

ABSTRACT

YONG, YICONG. Portfolio Optimization for Stock with Delays. (Under the direction of Tao Pang.)

We consider a portfolio optimization problem over an infinite time horizon. Our problem extends the classical Merton's model by capturing infinite historical memory. We formulate the problem as a stochastic control problem with delay where the goal is to choose the optimal investment and consumption controls that maximize total expected discounted logarithmic and hyperbolic absolute risk aversion (HARA) utility functions of consumption. Dynamic programming method is used to derive the Hamilton-Jacobi-Bellman (HJB) equation and we then prove the existence and uniqueness of a viscosity solution to the problem for each case. Finally, we show that the solution to each corresponding HJB equation is equivalent to each value function in the verification theorems.

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Portfolio Optimization for Stock with Delays

by
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DEDICATION

To my father, Jiongmin Yong.

BIOGRAPHY

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Chapter 1

Introduction

Optimal investment and consumption problems with delay capturing the entire historical performance of portfolios are studied in this dissertation. We use the utility function criterion to model the behavior of an investor who distributes his or her wealth between a risky and a riskless asset while also consuming part of his or her wealth. The investor seeks to select the optimal investment and consumption rate so that the total discounted expected utility is maximized. We look at two cases of utility function: logarithmic utility and hyperbolic absolute risk aversion (HARA) utility.

1.1 Background and Literature Review

In the classical Merton's portfolio problem, an investor's initial wealth is allocated between a risky asset and a riskless asset. At each point in time, he or she must decide the amount of his or her wealth to consume and the amount to invest in the two types of assets. The goal is to choose an optimal consumption rate that maximizes total expected utility over finite or, theoretically, infinite lifetime of the investor. The price of the risky asset is usually governed by a geometric Brownian motion, where historical price information is not described. Interest rate and volatility are held as fixed constants in the model.

A number of extensions to the Merton's model to make it more realistic have been done since the publication of Merton's papers in 1969 [43] and 1971 [44]. Stochastic interest rate, stochastic volatility, and/or stochastic returns are considered in Fleming and Hernandez-Hernandez [22], [23], as well as in Pang and Varga [57]. Stochastic volatility

with stochastic dividends is incorporated in Pang and Varga [58]. Fleming and Pang [24] and Pang [52] have developed models with correlated interest rate and risky asset price. Noh and Kim [47] combined both stochastic volatility and correlated interest rate into their model. Transaction costs are taken into account in Magill and Constantinides [42], Davis and Norman [13], and Shreve and Soner [62]. Bielecki and Pliska [3], as well as Fleming and Sheu [26] have considered models where the mean returns of individual assets are affected by underlying economic factors such as dividend yields, return on equity, interest rates, and unemployment rates. Karatzas, Lehoczky and Shreve [35] and Lehoczky, Sethi, and Shreve [39] have modeled bankruptcy in their extension of the Merton's model. Fleming and Pang [25] also applied the Merton's model to an economic production and consumption problem.

In these extensions, the models still only depend on the current information. This is not very realistic since historical performance of the risky asset could impact its future performance. In reality, the investor makes his or her decision not only based on the current price information but also on historical trend. For example, stocks that have performed well consistently in the past are more likely to be purchased than ones that did not perform so well. To remedy this shortcoming of the original Merton's model, many researchers have started to incorporate past information into the model by introducing a delay component that captures memory of past asset information. Many of these models involve memory variables modeled as an exponential moving average function of historical wealth and a function that contains the historical wealth at a specific point in time before the present time. This type of model gives rise to an infinite dimensional problem which poses some difficulty. However, Elsanosi and Larssen [14] as well as Larssen and Risebro [38] have shown that it is possible to reduce such a problem into finite dimensions under certain conditions. Most papers such as Chang, Pang, and Pemy [4],[5],[6], Chang, Pang and Yang [7], as well as Pang and Hussain [54],[56] have only considered delay variables that only capture finite or bounded memory. In Pang and Hussain [55], they have considered delay variables that capture the complete memory of past information in the wealth dynamics of an investment portfolio.

Models with delay have vast range of applications in the real world in addition to optimal investment and consumption problems in finance. Bauer and Rieder [1] studied a deterministic fluid problem with delay arising from admission control in ATM communication networks. Chen and Wu [8], Oksendal and Sulem [50] and Oksendal, Sulem and

Zhang [49] Oksendal and Sulem used maximum principle to analyze stochastic optimal control problems with delay. Elsanosi, Oksendal and Sulem [15] as well as Larssen and Risebro [38] studied harvesting problems with delayed dynamics. Federico [17] modeled the management of a pension fund with surplus depending on past performance of the fund's wealth. Gozzi and Marinelli [29] and Gozzi, Marinelli and Savin [30] studied a class of dynamic advertising problems allowing dynamics of the product goodwill to depend on past values and previous advertising levels. Many more references can be found in the references section and the references therein.

1.2 Contribution and Outline

In this dissertation, we consider an optimal investment and consumption problem incorporating delay that captures complete memory of the historical price and wealth information. We look at a model that is different and more complex than the one presented in Pang and Hussain [55], as we want to incorporate delay directly into the dynamics of the risky asset's price itself, rather than just into the dynamics of the wealth process.

In chapter 2, we present some mathematical preliminary knowledge before diving into the problem we will study.

Chapter 3 presents the formulation of our infinite delay problem. A derivation of the Hamilton-Jacobi-Bellman (HJB) equation with a general utility function of consumption is also included in this chapter.

The special case of logarithmic utility function of consumption is studied in chapter 4. Optimal controls are obtained and the corresponding HJB equation is transformed into a form with desirable properties. A perturbed elliptic equation is used to facilitate the establishment of existence and uniqueness of a viscosity solution. The method of sub/supersolutions is used to obtain the lower and upper bounds of the solution which are crucial pieces for the verification theorem. Lastly, the verification theorem is proved.

Chapter 6 continues with the case of HARA utility function of consumption, where existence and uniqueness of a viscosity solution are also presented, concluding with its verification theorem.

Chapter 2

Preliminaries

2.1 Merton's Portfolio Optimization Problem

We briefly present statement of the well-known classical Merton's portfolio optimization problem, which originally gave rise to this vast research area, and its main results. This problem is important as its results have many applications in continuous-time finance. In Merton's formulation, the investor invests his or her wealth in one riskless asset and one risky asset. The price of the risky asset $S(t)$ follows

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \quad (2.1)$$

where μ and σ are return and volatility of the risky asset respectively and $B(t)$ is a Brownian motion. The price of the riskless asset $P(t)$ follows

$$dP = rP(t)dt, \quad (2.2)$$

where r is the risk-free rate. It is assumed that $r < \mu$ and $\sigma > 0$. Let $\pi(t) \in [0, 1]$ be the proportion of investor's wealth in the risky asset and $c(t) \geq 0$ be the consumption rate. Then the investor's wealth $X(t) \geq 0$ follows

$$dX(t) = (1 - \pi(t))X(t)r dt + \pi(t)X(t)(\mu dt + \sigma dB(t)) - c(t)X(t)dt. \quad (2.3)$$

The control is $(\pi(t), c(t))$ and the investor looks to maximize his or her total discounted expected utility of consumption over a finite time horizon. The value function is given by

$$V(x) = \max_{(\pi(t), c(t))} J(x, \pi, c) = \max_{(\pi(t), c(t))} \mathbf{E} \left[\int_0^T e^{-\beta t} U(c(t)X(t)) dt \right], \quad (2.4)$$

where $U(\cdot)$ is the utility function. The Hamilton-Jacobi-Bellman (HJB) equation for this problem is of the form

$$\beta V = \max_{\pi} \left[\frac{\sigma^2 \pi^2 x^2}{2} V_{xx} + (\mu - r) \pi x V_x \right] + r x V_x + \max_{c \geq 0} (U(cx) - cx V_x). \quad (2.5)$$

Assuming hyperbolic absolute risk aversion (HARA) type utility function

$$U(C) = \frac{1}{\gamma} C^\gamma,$$

where $0 < \gamma < 1$ and C represents consumption, and suppose the solution is of form

$$V(x) = K x^\gamma,$$

the corresponding optimal controls are given by

$$\pi^* = \frac{(\mu - r)}{(1 - \gamma)\sigma^2}, \quad (2.6)$$

$$c^* = (\gamma K)^{\frac{1}{\gamma-1}}. \quad (2.7)$$

It turns out that K can be solved and has a positive solution if

$$\beta > \frac{(\mu - r)^2 \gamma}{2\sigma^2(1 - \gamma)} + r\gamma.$$

Merton's portfolio optimization problem is intriguing in that it unexpectedly has a closed form solution. However, the explicit solutions to many optimal control problems cannot be found analytically. Now we will discuss some of the recent developments that extend Merton's model to one that models delayed information.

2.2 Existing Delay Models

As previously mentioned, Merton's model does not capture the memory of historical price information, creating an overly simplistic view of the market in real life. In order to incorporate historical price information into the problem, a delay component needs to be considered. However, most existing models with delay give rise to infinite dimensional problems that are difficult to solve. One technique has been formulated to successfully reduce these infinite dimensional problems into finite dimensional, of which its ideas are presented in the next subsection.

2.2.1 Reduction from An Infinite Dimension Problem

Consider a general 1-dimensional stochastic control problem with delay of the following form

$$\begin{cases} dX(t) = \mu(t, X(t), Y(t), Z(t), \alpha(t))dt \\ \quad + \sigma(t, X(t), Y(t), Z(t), \alpha(t))dB(t), & t \in [0, T], \\ X(t) = \varphi(t), & t \in [-\delta, 0], \end{cases} \quad (2.8)$$

where

$$Y(t) = \int_{-\delta}^0 e^{\lambda\theta} X(t + \theta) d\theta, \quad (2.9)$$

$$Z(t) = X(t - \delta). \quad (2.10)$$

$Y(t)$ describes a weighted average of recent information while $Z(t)$ describes the historical value at a particular time in the past. In addition, $\delta > 0$, $\varphi \in C[-\delta, 0]$, $B(t)$ is a standard Brownian motion and α is the control in an admissible control space \mathcal{A} . Let the value function be

$$\begin{cases} V(t, \varphi) = \sup_{\alpha \in \mathcal{A}} J(t, \varphi, \alpha), & (t, \varphi) \in [0, T] \times C([-\delta, 0]), \\ V(T, \varphi) = g(X(0), Y(0)), & \varphi \in C([-\delta, 0]), \end{cases} \quad (2.11)$$

where

$$J(t, \varphi, \alpha) = \mathbf{E} \left[\int_t^T f(t, X(t), Y(t), \alpha(t)) dt + g(X(T), Y(T)) \right]. \quad (2.12)$$

Because $C[-\delta, 0]$ is infinite dimensional, it is very complicated to solve such a problem. However, if the value function only depends on φ through

$$X(0) = \varphi(0), \quad Y(0) = \int_{-\delta}^0 e^{\lambda\theta} \varphi(\theta) d\theta, \quad Z(0) = \varphi(-\delta),$$

then it is possible to derive the HJB equation for the value function, thus reducing the infinite dimension problem to a finite dimension problem. This is proved by Larssen and Risebro in [38]. Note that the above problem only contains finite memory information. For the problem that we consider in this dissertation, we capture the entire history of price information by using the delay variable

$$Y(t) = \int_{-\infty}^0 e^{\lambda\theta} X(t + \theta) d\theta.$$

Because complete memory is described in this delay variable, there is no need for the variable $Z(t)$. This type of model is also studied in [56]. However, we consider a model that is different from the one presented in their paper. We note here that an additional benefit of considering complete memory is that it is a generalization of the finite memory case where we can essentially obtain the finite memory case by adding an indicator,

$$Y(t) = \int_{-\infty}^0 e^{\lambda\theta} X(t + \theta) \mathbf{1}_{[-\delta, 0]} d\theta.$$

The original approach of considering a fixed window of $[-\delta, 0]$ was also very restricting in that if it was needed to start from a different historical value, the window would then be shifted. Our method provides much more flexibility in this regard. In the next section, we introduce the main method used to derive HJB equations for the type of problems we study.

2.3 Dynamic Programming Principle and HJB Equation

Bellman introduced the dynamic programming approach in the early 1950s to solve optimal control problems. We here consider a standard formulation of a stochastic control problem, which can be found in any standard textbook, such as [60]. Let

$$dX(t) = \mu(X(t), \alpha(t))dt + \sigma(X(t), \alpha(t))dB(t), \quad (2.13)$$

where $B(t)$ is a d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P; \mathbb{F})$ with $\mathbb{F} = \{\mathcal{F}^t, t \geq 0\}$ being the P -augmented natural filtration generated by $\{B(t), t \geq 0\}$. $\alpha = (\alpha(t))$ is the control and is \mathbb{F} -progressively measurable and takes its values in $A \subset \mathbb{R}^m$. Functions $\mu : \mathbb{R}^n \times A \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \rightarrow \mathbb{R}^{n \times d}$ are measurable and satisfy

$$|\mu(x, a) - \mu(\hat{x}, a)| + |\sigma(x, a) - \sigma(\hat{x}, a)| \leq K|x - \hat{x}|$$

for some $K \geq 0$ and $\forall x, \hat{x} \in \mathbb{R}^n$ and $\forall a \in A$.

Let us first consider a finite horizon problem where we have $0 < T < \infty$. Let \mathcal{A} be the set of controls α such that

$$\mathbf{E} \left[\int_0^T |\mu(0, \alpha(t))|^2 + |\sigma(0, \alpha(t))|^2 dt \right] < \infty. \quad (2.14)$$

Suppose $f : [0, T] \times \mathbb{R}^n \times A \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are two measurable functions where g is bounded from below or g satisfies

$$|g(x)| \leq C(1 + |x|^2), \quad \forall x \in \mathbb{R}^n$$

for some constant C . Let $\mathcal{A}(t, x) \subset \mathcal{A}$ be a subset of controls such that for $(t, x) \in [0, T] \times \mathbb{R}^n$

$$\mathbf{E} \left[\int_t^T |f(\tau, X(\tau), \alpha(\tau))| d\tau \right] < \infty. \quad (2.15)$$

Then for $\alpha \in \mathcal{A}(t, x)$, the value function is defined as

$$V(t, x) = \sup_{\alpha \in \mathcal{A}(t, x)} J(t, x, \alpha), \quad (2.16)$$

where

$$J(t, x, \alpha) = \mathbf{E} \left[\int_t^T f(\tau, X(\tau), \alpha(\tau)) d\tau + g(X(T)) \right].$$

The function g represents a terminal condition for the finite horizon problem. This function is not needed when we formulate the infinite horizon problem.

For an infinite horizon problem, (2.15) is replaced by

$$\mathbf{E} \left[\int_0^\infty e^{-\beta\tau} |f(X(\tau), \alpha(\tau))| d\tau \right] < \infty, \quad (2.17)$$

for $\beta > 0$ and the value function then changes to

$$V(x) = \sup_{\alpha \in \mathcal{A}(x)} J(x, \alpha), \quad (2.18)$$

where

$$J(x, \alpha) = \mathbf{E} \left[\int_0^\infty e^{-\beta\tau} f(X(\tau), \alpha(\tau)) d\tau \right].$$

Finally, dynamic programming principle for these problems are stated separately below.

Theorem 2.3.1. *For a finite horizon problem, let $(t, x) \in [0, T] \times \mathbb{R}^n$. Denote $\mathcal{T}_{t,T}$ as the set of stopping times taking values in $[t, T]$ and $\mathcal{T} = \mathcal{T}_{0,\infty}$. Then*

$$\begin{aligned} V(t, x) &= \sup_{\alpha \in \mathcal{A}(t, x)} \sup_{\theta \in \mathcal{T}_{t,T}} \mathbf{E} \left[\int_t^\theta f(\tau, X(\tau), \alpha(\tau)) d\tau + V(\theta, X(\theta)) \right] \\ &= \sup_{\alpha \in \mathcal{A}(t, x)} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbf{E} \left[\int_t^\theta f(\tau, X(\tau), \alpha(\tau)) d\tau + V(\theta, X(\theta)) \right]. \end{aligned} \quad (2.19)$$

For an infinite horizon problem, let $x \in \mathbb{R}^n$. Then

$$\begin{aligned} V(x) &= \sup_{\alpha \in \mathcal{A}(x)} \sup_{\theta \in \mathcal{T}} \mathbf{E} \left[\int_0^\theta e^{-\beta\tau} f(X(\tau), \alpha(\tau)) d\tau + e^{-\beta\theta} V(X(\theta)) \right] \\ &= \sup_{\alpha \in \mathcal{A}(x)} \inf_{\theta \in \mathcal{T}} \mathbf{E} \left[\int_0^\theta e^{-\beta\tau} f(X(\tau), \alpha(\tau)) d\tau + e^{-\beta\theta} V(X(\theta)) \right]. \end{aligned} \quad (2.20)$$

By applying dynamic programming principle to an optimal control problem, along with Ito's formula, we can get a Hamilton-Jacobi-Bellman Equation (HJB) equation which establishes relationship among a family of optimal control problems with different initial times and states. For the finite horizon problem, the HJB equation is a partial differential equation of the form

$$-\frac{\partial V}{\partial t}(t, x) - \sup_{a \in A} [L^a V(t, x) + f(t, x, a)] = 0,$$

which implies

$$\begin{cases} -\frac{\partial V}{\partial t}(t, x) - H(t, x, D_x V(t, x), D_x^2 V(t, x)) = 0, & \forall (t, x) \in [0, T] \times \mathbb{R}^n, \\ V(T, x) = g(x), & \forall x \in \mathbb{R}^n, \end{cases} \quad (2.21)$$

where

$$L^a V = \mu(x, a) D_x V + \frac{1}{2} \text{tr}(\sigma(x, a) \sigma(x, a)^\top D_x^2 V)$$

The function H is the Hamiltonian of the optimal control problem. And for an infinite horizon problem, the HJB equation is of the form

$$-\beta V(x) - \sup_{a \in A} [L^a V(x) + f(x, a)] = 0,$$

leading to

$$-\beta V(x) - H(x, D_x V(x), D_x^2 V(x)) = 0, \quad \forall x \in \mathbb{R}^n. \quad (2.22)$$

If the HJB equation is solvable, then verification technique obtains optimal control from the maximizer or minimizer of the Hamiltonian in the HJB equation and verifies that value function is the unique solution to the HJB equation. Note that the above theory

applies to stochastic control problems without delay. In [37], Larssen has proved that dynamic programming principle can also be applied to stochastic control problems for systems with delay. We now present the notion of a solution to the HJB equation in the next section.

2.4 Classical Solution

In the case of a stochastic optimal control problem, the HJB equation we study is a nonlinear second order partial differential equation, usually in the following form:

$$\mathbf{L}u = f(x, u), \quad (2.23)$$

where

$$\mathbf{L}u = \sum_{i,j=1}^n a^{ij}(x) D_{ij}u + \sum_{j=1}^n b^j(x) D_j u$$

and

$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2 \quad (2.24)$$

for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n - \{0\}$. The classical solution, if exists, solves the partial differential equation and satisfies regularity properties, which then gives the solution to the original stochastic optimal control problem. Subolutions and supersolutions play an important role in the existence of classical solutions and we will introduce them briefly in the next subsection.

2.4.1 Method of Subolutions and Supersolutions

The notion of having subsolution and supersolution facilitates the approximation of a classical solution by formulating a comparison theorem. We reference [28] and [59] for definition and some main results relating to sub/supersolutions.

Definition 2.4.1. *A function $u \in C_{loc}^2(\mathbb{R}^n)$ is a sub(super)solution of (2.23) if*

$$\mathbf{L}u \geq (\leq) f(x, u).$$

In addition, if \hat{u} is a subsolution and \tilde{u} is a supersolution, and $\hat{u} \leq \tilde{u}$, then $\langle \hat{u}, \tilde{u} \rangle$ is an ordered pair of sub/supersolutions.

Here, $C_{loc}^2(\mathbb{R}^n)$ denotes the class of functions in $C^2(\Omega)$ for every bounded domain $\Omega \in \mathbb{R}^n$. The operator \mathbf{L} is assumed to be uniformly elliptic, that is $\frac{\Lambda}{\lambda}$ is bounded in (2.24). The coefficients of \mathbf{L} are in $C^\alpha(\overline{\Omega})$ and f is locally Hölder continuous in x and Lipschitz continuous in u . In fact, a sequence of subsolutions converging to a minimal solution and a sequence of supersolutions converging to a maximal solution can be constructed. Details of such construction can be found in [59]. The following theorems are Theorem 5.1 and Theorem 5.3 from [59] respectively.

Theorem 2.4.1. *Let $\langle \hat{u}, \tilde{u} \rangle$ be an ordered pair of sub/supersolutions of (2.23) and let f be Lipschitz continuous in $\langle \hat{u}, \tilde{u} \rangle$. Then (2.23) has a maximal solution $\bar{u} \in C_{loc}^{2+\alpha}(\mathbb{R}^n)$ and a minimal solution $\underline{u} \in C_{loc}^{2+\alpha}(\mathbb{R}^n)$ such that*

$$\hat{u}(x) \leq \underline{u}(x) \leq \bar{u}(x) \leq \tilde{u}(x) \quad \text{in } \mathbb{R}^n.$$

Theorem 2.4.2. *Let $\langle \hat{u}, \tilde{u} \rangle$ be an ordered pair of sub/supersolutions of (2.23) and let f be Lipschitz continuous and nonincreasing in $\langle \hat{u}, \tilde{u} \rangle$. If the maximal and minimal solutions \bar{u}, \underline{u} possess the same limit, then $\bar{u} = \underline{u}$ and is the unique solution of (2.23) in $\langle \hat{u}, \tilde{u} \rangle$.*

The unique solution is the classical solution to (2.23). Thus we see that classical solution is bounded in between an arbitrary pair of sub/supersolutions. This will be very useful in the analysis of our problem.

2.5 Viscosity Solution

In the dynamic programming approach, the HJB equation ideally has a classical solution. A classical solution is one that not only satisfies the HJB equation, but is also smooth. However, for most problems, this cannot be achieved. For example, the problem that we will consider is of degenerate form, which means it may not have a classical solution at all. In the early 1980s, to overcome the difficulty of not being able to analytically solve for a regular classical solution, Crandall and Lions introduced the notion of viscosity solutions, which are allowed to be non-smooth and even discontinuous. Definition and main results on viscosity solutions below are taken from [27] and [65]. Consider the following HJB

equation for a stochastic optimal control problem over a finite horizon

$$\begin{cases} -v_t(t, x) + \sup_{u \in U} G(t, x, u, -v_x(t, x), -v_{xx}(t, x)) = 0, & (t, x) \in [0, T) \times \mathbb{R}^n, \\ v|_{t=T} = h(x), & x \in \mathbb{R}^n, \end{cases} \quad (2.25)$$

where

$$G(t, x, u, p, P) \equiv \frac{1}{2} \text{tr}(P\sigma(t, x, u)\sigma(t, x, u)^\top) + \langle p, b(t, x, u) \rangle - f(t, x, u),$$

$$\forall (t, x, u, p, P) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{S}^n.$$

U is a metric space and \mathcal{S}^n denotes the set of all $n \times n$ symmetric matrices. Here we give one way of characterizing viscosity solutions.

Definition 2.5.1. *A function $v \in C([0, T] \times \mathbb{R}^n)$ is a viscosity sub(super)solution to (2.25) if*

$$v(T, x) \leq (\geq) h(x), \quad \forall x \in \mathbb{R}^n,$$

and for any $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$, whenever $v - \varphi$ attains a local maximum (minimum) at $(t, x) \in [0, T) \times \mathbb{R}^n$, we have

$$-\varphi_t(t, x) + \sup_{u \in U} G(t, x, u, -\varphi_x(t, x), -\varphi_{xx}(t, x)) \leq 0.$$

The function v is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

The reason viscosity solution can be considered as a generalized solution to the HJB equation is due to the following proposition, which is Proposition 5.8 from [65].

Proposition 2.5.1. *$v \in C^{1,2}([0, T] \times \mathbb{R}^n)$ is a viscosity solution of (2.25) if and only if it is a classical solution of (2.25).*

2.5.1 Singular Perturbations and Vanishing Viscosity

Suppose (2.25) is a degenerate partial differential equation. We can turn it into non-degenerate form by perturbing and adding a viscosity term, which will eventually vanish.

To illustrate this, we consider

$$\begin{cases} -v_t^\varepsilon + \varepsilon \Delta v^\varepsilon + \sup_{u \in U} G(t, x, u, -v_x^\varepsilon, -v_{xx}^\varepsilon) = 0, & (t, x) \in [0, T) \times \mathbb{R}^n, \\ v^\varepsilon|_{t=T} = h(x), & x \in \mathbb{R}^n, \end{cases} \quad (2.26)$$

Now this equation is no longer degenerate, which means that it admits a unique classical solution under some conditions. The next proposition, which is Proposition 5.10 from [65], shows that we can actually approximate viscosity solution to the degenerate partial differential equation using a sequence of classical solutions to the corresponding non-degenerate partial differential equations as the viscosity term $\varepsilon \Delta v^\varepsilon$ vanishes.

Proposition 2.5.2. *Let v^ε be a classical solution to (2.26) and v^0 be a viscosity solution to (2.25). Then there exists a constant $K > 0$ such that*

$$|v^\varepsilon(t, x) - v^0(t, x)| \leq K\sqrt{\varepsilon}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, \varepsilon > 0.$$

The method of vanishing viscosity would prove to be a crucial tool to use in our study, due to the complexity of our formulated stochastic optimal control problem. Now that we have given a full overview of existing models and methods in this field, we will present our research problem in the next section.

Chapter 3

Problem Formulation

We begin the optimal portfolio problem by defining the return of a risky asset. At time $t \geq 0$, let

$$Y(t) = \ln(S(t)) \tag{3.1}$$

be the return of a risky asset, where $S(t)$ is the price of the risky asset. Let $\{B(t), t \geq 0\}$ be a 1-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbb{F})$, where $\mathbb{F} = \{\mathcal{F}^t, t \geq 0\}$ is the P -augmented natural filtration generated by $\{B(t), t \geq 0\}$. To capture historical price information, we include a delay component $Z(t)$ in the drift term of the price process. Assume the price process follows

$$dS(t) = \left(\mu_1 + \mu_2(Z(t)) + \frac{\sigma^2}{2} \right) S(t)dt + \sigma S(t)dB(t), \tag{3.2}$$

where $\mu_1 > 0$, $\sigma > 0$ and $|\mu_2(Z(t))| < M$ and Lipschitz continuous for all $t \geq 0$. Applying Itô's Lemma to $dY(t)$ and substituting (3.2) for $dS(t)$, we obtain

$$\begin{aligned} dY(t) &= d(\ln(S(t))) \\ &= \frac{1}{S(t)} \cdot \left[\left(\mu_1 + \mu_2(Z(t)) + \frac{\sigma^2}{2} \right) S(t)dt + \sigma S(t)dB(t) \right] \\ &\quad - \frac{1}{2} \cdot \frac{1}{(S(t))^2} \cdot \sigma^2 (S(t))^2 dt. \end{aligned}$$

This implies that the return process $Y(t)$ follows

$$\begin{cases} dY(t) = (\mu_1 + \mu_2(Z(t)))dt + \sigma dB(t), \\ Y(0) = y \in \mathbb{R}. \end{cases} \quad (3.3)$$

Initial value of the return can be positive or negative, depending on the price of the risky asset. The delay variable $Z(t)$ is defined as

$$\begin{cases} Z(t) = \int_{-\infty}^0 e^{\lambda\theta} Y(t + \theta) d\theta, \\ Z(0) = z > 0, \end{cases} \quad (3.4)$$

where $\lambda > 0$, giving more weight to the most recent history. This variable represents the weighted average of the past returns. Its dynamics follows

$$dZ(t) = (Y(t) - \lambda Z(t))dt. \quad (3.5)$$

The initial value of the delay variable is chosen to be positive so that it has an impact on the price process as soon as any historical information is captured. Now the bounded function containing the delay component is in the drift term of the return process. Direction of the risky asset's return will depend on this bounded function of delay variable. We consider a bounded function of delay to mean that the historical performance may only influence the return to a certain extent. In addition, the lower integral bound of the delay variable starts from negative infinity to incorporate the entire history of the risky asset price into our model.

Let $\pi(t)X(t)$ represent the proportion of investment on the risky asset and $(1 - \pi(t))X(t)$ be the proportion of investment on the riskless asset, where $X(t)$ is the total wealth of the investor. The proportion $\pi(t)$ is assumed to be finite and can take on values that are negative or greater than one. When $\pi(t) > 1$, the investor has borrowed money to invest. And when $\pi(t) < 0$, the investor is short selling his investments. The riskless assets earns a fixed interest rate $r > 0$ and a nonnegative but finite rate of consumption $c(t)$ is assumed to be from the riskless asset. Lower bound on the consumption assumes that our investor has a minimum consumption for everyday life necessary expenses such as paying for food, clothing, or housing. On the other hand, the upper bound means that the investor sets a limit on how much he or she spends at any time. Thus, riskless portion

of the investor's wealth follows

$$d((1 - \pi(t))X(t)) = [r(1 - \pi(t)) - c(t)]X(t)dt. \quad (3.6)$$

For the risky portion of the investor's wealth, we assume the change in this portion of wealth is proportional to the change in the risky asset's price. Using (3.2), we have

$$\frac{d(\pi(t)X(t))}{\pi(t)X(t)} = \frac{dS(t)}{S(t)} = \left(\mu_1 + \mu_2(Z(t)) + \frac{\sigma^2}{2} \right) dt + \sigma dB(t).$$

Thus, the risky portion follows

$$d(\pi(t)X(t)) = \pi(t)X(t) \left[\left(\mu_1 + \mu_2(Z(t)) + \frac{\sigma^2}{2} \right) dt + \sigma dB(t) \right]. \quad (3.7)$$

Finally, combining (3.6) and (3.7),

$$dX(t) = d((1 - \pi(t))X(t)) + d(\pi(t)X(t)),$$

the investor's total wealth follows

$$\begin{cases} dX(t) = X(t) \left[r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) \right] dt + \sigma \pi(t)X(t)dB(t), \\ X(0) = x > 0. \end{cases} \quad (3.8)$$

We note here that if we have positive initial wealth, that is $x > 0$, this leads to $X(t) > 0$ for all $t \geq 0$. This is easily seen by applying Itô's Lemma on $d \ln(X(t))$ with

$$\begin{aligned} d \ln(X(t)) &= \frac{1}{X(t)} dX(t) - \frac{1}{2(X(t))^2} (dX(t))^2 \\ &= \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) \right) dt \\ &\quad + \sigma \pi(t) dB(t) - \frac{\sigma^2 \pi(t)^2}{2} dt, \end{aligned}$$

and then integrating both sides and using properties of logarithm to get

$$\begin{aligned}
\ln(X(t)) - \ln(x) &= \int_0^t \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(\tau)) \right) \pi(\tau) - c(\tau) - \frac{\sigma^2 \pi(\tau)^2}{2} \right) d\tau \\
&\quad + \int_0^t \sigma \pi(\tau) dB(\tau), \\
\ln\left(\frac{X(t)}{x}\right) &= \int_0^t \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(\tau)) \right) \pi(\tau) - c(\tau) - \frac{\sigma^2 \pi(\tau)^2}{2} \right) d\tau \\
&\quad + \int_0^t \sigma \pi(\tau) dB(\tau).
\end{aligned}$$

After taking natural logarithm on both sides and rearranging the terms, we obtain

$$X(t) = x e^{\int_0^t \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(\tau)) \right) \pi(\tau) - c(\tau) - \frac{\sigma^2 \pi(\tau)^2}{2} \right) d\tau + \int_0^t \sigma \pi(\tau) dB(\tau)}, \quad (3.9)$$

which is always positive given that the initial wealth $x > 0$. This means that our investor will not go bankrupt.

Now let $U(\cdot)$ be a general utility function that measures utility of consumption of our investor and let Π be the admissible control space for $(\pi(\tau), c(\tau))$. Then the value function V is given by

$$V(x, y, z) = \sup_{(\pi(\tau), c(\tau)) \in \Pi} J(x, y, z, \pi, c), \quad (3.10)$$

with

$$J(x, y, z, \pi, c) = \mathbf{E} \left[\int_0^\infty e^{-\beta\tau} U(c(\tau)) X(\tau) d\tau \right]. \quad (3.11)$$

Assuming successors of the current investor can keep investing his total wealth, we wish to choose the optimal investment and consumption that maximizes the expected total discounted utility of consumption over infinite time horizon. Together with (3.3)-(3.4), (3.8) and (3.10)-(3.11), we have formulated our stochastic optimal control problem. We proceed to derive its HJB equation in the next section.

3.1 Derivation of HJB Equation

Now we derive the Hamilton-Jacobi-Bellman (HJB) equation for our stochastic optimal control problem. To proceed, we first fix x, y, z, π and c to find $dV(x, y, z)$, which is of the following form:

$$\begin{aligned} dV = & \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial x}dx + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(dx)^2 + \frac{\partial V}{\partial y}dy + \frac{1}{2}\frac{\partial^2 V}{\partial y^2}(dy)^2 + \frac{\partial V}{\partial z}dz + \frac{1}{2}\frac{\partial^2 V}{\partial z^2}(dz)^2 \\ & + \frac{\partial^2 V}{\partial x \partial y}(dx) \cdot (dy) + \frac{\partial^2 V}{\partial x \partial z}(dx) \cdot (dz) + \frac{\partial^2 V}{\partial y \partial z}(dy) \cdot (dz) \end{aligned} \quad (3.12)$$

Substituting the respective components from (3.8), (3.3), and (3.5) for dx , dy and dz into (3.12), then simplify and rearrange, we can get

$$\begin{aligned} dV = & 0 + V_x \cdot \left[x \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) \pi - c \right) dt + \sigma \pi x dB(t) \right] \\ & + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2dt + V_y \cdot [(\mu_1 + \mu_2(z))dt + \sigma dB(t)] + \frac{1}{2}V_{yy}\sigma^2dt + V_z \cdot (y - \lambda z)dt \\ & + 0 + V_{xy} \cdot \sigma^2\pi xdt + 0 + 0, \\ dV = & \left(\left[x \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) \pi - c \right) dt + \sigma \pi x dB(t) \right] V_x + (\mu_1 + \mu_2(z))V_y \right. \\ & + (y - \lambda z)V_z + \frac{1}{2}V_{yy}\sigma^2 + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2 + V_{xy}\sigma^2\pi x \left. \right) dt \\ & + (V_y\sigma + V_x\sigma\pi x) dB(t). \end{aligned} \quad (3.13)$$

Now note that dV appears in

$$\begin{aligned} d(e^{-\beta t}V) &= d(e^{-\beta t})V + e^{-\beta t}dV \\ &= (-\beta e^{-\beta t}dt)V + e^{-\beta t}dV. \end{aligned} \quad (3.14)$$

Substituting (3.13) into (3.14), we obtain

$$\begin{aligned} d(e^{-\beta t}V) = & \left[-\beta e^{-\beta t}V + e^{-\beta t} \left(\left[x \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) \pi - c \right) \right] V_x \right. \right. \\ & + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z + \frac{1}{2}V_{yy}\sigma^2 + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2 + V_{xy}\sigma^2\pi x \left. \right) dt \\ & \left. + e^{-\beta t}(V_y\sigma + V_x\sigma\pi x)dB(t) \right] \end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned}
d(e^{-\beta t}V) &= e^{-\beta t} \left(-\beta V + \left[x \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) \pi - c \right) \right] V_x \right. \\
&\quad \left. + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z + \frac{1}{2}V_{yy}\sigma^2 + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2 + V_{xy}\sigma^2\pi x \right) dt \\
&\quad + e^{-\beta t}(V_y\sigma + V_x\sigma\pi x)dB(t).
\end{aligned} \tag{3.15}$$

We see that when we take the following limit, we would have

$$\begin{aligned}
&\lim_{\delta \rightarrow 0} \frac{1}{\delta} \mathbf{E} \left[\int_0^\delta d(e^{-\beta\tau}V(X(\tau), Y(\tau), Z(\tau))) \right] \\
&= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \mathbf{E} [e^{-\beta\delta}V(X(\delta), Y(\delta), Z(\delta)) - V(X(0), Y(0), Z(0))] .
\end{aligned} \tag{3.16}$$

And also when (3.15) is substituted for $d(e^{-\beta t}V)$, the limit becomes

$$\begin{aligned}
&\lim_{\delta \rightarrow 0} \frac{1}{\delta} \mathbf{E} \left[\int_0^\delta d(e^{-\beta\tau}V(X(\tau), Y(\tau), Z(\tau))) \right] \\
&= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \mathbf{E} \left[\int_0^\delta e^{-\beta\tau} \left(-\beta V + \left[x \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) \pi - c \right) \right] V_x \right. \right. \\
&\quad \left. \left. + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z + \frac{1}{2}V_{yy}\sigma^2 + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2 + V_{xy}\sigma^2\pi x \right) d\tau \right. \\
&\quad \left. + \int_0^\delta e^{-\beta\tau}(V_y\sigma + V_x\sigma\pi x)dB(\tau) \right] \\
&= -\beta V + \left[x \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) \pi - c \right) \right] V_x \\
&\quad + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z + \frac{1}{2}V_{yy}\sigma^2 + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2 + V_{xy}\sigma^2\pi x.
\end{aligned} \tag{3.17}$$

In (3.17), we have assumed that

$$\mathbf{E} \left[\int_0^\delta e^{-\beta\tau}(V_y\sigma + V_x\sigma\pi x)dB(\tau) \right] = 0.$$

We will check the validity of this statement in the verification theorem sections. Combining (3.16) and (3.17),

$$\begin{aligned}
& \lim_{\delta \rightarrow 0} \frac{1}{\delta} \mathbf{E} [e^{-\beta\delta} V(X(\delta), Y(\delta), Z(\delta)) - V(X(0), Y(0), Z(0))] \\
&= -\beta V + \left[x \left(r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z) \right) \pi - c \right) \right] V_x \\
&\quad + (\mu_1 + \mu_2(z)) V_y + (y - \lambda z) V_z + \frac{1}{2} V_{yy} \sigma^2 + \frac{1}{2} V_{xx} \sigma^2 \pi^2 x^2 + V_{xy} \sigma^2 \pi x. \quad (3.18)
\end{aligned}$$

By the dynamic programming principle and by definition of the value function, we have that

$$\begin{aligned}
& V(x, y, z) \\
&= \sup_{(\pi(\tau), c(\tau)) \in \Pi} \left(\mathbf{E} \left[\int_0^\delta e^{-\beta\tau} U(c(\tau) X(\tau)) d\tau + \int_\delta^\infty e^{-\beta\tau} U(c(\tau) X(\tau)) d\tau \right] \right), \quad (3.19)
\end{aligned}$$

and using change of variable $\omega = \tau - \delta$ in (3.19), we get

$$\begin{aligned}
& V(X(0), Y(0), Z(0)) \\
&= \sup_{(\pi(\tau), c(\tau)) \in \Pi} \left(\mathbf{E} \left[\int_0^\delta e^{-\beta\tau} U(c(\tau) X(\tau)) d\tau + e^{-\beta\delta} \int_0^\infty e^{-\beta\omega} U(c(\omega) X(\omega)) d\omega \right] \right) \\
&= \sup_{(\pi(\tau), c(\tau)) \in \Pi} \left(\mathbf{E} \left[\int_0^\delta e^{-\beta\tau} U(c(\tau) X(\tau)) d\tau + e^{-\beta\delta} V(X(\delta), Y(\delta), Z(\delta)) \right] \right). \quad (3.20)
\end{aligned}$$

Dividing (3.20) by δ and taking the limit as δ approaches zero, then using (3.18), we obtain

$$\begin{aligned}
0 &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \sup_{(\pi(\tau), c(\tau)) \in \Pi} \left(\mathbf{E} \left[\int_0^\delta e^{-\beta\tau} U(c(\tau) X(\tau)) d\tau + e^{-\beta\delta} V(X(\delta), Y(\delta), Z(\delta)) \right. \right. \\
&\quad \left. \left. - V(X(0), Y(0), Z(0)) \right] \right), \\
0 &= \sup_{(\pi, c) \in \Pi} \left(U(cx) - \beta V + \left[x \left(r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z) \right) \pi - c \right) \right] V_x \right. \\
&\quad \left. + (\mu_1 + \mu_2(z)) V_y + (y - \lambda z) V_z + \frac{1}{2} V_{yy} \sigma^2 + \frac{1}{2} V_{xx} \sigma^2 \pi^2 x^2 + V_{xy} \sigma^2 \pi x \right).
\end{aligned}$$

Finally after rearranging the terms, we have

$$\begin{aligned} \beta V = & \sup_{\pi \in \Pi} \left(x\pi \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) V_x + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2 + V_{xy}\sigma^2\pi x \right) \\ & + \frac{1}{2}V_{yy}\sigma^2 + \sup_{c \in \Pi} (U(cx) - cxV_x) + xrV_x + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z. \end{aligned} \quad (3.21)$$

Equation (3.21) is our HJB equation for a general utility function $U(\cdot)$. Next, we will consider widely used special isoelastic utility functions that have constant relative risk aversion, namely the logarithmic utility function and the hyperbolic absolute risk aversion (HARA) utility function for the problem in (3.21).

Chapter 4

Logarithmic Utility

4.1 Logarithmic Utility

The logarithmic utility function is defined as

$$U(C) = \log(C), \quad (4.1)$$

where C represents consumption. It is a limiting case when $\gamma = 0$ in the HARA utility defined later in (5.2). In this special case, the investor has constant relative risk aversion, which is measured by the Arrow-Pratt relative risk aversion measure

$$R(C) = \frac{-CU''(C)}{U'(C)}. \quad (4.2)$$

We see that $R(C) = 1$ is a constant in the case of using the logarithmic utility function. We now define the admissible control space for the logarithmic utility case as follows:

Definition 4.1.1. *A control $(\pi(t), c(t))$ is said to be in the admissible control space Π if it satisfies*

1. $(\pi(t), c(t))$ is \mathbb{F}_t - progressively measurable;
2. $|\pi(t)| \leq \Gamma < \infty$ for all $t \geq 0$;
3. $0 < \varepsilon_c \leq c(t) \leq \Lambda < \infty$, for all $t \geq 0$.

As we will see in the subsequent sections, Γ and Λ are any constant that satisfy

$$\Gamma \geq \frac{\mu_1 + \frac{\sigma^2}{2} - r + M}{\sigma^2}, \quad \Lambda \geq \beta.$$

Here we note that the condition $\varepsilon_c \leq c(t)$ is necessary to ensure the logarithmic utility function will be defined. We now proceed to define the value function and HJB equation for the logarithmic utility case.

4.2 Value Function and HJB Equation

To derive the value function and HJB equation for the logarithmic utility case, we replace the generic utility function $U(\cdot)$ with the logarithmic utility function defined in (4.1). Then the value function defined in (3.10)-(3.11) becomes

$$\begin{aligned} J(x, y, z, \pi, c) &= \mathbf{E} \left[\int_0^\infty e^{-\beta\tau} \log(c(\tau)X(\tau)) d\tau \right], \\ V(x, y, z) &= \sup_{(\pi, c) \in \Pi} J(x, y, z, \pi, c). \end{aligned} \quad (4.3)$$

By applying dynamic programming principle to the value function, the resulting HJB equation transforms from (3.21) into

$$\begin{aligned} \beta V &= \sup_{\pi \in \Pi} \left(x\pi \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) V_x + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2 + V_{xy}\sigma^2\pi x \right) + \frac{1}{2}V_{yy}\sigma^2 \\ &\quad + \sup_{c \in \Pi} (\log(cx) - cxV_x) + xrV_x + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z. \end{aligned} \quad (4.4)$$

We observe that the HJB equation (4.4) currently depends on x , y , and z , which can be simplified to only exhibit dependence on y and z if we let

$$V(x, y, z) = \frac{1}{\beta} \log(x) + W(y, z). \quad (4.5)$$

Then using the fact that

$$\begin{aligned} V_x &= \frac{1}{\beta x}, & V_{xy} &= 0, & V_{xx} &= -\frac{1}{\beta x^2}, \\ V_y &= W_y, & V_{yy} &= W_{yy}, & V_z &= W_z, \end{aligned}$$

the HJB equation (4.4) becomes

$$\begin{aligned} \log(x) + \beta W &= \sup_{\pi \in \Pi} \left(\frac{1}{\beta} \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z) \right) \pi - \frac{\sigma^2 \pi^2}{2\beta} \right) + \frac{1}{2} W_{yy} \sigma^2 \\ &\quad + \sup_{c \in \Pi} \left(\log(c) + \log(x) - \frac{c}{\beta} \right) + \frac{r}{\beta} + (\mu_1 + \mu_2(z)) W_y + (y - \lambda z) W_z. \end{aligned} \quad (4.6)$$

Moving $\log(x)$ to the right-hand side of (4.6), we arrive at

$$\begin{aligned} \beta W &= \sup_{\pi \in \Pi} \left(\frac{1}{\beta} \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z) \right) \pi - \frac{\sigma^2 \pi^2}{2\beta} \right) + \frac{1}{2} W_{yy} \sigma^2 \\ &\quad + \sup_{c \in \Pi} \left(\log(c) - \frac{c}{\beta} \right) + \frac{r}{\beta} + (\mu_1 + \mu_2(z)) W_y + (y - \lambda z) W_z. \end{aligned} \quad (4.7)$$

From (4.7), we find extrema for the supremum expressions and determine the candidates for optimal controls to be

$$\pi^* = \frac{\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z)}{\sigma^2}, \quad c^* = \beta. \quad (4.8)$$

Plugging (4.8) in (4.7) and simplify we get

$$\begin{aligned} \beta W &= \frac{1}{\beta \sigma^2} \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z) \right)^2 - \frac{\sigma^2}{2\beta} \cdot \left(\frac{(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z))}{\sigma^2} \right)^2 \\ &\quad + \log(\beta) - 1 + \frac{r}{\beta} + (\mu_1 + \mu_2(z)) W_y + (y - \lambda z) W_z + \frac{\sigma^2}{2} W_{yy}, \\ &= \frac{1}{2\beta \sigma^2} \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z) \right)^2 + \log(\beta) - 1 + \frac{r}{\beta} \\ &\quad + (\mu_1 + \mu_2(z)) W_y + (y - \lambda z) W_z + \frac{\sigma^2}{2} W_{yy}. \end{aligned} \quad (4.9)$$

Rearranging the terms in (4.9), we get the following simplified equation:

$$\begin{aligned} \frac{\sigma^2}{2} W_{yy} &= \beta W - (\mu_1 + \mu_2(z)) W_y - (y - \lambda z) W_z \\ &\quad - \frac{1}{2\beta \sigma^2} \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(z) \right)^2 - \log(\beta) + 1 - \frac{r}{\beta}. \end{aligned} \quad (4.10)$$

If we define

$$f_1(z) = \frac{1}{2\beta\sigma^2} \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right)^2 + \log(\beta) - 1 + \frac{r}{\beta}, \quad (4.11)$$

$$f_2(z) = \mu_1 + \mu_2(z). \quad (4.12)$$

Then we can rewrite (4.10) as

$$\frac{\sigma^2}{2}W_{yy} = \beta W - f_1(z) - f_2(z)W_y - (y - \lambda z)W_z. \quad (4.13)$$

We will use the form in (4.13) throughout the sections below.

4.3 Perturbed Elliptic Equation and Viscosity Solution

Previously in (4.13), the term $y - \lambda z$ is not bounded, which leads to difficulty in analyzing the problem. Thus we would like the coefficients to be bounded. We will let

$$g(y, z) = y - \lambda z, \quad (4.14)$$

and do the following modification to (4.13):

$$\begin{aligned} 0 = & -\frac{\sigma^2 W_{yy}}{2(1 + g(y, z)^2)} + \frac{\beta W}{1 + g(y, z)^2} - \frac{f_1(z)}{1 + g(y, z)^2} \\ & - \frac{f_2(z)W_y}{1 + g(y, z)^2} - \frac{g(y, z)W_z}{1 + g(y, z)^2}. \end{aligned} \quad (4.15)$$

Equation (4.15) is essentially the same as equation (4.13) in that they should share the same solution. However, this equation has better properties than the original now that the coefficients are all bounded. We want to show existence of solution to (4.15). But with its current form, we have a degenerate elliptic partial differential equation, which still poses some difficulties. To overcome this, let us consider the following perturbed

non-degenerate equation that has an additional viscosity term $\varepsilon \Delta W^\varepsilon$ to (4.15):

$$0 = -\frac{\sigma^2 W_{yy}^\varepsilon}{2(1+g(y,z)^2)} - \varepsilon \Delta W^\varepsilon + \frac{\beta W^\varepsilon}{1+g(y,z)^2} - \frac{f_1(z)}{1+g(y,z)^2} - \frac{f_2(z)W_y^\varepsilon}{1+g(y,z)^2} - \frac{g(y,z)W_z^\varepsilon}{1+g(y,z)^2}, \quad (4.16)$$

where $\varepsilon > 0$. If we let

$$H(y, z, W_y^\varepsilon, W_z^\varepsilon) = \frac{f_1(z)}{1+g(y,z)^2} + \frac{f_2(z)}{1+g(y,z)^2} W_y^\varepsilon + \frac{g(y,z)}{1+g(y,z)^2} W_z^\varepsilon, \quad (4.17)$$

we can rewrite (4.16) as

$$0 = -\frac{\sigma^2}{2(1+g(y,z)^2)} W_{yy}^\varepsilon - \varepsilon \Delta W^\varepsilon + \frac{\beta}{1+g(y,z)^2} W^\varepsilon - H(y, z, W_y^\varepsilon, W_z^\varepsilon), \quad (4.18)$$

Here we briefly note that adding the viscosity term in (4.18) results in a very similar non-degenerate equation as the result from having the following approximate problem formulation to (3.3)-(3.8),

$$\begin{cases} dY(t) = (\mu_1 + \mu_2(Z(t)))dt + \sigma dB(t) + \varepsilon dB_1(t), \\ dZ(t) = (Y(t) - \lambda Z(t))dt + \varepsilon dB_2(t), \\ dX(t) = X(t) \left[r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) \right] dt \\ \quad + \pi(t)X(t)(\sigma dB(t) + \varepsilon dB_1(t)), \end{cases} \quad (4.19)$$

where $B_1(t)$ and $B_2(t)$ are 1-dimensional standard Brownian motions independent of $B(t)$. Thus, solving our problem is very similar to solving the approximate problem (4.19). In the subsequent sections, we will first study the existence of solution to (4.18) inside a ball in \mathbb{R}^2 and then extend it to the entire space of \mathbb{R}^2 , which eventually obtains the existence of solution to (4.15) when $\varepsilon \rightarrow 0$.

4.3.1 Existence and Uniqueness in B_R

We begin with a smaller problem in a small ball in \mathbb{R}^2 . For $R > 0$, let $B_R = \{(y, z) \in \mathbb{R}^2 : \|(y, z)\|_2 \leq R\}$. We consider the following problem

$$\begin{cases} -\frac{\sigma^2}{2(1+g(y,z)^2)}W_{yy}^\varepsilon - \varepsilon\Delta W^\varepsilon + \frac{\beta}{1+g(y,z)^2}W^\varepsilon - H(y, z, W_y^\varepsilon, W_z^\varepsilon) = 0, \\ W^\varepsilon|_{\partial B_R} = K, \end{cases} \quad (4.20)$$

where $K > 0$ is a bounded constant that is determined in the subsequent sections as

$$K = \frac{1}{2\beta^2\sigma^2} \left[\left(\mu_1 + \frac{\sigma^2}{2} - r \right)^2 - 2 \left(\mu_1 + \frac{\sigma^2}{2} - r \right) M \right] + \log(\beta) - 1 + \frac{r}{\beta},$$

to ensure that the solution to this problem is bounded below by K (this is needed in the verification theorem later on). We clarify here that the existence of the solution itself only requires K to be a positive constant. The problem in (4.20) is a non-degenerate partial differential equation boundary value problem with properties such that available results in the literature can be used to show the existence and uniqueness of its solution. To show this, we first define strict ellipticity.

Definition 4.3.1. Denote $Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u = f(x)$, where $a^{ij} = a^{ji}$ and coefficients and f are defined in an open set $\Omega \subset \mathbb{R}^n$. L is strictly elliptic if

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

for some positive constant λ .

Then, we know that Theorem 6.13 from [28] states

Theorem 4.3.1. Let L be strictly elliptic in a bounded domain Ω , with $c \leq 0$, and let f and the coefficients of L be bounded and belong to $C^\alpha(\Omega)$. Suppose that Ω satisfies an exterior sphere condition at every boundary point. Then, if φ is continuous on $\partial\Omega$, the Dirichlet problem,

$$Lu = f \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega,$$

has a unique solution $u \in C^0(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$.

This theorem then leads to our theorem.

Theorem 4.3.2. *The system (4.20) has a unique solution $W^\varepsilon(y, z) \in C^0(\overline{B}_R) \cap C^{2,\alpha}(B_R)$.*

Proof. For our problem, we have

$$\begin{aligned} LW^\varepsilon = & \left(\frac{\sigma^2}{2(1+g(y, z)^2)} + \varepsilon \right) W_{yy}^\varepsilon + \varepsilon W_{zz}^\varepsilon - \frac{\beta}{1+g(y, z)^2} W^\varepsilon \\ & + \frac{f_2(z)}{1+g(y, z)^2} W_y^\varepsilon + \frac{g(y, z)}{1+g(y, z)^2} W_z^\varepsilon, \end{aligned} \quad (4.21)$$

and

$$f = -\frac{f_1(z)}{1+g(y, z)^2}, \quad (4.22)$$

where f_1 , f_2 and g are as defined in (4.11)-(4.12) and (4.14). L is strictly elliptic if we take $\lambda = \varepsilon$ in Definition 4.3.1. We have

$$c(y, z) = -\frac{\beta}{1+g(y, z)^2} \leq 0. \quad (4.23)$$

Coefficients of L and f are clearly bounded. They also belong to $C^\alpha(B_R)$ since their partial derivatives are bounded and thus Lipschitz and α -Hölder continuous. The exterior sphere condition is satisfied since we are only considering $\Omega = B_R \subset \mathbb{R}^2$, where B_R is a ball of radius R . Our boundary condition K is a positive constant, thus continuous. Hence by Theorem 4.3.1, (4.20) has a unique solution $W^\varepsilon \in C^0(\overline{B}_R) \cap C^{2,\alpha}(B_R)$. \square

Now that we have shown existence in a small ball, we want to extend this result to \mathbb{R}^2 . First we need to show that W^ε and its first partial derivatives are uniformly bounded on \mathbb{R}^2 .

4.3.2 L^∞ Estimate of W^ε

We give the uniform upper bound of W^ε in the next lemma.

Lemma 4.3.3. *Let $f_1(z)$ and $g(y, z)$ be the functions defined in (4.11) and (4.14). Consider the problem in (4.18). If*

$$H(y, z, 0, 0) = \frac{f_1(z)}{1+g(y, z)^2} \in L^\infty(\mathbb{R}^2),$$

then

$$|W^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{\|f_1\|_{L^\infty(\mathbb{R}^2)}}{\beta} + K, \quad \forall (y, z) \in \mathbb{R}^2.$$

Proof. We first check that

$$H(y, z, 0, 0) = \frac{f_1(z)}{1 + g(y, z)^2} \in L^\infty(\mathbb{R}^2). \quad (4.24)$$

This is true since $\mu_2(z)$ is bounded and Lipschitz continuous for all $z \in \mathbb{R}^2$, we know that there exists $M_1 > 0$ such that $|f_1(z)| < M_1$. Also $f_1(z)$ is Lipschitz continuous. Then,

$$\frac{|f_1(z)|}{1 + g(y, z)^2} \leq |f_1(z)| \leq M_1 < \infty$$

and $H(y, z, 0, 0) \in L^\infty(\mathbb{R}^2)$. Note that when (4.24) is satisfied for all $(y, z) \in \mathbb{R}^2$, it is also satisfied for all $(y, z) \in B_R$. There exists $(y_0, z_0) \in B_R$ such that $W^\varepsilon(y_0, z_0) \geq W^\varepsilon(y, z)$ for all $(y, z) \in B_R$, then

$$\begin{cases} W_y^\varepsilon(y_0, z_0) = W_z^\varepsilon(y_0, z_0) = 0, \\ W_{yy}^\varepsilon(y_0, z_0) \leq 0, \\ W_{zz}^\varepsilon(y_0, z_0) \leq 0. \end{cases} \quad (4.25)$$

Using (4.25), we have

$$\begin{aligned} \frac{\beta}{1 + g(y_0, z_0)^2} W^\varepsilon(y_0, z_0) &= \frac{\sigma^2}{2(1 + g(y_0, z_0)^2)} W_{yy}^\varepsilon(y_0, z_0) + \varepsilon \Delta W^\varepsilon(y_0, z_0) \\ &\quad + H(y_0, z_0, W_y^\varepsilon(y_0, z_0), W_z^\varepsilon(y_0, z_0)) \\ &\leq |H(y_0, z_0, 0, 0)| \\ &= \frac{|f_1(z_0)|}{1 + g(y_0, z_0)^2} \\ &\leq \frac{\|f_1\|_{L^\infty(\mathbb{R}^2)}}{1 + g(y_0, z_0)^2}. \end{aligned}$$

This implies that

$$W^\varepsilon(y, z) \leq W^\varepsilon(y_0, z_0) \leq \frac{\|f_1\|_{L^\infty(\mathbb{R}^2)}}{\beta}. \quad (4.26)$$

Similarly, there exists $(y_1, z_1) \in B_R$ such that $W^\varepsilon(y_1, z_1) \leq W^\varepsilon(y, z)$ for all $(y, z) \in B_R$, then

$$\begin{cases} W_y^\varepsilon(y_1, z_1) = W_z^\varepsilon(y_1, z_1) = 0, \\ W_{yy}^\varepsilon(y_1, z_1) \geq 0, \\ W_{zz}^\varepsilon(y_1, z_1) \geq 0. \end{cases} \quad (4.27)$$

Then

$$\begin{aligned} \frac{\beta}{1 + g(y_1, z_1)^2} W^\varepsilon(y_1, z_1) &= \frac{\sigma^2}{2(1 + g(y_1, z_1)^2)} W_{yy}^\varepsilon(y_1, z_1) + \varepsilon \Delta W^\varepsilon(y_1, z_1) \\ &\quad + H(y_1, z_1, W_y^\varepsilon(y_1, z_1), W_z^\varepsilon(y_1, z_1)) \\ &\geq -|H(y_1, z_1, 0, 0)| \\ &= -\frac{|f_1(z_1)|}{1 + g(y_1, z_1)^2} \\ &\geq -\frac{\|f_1\|_{L^\infty(\mathbb{R}^2)}}{1 + g(y_1, z_1)^2}. \end{aligned}$$

This then implies

$$W^\varepsilon(y, z) \geq W^\varepsilon(y_1, z_1) \geq -\frac{\|f_1\|_{L^\infty(\mathbb{R}^2)}}{\beta}. \quad (4.28)$$

Combining (4.26) and (4.28), and then noting that $W^\varepsilon|_{\partial B_R} = K$, we would have

$$|W^\varepsilon(y, z)|_{L^\infty(B_R)} \leq \frac{\|f_1\|_{L^\infty(\mathbb{R}^2)}}{\beta} + K, \quad \forall (y, z) \in B_R. \quad (4.29)$$

We see that the upper bound is independent of R . Thus, it is true for all $(y, z) \in \mathbb{R}^2$ that

$$|W^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{\|f_1\|_{L^\infty(\mathbb{R}^2)}}{\beta} + K.$$

□

Now that we have shown that W^ε is bounded almost surely on \mathbb{R}^2 , we still need to show that its first order partial derivatives are also bounded.

4.3.3 Estimate of W_y^ε and W_z^ε

In this section, we denote

$$H^*(y, z, p_1, p_2) = f_1(z) + f_2(z)p_1 + g(y, z)p_2, \quad (4.30)$$

where f_1 , f_2 and g are as defined in (4.11)-(4.12) and (4.14).

Lemma 4.3.4. *Let $H^*(y, z, p_1, p_2)$ be as defined in (4.30), where $f_1(z)$ and $f_2(z)$ are as defined in (4.11)-(4.12). Consider the problem in (4.18). Assume $\beta > 1 + C_1$, where $C_1 = \max(L_1, L_2, \lambda)$ and L_1, L_2 are Lipschitz constants for $f_1(z)$, $f_2(z)$ respectively. If*

$$|H^*(y, z, p_1, p_2) - H^*(y, \hat{z}, p_1, p_2)| \leq C_1|z - \hat{z}|(1 + |p_1| + |p_2|),$$

and

$$|H^*(y, z, p_1, p_2) - H^*(\hat{y}, z, p_1, p_2)| \leq |y - \hat{y}||p_2|,$$

then

$$|W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R})} + |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R})} \leq \frac{C_1}{\beta - C_1 - 1}.$$

Proof. We first check

$$\begin{aligned} & |H^*(y, z, p_1, p_2) - H^*(y, \hat{z}, p_1, p_2)| \\ & \leq |f_1(z) - f_1(\hat{z})| + |f_2(z) - f_2(\hat{z})||p_1| + \lambda|z - \hat{z}||p_2| \\ & \leq L_1|z - \hat{z}| + L_2|z - \hat{z}||p_1| + \lambda|z - \hat{z}||p_2| \\ & \leq C_1|z - \hat{z}|(1 + |p_1| + |p_2|), \quad \text{where } C_1 = \max(L_1, L_2, \lambda). \end{aligned} \quad (4.31)$$

Next, define

$$\overline{W}^\varepsilon(y, z) = W^\varepsilon(y, z + \xi), \quad \forall (y, z) \in \mathbb{R}^2, \quad (4.32)$$

and also define

$$\overline{H}(y, z, p_1, p_2) = H^*(y, z + \xi, p_1, p_2), \quad \forall y, z, p_1, p_2 \in \mathbb{R}. \quad (4.33)$$

We consider the problem in (4.18) and let

$$\Phi(y, z) = W^\varepsilon(y, z) - \overline{W}^\varepsilon(y, z). \quad (4.34)$$

Since $W^\varepsilon, \overline{W}^\varepsilon \in L^\infty(\mathbb{R}^2)$, we know that $\Phi \in L^\infty(\mathbb{R}^2)$. Then for any $\delta > 0$ there exists a z_0 such that $\Phi(y, z_0) \geq \sup_{z \in \mathbb{R}} \Phi(y, z) - \delta$. Let $\zeta(\cdot) \in C_0^\infty(\mathbb{R})$ be a function such that

$$\begin{cases} \zeta(z_0) = 1, \\ 0 \leq \zeta(z) < 1, \quad \forall z \in \mathbb{R} \setminus \{z_0\}, \\ |\zeta_z(z)| \leq 1, \quad |\zeta_{zz}(z)| \leq 1, \quad \forall z \in \mathbb{R}. \end{cases} \quad (4.35)$$

It is easy to see that such a function does exist. Then let

$$\Psi(y, z) = \Phi(y, z) + 2\delta\zeta(z), \quad (4.36)$$

and let G be a bounded region such that $\zeta(z) = 0$ for $z \in \mathbb{R} \setminus G$. Then we would have

$$\begin{aligned} \Psi(y, z_0) &= \Phi(y, z_0) + 2\delta\zeta(z_0) \\ &= \Phi(y, z_0) + 2\delta \\ &\geq \sup_{z \in \mathbb{R}} \Phi(y, z) - \delta + 2\delta \\ &= \sup_{z \in \mathbb{R}} \Phi(y, z) + \delta \\ &> \sup_{z \in \mathbb{R}} \Phi(y, z) \\ &= \sup_{z \in \mathbb{R} \setminus G} \Phi(y, z). \end{aligned} \quad (4.37)$$

We know that on $\mathbb{R} \setminus G$, $\Psi(y, z) = \Phi(y, z)$. Having $\Psi(y, z_0) > \sup_{z \in \mathbb{R} \setminus G} \Phi(y, z)$ implies that $\sup_{z \in \mathbb{R}} \Psi(y, z) = \max_{z \in G} \Psi(y, z)$. Thus there exists a $z_1 \in \overline{G}$ such that $\Psi(y, z_1) \geq \Psi(y, z)$ for all $z \in \mathbb{R}$. This means

$$\begin{cases} \Psi_y(y, z_1) = 0, \quad \Psi_z(y, z_1) = 0, \\ \Psi_{yy}(y, z_1) \leq 0, \\ \Psi_{zz}(y, z_1) \leq 0, \\ \Psi_{yy}(y, z_1) + \Psi_{zz}(y, z_1) \leq 0, \end{cases} \quad (4.38)$$

and by the definition of Ψ from (4.36), these imply

$$\begin{cases} W_y^\varepsilon(y, z_1) - \overline{W}_y^\varepsilon(y, z_1) = 0, \\ W_z^\varepsilon(y, z_1) - \overline{W}_z^\varepsilon(y, z_1) + 2\delta\zeta_z(z_1) = 0, \\ W_{yy}^\varepsilon(y, z_1) - \overline{W}_{yy}^\varepsilon(y, z_1) \leq 0, \\ W_{zz}^\varepsilon(y, z_1) - \overline{W}_{zz}^\varepsilon(y, z_1) + 2\delta\zeta_{zz}(z_1) \leq 0, \\ \Delta W^\varepsilon(y, z_1) - \Delta \overline{W}^\varepsilon(y, z_1) + 2\delta\zeta_{zz}(z_1) \leq 0. \end{cases} \quad (4.39)$$

Since $\Psi(y, z) = W^\varepsilon(y, z) - \overline{W}^\varepsilon(y, z) + 2\delta\zeta(z)$, $\Psi(y, z_1) \geq \Psi(y, z)$ for all $z \in \mathbb{R}$ and $\zeta(z_1) \leq 1$, we then have

$$\begin{aligned} \beta(W^\varepsilon(y, z) - \overline{W}^\varepsilon(y, z)) &\leq \beta\Psi(y, z) \\ &\leq \beta\Psi(y, z_1) \\ &= \beta(W^\varepsilon(y, z_1) - \overline{W}^\varepsilon(y, z_1) + 2\delta\zeta(z_1)) \\ &\leq \beta(W^\varepsilon(y, z_1) - \overline{W}^\varepsilon(y, z_1)) + 2\delta\beta. \end{aligned} \quad (4.40)$$

Then using (4.16) to substitute for $\beta(W^\varepsilon(y, z_1) - \overline{W}^\varepsilon(y, z_1))$ on the right-hand side, we obtain

$$\begin{aligned} &\beta(W^\varepsilon(y, z) - \overline{W}^\varepsilon(y, z)) \\ &= \frac{\sigma^2}{2}W_{yy}^\varepsilon(y, z_1) - \frac{\sigma^2}{2}\overline{W}_{yy}^\varepsilon(y, z_1) \\ &\quad + \varepsilon(1 + g(y, z_1)^2)\Delta W^\varepsilon(y, z_1) + H^*(y, z_1, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) \\ &\quad - \varepsilon(1 + g(y, z_1 + \xi)^2)\Delta \overline{W}^\varepsilon(y, z_1) - \overline{H}(y, z_1, \overline{W}_y^\varepsilon(y, z_1), \overline{W}_z^\varepsilon(y, z_1)) + 2\delta\beta. \end{aligned} \quad (4.41)$$

From (4.39), we know that $W_{yy}^\varepsilon(y, z_1) - \overline{W}_{yy}^\varepsilon(y, z_1) \leq 0$ and $\Delta W^\varepsilon(y, z_1) - \Delta \overline{W}^\varepsilon(y, z_1) \leq -2\delta\zeta_{zz}(z_1)$. Thus equation (4.41) becomes

$$\begin{aligned} &\beta(W^\varepsilon(y, z) - \overline{W}^\varepsilon(y, z)) \\ &\leq \varepsilon(\max\{1 + g(y, z_1)^2, 1 + g(y, z_1 + \xi)^2\})(\Delta W^\varepsilon(y, z_1) - \Delta \overline{W}^\varepsilon(y, z_1)) \\ &\quad + H^*(y, z_1, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) - \overline{H}(y, z_1, \overline{W}_y^\varepsilon(y, z_1), \overline{W}_z^\varepsilon(y, z_1)) + 2\delta\beta \\ &\leq -2\varepsilon\delta\zeta_{zz}(z_1)(\max\{1 + g(y, z_1)^2, 1 + g(y, z_1 + \xi)^2\}) \\ &\quad + H^*(y, z_1, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) - \overline{H}(y, z_1, \overline{W}_y^\varepsilon(y, z_1), \overline{W}_z^\varepsilon(y, z_1)) + 2\delta\beta. \end{aligned}$$

Dividing both sides by β and taking the norm, we have

$$\begin{aligned}
& |W^\varepsilon(y, z) - \overline{W}^\varepsilon(y, z)| \\
& \leq \frac{1}{\beta} | -2\varepsilon\delta\zeta_{zz}(z_1)(\max\{1 + g(y, z_1)^2, 1 + g(y, z_1 + \xi)^2\}) \\
& \quad + H^*(y, z_1, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) - \overline{H}(y, z_1, \overline{W}_y^\varepsilon(y, z_1), \overline{W}_z^\varepsilon(y, z_1)) | + 2\delta \\
& \leq \frac{2\varepsilon\delta}{\beta} \cdot \max\{1 + g(y, z_1)^2, 1 + g(y, z_1 + \xi)^2\} \\
& \quad + \frac{1}{\beta} | H^*(y, z_1, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) - \overline{H}(y, z_1, \overline{W}_y^\varepsilon(y, z_1), \overline{W}_z^\varepsilon(y, z_1)) | + 2\delta.
\end{aligned} \tag{4.42}$$

From (4.39) we know that $W_y^\varepsilon(y, z_1) = \overline{W}_y^\varepsilon(y, z_1)$ and $\overline{W}_z^\varepsilon(y, z_1) = W_z^\varepsilon(y, z_1) + 2\delta\zeta_z(z_1)$ and from (4.33) we can rewrite

$$\overline{H}(y, z_1, \overline{W}_y^\varepsilon(y, z_1), \overline{W}_z^\varepsilon(y, z_1)) = H^*(y, z_1 + \xi, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1) + 2\delta\zeta_z(z_1)). \tag{4.43}$$

Adding and subtracting $H^*(y, z_1 + \xi, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1))$ on the right-hand side of (4.42) then using the verified assumption in (4.31), the inequality in (4.42) becomes

$$\begin{aligned}
& |W^\varepsilon(y, z) - \overline{W}^\varepsilon(y, z)| \\
& \leq 2\delta \left(\frac{\varepsilon}{\beta} \cdot \max\{1 + g(y, z_1)^2, 1 + g(y, z_1 + \xi)^2\} + 1 \right) \\
& \quad + \frac{1}{\beta} | H^*(y, z_1, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) - H^*(y, z_1 + \xi, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) \\
& \quad + H^*(y, z_1 + \xi, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) \\
& \quad - H^*(y, z_1 + \xi, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1) + 2\delta\zeta_z(z_1)) | \\
& \leq 2\delta \left(\frac{\varepsilon}{\beta} \cdot \max\{1 + g(y, z_1)^2, 1 + g(y, z_1 + \xi)^2\} + 1 \right) \\
& \quad + \frac{1}{\beta} C_1 |\xi| (1 + |W_y^\varepsilon(y, z_1)| + |W_z^\varepsilon(y, z_1)|) \\
& \quad + \frac{1}{\beta} | H^*(y, z_1 + \xi, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1)) \\
& \quad - H^*(y, z_1 + \xi, W_y^\varepsilon(y, z_1), W_z^\varepsilon(y, z_1) + 2\delta\zeta_z(z_1)) |.
\end{aligned} \tag{4.44}$$

Let $\delta \rightarrow 0$ and we have

$$|W^\varepsilon(y, z) - W^\varepsilon(y, z + \xi)| \leq \frac{C_1}{\beta} (1 + |W_y^\varepsilon(y, z_1)| + |W_z^\varepsilon(y, z_1)|) |\xi|. \tag{4.45}$$

Now, we go through a similar procedure for the y variable by letting

$$\underline{W}^\varepsilon(y, z) = W^\varepsilon(y + \xi, z), \quad \forall (y, z) \in \mathbb{R}^2, \quad (4.46)$$

and

$$\underline{H}(y, z, p_1, p_2) = H^*(y + \xi, z, p_1, p_2), \quad \forall y, z, p_1, p_2 \in \mathbb{R}. \quad (4.47)$$

Similarly, we check the condition

$$\begin{aligned} |H^*(y, z, p_1, p_2) - H^*(\hat{y}, z, p_1, p_2)| &= |(y - \hat{y})p_2| \\ &\leq |y - \hat{y}||p_2|. \end{aligned} \quad (4.48)$$

Let

$$\hat{\Phi}(y, z) = W^\varepsilon(y, z) - \underline{W}^\varepsilon(y, z). \quad (4.49)$$

Again since $W^\varepsilon, \underline{W}^\varepsilon \in L^\infty(\mathbb{R}^2)$, we know that $\hat{\Phi} \in L^\infty(\mathbb{R}^2)$ and for any $\delta > 0$ there exists a y_0 such that $\hat{\Phi}(y_0, z) \geq \sup_{y \in \mathbb{R}} \hat{\Phi}(y, z) - \delta$. Let $\eta(\cdot) \in C_0^\infty(\mathbb{R})$ be a function that satisfies the following:

$$\begin{cases} \eta(y_0) = 1, \\ 0 \leq \eta(y) < 1, \quad \forall y \in \mathbb{R} \setminus \{y_0\}, \\ |\eta_y(y)| \leq 1, \quad |\eta_{yy}(y)| \leq 1, \quad \forall y \in \mathbb{R}. \end{cases} \quad (4.50)$$

Again, it is easy to see that such a function does exist. Then let

$$\hat{\Psi}(y, z) = \hat{\Phi}(y, z) + 2\delta\eta(y), \quad (4.51)$$

and let G' be a bounded region such that $\eta(y) = 0$ for $y \in \mathbb{R} \setminus G'$.

Then we would have

$$\begin{aligned}
\hat{\Psi}(y_0, z) &= \hat{\Phi}(y_0, z) + 2\delta\eta(y_0) \\
&= \hat{\Phi}(y_0, z) + 2\delta \\
&\geq \sup_{y \in \mathbb{R}} \hat{\Phi}(y, z) - \delta + 2\delta \\
&= \sup_{y \in \mathbb{R}} \hat{\Phi}(y, z) + \delta \\
&> \sup_{y \in \mathbb{R}} \hat{\Phi}(y, z) \\
&= \sup_{y \in \mathbb{R} \setminus G'} \hat{\Phi}(y, z).
\end{aligned} \tag{4.52}$$

Inequality (4.52) implies that $\sup_{y \in \mathbb{R}} \hat{\Psi}(y, z) = \max_{y \in G'} \hat{\Psi}(y, z)$. Thus there exists a $y_1 \in \overline{G'}$ such that $\hat{\Psi}(y_1, z) \geq \hat{\Psi}(y, z)$ for all $y \in \mathbb{R}$. This means

$$\begin{cases} \hat{\Psi}_y(y_1, z) = 0, \\ \hat{\Psi}_z(y_1, z) = 0, \\ \hat{\Psi}_{yy}(y_1, z) \leq 0, \\ \hat{\Psi}_{zz}(y_1, z) \leq 0, \\ \hat{\Psi}_{yy}(y_1, z) + \hat{\Psi}_{zz}(y_1, z) \leq 0, \end{cases} \tag{4.53}$$

which then implies

$$\begin{cases} W_y^\varepsilon(y_1, z) - \underline{W}_y^\varepsilon(y_1, z) + 2\delta\eta(y_1) = 0, \\ W_z^\varepsilon(y_1, z) - \underline{W}_z^\varepsilon(y_1, z) = 0, \\ W_{yy}^\varepsilon(y_1, z) - \underline{W}_{yy}^\varepsilon(y_1, z) + 2\delta\eta_{yy}(y_1) \leq 0, \\ W_{zz}^\varepsilon(y_1, z) - \underline{W}_{zz}^\varepsilon(y_1, z) \leq 0, \\ \Delta W^\varepsilon(y_1, z) - \Delta \underline{W}^\varepsilon(y_1, z) + 2\delta\eta_{yy}(y_1) \leq 0. \end{cases} \tag{4.54}$$

Since $\hat{\Psi}(y, z) = W^\varepsilon(y, z) - \underline{W}^\varepsilon(y, z) + 2\delta\eta(y)$, $\hat{\Psi}(y_1, z) \geq \hat{\Psi}(y, z)$ for all $y \in \mathbb{R}$ and

$\eta(y_1) \leq 1$, we have

$$\begin{aligned}
& \beta(W^\varepsilon(y, z) - \underline{W}^\varepsilon(y, z)) \\
& \leq \beta\hat{\Psi}(y, z) \\
& \leq \beta\hat{\Psi}(y_1, z) \\
& = \beta(W^\varepsilon(y_1, z) - \underline{W}^\varepsilon(y_1, z) + 2\delta\eta(y_1)) \\
& \leq \beta(W^\varepsilon(y_1, z) - \underline{W}^\varepsilon(y_1, z)) + 2\delta\beta.
\end{aligned} \tag{4.55}$$

Then using (4.16) to substitute for $\beta(W^\varepsilon(y_1, z) - \underline{W}^\varepsilon(y_1, z))$ on the right-hand side, we obtain

$$\begin{aligned}
& \beta(W^\varepsilon(y, z) - \underline{W}^\varepsilon(y, z)) \\
& = \frac{\sigma^2}{2}W_{yy}^\varepsilon(y_1, z) - \frac{\sigma^2}{2}\underline{W}_{yy}^\varepsilon(y_1, z) \\
& \quad + \varepsilon(1 + g(y_1, z)^2)\Delta W^\varepsilon(y_1, z) + H^*(y_1, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) \\
& \quad - \varepsilon(1 + g(y_1 + \xi, z)^2)\Delta \underline{W}^\varepsilon(y_1, z) - \underline{H}(y_1, z, \underline{W}_y^\varepsilon(y_1, z), \underline{W}_z^\varepsilon(y_1, z)) + 2\delta\beta.
\end{aligned} \tag{4.56}$$

From (4.54), we know that $W_{yy}^\varepsilon(y_1, z) - \underline{W}_{yy}^\varepsilon(y_1, z) \leq -2\delta\eta_{yy}(y_1)$ and $\Delta W^\varepsilon(y_1, z) - \Delta \underline{W}^\varepsilon(y_1, z) \leq -2\delta\eta_{yy}(y_1)$. Thus equation (4.56) becomes

$$\begin{aligned}
& \beta(W^\varepsilon(y, z) - \underline{W}^\varepsilon(y, z)) \\
& \leq -2\delta\eta_{yy}(y_1) \left(\frac{\sigma^2}{2} \right) \\
& \quad + \varepsilon(\max\{1 + g(y_1, z)^2, 1 + g(y_1 + \xi, z)^2\})(\Delta W^\varepsilon(y_1, z) - \Delta \underline{W}^\varepsilon(y_1, z)) \\
& \quad + H^*(y_1, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) - \underline{H}(y_1, z, \underline{W}_y^\varepsilon(y_1, z), \underline{W}_z^\varepsilon(y_1, z)) + 2\delta\beta \\
& \leq -2\delta\eta_{yy}(y_1) \left(\frac{\sigma^2}{2} \right) - 2\varepsilon\delta\eta_{yy}(y_1)(\max\{1 + g(y_1, z)^2, 1 + g(y_1 + \xi, z)^2\}) \\
& \quad + H^*(y_1, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) - \underline{H}(y_1, z, \underline{W}_y^\varepsilon(y_1, z), \underline{W}_z^\varepsilon(y_1, z)) + 2\delta\beta \\
& = -2\delta \left(\frac{\sigma^2}{2} + \varepsilon \cdot \max\{1 + g(y_1, z)^2, 1 + g(y_1 + \xi, z)^2\} \right) \eta_{yy}(y_1) \\
& \quad + H^*(y_1, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) - \underline{H}(y_1, z, \underline{W}_y^\varepsilon(y_1, z), \underline{W}_z^\varepsilon(y_1, z)) + 2\delta\beta.
\end{aligned}$$

Dividing both sides by β and taking the norm, we obtain

$$\begin{aligned}
& |W^\varepsilon(y, z) - \underline{W}^\varepsilon(y, z)| \\
\leq & \frac{1}{\beta} | -\delta(\sigma^2 + 2\varepsilon \cdot \max\{1 + g(y_1, z)^2, 1 + g(y_1 + \xi, z)^2\})\eta_{yy}(y_1) \\
& + H^*(y_1, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) - \underline{H}(y_1, z, \underline{W}_y^\varepsilon(y_1, z), \underline{W}_z^\varepsilon(y_1, z)) | + 2\delta.
\end{aligned} \tag{4.57}$$

From (4.54) we know that $W_z^\varepsilon(y_1, z) = \underline{W}_z^\varepsilon(y_1, z)$ and $\underline{W}_y^\varepsilon(y_1, z) = W_y^\varepsilon(y_1, z) + 2\delta\eta_y(y_1)$ and from (4.47) we can rewrite

$$\underline{H}(y_1, z, \underline{W}_y^\varepsilon(y_1, z), \underline{W}_z^\varepsilon(y_1, z)) = H^*(y_1 + \xi, z, W_y^\varepsilon(y_1, z) + 2\delta\eta_y(y_1), W_z^\varepsilon(y_1, z)). \tag{4.58}$$

Adding and subtracting $H^*(y_1 + \xi, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z))$ on the right-hand side of (4.57) then using the verified assumption in (4.48), the inequality in (4.57) becomes

$$\begin{aligned}
& |W^\varepsilon(y, z) - \underline{W}^\varepsilon(y, z)| \\
\leq & \frac{\delta(\sigma^2 + 2\varepsilon \cdot \max\{1 + g(y_1, z)^2, 1 + g(y_1 + \xi, z)^2\})}{\beta} \\
& + \frac{1}{\beta} |H^*(y_1, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) - H^*(y_1 + \xi, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) \\
& + H^*(y_1 + \xi, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) \\
& - H^*(y_1 + \xi, z, W_y^\varepsilon(y_1, z) + 2\delta\eta_y(y_1), W_z^\varepsilon(y_1, z))| + 2\delta \\
\leq & \delta \left(\frac{\sigma^2 + 2\varepsilon \cdot \max\{1 + g(y_1, z)^2, 1 + g(y_1 + \xi, z)^2\}}{\beta} + 2 \right) \\
& + \frac{1}{\beta} |W_z^\varepsilon(y_1, z)| |\xi| + |H^*(y_1 + \xi, z, W_y^\varepsilon(y_1, z), W_z^\varepsilon(y_1, z)) \\
& - H^*(y_1 + \xi, z, W_y^\varepsilon(y_1, z) + 2\delta\eta_y(y_1), W_z^\varepsilon(y_1, z))|.
\end{aligned} \tag{4.59}$$

Let $\delta \rightarrow 0$ and we have

$$|W^\varepsilon(y, z) - W^\varepsilon(y + \xi, z)| \leq \frac{1}{\beta} |W_z^\varepsilon(y_1, z)| |\xi|. \tag{4.60}$$

Adding (4.45) and (4.60) then simplify, we get

$$\begin{aligned}
& |W^\varepsilon(y, z) - W^\varepsilon(y, z + \xi)| + |W^\varepsilon(y, z) - W^\varepsilon(y + \xi, z)| \\
& \leq \left(\frac{C_1}{\beta} (1 + |W_y^\varepsilon(y, z_1)| + |W_z^\varepsilon(y, z_1)|) + \frac{1}{\beta} |W_z^\varepsilon(y_1, z)| \right) |\xi| \\
& \leq \left(\frac{C_1}{\beta} (1 + |W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} + |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)}) + \frac{1}{\beta} |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} \right) |\xi| \\
& \leq \left(\frac{C_1}{\beta} + \frac{1 + C_1}{\beta} (|W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} + |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)}) \right) |\xi|. \tag{4.61}
\end{aligned}$$

Dividing $|\xi|$ on both sides of (4.61) and take the limit as ξ goes to zero, we arrive at

$$|W_z^\varepsilon(y, z)| + |W_y^\varepsilon(y, z)| \leq \frac{C_1}{\beta} + \frac{1 + C_1}{\beta} (|W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} + |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)}), \tag{4.62}$$

for all $(y, z) \in \mathbb{R}^2$, which leads to

$$|W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} + |W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{C_1}{\beta} + \frac{1 + C_1}{\beta} (|W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} + |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)}).$$

Then rearranging the terms, we finally get

$$\begin{aligned}
\frac{\beta - C_1 - 1}{\beta} (|W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} + |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)}) & \leq \frac{C_1}{\beta}, \\
|W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} + |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} & \leq \frac{C_1}{\beta - C_1 - 1}, \tag{4.63}
\end{aligned}$$

for $\beta > 1 + C_1$. □

Now we have shown that both W^ε and its first partial derivatives are uniformly bounded in \mathbb{R}^2 . These bounds will be needed in proving the existence result.

4.3.4 Existence and Uniqueness of Viscosity Solution in \mathbb{R}^2

Before we proceed to prove the existence and uniqueness of a viscosity solution to (4.15), we first formally define what it means to be a viscosity solution to our problem.

Definition 4.3.2. *W is a **viscosity subsolution (supersolution)** to the problem*

(4.15) if for every $(y_0, z_0) \in \mathbb{R}^2$ and every $\varphi \in C^2(\mathbb{R}^2)$ such that

$$\varphi \geq (\leq) W, \quad \varphi(y_0, z_0) = W(y_0, z_0)$$

in a neighborhood of (y_0, z_0) , we have

$$\begin{aligned} 0 \geq (\leq) & -\frac{\sigma^2 \varphi_{yy}}{2(1+g(y, z)^2)} + \frac{\beta \varphi}{1+g(y, z)^2} - \frac{f_1(z)}{1+g(y, z)^2} \\ & -\frac{f_2(z) \varphi_y}{1+g(y, z)^2} - \frac{g(y, z) \varphi_z}{1+g(y, z)^2}. \end{aligned}$$

In addition, when W is both a viscosity subsolution and supersolution, W is called a **viscosity solution** to (4.15).

Now we are ready to present the following main result.

Theorem 4.3.5. Assume $\beta > 1 + C_1$ as in the assumption of Lemma 4.3.4, then equation (4.15) has a unique viscosity solution (weak solution) $W(y, z)$ in the Sobolev space $W^{1,\infty}(\mathbb{R}^2)$.

Proof. Having

$$|W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} + |W_z^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{C_1}{\beta - C_1 - 1}$$

implies that $|\nabla W^\varepsilon(y, z)|$ is also bounded and this bound is independent of ε . Then W^ε is Lipschitz continuous and thus uniformly continuous. W^ε is uniformly bounded by $\frac{\|f_1\|_{L^\infty(\mathbb{R}^2)}}{\beta} + K$, independent of ε . And $W^\varepsilon \in W^{1,\infty}(\overline{B}_R)$, which is compactly embedded in $C^\alpha(\overline{B}_R)$ for $0 < \alpha < 1$. For ease of notation, we regard W_n as the same sequence as W^ε except now that it is indexed by n ($n \rightarrow \infty$ and $\varepsilon \rightarrow 0$). We further denote W_{n_R} as the sequence in \overline{B}_R . Since W_{n_R} is bounded, for a fixed $R > 0$, n_R has a subsequence $n_R(j)$ such that $W_{n_R(j)}$, which is a subsequence of W_{n_R} in \overline{B}_R , converges uniformly to $W \in W^{1,\infty}(\overline{B}_R)$ as $j \rightarrow \infty$. Now consider the following diagonal argument:

Since W_{n_1} is bounded, n_1 has a subsequence $n_1(j)$ such that $W_{n_1(j)}$ (a subsequence of W_{n_1} in \overline{B}_1) converges uniformly to $W \in W^{1,\infty}(\overline{B}_1)$ as $j \rightarrow \infty$. Note that \overline{B}_1 is contained in \overline{B}_2 . We pick a subsequence $n_2(j)$ of $n_1(j)$ such that $W_{n_2(j)}$, which is a subsequence in \overline{B}_2 , converges uniformly to $W \in W^{1,\infty}(\overline{B}_2)$. We proceed and obtain subsequences $n_k(j)$ of $n_{k-1}(j)$ such that $W_{n_k(j)}$, a subsequence in \overline{B}_k , converges uniformly to $W \in W^{1,\infty}(\overline{B}_k)$. We pick $W_{n_j(j)}$ from $W_{n_k(j)}$ and this sequence converges uniformly to W in every \overline{B}_j in

\mathbb{R}^2 as $j \rightarrow \infty$. Hence the subsequence $W_{n_j(j)}$ of the original sequence W_n converges uniformly to W in \mathbb{R}^2 as $j \rightarrow \infty$. Translating the notation back, we have that there exists a subsequence of W^ε that converges uniformly to $W \in W^{1,\infty}(\mathbb{R}^2)$ as $\varepsilon \rightarrow 0$. Now let us denote this subsequence of W^ε as itself. Then we have that W^ε that converges uniformly to $W \in W^{1,\infty}(\mathbb{R}^2)$.

We can show that W is the viscosity solution to (4.15). Let $\varphi(\cdot, \cdot) \in C^2(\mathbb{R}^2)$ where $\varphi \geq W$ and $W(y_0, z_0) = \varphi(y_0, z_0)$. Thus $W - \varphi$ attains its local maximum at (y_0, z_0) . Let η be a function that satisfies

$$\begin{cases} \eta(y_0, z_0) = 1, \\ 0 \leq \eta(y, z) < 1, \quad (y, z) \neq (y_0, z_0). \end{cases} \quad (4.64)$$

Note that the local maximum of $W - (\varphi - \eta)$ occurs at (y_0, z_0) and is now unique. Then since W^ε converges to W uniformly, there exists a sequence $(y_0^\varepsilon, z_0^\varepsilon)$ that converges to (y_0, z_0) as ε goes to zero where $W^\varepsilon - (\varphi - \eta)$ attains its local maximum at $(y_0^\varepsilon, z_0^\varepsilon)$. Thus we must have

$$\begin{cases} \frac{\partial}{\partial y}(W^\varepsilon - (\varphi - \eta))|_{y=y_0^\varepsilon} = 0, \\ \frac{\partial}{\partial z}(W^\varepsilon - (\varphi - \eta))|_{z=z_0^\varepsilon} = 0, \\ \frac{\partial^2}{\partial y^2}(W^\varepsilon - (\varphi - \eta))|_{y=y_0^\varepsilon} \leq 0, \\ \frac{\partial^2}{\partial z^2}(W^\varepsilon - (\varphi - \eta))|_{z=z_0^\varepsilon} \leq 0, \\ \Delta(W^\varepsilon - (\varphi - \eta))|_{(y,z)=(y_0^\varepsilon, z_0^\varepsilon)} \leq 0. \end{cases} \quad (4.65)$$

And since the local maximum of η also occurs at $(y_0^\varepsilon, z_0^\varepsilon)$,

$$\begin{cases} \eta_y(y_0^\varepsilon, z_0^\varepsilon) = \eta_z(y_0^\varepsilon, z_0^\varepsilon) = 0, \\ \eta_{yy}(y_0^\varepsilon, z_0^\varepsilon) \leq 0, \\ \eta_{zz}(y_0^\varepsilon, z_0^\varepsilon) \leq 0. \end{cases} \quad (4.66)$$

Additionally note that $W^\varepsilon - \varphi$ attains its local maximum at $(y_0^\varepsilon, z_0^\varepsilon)$, so we also have

$$(W^\varepsilon - \varphi)_{yy}(y_0^\varepsilon, z_0^\varepsilon) \leq 0. \quad (4.67)$$

Using (4.65), (4.67) and (4.18), we see that after adding and subtracting $\Delta\varepsilon(\varphi - \eta)(y_0^\varepsilon, z_0^\varepsilon)$

and $\frac{\sigma^2}{2(1+g(y_0^\varepsilon, z_0^\varepsilon)^2)}\varphi_{yy}(y_0^\varepsilon, z_0^\varepsilon)$ we get

$$\begin{aligned}
& -H^\varepsilon(y_0^\varepsilon, z_0^\varepsilon, W_y^\varepsilon(y_0^\varepsilon, z_0^\varepsilon), W_z^\varepsilon(y_0^\varepsilon, z_0^\varepsilon)) + \frac{\beta}{1+g(y_0^\varepsilon, z_0^\varepsilon)^2}W^\varepsilon(y_0^\varepsilon, z_0^\varepsilon) \\
&= \frac{\sigma^2}{2(1+g(y_0^\varepsilon, z_0^\varepsilon)^2)}W_{yy}^\varepsilon(y_0^\varepsilon, z_0^\varepsilon) + \varepsilon\Delta W^\varepsilon(y_0^\varepsilon, z_0^\varepsilon) \\
&= \frac{\sigma^2}{2(1+g(y_0^\varepsilon, z_0^\varepsilon)^2)}(W^\varepsilon - \varphi)_{yy}(y_0^\varepsilon, z_0^\varepsilon) + \frac{\sigma^2}{2(1+g(y_0^\varepsilon, z_0^\varepsilon)^2)}(\varphi)_{yy}(y_0^\varepsilon, z_0^\varepsilon) \\
&\quad + \varepsilon\Delta(W^\varepsilon - (\varphi - \eta))(y_0^\varepsilon, z_0^\varepsilon) + \Delta\varepsilon(\varphi - \eta)(y_0^\varepsilon, z_0^\varepsilon) \\
&\leq \frac{\sigma^2}{2(1+g(y_0^\varepsilon, z_0^\varepsilon)^2)}\varphi_{yy}(y_0^\varepsilon, z_0^\varepsilon) + \varepsilon\Delta(\varphi - \eta)(y_0^\varepsilon, z_0^\varepsilon).
\end{aligned} \tag{4.68}$$

Also by (4.65) and (4.66), we have

$$\begin{aligned}
W_y^\varepsilon(y_0^\varepsilon, z_0^\varepsilon) &= \varphi_y(y_0^\varepsilon, z_0^\varepsilon), \\
W_z^\varepsilon(y_0^\varepsilon, z_0^\varepsilon) &= \varphi_z(y_0^\varepsilon, z_0^\varepsilon).
\end{aligned}$$

Hence, (4.68) becomes

$$\begin{aligned}
& -H^\varepsilon(y_0^\varepsilon, z_0^\varepsilon, \varphi_y(y_0^\varepsilon, z_0^\varepsilon), \varphi_z(y_0^\varepsilon, z_0^\varepsilon)) + \frac{\beta}{1+g(y_0^\varepsilon, z_0^\varepsilon)^2}W^\varepsilon(y_0^\varepsilon, z_0^\varepsilon) \\
&\leq \frac{\sigma^2}{2(1+g(y_0^\varepsilon, z_0^\varepsilon)^2)}\varphi_{yy}(y_0^\varepsilon, z_0^\varepsilon) + \varepsilon\Delta(\varphi - \eta)(y_0^\varepsilon, z_0^\varepsilon),
\end{aligned}$$

and let $\varepsilon \rightarrow 0$, we obtain

$$\begin{aligned}
& -H(y_0, z_0, \varphi_y(y_0, z_0), \varphi_z(y_0, z_0)) + \frac{\beta}{1+g(y_0, z_0)^2}\varphi(y_0, z_0) \\
&\leq \frac{\sigma^2}{2(1+g(y_0, z_0)^2)}\varphi_{yy}(y_0, z_0).
\end{aligned}$$

By definition, W is a viscosity subsolution. By a similar argument we can also show that W is a viscosity supersolution. Therefore, W is a viscosity solution.

We can also show that the solution W is unique. Suppose that there are two different solutions to (4.15), W_1 and W_2 , where $\sup_{(y,z) \in \mathbb{R}^2}(W_1 - W_2) > 0$. Thus $W_1 - W_2$ attains its maximum at some point $(y_1, z_1) \in \mathbb{R}^2$. With W_1 and W_2 being solutions to (4.15), we

must have that

$$\begin{aligned}
0 = & -\frac{\sigma^2(W_1 - W_2)_{yy}}{2(1 + g(y, z)^2)} + \frac{\beta(W_1 - W_2)}{1 + g(y, z)^2} - \frac{f_2(z)(W_1 - W_2)_y}{1 + g(y, z)^2} \\
& - \frac{g(y, z)(W_1 - W_2)_z}{1 + g(y, z)^2}.
\end{aligned} \tag{4.69}$$

But since $W_1 - W_2$ attains its maximum at (y_1, z_1) , we must also have that

$$\begin{cases} (W_1 - W_2)_{yy}(y_1, z_1) < 0, \\ (W_1 - W_2)_y(y_1, z_1) = 0, \\ (W_1 - W_2)_z(y_1, z_1) = 0. \end{cases}$$

Thus evaluating the right-hand side of (4.69) at (y_1, z_1) we have

$$\left. \frac{\beta(W_1 - W_2)}{1 + g(y, z)^2} \right|_{(y, z) = (y_1, z_1)} < 0,$$

implying that $W_1 - W_2 < 0$ at (y_1, z_1) . This is a contradiction since $\sup_{(y, z) \in \mathbb{R}^2} (W_1 - W_2)$ is greater than zero. Therefore W must be unique and we have completed the proof. \square

We have now shown that a viscosity solution exists. In the next section, we find the subsolution and supersolution bounds for the solution in the classical sense, which will be useful in the verification theorem.

4.4 Subsolution and Supersolution

From preliminary knowledge, we know that if there was a classical solution, it would be sandwiched between its subsolutions and supersolutions. Here, we give definitions of a subsolution and a supersolution in the context of our problem. We first define subsolution and supersolution in \mathbb{R}^2 .

Definition 4.4.1. $W^\varepsilon(y, z)$ is a **subsolution (supersolution)** to the problem (4.18) in \mathbb{R}^2 if

$$\frac{\sigma^2}{2(1 + g(y, z)^2)} W_{yy}^\varepsilon + \varepsilon \Delta W^\varepsilon \geq (\leq) \frac{\beta}{1 + g(y, z)^2} W^\varepsilon - H(y, z, W_y^\varepsilon, W_z^\varepsilon).$$

In addition, if \hat{W} is a subsolution and \bar{W} is a supersolution, and $\hat{W} \leq \bar{W}$, then $\langle \hat{W}, \bar{W} \rangle$ is an **ordered pair of sub/supersolutions**.

Similarly, we can define subsolution and supersolution in the ball B_R .

Definition 4.4.2. $W^\varepsilon(y, z)$ is a **subsolution (supersolution)** to (4.20) on B_R if

$$\begin{cases} \frac{\sigma^2}{2(1+g(y,z)^2)} W_{yy}^\varepsilon + \varepsilon \Delta W^\varepsilon \geq (\leq) \frac{\beta}{1+g(y,z)^2} W^\varepsilon - H(y, z, W_y^\varepsilon, W_z^\varepsilon), & \text{on } B_R, \\ W^\varepsilon \leq (\geq) K, & \text{on } \partial B_R. \end{cases}$$

In addition, if \hat{W} is a subsolution and \bar{W} is a supersolution, and $\hat{W} \leq \bar{W}$ for all $(y, z) \in \bar{B}_R$, then $\langle \hat{W}, \bar{W} \rangle$ is an **ordered pair of sub/supersolutions**.

The next few lemmas give explicit forms to the subsolution and supersolution that we seek.

Lemma 4.4.1. Assume $\mu_1 + \frac{1}{2}\sigma^2 - r > 0$. Define

$$K_1 \equiv \frac{1}{2\beta\sigma^2} \left[\left(\mu_1 + \frac{\sigma^2}{2} - r \right)^2 - 2 \left(\mu_1 + \frac{\sigma^2}{2} - r \right) M \right] + \log(\beta) - 1 + \frac{r}{\beta}. \quad (4.70)$$

Then any constant $K \leq \frac{K_1}{\beta} = \tilde{K}_1$ is a subsolution of (4.18).

Proof. Let f_1 be as defined in (4.11). With $W = K \leq \tilde{K}_1$, we have $W_{yy} = W_{zz} = W_y = W_z = 0$. Since $\mu_1 + \frac{1}{2}\sigma^2 - r > 0$ and $-M \leq \mu_2(z) \leq M$ implying $0 \leq \mu_2(z) + M$, we then have

$$\begin{aligned} & \frac{\beta}{1+g(y,z)^2} \cdot K - \frac{f_1(z)}{1+g(y,z)^2} \\ & \leq \frac{K_1}{1+g(y,z)^2} - \frac{f_1(z)}{1+g(y,z)^2} \\ & = \frac{1}{1+g(y,z)^2} \cdot \left(\frac{1}{2\beta\sigma^2} \left[\left(\mu_1 + \frac{\sigma^2}{2} - r \right)^2 - 2 \left(\mu_1 + \frac{\sigma^2}{2} - r \right) M \right] + \log(\beta) - 1 + \frac{r}{\beta} \right. \\ & \quad \left. - \frac{1}{2\beta\sigma^2} \left[\left(\mu_1 + \frac{\sigma^2}{2} - r \right)^2 + 2 \left(\mu_1 + \frac{\sigma^2}{2} - r \right) \mu_2(z) + (\mu_2(z))^2 \right] + \log(\beta) - 1 + \frac{r}{\beta} \right) \\ & = -\frac{1}{2\beta\sigma^2(1+g(y,z)^2)} \left(2 \left(\mu_1 + \frac{\sigma^2}{2} - r \right) [M + \mu_2(z)] + (\mu_2(z))^2 \right) \\ & \leq 0. \end{aligned}$$

□

Lemma 4.4.2. Assume $\mu_1 + \frac{1}{2}\sigma^2 - r > 0$. Define

$$K_2 \equiv \frac{1}{2\beta\sigma^2} \left(\mu_1 + \frac{\sigma^2}{2} - r + M \right)^2 + \log(\beta) - 1 + \frac{r}{\beta}. \quad (4.71)$$

Then any constant $K \geq \frac{K_2}{\beta} = \tilde{K}_2$ is a supersolution of (4.18).

Proof. Let f_1 be as defined in (4.11). Similarly, with $W = K \geq \tilde{K}_2$, we have $W_{yy} = W_{zz} = W_y = W_z = 0$. Since $\mu_1 + \frac{1}{2}\sigma^2 - r > 0$ and $-M \leq \mu_2(z) \leq M$ (This implies $0 \leq \mu_2(z) + M$ and $0 \leq M - \mu_2(z)$), we then have

$$\begin{aligned} & \frac{\beta}{1 + g(y, z)^2} \cdot K - \frac{f_1(z)}{1 + g(y, z)^2} \\ \geq & \frac{K_2}{1 + g(y, z)^2} - \frac{f_1(z)}{1 + g(y, z)^2} \\ = & \frac{1}{1 + g(y, z)^2} \cdot \left(\frac{1}{2\beta\sigma^2} \left(\mu_1 + \frac{\sigma^2}{2} - r + M \right)^2 + \log(\beta) - 1 + \frac{r}{\beta} \right. \\ & \left. - \frac{1}{2\beta\sigma^2} \left(\mu_1 + \frac{\sigma^2}{2} - r + \mu_2(z) \right)^2 + \log(\beta) - 1 + \frac{r}{\beta} \right) \\ = & \frac{1}{2\beta\sigma^2(1 + g(y, z)^2)} \left[2 \left(\mu_1 + \frac{\sigma^2}{2} - r \right) + M + \mu_2(z) \right] [M - \mu_2(z)] \\ \geq & 0. \end{aligned}$$

□

Lemma 4.4.3. Assume $\mu_1 + \frac{1}{2}\sigma^2 - r > 0$. Then $\langle \tilde{K}_1, \tilde{K}_2 \rangle$ is an ordered pair of sub/supersolutions to (4.18).

Proof. It is clear that $K_1 \leq K_2$. Therefore, $\tilde{K}_1 \leq \tilde{K}_2$ and $\langle \tilde{K}_1, \tilde{K}_2 \rangle$ is an ordered pair of sub/supersolutions to (4.18). □

We note here that if we take $K = \tilde{K}_1$ on ∂B_R in (4.20), then any constant less than or equal to \tilde{K}_1 is a subsolution of (4.20) and any constant greater than or equal to \tilde{K}_2 (since $\tilde{K}_2 \geq \tilde{K}_1$) is a supersolution of (4.20). This confirms the fact that solution to (4.20) is indeed bounded by the sub/supersolution pair. In addition, since the solution to (4.20) converges uniformly to the solution to (4.15) and that it is easy to see $\langle \tilde{K}_1, \tilde{K}_2 \rangle$ is also an

ordered pair of sub/supersolutions to (4.15), hence the ordered pair serve as lower and upper bounds for the solution on \mathbb{R}^2 as well. This will be an important property needed in the verification theorem in the next section. We now proceed to verifying the solution to our problem.

4.5 Verification Theorem

Now let $\tilde{W}(y, z)$ be the classical solution to (4.15). Then

$$\tilde{V}(x, y, z) \equiv \frac{1}{\beta} \log(x) + \tilde{W}(y, z) \quad (4.72)$$

would be the corresponding classical solution to (4.4). In the original derivation of the HJB equation in (3.17), it requires that

$$\mathbf{E} \left[\int_0^T \sigma e^{-\beta t} (\tilde{V}_y + x\pi(t)\tilde{V}_x) dB(t) \right] = 0.$$

Here we verify this is indeed true.

Lemma 4.5.1. *Assume $\beta > 1 + C_1$ as in the assumption of Lemma 4.3.4.*

If $\tilde{V}(X(t), Y(t), Z(t)) \equiv \frac{1}{\beta} \log(X(t)) + \tilde{W}(Y(t), Z(t))$, where \tilde{W} is the classical solution to (4.15), then

$$\mathbf{E} \left[\int_0^T \sigma e^{-\beta t} (\tilde{V}_y + X(t)\pi(t)\tilde{V}_x) dB(t) \right] = 0.$$

Proof. Note that $\tilde{V}_x = \frac{1}{\beta X(t)}$. So for this condition to be valid, we must show that

$$\begin{aligned} & \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} (\tilde{V}_y + X(t)\pi(t)\tilde{V}_x)^2 dt \right] \\ &= \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} \left(\tilde{V}_y + \frac{\pi(t)}{\beta} \right)^2 dt \right] \\ &< \infty. \end{aligned} \quad (4.73)$$

From our previous proofs in Lemma 4.3.4, we have

$$|W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{C_1}{\beta - C_1 - 1}.$$

Note that the upper bounds are independent of ε . If we take $\varepsilon \rightarrow 0$, we would still have

$$|W_y(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{C_1}{\beta - C_1 - 1}.$$

Thus, our classical solution \tilde{V} satisfies

$$|\tilde{V}_y|_{L^\infty(\mathbb{R}^2)} = |\tilde{W}_y|_{L^\infty(\mathbb{R}^2)} \leq \frac{C_1}{\beta - C_1 - 1}. \quad (4.74)$$

Then for a fixed T , using (4.74) and the property that

$$(a + b)^2 \leq 2a^2 + 2b^2, \quad (4.75)$$

for any functions a and b , we have

$$\begin{aligned} & \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} \left(\tilde{V}_y + \frac{\pi(t)}{\beta} \right)^2 dt \right] \\ & \leq \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} \left(2 \cdot \tilde{V}_y^2 + 2 \cdot \frac{\pi(t)^2}{\beta^2} \right) dt \right] \\ & \leq \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} \left(2 \cdot |\tilde{V}_y|^2 + 2 \cdot \frac{\pi(t)^2}{\beta^2} \right) dt \right] \\ & \leq \mathbf{E} \left[\int_0^T 2\sigma^2 e^{-2\beta t} \left(\frac{C_1^2}{(\beta - C_1 - 1)^2} + \frac{\Gamma^2}{\beta^2} \right) dt \right] \\ & = \mathbf{E} \left[\sigma^2 \left(\frac{C_1^2}{(\beta - C_1 - 1)^2} + \frac{\Gamma^2}{\beta^2} \right) \left(\frac{1 - e^{-2\beta T}}{\beta} \right) \right] < \infty, \end{aligned}$$

as desired in (4.73). □

We now want to verify that $\tilde{V}(x, y, z) = V(x, y, z)$, our value function.

Theorem 4.5.2 (Verification Theorem). *Assume $\beta > 1 + C_1$ as in the assumption of Lemma 4.3.4 and assume additionally that $\mu_1 + \frac{1}{2}\sigma^2 - r > 0$. Let $\tilde{W}(y, z)$ be the classical solution to (4.15) and $\tilde{V}(x, y, z) \equiv \frac{1}{\beta} \log(x) + \tilde{W}(y, z)$. Then $\tilde{V}(x, y, z) = V(x, y, z)$, where $V(x, y, z)$ is the value function in (4.3). Moreover, the optimal control policy is given by*

$$\pi^*(z) = \frac{\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)}{\sigma^2}, \quad c^* = \beta.$$

Proof. From (4.4) we know that for any $(\pi(t), c(t)) \in \Pi$ we have

$$\begin{aligned}
-\log(c(t)X(t)) &\geq -\beta\tilde{V} + X(t)\pi(t) \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(t)) \right) \tilde{V}_x \\
&\quad + \frac{\sigma^2\pi(t)^2}{2} X(t)^2 \tilde{V}_{xx} + \sigma^2\pi(t)X(t) \tilde{V}_{xy} + \frac{\sigma^2}{2} \tilde{V}_{yy} - c(t)X(t) \tilde{V}_x \\
&\quad + rX(t) \tilde{V}_x + (\mu_1 + \mu_2(Z(t))) \tilde{V}_y + (Y(t) - \lambda Z(t)) \tilde{V}_z. \tag{4.76}
\end{aligned}$$

Combining (4.76) with a similar calculation as in (3.15), we can get

$$\begin{aligned}
d(e^{-\beta t} \tilde{V}) &= e^{-\beta t} \left(-\beta\tilde{V} + \left[X(t) \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) \right) \right] \tilde{V}_x \right. \\
&\quad + (\mu_1 + \mu_2(Z(t))) \tilde{V}_y + (Y(t) - \lambda Z(t)) \tilde{V}_z + \frac{\sigma^2}{2} \tilde{V}_{yy} + \frac{\sigma^2\pi(t)^2}{2} X(t)^2 \tilde{V}_{xx} \\
&\quad + \left. \sigma^2\pi(t)X(t) \tilde{V}_{xy} \right) dt + e^{-\beta t} (\tilde{V}_y \sigma + \tilde{V}_x \sigma \pi(t) X(t)) dB(t) \\
&\leq e^{-\beta t} (-\log(c(t)X(t))) dt + e^{-\beta t} (\tilde{V}_y \sigma + \tilde{V}_x \sigma \pi(t) X(t)) dB(t).
\end{aligned}$$

Integrating both sides and taking expectation then using Lemma 4.5.1 we have

$$\begin{aligned}
&\mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T)) - \tilde{V}(x, y, z)] \\
&\leq \mathbf{E} \left[\int_0^T e^{-\beta t} (-\log(c(t)X(t))) dt + \int_0^T e^{-\beta t} (\tilde{V}_y \sigma + \tilde{V}_x \sigma \pi(t) X(t)) dB(t) \right] \\
&= -\mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c(t)X(t))) dt \right].
\end{aligned}$$

This implies

$$\begin{aligned}
\tilde{V}(x, y, z) &\geq \mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c(t)X(t))) dt \right] \\
&\quad + \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))]. \tag{4.77}
\end{aligned}$$

We need to show that

$$\limsup_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \geq 0. \tag{4.78}$$

Note that from (4.72) we have

$$\mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] = \mathbf{E} \left[e^{-\beta T} \left(\frac{1}{\beta} \log(X(T)) + \tilde{W}(Y(T), Z(T)) \right) \right] \quad (4.79)$$

and by Itô's Lemma,

$$\begin{aligned} d(\log(X(t))) &= \frac{1}{X(t)} dX(t) - \frac{1}{2} \cdot \frac{1}{X(t)^2} (dX(t))^2 \\ &= \left[r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) \right] dt \\ &\quad + \sigma \pi(t) dB(t) - \frac{1}{2} \sigma^2 \pi(t)^2 dt. \end{aligned} \quad (4.80)$$

Then integrating both sides of (4.80) and take expectation to obtain

$$\begin{aligned} &\mathbf{E}[\log(X(T))] \\ &= \mathbf{E} \left[\int_0^T \left[r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) - \frac{1}{2} \sigma^2 \pi(t)^2 \right] dt \right] + \log(x). \end{aligned} \quad (4.81)$$

Clearly,

$$\limsup_{T \rightarrow \infty} e^{-\beta T} \log(x) = 0,$$

and since $-\Gamma \leq \pi(t) \leq \Gamma$, $\varepsilon_c \leq c(t) \leq \Lambda$ and $\mu_1 + \frac{1}{2} \sigma^2 - r > 0$,

$$\begin{aligned} &\limsup_{T \rightarrow \infty} e^{-\beta T} \mathbf{E} \left[\int_0^T \left[r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) - \frac{1}{2} \sigma^2 \pi(t)^2 \right] dt \right] \\ &\geq \limsup_{T \rightarrow \infty} e^{-\beta T} \mathbf{E} \left[\int_0^T - \left(\mu_1 + \frac{\sigma^2}{2} - r + M \right) \Gamma - \Lambda - \frac{\sigma^2 \Gamma^2}{2} dt \right] \\ &= 0. \end{aligned}$$

Thus,

$$\limsup_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \log(X(T))] = \limsup_{T \rightarrow \infty} e^{-\beta T} \mathbf{E}[\log(X(T))] \geq 0. \quad (4.82)$$

Next notice that \tilde{W} is bounded below by its subsolution \tilde{K}_1 defined in (4.70). Thus,

$$\begin{aligned}
\limsup_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{W}] &= \limsup_{T \rightarrow \infty} e^{-\beta T} \mathbf{E}[\tilde{W}] \\
&\geq \limsup_{T \rightarrow \infty} e^{-\beta T} \mathbf{E}[\tilde{K}_1] \\
&= \limsup_{T \rightarrow \infty} e^{-\beta T} \tilde{K}_1 \\
&= 0.
\end{aligned} \tag{4.83}$$

Hence combining (4.82) and (4.83),

$$\begin{aligned}
&\limsup_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \\
&= \mathbf{E} \left[e^{-\beta T} \left(\frac{1}{\beta} \log(X(T)) + \tilde{W}(Y(T), Z(T)) \right) \right] \geq 0,
\end{aligned}$$

and we have shown (4.78). Now we need to show

$$\limsup_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(c(t)X(t)) dt \right] \geq \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(c(t)X(t)) dt \right]. \tag{4.84}$$

Since $\varepsilon_c \leq c(t) \leq \Lambda$, then $|\log(c(t))| \leq \max\{|\log(\varepsilon_c)|, |\log(\Lambda)|\}$, a clearly integrable function, for all $t \geq 0$. Using dominated convergence theorem, we have

$$\begin{aligned}
\limsup_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c(t))) dt \right] &\geq \lim_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c(t))) dt \right] \\
&= \mathbf{E} \left[\lim_{T \rightarrow \infty} \int_0^T e^{-\beta t} (\log(c(t))) dt \right] \\
&= \mathbf{E} \left[\int_0^\infty e^{-\beta t} (\log(c(t))) dt \right].
\end{aligned} \tag{4.85}$$

Next we want to check the conditions of stochastic Fubini theorem, which will be used in the subsequent part. The integrability conditions we check for are taken from (1.3) in [63]. First, it is clear that $\mu(\Omega) = 1 < \infty$ and also the measure defined by

$$\int_0^\infty e^{-\beta t} dt = \frac{1}{\beta} < \infty.$$

Second, we use (3.9) to get

$$\begin{aligned}
& \int_0^T \mathbf{E}[|e^{-\beta t} \log(X(t))|^2] dt \\
& \leq \int_0^\infty \mathbf{E}[|e^{-\beta t} \log(X(t))|^2] dt \\
& = \int_0^\infty \mathbf{E} \left[e^{-2\beta t} \left| \int_0^t \left(r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(Z(\tau)) \right) \pi(\tau) - c(\tau) - \frac{\sigma^2 \pi(\tau)^2}{2} \right) d\tau \right. \right. \\
& \quad \left. \left. + \int_0^t \sigma \pi(\tau) dB(\tau) + \log(x) \right|^2 \right] dt \\
& \leq \int_0^\infty \mathbf{E} \left[e^{-2\beta t} \left| \left(r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + M \right) \Gamma \right) t + \log(x) + \int_0^t \sigma \pi(\tau) dB(\tau) \right|^2 \right] dt.
\end{aligned} \tag{4.86}$$

Denote

$$h(x, t) = \left(r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + M \right) \Gamma \right) t + \log(x). \tag{4.87}$$

Then using Ito's isometry and (4.75), for a fixed x we have

$$\begin{aligned}
& \int_0^\infty \mathbf{E} \left[e^{-2\beta t} \left| h(x, t) + \int_0^t \sigma \pi(\tau) dB(\tau) \right|^2 \right] dt \\
& \leq \int_0^\infty \mathbf{E} \left[e^{-2\beta t} \left(2h(x, t)^2 + 2 \left(\int_0^t \sigma \pi(\tau) dB(\tau) \right)^2 \right) \right] dt \\
& = \int_0^\infty 2e^{-2\beta t} \left(h(x, t)^2 + \mathbf{E} \left[\int_0^t \sigma^2 \pi(\tau)^2 d\tau \right] \right) dt \\
& \leq \int_0^\infty 2e^{-2\beta t} \left(h(x, t)^2 + \int_0^t \sigma^2 \Gamma^2 d\tau \right) dt \\
& = \int_0^\infty 2e^{-2\beta t} h(x, t)^2 + 2e^{-2\beta t} \sigma^2 \Gamma^2 t dt \\
& = \int_0^\infty 2e^{-2\beta t} \left(\left(r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + M \right) \Gamma \right) t + \log(x) \right)^2 + 2e^{-2\beta t} \sigma^2 \Gamma^2 t dt \\
& < \infty.
\end{aligned} \tag{4.88}$$

Combining (4.86) and (4.88) also implies that we can have

$$\int_0^\infty \mathbf{E}[e^{-\beta t} |\log(X(t))|^2] dt < \infty. \quad (4.89)$$

We use the fact that

$$|\log(X(t))| \leq \frac{1 + |\log(X(t))|^2}{2}, \quad (4.90)$$

to get a dominating function

$$\begin{aligned} & \left| \int_0^T e^{-\beta t} \log(X(t)) dt \right| \\ & \leq \int_0^T e^{-\beta t} |\log(X(t))| dt \\ & \leq \int_0^T e^{-\beta t} \cdot \frac{1 + |\log(X(t))|^2}{2} dt \\ & \leq \int_0^\infty e^{-\beta t} \cdot \frac{1 + |\log(X(t))|^2}{2} dt. \end{aligned}$$

And using stochastic Fubini theorem, we can show that this dominating function is integrable

$$\begin{aligned} & \mathbf{E} \left[\int_0^\infty e^{-\beta t} \cdot \frac{1 + |\log(X(t))|^2}{2} dt \right] \\ & = \int_0^\infty \mathbf{E} \left[e^{-\beta t} \cdot \frac{1 + |\log(X(t))|^2}{2} \right] dt \\ & = \int_0^\infty \frac{e^{-\beta t}}{2} + \frac{1}{2} \mathbf{E} [e^{-\beta t} |\log(X(t))|^2] dt \\ & = \frac{1}{2\beta} + \frac{1}{2} \int_0^\infty \mathbf{E} [e^{-\beta t} |\log(X(t))|^2] dt < \infty. \end{aligned}$$

Hence, by dominated convergence theorem, we get

$$\lim_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(X(t)) dt \right] = \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(X(t)) dt \right]. \quad (4.91)$$

And so finally we take limit superior on both sides of (4.77) and using (4.79), (4.85) and

(4.91) to reach

$$\begin{aligned}
\tilde{V}(x, y, z) &\geq \limsup_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(c(t)X(t)) dt \right] \\
&\quad + \limsup_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \\
&\geq \limsup_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(c(t)) dt \right] + \limsup_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(X(t)) dt \right] \\
&\geq \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(c(t)) dt \right] + \lim_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(X(t)) dt \right] \\
&= \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(c(t)) dt \right] + \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(X(t)) dt \right] \\
&= \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(c(t)X(t)) dt \right]. \tag{4.92}
\end{aligned}$$

The inequality in (4.92) holds for all admissible controls $c(t)$ and $\pi(t)$. Therefore

$$\tilde{V}(x, y, z) \geq \sup_{(\pi(t), c(t)) \in \Pi} \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(c(t)X(t)) dt \right] = V(x, y, z).$$

For the reverse inequality $V(x, y, z) \leq \tilde{V}(x, y, z)$, we first show that $(\pi^*, c^*) \in \Pi$. Both are progressively measurable since their values are determined for all t . Since

$$\begin{aligned}
|\pi^*| &= \left| \frac{\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)}{\sigma^2} \right| \\
&\leq \frac{\mu_1 + \frac{1}{2}\sigma^2 - r + M}{\sigma^2},
\end{aligned}$$

If we choose

$$\Gamma \geq \frac{\mu_1 + \frac{1}{2}\sigma^2 - r + M}{\sigma^2}, \tag{4.93}$$

then $|\pi^*| \leq \Gamma$. Also $c^* = \beta > 0$ for all $t \geq 0$. Similarly, we can choose

$$\Lambda \geq \beta. \tag{4.94}$$

Thus, $(\pi^*, c^*) \in \Pi$. Now using (π^*, c^*) , we would have the following equality

$$\tilde{V}(x, y, z) = \mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c^* X(t))) dt \right] + \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))]. \quad (4.95)$$

Thus, we need to show

$$\liminf_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \leq 0. \quad (4.96)$$

Again using (4.79) and (4.81), we have that for a fixed x ,

$$\liminf_{T \rightarrow \infty} e^{-\beta T} \log(x) = 0,$$

and

$$\begin{aligned} & \liminf_{T \rightarrow \infty} e^{-\beta T} \mathbf{E} \left[\int_0^T \left[r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) - \frac{1}{2} \sigma^2 (\pi(t))^2 \right] dt \right] \\ & \leq \liminf_{T \rightarrow \infty} e^{-\beta T} \mathbf{E} \left[\int_0^T \left[r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + M \right) \Gamma \right] dt \right] \\ & = 0, \end{aligned}$$

Thus,

$$\liminf_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \log(X(T))] = \liminf_{T \rightarrow \infty} e^{-\beta T} \mathbf{E}[\log(X(T))] \leq 0. \quad (4.97)$$

Next notice that \tilde{W} is bounded above by its supersolution \tilde{K}_2 defined in (4.71). Thus,

$$\begin{aligned} \liminf_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{W}] &= \liminf_{T \rightarrow \infty} e^{-\beta T} \mathbf{E}[\tilde{W}] \\ &\leq \liminf_{T \rightarrow \infty} e^{-\beta T} \mathbf{E}[\tilde{K}_2] \\ &= \liminf_{T \rightarrow \infty} e^{-\beta T} \tilde{K}_2 \\ &= 0. \end{aligned} \quad (4.98)$$

Hence by (4.97), (4.98) and (4.79),

$$\liminf_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \leq 0,$$

and taking limit inferior on both sides of (4.95) we obtain

$$\begin{aligned}
& \tilde{V}(x, y, z) \\
&= \liminf_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c^* X(t))) dt \right] + \liminf_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \\
&\leq \liminf_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c^* X(t))) dt \right]. \tag{4.99}
\end{aligned}$$

From assumptions in Lemma 4.3.4, we know that $c^* = \beta > 1 + C_1 > 1$. Thus, $0 < \log(c^*)$ and by monotone convergence theorem,

$$\begin{aligned}
\liminf_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c^*)) dt \right] &\leq \lim_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} (\log(c^*)) dt \right] \\
&= \mathbf{E} \left[\lim_{T \rightarrow \infty} \int_0^T e^{-\beta t} (\log(c^*)) dt \right] \\
&= \mathbf{E} \left[\int_0^\infty e^{-\beta t} (\log(c^*)) dt \right]. \tag{4.100}
\end{aligned}$$

Combining (4.99)-(4.100) with (4.91),

$$\begin{aligned}
\tilde{V}(x, y, z) &\leq \liminf_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(c^* X(t)) dt \right] + \liminf_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \\
&\leq \liminf_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(c^*) dt \right] + \liminf_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(X(t)) dt \right] \\
&\leq \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(c^*) dt \right] + \lim_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \log(X(t)) dt \right] \\
&= \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(c^*) dt \right] + \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(X(t)) dt \right] \\
&= \mathbf{E} \left[\int_0^\infty e^{-\beta t} \log(c^* X(t)) dt \right] \\
&= V(x, y, z).
\end{aligned}$$

Hence, we showed that $\tilde{V}(x, y, z) = V(x, y, z)$. □

Chapter 5

HARA Utility

5.1 HARA Utility

We now turn to the HARA utility case that also models constant risk aversion, but for different values of the parameter γ . The HARA utility function is defined as

$$U(C) = \frac{1}{\gamma}C^\gamma - \frac{1}{\gamma}, \quad (5.1)$$

where C denotes consumption and $\gamma \neq 0$. The Arrow-Pratt measure of relative risk aversion in (4.2) for the HARA utility function is $R(C) = 1 - \gamma$, a constant. Note that for the logarithmic utility case, measure of relative risk aversion is one as $\gamma = 0$. When $\gamma = 1$, the investor is risk neutral; and when $\gamma \rightarrow -\infty$, the investor has infinite risk aversion. We note here that the constant $-\frac{1}{\gamma}$ does not actually have an impact on obtaining the optimal solutions, thus it is usually omitted for technical simplicity. The HARA utility to be used throughout the rest of this dissertation is defined as

$$U(C) = \frac{1}{\gamma}C^\gamma, \quad (5.2)$$

where C denotes consumption, $-\infty < \gamma < 1$ and $\gamma \neq 0$. In this dissertation, we analyze the case for when $0 < \gamma < 1$. We will see in this chapter that using the HARA utility function leads to a different problem than in the logarithmic utility case. We define the admissible control space for the HARA utility case as follows:

Definition 5.1.1. *A control $(\pi(t), c(t))$ is said to be in the admissible control space Π if*

it satisfies

1. $(\pi(t), c(t))$ is \mathbb{F}_t - progressively measurable;
2. $|\pi(t)| \leq \Gamma < \infty$ for all $t \geq 0$;
3. $0 \leq c(t) \leq \Lambda < \infty$, for all $t \geq 0$.

The thresholds for positive constants Γ and Λ are given in the proof of the verification theorem. Next we present the value function and HJB equation for the HARA utility case.

5.2 Value Function and HJB Equation

Similar to the logarithmic utility case, we now replace the generic utility function $U(\cdot)$ with the HARA utility function defined in (5.2). Then value function defined in (3.10)-(3.11) becomes

$$\begin{aligned} J(x, y, z, \pi, c) &= \mathbf{E} \left[\int_0^\infty e^{-\beta\tau} \frac{1}{\gamma} (c(\tau)X(\tau))^\gamma d\tau \right], \\ V(x, y, z) &= \sup_{(\pi, c) \in \Pi} J(x, y, z, \pi, c), \end{aligned} \quad (5.3)$$

where $0 < \gamma < 1$. Applying dynamic programming principle, the corresponding HJB equation then evolves into

$$\begin{aligned} \beta V &= \sup_{\pi \in \Pi} \left(x\pi \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) V_x + \frac{1}{2}V_{xx}\sigma^2\pi^2x^2 + V_{xy}\sigma^2\pi x \right) + \frac{1}{2}V_{yy}\sigma^2 \\ &\quad + \sup_{c \in \Pi} \left(\frac{1}{\gamma}(cx)^\gamma - cxV_x \right) + xrV_x + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z. \end{aligned} \quad (5.4)$$

We then find the candidates for optimal controls

$$\pi^* = \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x - \sigma^2V_{xy}}{\sigma^2xV_{xx}}, \quad c^* = \frac{V_x^{\frac{1}{\gamma-1}}}{x}, \quad (5.5)$$

by finding extrema of the supremum expressions in (5.4).

Now plugging (5.5) in (5.4), we get

$$\begin{aligned}
\beta V &= x \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z) \right) V_x \cdot \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x - \sigma^2 V_{xy}}{\sigma^2 x V_{xx}} \\
&\quad + \frac{1}{2} V_{xx} \sigma^2 x^2 \left(\frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x - \sigma^2 V_{xy}}{\sigma^2 x V_{xx}} \right)^2 \\
&\quad + V_{xy} \sigma^2 x \cdot \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x - \sigma^2 V_{xy}}{\sigma^2 x V_{xx}} + \frac{1}{2} V_{yy} \sigma^2 \\
&\quad + \frac{1}{\gamma} \left(\frac{V_x^{\frac{1}{\gamma-1}}}{x} \cdot x \right)^\gamma - \frac{V_x^{\frac{1}{\gamma-1}}}{x} \cdot x V_x + x r V_x + (\mu_1 + \mu_2(z)) V_y + (y - \lambda z) V_z.
\end{aligned}$$

Simplifying and rewriting leads to

$$\begin{aligned}
\beta V &= \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))^2 V_x^2 - \sigma^2 (\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)) V_{xy} V_x}{\sigma^2 V_{xx}} \\
&\quad + \frac{1}{2} \cdot \frac{(-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x - \sigma^2 V_{xy})^2}{\sigma^2 V_{xx}} \\
&\quad + \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x \cdot V_{xy} \sigma^2 - \sigma^4 V_{xy}^2}{\sigma^2 V_{xx}} \\
&\quad + \frac{1}{2} V_{yy} \sigma^2 + \frac{1}{\gamma} V_x^{\frac{\gamma}{\gamma-1}} - V_x^{\frac{\gamma}{\gamma-1}} + x r V_x + (\mu_1 + \mu_2(z)) V_y + (y - \lambda z) V_z. \quad (5.6)
\end{aligned}$$

Note that in (5.6) we can complete the square and get

$$\begin{aligned}
\beta V &= \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))^2 V_x^2 - 2\sigma^2 (\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)) V_{xy} V_x - \sigma^4 V_{xy}^2}{\sigma^2 V_{xx}} \\
&\quad + \frac{1}{2} \cdot \frac{(-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x - \sigma^2 V_{xy})^2}{\sigma^2 V_{xx}} + \frac{1}{2} V_{yy} \sigma^2 \\
&\quad + \frac{1-\gamma}{\gamma} V_x^{\frac{\gamma}{\gamma-1}} + x r V_x + (\mu_1 + \mu_2(z)) V_y + (y - \lambda z) V_z \\
&= -\frac{((\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x + \sigma^2 V_{xy})^2}{\sigma^2 V_{xx}} \\
&\quad + \frac{1}{2} \cdot \frac{((\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x + \sigma^2 V_{xy})^2}{\sigma^2 V_{xx}} + \frac{1}{2} V_{yy} \sigma^2 \\
&\quad + \frac{1-\gamma}{\gamma} V_x^{\frac{\gamma}{\gamma-1}} + x r V_x + (\mu_1 + \mu_2(z)) V_y + (y - \lambda z) V_z. \quad (5.7)
\end{aligned}$$

Combining like terms in (5.7) we finally reach

$$\begin{aligned}\beta V &= -\frac{1}{2} \cdot \frac{((\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))V_x + \sigma^2 V_{xy})^2}{\sigma^2 V_{xx}} + \frac{1}{2} V_{yy} \sigma^2 \\ &\quad + \frac{1-\gamma}{\gamma} V_x^{\frac{\gamma}{\gamma-1}} + xrV_x + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z.\end{aligned}\quad (5.8)$$

We see that there are many nonlinear terms on the right-hand side of (5.8). We would like to simplify our equation by some transformations. Let

$$V(x, y, z) = x^\gamma U(y, z). \quad (5.9)$$

Then with

$$\begin{aligned}V_x &= \gamma x^{\gamma-1} U, & V_{xy} &= \gamma x^{\gamma-1} U_y, & V_{xx} &= \gamma(\gamma-1)x^{\gamma-2} U, \\ V_y &= x^\gamma U_y, & V_{yy} &= x^\gamma U_{yy}, & V_z &= x^\gamma U_z,\end{aligned}$$

equation (5.8) transforms into

$$\begin{aligned}\beta x^\gamma U &= -\frac{1}{2} \cdot \frac{((\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)) \cdot \gamma x^{\gamma-1} U + \sigma^2 \gamma x^{\gamma-1} U_y)^2}{\sigma^2 \gamma (\gamma-1) x^{\gamma-2} U} + \frac{1}{2} x^\gamma U_{yy} \sigma^2 \\ &\quad + \frac{1-\gamma}{\gamma} (\gamma x^{\gamma-1} U)^{\frac{\gamma}{\gamma-1}} + xr \gamma x^{\gamma-1} U + x^\gamma (\mu_1 + \mu_2(z)) U_y + x^\gamma (y - \lambda z) U_z.\end{aligned}\quad (5.10)$$

Dividing x^γ on both sides of (5.10) and rearrange then simplify, we have

$$\begin{aligned}\beta U &= -\frac{((\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)) \cdot \gamma x^{\gamma-1} U + \sigma^2 \gamma x^{\gamma-1} U_y)^2}{2\sigma^2 \gamma (\gamma-1) x^2 \gamma^{\gamma-2} U} + \frac{1}{2} U_{yy} \sigma^2 \\ &\quad + \frac{1-\gamma}{\gamma} (\gamma U)^{\frac{\gamma}{\gamma-1}} + r \gamma U + (\mu_1 + \mu_2(z)) U_y + (y - \lambda z) U_z \\ 0 &= -\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))^2 \gamma^2 U^2 + 2(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)) \gamma U \sigma^2 \gamma U_y + \sigma^4 \gamma^2 U_y^2}{2\sigma^2 \gamma (\gamma-1) U} \\ &\quad + \frac{1}{2} U_{yy} \sigma^2 + \frac{1-\gamma}{\gamma} (\gamma U)^{\frac{\gamma}{\gamma-1}} + r \gamma U + (\mu_1 + \mu_2(z)) U_y + (y - \lambda z) U_z - \beta U \\ 0 &= -\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))^2 \gamma U}{2\sigma^2 (\gamma-1)} - \frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)) \gamma U_y}{\gamma-1} - \frac{\sigma^2 \gamma U_y^2}{2(\gamma-1) U} \\ &\quad + \frac{1}{2} U_{yy} \sigma^2 + (1-\gamma) \gamma^{\frac{1}{\gamma-1}} U^{\frac{\gamma}{\gamma-1}} + r \gamma U + (\mu_1 + \mu_2(z)) U_y + (y - \lambda z) U_z - \beta U.\end{aligned}$$

Then grouping like terms to get

$$\begin{aligned}
0 = & \left(r\gamma - \beta - \frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))^2\gamma}{2\sigma^2(\gamma - 1)} \right) U \\
& + \left(\mu_1 + \mu_2(z) - \frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))\gamma}{\gamma - 1} \right) U_y \\
& - \frac{\sigma^2\gamma U_y^2}{2(\gamma - 1)U} + \frac{1}{2}U_{yy}\sigma^2 + \frac{1 - \gamma}{\gamma^{\frac{1}{1-\gamma}}U^{\frac{\gamma}{1-\gamma}}} + (y - \lambda z)U_z.
\end{aligned} \tag{5.11}$$

To simplify (5.11) further, we consider the system

$$\begin{cases} z_\theta = \frac{\partial z}{\partial \theta} = y - \lambda z, \\ z(y, 0) = 0. \end{cases} \tag{5.12}$$

Solving the differential equation (5.12) yields the solution

$$z(y, \theta) = \frac{y}{\lambda}(1 - e^{-\lambda\theta}). \tag{5.13}$$

Note that

$$U_\theta = U_z \cdot z_\theta = U_z \cdot (y - \lambda z). \tag{5.14}$$

Plugging (5.14) into (5.11), we obtain

$$\begin{aligned}
0 = & \left(r\gamma - \beta - \frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2\gamma}{2\sigma^2(\gamma - 1)} \right) U \\
& + \left(\mu_1 + \mu_2(y, \theta) - \frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))\gamma}{\gamma - 1} \right) U_y - \frac{\sigma^2\gamma U_y^2}{2(\gamma - 1)U} \\
& + \frac{1}{2}U_{yy}\sigma^2 + \frac{1 - \gamma}{\gamma^{\frac{1}{1-\gamma}}U^{\frac{\gamma}{1-\gamma}}} + U_\theta \\
U_\theta = & \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2\gamma}{2\sigma^2(\gamma - 1)} - r\gamma + \beta \right) U \\
& + \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))\gamma}{\gamma - 1} - \mu_1 - \mu_2(y, \theta) \right) U_y + \frac{\sigma^2\gamma U_y^2}{2(\gamma - 1)U} \\
& - \frac{1}{2}U_{yy}\sigma^2 - \frac{1 - \gamma}{\gamma^{\frac{1}{1-\gamma}}U^{\frac{\gamma}{1-\gamma}}}.
\end{aligned} \tag{5.15}$$

Now $U(\cdot, \cdot)$ is a function of y and θ . Note that

$$(U^{\frac{1}{1-\gamma}})_y = \frac{1}{1-\gamma} U^{\frac{\gamma}{1-\gamma}} U_y, \quad (5.16)$$

$$(U^{\frac{1}{1-\gamma}})_{yy} = \frac{1}{1-\gamma} U^{\frac{\gamma}{1-\gamma}} U_{yy} + \frac{\gamma}{(1-\gamma)^2} U^{\frac{2\gamma-1}{1-\gamma}} U_y^2. \quad (5.17)$$

Using (5.15), we have

$$\begin{aligned} (U^{\frac{1}{1-\gamma}})_\theta &= \frac{1}{1-\gamma} U^{\frac{\gamma}{1-\gamma}} U_\theta \\ &= \frac{1}{1-\gamma} U^{\frac{\gamma}{1-\gamma}} \cdot \left[\left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2 \gamma}{2\sigma^2(\gamma-1)} - r\gamma + \beta \right) U \right. \\ &\quad \left. + \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))\gamma}{\gamma-1} - \mu_1 - \mu_2(y, \theta) \right) U_y + \frac{\sigma^2 \gamma U_y^2}{2(\gamma-1)U} \right. \\ &\quad \left. - \frac{1}{2} U_{yy} \sigma^2 - \frac{1-\gamma}{\gamma^{\frac{1}{1-\gamma}} U^{\frac{\gamma}{1-\gamma}}} \right]. \end{aligned} \quad (5.18)$$

Multiplying through all terms and then plugging (5.16)-(5.17) into (5.18), we obtain

$$\begin{aligned} (U^{\frac{1}{1-\gamma}})_\theta &= \frac{1}{1-\gamma} \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2 \gamma}{2\sigma^2(\gamma-1)} - r\gamma + \beta \right) U^{\frac{1}{1-\gamma}} \\ &\quad + \frac{1}{1-\gamma} U^{\frac{\gamma}{1-\gamma}} U_y \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))\gamma}{\gamma-1} - \mu_1 - \mu_2(y, \theta) \right) \\ &\quad - \frac{\sigma^2}{2} \cdot \frac{\gamma}{(1-\gamma)^2} U^{\frac{2\gamma-1}{1-\gamma}} U_y^2 - \frac{\sigma^2}{2} \cdot \frac{1}{1-\gamma} U^{\frac{\gamma}{1-\gamma}} U_{yy} - \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \\ &= \frac{1}{1-\gamma} \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2 \gamma}{2\sigma^2(\gamma-1)} - r\gamma + \beta \right) U^{\frac{1}{1-\gamma}} \\ &\quad + (U^{\frac{1}{1-\gamma}})_y \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))\gamma}{\gamma-1} - \mu_1 - \mu_2(y, \theta) \right) \\ &\quad - \frac{\sigma^2}{2} \cdot (U^{\frac{1}{1-\gamma}})_{yy} - \frac{1}{\gamma^{\frac{1}{1-\gamma}}}. \end{aligned} \quad (5.19)$$

Lastly, set

$$W(y, \theta) = U(y, \theta)^{\frac{1}{1-\gamma}}. \quad (5.20)$$

Then we can rewrite and simplify (5.19) as

$$\begin{aligned}
W_\theta &= \frac{1}{1-\gamma} \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2 \gamma}{2\sigma^2(\gamma-1)} - r\gamma + \beta \right) W \\
&\quad + \left(\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))\gamma}{\gamma-1} - \mu_1 - \mu_2(y, \theta) \right) W_y \\
&\quad - \frac{\sigma^2}{2} W_{yy} - \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \\
&= \left(-\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2 \gamma}{2\sigma^2(1-\gamma)^2} - \frac{r\gamma - \beta}{1-\gamma} \right) W \\
&\quad + \left(-\left(\mu_1 + \frac{1}{2}\sigma^2 - r \right) \cdot \frac{\gamma}{1-\gamma} - \mu_1 - \mu_2(y, \theta) \cdot \left(\frac{\gamma}{1-\gamma} + 1 \right) \right) W_y \\
&\quad - \frac{\sigma^2}{2} W_{yy} - \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \\
&= \left(-\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2 \gamma}{2\sigma^2(1-\gamma)^2} - \frac{r\gamma - \beta}{1-\gamma} \right) W \\
&\quad + \left(-\left(\mu_1 + \frac{1}{2}\sigma^2 - r \right) \cdot \frac{\gamma}{1-\gamma} - \mu_1 - \mu_2(y, \theta) \cdot \frac{1}{1-\gamma} \right) W_y \\
&\quad - \frac{\sigma^2}{2} W_{yy} - \frac{1}{\gamma^{\frac{1}{1-\gamma}}}. \tag{5.21}
\end{aligned}$$

Rearrange (5.21) into standard form, we get

$$\begin{aligned}
\frac{\sigma^2}{2} W_{yy} &= \left(-\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2 \gamma}{2\sigma^2(1-\gamma)^2} + \frac{\beta - r\gamma}{1-\gamma} \right) W - W_\theta \\
&\quad + \left(-\left(\mu_1 + \frac{1}{2}\sigma^2 - r \right) \cdot \frac{\gamma}{1-\gamma} - \mu_1 - \mu_2(y, \theta) \cdot \frac{1}{1-\gamma} \right) W_y - \frac{1}{\gamma^{\frac{1}{1-\gamma}}}. \tag{5.22}
\end{aligned}$$

Let

$$f_1(y, \theta) = -\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(y, \theta))^2 \gamma}{2\sigma^2(1-\gamma)^2} + \frac{\beta - r\gamma}{1-\gamma}, \tag{5.23}$$

$$f_2(y, \theta) = -\left(\mu_1 + \frac{1}{2}\sigma^2 - r \right) \cdot \frac{\gamma}{1-\gamma} - \mu_1 - \mu_2(y, \theta) \cdot \frac{1}{1-\gamma}, \tag{5.24}$$

$$H(y, \theta, W_y, W_\theta) = W_\theta - f_2(y, \theta) W_y + \frac{1}{\gamma^{\frac{1}{1-\gamma}}}. \tag{5.25}$$

So we can write (5.22) as

$$\begin{aligned}\frac{\sigma^2}{2}W_{yy} &= f_1(y, \theta)W - W_\theta + f_2(y, \theta)W_y - \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \\ &= f_1(y, \theta)W - H(y, \theta, W_y, W_\theta),\end{aligned}\tag{5.26}$$

which is in a much simpler form than before and we will use this form throughout the next sections.

5.3 Perturbed Elliptic Equation and Viscosity Solution

Like in the logarithmic utility case, the HARA utility case also leads to a degenerate elliptic partial differential equation. Thus, we employ a similar approach as in the logarithmic utility case and consider the following perturbed non-degenerate equation with an additional viscosity term $\varepsilon\Delta W^\varepsilon$,

$$0 = -\frac{\sigma^2}{2}W_{yy}^\varepsilon - \varepsilon\Delta W^\varepsilon + f_1(y, \theta)W^\varepsilon - H(y, \theta, W_y^\varepsilon, W_\theta^\varepsilon),\tag{5.27}$$

where $\varepsilon > 0$ and H , f_1 and f_2 are as defined in (5.23)-(5.25). Similar to the logarithmic utility case, we will begin with the existence of a solution to (5.27) in a ball in \mathbb{R}^2 .

5.3.1 Existence and Uniqueness in B_R

For $R > 0$, let $B_R = \{(y, \theta) \in \mathbb{R}^2 : \|(y, \theta)\|_2 \leq R\}$. We consider the following problem

$$\begin{cases} -\frac{\sigma^2}{2}W_{yy}^\varepsilon - \varepsilon\Delta W^\varepsilon + f_1(y, \theta)W^\varepsilon - H(y, \theta, W_y^\varepsilon, W_\theta^\varepsilon) = 0, \\ W^\varepsilon|_{\partial B_R} = K, \end{cases}\tag{5.28}$$

where $K > 0$ is a bounded constant that is determined in the subsequent sections as

$$K = \frac{1 - \gamma}{(\beta - r\gamma)\gamma^{\frac{1}{1-\gamma}}}.$$

Again, the restriction on K is not required to ensure the existence of a solution, but to ensure that the solution to this problem is bounded below by K , which is needed in the verification theorem. Likewise, the problem in (5.28) is a standard non-degenerate elliptic partial differential equation in which there are existing results we can use to show the following existence theorem.

Theorem 5.3.1. *Assume $\mu_1 + \frac{1}{2}\sigma^2 - r > 0$ and β is large enough such that*

$$M_1 = -\frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + M)^2\gamma}{2\sigma^2(1-\gamma)^2} + \frac{\beta - r\gamma}{1-\gamma} > 0. \quad (5.29)$$

Then the system (5.28) has a unique solution $W^\varepsilon(y, \theta) \in C^0(\overline{B_R}) \cap C^{2,\alpha}(B_R)$.

Proof. Again using Theorem 4.3.1 we check that for

$$LW^\varepsilon = \left(\frac{\sigma^2}{2} + \varepsilon\right) W_{yy}^\varepsilon + \varepsilon W_{\theta\theta}^\varepsilon - f_1(y, \theta)W^\varepsilon - f_2(y, \theta)W_y^\varepsilon + W_\theta^\varepsilon \quad (5.30)$$

and

$$f = -\frac{1}{\gamma^{\frac{1}{1-\gamma}}}, \quad (5.31)$$

L is strictly elliptic if we take $\lambda = \varepsilon$ in Definition 4.3.1. Since $\mu_1 + \frac{1}{2}\sigma^2 - r > 0$, we have

$$c(y, \theta) = -f_1(y, \theta) \leq -M_1 < 0. \quad (5.32)$$

Coefficients of L and f are clearly bounded. They also belong to $C^\alpha(B_R)$ since their partial derivatives are bounded and thus Lipschitz and α -Hölder continuous. The exterior sphere condition is satisfied since we are only considering $\Omega = B_R \in \mathbb{R}^2$, where B_R is a ball of radius R . Our boundary condition K is a positive constant, thus continuous. Hence (5.28) also has a unique solution $W^\varepsilon \in C^0(\overline{B_R}) \cap C^{2,\alpha}(B_R)$. \square

To extend the existence result to \mathbb{R}^2 , we again need to find uniform bounds for W^ε and its first order partial derivatives on \mathbb{R}^2 .

5.3.2 L^∞ Estimate of W^ε

We now show W^ε is uniformly bounded with the following lemma. Note that M_1 defined here is the same as in (5.29).

Lemma 5.3.2. *Let H be as defined in (5.25) and consider the problem in (5.27). Additionally define the following:*

$$A = \mu_1 + \frac{1}{2}\sigma^2 - r, \quad (5.33)$$

$$B = \frac{\gamma}{1-\gamma}, \quad (5.34)$$

$$C = \frac{\beta - r\gamma}{1-\gamma}, \quad (5.35)$$

$$D = \frac{1}{\gamma^{\frac{1}{1-\gamma}}}, \quad (5.36)$$

$$M_1 = -\frac{(A+M)^2 B}{2\sigma^2(1-\gamma)} + C. \quad (5.37)$$

Assume $A > 0$ and that β is large enough so that $M_1 > 0$. If

$$H(y, \theta, 0, 0) \in L^\infty(\mathbb{R}^2),$$

then

$$|W^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} \leq \frac{D}{M_1} + K, \quad \forall (y, \theta) \in \mathbb{R}^2.$$

Proof. Similar to the logarithm utility case, we check that

$$H(y, \theta, 0, 0) = \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \in L^\infty(\mathbb{R}^2),$$

which is clearly true. We know that $H(y, \theta, 0, 0) \in L^\infty(\mathbb{R}^2)$ for all $(y, \theta) \in B_R$. Then there exists $(y_0, \theta_0) \in B_R$ such that $W^\varepsilon(y_0, \theta_0) \geq W^\varepsilon(y, \theta)$ for all $(y, \theta) \in B_R$, then

$$\begin{cases} W_y^\varepsilon(y_0, \theta_0) = W_\theta^\varepsilon(y_0, \theta_0) = 0, \\ W_{yy}^\varepsilon(y_0, \theta_0) \leq 0, \\ W_{\theta\theta}^\varepsilon(y_0, \theta_0) \leq 0. \end{cases} \quad (5.38)$$

Note that additionally from our assumptions, since $A = \mu_1 + \frac{1}{2}\sigma^2 - r > 0$ and $|\mu_2(y, \theta)| <$

M we have $f_1(y, \theta) \geq M_1 > 0$ for all $(y, \theta) \in \mathbb{R}^2$.

Then using (5.38), we obtain

$$\begin{aligned} f_1(y_0, \theta_0)W^\varepsilon(y_0, \theta_0) &= \frac{\sigma^2}{2}W_{yy}^\varepsilon(y_0, \theta_0) + \varepsilon\Delta W^\varepsilon(y_0, \theta_0) + H(y, \theta, W_y^\varepsilon(y_0, \theta_0), W_\theta^\varepsilon(y_0, \theta_0)) \\ &\leq |H(y_0, \theta_0, 0, 0)| \\ &= \frac{1}{\gamma^{\frac{1}{1-\gamma}}}, \end{aligned}$$

which implies

$$W^\varepsilon(y, \theta) \leq W^\varepsilon(y_0, \theta_0) \leq \frac{1}{f_1(y_0, \theta_0)\gamma^{\frac{1}{1-\gamma}}} \leq \frac{1}{M_1\gamma^{\frac{1}{1-\gamma}}}. \quad (5.39)$$

Similarly, there exists $(y_1, \theta_1) \in B_R$ such that $W^\varepsilon(y_1, \theta_1) \leq W^\varepsilon(y, \theta)$ for all $(y, \theta) \in B_R$, then

$$\begin{cases} W_y^\varepsilon(y_1, \theta_1) = W_\theta^\varepsilon(y_1, \theta_1) = 0, \\ W_{yy}^\varepsilon(y_1, \theta_1) \geq 0, \\ W_{\theta\theta}^\varepsilon(y_1, \theta_1) \geq 0. \end{cases} \quad (5.40)$$

Then we have

$$\begin{aligned} f_1(y_1, \theta_1)W^\varepsilon(y_1, \theta_1) &= \frac{\sigma^2}{2}W_{yy}^\varepsilon(y_1, \theta_1) + \varepsilon\Delta W^\varepsilon(y_1, \theta_1) + H(y, \theta, W_y^\varepsilon(y_1, \theta_1), W_\theta^\varepsilon(y_1, \theta_1)) \\ &\geq -|H(y_1, \theta_1, 0, 0)| \\ &= -\frac{1}{\gamma^{\frac{1}{1-\gamma}}}, \end{aligned}$$

which implies

$$W^\varepsilon(y, \theta) \geq W^\varepsilon(y_1, \theta_1) \geq -\frac{1}{f_1(y_1, \theta_1)\gamma^{\frac{1}{1-\gamma}}} \geq \frac{1}{M_1\gamma^{\frac{1}{1-\gamma}}}. \quad (5.41)$$

Combining (5.39) and (5.41), then noting that $W^\varepsilon|_{\partial B_R} = K$, we would have

$$|W^\varepsilon(y, \theta)|_{L^\infty(B_R)} \leq \frac{1}{M_1\gamma^{\frac{1}{1-\gamma}}} + K, \quad \forall (y, \theta) \in B_R. \quad (5.42)$$

We see that in (5.42) the upper bound is independent of R . Thus, it is true for all $(y, \theta) \in \mathbb{R}^2$ that

$$|W^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{M_1 \gamma^{\frac{1}{1-\gamma}}} + K = \frac{D}{M_1} + K.$$

□

Next, we need to find uniform upper bounds for the first order partial derivatives of W^ε in order to proceed to proving the existence of solution.

5.3.3 Estimate of W_y^ε and W_θ^ε

The next lemma sets upper bounds for the first partials of W^ε .

Lemma 5.3.3. *Let f_1 , f_2 and H be as defined in (5.23)-(5.25) and consider the problem in (5.27). Assume $A > 0$ and that β is large enough so that $M_1 > 2L > 0$, where L is the Lipschitz constant for $f_2(y, \theta)$ and constants A and M_1 are as defined in Lemma 5.3.2. If*

$$|H(y, \theta, p_1, p_2) - H(\hat{y}, \theta, p_1, p_2)| \leq L|y - \hat{y}|(1 + |p_1| + |p_2|),$$

and

$$|H(y, \theta, p_1, p_2) - H(y, \hat{\theta}, p_1, p_2)| \leq L|\theta - \hat{\theta}|(1 + |p_1| + |p_2|),$$

then

$$|W_\theta^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} + |W_y^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} \leq \frac{2L}{M_1 - 2L}.$$

Proof. Before we start, using the definition of H , we check

$$\begin{aligned} & |H(y, \theta, p_1, p_2) - H(\hat{y}, \theta, p_1, p_2)| \\ &= \left| \left(p_2 - f_2(y, \theta)p_1 + \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \right) - \left(p_2 - f_2(\hat{y}, \theta)p_1 + \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \right) \right| \\ &= |f_2(y, \theta) - f_2(\hat{y}, \theta)||p_1| \\ &= |\mu_2(y, \theta) - \mu_2(\hat{y}, \theta)||p_1| \cdot \frac{1}{1-\gamma} \\ &\leq L|y - \hat{y}||p_1| \\ &\leq L|y - \hat{y}|(1 + |p_1| + |p_2|), \end{aligned} \tag{5.43}$$

where L is the Lipschitz constant for $f_2(y, \theta)$.

Similarly we also have

$$\begin{aligned}
& |H(y, \theta, p_1, p_2) - H(y, \hat{\theta}, p_1, p_2)| \\
&= \left| \left(p_2 - f_2(y, \theta) p_1 + \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \right) - \left(p_2 - f_2(y, \hat{\theta}) p_1 + \frac{1}{\gamma^{\frac{1}{1-\gamma}}} \right) \right| \\
&= |f_2(y, \theta) - f_2(y, \hat{\theta})| |p_1| \\
&= |\mu_2(y, \theta) - \mu_2(y, \hat{\theta})| |p_1| \cdot \frac{1}{1-\gamma} \\
&\leq L|\theta - \hat{\theta}| |p_1| \\
&\leq L|\theta - \hat{\theta}| (1 + |p_1| + |p_2|).
\end{aligned} \tag{5.44}$$

Using a similar construction of $\overline{W}^\varepsilon, \underline{W}^\varepsilon, \Phi, \hat{\Phi}, \Psi, \hat{\Psi}, \zeta, \eta, \overline{H}, \underline{H}$ and regions G, G' with θ and H in place of z and H^* respectively as in estimating first partials in the proof of Lemma 4.3.4, we can also arrive at the following (borrowing the same notations but for the current context):

1. Let \overline{G} is a bounded region such that $\zeta(\theta) = 0$ for $\theta \in \mathbb{R} \setminus \overline{G}$. There exists a $\theta_1 \in \overline{G}$ such that $\Psi(y, \theta_1) \geq \Psi(y, \theta)$ for all $\theta \in \mathbb{R}$, where

$$\Psi(y, \theta) = W^\varepsilon(y, \theta) - \overline{W}^\varepsilon(y, \theta) + 2\delta\zeta(\theta), \tag{5.45}$$

and

$$\begin{cases} \zeta(\theta_0) = 1, \\ 0 \leq \zeta(\theta) < 1, \quad \forall \theta \in \mathbb{R} \setminus \{\theta_0\}, \\ |\zeta_\theta(\theta)| \leq 1, \quad |\zeta_{\theta\theta}(\theta)| \leq 1, \quad \forall \theta \in \mathbb{R}. \end{cases} \tag{5.46}$$

2. Let \overline{G}' be a bounded region such that $\eta(y) = 0$ for $y \in \mathbb{R} \setminus \overline{G}'$. There exists a $y_1 \in \overline{G}'$ such that $\hat{\Psi}(y_1, \theta) \geq \hat{\Psi}(y, \theta)$ for all $y \in \mathbb{R}$, where

$$\hat{\Psi}(y, \theta) = W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta) + 2\delta\eta(y), \tag{5.47}$$

and

$$\begin{cases} \eta(y_0) = 1, \\ 0 \leq \eta(y) < 1, \quad \forall y \in \mathbb{R} \setminus \{y_0\}, \\ |\eta_y(y)| \leq 1, \quad |\eta_{yy}(y)| \leq 1, \quad \forall y \in \mathbb{R}. \end{cases} \tag{5.48}$$

Hence we have

$$\begin{cases} \Psi_y(y, \theta_1) = 0, \\ \Psi_\theta(y, \theta_1) = 0, \\ \Psi_{yy}(y, \theta_1) \leq 0, \\ \Psi_{\theta\theta}(y, \theta_1) \leq 0, \\ \Psi_{yy}(y, \theta_1) + \Psi_{\theta\theta}(y, \theta_1) \leq 0, \end{cases} \quad (5.49)$$

which implies

$$\begin{cases} W_y^\varepsilon(y, \theta_1) - \overline{W}_y^\varepsilon(y, \theta_1) = 0, \\ W_\theta^\varepsilon(y, \theta_1) - \overline{W}_\theta^\varepsilon(y, \theta_1) + 2\delta\zeta_\theta(\theta_1) = 0, \\ W_{yy}^\varepsilon(y, \theta_1) - \overline{W}_{yy}^\varepsilon(y, \theta_1) \leq 0, \\ W_{\theta\theta}^\varepsilon(y, \theta_1) - \overline{W}_{\theta\theta}^\varepsilon(y, \theta_1) + 2\delta\zeta_{\theta\theta}(\theta_1) \leq 0, \\ \Delta W^\varepsilon(y, \theta_1) - \Delta \overline{W}^\varepsilon(y, \theta_1) + 2\delta\zeta_{\theta\theta}(\theta_1) \leq 0, \end{cases} \quad (5.50)$$

and

$$\begin{cases} \hat{\Psi}_y(y_1, \theta) = 0, \\ \hat{\Psi}_\theta(y_1, \theta) = 0, \\ \hat{\Psi}_{yy}(y_1, \theta) \leq 0, \\ \hat{\Psi}_{\theta\theta}(y_1, \theta) \leq 0, \\ \hat{\Psi}_{yy}(y_1, \theta) + \hat{\Psi}_{\theta\theta}(y_1, \theta) \leq 0, \end{cases} \quad (5.51)$$

which implies

$$\begin{cases} W_y^\varepsilon(y_1, \theta) - \underline{W}_y^\varepsilon(y_1, \theta) + 2\delta\eta_y(y_1) = 0, \\ W_\theta^\varepsilon(y_1, \theta) - \underline{W}_\theta^\varepsilon(y_1, \theta) = 0, \\ W_{yy}^\varepsilon(y_1, \theta) - \underline{W}_{yy}^\varepsilon(y_1, \theta) + 2\delta\eta_{yy}(y_1) \leq 0, \\ W_{\theta\theta}^\varepsilon(y_1, \theta) - \underline{W}_{\theta\theta}^\varepsilon(y_1, \theta) \leq 0, \\ \Delta W^\varepsilon(y_1, \theta) - \Delta \underline{W}^\varepsilon(y_1, \theta) + 2\delta\eta_{yy}(y_1) \leq 0. \end{cases} \quad (5.52)$$

Additionally note that with the assumption $A > 0$, we would have

$$0 < M_1 \leq f_1(y, \theta) \leq \frac{\beta - r\gamma}{1 - \gamma} = C,$$

for all $(y, \theta) \in \mathbb{R}^2$, where A and C are as defined in Lemma 5.3.2. Then since $\Psi(y, \theta_1) \geq$

$\Psi(y, \theta)$ for all $\theta \in \mathbb{R}$ and $\zeta(\theta_1) \leq 1$, we would get

$$\begin{aligned}
& f_1(y, \theta_1)(W^\varepsilon(y, \theta) - \overline{W}^\varepsilon(y, \theta)) \\
& \leq f_1(y, \theta_1)(W^\varepsilon(y, \theta) - \overline{W}^\varepsilon(y, \theta) + 2\delta\zeta(\theta)) = f_1(y, \theta_1)\Psi(y, \theta) \\
& \leq f_1(y, \theta_1)(W^\varepsilon(y, \theta_1) - \overline{W}^\varepsilon(y, \theta_1) + 2\delta\zeta(\theta_1)) = f_1(y, \theta_1)\Psi(y, \theta_1) \\
& \leq f_1(y, \theta_1)(W^\varepsilon(y, \theta_1) - \overline{W}^\varepsilon(y, \theta_1)) + 2\delta C.
\end{aligned} \tag{5.53}$$

Substituting (5.27) into the right-hand side of (5.53) and using the third and the last properties in (5.50),

$$\begin{aligned}
& f_1(y, \theta_1)(W^\varepsilon(y, \theta) - \overline{W}^\varepsilon(y, \theta)) \\
& \leq \frac{\sigma^2}{2}W_{yy}^\varepsilon(y, \theta_1) + \varepsilon\Delta W^\varepsilon(y, \theta_1) + H(y, \theta_1, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) \\
& \quad - \frac{\sigma^2}{2}\overline{W}_{yy}^\varepsilon(y, \theta_1) - \varepsilon\Delta\overline{W}^\varepsilon(y, \theta_1) - \overline{H}(y, \theta_1, \overline{W}_y^\varepsilon(y, \theta_1), \overline{W}_\theta^\varepsilon(y, \theta_1)) + 2\delta C \\
& = \frac{\sigma^2}{2}(W_{yy}^\varepsilon(y, \theta_1) - \overline{W}_{yy}^\varepsilon(y, \theta_1)) + \varepsilon(\Delta W^\varepsilon(y, \theta_1) - \Delta\overline{W}^\varepsilon(y, \theta_1)) \\
& \quad + H(y, \theta_1, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) - \overline{H}(y, \theta_1, \overline{W}_y^\varepsilon(y, \theta_1), \overline{W}_\theta^\varepsilon(y, \theta_1)) + 2\delta C \\
& \leq -2\varepsilon\delta\zeta_{\theta\theta}(\theta_1) + H(y, \theta_1, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) \\
& \quad - \overline{H}(y, \theta_1, \overline{W}_y^\varepsilon(y, \theta_1), \overline{W}_\theta^\varepsilon(y, \theta_1)) + 2\delta C.
\end{aligned} \tag{5.54}$$

Divide both sides of (5.54) by $f_1(y, \theta_1)$ and take the norm, we have

$$\begin{aligned}
& |W^\varepsilon(y, \theta) - \overline{W}^\varepsilon(y, \theta)| \\
& \leq \frac{1}{f_1(y, \theta_1)}|-2\varepsilon\delta\zeta_{\theta\theta}(\theta_1) + H(y, \theta_1, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) \\
& \quad - \overline{H}(y, \theta_1, \overline{W}_y^\varepsilon(y, \theta_1), \overline{W}_\theta^\varepsilon(y, \theta_1))| + \frac{2\delta C}{f_1(y, \theta_1)} \\
& \leq \frac{2\varepsilon\delta}{M_1} + \frac{1}{M_1}|H(y, \theta_1, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) - \overline{H}(y, \theta_1, \overline{W}_y^\varepsilon(y, \theta_1), \overline{W}_\theta^\varepsilon(y, \theta_1))| + \frac{2\delta C}{M_1}.
\end{aligned}$$

Adding and subtracting $H(y, \theta_1 + \xi, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1))$ on the right-hand side and using

the first two properties in (5.50), we get

$$\begin{aligned}
& |W^\varepsilon(y, \theta) - \overline{W}^\varepsilon(y, \theta)| \\
\leq & \frac{2\delta(\varepsilon + C)}{M_1} + \frac{1}{M_1} |H(y, \theta_1, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) \\
& - H(y, \theta_1 + \xi, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) + H(y, \theta_1 + \xi, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) \\
& - H(y, \theta_1 + \xi, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1) + 2\delta\zeta_\theta(\theta_1))| \\
\leq & \frac{2\delta(\varepsilon + C)}{M_1} + \frac{1}{M_1} \cdot L(1 + |W_y^\varepsilon(y, \theta_1)| + |W_\theta^\varepsilon(y, \theta_1)|)|\xi| \\
& + \frac{1}{M_1} |H(y, \theta_1 + \xi, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1)) \\
& - H(y, \theta_1 + \xi, W_y^\varepsilon(y, \theta_1), W_\theta^\varepsilon(y, \theta_1) + 2\delta\zeta_\theta(\theta_1))|.
\end{aligned}$$

Now let $\delta \rightarrow 0$ and we have

$$|W^\varepsilon(y, \theta) - \overline{W}^\varepsilon(y, \theta)| \leq \frac{L}{M_1} (1 + |W_y^\varepsilon(y, \theta_1)| + |W_\theta^\varepsilon(y, \theta_1)|)|\xi|. \quad (5.55)$$

For the second part, since $\hat{\Psi}(y_1, \theta) \geq \hat{\Psi}(y, \theta)$ for all $y \in \mathbb{R}$ and $\eta(y_1) \leq 1$ we have

$$\begin{aligned}
& f_1(y_1, \theta)(W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta)) \\
\leq & f_1(y_1, \theta)(W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta) + 2\delta\eta(y)) = f_1(y_1, \theta)\hat{\Psi}(y, \theta) \\
= & f_1(y_1, \theta)(W^\varepsilon(y_1, \theta) - \underline{W}^\varepsilon(y_1, \theta) + 2\delta\eta(y_1)) = f_1(y_1, \theta)\hat{\Psi}(y_1, \theta) \\
\leq & f_1(y_1, \theta)(W^\varepsilon(y_1, \theta) - \underline{W}^\varepsilon(y_1, \theta)) + 2\delta C.
\end{aligned}$$

Similarly substituting (5.27) into the right-hand side, we get

$$\begin{aligned}
& f_1(y_1, \theta)(W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta)) \\
\leq & \frac{\sigma^2}{2} W_{yy}^\varepsilon(y_1, \theta) + \varepsilon \Delta W^\varepsilon(y_1, \theta) + H(y_1, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) \\
& - \frac{\sigma^2}{2} \underline{W}_{yy}^\varepsilon(y_1, \theta) - \varepsilon \Delta \underline{W}^\varepsilon(y_1, \theta) - \underline{H}(y_1, \theta, \underline{W}_y^\varepsilon(y_1, \theta), \underline{W}_\theta^\varepsilon(y_1, \theta)) + 2\delta C \\
= & \frac{\sigma^2}{2} (W_{yy}^\varepsilon(y_1, \theta) - \underline{W}_{yy}^\varepsilon(y_1, \theta)) + \varepsilon (\Delta W^\varepsilon(y_1, \theta) - \Delta \underline{W}^\varepsilon(y_1, \theta)) \\
& + H(y_1, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) - \underline{H}(y_1, \theta, \underline{W}_y^\varepsilon(y_1, \theta), \underline{W}_\theta^\varepsilon(y_1, \theta)) + 2\delta C.
\end{aligned}$$

So it follows from using the third and the last properties in (5.52) that

$$\begin{aligned}
& f_1(y_1, \theta)(W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta)) \\
& \leq -2\delta\eta_{yy}(y_1) \left(\frac{\sigma^2}{2} \right) - 2\varepsilon\delta\eta_{yy}(y_1) + H(y_1, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) \\
& \quad - \underline{H}(y_1, \theta, \underline{W}_y^\varepsilon(y_1, \theta), \underline{W}_\theta^\varepsilon(y_1, \theta)) + 2\delta C \\
& = -2\delta\eta_{yy}(y_1) \left(\frac{\sigma^2}{2} + \varepsilon \right) + H(y_1, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) \\
& \quad - \underline{H}(y_1, \theta, \underline{W}_y^\varepsilon(y_1, \theta), \underline{W}_\theta^\varepsilon(y_1, \theta)) + 2\delta C,
\end{aligned}$$

which leads to

$$\begin{aligned}
& |W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta)| \\
& \leq \frac{1}{f_1(y_1, \theta)} \left| -2\delta\eta_{yy}(y_1) \left(\frac{\sigma^2}{2} + \varepsilon \right) \right| + \frac{1}{f_1(y_1, \theta)} |H(y_1, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) \\
& \quad - \underline{H}(y_1, \theta, \underline{W}_y^\varepsilon(y_1, \theta), \underline{W}_\theta^\varepsilon(y_1, \theta))| + \frac{2\delta C}{f_1(y_1, \theta)}.
\end{aligned}$$

Again adding and subtracting $H(y_1 + \xi, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta))$ on the right-hand side and using the first two properties in (5.52), we get

$$\begin{aligned}
& |W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta)| \\
& \leq \frac{2\delta}{M_1} \left(\frac{\sigma^2}{2} + \varepsilon \right) + \frac{2\delta C}{M_1} + \frac{1}{M_1} |H(y_1, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) \\
& \quad - H(y_1 + \xi, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) + H(y_1 + \xi, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) \\
& \quad - H(y_1 + \xi, \theta, W_y^\varepsilon(y_1, \theta) + 2\delta\eta_y(y_1), W_\theta^\varepsilon(y_1, \theta))| \\
& \leq \frac{2\delta}{M_1} \left(\frac{\sigma^2}{2} + \varepsilon + C \right) + \frac{1}{M_1} \cdot L(1 + |W_y^\varepsilon(y_1, \theta)| + |W_\theta^\varepsilon(y_1, \theta)|)|\xi| \\
& \quad + \frac{1}{M_1} |H(y_1 + \xi, \theta, W_y^\varepsilon(y_1, \theta), W_\theta^\varepsilon(y_1, \theta)) \\
& \quad - H(y_1 + \xi, \theta, W_y^\varepsilon(y_1, \theta) + 2\delta\eta_y(y_1), W_\theta^\varepsilon(y_1, \theta))|.
\end{aligned}$$

Let $\delta \rightarrow 0$ and we have

$$|W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta)| \leq \frac{L}{M_1} (1 + |W_y^\varepsilon(y_1, \theta)| + |W_\theta^\varepsilon(y_1, \theta)|)|\xi|. \quad (5.56)$$

Adding (5.55) and (5.56), we get

$$\begin{aligned}
& |W^\varepsilon(y, \theta) - \overline{W}^\varepsilon(y, \theta)| + |W^\varepsilon(y, \theta) - \underline{W}^\varepsilon(y, \theta)| \\
& \leq \frac{L}{M_1}(1 + |W_y^\varepsilon(y, \theta_1)| + |W_\theta^\varepsilon(y, \theta_1)|)|\xi| + \frac{L}{M_1}(1 + |W_y^\varepsilon(y_1, \theta)| + |W_\theta^\varepsilon(y_1, \theta)|)|\xi| \\
& \leq \frac{2L}{M_1}(1 + |W_y^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} + |W_\theta^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)})|\xi|.
\end{aligned}$$

This implies

$$|W_\theta^\varepsilon(y, \theta)| + |W_y^\varepsilon(y, \theta)| \leq \frac{2L}{M_1}(1 + |W_y^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} + |W_\theta^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)}),$$

leading to

$$|W_\theta^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} + |W_y^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} \leq \frac{2L}{M_1}(1 + |W_y^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} + |W_\theta^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)}).$$

Then

$$\begin{aligned}
\frac{M_1 - 2L}{M_1} \cdot (|W_\theta^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} + |W_y^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)}) & \leq \frac{2L}{M_1} \\
|W_\theta^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} + |W_y^\varepsilon(y, \theta)|_{L^\infty(\mathbb{R}^2)} & \leq \frac{2L}{M_1 - 2L}.
\end{aligned}$$

provided that $M_1 > 2L$. □

We are now ready to show the existence of a unique solution in \mathbb{R}^2 .

5.3.4 Existence and Uniqueness in of Viscosity Solution in \mathbb{R}^2

Like in the logarithmic utility case, we first present the formal definition of a viscosity solution to our problem.

Definition 5.3.1. *W is a **viscosity subsolution (supersolution)** to the problem (5.26) if for every $(y_0, \theta_0) \in \mathbb{R}^2$ and every $\varphi \in C^2(\mathbb{R}^2)$ such that*

$$\varphi \geq (\leq) W, \quad \varphi(y_0, \theta_0) = W(y_0, \theta_0)$$

in a neighborhood of (y_0, θ_0) , we have

$$\frac{\sigma^2}{2} \varphi_{yy} \geq (\leq) f_1(y, \theta) \varphi - H(y, \theta, \varphi_y, \varphi_\theta).$$

In addition, when W is both a viscosity subsolution and supersolution, W is called a **viscosity solution** to (5.26).

Theorem 5.3.4. *Let A and M_1 be as defined in (5.33) and (5.37) and let L be the Lipschitz constant for f_2 as defined in (5.23). Assume $A > 0$ and that β is large enough so that $M_1 > 2L > 0$. Then equation (5.26) has a unique viscosity solution (weak solution) $W(y, \theta)$ in the Sobolev space $W^{1,\infty}(\mathbb{R}^2)$.*

Proof. Now that we have bounded partial derivatives with a bound that is independent of ε , by a similar argument as in the logarithm utility case, the result holds. \square

We have now established the existence of a unique viscosity solution. In the next section, we will find the subsolution and supersolution bounds for the solution in the classical sense just as we did in the logarithmic utility case, which will be essential in the verification theorem.

5.4 Subsolution and Supersolution

We first give formal definitions of a subsolution and a supersolution for our problem on the ball B_R and on \mathbb{R}^2 .

Definition 5.4.1. $W^\varepsilon(y, \theta)$ is a **subsolution (supersolution)** to the problem (5.27) in \mathbb{R}^2 if

$$\frac{\sigma^2}{2} W_{yy}^\varepsilon + \varepsilon \Delta W^\varepsilon \geq (\leq) f_1(y, \theta) W^\varepsilon - H(y, \theta, W_y^\varepsilon, W_\theta^\varepsilon).$$

In addition, if \hat{W} is a subsolution and \bar{W} is a supersolution, and $\hat{W} \leq \bar{W}$, then $\langle \hat{W}, \bar{W} \rangle$ is an **ordered pair of sub/supersolutions**.

Definition 5.4.2. $W^\varepsilon(y, \theta)$ is a **subsolution (supersolution)** to (5.28) on B_R if

$$\begin{cases} \frac{\sigma^2}{2} W_{yy}^\varepsilon + \varepsilon \Delta W^\varepsilon \geq (\leq) f_1(y, \theta) W^\varepsilon - H(y, \theta, W_y^\varepsilon, W_\theta^\varepsilon), & \text{on } B_R, \\ W^\varepsilon \leq (\geq) K, & \text{on } \partial B_R. \end{cases}$$

In addition, if \hat{W} is a subsolution and \bar{W} is a supersolution, and $\hat{W} \leq \bar{W}$ for all $(y, \theta) \in \bar{B}_R$, then $\langle \hat{W}, \bar{W} \rangle$ is an **ordered pair of sub/supersolutions**.

The following lemmas show explicit forms of the sub/supersolution pair in which we look for.

Lemma 5.4.1. *Let A, B, C, D and M_1 be defined as in Lemma 5.3.2 and also define*

$$K_1 \equiv \frac{1 - \gamma}{(\beta - r\gamma)\gamma^{\frac{1}{1-\gamma}}} = \frac{D}{C}. \quad (5.57)$$

Assume $A > 0$ and β is large enough so that $M_1 > 0$. Then any constant $K \leq K_1$ is a subsolution of (5.27).

Proof. Let $W = K \leq K_1$. Note that A, B, C, D, M_1 from (5.33)-(5.37) are all positive. We have $W_\theta = W_y = W_{yy} = W_{\theta\theta} = 0$ and $H(y, \theta, 0, 0) = D$. Also, with $A > 0$, we have

$$f_1(y, \theta) = -\frac{(A + \mu_2(y, \theta))^2 B}{2\sigma^2(1 - \gamma)} + C \geq M_1 > 0.$$

Then,

$$\begin{aligned} & f_1(y, \theta)K - H(y, \theta, 0, 0) \\ & \leq f_1(y, \theta) \cdot \frac{D}{C} - D \\ & = \left(-\frac{(A + \mu_2(y, \theta))^2 B}{2\sigma^2(1 - \gamma)} + C \right) \cdot \frac{D}{C} - D \\ & = \left(-\frac{(A + \mu_2(y, \theta))^2 BD}{2\sigma^2(1 - \gamma)C} + D \right) - D \\ & = -\frac{(A + \mu_2(y, \theta))^2 BD}{2\sigma^2(1 - \gamma)C} \\ & < 0. \end{aligned}$$

□

Lemma 5.4.2. *Let A, B, C, D and M_1 be defined as in Lemma 5.3.2 and define*

$$K_2 \equiv \frac{D}{M_1}, \quad (5.58)$$

Assume $A > 0$ and β is large enough so that $M_1 > 0$. Then any constant $K \geq K_2$ is a supersolution of (5.27).

Proof. Let $W = K \geq K_2$. Then we have $W_\theta = W_{\theta\theta} = W_y = W_{yy} = 0$. Since $A > 0$ implies that $f_1(y, \theta) \geq M_1 > 0$,

$$\begin{aligned}
& f_1(y, \theta)K - H(y, \theta, 0, 0) \\
& \geq f_1(y, \theta) \cdot \frac{D}{M_1} - D \\
& = \left(-\frac{(A + \mu_2(y, \theta))^2 B}{2\sigma^2(1 - \gamma)} + C \right) \cdot \frac{D}{M_1} - D \\
& \geq \left(-\frac{(A + M)^2 B}{2\sigma^2(1 - \gamma)} + C \right) \cdot \frac{D}{M_1} - D \\
& = M_1 \cdot \frac{D}{M_1} - D \\
& = 0.
\end{aligned}$$

□

Lemma 5.4.3. *Let A and M_1 be defined as in Lemma 5.3.2. Assume $A > 0$ and β is large enough so that $M_1 > 0$. Then $\langle K_1, K_2 \rangle$ is an ordered pair of sub/supersolutions to (5.27).*

Proof. Since $A > 0$ and $0 < \gamma < 1$, we must have $M_1 < C$. Then it is clear that $K_2 > K_1$. Therefore, $\langle K_1, K_2 \rangle$ is a pair of subsolution and supersolution (5.27). □

We again note here that if we take $K = K_1$ on ∂B_R in (5.28), then any constant less than or equal to K_1 is a subsolution of (5.28) and any constant greater than or equal to K_2 is a supersolution of (5.28). This confirms that the solution to (5.28) is sandwiched between the sub/supersolution pair found. Since the solution to (5.28) converges uniformly to the solution of (5.26) and we can easily verify that $\langle K_1, K_2 \rangle$ is also an ordered pair of sub/supersolutions to (5.26), this ensures the solution to (5.26) is also sandwiched between the sub/supersolution pair. Knowing this, we can now verify our solution is indeed the value function.

5.5 Verification Theorem

To begin the verifying process, we let $\tilde{W}(y, z)$ be the classical solution to (5.26). Then

$$\tilde{V}(x, y, z) \equiv x^\gamma \tilde{W}(y, z)^{1-\gamma} \quad (5.59)$$

would be the corresponding classical solution to (5.8). We want to show that $\tilde{V}(x, y, z) = V(x, y, z)$, our value function. We clarify here that in the inverse transformations, we have

$$\theta(y, z) = \frac{\ln(1 - \frac{\lambda z}{y})}{-\lambda}. \quad (5.60)$$

So $\tilde{W}(y, \theta) \equiv \tilde{W}(y, z)$. Like in the logarithm utility case, the original derivation of the HJB equation in (3.17) requires

$$\mathbf{E} \left[\int_0^T \sigma e^{-\beta t} (\tilde{V}_y + x \pi(t) \tilde{V}_x) dB(t) \right] = 0.$$

We will show this also true for the HARA utility case with help of the following lemma.

Lemma 5.5.1. *Let $(\pi(t), c(t)) \in \Pi$. Then $\mathbf{E}[X(t)^m] < \infty$ for all fixed $t \geq 0$, $m > 0$ and $x > 0$.*

Proof. From (3.9), since $X(t) = x e^{\int_0^t \left(r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(Z(\tau)) \right) \pi(\tau) - c(\tau) - \frac{\sigma^2 \pi(\tau)^2}{2} \right) d\tau + \int_0^t \sigma \pi(\tau) dB(\tau)}$ and $\pi(t)$, $c(t)$ and $\mu_2(Z(t))$ are all bounded by constants,

$$\mathbf{E} \left[e^{m \int_0^t \left(r + \left(\mu_1 + \frac{1}{2} \sigma^2 - r + \mu_2(Z(\tau)) \right) \pi(\tau) - c(\tau) - \frac{\sigma^2 \pi(\tau)^2}{2} \right) d\tau} \right] \leq e^{m \Lambda_1 t},$$

for some constant Λ_1 . In addition, by Lemma 2.1 (ii) in [64],

$$\mathbf{E} \left[e^{\int_0^t \sigma \pi(\tau) dB(\tau)} \right] < \infty.$$

Hence, fixing x , t and m , the result holds. □

Lemma 5.5.2. *Assume the same set of assumptions as in Theorem 5.3.4.*

If $\tilde{V}(X(t), Y(t), Z(t)) \equiv X(t)^\gamma \tilde{W}(Y(t), Z(t))^{1-\gamma}$, where \tilde{W} is the classical solution to

(5.26), then

$$\mathbf{E} \left[\int_0^T \sigma e^{-\beta t} (\tilde{V}_y + X(t)\pi(t)\tilde{V}_x) dB(t) \right] = 0.$$

Proof. For the result to be hold, we must show that

$$\mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} (\tilde{V}_y + X(t)\pi(t)\tilde{V}_x)^2 dt \right] < \infty.$$

Note here by the definition of \tilde{V} , we have

$$\tilde{V}_x = \gamma X(t)^{\gamma-1} \tilde{W}^{1-\gamma} \quad (5.61)$$

$$\tilde{V}_y = (1-\gamma)X(t)^\gamma \tilde{W}_y \tilde{W}^{-\gamma}. \quad (5.62)$$

Additionally, we know from the proof of Lemma 5.3.2 and Lemma 5.3.3 that

$$|W^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{D}{M_1} + K, \quad |W_y^\varepsilon(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{2L}{M_1 - 2L}.$$

Again the upper bounds are independent of ε . If we take $\varepsilon \rightarrow 0$, we would still have

$$|W(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{D}{M_1} + K, \quad |W_y(y, z)|_{L^\infty(\mathbb{R}^2)} \leq \frac{2L}{M_1 - 2L}.$$

Thus, our classical solution \tilde{V} satisfies

$$\begin{aligned} |\tilde{V}_x|_{L^\infty(\mathbb{R}^2)} &= |\gamma X(t)^{\gamma-1} \tilde{W}^{1-\gamma}|_{L^\infty(\mathbb{R}^2)} \\ &\leq \gamma X(t)^{\gamma-1} \left(\frac{D}{M_1} + K \right)^{1-\gamma}, \end{aligned} \quad (5.63)$$

$$\begin{aligned} |\tilde{V}_y|_{L^\infty(\mathbb{R}^2)} &= |(1-\gamma)X(t)^\gamma \tilde{W}_y \tilde{W}^{-\gamma}|_{L^\infty(\mathbb{R}^2)} \\ &\leq \frac{2L(1-\gamma)X(t)^\gamma \left(\frac{D}{M_1} + K \right)^{-\gamma}}{M_1 - 2L}. \end{aligned} \quad (5.64)$$

Since $-\Gamma \leq \pi(t) \leq \Gamma$ and $0 < \gamma < 1$, for any fixed T , using (4.75) and Lemma 5.5.1, we

have

$$\begin{aligned}
& \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} (\tilde{V}_y + X(t)\pi(t)\tilde{V}_x)^2 dt \right] \\
& \leq \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} (2 \cdot \tilde{V}_y^2 + 2 \cdot X(t)^2 \pi(t)^2 \tilde{V}_x^2) dt \right] \\
& = \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} (2 \cdot |\tilde{V}_y|^2 + 2 \cdot X(t)^2 \pi(t)^2 |\tilde{V}_x|^2) dt \right] \\
& \leq \mathbf{E} \left[\int_0^T \sigma^2 e^{-2\beta t} \left(\frac{2^3 \cdot L^2 (1-\gamma)^2 X(t)^{2\gamma} \left(\frac{D}{M_1} + K \right)^{-2\gamma}}{(M_1 - 2L)^2} \right. \right. \\
& \quad \left. \left. + 2 \cdot \Gamma^2 \gamma^2 X(t)^{2\gamma} \left(\frac{D}{M_1} + K \right)^{2-2\gamma} \right) dt \right] \\
& = \sigma^2 \left(\frac{2^3 \cdot L^2 (1-\gamma)^2 \left(\frac{D}{M_1} + K \right)^{-2\gamma}}{(M_1 - 2L)^2} \right. \\
& \quad \left. + 2 \cdot \Gamma^2 \gamma^2 \left(\frac{D}{M_1} + K \right)^{2-2\gamma} \right) \int_0^T e^{-2\beta t} \mathbf{E}[X(t)^{2\gamma}] dt \\
& < \infty.
\end{aligned}$$

Hence the result holds. \square

Now we prove the main result of this section.

Theorem 5.5.3 (Verification Theorem). *Assume the same set of assumptions as in Theorem 5.3.4. Let $\tilde{W}(y, z)$ be the classical solution to (5.26) and $\tilde{V}(x, y, z) \equiv x^\gamma \tilde{W}(y, z)^{1-\gamma}$. Then $\tilde{V}(x, y, z) = V(x, y, z)$, where $V(x, y, z)$ is the value function in (5.3). Moreover, the optimal control policy is given by*

$$\pi^*(x, y, z) = \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))\tilde{V}_x - \sigma^2 \tilde{V}_{xy}}{\sigma^2 x \tilde{V}_{xx}}, \quad c^*(x, y, z) = \frac{\tilde{V}_x^{\frac{1}{\gamma-1}}}{x}.$$

Proof. From (5.4) we know that for any $(\pi(t), c(t)) \in \Pi$ we have

$$\begin{aligned}
-\frac{1}{\gamma}(c(t)X(t))^\gamma &\geq -\beta\tilde{V} + X(t)\pi(t) \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(t)) \right) \tilde{V}_x + \frac{\sigma^2\pi(t)^2}{2}X(t)^2\tilde{V}_{xx} \\
&\quad + \sigma^2\pi(t)X(t)\tilde{V}_{xy} + \frac{\sigma^2}{2}\tilde{V}_{yy} - c(t)X(t)\tilde{V}_x + rX(t)\tilde{V}_x \\
&\quad + (\mu_1 + \mu_2(Z(t)))\tilde{V}_y + (Y(t) - \lambda Z(t))\tilde{V}_z.
\end{aligned} \tag{5.65}$$

By a similar calculation as in (3.15) and using (5.65), we also have the relation

$$\begin{aligned}
&d(e^{-\beta t}\tilde{V}) \\
&= e^{-\beta t} \left(-\beta\tilde{V} + \left[X(t) \left(r + \left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(t)) \right) \pi(t) - c(t) \right) \right] \tilde{V}_x \right. \\
&\quad + (\mu_1 + \mu_2(Z(t)))\tilde{V}_y + (Y(t) - \lambda Z(t))\tilde{V}_z + \frac{\sigma^2}{2}\tilde{V}_{yy} + \sigma^2\pi(t)X(t)\tilde{V}_{xy} \\
&\quad \left. + \frac{\sigma^2\pi(t)^2}{2}X(t)^2\tilde{V}_{xx} \right) dt + e^{-\beta t}(\tilde{V}_y\sigma + \tilde{V}_x\sigma\pi(t)X(t))dB(t) \\
&\leq e^{-\beta t} \left(-\frac{1}{\gamma}(c(t)X(t))^\gamma \right) dt + e^{-\beta t}(\tilde{V}_y\sigma + \tilde{V}_x\sigma\pi(t)X(t))dB(t).
\end{aligned}$$

Integrating both sides and taking expectation then using Lemma 5.5.2 we have

$$\begin{aligned}
&\mathbf{E}[e^{-\beta T}\tilde{V}(X(T), Y(T), Z(T)) - \tilde{V}(x, y, z)] \\
&\leq \mathbf{E} \left[\int_0^T e^{-\beta t} \left(-\frac{1}{\gamma}(c(t)X(t))^\gamma \right) dt + \int_0^T e^{-\beta t}(\tilde{V}_y\sigma + \tilde{V}_x\sigma\pi(t)X(t))dB(t) \right],
\end{aligned}$$

which implies

$$\begin{aligned}
\tilde{V}(x, y, z) &\geq \mathbf{E} \left[\int_0^T e^{-\beta t} \left(\frac{1}{\gamma}(c(t)X(t))^\gamma \right) dt \right] \\
&\quad + \mathbf{E}[e^{-\beta T}\tilde{V}(X(T), Y(T), Z(T))].
\end{aligned} \tag{5.66}$$

Note that by definition we have

$$\mathbf{E}[e^{-\beta T}\tilde{V}(X(T), Y(T), Z(T))] = \mathbf{E} \left[e^{-\beta T} \left(X(T)^\gamma \tilde{W}(Y(T), Z(T))^{1-\gamma} \right) \right].$$

From (3.9), we know $X(T) > 0$ since $x > 0$. Additionally, $\tilde{W}(y, z)$ is bounded below by

its subsolution, which is a positive constant. It follows that

$$\limsup_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \geq 0. \quad (5.67)$$

With $c(t)$ also being nonnegative, by monotone convergence theorem

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c(t)X(t))^\gamma \right) dt \right] \\ & \geq \lim_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c(t)X(t))^\gamma \right) dt \right] \\ & = \mathbf{E} \left[\lim_{T \rightarrow \infty} \int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c(t)X(t))^\gamma \right) dt \right] \\ & = \mathbf{E} \left[\int_0^\infty e^{-\beta t} \left(\frac{1}{\gamma} (c(t)X(t))^\gamma \right) dt \right]. \end{aligned} \quad (5.68)$$

Hence taking limit superior on both sides of (5.66) and using (5.67)-(5.68) gives us

$$\begin{aligned} & \tilde{V}(x, y, z) \\ & \geq \limsup_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c(t)X(t))^\gamma \right) dt \right] + \limsup_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \\ & \geq \mathbf{E} \left[\int_0^\infty e^{-\beta t} \left(\frac{1}{\gamma} (c(t)X(t))^\gamma \right) dt \right]. \end{aligned} \quad (5.69)$$

Inequality (5.69) holds for all admissible controls $c(t)$ and $\pi(t)$. Therefore

$$\tilde{V}(x, y, z) \geq \sup_{(\pi(t), c(t)) \in \Pi} \mathbf{E} \left[\int_0^\infty e^{-\beta t} \frac{1}{\gamma} (c(t)X(t))^\gamma dt \right] = V(x, y, z).$$

For the reverse inequality $V(x, y, z) \leq \tilde{V}(x, y, z)$, we first show that $(\pi^*, c^*) \in \Pi$. Both are progressively measurable since their values are determined for all t . Note by definition

$$\begin{cases} \tilde{V}_x(x, y, z) = \gamma x^{\gamma-1} \tilde{W}^{1-\gamma}, \\ \tilde{V}_{xx}(x, y, z) = \gamma(\gamma-1) x^{\gamma-2} \tilde{W}^{1-\gamma}, \\ \tilde{V}_{xy}(x, y, z) = \gamma(1-\gamma) x^{\gamma-1} \tilde{W}_y \tilde{W}^{-\gamma}. \end{cases} \quad (5.70)$$

Again by Lemma 5.3.3 and since \tilde{W} is bounded above and below by its subsolution and

supersolution given in Lemma 5.4.1 and Lemma 5.4.2,

$$0 < \frac{D}{C} \leq \tilde{W} \leq \frac{D}{M_1},$$

$$|\tilde{W}_y| \leq \frac{2L}{M_1 - 2L},$$

we have

$$\begin{aligned}
|\pi^*| &= \left| \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))\tilde{V}_x - \sigma^2\tilde{V}_{xy}}{\sigma^2 x \tilde{V}_{xx}} \right| \\
&= \left| \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))\gamma x^{\gamma-1}\tilde{W}^{1-\gamma} - \sigma^2\gamma(1-\gamma)x^{\gamma-1}\tilde{W}^{-\gamma}\tilde{W}_y}{\sigma^2\gamma(\gamma-1)x^{\gamma-1}\tilde{W}^{1-\gamma}} \right| \\
&= \left| \frac{-(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))\tilde{W} - \sigma^2(1-\gamma)\tilde{W}_y}{\sigma^2(\gamma-1)\tilde{W}} \right| \\
&\leq \frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + M)(\frac{D}{M_1}) + \sigma^2(1-\gamma) \cdot \frac{2L}{M_1-2L}}{\sigma^2(1-\gamma)(\frac{D}{C})} \\
&= \Gamma_1.
\end{aligned} \tag{5.71}$$

We pick Γ such that $\Gamma > \Gamma_1$. Thus $|\pi^*| \leq \Gamma$. Similarly,

$$\begin{aligned}
c^* &= \frac{\tilde{V}_x^{\frac{1}{\gamma-1}}}{x} \\
&= \frac{(\gamma x^{\gamma-1}\tilde{W}^{1-\gamma})^{\frac{1}{\gamma-1}}}{x} \\
&= (\gamma\tilde{W}^{1-\gamma})^{\frac{1}{\gamma-1}} \\
&\leq \left(\gamma \left(\frac{D}{M_1} \right)^{1-\gamma} \right)^{\frac{1}{\gamma-1}} \\
&= (\gamma^{\frac{1}{1-\gamma}})^{-1} \left(\frac{D}{M_1} \right)^{-1} \\
&= D \cdot \frac{M_1}{D} \\
&= M_1.
\end{aligned} \tag{5.72}$$

We pick Λ such that $\Lambda > M_1$. In addition, we see that

$$c^* = (\gamma \tilde{W}^{1-\gamma})^{\frac{1}{\gamma-1}} \geq \left(\gamma \left(\frac{D}{C} \right)^{1-\gamma} \right)^{\frac{1}{\gamma-1}} = C > 0.$$

Thus $0 \leq c^* \leq \Lambda$. Now using (π^*, c^*) , we would have the following equality

$$\tilde{V}(x, y, z) = \mathbf{E} \left[\int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c^* X(t))^\gamma \right) dt \right] + \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))]. \quad (5.73)$$

By a similar reasoning as for checking (5.67), the fact that

$$\mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] = \mathbf{E} \left[e^{-\beta T} \left(X(T)^\gamma \tilde{W}(Y(T), Z(T))^{1-\gamma} \right) \right],$$

where $\tilde{W}(Y(T), Z(T))$ is bounded above by its supersolution and $\mathbf{E}[X(T)^\gamma] < \infty$ also implies

$$\liminf_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \leq 0. \quad (5.74)$$

Having both $X(t) > 0$ and $c^* > 0$, we can apply the monotone convergence theorem to get

$$\begin{aligned} & \liminf_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c^* X(t))^\gamma \right) dt \right] \\ & \leq \lim_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c^* X(t))^\gamma \right) dt \right] \\ & = \mathbf{E} \left[\lim_{T \rightarrow \infty} \int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c^* X(t))^\gamma \right) dt \right] \\ & = \mathbf{E} \left[\int_0^\infty e^{-\beta t} \left(\frac{1}{\gamma} (c^* X(t))^\gamma \right) dt \right]. \end{aligned} \quad (5.75)$$

Hence taking limit inferior on both sides of (5.73) and using (5.74)-(5.75) along with the

definition of our value function gives us

$$\begin{aligned}
& \tilde{V}(x, y, z) \\
&= \liminf_{T \rightarrow \infty} \mathbf{E} \left[\int_0^T e^{-\beta t} \left(\frac{1}{\gamma} (c^* X(t))^\gamma \right) dt \right] + \liminf_{T \rightarrow \infty} \mathbf{E}[e^{-\beta T} \tilde{V}(X(T), Y(T), Z(T))] \\
&\leq \mathbf{E} \left[\int_0^\infty e^{-\beta t} \left(\frac{1}{\gamma} (c^* X(t))^\gamma \right) dt \right] \\
&= V(x, y, z).
\end{aligned}$$

Hence, we showed that $\tilde{V}(x, y, z) = V(x, y, z)$. □

Chapter 6

Conclusion and Future Work

In this dissertation, we have shown the existence and uniqueness of viscosity and classical solutions to the problem presented for the case of logarithmic utility function of consumption and HARA utility function of consumption with $0 < \gamma < 1$. We have not considered the case of HARA utility function with $-\infty < \gamma < 0$ and the exponential utility function. In addition, variables such as interest rate and volatility are assumed to be fixed values, which is not realistic in practice. Future works can include extending the current model by considering stochastic interest rates and volatility, possibly involving delay in those variables, as well as adding stochastic dividends to the risky asset. Numerical results for the solution can also be developed for the cases studied in this dissertation.

REFERENCES

- [1] Bauer, H. and Rieder, U. (2005) *Stochastic Control Problems with Delay*, Mathematical Methods of Operations Research, 62(3):411-427.
- [2] Bellman, R. (1957) *Dynamic Programming*, Princeton University Press.
- [3] Bielecki, T. and Pliska, S. (1999) *Risk-Sensitive Dynamic Asset Management*, Applied Mathematics and Optimization, 39(3):337-360.
- [4] Chang, M.H., Pang, T., and Pemy, M. (2006) *Stochastic Optimal Control Problems with a Bounded Memory*, Operations Research and Its Applications, Lecture Notes in Operations Research, 6:82-94.
- [5] Chang, M.H., Pang, T., and Pemy, M. (2008) *Optimal Control of Stochastic Functional Differential Equations with a Bounded Memory*, Stochastics: An International Journal of Probability and Stochastic Processes, 80(1):69-96.
- [6] Chang, M.H., Pang, T., and Pemy, M. (2012) *Viscosity Solutions of Optimal Stopping Problem for Stochastic Systems with Delays*, Stochastics Analysis and Applications, 30(6):1102-1305.
- [7] Chang, M.H., Pang, T., and Yang, Y. (2011) *A Stochastic Portfolio Optimization Model with Bounded Memory*, Mathematics of Operations Research, 36(4):604-619.
- [8] Chen, L. and Wu, Z. (2010) *Maximum Principle for the Stochastic Optimal Control Problem with Delay and Application*, Automatica, 46(6):1074-1080.
- [9] Crandall, M., Evans, L., and Lions, P.L. (1984) *Some Properties of Viscosity Solutions of Hamilton-Jacobi equations*, Transactions of the American Mathematical Society 282(2):487-502.
- [10] Crandall, M., Ishii, H., and Lions, P.L. (1992) *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations*, Bulletin of the American Mathematical Society, 27:1-67.
- [11] Crandall, M., Lions, P.L. (1983) *Viscosity Solutions of Hamilton-Jacobi Equations*, Transactions of the American Mathematical Society 277(1):142.
- [12] Da Prato, G. and Zabczyk, J. (2014) *Stochastic Equations in Infinite Dimensions*, Volume 152, Encyclopedia of Mathematics and Its Applications.
- [13] Davis, M.H., Norman, A. (1990) *Portfolio Selection with Transaction Costs*, Mathematics of Operations Research, 15(4):676-713.

- [14] Elsanosi, I. and Larssen, B. (2001) *Optimal Consumption under Partial Observations for a Stochastic System with Delay*, Preprint Series. Pure mathematics <http://urn.nb.no/URN:NBN:no-8076>.
- [15] Elsanosi, I., Øksendal, B., and Sulem, A. (2000) *Some Solvable Stochastic Control Problems with Delay*, Stochastics: An International Journal of Probability and Stochastic Processes, 71(1-2):69-89.
- [16] Evans, L. (2010) *Partial Differential Equations*, American Mathematical Society.
- [17] Federico, S. (2011) *A Stochastic Control Problem with Delay Arising in a Pension Fund Model*, Finance and Stochastics, 15(3):421-459.
- [18] Federico, S., Goldys, B., and Gozzi, F. (2010) *HJB Equations for the Optimal Control of Differential Equations with Delays and State Constraints, I: Regularity of Viscosity Solutions*, SIAM Journal on Control and Optimization, 48(8):4910-4937.
- [19] Federico, S., Goldys, B., and Gozzi, F. (2011) *HJB Equations for the Optimal Control of Differential Equations with Delays and State Constraints, II: Verification and Optimal Feedbacks*, SIAM Journal on Control and Optimization, 49(6):2378-2414.
- [20] Federico, S. and Tacconi, E. (2014) *Dynamic Programming for Optimal Control Problems with Delays in the Control Variable*, SIAM Journal on Control and Optimization, 52(2):1203-1236.
- [21] Federico, S. and Tankov, P. (2015) *Finite-Dimensional Representations for Controlled Diffusions with Delay*, Applied Mathematics and Optimization, 71(1):165-194.
- [22] Fleming, W. and Hernández-Hernández, D. (2003) *An Optimal Consumption Model with Stochastic Volatility*, Finance and Stochastics, 7(2):245-262.
- [23] Fleming, W. and Hernández-Hernández, D. (2005) *The Tradeoff Between Consumption and Investment in Incomplete Financial Markets*, Applied Mathematics And Optimization, 52(2):219-235.
- [24] Fleming, W. and Pang, T. (2004) *An Application of Stochastic Control Theory to Financial Economics*, SIAM Journal on Control and Optimization, 43(2):502-531.
- [25] Fleming, W. and Pang, T. (2005) *A Stochastic Control Model of Investment, Production and Consumption*, Quarterly of Applied Mathematics, 63(1):71-87.
- [26] Fleming, W. and Sheu, S. (2000) *Risk-Sensitive Control and An Optimal Investment Model*, Mathematical Finance, 10(2):197-213.
- [27] Fleming, W. and Soner, H. (2006) *Controlled Markov Processes and Viscosity Solutions (2nd ed.)*, Volume 25, Springer Science and Business Media.

- [28] Gilbarg D. and Trudinger N.S. (2001) *Elliptic Partial Differential Equations of Second Order*, Springer.
- [29] Gozzi, F. and Marinelli, C. (2005) *Stochastic Optimal Control of Delay Equations Arising in Advertising Models*, Stochastic Partial Differential Equations and Applications VII, 133148.
- [30] Gozzi, F., Marinelli, C., and Savin, S. (2009) *On Controlled Linear Diffusions with Delay in a Model of Optimal Advertising under Uncertainty with Memory Effects*, Journal of Optimization Theory and Applications, 142(2):291-321.
- [31] Hata, H. and Sheu, S.J. (2012) *On the Hamilton-Jacobi-Bellman Equation for An Optimal Consumption Problem: I. Existence of Solution*, SIAM Journal on Control and Optimization, 50(4):2373-2400.
- [32] Hata, H. and Sheu, S.J. (2012) *On the Hamilton-Jacobi-Bellman Equation for An Optimal Consumption Problem: I. Verification Theorem*, SIAM Journal on Control and Optimization, 50(4):2401-2430.
- [33] Ivanov, A., Swishchuk, A. (2008) *Optimal Control of Stochastic Differential Delay Equations with Application in Economics*, International Journal of Qualitative Theory of Differential Equations and Applications, 2(2):201-213.
- [34] Karatzas, I., Lehoczky, J., Sethi, S., and Shreve, S. (1986) *Explicit Solution of a General Consumption Investment Problem*, Mathematics of Operations Research, 11(2):261-294.
- [35] Karatzas, I., Lehoczky, J., and Shreve, S. (1987) *Optimal Portfolio and Consumption Decisions for a "Small Investor" on a Finite Horizon*, Siam Journal On Control And Optimization, 25(6):1557-1586.
- [36] Karatzas, I. and Shreve, S. (2012) *Brownian Motion and Stochastic Calculus*, Volume 113, Springer Science and Business Media.
- [37] Larssen, B. (2002) *Dynamic Programming in Stochastic Control of Systems with Delay*, Stochastics: An International Journal of Probability and Stochastic Processes, 74(3-4):651-673.
- [38] Larssen, B. and Risebro, N.H. (2003) *When are HJB-Equations in Stochastic Control of Delay Systems Finite Dimensional?*, Stochastic Analysis and Applications, 21(3):643-671.
- [39] Lehoczky, J., Sethi, S., Shreve, S. (1983) *Optimal Consumption and Investment Policies Allowing Consumption Constraints and Bankruptcy*, Mathematics Of Operations Research, 8(4):613-636.

- [40] Lindquist, A. (1973) *On Feedback Control of Linear Stochastic Systems*, SIAM Journal on Control, 11(2):323-343.
- [41] Lindquist, A. (1973) *Optimal Control of Linear Stochastic Systems with Applications to Time Lag Systems*, Information Sciences, 5:81-124.
- [42] Magill, M. and Constantinides, G. (1976) *Portfolio Selection with Transaction Costs*, Journal of Economic Theory, 13(2):245-263.
- [43] Merton, C. (1969) *Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case*, The Review of Economics and Statistics, 247-257.
- [44] Merton, C. (1971) *Optimum Consumption and Portfolio Rules in a Continuous-Time Model*, Journal of Economic Theory, 3(4):373-413.
- [45] Mohammed, S. (1998) *Stochastic Differential Systems with Memory: Theory, Examples and Applications*, Stochastic Analysis and Related Topics IV, 1-77.
- [46] Nagai, H. (2015) *HJB Equations of Optimal Consumption-Investment and Verification Theorems*, Applied Mathematics and Optimization, 71(2):279-311.
- [47] Noh, E.J. and Kim, J.H. (2011) *An Optimal Portfolio Model with Stochastic Volatility and Stochastic Interest Rate*, Journal of Mathematical Analysis and Applications, 375(2):510-522.
- [48] Øksendal, B. (2007) *Stochastic Differential Equations: An Introduction with Applications (6th ed.)*, Springer.
- [49] Øksendal, B., Sulem, A., Zhang, T., et al. (2011) *Optimal Control of Stochastic Delay Equations and Time-Advanced Backward Stochastic Differential Equations*, Advances in Applied Probability, 43(2):572-596.
- [50] Øksendal, B. and Sulem, A. (2001) *A Maximum Principle for Optimal Control of Stochastic Systems with Delay with Applications to Finance*, In: Menaldi, J.L., Roman, E., Sulem, A. (eds.) Proceedings of the Conference on Optimal Control and Partial Differential Equations, Paris, December 2000, 6479.
- [51] Pang, T. (2004) *Portfolio Optimization Models on Infinite-Time Horizon*, Journal of Optimization Theory and Applications, 122(3):573-597.
- [52] Pang, T. (2006) *Stochastic Portfolio Optimization with Log Utility*, International Journal of Theoretical and Applied Finance, 9(06):869-887.
- [53] Pang, T. (2014) *A Stochastic Investment Model on Finite Time Horizon*, Research on Finance and Management, 2(1):1-26.

- [54] Pang, T. and Hussain, A. (2015) *An Application of Functional Ito's Formula to Stochastic Portfolio Optimization with Bounded Memory*, Proceedings of 2015 SIAM Conference on Control and Its Applications (CT15), 159-166.
- [55] Pang, T. and Hussain, A. (2016) *An Infinite Time Horizon Portfolio Optimization Model with Delays*, Mathematical Control and Related Fields, 6(4):629-651.
- [56] Pang, T. and Hussain, A. (2017) *A Stochastic Portfolio Optimization Model with Complete Memory*, Stochastic Analysis and Applications, 35(4):742-766.
- [57] Pang, T. and Varga, K. (2015) *Optimal Investment and Consumption for a Portfolio with Stochastic Dividends*, Journal of Finance and Management Research, 1(2):1-22.
- [58] Pang, T. and Varga, K. (2017) *Portfolio Optimization with Stochastic Dividends and Stochastic Volatility*, Preprint.
- [59] Pao, C.V. (2012) *Nonlinear Parabolic and Elliptic Equations*, Springer Science and Business Media.
- [60] Pham, H. (2009) *Continuous-Time Stochastic Control and Optimization with Financial Applications*, Springer-Verlag.
- [61] Protter, P. (2005) *Stochastic Integration and Differential Equations*, Springer-Verlag.
- [62] Shreve, S. and Soner, H. (1994) *Optimal Investment and Consumption with Transaction Costs*, The Annals of Applied Probability, 4(3):609-692.
- [63] Veraar, M. (2012) *The Stochastic Fubini Theorem Revisited*, Stochastics: An International Journal of Probability and Stochastic Process, 84(4):543-551.
- [64] Yong, J. (2006) *Remarks on Some Short Rate Term Structure Models*, Journal of Industrial and Management Optimization, 2(2):119.
- [65] Yong, J. and Zhou, X.Y. (1999) *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Volume 43, Springer Science and Business Media.