thus H is a SCAP-subalgebra. □

Lemma 5.1.5. *If H is a SCAP-subalgebra of A, then H/I is a SCAP-subalgebra of A/I for any ideal $I \subset H$.*

Proof. Since the ideals of A/I are in one-to-one correspondence with the ideals of A containing I, any chief series of A/I corresponds to a chief series in A possibly with some terms mapping to zero in A/I. If H covers the chief series in A, H/I covers the chief series mapped into A/I and is thus a SCAP-subalgebra. □

Lemma 5.1.6. *A is simple if and only if it has no nonzero proper SCAP-subalgebras.*

Proof. ($\Longrightarrow$) Suppose A is simple. Then the only chief factor of A is $A/0$. Suppose H is a SCAP subalgebra. The only chief series is $0 \subset A$. Suppose H covers or avoids $A/0$. Then $H+A = H+0$ implies $H = 0$ or $H = A$, or $H \cap A = H \cap 0$ implies $H = 0$.

($\Longleftarrow$) Suppose A has no nonzero proper SCAP-subalgebras. Then it has no c-ideals, and hence no nonzero proper ideals. □

Lemma 5.1.7. *Suppose I is a minimal Abelian ideal of A such that A/I is nilpotent. Then all maximal subalgebras of Cartan subalgebras of A are SCAP-subalgebras of A.*

Proof. A/I is nilpotent and thus solvable, I is Abelian and thus solvable, so A is solvable. Hence there is a Cartan subalgebra C of A and $(C+I)/I$ is a Cartan subalgebra of A/I. Let C_1 be a maximal subalgebra of C. A/I is nilpotent so $(C+I)/I = A/I$ and hence $A = C+I$. Therefore $(C_1+I)/I$ is a maximal subalgebra of A/I, hence an ideal since A/I is nilpotent.

Consider the Fitting decomposition of $A = C + L_1(C)$ with respect to u. Any element of $L_1(C)$, say $x = c+i$ where $c \in C, i \in I$ since $A = C+I$. C is Cartan so there exists a k so that $ad_u^k(x) = ad_u^k(i) \in I$. Hence $L_1(C) \in I$, a minimal ideal. $[A, L_1(C)] = [C + L_1(C), L_1(C)] \subseteq L_1(C)$. Hence $L_1(C)$ is a left ideal. Thus by Lemma 2.1.1 it is a right ideal as well since i tis contained in a minimal ideal.

But $L_1(C)$ is then an Abelian ideal contained in I, a minimal ideal. This implies $L_1(C)$ is zero or I. If $L_1(C)$ is zero then $A = C$ and so A is nilpotent. So $L_1(C) = I$ and thus $0 \subset I \subset C_1 + I \subset A$ is an ideal series of A. C_1 avoids $I/0$ and $(C_1+I)/I$. It covers $A/(C_1+I)$. By Lemma 5.1.3 C_1 is a SCAP-subalgebra. □

5.2 Solvability

Theorem 5.2.1. *Let A be a Leibniz algebra. If every maximal subalgebra H of A satisfying $A = H + I$ for some ideal I of A is a SCAP-subalgebra of A, then I is solvable.*

Proof. This theorem is true for Lie algebras by Theorem 3.1 of [**5**], so assume A is non-Lie and so Leib(A) is not zero. Assume A is the smallest Leibniz algebra such that every maximal subalgebra H satisfying $A = H+I$ for some ideal I is a SCAP-subalgebra, but I is not solvable. Consider $A/$Leib(A), a Lie algebra. Let $M/$Leib(A) be a maximal subalgebra of $A/$Leib(A), then M is maximal in A. Suppose there is an ideal $K/$Leib(A) such that $M/$Leib$(A) + K/$Leib$(A) = A/$Leib(A), then $M + K = A$ and hence M is a SCAP-subalgbera of A. By Lemma 5.1.5, $M/$Leib(A) is a SCAP-subalgebra of $A/$Leib(A). Hence $K/$Leib(A) is solvable by the induction hypothesis. Leib(A) is Abeilan and thus solvable, so we have K is solvable. □

Theorem 5.2.2. *A Leibniz algebra A is solvable if and only if every maximal subalgebra of A is a SCAP-subalgebra of A.*

Proof. ($\Longrightarrow$) Suppose A is solvable. Then, by Theorem 4.4.1 in [**16**], every maximal subalgebra is a CAP-subalgebra and thus a SCAP-subalgebra.

($\Longleftarrow$) Suppose every maximal subalgebra of A is a SCAP-subalgebra. Then by Theorem 5.2.1 A is solvable. □

Theorem 5.2.3. *Let A be a Leibniz algebra over a field that has either characteristic zero or is algebraically closed with characteristic greater than 5. Then A is solvable if and only if there is a solvable maximal subalgebra H of A such that H is a SCAP-subalgebra of A.*

Proof. ($\Longrightarrow$) Suppose A is solvable. Then by Theorem 5.2.2, every maximal subalgebra is solvable and an SCAP-subalgebra.

($\Longleftarrow$) Suppose by induction A is the smallest non-Lie Leibniz algebra such that there is a solvable maximal subalgebra H of A such that H is a SCAP-subalgebra of A. but A is not solvable. If $\mathrm{Leib}(A) \subset H$ then $H/\mathrm{Leib}(A)$ is a SCAP-subalgebra of $A/\mathrm{Leib}(A)$. H and $\mathrm{Leib}(A)$ are solvable, so $H/\mathrm{Leib}(A)$ is as well. Hence by induction $A/\mathrm{Leib}(A)$ and thus A are solvable.

Now suppose $\mathrm{Leib}(A)$ is not contained in H. H is maximal, so we have $H+\mathrm{Leib}(A) = A$, but H and $\mathrm{Leib}(A)$ are both solvable, so we have A is solvable. □

Theorem 5.2.4. *Let A be a Leibniz algebra over a field with at least $\dim A$ elements. If every maximal subalgebra of A that contains the normalizer of a maximal nilpotent subalgebra of A is a SCAP-subalgebra of A, then A is solvable.*

Proof. Assume A is a non-Lie Leibniz algebra. If A is Lie, it is true by Theorem 3.4 of [**5**]. Let $H/\mathrm{Leib}(A)$ be a maximal subalgebra of $A/\mathrm{Leib}(A)$ that contains the normalizer of a maximal nilpotent subalgebra $N/\mathrm{Leib}(A)$ of $A/\mathrm{Leib}(A)$. By Lemma 5.1.2 of [**16**], $N = C + \mathrm{Leib}(A)$ where C is a maximal nilpotent subalgebra of A.

First, $N_A(C) \subseteq N_A(N)$ and $(N_A(N) + \mathrm{Leib}(A))/\mathrm{Leib}(A) \subseteq N_{A/\mathrm{Leib}(A)}(Q/\mathrm{Leib}(A)) \subseteq H/\mathrm{Leib}(A)$ implies $N_A(C) \subseteq H$. Thus H is a SCAP-subalgebra of A and so by Lemma 5.1.5 $H/\mathrm{Leib}(A)$ is a SCAP-subalgebra of $A/\mathrm{Leib}(A)$. By induction, $A/\mathrm{Leib}(A)$ is solvable, and since $\mathrm{Leib}(A)$ is Abelian and hence solvable, so is A. $\square$

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