ABSTRACT

CASTLE, LUCAS CULLEN. Well-Posedness and Control in Moving Boundary Fluid-Structure Interactions. (Under the direction of Lorena Bociu.)

In the field of partial differential equations (PDEs), there are many problems in which the domain is unknown and must be determined as part of the solution process. These PDE systems, known as moving boundary problems in the unsteady case, present themselves in a wide variety of applications ranging from engineering to biology. The particular class of moving boundary problems we consider in this work are fluid-structure interactions (FSIs), which describe the evolution of a movable or deformable solid as it interacts with a surrounding fluid flow. In this work, we discuss two PDE-constrained optimization problems: one in which the solid equation is governed by the equations of elastodynamics and another where the solid is modeled by a damped linear wave equation.

Due to the strong nonlinearity and the coupling of hyperbolic and parabolic phases in FSIs, the basic question of existence of solutions had been unresolved until recently. We build upon the work of [31, 32, 33] in order to construct a unique, global-in-time solution for a FSI modeled by the incompressible Navier-Stokes equations coupled with a damped wave equation. More specifically, we use a series of fixed point arguments to construct local solutions with sufficient regularity to take advantage of decay estimates on a-priori solutions. These decay estimates permit us to extend the time of existence up to any time $T > 0$.

Using this well-posedness theory, we establish the existence of an optimal distributed control acting on the system in order to minimize (within a specified control class) the vorticity of the fluid flow surrounding the elastic solid. A similar analysis is also done using the local well-posedness theory developed in [18] for the case of quasilinear elasticity. In both scenarios, the existence of the control is not limited to the minimization of turbulence in the fluid flow. The theory holds for a general cost functional provided it is similar in structure to that considered in this thesis.

Well-Posedness and Control in Moving Boundary Fluid- Structure Interactions

by
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BIOGRAPHY

Lucas Castle was born and raised in southeast Texas. He earned his Bachelor of Science degree in mathematics at Lamar University in Beaumont, Texas, in 2012. Immediately afterward, he entered the Mathematics Ph.D. program at North Carolina State University, and received his Master of Science degree in mathematics in 2014. He plans to graduate with his Ph.D. in August 2018.

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Chapter 1

Introduction

1.1 Fluid-Structure Interactions

A fluid-structure interaction (FSI) describes the interplay between some movable and/or deformable structure with either an internal or surrounding fluid flow. FSIs, which are typically modeled by partial differential equations (PDEs), are ubiquitous in medical and engineering applications, ranging from blood flow in stenosed arteries and heart valve dynamics to the design of small-scale unmanned aircrafts and morphing aircraft wings. Due to the fact that the shared interface between the fluid and solid is unknown in advance, FSIs fall into the category of moving boundary problems. Consequently, the unknown boundary must be determined alongside any state variables as part of the solution process.

From a mathematical viewpoint, fluid-structure interactions involve the coupling of (possibly) nonlinear equations describing the motion of the fluid with the displacement of an elastic solid. Therefore, these coupled systems are highly nonlinear due to the nonlinear state equations and the presence of the moving boundary. Furthermore, a PDE system modeling an FSI is characterized by a mismatch in regularity at the common interface, a feature resulting from the coupling of parabolic (fluid) and hyperbolic (solid) phases. These challenges have left the questions of the well-posedness in FSI systems (existence and uniqueness of solutions, and continuous dependence of solutions on initial and boundary data) open for a long time. Furthermore, problems of controllability, which depend on the aforementioned solutions and their properties, also were unattainable as a result. Many advances have been made recently in terms of the solvability [9, 10, 11, 15, 17, 18, 27, 28, 29, 31, 32, 33, 34, 37, 38, 39, 40, 48, 49, 50, 54] and the numerical study [20, 24, 25, 26, 30, 52, 53] of fluid-structure interactions.

The first breakthrough in moving-boundary parabolic-hyperbolic interactions was provided in [17, 18], where the local well-posedness of smooth solutions was obtained in the case of linear and quasilinear hydro-dynamics for small, highly regular initial data and sufficiently regular distributed sources. Subsequent results on local theory with improved requirements on the smoothness of initial data were obtained in [31, 38, 39] for the case of an incompressible flow, and similar results appeared in [11, 12, 40] for the case of compressible flows in the absence of source terms.

More recently, the first global in time existence result for an FSI system was provided in [31] in the case of an incompressible fluid coupled with a linear, damped wave equation under the assumption of small initial data and homogeneous source terms. For quasilinear dynamics, it is known that global existence is connected to the asymptotic decay of solutions, which depends on damping mechanisms. The authors of [31] focussed first on the case of boundary interface damping and star shaped domains. In the subsequent work [33], they showed that frictional damping in the structure permits the removal of restrictions based on geometry. The proof is a clever combination of a-priori estimates that close at the desired level of regularity and a construction of global solutions that depends heavily on hidden trace regularity results for the wave equation [41] and maximal regularity for the Stokes operator [51], which allow the transfer of regularity (without any loss of derivatives) from the fluid to the solid across the common interface.

From the perspective of controllability, we limit our focus on distributed forces acting on the body of the system (such as gravity or an electromagnetic field). Electromagnetic fields have been shown to mitigate turbulent fluid flows through a 2D channel (or 3D duct) [4, 13, 19, 21], which is of interest in optimizing pumping power in transporting fluid through pipelines, the design of non-mechanical mamicropumps, and the laminarization of submerged jets. Electromagnetic fields are also of interest in using magnetic nanoparticles to target tumors, and studies on the influence of such fields on blood flow revealed an appreciable effect on the flow velocity, fluid temperature, and the skin friction and rate of heat transfer at the vessel walls in both laminar and turbulent flows [61, 62]. These effects have also been demonstrated in the presence of multiple stenoses [56, 59, 63].

In this thesis, we consider the scenario in which an elastic body is submerged in a viscous fluid subjected to the influence of a body force. This system is governed by the fluid-structure model considered in [33] with the addition of a source term on the body of the system. Our goals are as follows: first, we seek to prove the global well-posedness of the FSI in the case of small initial data and small distributed sources (i.e. the body forces applied to the system). Secondly, we aim to optimize a given quantity of interest using the distributed sources as our control. For example, we can establish the existence of a distributed control that minimizes turbulence in the fluid flow induced by the motion of an elastic solid submerged within. Though we focus primarily on the case of minimizing turbulence, this work is largely independent of the desired quantity to be optimized. The results for optimal controllability therefore hold in a variety of settings and applications.

The contributions of this thesis can be summarized as follows:

1. The existence, uniqueness, and decay of global-in-time solutions for a non-homogeneous moving boundary fluid-structure interaction under sufficiently regular initial and forcing data. Tools utilized include the use of the flow mapping and Lagrangian framework to deal with the moving domains, multiplier and barrier methods to obtain decay of a-priori solutions, and the use of the contraction mapping theorem in the construction of solutions. Several regularity results, such as maximal parabolic regularity and the sharp trace regularity of the Dirichlet-Neumann map, were used in the fixed point arguments described herein.

2. The existence of an optimal distributed control acting upon the body of two separate FSI systems: one in which the solid is governed by a quasilinear elasticity model and another in which it is described by a damped linear wave equation. In both scenarios, we use weak limits of a minimizing sequence of controls for a given cost functional to reconstruct the functional evaluated at its minimizer. We require strong convergence of our weakly convergent subsequences in the nonlinear terms that appear in our coupled systems. Lions-Aubin compactness results are used to provide this convergence.

1.2 Notation

Throughout this thesis, we use the following notation and conventions:

- For a given matrix $a \in \mathbb{M}^3$, the ij-component of a is denoted by a^i_j, $i, j = 1, 2, 3$.

- The Einstein notation for double indices is used. For example, the divergence of a scalar field $f : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted by $\text{div} f = \partial_i f_i$.
- For a given matrix a, the transpose of a is denoted by a^T.
- For a given tensor field $T : \mathbb{R}^3 \to \mathbb{M}^3$, the divergence of T is denoted by $\text{Div}T$. Furthermore, the i-th component of $\text{Div}T$ is given by $\partial_j T^i_j$.
- Bold-face symbols will indicate spaces of $\mathbb{R}^3$-valued functions with the components in the indicated topology.
- In estimates, it is assumed that any constant C adjusts line-to-line.
- We suppress the notation $L^2(\Omega_f)$ and $L^2(\Omega_e)$ to L^2 when the spaces are obvious based on context. We use a similar notation for various Sobolev spaces.

1.3 Outline

The rest of this thesis is outlined below:

- Chapter 2 describes the relationship between a given fixed reference configuration of the FSI system and a deformed configuration at time $t > 0$ and the derivation of the PDE system considered from conservation laws within this framework.
- Chapter 3 provides the energy estimates and decay of energy norms necessary for the construction of global solutions in a fluid-structure interaction in which the solid is governed by a damped linear wave equation.
- Chapter 4 investigates the well-posedness of the coupled systems described in Chapter 2. It describes in detail the construction of a local-in-time solution (which can be extended to be global-in-time) to the PDE system via a series of fixed point arguments. In this process, we address the mismatch of regularity using sharp trace regularity and maximal parabolic regularity results.
- Chapter 5 proves the existence of an optimal control in that minimizes the turbulence in the fluid flow in the FSI systems considered.

- Chapter 6 summarizes the work found in the previous chapters and provides a synopsis of ideas for future work.
- The appendix contains proofs of auxillary lemmas used throughout this thesis.

Chapter 2

The Equations of FSI

In this chapter, we derive the state equations for a general fluid-structure interaction system. We follow the work of [26] in our derivation.

2.1 The Deformation of a Static Domain

Consider a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$ filled with a continuum medium, such as a viscous fluid or elastic solid. This domain, which is known, will be referred to as the reference configuration for our fluid-structure system. We define a deformation of Ω as a smooth, injective (one-to-one) map $\eta : \Omega \to \eta(\Omega)$, where $\eta(\Omega)$ is known as the deformed or current configuration of our medium, such that

$$x \mapsto \eta(x) \in \Omega.$$

In words, the mapping η associates each point x in the current configuration $\eta(\Omega)$ with a unique point x in the reference domain Ω. Often, we refer to the points x and $\eta(x)$ as the material points and spatial points of the medium, respectively. In some calculations, we are concerned with with the displacement $w : \Omega \to \eta(\Omega)$ of the material point x, which is defined in terms of the deformation via the relation

$$w(x) = \eta(x) - x, \;\; x \in \Omega. \tag{2.1.1}$$

Another quantity of interest is the deformation gradient (with respect to the material coordinates) $D\eta : \hat{\Omega} \to \mathbb{R}^{3\times 3}$, which is the second order tensor field given by

$$D\eta_i^j = \frac{\partial\big(\eta(x)_i\big)}{\partial x_j}.$$

We assume that the determinant of the deformation gradient, known as the Jacobian of the deformation, has the property that $J(x) = \det D\eta(x) > 0$ for all $x \in \Omega$, i.e. the mapping η is orientation preserving.

2.2 The Motion of a Time-dependent Domain

Up to this stage, we have only considered a static domain Ω. In many applications, the medium in question moves and deforms in time. To describe this phenomenon, we define a motion, which is a smooth map $\eta : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$,

$$(x, t) \mapsto \eta(x, t)$$

with the property that at any $t \geq 0$, $\eta_t = \eta(\cdot, t)$ is a deformation. That is, a motion is a one-parameter family of deformations (in time). For a given material point x, the point $\eta(x, t)$ is its position at time t, and $\eta(x, t) = \eta(x, t) - x$ is its displacement at time t. Furthermore, the domain $\Omega(t) = \eta(\Omega, t)$ is the current configuration of the medium at

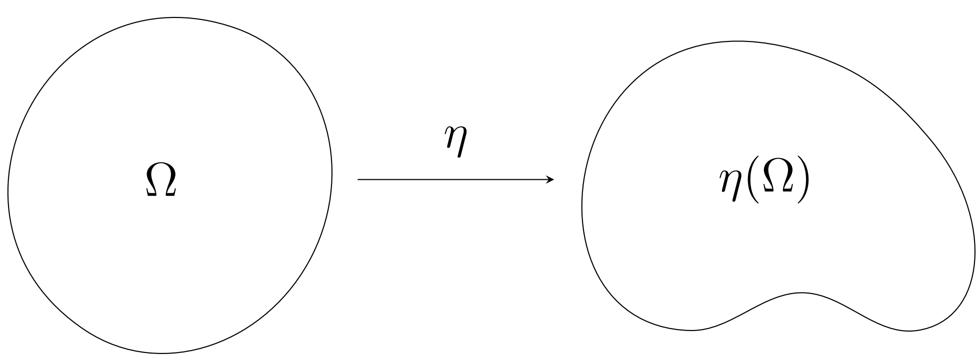

Figure 2.1 Transformation of a material volume through deformation.

time $t \geq 0$. In applications, we often choose the initial configuration $\Omega = \Omega(0)$ as the prescribed reference configuration (though it can be selected arbitrarily). As defined in the static case, we define $D\eta(x,t)$ to be the deformation gradient at time t with Jacobian $J(x,t) = \det\big(D\eta(x,t)\big)$. Finally, the velocity v of a given medium is given as follows:

$$v(x,t) = \frac{\partial}{\partial t}\eta(x,t) = \frac{\partial}{\partial t}w(x,t). \tag{2.2.1}$$

2.3 The Lagrangian and Eulerian Formulations

All physical quantities in an FSI system can be defined either on the reference or on the current configuration. When working in the reference configuration Ω, we focus on the evolution of an individual particle of medium x. This framework is known as the Lagrangian formulation of the system. By viewing the system from the current configuration $\Omega(t)$, we observe particles of a continuum medium flow through a point $\eta(x,\cdot)$ in space, and we work in what is known as the Eulerian formulation of the system. The use of a particular framework depends on the physical properties and desired method of analyzing a given system. For example, in fluids, displacements of individual particles are large and usually irrelevant. Instead, we are more often interested in the velocity field of the fluid and choose to work in the Eulerian frame. On the other hand, displacements of particles in solids are, in general, relatively small. Working in the the Lagrangian formulation is thus more natural in this setting.

To resolve the difference between the descriptions of the fluid and solid dynamics in FSI systems, we utilize the deformation η_t and its inverse η_t^{-1} to move between the material space Ω and physical space $\Omega(t)$ as needed.

For example, consider the material particle occupying the point x in the reference domain Ω. Suppose that $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$ describes the pressure at the point $\eta_t(x) \in \Omega(t)$ occupied by the material particle x at time t. Then, we can express the pressure at time t in the material point x by composing p with the deformation mapping η_t, i.e.

$$q(x,t) = p \circ \eta_t = p(\eta_t(x), t).$$

In this example, we call p and q the Eulerian pressure and Lagrangian pressure, respectively. We will adopt this convention when describing any state variables in this thesis.

2.4 Eulerian and Material Derivatives

In this section, we consider an Eulerian field q with associated Lagrangian description p; that is, $p = q \circ \eta$. Then the Eulerian time derivative of q is given by

$$\frac{\partial q}{\partial t}(x,t),\ \ x \in \Omega(t). \tag{2.4.1}$$

For p, the material time derivative of p is defined by

$$\frac{Dq}{Dt}(\cdot,t) = \frac{\partial p}{\partial t}(\cdot,t) \circ \eta_t^{-1}, \tag{2.4.2}$$

By rewriting (2.4.2) and applying the chain rule, we can derive the relationship between the Eulerian and material derivatives, that is

$$\frac{Dq}{Dt} = u \cdot \nabla q + \frac{\partial q}{\partial t}, \tag{2.4.3}$$

where u is the velocity of the medium expressed in Eulerian coordinates.

2.4.1 The Reynolds Transport Formula

Recall that for a given deformation η, $J = \det(D\eta)$ is the Jacobian of the deformation written in Lagrangian coordinates. If we define J_E to be J written in the Eulerian frame,

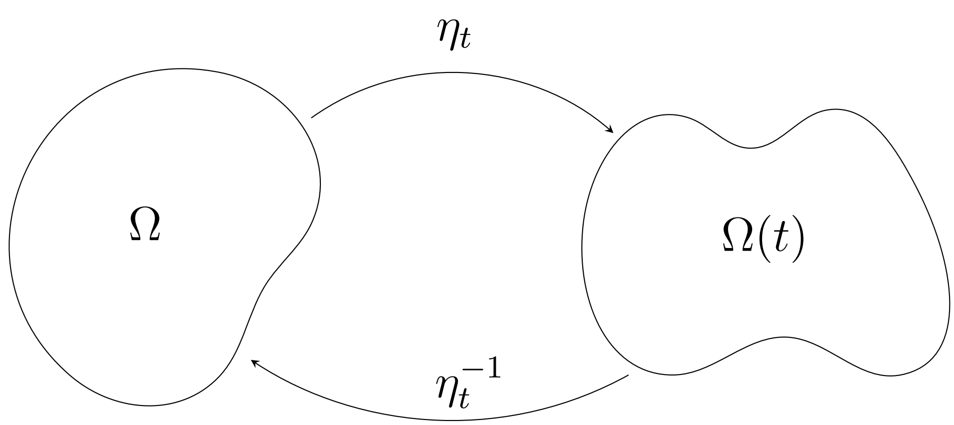

Figure 2.2 Transport between Lagrangian and Eulerian Configurations.

then we the following expression for the material derivative of J_E (the so-called Euler expansion formula):

$$\frac{DJ_E}{Dt} = J_E \text{div} u. \tag{2.4.4}$$

Formula (2.4.4) helps us to differentiate integrals of Eulerian fields over a material domains $\Omega(t)$ with reference configuration Ω. If f is a continuously differentiable material field, then

$$\frac{d}{dt} \int\limits_{\Omega(t)} f \, dx = \int\limits_{\Omega(t)} \left(\frac{Df}{Dt} + f \text{div} u \right) dx = \int\limits_{\Omega(t)} \left(\frac{\partial f}{\partial t} + \text{div}(fu) \right) dx. \tag{2.4.5}$$

This useful result is known as the the Reynolds transport formula.

2.5 Derivation of Equations

2.5.1 Conservation of Mass

Consider a continuum medium occupying an arbitrary domain $\Omega(t)$ with density ρ. Then the mass of $\Omega(t)$ at time t is given by

$$\int\limits_{\Omega(t)} \rho \, dx. \tag{2.5.1}$$

The principle of mass conservation states that the mass of a body cannot change during motion, that is

$$\frac{d}{dt} \int\limits_{\Omega(t)} \rho \, dx = 0, \tag{2.5.2}$$

at any time $t \geq 0$. Applying the Reynolds transport formula (2.4.5) in (2.5.2), we have that

$$\int\limits_{\Omega(t)} \left(\frac{\partial \rho}{\partial t} + \text{div}(\rho u) \right) dx = 0. \tag{2.5.3}$$

Furthermore, since $\Omega(t)$ was arbitrary, we can express (2.5.3) in a pointwise form, that is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \tag{2.5.4}$$

in $\Omega(t)$ for all $t > 0$.

2.5.2 Conservation of Momentum

The momentum of an arbitrary domain $\Omega(t)$ at time t is given by

$$\int_{\Omega(t)} \rho u \, dx. \tag{2.5.5}$$

The law of conservation of linear momentum states that the rate of change in the momentum of an object is equivalent to the resultant of the external forces F acting upon it. In terms of our material domain $\Omega(t)$, this can be expressed as

$$\frac{d}{dt} \int_{\Omega(t)} \rho u \, dx = F. \tag{2.5.6}$$

The external force F can be decomposed into a volume force F_v, which acts throughout the body of the material domain (such as gravity or an electromagnetic field), and a surface force F_s acting through the boundary $\Gamma(t) = \partial\Omega(t)$. The volume force is given in terms of a specific force f through the relation

$$F_v = \int_{\Omega(t)} \rho f \, dx,$$

and the surface force is given in terms of the Cauchy stress tensor $\sigma : \Omega(t) \to \mathbb{R}^{3\times 3}$ via

$$F_s = \int_{\Gamma(t)} \sigma n \, d\Gamma(t).$$

Using the tools we've built up so far and the divergence theorem, we can express the momentum conservation law (2.5.6) as

$$\frac{d}{dt}\int\limits_{\Omega(t)} \rho u \; dx = \int\limits_{\Omega(t)} \rho f \; dx + \int\limits_{\Gamma(t)} \sigma n \; d\Gamma(t) = \int\limits_{\Omega(t)} \rho f + \mathrm{Div}\sigma \; dx. \tag{2.5.7}$$

Since $\Omega(t)$ is arbitrary, we can conclude the following pointwise representation of (2.5.7):

$$\rho\frac{\partial u}{\partial t} + \rho(Du)u - \mathrm{Div}\sigma = \rho f. \tag{2.5.8}$$

2.5.3 The Fluid Equations

As mentioned previously, fluid displacements are large and usually irrelevant. As a result, one is mostly interested in the velocity field. To describe the fluid, we work from the current configuration $\Omega(t)$ in Eulerian coordinates. We now consider an incompressible Newtonian fluid occupying $\Omega(t)$ that has constant density. By applying the mass conservation law (2.5.4), we can conclude that $\mathrm{div}u = 0$ in $\Omega(t)$ (since the density ρ independent of time). This characterization is known as the divergence-free or incompressibility condition.

Next, we apply the law of conservation of momentum. The Cauchy stress tensor σ for a Newtonian incompressible fluid depends linearly on the rate of deformation rather than the deformation itself. This is due to the fact that fluids can fill any shape; however, the time it takes to fill a given shape depends on the fluid itself (the higher the viscosity, the more time it takes). As a result, the Cauchy stress tensor σ depends linearly strain rate tensor $D(u) = \frac{1}{2}(Du + Du^T)$ and the fluid pressure p, i.e.

$$\sigma = \sigma(u, P) = -PI + 2\mu D(u) = -PI + \mu(Du + Du^T). \tag{2.5.9}$$

To proceed, we compute the divergence of the stress tensor (2.5.9) term in (2.5.8). To accomplish this, we require the following formula that describes the divergence of the product of a scalar function P with a tensor field A:

$$\mathrm{Div}(PA) = A\nabla P + P\mathrm{Div}A. \tag{2.5.10}$$

Applying (2.5.10) to (2.5.9), we obtain

$$\begin{aligned}\mathrm{Div}\big(\sigma(P,u)\big) &= \mathrm{Div}\big(-PI + 2\mu D(u)\big)\\ &= -\mathrm{Div}(PI) + \mu\mathrm{Div}(Du + Du^T)\\ &= -\nabla P + \mu\Delta u + \mathrm{Div}(Du^T)\\ &= -\nabla P + \mu\Delta u,\end{aligned}$$

where the divergence-free property was used in the final step above. We denote the fluid source term by f_f and conclude that

$$\rho\frac{\partial u}{\partial t} - \mu\mathrm{Div}(Du) + \rho(Du)u + \nabla P = \rho f_f. \tag{2.5.11}$$

We now rescale. Let $p = \frac{1}{\rho}P$ and $\nu = \frac{1}{\rho}\mu$. Here, p is the scaled pressure and ν is known as the kinematic viscosity. Then,

$$\rho\frac{\partial u}{\partial t} - \rho\nu\mathrm{Div}(Du) + \rho(Du)u + \rho\nabla p = \rho f_f,$$

which implies that

$$\frac{\partial u}{\partial t} - \nu\mathrm{Div}(Du) + (Du)u + \nabla p = f_f. \tag{2.5.12}$$

Equation (2.5.12) is the classical Navier-Stokes equation. Taken together with the divergence-free condition, we have the following equations that characterize an incompressible fluid occupying a domain $\Omega(t)$:

$$\begin{cases}\dfrac{\partial u}{\partial t} - \nu\mathrm{Div}(Du) + (Du)u + \nabla p = f_f, & \Omega(t),\\ \mathrm{div}u = 0, & \Omega(t).\end{cases} \tag{2.5.13}$$

2.5.4 The Solid Equation

In solids, the displacement η is often relatively small. We thus work from the perspective of the reference configuration Ω in Lagrangian coordinates. We first write the law of conservation of mass (2.5.2) on the reference configuration Ω.

$$\frac{d}{dt}\int\limits_{\Omega(t)} \rho\ dx = \frac{d}{dt}\int\limits_{\Omega} \rho J\ dx = \int\limits_{\Omega} \frac{d}{dt}(\rho J)\ dx = 0.$$

Due to the fact that Ω is arbitrary, we conclude that $\dfrac{d}{dt}(\rho J) = 0$. We now use this fact in the momentum conservation law (2.5.7). First, we transport each side to the reference domain Ω. Let $v = u \circ \eta$ be the Lagrangian velocity and $\hat{\rho} = \rho \circ \eta$ be the Lagrangian density of the solid material. Then $\frac{\partial \eta}{\partial t} = v$, and we have that

$$\begin{aligned} \frac{d}{dt}\int\limits_{\Omega} \hat{\rho} v J\ dx &= \frac{d}{dt}\int\limits_{\Omega} \hat{\rho}\frac{\partial \eta}{\partial t} J dx \\ &= \int\limits_{\Omega} \frac{d}{dt}(\hat{\rho} J)\frac{\partial \eta}{\partial t} + \hat{\rho} J \frac{\partial^2 \eta}{\partial t^2}\ dx \\ &= \int\limits_{\Omega} \hat{\rho} J \frac{\partial^2 \eta}{\partial t^2}\ dx. \end{aligned} \tag{2.5.14}$$

Furthermore, we express the source term $\hat{f} = f \circ \eta$ in Lagrangian coordinates and write the complete law of conservation of momentum.

$$\int\limits_{\Omega} \hat{\rho} J \frac{\partial^2 \eta}{\partial t^2} - J\big((\mathrm{Div}\sigma) \circ \eta\big)\ dx = \int\limits_{\Omega} \hat{\rho} J \hat{f}\ dx,$$

which allows us to conclude that

$$\hat{\rho} J \frac{\partial^2 \eta}{\partial t^2} - J\big((\mathrm{Div}\sigma) \circ \eta\big) = \hat{\rho} J \hat{f}. \tag{2.5.15}$$

We now observe that the divergence taken in (2.5.15) is carried out in Eulerian coordinates before it is transported to the Lagrangian frame via composition with the deformation mapping. As a result, the solid equation is in a "mixed" form. To resolve this issue, we use the Piola transformation $\mathcal{P} : \Omega \to \mathbb{R}^{3\times 3}$ of the second order tensor σ

associated with the deformation η, given by

$$\mathcal{P} = \big(\mathcal{P}_\eta(\sigma)\big)(x) = J(x)(\sigma \circ \phi(x))(D\eta(x))^{-T}. \tag{2.5.16}$$

We now use the following relationship between $\mathcal{P}$ and σ:

$$\mathrm{Div}_\eta \mathcal{P} = J\big((\mathrm{Div}\sigma) \circ \eta\big), \tag{2.5.17}$$

where $\mathrm{Div}_\eta \mathcal{P}$ denotes the divergence of $\mathcal{P}$ computed with respect to the Lagrangian coordinates. Substituting (2.5.17) into (2.5.15) and setting $\rho_0 = \hat{\rho} J$, we obtain

$$\rho_0 \frac{\partial^2 \eta}{\partial t^2} - \mathrm{Div}_\eta \mathcal{P} = \rho_0 \hat{f} \text{ in } \Omega, \tag{2.5.18}$$

which is known as the equation of elastodynamics. We observe that $\mathcal{P}$ is in fact not symmetric. To remedy this, we introduce the symmetric second Piola-Kirchhoff stress tensor

$$\Sigma = D\eta^{-1}\mathcal{P}. \tag{2.5.19}$$

On Ω, (2.5.18) can be re-expressed as

$$\rho_0 \frac{\partial^2 \eta}{\partial t^2} - \mathrm{Div}_\eta \big(D\eta \Sigma\big) = \rho_0 \hat{f}, \tag{2.5.20}$$

In this work, we consider a homogeneous and isotropic hyperelastic material. In an elastic body, the stress depends on the deformation but is independent of time. A homogeneous and isotropic material has mechanical properties that do not depend on space and respond to deformation independent of direction. In this case, the constitutive law is typically written in terms of the Green-Lagrange strain tensor

$$E = \frac{1}{2}\big(D\eta^T D\eta - I\big). \tag{2.5.21}$$

We can use (2.1.1) to express E in terms of the displacement w, i.e.

$$E = \frac{1}{2}\big(Dw + Dw^T\big) + \frac{1}{2} Dw^T Dw. \tag{2.5.22}$$

The constitutive relation for a hyperelastic material is given in terms of a given density of elastic energy $W : \mathbb{R}^{3\times 3} \to \mathbb{R}^+$ through the relation

$$\Sigma(E) = \frac{\partial W}{\partial E}(E). \tag{2.5.23}$$

A homogeneous isotropic material with the natural state (i.e. a configuration in which the Cauchy stress tensor is identically zero everywhere) as its initial configuration can be described using the Saint-Venant Kirchoff model. In this case, the elastic energy density is given by

$$W(E) = \frac{\lambda}{2}(\mathrm{Tr}(E))^2 + \mu \mathrm{Tr}(E)^2, \tag{2.5.24}$$

where λ and μ denote the Lamé parameters. Using (2.5.24) in (2.5.23), we obtain the following expression for the second Piola-Kirchhoff tensor:

$$\Sigma(E) = \lambda \mathrm{Tr}(E) + 2\mu \mathrm{Tr}(E). \tag{2.5.25}$$

Equation (3.2.1) supplemented with the constitutive relation (2.5.25) provide a complete characterization of the displacement of a homogeneous isotropic material.

2.6 Configuration and Domain

In this thesis, we consider a fluid-structure interaction governed by the coupling of the Navier-Stokes equation (2.5.13) with an equation governing the motion of a moving and deforming elastic solid under the influence of a distributed control. Our goals are to (1) establish global well-posedness of the fluid-structure interaction in the case of small distributed sources and small initial data and (2) minimize the turbulence inside the fluid induced by the motion of the solid moving and deforming within it. The latter will be achieved using a body force applied to the system, which appears as a source term on the right-hand side of the fluid and solid equations.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain containing both the fluid and the elastic structure. At time $t \geq 0$, the elastic body is located in domain $\Omega_e(t)$, with boundary $\Gamma_c(t)$. The structure is surrounded by fluid, which occupies domain $\Omega_f(t) = \Omega \setminus \overline{\Omega_e}(t)$, with smooth boundary $\Gamma_c(t) \cup \Gamma_f$, where Γ_f is fixed in time. We assume that $\Gamma_c(t) \cap \Gamma_f = \emptyset$, and we define the control volume $\mathcal{D} = \Omega_f(t) \cup \overline{\Omega_e}(t)$ for all $t \geq 0$.

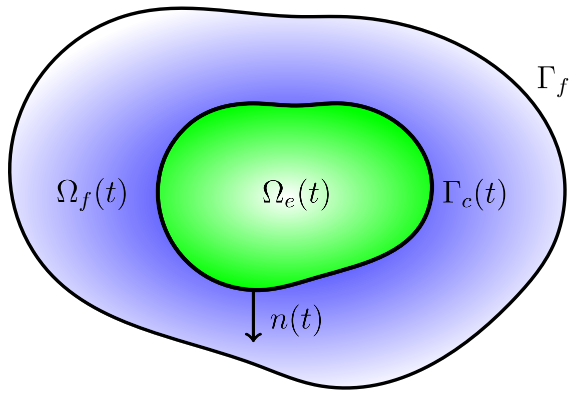

Figure 2.3 Current configuration for a general FSI with the solid immersed in the fluid on a bounded domain.

When coupling fluids and solids, one has to accommodate both Lagrangian and Eulerian frameworks. In order to match the two frameworks, we transport the fluid equation to the reference configuration using the following set-up [33]: Let $\eta(\cdot, t) : \Omega \to \Omega$ be the flow map under which the reference configurations $\Omega_f = \Omega_f(0)$ and $\Omega_e = \Omega_e(0)$ evolve in time, i.e. $\eta(\Omega_f, t) = \Omega_f(t)$ and $\eta(\Omega_e, t) = \Omega_e(t)$.

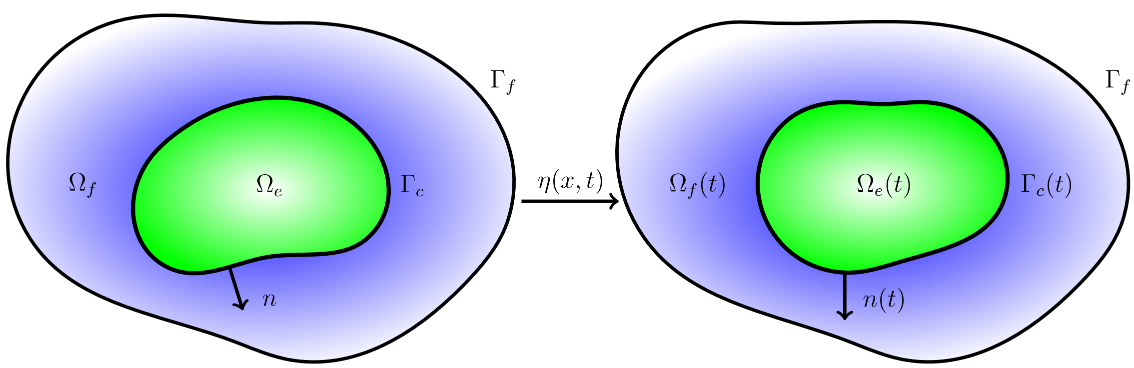

Figure 2.4 The evolution of the fluid and solid domains via the flow mapping η.

2.7 Equations of Fluid and Structure

We now write the fully coupled system for an elastic solid immersed in a viscous fluid. To describe the interaction between the fluid and solid components at the common interface $\Gamma_c(t)$, we use the flow mapping η to express the fluid in terms of the reference configuration Ω_f. With this in mind, we require the continuity of the fluid and solid velocities and normal stresses across the reference interface Γ_c:

$$\begin{cases} u \circ \eta = w_t, & \Gamma_c \\ \Sigma \cdot n = [\sigma(u,p) \circ \eta] D\eta^{-T} n, & \Gamma_c, \end{cases}, \tag{2.7.1}$$

where n is the outward unit normal to Γ_c with respect to Ω_s.

In summary, we arrive at the following PDE model for a fluid-structure interaction:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \mathrm{Div}(Du) + (Du)u + \nabla p = f_f, & \Omega_f(t), \\ \mathrm{div} u = 0, & \Omega_f(t), \\ \rho_0 \frac{\partial^2 \eta}{\partial t^2} - \mathrm{Div}_\eta\big(D\eta\Sigma\big) = \rho_0 \hat{f}, & \Omega_e, \\ u \circ \eta = w_t, & \Gamma_c, \\ \Sigma \cdot n = [\sigma(u,p) \circ \eta] D\eta^{-T} n, & \Gamma_c, \\ u = 0, & \Gamma_f. \end{cases} \tag{2.7.2}$$

Chapter 3

Energy Estimates

In this chapter, we use multiplier methods to derive a-priori estimates at six energetic levels for the case of a solid modeled by a damped linear wave equation. Using these estimates, we demonstrate the decay of solutions, which hinges on the damping terms in the system. This decay will be used in later sections to extend local solutions for the non-linear problem for all times $t > 0$. We begin by expressing the fluid equations in Lagrangian coordinates.

3.1 Lagrangian Formulation of the PDE System

Recall from (2.7.2) that the only components of the coupled system written on the current configuration $\Omega(t)$ are the fluid equations. Let $v(x,t) = \eta_t(x,t) = u(\eta(x,t),t)$ and $q(x,t) = p(\eta(x,t),t)$ denote the Lagrangian fluid velocity and pressure in $\Omega_f \times (0,T)$. The elastic displacement is given as the difference between the flow map and identity: $w(x,t) = \eta(x,t) - x$. We define the matrix $a(x,t) = (D\eta)^{-1}$, and use a_i^j as notation for the ij entry of a for $i,j = 1,2,3$. The matrix a solves the following initial value problem (IVP):

$$\begin{cases} a_t = -a(Dv)a, & \Omega_f \times (0,T), \\ a(x,0) = I, & \Omega_f. \end{cases} \tag{3.1.1}$$

Assuming the no-slip condition for the fluid velocity on the fixed boundary Γ_f, the fluid equation can be expressed in Lagrangian coordinates as follows:

$$\begin{cases} v_t - \mathrm{Div}((Dv)aa^T) + a^T\nabla q = f_f \circ \eta, & \Omega_f \times (0,T), \\ \mathrm{div}(av) = 0, & \Omega_f \times (0,T), \\ v = 0, & \Gamma_f \times (0,T). \end{cases} \tag{3.1.2}$$

Due to the composition with η, we define f_f on the whole domain Ω and write $f_f \circ \eta$ in Ω_f.

3.2 Energy Estimates: Global Theory

In studying a global-in-time solution for an FSI system, we choose to work with a simplified model for the solid. Instead of using the elastodynamics equation (2.5.18), we utilize a damped linear wave equation under the influence of a distributed source f_e given below:

$$w_{tt} - \Delta w + \alpha w_t + \beta w = f_e \text{ in } \Omega_e. \tag{3.2.1}$$

In order to further simplify the analysis, we will consider the case of initially flat subdomains, i.e.

$$\Omega_f = \{x = (x_1, x_2, x_3) \mid h_1 \leq x_3 \leq h_2 \text{ or } h_3 \leq x_3 \leq h_4\}$$

and

$$\Omega_e = \{x = (x_1, x_2, x_3) \mid h_2 \leq x_3 \leq h_3\},$$

with periodic boundary conditions with period 1 in the lateral directions. See Figure 3.1 below for a schematic 2D cross section representation of the interaction.

Remark 3.2.1. *The assumption of initially flat subdomains, which is not essential, is for the sake of exposition and cleaner proofs. It allows us to bypass the use of commutators in the a-priori tangential estimates. The proofs can be adjusted to general initial configurations (such as in Figure 2.4) using the strategy of [37].*

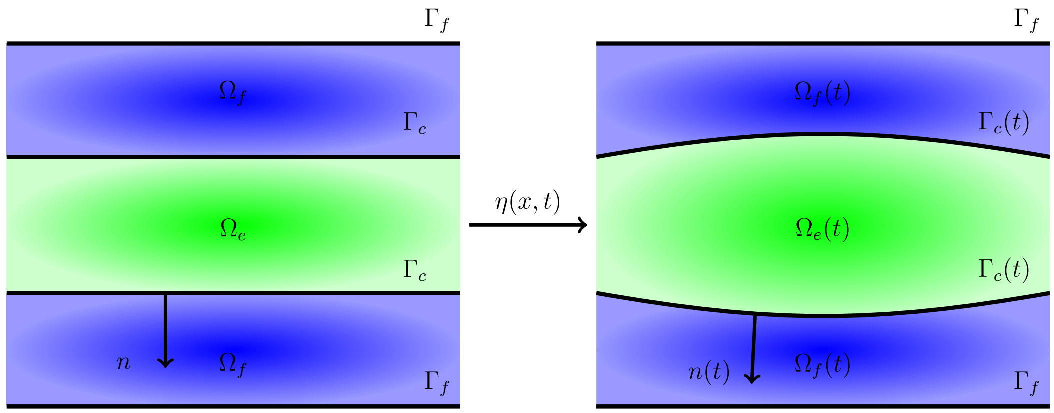

Figure 3.1 The evolution of the fluid and solid domains via the flow mapping η in the case of initially flat domains.

We again assume continuity of the velocity and normal stresses across Γ_c, i.e.

$$\begin{cases} w_t = v, & \Gamma_c \times (0,T), \\ \dfrac{\partial w}{\partial n} = (Dv)aa^T n - qa^T n, & \Gamma_c \times (0,T), \end{cases} \tag{3.2.2}$$

where n denotes the unit outward normal vector to Ω_e. For simplicity, we use the strain tensor Dv instead of $\varepsilon(v)$ in the matching of the stress tensors on the common boundary. We also use the classical spaces $H = \{v \in L^2(\Omega_f) \mid \mathrm{div} v = 0, v \cdot n|_{\Gamma_f} = 0\}$ and $V = \{v \in H^1(\Omega_f) \mid \mathrm{div} v = 0,\ v|_{\Gamma_f} = 0\}$ for the fluid velocity. The Lagrangian Formulation of the full coupled system can be expressed as follows:

$$\begin{cases} v_t - \mathrm{Div}((Dv)aa^T) + a^T \nabla q = f_f \circ \eta, & \Omega_f \times (0,T), \\ \mathrm{div}(av) = 0, & \Omega_f \times (0,T), \\ v = 0, & \Gamma_f \times (0,T), \\ w_{tt} - \Delta w + \alpha w_t + \beta w = f_e, & \Omega_e \times (0,T), \\ w_t = v, & \Gamma_c \times (0,T), \\ \frac{\partial w}{\partial n} = (Dv)aa^T n - qa^T n, & \Gamma_c \times (0,T). \end{cases} \tag{3.2.3}$$

In the analysis, we will make extensive use of the component-wise version of (3.2.3) given below:

$$\begin{cases} v_t^i - \partial_j(a_\ell^j a_\ell^k \partial_k v^i) + \partial_k(a_i^k q) = f_f^i \circ \eta, & \Omega_f \times (0,T), \\ a_i^k \partial_k v^i = 0, & \Omega_f \times (0,T), \\ w_{tt}^i - \Delta w^i + \alpha w_t^i + \beta w^i = f_e^i, & \Omega_e \times (0,T), \\ w_t^i = v^i & \Gamma_c \times (0,T), \\ \partial_j w^i N_j = a_\ell^j a_\ell^k \partial_k v^i N_j - a_i^k q N_k, & \Gamma_c \times (0,T), \\ v^i = 0, & \Gamma_f \times (0,T). \end{cases} \tag{3.2.4}$$

Here, $i = 1,2,3$ indicates the i-th entry of a vector in $\mathbb{R}^3$.

Our well-posedness analysis for (3.2.4) relies heavily on a priori estimates on the energy for the original system and its temporal and tangential derivatives. We will use the following lemma given in [31] in our analysis.

Lemma 3.2.2. *Assume that $\|Dv\|_{L^\infty([0,T];\ H^2(\Omega_f))} < \infty$. With $T \in [0, \frac{1}{CM}]$, where C is a sufficiently large constant, the following statements hold:*

(i) $\|\nabla\eta\|_{H^2(\Omega_f)} \le C$ *for* $t \in [0,T]$.

(ii) $\|a\|_{H^2(\Omega_f)} \le C$ *for* $t \in [0,T]$.

(iii) *for every* $\epsilon \in [0, \frac{1}{2}]$ *and all* $t \le T^* = \min\{\frac{\epsilon}{CM^2}, T\}$, *we have*

$$\|\delta_{jk} - a_\ell^j a_\ell^k\|_{H^2(\Omega_f)}^2 \le \epsilon,\ \ j,k = 1,2,3 \tag{3.2.5}$$

and

$$\|\delta_{jk} - a_k^j\|_{H^2(\Omega_f)}^2 \le \epsilon,\ \ j,k = 1,2,3. \tag{3.2.6}$$

In particular, the form $a_\ell^j a_\ell^k \xi_j^k \xi_k^i$ *satisfies the ellipticity estimate*

$$a_\ell^j a_\ell^k \xi_j^k \xi_k^i \ge \frac{1}{C}|\xi|^2,\ \ \xi \in \mathbb{R}^{n^2}, \tag{3.2.7}$$

for all $t \in [0, T^*]$ *and* $x \in \Omega_f$, *provided* $\epsilon < \frac{1}{C}$ *with* C *sufficiently large.*

3.2.1 First Level Estimates

Define the first level energy $E(t)$ and dissipation $D(t)$ by

$$E(t) = \frac{1}{2}\Big(\|v(t)\|_{L^2}^2 + \beta\|w(t)\|_{L^2}^2 + \|w_t(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\Big), \tag{3.2.8}$$

$$D(t) = \alpha\|w_t(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla v(t)\|_{L^2}^2, \tag{3.2.9}$$

where α and β are as in (3.2.4). Our objective is to derive an estimate on the first level energy for the coupled system. We have the following lemma:

Lemma 3.2.3. *The energy inequality*

$$E(t) + \int_0^t D(s)\ ds \leq \ E(0) + C_\epsilon\|f_f \circ \eta\|_{L^2(\Omega_f\times[0,T])}^2 + C_\epsilon\|f_e\|_{L^2(\Omega_e\times[0,T])}^2 + \epsilon\int_0^t \|v\|_{L^2}^2\ ds, \tag{3.2.10}$$

holds for all $t \in [0,T]$.

Proof. Multiply the fluid equation by v, the solid equation by w_t, and integrate by parts in space as follows:

$$\int_{\Omega_f} v_t^i v^i - \partial_j(a_\ell^j a_\ell^k \partial_k v^i)v^i + \partial_k(a_i^k q)v^i\ d\Omega_f \ = \ \int_{\Omega_f} (f_f^i \circ \eta)v^i\ d\Omega_f. \tag{3.2.11}$$

Focusing on the left-hand side of (3.2.11), we have for the first term

$$\begin{aligned} \int_{\Omega_f} v_t^i v^i\ d\Omega_f \ &= \ \int_{\Omega_f} \frac{1}{2}\partial_t\big(v^i\big)^2\ d\Omega_f \\ &= \ \partial_t \frac{1}{2}\|v^i\|_{L^2}^2. \end{aligned} \tag{3.2.12}$$

In the second term on the left-hand side, we apply the stress-matching condition to obtain

$$\int_{\Omega_f} \partial_j(a_\ell^j a_\ell^k \partial_k v^i)v^i\ d\Omega_f \ = \ -\int_{\Omega_f} a_\ell^j a_\ell^k \partial_k v^i \partial_j v^i\ d\Omega_f - \int_{\Gamma_c} a_\ell^j a_\ell^k \partial_k v^i N_j v^i\ d\Gamma_c$$

$$= -\int_{\Omega_f} a^j_\ell a^k_\ell \partial_k v^i \partial_j v^i \; d\Omega_f - \int_{\Gamma_c} \Big(\partial_j w^i N_j + a^k_i q N_k \Big) v^i \; d\Gamma_c. \tag{3.2.13}$$

In the third term, we apply the divergence-free condition to obtain

$$\begin{aligned}\int_{\Omega_f} \partial_k (a^k_i q) v^i \; d\Omega_f &= -\int_{\Omega_f} a^k_i q \partial_k v^i d\Omega_f - \int_{\Gamma_c} a^k_i q N_k v^i \; d\Gamma_c \\ &= -\int_{\Gamma_c} a^k_i q N_k v^i \; d\Gamma_c. \end{aligned} \tag{3.2.14}$$

Using (3.2.12), (3.2.13), and (3.2.14) in (3.2.11), we obtain

$$\partial_t \frac{1}{2} \|v^i\|^2_{L^2} + \int_{\Omega_f} a^j_\ell a^k_\ell \partial_k v^i \partial_j v^i \; d\Omega_f + \int_{\Gamma_c} \partial_j w^i N_j v^i d\Gamma_c = \int_{\Omega_f} (f^i_f \circ \eta) v^i \; d\Omega_f. \tag{3.2.15}$$

By the ellipticity of a (A.1.6), we have

$$\partial_t \frac{1}{2} \|v^i\|^2_{L^2} + \frac{1}{C} \|\nabla v\|_{L^2} + \int_{\Gamma_c} \partial_j w^i N_j v^i d\Gamma_c \leq \int_{\Omega_f} (f^i_f \circ \eta) v^i \; d\Omega_f. \tag{3.2.16}$$

We follow a similar approach with the solid to obtain

$$\partial_t \frac{1}{2} \Big(\|w^i_t\|^2_{L^2} + \|\nabla w^i\|^2_{L^2} + \frac{\beta}{2} \|w^i_t\|^2_{L^2} \Big) - \int_{\Gamma_c} \frac{\partial w^i}{\partial N} w^i_t \; d\Gamma_c + \alpha \|w^i_t\|^2_{L^2} = \int_{\Omega_e} f^i_e w^i_t \; d\Omega_e. \tag{3.2.17}$$

Applying the velocity matching condition to (3.2.17) yields

$$\partial_t \frac{1}{2} \Big(\|w^i_t\|^2_{L^2} + \|\nabla w^i\|^2_{L^2} + \frac{\beta}{2} \|w^i_t\|^2_{L^2} \Big) - \int_{\Gamma_c} \frac{\partial w^i}{\partial N} v^i \; d\Gamma_c + \alpha \|w^i_t\|^2_{L^2} = \int_{\Omega_e} f^i_e w^i_t \; d\Omega_e. \tag{3.2.18}$$

Adding (3.2.16) and (3.2.18), summing in i, and integrating in time yields

$$E(t) + \int_0^t D(s)\ ds \quad \leq \quad E(0) + \int_0^t \int_{\Omega_f} (f_f \circ \eta) \cdot v\ dx\ ds + \int_0^t \int_{\Omega_e} f_e \cdot w_t\ dx\ ds. \tag{3.2.19}$$

Note that the velocity matching condition was used to cancel the terms on the boundary. Applying the Cauchy-Schwartz and Young's inequality, we have that

$$\begin{aligned} E(t) + \int_0^t D(s)\ ds \quad &\leq \quad E(0) + \int_0^t \|f_f \circ \eta\|_{L^2} \|v\|_{L^2}\ ds + \int_0^t \|f_e\|_{L^2} \|w_t\|_{L^2}\ ds \\ &\leq \quad E(0) + C_\epsilon \|f_f \circ \eta\|^2_{L^2(\Omega_f \times [0,T])} + \epsilon \int_0^t \|v\|^2_{L^2}\ ds + C_\epsilon \|f_e\|^2_{L^2(\Omega_e \times [0,T])} \\ &\qquad + \epsilon \int_0^t \|w_t\|^2_{L^2}\ ds. \end{aligned} \tag{3.2.20}$$

Absorbing the last term into the integral on the left-hand side, we obtain:

$$E(t) + \int_0^t D(s)\ ds \quad \leq \quad E(0) + C_\epsilon \|f_f \circ \eta\|^2_{L^2(\Omega_f \times [0,T])} + C_\epsilon \|f_e\|^2_{L^2(\Omega_e \times [0,T])} + \epsilon \int_0^t \|v\|^2_{L^2}\ ds. \tag{3.2.21}$$

□

In order to obtain a complete characterization of the first level energy norms, we require an estimate for ∇w.

Lemma 3.2.4 (Equipartition of Energy)**.** *We have*

$$\begin{aligned} &\int_0^t \|\nabla w\|^2_{L^2}\ ds + \beta \int_0^t \|w\|^2_{L^2}\ ds + \frac{\alpha}{4} \|w(t)\|^2_{L^2} + \frac{1}{C} \|\nabla(\eta(\cdot,t) - x)\|^2_{L^2} \\ &\quad \leq CE(0) + \int_0^t \|w_t\|^2_{L^2}\ ds + C\|w_t(t)\|^2_{L^2} + C\|v(t)\|^2_{L^2} + \int_0^t \|v\|^2_{L^2}\ ds \end{aligned}$$

$$+\int_0^t (\tilde{R}(s), \eta(x,s) - x)\ ds + C_\epsilon \|f_f \circ \eta\|^2_{L^2(\Omega_f \times [0,T])} + C_\epsilon \|f_e\|^2_{L^2(\Omega_e \times [0,T])}$$

$$+\epsilon \int_0^t \|\eta^i - x^i\|^2_{L^2(\Omega_f)}\ ds \tag{3.2.22}$$

for all $t \in [0,T]$, where

$$\begin{aligned}\int_0^t (\tilde{R}(s), \eta(x,s) - x)\ ds \quad &= \quad \frac{1}{2}\int_0^t \int_{\Omega_f} \partial_t(a^j_\ell a^k_\ell)\partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i)\ d\Omega_f\ ds \\ &\quad + \int_0^t \int_{\Omega_f} q(s) \int_0^s \partial_t a^k_i \partial_k(\eta^i - x^i)\ d\tau\ d\Omega_f\ ds. \end{aligned} \tag{3.2.23}$$

Proof. We multiply the wave equation in (3.2.4) by w and integrate over Ω_e.

$$\int_{\Omega_e} w^i_{tt} w^i - \Delta w^i w^i + \alpha w^i_t w^i + \beta w^i w^i\ d\Omega_e \quad = \quad \int_{\Omega_e} f^i_e w^i\ d\Omega_e.$$

Integrating by parts and noting that

$$\partial_t(w_t w) = w_{tt} w + w_t^2 \quad \Longrightarrow \quad w_{tt} w = \partial_t(w_t w) - w_t^2,$$

we obtain

$$\begin{aligned}\int_{\Omega_e} \partial_t(w^i_t w^i) - w^i_t w^i_t + \nabla w^i \nabla w^i + \frac{\alpha}{2}\partial_t(w^i)^2 + \beta w^i w^i\ d\Omega_e \quad &= \quad \int_{\Gamma_c} \frac{\partial w^i}{\partial N} w^i\ d\Gamma_c \\ &\quad + \int_{\Omega_e} f^i_e w^i\ d\Omega_e.\end{aligned}$$

Integrating in time yields

$$\int_{\Omega_e} \left(\int_0^t |\nabla w|^2 + \beta |w|^2\ ds \right) + \frac{\alpha}{2}|w|^2\ d\Omega_e \quad = \quad \int_{\Omega_e} \frac{\alpha}{2}|w_0|^2\ d\Omega_e + \int_0^t \int_{\Omega_e} |w_t|^2\ d\Omega\ ds$$

$$- \int_{\Omega_e} w_t^i w^i \; d\Omega_e + \int_{\Omega_e} w_1^i w_0^i \; d\Omega_e$$
$$+ \int_0^t \int_{\Gamma_c} \frac{\partial w^i}{\partial N} w^i \; d\Gamma_c \; ds$$
$$+ \int_0^t \int_{\Omega_e} f_e^i w^i \; d\Omega_e \; ds. \qquad (3.2.24)$$

To estimate the boundary term, we multiply the stress-matching condition by $w^i = \eta^i - x^i$, and integrate in time and space.

$$\int_0^t \int_{\Gamma_c} \partial_j w^i w^i N_j - a_\ell^j a_\ell^k v^i N_j(\eta^i - x^i) + a_i^k q N_k(\eta^i - x^i) \; d\Gamma_c \; ds \;\; = \;\; 0, \qquad (3.2.25)$$

which implies

$$\int_0^t \int_{\Gamma_c} \partial_j w^i w^i N_j \; d\Gamma_c \; ds \;\; = \;\; \int_0^t \int_{\Gamma_c} a_\ell^j a_\ell^k v^i N_j(\eta^i - x^i) - a_i^k q N_k(\eta^i - x^i) \; d\Gamma_c \; ds. \qquad (3.2.26)$$

Next, we multiply the fluid equation by $(\eta^i - x^i)$ and integrate over Ω_f.

$$\int_{\Omega_f} v_t^i(\eta^i - x^i) - \partial_j(a_\ell^j a_\ell^k \partial_k v^i)(\eta^i - x^i) + \partial_k(a_i^k q)(\eta^i - x^i) \; d\Omega_f$$
$$= \int_{\Omega_f} (f_f^i \circ \eta)(\eta^i - x^i) \; d\Omega_f.$$

We write $\partial_t(v^i(\eta^i - x^i)) = v_t^i(\eta^i - x^i) + v^i \eta_t^i$ to obtain

$$\int_{\Omega_f} \partial_t(v^i(\eta^i - x^i)) - v^i \eta_t^i + a_\ell^j a_\ell^k \partial_k v^i \partial_j(\eta^i - x^i) - a_i^k q \partial_k(\eta^i - x^i) \; d\Omega_f$$
$$+ \int_{\Gamma_c} a_\ell^j a_\ell^k \partial_k v^i (\eta^i - x^i) N_j - a_i^k q(\eta^i - x^i) N_k \; d\Gamma_c = \int_{\Omega_f} (f_f^i \circ \eta)(\eta^i - x^i) \; d\Omega_f.$$

Integrating in time yields

$$\int_{\Omega_f} v^i(\eta^i - x^i)\Big|_0^t - \int_0^t v^i\eta^i_t + a^j_\ell a^k_\ell \partial_k v^i \partial_j(\eta^i - x^i) - a^k_i q \partial_k(\eta^i - x^i)\ d\Omega_f\ ds$$

$$+ \int_0^t \int_{\Gamma_c} a^j_\ell a^k_\ell \partial_k v^i (\eta^i - x^i) N_j - a^k_i q(\eta^i - x^i) N_k\ d\Gamma_c\ ds$$

$$= \int_0^t \int_{\Omega_f} (f^i_f \circ \eta)(\eta^i - x^i)\ d\Omega_f\ ds.$$

Recall that $\eta_t = v$ and $\eta(x, 0) = x$. These facts imply

$$\int_{\Omega_f} v^i(\eta^i - x^i)\ d\Omega_f - \int_0^t \int_{\Omega_f} |v^i|^2 + a^j_\ell a^k_\ell \partial_k v^i \partial_j(\eta^i - x^i) - a^k_i q \partial_k(\eta^i - x^i)\ d\Omega_f\ ds$$

$$+ \int_0^t \int_{\Gamma_c} a^j_\ell a^k_\ell \partial_k v^i (\eta^i - x^i) N_j - a^k_i q(\eta^i - x^i) N_k\ d\Gamma_c\ ds$$

$$= \int_0^t \int_{\Omega_f} (f^i_f \circ \eta)(\eta^i - x^i)\ d\Omega_f\ ds.$$

Moving all terms on the interior of the domain to the right-hand side, we have that

$$\int_0^t \int_{\Gamma_c} a^j_\ell a^k_\ell \partial_k v^i (\eta^i - x^i) N_j - a^k_i q(\eta^i - x^i) N_k\ d\Gamma_c\ ds$$

$$= \int_0^t \int_{\Omega_f} (f^i_f \circ \eta)(\eta^i - x^i)\ d\Omega_f\ ds - \int_{\Omega_f} v^i(\eta^i - x^i)\ d\Omega_f$$

$$+ \int_0^t \int_{\Omega_f} \left(|v|^2 - a^j_\ell a^k_\ell \partial_k v^i \partial_j(\eta^i - x^i) + a^k_i q \partial_k(\eta^i - x^i) \right) ds\ d\Omega_f. \quad (3.2.27)$$

Plugging (3.2.27) into (3.2.26) yields

$$\begin{aligned}
&\int_0^t \int_{\Gamma_c} \partial_j w^i w^i N_j \; d\Gamma_c \; ds \\
&\quad = \int_0^t \int_{\Omega_f} (f_f^i \circ \eta)(\eta^i - x^i) \; d\Omega_f \; ds - \int_{\Omega_f} v^i(\eta^i - x^i) \; d\Omega_f \\
&\qquad + \int_0^t \int_{\Omega_f} |v|^2 - a_\ell^j a_\ell^k \partial_k v^i \partial_j(\eta^i - x^i) + a_i^k q \partial_k(\eta^i - x^i) \;\; ds \; d\Omega_f. \qquad (3.2.28)
\end{aligned}$$

Observe that $v = \eta_t = (\eta - x)_t$. Furthermore,

$$\begin{aligned}
\partial_t\big[a_\ell^j a_\ell^k \partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i)\big] &= \partial_t(a_\ell^j a_\ell^k)\partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i) \\
&\quad + a_\ell^j a_\ell^k \partial_k(\eta^i - x^i)_t \partial_j(\eta^i - x^i) \\
&\quad + a_\ell^j a_\ell^k \partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i)_t \\
&= \partial_t(a_\ell^j a_\ell^k)\partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i) \\
&\quad + 2a_\ell^j a_\ell^k \partial_k(\eta^i - x^i)_t \partial_j(\eta^i - x^i)
\end{aligned}$$

Solving for $a_\ell^j a_\ell^k \partial_k(\eta^i - x^i)_t \partial_j(\eta^i - x^i) = a_\ell^j a_\ell^k \partial_k v^i \partial_j(\eta^i - x^i)$, we have

$$\begin{aligned}
a_\ell^j a_\ell^k \partial_k v^i \partial_j(\eta^i - x^i) &= \frac{1}{2}\partial_t\big[a_\ell^j a_\ell^k \partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i)\big] \\
&\quad - \frac{1}{2}\partial_t(a_\ell^j a_\ell^k)\partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i)
\end{aligned}$$

This implies that

$$\begin{aligned}
\int_0^t \int_{\Omega_f} a_\ell^j a_\ell^k \partial_k v^i \partial_j(\eta^i - x^i) ds \; d\Omega_f &= \int_0^t \int_{\Omega_f} \bigg(\frac{1}{2}\partial_t\big[a_\ell^j a_\ell^k \partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i)\big] \\
&\quad - \frac{1}{2}\partial_t(a_\ell^j a_\ell^k)\partial_k(\eta^i - x^i)\bigg) \; ds \; d\Omega_f \\
&= \int_{\Omega_f} \frac{1}{2} a_\ell^j a_\ell^k \partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i) \; d\Omega_f
\end{aligned}$$

$$-\int_0^t \int_{\Omega_f} \frac{1}{2}\partial_t(a^j_\ell a^k_\ell)\partial_k(\eta^i - x^i) ds\ d\Omega_f.$$

Substituting into (3.2.28), we obtain

$$\begin{aligned} \int_0^t \int_{\Gamma_c} \partial_j w^i w^i N_j\ d\Gamma_c\ ds \ &= \int_0^t \int_{\Omega_f} (f^i_f \circ \eta)(\eta^i - x^i)\ d\Omega_f\ ds - \int_{\Omega_f} v^i(\eta^i - x^i)\ d\Omega_f \\ &\quad - \int_{\Omega_f} \frac{1}{2} a^j_\ell a^k_\ell \partial_k(\eta^i - x^i)\partial_j(\eta^i + x^i)\ d\Omega_f + \int_0^t \int_{\Omega_f} \Bigg(|v|^2 \\ &\quad + \frac{1}{2}\partial_t(a^j_\ell a^k_\ell)\partial_k(\eta^i - x^i) ds + a^k_i q \partial_k(\eta^i - x^i) \Bigg)\ ds\ d\Omega_f. \end{aligned} \tag{3.2.29}$$

We apply (3.2.29) to the boundary term in (3.2.24) to obtain

$$\begin{aligned} &\int_{\Omega_e} \left(\int_0^t |\nabla w|^2 + \beta |w|^2\ ds \right) + \frac{\alpha}{2}|w|^2\ d\Omega_e \\ &= \int_{\Omega_e} \frac{\alpha}{2}|w_0|^2\ d\Omega_e + \int_0^t \int_{\Omega_e} |w_t|^2\ d\Omega_e\ ds - \int_{\Omega_e} w^i_t w^i\ d\Omega_e + \int_{\Omega_e} w^i_1 w^i_0\ d\Omega_e \\ &+ \int_0^t \int_{\Omega_f} (f^i_f \circ \eta)(\eta^i - x^i)\ d\Omega_f\ ds - \int_{\Omega_f} v^i(\eta^i - x^i)\ d\Omega_f \\ &- \int_{\Omega_f} \frac{1}{2} a^j_\ell a^k_\ell \partial_k(\eta^i - x^i)\partial_j(\eta^i + x^i)\ d\Omega_f \\ &+ \int_0^t \int_{\Omega_f} |v|^2 + \frac{1}{2}\partial_t(a^j_\ell a^k_\ell)\partial_k(\eta^i - x^i) + a^k_i q \partial_k(\eta^i - x^i)\ ds\ d\Omega_f \\ &+ \int_0^t \int_{\Omega_e} f^i_e w^i\ d\Omega_e\ ds, \end{aligned}$$

which can be used in (3.2.28) to write

$$
\begin{aligned}
&\int_0^t \int_{\Omega_e} |\nabla w|^2 + \beta |w|^2 \; d\Omega_e \; ds + \frac{\alpha}{2} \int_{\Omega_e} |w|^2 \; d\Omega_e \\
&= \frac{\alpha}{2} \int_{\Omega_e} |w_0|^2 \; d\Omega_e + \int_0^t \int_{\Omega_e} |w_t|^2 \; d\Omega_e \; ds - \int_{\Omega_e} w_t^i w^i \; d\Omega_e + \int_{\Omega_e} w_1^i w_0^i \; d\Omega_e \\
&\quad - \int_{\Omega_f} v^i(\eta^i - x^i) \; d\Omega_f + \int_0^t \int_{\Omega_f} |v|^2 \; d\Omega_f \; ds - \frac{1}{2} \int_{\Omega_f} a_\ell^j a_\ell^k \partial_k(\eta^i - x^i) \partial_j(\eta^i + x^i) \; d\Omega_f \\
&\quad + \frac{1}{2} \int_0^t \int_{\Omega_f} \partial_t(a_\ell^j a_\ell^k) \partial_k(\eta^i - x^i) ds + \int_0^t \int_{\Omega_f} a_i^k q \partial_k(\eta^i - x^i) \;\; d\Omega_f \; ds \\
&\quad + \int_0^t \int_{\Omega_f} (f_f^i \circ \eta)(\eta^i - x^i) \; d\Omega_f \; ds + \int_0^t \int_{\Omega_e} f_e^i w^i \; d\Omega_e \; ds. \qquad (3.2.30)
\end{aligned}
$$

Next, we differentiate $a_i^k \partial_k(\eta^i - x^i)$ in time to obtain

$$\partial_t[a_i^k \partial_k(\eta^i - x^i)] \;\; = \;\; \partial_t a_i^k \partial_k(\eta^i - x^i) + a_i^k \partial_k(\eta^i - x^i)_t.$$

Solving for $a_i^k \partial_k(\eta^i - x^i)$ yields

$$a_i^k \partial_k(\eta^i - x^i) \;\; = \;\; \int_0^s \partial_t a_i^k \partial_k(\eta^i - x^i) + a_i^k \partial_k(\eta^i - x^i)_t \; ds.$$

Integrating in time and space, we have

$$
\begin{aligned}
&\int_0^t \int_{\Omega_f} a_i^k q \partial_k(\eta^i - x^i) \;\; d\Omega_f \; ds \\
&\quad = \int_0^t \int_{\Omega_f} q(s) a_i^k \partial_k(\eta^i - x^i) \;\; d\Omega_f \; ds
\end{aligned}
$$

$$= \int_0^t \int_{\Omega_f} q(s) \Big(\int_0^s \partial_t a_i^k \partial_k (\eta^i - x^i) + a_i^k \partial_k (\eta^i - x^i)_t \; d\tau \quad d\Omega_f \Big) \; ds$$

$$= \int_0^t \int_{\Omega_f} q(s) \int_0^s \partial_t a_i^k \partial_k (\eta^i - x^i) \; d\tau \; d\Omega_f \; ds,$$

where the divergence-free condition was used. Substituting into (3.2.30), we obtain

$$\begin{aligned}
&\int_0^t \int_{\Omega_e} |\nabla w|^2 + \beta \int_0^t \int_{\Omega_e} |w|^2 \; d\Omega_e \; ds + \frac{\alpha}{2} \int_{\Omega_e} |w|^2 \; d\Omega_e \\
&\quad + \frac{1}{2} \int_{\Omega_f} a_\ell^j a_\ell^k \partial_k (\eta^i - x^i) \partial_j (\eta^i + x^i) \; d\Omega_f \\
&\quad = \frac{\alpha}{2} \int_{\Omega_e} |w_0|^2 \; d\Omega_e + \int_0^t \int_{\Omega_e} |w_t|^2 \; d\Omega_e \; ds - \int_{\Omega_e} w_t^i w^i \; d\Omega_e + \int_{\Omega_e} w_1^i w_0^i \; d\Omega_e \\
&\quad - \int_{\Omega_f} v^i (\eta^i - x^i) \; d\Omega_f + \int_0^t \int_{\Omega_f} |v|^2 \; d\Omega_f \; ds + \frac{1}{2} \int_0^t \int_{\Omega_f} \partial_t (a_\ell^j a_\ell^k) \partial_k (\eta^i - x^i) ds \\
&\quad + \int_0^t \int_{\Omega_f} q(s) \int_0^s \partial_t a_i^k \partial_k (\eta^i - x^i) \; d\tau \; d\Omega_f \; ds + \int_0^t \int_{\Omega_f} (f_f^i \circ \eta)(\eta^i - x^i) \; d\Omega_f \; ds \\
&\quad + \int_0^t \int_{\Omega_e} f_e^i w^i \; d\Omega_e \; ds. \qquad (3.2.31)
\end{aligned}$$

For the fourth term on the left side of (3.2.31), we use the ellipticity of a to obtain

$$\begin{aligned}
&\int_0^t \int_{\Omega_e} |\nabla w|^2 + \beta \int_0^t \int_{\Omega_e} |w|^2 \; d\Omega_e \; ds + \frac{\alpha}{2} \int_{\Omega_e} |w|^2 \; d\Omega_e + \frac{1}{2} \int_{\Omega_f} |\nabla (\eta - x)|^2 \; d\Omega_f \\
&\quad \leq \frac{\alpha}{2} \int_{\Omega_e} |w_0|^2 \; d\Omega_e + \int_0^t \int_{\Omega_e} |w_t|^2 \; d\Omega_e \; ds - \int_{\Omega_e} w_t^i w^i \; d\Omega_e + \int_{\Omega_e} w_1^i w_0^i \; d\Omega_e
\end{aligned}$$

$$-\int_{\Omega_f} v^i(\eta^i - x^i)\ d\Omega_f + \int_0^t \int_{\Omega_f} |v|^2\ d\Omega_f\ ds + \frac{1}{2}\int_0^t \int_{\Omega_f} \partial_t(a^j_\ell a^k_\ell)\partial_k(\eta^i - x^i) ds$$

$$+ \int_0^t \int_{\Omega_f} q(s) \int_0^s \partial_t a^k_i \partial_k(\eta^i - x^i)\ d\tau\ d\Omega_f\ ds + \int_0^t \int_{\Omega_f} (f^i_f \circ \eta)(\eta^i - x^i)\ d\Omega_f\ ds$$

$$+ \int_0^t \int_{\Omega_e} f^i_e w^i\ d\Omega_e\ ds. \tag{3.2.32}$$

For the pointwise terms on the right-hand side, we use

$$-\int_{\Omega_e} w^i_t w^i\ d\Omega_e \ \leq\ \frac{\alpha}{4}\|w(t)\|^2_{L^2} + \frac{1}{\alpha}\|w_t(t)\|^2_{L^2}. \tag{3.2.33}$$

and

$$-\int_{\Omega_f} v^i(\eta^i - x^i)\ d\Omega_e \ \leq\ \epsilon\|\eta(\cdot,t) - x\|^2_{L^2} + C_\epsilon\|v(t)\|^2_{L^2}. \tag{3.2.34}$$

Since $\eta - x = 0$ on Γ_f, we can apply the Poincaré inequality to absorb the first term on the right side of (3.2.34) into the left side of (3.2.32). Thus,

$$\int_0^t \|\nabla w\|^2_{L^2}\ ds + \beta\int_0^t \|w\|^2_{L^2}\ ds + \frac{\alpha}{4}\|w(t)\|^2_{L^2} + \frac{1}{C}\|\nabla(\eta(\cdot,t) - x)\|^2_{L^2}$$

$$\leq CE(0) + \int_0^t \|w_t\|^2_{L^2}\ ds + C\|w_t(t)\|^2_{L^2} + C\|v(t)\|^2_{L^2} + \int_0^t \|v\|^2_{L^2}\ ds$$

$$+ \int_0^t (\tilde{R}(s), \eta(x,s) - x)\ ds + \int_0^t \int_{\Omega_f} (f^i_f \circ \eta)(\eta^i - x^i)\ d\Omega_f\ ds + \int_0^t \int_{\Omega_e} f^i_e w^i\ d\Omega_e\ ds. \tag{3.2.35}$$

Using Cauchy-Schwarz and Young's inequality, and then absorbing terms, we obtain the desired result

$$
\begin{aligned}
&\int_0^t \|\nabla w\|_{L^2}^2 \, ds + \beta \int_0^t \|w\|_{L^2}^2 \, ds + \frac{\alpha}{4}\|w(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla(\eta(\cdot,t)-x)\|_{L^2}^2 \\
&\quad \le CE(0) + \int_0^t \|w_t\|_{L^2}^2 \, ds + C\|w_t(t)\|_{L^2}^2 + C\|v(t)\|_{L^2}^2 + \int_0^t \|v\|_{L^2}^2 \, ds \\
&\qquad + \int_0^t (\tilde{R}(s), \eta(x,s)-x) \, ds + C_\epsilon \|f_f \circ \eta\|_{L^2(\Omega_f \times [0,T])}^2 + C_\epsilon \|f_e\|_{L^2(\Omega_e \times [0,T])}^2 \\
&\qquad + \epsilon \int_0^t \|\eta^i - x^i\|_{L^2(\Omega_f)}^2 \, ds. \qquad (3.2.36)
\end{aligned}
$$

Remark 3.2.5. *The equipartition estimate* (3.2.36) *will be necessary in establishing the decay of solutions required to extend the time of existence for all $t > 0$. We note that Lemma 3.2.4 is given for the interval $[0,t]$, and a change is required for the general case of $[t_0,t]$ due to the fact that the initial data differs from the data at $t_0 > 0$. We refer the reader to [36] for the adjustments for the general case.*

□

In order to have a complete characterization of the first energy level, we multiply (3.2.22) by a small parameter $\overline{\epsilon} > 0$ and add it to (3.2.10) to give

$$
\begin{aligned}
&E(t) + \int_0^t D(s) \, ds + \overline{\epsilon} \int_0^t \|\nabla w\|_{L^2}^2 \, ds + \overline{\epsilon}\beta \int_0^t \|w\|_{L^2}^2 \, ds + \overline{\epsilon}\frac{\alpha}{4}\|w(t)\|_{L^2}^2 \\
&+ \frac{\overline{\epsilon}}{C}\|\nabla(\eta(\cdot,t)-x)\|_{L^2}^2 \\
&\quad \le CE(0) + \overline{\epsilon} \int_0^t \|w_t\|_{L^2}^2 \, ds + C\overline{\epsilon}\|w_t(t)\|_{L^2}^2 + C\overline{\epsilon}\|v(t)\|_{L^2}^2 + \overline{\epsilon} \int_0^t \|v\|_{L^2}^2 \, ds \\
&\qquad + C\int_0^t (\tilde{R}(s), \eta(x,s)-x) \, ds + C_\epsilon \|f_f \circ \eta\|_{L^2(\Omega_f \times [0,T])}^2 + C_\epsilon \|f_e\|_{L^2(\Omega_e \times [0,T])}^2
\end{aligned}
$$

$$+\epsilon \int_0^t \|\eta - x\|^2_{L^2(\Omega_f)}\ ds + \epsilon \int_0^t \|v\|^2_{L^2}\ ds. \tag{3.2.37}$$

We apply the Poincaré inequality to bound $D(t)$ below, i.e.

$$\int_0^t \frac{1}{C}\|v\|^2_{L^2} + \alpha\|w_t\|^2_{L^2}\ ds \le \int_0^t D(s)\ ds.$$

We also absorb the energy norms appearing on the right-hand side into the energy terms on the left-hand side. This yields

$$\begin{aligned} E(t) + \int_0^t E(s)\ ds \quad &\le \quad CE(0) + C\int_0^t (\tilde{R}(s), \eta(x,s) - x)\ ds + C_\epsilon \|f_f \circ \eta\|^2_{L^2(\Omega_f \times [0,T])} \\ &\qquad + C_\epsilon \|f_e\|^2_{L^2(\Omega_e \times [0,T])} + \epsilon \int_0^t \|\eta - x\|^2_{L^2(\Omega_f)}\ ds. \end{aligned} \tag{3.2.38}$$

Summing (3.2.21) and (3.2.38) and absorbing the integral of the velocity term, we obtain

$$\begin{aligned} &E(t) + \int_0^t E(s)\ ds + \int_0^t D(s)\ ds \\ &\quad \le CE(0) + C\int_0^t (\tilde{R}(s), \eta(x,s) - x)\ ds + C_\epsilon \|f_f \circ \eta\|^2_{L^2(\Omega_f \times [0,T])} + C_\epsilon \|f_e\|^2_{L^2(\Omega_e \times [0,T])} \\ &\quad\quad + \epsilon \int_0^t \|\eta - x\|^2_{L^2}\ ds. \end{aligned} \tag{3.2.39}$$

3.2.2 Second Level Estimates

We repeat the above analysis at the second level energy. Differentiate (3.2.4) in time to obtain

$$\begin{cases} v^i_{tt} - \partial_t\partial_j(a^j_\ell a^k_\ell \partial_k v^i) + \partial_t\partial_k(a^k_i q) = \partial_t(f^i_f \circ \eta), & \Omega_f \times (0,T), \\ a^k_i \partial_k v^i_t + \partial_t a^k_i \partial_k v^i = 0, & \Omega_f \times (0,T), \\ w^i_{ttt} - \Delta w^i_t + \alpha w^i_{tt} + \beta w^i_t = \partial_t f^i_e, & \Omega_e \times (0,T). \end{cases} \tag{3.2.40}$$

Define the second level energy $E^{(1)}(t)$ and dissipation $D^{(1)}(t)$ as follows:

$$E^{(1)}(t) = \frac{1}{2}\Big(\|v_t(t)\|^2_{L^2} + \beta\|w_t(t)\|^2_{L^2} + \beta\|w_{tt}(t)\|^2_{L^2} + \|\nabla w_t(t)\|^2_{L^2}\Big), \tag{3.2.41}$$

$$D^{(1)}(t) = \alpha\|w_{tt}(t)\|^2_{L^2} + \frac{1}{C}\|\nabla v_t(t)\|^2_{L^2}. \tag{3.2.42}$$

Lemma 3.2.6. *The energy inequality*

$$\begin{aligned} E^{(1)}(t) + \int_0^t D^{(1)}(s)\ ds \quad &\leq \quad E^{(1)}(0) + \int_0^t (R^{(1)}(s), v_t(s))\ ds + C_\epsilon\|\partial_t(f_f \circ \eta)\|^2_{L^2(\Omega_f\times[0,T])} \\ &\quad + C_\epsilon\|\partial_t f_e\|^2_{L^2(\Omega_e\times[0,T])} + \epsilon\int_0^t \|v_t\|^2_{L^2}\ ds, \end{aligned} \tag{3.2.43}$$

holds for all $t \in [0,T]$, where

$$\begin{aligned} \int_0^t (R^{(1)}(s), v(s))\ ds \quad &= \quad -\int_0^t\int_{\Omega_f} \partial_t(a^j_\ell a^k_\ell)\partial_k v^i \partial_j v^i_t\ dx\ ds + \int_0^t\int_{\Omega_f} \partial_t a^k_i q \partial_k v^i_t\ dx\ ds \\ &\quad - \int_0^t\int_{\Omega_f} \partial_t a^k_i \partial_t q \partial_k v^i\ dx\ ds. \end{aligned} \tag{3.2.44}$$

Proof. In (3.2.40), multiply the fluid equation by v^i_t, the solid equation by w^i_{tt}, and integrate in time and space. Following the strategy used in the first level energy estimates, we obtain

$$
\begin{aligned}
E^{(1)}(t)+\int_0^t D^{(1)}(s)\ ds \quad &\leq \quad E^{(1)}(0)-\int_0^t\int_{\Omega_f}\partial_t(a_\ell^j a_\ell^k)\partial_k v^i\partial_j v_t^i\ dx\ ds \\
&\quad +\int_0^t\int_{\Omega_f}\partial_t(a_i^k q)\partial_k v_t^i\ dx\ ds+C_\epsilon\|\partial_t(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])} \\
&\quad +C_\epsilon\|\partial_t f_e\|^2_{L^2(\Omega_e\times[0,T])}+\epsilon\int_0^t\|v_t\|^2_{L^2}\ ds \\
&\leq \quad E^{(1)}(0)-\int_0^t\int_{\Omega_f}\partial_t(a_\ell^j a_\ell^k)\partial_k v^i\partial_j v_t^i\ dx\ ds \\
&\quad +\int_0^t\int_{\Omega_f}\partial_t a_i^k q\partial_k v_t^i\ dx\ ds+\int_0^t\int_{\Omega_f}a_i^k q_t\partial_k v_t^i\ dx\ ds \\
&\quad +C_\epsilon\|\partial_t(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])}+C_\epsilon\|\partial_t f_e\|^2_{L^2(\Omega_e\times[0,T])} \\
&\quad +\epsilon\int_0^t\|v_t\|^2_{L^2}\ ds. \qquad (3.2.45)
\end{aligned}
$$

Using the divergence-free condition, we can express

$$a_i^k\partial_k v_t^i=-\partial_t a_i^k\partial_k v^i,$$

which allows us to conclude

$$
\begin{aligned}
E^{(1)}(t)+\int_0^t D^{(1)}(s)\ ds \quad &\leq \quad E^{(1)}(0)+\int_0^t(R^{(1)}(s),v_t(s))\ ds+\ C_\epsilon\|\partial_t(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])} \\
&\quad +\ C_\epsilon\|\partial_t f_e\|^2_{L^2(\Omega_e\times[0,T])}+\epsilon\int_0^t\|v_t\|^2_{L^2}\ ds. \qquad (3.2.46)
\end{aligned}
$$

□

Next we provide estimates for w_t and ∇w_t to completely characterize the second level energy.

Lemma 3.2.7. *We have*

$$\int_0^t \|\nabla w_t\|_{L^2}^2 \ ds + \int_0^t \|w_t\|_{L^2}^2 \ ds + \frac{\alpha}{4}\|w_t(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla v(t)\|_{L^2}^2$$
$$\leq CE^{(1)}(0) + C\|w_{tt}(t)\|_{L^2}^2 + C\|v_t(t)\|_{L^2}^2 + \int_0^t \|w_{tt}\|_{L^2}^2 \ ds + \int_0^t \|v_t\|_{L^2}^2 \ ds$$
$$+ \int_0^t (\tilde{R}^{(1)}(s), v_t(s)) \ ds + C\|\nabla v(0)\|_{L^2}^2 + C_\epsilon \|\partial_t f_e\|_{L^2(\Omega_e\times[0,T])}^2,$$
$$+C_\epsilon \|\partial_t (f_f \circ \eta)\|_{L^2(\Omega_f\times[0,T])}^2 + \epsilon \int_0^t \|v\|_{L^2}^2, \tag{3.2.47}$$

where

$$\int_0^t (\tilde{R}^{(1)}(s), v_t(s)) \ ds \ = \ -\frac{1}{2}\int_0^t \int_{\Omega_f} \partial_t(a_\ell^j a_\ell^k)\partial_k v^i \partial_j v^i \ d\Omega_f \ ds + \int_0^t \int_{\Omega_f} \partial_t a_i^k q \partial_k v^i \ d\Omega_f \ ds. \tag{3.2.48}$$

The constant C depends on α.

Proof. We multiply the solid equation in (3.2.40) by w_t^i, sum in i, and integrate in time and space to obtain

$$\int_0^t \int_{\Omega_e} w_{ttt}w_t - \Delta w_t w_t + \alpha w_{tt} w_t + \beta w_t^2 \ d\Omega_e \ ds \ = \ \int_0^t \int_{\Omega_e} \partial_t f_e w_t \ d\Omega_e \ ds. \tag{3.2.49}$$

We write $\partial_t(w_{tt}w_t) = w_{ttt}w_t + w_{tt}^2$, which implies

$$w_{ttt}w_t = \partial_t(w_{tt}w_t) - w_{tt}^2.$$

Using this fact in (3.2.49), we obtain

$$\int_0^t \int_{\Omega_e} \partial_t(w_{tt}w_t) - w_{tt}^2 - \Delta w_t w_t + \alpha w_{tt} w_t + \beta w_t^2 \; d\Omega_e \; ds \;\; = \;\; \int_0^t \int_{\Omega_e} \partial_t f_e w_t \; d\Omega_e \; ds.$$

Integrating by parts in space, we arrive at the following:

$$\begin{aligned}
&\int_0^t \|\nabla w_t\|_{L^2}^2 \; ds + \beta \int_0^t \|w_t\|^2 \; ds + \frac{\alpha}{2}\|w_t(t)\|_{L^2}^2 \\
&\quad = \frac{\alpha}{2}\|w_1\|_{L^2}^2 + \int_0^t \|w_{tt}\|_{L^2}^2 \; ds - \int_{\Omega_e} w_{tt}w_t\Big|_0^t \; d\Omega_e + \int_0^t \int_{\Gamma_c} \frac{\partial w_t}{\partial N} w_t \; d\Gamma_c \; ds \\
&\qquad + \int_0^t \int_{\Omega_e} \partial_t f_e w_t \; d\Omega_e \; ds. \qquad (3.2.50)
\end{aligned}$$

Next, we multiply the fluid component of (3.2.40) by v^i, sum in i, and integrate in time and space, i.e.

$$\int_0^t \int_{\Omega_f} v_{tt}^i v^i - \partial_t \partial_j (a_\ell^j a_\ell^k \partial_k v^i) v^i + \partial_t \partial_k (a_i^k q) v^i \; d\Omega_f \; ds \;\; = \;\; \int_0^t \int_{\Omega_f} \partial_t (f_f^i \circ \eta) v^i \; d\Omega_f \; ds.$$

Using a similar approach as with the wave equation, we have

$$\begin{aligned}
&\int_{\Omega_f} v_t^i v^i \Big|_0^t \; ds - \int_0^t \int_{\Omega_f} (v_t^i)^2 \; d\Omega_f \; ds + \int_0^t \int_{\Omega_f} \partial_t (a_\ell^j a_\ell^k \partial_k v^i) \partial_j v^i - \partial_t (a_i^k q) \partial_k v^i \; d\Omega_f \; ds \\
&- \int_0^t \int_{\Gamma_c} \partial_t (a_\ell^j a_\ell^k \partial_k v^i) v^i N_j \; d\Gamma_c \; ds + \int_0^t \int_{\Gamma_c} \partial_t (a_i^k q) v^i N_k \; d\Gamma_c \; ds \\
&\qquad = \int_0^t \int_{\Omega_f} \partial_t (f_f^i \circ \eta) v^i \; d\Omega_f \; ds.
\end{aligned}$$

We observe that $\partial_t (a_\ell^j a_\ell^k \partial_k v^i \partial_j v^i) = \partial_t (a_\ell^j a_\ell^k \partial_k v^i) \partial_j v^i + (a_\ell^j a_\ell^k \partial_k v^i) \partial_j v_t^i$, which allows us to write

$$\partial_t(a^j_\ell a^k_\ell \partial_k v^i)\partial_j v^i \;=\; \partial_t(a^j_\ell a^k_\ell \partial_k v^i \partial_j v^i) - a^j_\ell a^k_\ell \partial_k v^i \partial_j v^i_t, \tag{3.2.51}$$

and

$$\partial_t(a^j_\ell a^k_\ell \partial_k v^i)\partial_j v^i \;=\; \partial_t(a^j_\ell a^k_\ell)\partial_k v^i \partial_j v^i + a^j_\ell a^k_\ell \partial_k v^i_t \partial_j v^i. \tag{3.2.52}$$

Taking (3.2.51) and (3.2.52)

$$\partial_t(a^j_\ell a^k_\ell \partial_k v^i)\partial_j v^i \;=\; \frac{1}{2}\partial_t(a^j_\ell a^k_\ell \partial_k v^i \partial_j v^i) + \frac{1}{2}\partial_t(a^j_\ell a^k_\ell)\partial_k v^i \partial_j v^i.$$

Also,

$$\partial_t(a^k_i q)\partial_k v^i \;=\; \partial_t a^k_i q \partial_k v^i$$

by the divergence-free property of v. Therefore,

$$\begin{aligned}
&\int\limits_{\Omega_f} v^i_t v^i \Big|_0^t \, ds - \int\limits_0^t \int\limits_{\Omega_f} (v^i_t)^2 + \frac{1}{2}\partial_t(a^j_\ell a^k_\ell \partial_k v^i \partial_j v^i) + \frac{1}{2}\partial_t(a^j_\ell a^k_\ell)\partial_k v^i \partial_j v^i - \partial_t a^k_i q \partial_k v^i \; d\Omega_f \; ds \\
&- \int\limits_0^t \int\limits_{\Gamma_c} \partial_t(a^j_\ell a^k_\ell \partial_k v^i) v^i N_j \; d\Gamma_c \; ds + \int\limits_0^t \int\limits_{\Gamma_c} \partial_t a^k_i q v^i N_k \; d\Gamma_c \; ds = \int\limits_0^t \int\limits_{\Omega_f} \partial_t(f^i_f \circ \eta) v^i \; d\Omega_f \; ds.
\end{aligned} \tag{3.2.53}$$

Next, we sum (3.2.53) and (3.2.50). First, we focus on the boundary terms. Using the velocity and stress matching conditions, we have

$$\begin{aligned}
&\int\limits_0^t \partial_j w^i_t w^i_t N_j \; d\Gamma_c \; ds - \int\limits_0^t \int\limits_{\Gamma_c} \partial_t(a^j_\ell a^k_\ell \partial_k v^i) v^i N_j \; d\Gamma_c \; ds + \int\limits_0^t \int\limits_{\Gamma_c} \partial_t a^k_i q v^i N_k \; d\Gamma_c \; ds \\
&\quad = \int\limits_0^t \partial_j w^i_t v^i N_j \; d\Gamma_c \; ds - \int\limits_0^t \int\limits_{\Gamma_c} \partial_t(a^j_\ell a^k_\ell \partial_k v^i) v^i N_j \; d\Gamma_c \; ds + \int\limits_0^t \int\limits_{\Gamma_c} \partial_t a^k_i q v^i N_k \; d\Gamma_c \; ds \\
&\quad = 0.
\end{aligned} \tag{3.2.54}$$

Therefore, we obtain

$$\int_0^t \|\nabla w_t\|_{L^2}^2 \, ds + \beta \int_0^t \|w_t\|^2 \, d\Omega_e \, ds + \frac{\alpha}{2}\|w_t(t)\|_{L^2}^2 + \frac{1}{2}\int_{\Omega_f} \partial_t(a_\ell^j a_\ell^k \partial_k v^i \partial_j v^i) \, d\Omega_f$$
$$= \frac{\alpha}{2}\|w_1\|_{L^2}^2 + \int_0^t \|w_{tt}\|_{L^2}^2 \, ds - \int_{\Omega_e} w_{tt}^i w_t^i \Big|_0^t \, d\Omega_e + \int_{\Omega_f} v_t^i v^i \Big|_0^t \, ds + \int_0^t \int_{\Omega_f} \|v_t\|^2 \, ds$$
$$+ \frac{1}{2}\int_{\Omega_f} a_\ell^j a_\ell^k \partial_k v^i \partial_j v^i \, d\Omega_f \Big|_{t=0} - \frac{1}{2}\int_{\Omega_f} \partial_t(a_\ell^j a_\ell^k)\partial_k v^i \partial_j v^i \, d\Omega_f \, ds$$
$$+ \int_0^t \int_{\Omega_f} \partial_t a_i^k q \partial_k v^i \, d\Omega_f \, ds + \int_0^t \int_{\Omega_e} \partial_t f_e w_t \, d\Omega_e \, ds + \int_0^t \int_{\Omega_f} \partial_t(f_f^i \circ \eta) v^i \, d\Omega_f \, ds.$$

We also use the following estimates for the two piecewise terms on the right-hand side:

$$-\int_{\Omega_e} w_{tt}^i w_t^i \, d\Omega_e \le \frac{\alpha}{4}\|w_t(t)\|_{L^2}^2 + \frac{1}{\alpha}\|w_{tt}\|_{L^2}^2$$

and

$$-\int_{\Omega_f} v_t^i v^i \, d\Omega_f \le \epsilon\|\nabla v(t)\|_{L^2}^2 + C_\epsilon \|v_t\|_{L^2}^2,$$

where $\epsilon > 0$ is sufficiently small. Using ellipticity of a, estimating forcing terms, and absorbing terms onto the left-hand side, we obtain the desired result, i.e.

$$\int_0^t \|\nabla w_t\|_{L^2}^2 \, ds + \int_0^t \|w_t\|_{L^2}^2 \, ds + \frac{\alpha}{4}\|w_t(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla v(t)\|_{L^2}^2$$
$$\le CE^{(1)}(0) + C\|w_{tt}(t)\|_{L^2}^2 + C\|v_t(t)\|_{L^2}^2 + \int_0^t \|w_{tt}\|_{L^2}^2 \, ds + \int_0^t \|v_t\|_{L^2}^2 \, ds$$
$$+ \int_0^t (\tilde{R}^{(1)}(s), v_t(s)) \, ds + C\|\nabla v(0)\|_{L^2}^2 + C\|\partial_t f_e\|_{L^2(\Omega_e \times [0,T])}^2$$

$$+C\|\partial_t(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])}+\epsilon\int_0^t \|v\|^2_{L^2}. \tag{3.2.55}$$

□

Multiplying (3.2.47) by a small parameter and adding it to (3.2.43) (similarly to how we derived (3.2.38)), we obtain

$$\begin{aligned} E^{(1)}(t)+\int_0^t E^{(1)}(s)\ ds \ \ \leq\ \ & CE^{(1)}(0)+C\|\nabla v(0)\|^2_{L^2}+\int_0^t (R^{(1)}(s),v_t(s))\ ds \\ & +\int_0^t (\tilde{R}^{(1)}(s),v(s))\ ds + C_\epsilon\|\partial_t(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])} \\ & +C_\epsilon\|\partial_t f_e\|^2_{L^2(\Omega_e\times[0,T])}+\epsilon\int_0^t \|v\|^2_{L^2}\ ds. \end{aligned} \tag{3.2.56}$$

Summing (3.2.43) and (3.2.56) and absorbing the integral of the velocity term, we obtain

$$\begin{aligned} & E^{(1)}(t)+\int_0^t E^{(1)}(s)\ ds+\int_0^t D^{(1)}(s)\ ds \\ & \leq CE^{(1)}(0)+C\|\nabla v(0)\|^2_{L^2}+C\int_0^t (R^{(1)}(s),v_t(s))\ ds+\int_0^t (\tilde{R}^{(1)}(s),v(s))\ ds \\ & \quad + C_\epsilon\|\partial_t(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])}+\ C_\epsilon\|\partial_t f_e\|^2_{L^2(\Omega_e\times[0,T])}+\epsilon\int_0^t \|v_t\|^2_{L^2}\ ds \\ & \quad +\epsilon\int_0^t \|v\|^2_{L^2}\ ds. \end{aligned} \tag{3.2.57}$$

3.2.3 Third Level Estimates

We now differentiate (3.2.4) twice in time to obtain

$$\begin{cases} v^i_{ttt} - \partial_{tt}\partial_j(a^j_\ell a^k_\ell \partial_k v^i) + \partial_{tt}\partial_k(a^k_i q) = \partial_{tt} f^i_f, & \Omega_f \times (0,T), \\ a^k_i \partial_k v^i_{tt} + 2\partial_t a^k_i \partial_k v^i_t + \partial_{tt} a^k_i \partial_k v^i = 0, & \Omega_f \times (0,T), \\ w^i_{tttt} - \Delta w^i_{tt} + \alpha w^i_{ttt} + \beta w^i_{tt} = \partial_{tt} f^i_e, & \Omega_e \times (0,T). \end{cases} \tag{3.2.58}$$

Define the third level energy $E^{(2)}(t)$ and dissipation $D^{(2)}(t)$ as follows:

$$E^{(2)}(t) = \frac{1}{2}\Big(\|v_{tt}(t)\|^2_{L^2} + \beta\|w_{tt}(t)\|^2_{L^2} + \beta\|w_{ttt}(t)\|^2_{L^2} + \|\nabla w_{tt}(t)\|^2_{L^2}\Big), \tag{3.2.59}$$

$$D^{(2)}(t) = \alpha\|w_{ttt}(t)\|^2_{L^2} + \frac{1}{C}\|\nabla v_{tt}(t)\|^2_{L^2}. \tag{3.2.60}$$

Then we have the following energy inequality.

Lemma 3.2.8. *The energy inequality*

$$\begin{aligned} E^{(2)}(t) + \int_0^t D^{(2)}(s)\ ds \quad &\le \quad E^{(2)}(0) + \int_0^t (R^{(2)}(s), v_t(s))\ ds + C_\epsilon \|\partial_{tt}(f_f \circ \eta)\|^2_{L^2(\Omega_f \times [0,T])} \\ &+ C_\epsilon \|\partial_{tt} f_e\|^2_{L^2(\Omega_e \times [0,T])} + \epsilon \int_0^t \|v_{tt}\|^2_{L^2}\ ds, \end{aligned} \tag{3.2.61}$$

holds for all $t \in [0,T]$, *where*

$$\begin{aligned} \int_0^t (R^{(2)}(s), v_{tt}(s))\ ds \quad &= \quad 2\int_0^t \int_{\Omega_f} \partial_t(a^j_\ell a^k_\ell)\partial_k v^i_t \partial_j v^i_{tt}\ dx\ ds + \int_0^t \int_{\Omega_f} \partial_{tt}(a^j_\ell a^k_\ell)\partial_k v^i \partial_j v^i_{tt}\ dx\ ds \\ &- \int_0^t \int_{\Omega_f} \partial_{tt}(a^k_i q)\partial_k v^i_{tt}\ dx\ ds. \end{aligned} \tag{3.2.62}$$

Proof. In (3.2.58), we multiply the fluid equation by v^i_{tt} and the solid equation by w^i_{ttt} and integrate in time and space. The desired estimate is obtained via a similar argument as the previous energy levels. $\square$

As before, we also require estimates on w_{tt} and its gradient.

Lemma 3.2.9. *We have*

$$\int_0^t \|\nabla w_{tt}\|_{L^2}^2 \; ds + \beta \int_0^t \|w_{tt}\|_{L^2}^2 \; ds + \frac{\alpha}{4}\|w_{tt}(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla v_t(t)\|_{L^2}^2$$
$$\le CE^{(2)}(0) + C\|w_{ttt}(t)\|_{L^2}^2 + C\|v_{tt}(t)\|_{L^2}^2 + \int_0^t (\tilde{R}^{(2)}(s), v_t(s)) \; ds + C\|\nabla v_t(0)\|_{L^2}^2$$
$$+C\|\partial_{tt} f_e\|_{L^2(\Omega_e \times [0,T])}^2 + C\|\partial_{tt}(f_f \circ \eta)\|_{L^2(\Omega_f \times [0,T])}^2 + \epsilon \int_0^t \|v_t\|_{L^2}^2, \tag{3.2.63}$$

where

$$\begin{aligned}\int_0^t (\tilde{R}^{(2)}(s), v_t(s)) \; ds \;\; &= \;\; -\frac{3}{2}\int_0^t \int_{\Omega_f} \partial_t(a_\ell^j a_\ell^k)\partial_k v_t^i \partial_j v_t^i \; d\Omega_f \; ds \\ &\quad - \int_0^t \int_{\Omega_f} \partial_{tt}(a_\ell^j a_\ell^k)\partial_k v^i \partial_j v_t^i \; d\Omega_f \; ds \\ &\quad + \int_0^t \int_{\Omega_f} \partial_{tt}(a_i^k q)\partial_k v_t^i \; d\Omega_f \; ds. \end{aligned} \tag{3.2.64}$$

Proof. In (3.2.58), we multiply the fluid equation by v_t^i and the solid equation by w_{tt}^i and integrate in time and space. The derivation of the estimate follows a similar argument to the previous energy levels. $\square$

Multiplying (3.2.63) by a small parameter and adding to (3.2.61) yields

$$\begin{aligned} E^{(2)}(t) + \int_0^t E^{(2)}(s) \; ds \;\; &\le \;\; CE^{(2)}(0) + C\|\nabla v_t(0)\|_{L^2}^2 + \int_0^t (R^{(2)}(s), v_{tt}(s)) \; ds \\ &\quad +C\int_0^t (\tilde{R}^{(2)}(s), v_t(s)) \; ds \\ &\quad +C\|\partial_{tt}(f_f \circ \eta)\|_{L^2(\Omega_f \times [0,T])}^2 \end{aligned}$$

$$+C\|\partial_{tt} f_e\|^2_{L^2(\Omega_e\times[0,T])} + \epsilon \int\limits_0^t \|v_t\|^2_{L^2}\ ds. \tag{3.2.65}$$

Summing (3.2.61) and (3.2.65) and absorbing second time derivative of the velocity term, we obtain

$$\begin{aligned}
&E^{(2)}(t) + \int\limits_0^t E^{(2)}(s)\ ds + \int\limits_0^t D^{(2)}(s)\ ds \\
&\quad \le CE^{(2)}(0) + C\|\nabla v_t(0)\|^2_{L^2} + \int\limits_0^t (R^{(2)}(s), v_t(s))\ ds + C\int\limits_0^t (\tilde{R}^{(2)}(s), v_t(s))\ ds \\
&\quad\quad + C_\epsilon\|\partial_{tt}(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])} + C\|\partial_{tt} f_e\|^2_{L^2(\Omega_e\times[0,T])} + \epsilon\int\limits_0^t \|v_t\|^2_{L^2}\ ds.
\end{aligned} \tag{3.2.66}$$

3.2.4 First Order Tangential Energy Estimates

We differentiate (3.2.4) in the lateral spatial directions and define the first order tangential energies $E_m(t)$ and dissipations $D_m(t)$ for $m = 1, 2$ as follows:

$$E_m(t) = \frac{1}{2}\Big(\|\partial_m v(t)\|^2_{L^2} + \beta\|\partial_m w(t)\|^2_{L^2} + \|\partial_m w_t(t)\|^2_{L^2} + \|\nabla\partial_m w(t)\|^2_{L^2}\Big), \tag{3.2.67}$$

$$D_m(t) = \alpha\|\partial_m w_t(t)\|^2_{L^2} + \frac{1}{C}\|\nabla\partial_m v(t)\|^2_{L^2} + \gamma\left\|\frac{\partial(\partial_m w)}{\partial N}(t)\right\|^2_{L^2(\Gamma_c)}. \tag{3.2.68}$$

We have the following first level tangential energy bound:

Lemma 3.2.10.

$$\begin{aligned}
E_m(t) + \int\limits_0^t D_m(s)\ ds \quad &\le \quad E_m(0) + \int\limits_0^t (R_m(s),\ \partial_m v(s))\ ds + C_\epsilon\|\partial_m(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])} \\
&\quad + C_\epsilon\|\partial_m f_e\|^2_{L^2(\Omega_e\times[0,T])} + \epsilon\int\limits_0^t \|\partial_m v\|^2_{L^2}\ ds,
\end{aligned} \tag{3.2.69}$$

for all $t \in [0,T]$, *where*

$$\int_0^t (R_m(s), \partial_m v(s))\ ds = -\int_0^t \int_{\Omega_f} \partial_m (a^j_\ell a^k_\ell - \delta_{jk}) \partial_k v^i \partial_j \partial_m v^i\ dx\ ds$$
$$-\int_0^t \int_{\Omega_f} \partial_m (a^k_i - \delta_{ik}) q \partial_k v^i\ dx\ ds$$
$$+\int_0^t \int_{\Omega_f} \partial_m (a^k_i - \delta_{ik}) q \partial_k \partial_m v^i\ dx\ ds. \tag{3.2.70}$$

Proof. Apply ∂_m (3.2.58) and multiply the fluid equation by $\partial_m v^i$ and the wave equation by $\partial_m w^i_t$. Integrate by parts, estimate, and sum the resulting inequalities to obtain the desired result. □

Next we provide an estimate for $\nabla \partial_m w$ to complete the characterization of the first order tangential energy.

Lemma 3.2.11. *We have*

$$\int_0^t \|\nabla \partial_m w\|^2_{L^2}\ ds + \int_0^t \|\partial_m w\|^2_{L^2}\ ds + \frac{\alpha}{4}\|\partial_m w(t)\|^2_{L^2} + \frac{1}{C}\|\nabla \partial_m (\eta(x,t) - x)\|^2_{L^2}$$
$$\leq CE_m(0) + \int_0^t \|\partial_m w_t\|^2_{L^2}\ ds + C\|\partial_m w_t(t)\|^2_{L^2} + C\|\partial_m v(t)\|^2_{L^2}$$
$$+\int_0^t \|\partial_m v\|^2_{L^2}\ ds + \int_0^t (\tilde{R}_m(s), \partial_m(\eta(x,s) - x))\ ds + C_\epsilon \|\partial_m f_e\|^2_{L^2(\Omega_e \times [0,T])}$$
$$+C_\epsilon \|\partial_m (f_f \circ \eta)\|^2_{L^2(\Omega_f \times [0,T])} + \epsilon \int_0^t \|\partial_m(\eta^i - x^i)\|^2_{L^2}\ ds \tag{3.2.71}$$

for $t \in [0,T]$, *where*

$$\int_0^t (\tilde{R}_m(s), \partial_m(\eta(x,s) - x))\ ds$$

$$= -\int_0^t \int_{\Omega_f} \partial_m (a_\ell^j a_\ell^k - \delta_{jk}) \partial_k v^i \partial_j \partial_m (\eta^i - x^i) \ d\Omega_f \ ds$$

$$+ \frac{1}{2} \int_0^t \int_{\Omega_f} \partial_t (a_\ell^j a_\ell^k) \partial_k \partial_m (\eta^i - x^i) \partial_j \partial_m (\eta^i - x^i) \ d\Omega_f \ ds$$

$$+ \int_0^t \int_{\Omega_f} \partial_m (a_i^k - \delta_{jk}) q \partial_k \partial_m (\eta^i - x^i) \ d\Omega_f \ ds$$

$$+ \int_0^t \int_{\Omega_f} a_i^k \partial_m q \partial_k \partial_m (\eta^i - x^i) \ d\Omega_f \ ds. \tag{3.2.72}$$

This result is proven similarly to the previous equipartition estimates.

Using a small parameter $\bar{\epsilon}$, we can use (3.2.69) and (3.2.71). to obtain

$$\begin{aligned} E_m(t) + \int_0^t E_m(s) \ ds \quad \leq \quad & C E_m(0) + \int_0^t (R_m(s), \partial_m v(s)) \ ds + \int_0^t (\tilde{R}_m(s), \partial_m (\eta - x)) \ ds \\ & + C_\epsilon \| \partial_m (f_f \circ \eta) \|^2_{L^2(\Omega_f \times [0,T])} + \ C_\epsilon \| \partial_m f_e \|^2_{L^2(\Omega_e \times [0,T])} \\ & + \epsilon \int_0^t \| \partial_m (\eta^i - x^i) \|^2_{L^2} \ ds. \end{aligned} \tag{3.2.73}$$

Proof. The proof is analogous to that of Lemma 3.2.4. $\square$

Summing (3.2.71) and (3.2.73), we obtain

$$\begin{aligned} & E_m(t) + \int_0^t E_m(s) \ ds + \int_0^t D_m(s) \ ds \\ & \leq C E_m(0) + C \int_0^t (R_m(s), \ \partial_m v(s)) \ ds + C \int_0^t (\tilde{R}_m(s), \partial_m (\eta(x,s) - x)) \ ds \\ & \quad + C_\epsilon \| \partial_m (f_f \circ \eta) \|^2_{L^2(\Omega_f \times [0,T])} + C_\epsilon \| \partial_m f_e \|^2_{L^2(\Omega_e \times [0,T])} + C\epsilon \int_0^t \| \partial_m (\eta^i - x^i) \|^2_{L^2} \ ds. \end{aligned} \tag{3.2.74}$$

3.2.5 Second Order Tangential Energy Estimates

We differentiate (3.2.4) twice in the lateral spatial directions and define the first order tangential energies $E_{mm}(t)$ and dissipations $D_{mm}(t)$ for $m = 1, 2$ as follows:

$$E_{mm}(t) = \frac{1}{2}\Big(\|\partial_{mm} v(t)\|^2_{L^2} + \beta\|\partial_{mm} w(t)\|^2_{L^2} + \|\partial_{mm} w_t(t)\|^2_{L^2} + \|\nabla\partial_{mm} w(t)\|^2_{L^2}\Big), \tag{3.2.75}$$

$$D_{mm}(t) = \alpha\|\partial_{mm} w_t(t)\|^2_{L^2} + \frac{1}{C}\|\nabla\partial_{mm} v(t)\|^2_{L^2}. \tag{3.2.76}$$

We have the following bound on the second order tangential energy:

Lemma 3.2.12.

$$\begin{aligned} E_{mm}(t) + \int_0^t D_{mm}(s)\, ds \quad \leq \quad & E_{mm}(0) + \int_0^t (R_{mm}(s),\ \partial_{mm} v(s))\, ds \\ & + C_\epsilon \|\partial_{mm}(f_f \circ \eta)\|^2_{L^2(\Omega_f \times [0,T])} + C_\epsilon \|\partial_{mm} f_e\|^2_{L^2(\Omega_e \times [0,T])} \\ & + \epsilon \int_0^t \|\partial_{mm} v\|^2_{L^2}\, ds, \end{aligned} \tag{3.2.77}$$

for all $t \in [0,T]$, *where*

$$\begin{aligned} \int_0^t (R_{mm}(s), \partial_{mm} v(s))\, ds \quad = \quad & -2\int_0^t \int_{\Omega_f} \partial_m (a^j_\ell a^k_\ell - \delta_{jk})\partial_k \partial_m v^i \partial_j \partial_{mm} v^i\, dx\, ds \\ & - \int_0^t \int_{\Omega_f} \partial_{mm} (a^j_\ell a^k_\ell - \delta_{jk})\partial_k v^i \partial_j \partial_{mm} v^i\, dx\, ds \\ & + \int_0^t \int_{\Omega_f} \partial_{mm}(a^k_i - \delta_{ik}) q \partial_k \partial_{mm} v^i\, dx\, ds \\ & + 2\int_0^t \int_{\Omega_f} \partial_m (a^k_i - \delta_{ik}) \partial_m q \partial_k \partial_{mm} v^i\, dx\, ds \end{aligned}$$

$$+\int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm} v^i \ dx \ ds. \qquad (3.2.78)$$

Proof. Apply ∂_{mm} to (3.2.4) and multiply the fluid equation by $\partial_m v^i$ and the wave equation by $\partial_m w_t^i$. Integrate by parts, estimate, and sum the resulting inequalities. $\square$

Next we provide an estimate for $\nabla \partial_{mm} w$, which is obtained similarly as (3.2.71)

Lemma 3.2.13. *We have*

$$\begin{aligned}
&\int_0^t \|\nabla \partial_{mm} w\|_{L^2}^2 \ ds + \int_0^t \|\partial_{mm} w\|_{L^2}^2 \ ds + \frac{\alpha}{4} \|\partial_{mm} w_t(t)\|_{L^2}^2 + \frac{1}{C} \|\nabla \partial_{mm} \eta(x,t)\|_{L^2}^2 \\
&\quad \leq C E_{mm}(0) + \int_0^t \|\partial_{mm} w_t\|_{L^2}^2 \ ds + C\|\partial_{mm} w_t(t)\|_{L^2}^2 + C\|\partial_{mm} v(t)\|_{L^2}^2 \\
&\qquad + \int_0^t \|\partial_{mm} v\|_{L^2}^2 \ ds + \int_0^t (\tilde{R}_{mm}(s), \partial_m(\eta(x,s) - x)) \ ds + \ C_\epsilon \|\partial_{mm} f_e\|_{L^2(\Omega_e \times [0,T])}^2 \\
&\qquad + C_\epsilon \|\partial_{mm}(f_f \circ \eta)\|_{L^2(\Omega_f \times [0,T])}^2 + \ \epsilon \int_0^t \|\partial_{mm}(\eta^i - x^i)\|_{L^2}^2 \ ds
\end{aligned} \qquad (3.2.79)$$

for $t \in [0,T]$, *where*

$$\begin{aligned}
&\int_0^t (\tilde{R}_{mm}(s), \partial_{mm}(\eta(x,s) - x)) \ ds \\
&\quad = -\int_0^t \int_{\Omega_f} \partial_{mm}(a_\ell^j a_\ell^k - \delta_{jk}) \partial_k v^i \partial_j \partial_{mm}(\eta^i - x^i) \ d\Omega_f \ ds \\
&\qquad -2 \int_0^t \int_{\Omega_f} \partial_m(a_\ell^j a_\ell^k - \delta_{jk}) \partial_k v^i \partial_j \partial_{mm}(\eta^i - x^i) \ d\Omega_f \ ds \\
&\qquad + \frac{1}{2} \int_0^t \int_{\Omega_f} \partial_t(a_\ell^j a_\ell^k) \partial_k \partial_{mm}(\eta^i - x^i) \partial_j \partial_{mm}(\eta^i - x^i) \ d\Omega_f \ ds
\end{aligned}$$

$$+\int_0^t \int_{\Omega_f} \partial_{mm}(a_i^k - \delta_{jk}) q \partial_k \partial_{mm}(\eta^i - x^i) \, d\Omega_f \, ds$$
$$+2\int_0^t \int_{\Omega_f} \partial_m(a_i^k - \delta_{ik}) \partial_m q \partial_k \partial_{mm}(\eta^i - x^i) \, d\Omega_f \, ds$$
$$+\int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm}(\eta^i - x^i) \, d\Omega_f \, ds. \tag{3.2.80}$$

Using a small parameter $\bar{\epsilon}$, (3.2.69) and (3.2.71) imply

$$\begin{aligned} E_{mm}(t) + \int_0^t E_{mm}(s) \, ds \quad \leq \quad & CE_{mm}(0) + \int_0^t (R_{mm}(s), \partial_{mm} v(s)) \, ds \\ & + \int_0^t (\tilde{R}_{mm}(s), \partial_{mm}(\eta - x)) \, ds \\ & + C_\epsilon \|\partial_{mm}(f_f \circ \eta)\|^2_{L^2(\Omega_f \times [0,T])} \\ & + C_\epsilon \|\partial_{mm} f_e\|^2_{L^2(\Omega_e \times [0,T])} \\ & + \epsilon \int_0^t \|\partial_{mm}(\eta^i - x^i)\|^2_{L^2} \, ds. \end{aligned} \tag{3.2.81}$$

Summing (3.2.79) and (3.2.82), we obtain

$$\begin{aligned} & E_{mm}(t) + \int_0^t E_{mm}(s) \, ds + \int_0^t D_{mm}(s) \, ds \\ & \leq CE_{mm}(0) + \int_0^t (R_{mm}(s), \partial_{mm} v(s)) \, ds + \int_0^t (\tilde{R}_{mm}(s), \partial_{mm}(\eta - x)) \, ds \\ & \quad + C_\epsilon \|\partial_{mm}(f_f \circ \eta)\|^2_{L^2(\Omega_f \times [0,T])} + C_\epsilon \|\partial_{mm} f_e\|^2_{L^2(\Omega_e \times [0,T])} \\ & \quad + C\epsilon \int_0^t \|\partial_{mm}(\eta^i - x^i)\|^2_{L^2} \, ds. \end{aligned} \tag{3.2.82}$$

3.2.6 Mixed Time Tangential Energy Estimates

We now differentiate (3.2.4) once in the lateral spatial directions and once in time. Define the mixed tangential energies $E_{tm}(t)$ and dissipations $D_{tm}(t)$ for $m = 1, 2$ as follows:

$$E_{tm}(t) = \frac{1}{2}\Big(\|\partial_t\partial_m v(t)\|^2_{L^2} + \beta\|\partial_t\partial_m w(t)\|^2_{L^2} + \|\partial_t\partial_m w_t(t)\|^2_{L^2} + \|\nabla\partial_t\partial_m w(t)\|^2_{L^2}\Big), \tag{3.2.83}$$

$$D_{tm}(t) = \alpha\|\partial_t\partial_m w_t(t)\|^2_{L^2} + \frac{1}{C}\|\nabla\partial_t\partial_m v(t)\|^2_{L^2}. \tag{3.2.84}$$

We have the following mixed tangential energy bound:

Lemma 3.2.14.

$$\begin{aligned} E_{tm}(t) + \int_0^t D_{tm}(s)\ ds \quad \leq \quad & E_{tm}(0) + \int_0^t (R_{tm}(s),\ \partial_t\partial_m v(s))\ ds \\ & + C_\epsilon\|\partial_t\partial_m(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])} + C_\epsilon\|\partial_t\partial_m f_e\|^2_{L^2(\Omega_e\times[0,T])} \\ & + \epsilon\int_0^t \|\partial_t\partial_m v\|^2_{L^2}\ ds, \end{aligned} \tag{3.2.85}$$

for all $t \in [0, T]$, where

$$\begin{aligned} \int_0^t (R_{tm}(s), \partial_t\partial_m v(s))\ ds \quad = \quad & -\int_0^t\int_{\Omega_f} \partial_t(a^j_\ell a^k_\ell)\partial_m\partial_k v^i \partial_t\partial_m\partial_j v^i\ dx\ ds \\ & -\int_0^t\int_{\Omega_f} \partial_m(a^j_\ell a^k_\ell - \delta_{jk})\partial_t\partial_k v^i\partial_t\partial_m\partial_j v^i\ dx\ ds \\ & -\int_0^t\int_{\Omega_f} \partial_t\partial_m(a^j_\ell a^k_\ell)\partial_k v^i\partial_t\partial_m\partial_j v^i\ dx\ ds \\ & +\int_0^t\int_{\Omega_f} \partial_t\partial_m a^k_i q\partial_t\partial_m\partial_k v^i\ dx\ ds \end{aligned}$$

$$+\int_0^t \int_{\Omega_f} \partial_t a_i^k \partial_m q \partial_t \partial_m \partial_k v^i \ dx \ ds$$
$$+\int_0^t \int_{\Omega_f} \partial_m (a_i^k - \delta_{ik}) \partial_t q \partial_t \partial_m \partial_k v^i \ dx \ ds$$
$$+\int_0^t \int_{\Omega_f} a_i^k \partial_t \partial_m q \partial_t \partial_m \partial_k v^i \ dx \ ds. \tag{3.2.86}$$

Next we provide estimates for $\nabla \partial_t \partial_m w$.

Lemma 3.2.15. *We have*

$$\begin{aligned}
&\int_0^t \|\nabla \partial_m w_t\|_{L^2}^2 \ ds + \int_0^t \|\partial_m w_t\|_{L^2}^2 \ ds + \frac{\alpha}{4}\|\partial_m w_t(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla_t \partial_m v_t(t)\|_{L^2}^2 \\
&\leq C E_{tm}(0) + \frac{1}{\alpha}\|\partial_m w_{tt}(t)\|_{L^2}^2 + C\|\partial_m v_t(t)\|_{L^2}^2 + \int_0^t \|\partial_m w_{tt}\|_{L^2}^2 \ ds \\
&\quad + \int_0^t \|\partial_m v_t\|_{L^2}^2 \ ds + \int_0^t (\tilde{R}_{tm}(s), \partial_m v(s)) \ ds + C\|\nabla \partial_m v(0)\|_{L^2}^2 \\
&\quad + C_\epsilon \||\partial_t \partial_m f_e\|_{L^2(\Omega_e \times [0,T])}^2 + C_\epsilon \|\partial_t \partial_m (f_f \circ \eta)\|_{L^2(\Omega_f \times [0,T])}^2 \\
&\quad + \epsilon \int_0^t \|\partial_m v\|_{L^2}^2 \ ds
\end{aligned} \tag{3.2.87}$$

for $t \in [0,T]$*, where*

$$\begin{aligned}
\int_0^t (\tilde{R}_{tm}(s), \partial_m v(s)) \ ds \quad = \quad & -\int_0^t \int_{\Omega_f} \partial_t \partial_m (a_\ell^j a_\ell^k) \partial_k v^i \partial_m \partial_j v^i \ d\Omega_f \ ds \\
& -\int_0^t \int_{\Omega_f} \partial_m (a_\ell^j a_\ell^k - \delta_{jk}) \partial_t \partial_k v^i \partial_m \partial_j v^i \ d\Omega_f \ ds \\
& -\frac{1}{2}\int_0^t \int_{\Omega_f} \partial_t (a_\ell^j a_\ell^k) \partial_m \partial_k v^i \partial_m \partial_j v^i \ d\Omega_f \ ds
\end{aligned}$$

$$+\int_0^t\int_{\Omega_f}\partial_t\partial_m a_i^k q\partial_m\partial_k v^i\ d\Omega_f\ ds$$
$$+\int_0^t\int_{\Omega_f}\partial_t a_i^k\partial_m q\partial_m\partial_k v^i\ d\Omega_f\ ds$$
$$+\int_0^t\int_{\Omega_f}\partial_m(a_i^k-\delta_{ik})\partial_t q\partial_m\partial_k v^i\ d\Omega_f\ ds$$
$$+\int_0^t\int_{\Omega_f}a_i^k\partial_m\partial_t q\partial_m\partial_k v^i\ d\Omega_f\ ds. \tag{3.2.88}$$

Using a small parameter $\bar{\epsilon}$, we can use (3.2.85) and (3.2.87). to obtain

$$\begin{aligned}E_{tm}(t)+\int_0^t E_{tm}(s)\ ds \quad\le\quad & CE_{tm}(0)+C\|\nabla\partial_m v(0)\|_{L^2}^2+\int_0^t(R_{tm}(s),\partial_t\partial_m v(s))\ ds\\ &+\int_0^t(\tilde{R}_{tm}(s),\partial_m v(s))\ ds+C_\epsilon\|\partial_t\partial_m(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])}\\ &+C_\epsilon\|\partial_t\partial_m f_e\|^2_{L^2(\Omega_e\times[0,T])}+\epsilon\int_0^t\|\partial_m v\|^2_{L^2}\ ds.\end{aligned}\tag{3.2.89}$$

Summing (3.2.87) and (3.2.89), we obtain

$$\begin{aligned}&E_{tm}(t)+\int_0^t E_{tm}(s)\ ds+\int_0^t D_{tm}(s)\ ds\\ &\le CE_{tm}(0)+C\|\nabla\partial_m v(0)\|_{L^2}^2+C\int_0^t(R_{tm}(s),\partial_t\partial_m v(s))\ ds+\ C\int_0^t(\tilde{R}_{tm}(s),\partial_m v)\ ds\\ &\quad+C_\epsilon\|\partial_t\partial_m(f_f\circ\eta)\|^2_{L^2(\Omega_f\times[0,T])}+C_\epsilon\|\partial_t\partial_m f_e\|^2_{L^2(\Omega_e\times[0,T])}+C\epsilon\int_0^t\|\partial_m v\|^2_{L^2}\ ds.\end{aligned}\tag{3.2.90}$$

3.3 Energy Estimates with a Close to Identity

We now construct an energy norm by summing the energy estimates obtained in the previous section. Denote

$$
\begin{aligned}
X(t) \;&=\; E(t) + E^{(1)}(t) + E^{(2)}(t) + \sum_{m=1}^{2} E_m(t) + \sum_{m=1}^{2} E_{mm}(t) + \sum_{m=1}^{2} E_{tm}(t) \\
&\quad + \tilde{\epsilon}\|\nabla v(t)\|_{L^2}^2 + \tilde{\epsilon}\|\nabla v_t(t)\|_{L^2}^2 + \tilde{\epsilon}\|\nabla \partial_m v(t)\|_{L^2}^2, \qquad (3.3.1)
\end{aligned}
$$

where $\tilde{\epsilon} > 0$ is a small parameter. We also define the dissipative term

$$
\hat{D}(t) \;=\; D(t) + D^{(1)}(t) + D^{(2)}(t) + \sum_{m=1}^{2} D_m(t) + \sum_{m=1}^{2} D_{mm}(t) + \sum_{m=1}^{2} D_{tm}(t). \qquad (3.3.2)
$$

The fluid-velocity terms are controlled by the dissipation terms

$$
\|\nabla v(t)\|_{L^2}^2 \le \|\nabla v(0)\|_{L^2}^2 + C \int_0^t \left(D(s) + D^{(1)}(s)\right)\, ds,
$$

$$
\|\nabla v_t(t)\|_{L^2}^2 \le \|\nabla v_t(0)\|_{L^2}^2 + C \int_0^t \left(D^{(1)}(s) + D^{(2)}(s)\right)\, ds,
$$

and

$$
\|\nabla \partial_m v(t)\|_{L^2}^2 \le \|\nabla \partial_m v(0)\|_{L^2}^2 + C \int_0^t \left(D_m(s) + D_{tm}(s)\right)\, ds
$$

arise from integrating ∇v_t, ∇v_{tt}, and $\nabla \partial_m v_t$ with respect to time and applying the triangle inequality. We also have the following estimates on η and the matrix a:

$$
\|\nabla(\eta - x)\|_{H^s}^2 \le t \int_0^t \|\nabla \eta_t\|_{H^s}^2\, ds \le Ct \int_0^t \|v\|_{H^{s+1}}^2\, ds, \qquad (3.3.3)
$$

$$\|a_i^k - \delta_{ik}\|_{H^s}^2 \le t \int_0^t \|\partial_t a_i^k\|_{H^s}^2 \, ds \le Ct \int_0^t \|v\|_{H^{s+1}}^2 \, ds, \ i,k = 1,2,3, \tag{3.3.4}$$

with $s = 0, 1, 2$. Similarly,

$$\|a_\ell^j a_\ell^k - \delta_{jk}\|_{H^2}^2 \le Ct \int_0^t \|v\|_{H^3}^2 \, ds, \ i,k = 1,2,3. \tag{3.3.5}$$

We now seek an a priori bound on $X(t)$ in the scenario where $a = a(x,t)$ is close to identity in the following sense:

$$a(0) = I, \ \partial_k a_j^k = 0, \ k,j = 1,2,3, \tag{3.3.6}$$

in addition to the conditions

$$\begin{gathered} \|a - I\|_{L^\infty([0,T];H^2(\Omega_f))}, \ \|aa^T - I\|_{L^\infty([0,T];H^2(\Omega_f))}, \ \|\partial_t(aa^T)\|_{L^\infty([0,T];H^2(\Omega_f))}, \\ \|a_t\|_{L^\infty([0,T];H^2(\Omega_f))}, \ \|a_{tt}\|_{L^\infty([0,T];H^1(\Omega_f))}, \ \|a_{ttt}\|_{L^\infty([0,T];L^2(\Omega_f))} \le \epsilon. \end{gathered} \tag{3.3.7}$$

Under these assumptions, we refine our energy bounds to be in terms of the norm $X(t)$

Lemma 3.3.1. *With $\tilde{R}$ defined in* (3.2.23)*, we have*

$$\int_0^t |(\tilde{R}, \eta - x)| \, ds \quad \le \quad C\epsilon t^2 \int_0^t X(s) \, ds + C\epsilon t^2 \int_0^t \|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 \, ds \tag{3.3.8}$$

for $t \in [0,T]$.

Proof. Recall $(\tilde{R}, \eta - x)$:

$$\begin{aligned} (\tilde{R}(t), \eta(x,t) - x) \quad &= \quad \frac{1}{2} \int_{\Omega_f} \partial_t(a_\ell^j a_\ell^k) \partial_k(\eta^i - x^i) \partial_j(\eta^i - x^i) \, d\Omega_f \\ &\quad + \int_{\Omega_f} q(t) \int_0^t \partial_t a_i^k \partial_k(\eta^i - x^i) \, ds \, d\Omega_f. \end{aligned}$$

We estimate using Holder's inequality

$$
\begin{aligned}
|(\tilde{R}(t), \eta(x,t) - x)| &= \left| \frac{1}{2} \int\limits_{\Omega_f} \partial_t (a^j_\ell a^k_\ell) \partial_k (\eta^i - x^i) \partial_j (\eta^i - x^i) \; d\Omega_f \right. \\
&\quad \left. + \int\limits_{\Omega_f} q(t) \int\limits_0^t \partial_t a^k_i \partial_k (\eta^i - x^i) \; ds \; d\Omega_f \right| \\
&\le \frac{1}{2} \|\partial_t (a^j_\ell a^k_\ell)\|_{L^3} \|\partial_k (\eta^i - x^i)\|_{L^3} \|\partial_j (\eta^i - x^i)\|_{L^3} \\
&\quad + \left| \int\limits_0^t \int\limits_{\Omega_f} q(t) \partial_t a^k_i \partial_k (\eta^i - x^i) \; ds \; d\Omega_f \right| \\
&\le C \|\partial_t (a^j_\ell a^k_\ell)\|_{H^1} \|\nabla(\eta - x)\|^2_{H^1} \\
&\quad + \int\limits_0^t \|q(t)\|_{L^6} \|\partial_t a^k_i\|_{L^3} \|\partial_k (\eta^i - x^i)\|_{L^2} \; ds \\
&\le C \|\partial_t (a^j_\ell a^k_\ell)\|_{H^1} \|\nabla(\eta - x)\|^2_{H^1} \\
&\quad + C \int\limits_0^t \|q(t)\|_{H^1} \|a_t\|_{H^1} \|\nabla(\eta - x)\|_{L^2} \; ds. \qquad (3.3.9)
\end{aligned}
$$

Consider the $\nabla(\eta - x)$ terms. By the Holder, Sobolev, Bochner, and Cauchy-Schwarz Inequalities, we have

$$
\begin{aligned}
\|\nabla(\eta - x)\|^2_{H^s} &= \left\| \int\limits_0^t \nabla v \; d\tau \right\|^2_{H^s} \\
&\le \left(\int\limits_0^t \|\nabla v\|_{H^s} \; d\tau \right)^2 \\
&\le \left(\int\limits_0^t d\tau \right) \left(\int\limits_0^t \|\nabla v\|^2_{H^s} \; d\tau \right) \\
&\le t \int\limits_0^t \|\nabla v\|^2_{H^s} \; d\tau. \qquad (3.3.10)
\end{aligned}
$$

Using (3.3.10) with (3.3.7) n (3.3.9), we have

$$|(\tilde{R}(t), \eta(x,t) - x)| \;\; \leq \;\; C\epsilon t \int_0^t \|\nabla v\|_{H^1}^2 \; ds + C\epsilon \int_0^t s^{\frac{1}{2}} \|q(t)\|_{H^1} \Bigg(\int_0^s \|\nabla v\|_{L^2}^2 \; d\tau \Bigg)^{\frac{1}{2}} \; ds. \tag{3.3.11}$$

Integrating in time, we have

$$\begin{aligned}
\int_0^t |(\tilde{R}(t), \eta(x,t) - x)| \; ds \;\; &\leq \;\; C\epsilon \int_0^t s \int_0^s \|v\|_{H^2}^2 \; d\tau \; ds \\
&\quad + C\epsilon \int_0^t \|q(s)\|_{H^1} \int_0^s \tau^{\frac{1}{2}} \Bigg(\int_0^\tau \|v\|_{H^1}^2 \; d\mu \Bigg)^{\frac{1}{2}} \; d\tau \; ds \\
&\leq \;\; C\epsilon \int_0^t s \int_0^t \|v\|_{H^2}^2 \; d\tau \; ds \\
&\quad + C\epsilon \int_0^t \|q(s)\|_{H^1} \int_0^s s^{\frac{1}{2}} \Bigg(\int_0^s \|v\|_{H^1}^2 \; d\mu \Bigg)^{\frac{1}{2}} \; d\tau \; ds \\
&\leq \;\; C\epsilon \Bigg(\int_0^t s \; ds \Bigg) \Bigg(\int_0^s \|v\|_{H^2}^2 \; d\tau \Bigg) \\
&\quad + C\epsilon \int_0^t s^{\frac{1}{2}} \|q(s)\|_{H^1} \Bigg(\int_0^s \|v\|_{H^1}^2 \; d\mu \Bigg)^{\frac{1}{2}} \Bigg(\int_0^s \; d\tau \Bigg) \; ds \\
&\leq \;\; C\epsilon t^2 \int_0^t \|v\|_{H^2}^2 \; d\tau \; ds \\
&\quad + C\epsilon \int_0^t s^{\frac{3}{2}} \|q(s)\|_{H^1} \Bigg(\int_0^s \|v\|_{H^1}^2 \; d\mu \Bigg)^{\frac{1}{2}} \; ds \\
&\leq \;\; C\epsilon t^2 \int_0^t \|v\|_{H^2}^2 \; d\tau \; ds
\end{aligned}$$

$$
\begin{aligned}
& +C\epsilon \int_0^t s\|q(s)\|_{H^1} s^{\frac{1}{2}} \left(\int_0^s \|v\|_{H^1}^2 \, d\mu \right)^{\frac{1}{2}} \, ds \\
\leq \;& C\epsilon t^2 \int_0^t \|v\|_{H^2}^2 \, d\tau \, ds + C\epsilon \int_0^t s^2 \|q(s)\|_{H^1}^2 \, ds \\
& +C\epsilon \int_0^t s \int_0^s \|v\|_{H^1}^2 \, d\mu \, ds \\
\leq \;& C\epsilon t^2 \int_0^t \|v\|_{H^2}^2 \, d\tau \, ds + C\epsilon t^2 \int_0^t \|q(s)\|_{H^1}^2 \, ds \\
& +C\epsilon \int_0^t s \int_0^t \|v\|_{H^1}^2 \, d\mu \, ds \\
\leq \;& C\epsilon t^2 \int_0^t \|v\|_{H^2}^2 \, d\tau \, ds + C\epsilon t^2 \int_0^t \|q(s)\|_{H^1}^2 \, ds \\
& +C\epsilon \left(\int_0^t s \, ds \right) \left(\int_0^t \|v\|_{H^1}^2 \, d\mu \right) \\
\leq \;& C\epsilon t^2 \int_0^t \|v\|_{H^2}^2 \, d\tau \, ds + C\epsilon t^2 \int_0^t \|q(s)\|_{H^1}^2 \, ds \\
& +C\epsilon t^2 \int_0^t \|v\|_{H^1}^2 \, d\mu. \qquad (3.3.12)
\end{aligned}
$$

Applying (A.2.14) to (3.3.12) completes the proof. $\square$

Combining (3.2.39), (3.3.8), and (3.3.10), we have the following:

$$
\begin{aligned}
& E(t) + \int_0^t E(s) \, ds + \int_0^t D(s) \, ds \\
& \leq CE(0) + C\epsilon t^2 \int_0^t (X(s) + \|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2) \, ds + C_\epsilon \|f_f \circ \eta\|_{L^2(\Omega_f \times [0,T])}^2 \\
& \quad +C_\epsilon \|f_e\|_{L^2(\Omega_e \times [0,T])}^2. \qquad (3.3.13)
\end{aligned}
$$

We repeat this analysis for the second, third, first and second order tangential, and mixed order tangential energy estimates.

Lemma 3.3.2. *With $R^{(1)}$ defined in* (3.2.48)*, we have*

$$\int_0^t |(R^{(1)}, v_t)| \; ds \;\; \leq \;\; C\epsilon \int_0^t X(s) \; ds + C\epsilon \int_0^t \Big(\|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 + \|\partial_t (f_f \circ \eta)\|_{L^2}^2 + \|f_e\|_{L^2}^2 \Big) \; ds \qquad (3.3.14)$$

for $t \in [0,T]$.

Proof. Recall $(R^{(1)}, v)$:

$$|(R^{(1)}(s), v(s))| \;\; = \;\; -\int_{\Omega_f} \partial_t (a_\ell^j a_\ell^k) \partial_k v^i \partial_j v_t^i \; dx + \int_{\Omega_f} \partial_t a_i^k q \partial_k v_t^i \; dx - \int_{\Omega_f} \partial_t a_i^k \partial_t q \partial_k v^i \; dx. \qquad (3.3.15)$$

We estimate the right-hand side of (3.3.15) using Holder's, Sobolev, Young's inequality, and the fact that a is close to identity. This yields the following inequality:

$$\begin{aligned} |(R^{(1)}(s), v(s))| \;\; &\leq \;\; \left| \int_{\Omega_f} \partial_t (a_\ell^j a_\ell^k) \partial_k v^i \partial_j v_t^i \; dx \right| + \left| \int_{\Omega_f} \partial_t a_i^k q \partial_k v_t^i \; dx \right| \\ &\quad + \left| \int_{\Omega_f} \partial_t a_i^k \partial_t q \partial_k v^i \; dx \right| \\ &\leq \;\; \|\partial_t (a_\ell^j a_\ell^k)\|_{L^3} \|\partial_k v^i\|_{L^3} \|\partial_j v_t^i\|_{L^3} + \|\partial_t a_i^k\|_{L^3} \|q\|_{L^3} \|\partial_k v_t^i\|_{L^3} \\ &\quad + \|\partial_t a_i^k\|_{L^3} \|\partial_t q\|_{L^3} \|\partial_k v^i\|_{L^3} \\ &\leq \;\; C\|\partial_t a_\ell^j a_\ell^k + a_\ell^j \partial_t a_\ell^k\|_{H^1} \|\nabla v\|_{H^1} \|\nabla v_t\|_{H^1} + C\|a_t\|_{H^1} \|q\|_{H^1} \|\nabla v_t\|_{H^1} \\ &\quad + C\|a_t\|_{H^1} \|q_t\|_{H^1} \|\nabla v\|_{H^1} \\ &\leq \;\; C\|a\|_{H^2} \|a_t\|_{H^1} \|v\|_{H^2} \|v_t\|_{H^2} + C\|a_t\|_{H^1} \|q\|_{H^1} \|v_t\|_{H^2}. \qquad (3.3.16) \end{aligned}$$

We now integrate in time, use (3.3.7), and apply Young's inequality and Stokes estimates

(A.2.14)-(A.2.15) to obtain

$$\begin{aligned}
\int_0^t |(R^{(1)}(t), v)| \; ds &\leq C\epsilon \int_0^t \|v\|^2_{H^2} + \|q\|^2_{H^1} + \|v_t\|^2_{H^2} + \|q_t\|^2_{H^1} \; ds \\
&\leq C\epsilon \int_0^t X(s) \; ds + C\epsilon \int_0^t \bigg(\|f_f \circ \eta\|^2_{L^2} + \|f_e\|^2_{L^2} \\
&\quad + \|\partial_t (f_f \circ \eta)\|^2_{L^2} + \|f_e\|^2_{L^2} \bigg) \; ds. \qquad (3.3.17)
\end{aligned}$$

□

Lemma 3.3.3. *With* $\tilde{R}_1$ *defined in* (3.2.48), *we have*

$$\int_0^t |(\tilde{R}^{(1)}, v_t)| \; ds \;\; \leq \;\; C\epsilon \int_0^t X(s) \; ds + C\epsilon \int_0^t \|f_f \circ \eta\|^2_{L^2} + \|f_e\|^2_{L^2} \; ds \qquad (3.3.18)$$

for $t \in [0, T]$.

Proof. Recall $(\tilde{R}^{(1)}, v_t)$:

$$(\tilde{R}^{(1)}(t), v_t) \;\; = \;\; -\frac{1}{2} \int_{\Omega_f} \partial_t (a^j_\ell a^k_\ell) \partial_k v^i \partial_j v^i \; d\Omega_f + \int_{\Omega_f} \partial_t a^k_i q \partial_k v^i \; d\Omega_f. \qquad (3.3.19)$$

We estimate the right-hand side of (3.3.19) using Holder's, Sobolev, Young's inequality, and the fact that a is close to identity, i.e.

$$\begin{aligned}
|(\tilde{R}^{(1)}(t), v_t)| &= \left| -\frac{1}{2} \int_{\Omega_f} \partial_t (a^j_\ell a^k_\ell) \partial_k v^i \partial_j v^i \; d\Omega_f + \int_{\Omega_f} \partial_t a^k_i q \partial_k v^i \; d\Omega_f \right| \\
&\leq \frac{1}{2} \|\partial_t (a^j_\ell a^k_\ell)\|_{L^3} \|\partial_k v^i\|_{L^3} \|\partial_j v^i\|_{L^3} + \|\partial_t a^k_i\|_{L^3} \|q\|_{L^3} \|\partial_k v^i\|_{L^3} \\
&\leq C \|\partial_t (a^j_\ell a^k_\ell)\|_{H^1} \|\nabla v\|^2_{H^1} + C \|\partial_t a\|_{H^1} \|q\|_{H^1} \|\nabla v\|_{H^1} \\
&\leq C\epsilon \|v\|^2_{H^2} + C\epsilon \|q\|^2_{H^1}. \qquad (3.3.20)
\end{aligned}$$

Note that (3.3.7) is used in the last line of (3.3.20). Integrating in time yields

$$\begin{aligned}\int_0^t |(\tilde{R}^{(1)}(t), v_t)|\ ds &\leq C\epsilon \int_0^t \|v\|_{H^2}^2 + \|q\|_{H^1}^2\ ds \\ &\leq C\epsilon \int_0^t X(s)\ ds + C\epsilon \int_0^t \|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2\ ds, \quad (3.3.21)\end{aligned}$$

where (A.2.14) was used.

□

Combining (3.2.43), (3.3.14), and (3.3.20), we have the following second level energy bound:

$$\begin{aligned}&E^{(1)}(t) + \int_0^t E^{(1)}(s)\ ds + \int_0^t D^{(1)}(s)\ ds \\ &\quad\leq CE^{(1)}(0) + C\epsilon \int_0^t (X(s) + \|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2)\ ds + C_\epsilon \|\partial_t (f_f \circ \eta)\|_{L^2(\Omega_f \times [0,T])}^2 \\ &\qquad + C_\epsilon \|\partial_t f_e\|_{L^2(\Omega_e \times [0,T])}^2. \quad (3.3.22)\end{aligned}$$

Lemma 3.3.4. *With $R^{(2)}$ defined in* (3.2.62), *we have*

$$\begin{aligned}\int_0^t |(R^{(2)}, v_{tt})|\ ds \leq\ & C\epsilon X(t) + C\epsilon (\|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 + \|\partial_t (f_f \circ \eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2) \\ & + C\epsilon X(0) + C\epsilon (\|(f_f \circ \eta)(0)\|_{H^1}^2 + \|f_e(0)\|_{H^1}^2 + \|\partial_t (f_f \circ \eta)(0)\|_{L^2}^2 \\ & + \|\partial_t f_e(0)\|_{L^2}^2) + C\epsilon \int_0^t X(s)\ ds + C\epsilon \int_0^t (\|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2)\ ds \\ & + C\epsilon \int_0^t (\|\partial_t (f_f \circ \eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2)\ ds + C\epsilon \int_0^t D^{(2)}(s)\ ds. \quad (3.3.23)\end{aligned}$$

for $t \in [0,T]$.

Proof. We estimate (3.2.62) using Holder's, Sobolev, Young's inequality, and the fact that a is close to identity:

$$
\begin{aligned}
\int_0^t (R^{(2)}(s), v(s))\, ds \;\leq\;& 2\int_0^t \|\partial_t(a^j_\ell a^k_\ell)\|_{L^6}\|\partial_k v^i_t\|_{L^3}\|\partial_j v^i_{tt}\|_{L^2}\, ds \\
&+ \int_0^t \|\partial_{tt}(a^j_\ell a^k_\ell)\|_{L^6}\|\|\partial_k v^i\|_{L^3}\|\partial_j v^i_{tt}\|_{L^2}\, ds \\
&+ \left| \int_0^t \int_{\Omega_f} \partial_{tt}(a^k_i q)\partial_k v^i_{tt}\, dx\, ds \right| \\
\leq\;& C\int_0^t \|\partial_t(a^j_\ell a^k_\ell)\|_{H^1}\|\nabla v_t\|_{H^1}\|\nabla v_{tt}\|_{L^2}\, ds \\
&+ C\int_0^t \|\partial_{tt}a^j_\ell a^k_\ell + 2\partial_t a^j_\ell \partial_t a^k_\ell + \partial_{tt}a^j_\ell a^k_\ell\|_{H^1}\|\nabla v\|_{H^1}\|\nabla v_{tt}\|_{L^2}\, ds \\
&+ \left| \int_0^t \int_{\Omega_f} \partial_{tt}(a^k_i q)\partial_k v^i_{tt}\, dx\, ds \right| \\
\leq\;& C\int_0^t \|\partial_t(a^j_\ell a^k_\ell)\|_{H^1}\|\nabla v_t\|_{H^1}\|\nabla v_{tt}\|_{L^2}\, ds \\
&+ C\int_0^t (\|a\|_{H^2}\|a_{tt}\|_{H^1} + C\|a_t\|^2_{H^1})\|v\|_{H^2}\|\nabla v_{tt}\|_{L^2}\, ds \\
&+ \left| \int_0^t \int_{\Omega_f} \partial_{tt}(a^k_i q)\partial_k v^i_{tt}\, dx\, ds \right| \\
\leq\;& C\epsilon\int_0^t \|v_t\|_{H^2}\|\nabla v_{tt}\|_{L^2} + \|v\|_{H^2}\|\nabla v_{tt}\|_{L^2}\, ds \\
&+ \left| \int_0^t \int_{\Omega_f} \partial_{tt}(a^k_i q)\partial_k v^i_{tt}\, dx\, ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \;\; C\epsilon \int_0^t \|v_t\|^2_{H^2} + \|\nabla v_{tt}\|^2_{L^2} + \|v\|^2_{H^2} \; ds \\
&\qquad + \left| \int_0^t \int_{\Omega_f} \partial_{tt}(a_i^k q)\partial_k v_{tt}^i \; dx \; ds \right|.
\end{aligned} \tag{3.3.24}
$$

Consider the last term in (3.3.24).

$$
\begin{aligned}
\int_0^t \int_{\Omega_f} \partial_{tt}(a_i^k q)\partial_k v_{tt}^i \; dx \; ds &= \int_0^t \int_{\Omega_f} \partial_{tt} a_i^k q \partial_k v_{tt}^i \; dx \; ds + \int_0^t \int_{\Omega_f} \partial_t a_i^k q_t \partial_k v_{tt}^i \; dx \; ds \\
&\quad + \int_0^t \int_{\Omega_f} a_i^k q_{tt} \partial_k v_{tt}^i \; dx \; ds.
\end{aligned} \tag{3.3.25}
$$

The first two terms in (3.3.25) can be estimated similarly as (3.3.24), i.e.

$$
\begin{aligned}
&\int_0^t \int_{\Omega_f} \partial_{tt} a_i^k q \partial_k v_{tt}^i \; dx \; ds + \int_0^t \int_{\Omega_f} \partial_t a_i^k q_t \partial_k v_{tt}^i \; dx \; ds \\
&\quad \leq C\epsilon \int_0^t \|q\|^2_{H^1} + \|q_t\|^2_{H^1} + \|\nabla v_{tt}\|^2_{L^2} \; ds.
\end{aligned} \tag{3.3.26}
$$

We focus on the last term on the right side of (3.3.25). Using the time-differentiated divergence-free condition

$$
\partial_{tt} a_i^k \partial_k v^i + 2\partial_t a_i^k \partial_k v_t^i + a_i^k \partial_k v_{tt}^i = 0,
$$

and integrating by parts (to remove a time derivative from the pressure terms), we have that

$$
\begin{aligned}
\int_0^t \int_{\Omega_f} a_i^k q_{tt} \partial_k v_{tt}^i \; dx \; ds &= -2\int_0^t \int_{\Omega_f} \partial_t a_i^k q_{tt} \partial_k v_t^i \; dx \; ds - \int_0^t \int_{\Omega_f} \partial_{tt} a_i^k q_{tt} \partial_k v^i \; dx \; ds \\
&= 2\int_{\Omega_f} \partial_t a_i^k(t) q_t(t) \partial_k v_t^i(t) \; dx + \int_{\Omega_f} \partial_{tt} a_i^k(t) q_t(t) \partial_k v^i(t) \; dx
\end{aligned}
$$

$$-2\int_{\Omega_f} \partial_t a_i^k(0) q_t(0)\partial_k v^i(0)\ dx - \int_{\Omega_f} \partial_{tt} a_i^k(0) q_t(0)\partial_k v^i(0)\ dx$$
$$+3\int_0^t\int_{\Omega_f} \partial_{tt} a_i^k q_t \partial_k v_t^i\ dx\ ds + 2\int_0^t\int_{\Omega_f} \partial_t a_i^k q_t\partial_k v_{tt}^i\ dx\ ds$$
$$+\int_0^t\int_{\Omega_f} \partial_{ttt} a_i^k q_t\partial_k v^i\ dx\ ds. \tag{3.3.27}$$

Every term in the inequality (3.3.27) can be estimated similarly as (3.3.24), i.e.

$$\begin{aligned}
\int_0^t\int_{\Omega_f} a_i^k q_{tt}\partial_k v_{tt}^i\ dx\ ds \quad \leq \quad & C\epsilon\|q_t\|_{H^1}^2 + C\epsilon\|v_t\|_{H^2}^2 + C\epsilon\|v\|_{H^1}^2 \\
& +C\epsilon\|q_t(0)\|_{H^1}^2 + C\epsilon\|v_t(0)\|_{H^2}^2 + C\epsilon\|v(0)\|_{H^1}^2 \\
& +C\epsilon\int_0^t \|q_t\|_{H^1}^2 + \|v_t\|_{H^1}^2 + \|\nabla v_{tt}\|_{L^2}^2\ ds \\
& +C\int_0^t \|\partial_{ttt} a\|_{L^2}\|q_t\|_{L^6}\|\partial_k v\|_{L^3}\ ds.
\end{aligned} \tag{3.3.28}$$

Combining (3.3.24), (3.3.26), and (3.3.28), and using (A.2.14)-(A.2.15), we obtain the following result:

$$\begin{aligned}
\int_0^t |(R^{(2)}, v_{tt})|\ ds \quad \leq \quad & C\epsilon X(t) + C\epsilon(\|f_f\circ\eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 + \|\partial_t(f_f\circ\eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2) \\
& +C\epsilon X(0) + C\epsilon(\|(f_f\circ\eta)(0)\|_{L^2}^2 + \|f_e(0)\|_{L^2}^2 + \|\partial_t(f_f\circ\eta)(0)\|_{L^2}^2 \\
& +\|\partial_t f_e(0)\|_{L^2}^2) + C\epsilon\int_0^t (X(s) + \|f_f\circ\eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 \\
& +\|\partial_t(f_f\circ\eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2)\ ds.
\end{aligned} \tag{3.3.29}$$

□

Lemma 3.3.5. *With $\tilde{R}_2$ defined in* (3.2.64), *we have*

$$\begin{aligned}\int_0^t |(\tilde{R}^{(2)}, v_{tt})| \; ds \;\; &\leq \;\; C\epsilon X(t) + C\epsilon(\|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 + \|\partial_t(f_f \circ \eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2) \\ &\quad + C\epsilon X(0) + C\epsilon(\|(f_f \circ \eta)(0)\|_{L^2}^2 + \|f_e(0)\|_{L^2}^2 + \|\partial_t(f_f \circ \eta)(0)\|_{L^2}^2 \\ &\quad + \|\partial_t f_e(0)\|_{L^2}^2) + C\epsilon \int_0^t (X(s) + \|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 \\ &\quad + \|\partial_t(f_f \circ \eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2) \; ds \qquad (3.3.30)\end{aligned}$$

for $t \in [0, T]$.

Proof. We follow a similar argument as Lemma 3.3.4. Analogues to (3.3.24) and (3.3.25) are given by

$$\begin{aligned}\int_0^t |(\tilde{R}^{(2)}, v_t)| \; ds \;\; &\leq \;\; C\epsilon \int_0^t \|v_t\|_{H^2}^2 + \|v\|_{H^2}^2 \; ds \\ &\quad + \left| \int_0^t \int_{\Omega_f} \partial_{tt}(a_i^k q) \partial_k v_{tt}^i \; dx \; ds \right|. \qquad (3.3.31)\end{aligned}$$

$$\begin{aligned}\int_0^t \int_{\Omega_f} \partial_{tt}(a_i^k q) \partial_k v_t^i \; dx \; ds \;\; &= \;\; \int_0^t \int_{\Omega_f} \partial_{tt} a_i^k q \partial_k v_t^i \; dx \; ds + \int_0^t \int_{\Omega_f} \partial_t a_i^k q_t \partial_k v_{tt}^i \; dx \; ds \\ &\quad + \int_0^t \int_{\Omega_f} a_i^k q_{tt} \partial_k v_t^i \; dx \; ds. \qquad (3.3.32)\end{aligned}$$

The first two terms in (3.3.32) can be estimated similarly as (3.3.26), i.e.

$$\int_0^t \int_{\Omega_f} \partial_{tt} a_i^k q \partial_k v_t^i \; dx \; ds + \int_0^t \int_{\Omega_f} \partial_t a_i^k q_t \partial_k v_t^i \; dx \; ds \;\; \leq \;\; C\epsilon \int_0^t \|q\|_{H^1}^2 + \|q_t\|_{H^1}^2 + \|v_t\|_{H^2}^2 \; ds. \qquad (3.3.33)$$

For the last term in (3.3.32), we integrate by parts to obtain

$$\begin{aligned}\int_0^t\int_{\Omega_f} a_i^k q_{tt}\partial_k v_t^i\ dx\ ds &= \int_{\Omega_f} a_i^k q_t\partial_k v_t^i\ dx - \int_{\Omega_f} a_i^k(0) q_t(0)\partial_k v_t^i(0)\ dx \\ &\quad - \int_0^t\int_{\Omega_f} \partial_t a_i^k q_t\partial_k v_t^i\ dx\ ds - \int_0^t\int_{\Omega_f} a_i^k q_t\partial_k v_t^i\ dx\ ds.\end{aligned} \tag{3.3.34}$$

By the time differentiated divergence-free condition

$$\partial_t a_i^k\partial_k v^i + a_i^k\partial_k v_t^i = 0, \tag{3.3.35}$$

we have

$$\begin{aligned}\int_0^t\int_{\Omega_f} a_i^k q_{tt}\partial_k v_t^i\ dx\ ds &= -\int_{\Omega_f} \partial_t a_i^k q_t\partial_k v^i\ dx + \int_{\Omega_f} \partial_t a_i^k(0) q_t(0)\partial_k v^i(0)\ dx \\ &\quad - \int_0^t\int_{\Omega_f} \partial_t a_i^k q_t\partial_k v_t^i\ dx\ ds + \int_0^t\int_{\Omega_f} \partial_t a_i^k q_t\partial_k v^i\ dx\ ds.\end{aligned} \tag{3.3.36}$$

Estimating, we obtain

$$\begin{aligned}\int_0^t\int_{\Omega_f} a_i^k q_{tt}\partial_k v_t^i\ dx\ ds &\leq C\|\partial_t a_i^k\|_{L^3}\|q_t\|_{L^3}\|\partial_k v^i\|_{L^3} + C\|\partial_t a_i^k(0)\|_{L^3}\|q_t(0)\|_{L^3}\|\partial_k v^i(0)\|_{L^3} \\ &\quad + C\int_0^t \|\partial_t a_i^k\|_{L^3}\|q_t\|_{L^3}\|\partial_k v_t^i\|_{L^3}\ ds \\ &\quad + C\int_0^t \|\partial_t a_i^k\|_{L^3}\|q_t\|_{L^3}\|\partial_k v^i\|_{L^3}\ ds \\ &\leq C\|a_t\|_{H^1}\|q_t\|_{H^1}\|v\|_{H^2} + C\|a_t(0)\|_{H^2}\|q_t(0)\|_{H^1}\|v(0)\|_{H^2}\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t \|a_t\|_{H^1}\|q_t\|_{H^1}\|v_t\|_{H^2}\ ds + C\int_0^t \|a_t\|_{H^1}\|q_t\|_{H^1}\|v\|_{H^2}\ ds \\
\leq\ & C\epsilon\|q_t\|_{H^1}^2 + C\epsilon\|v\|_{H^2}^2 + C\epsilon\|q_t(0)\|_{H^1}^2 + C\epsilon\|v(0)\|_{H^2}^2 \\
&+C\epsilon\int_0^t \|q_t\|_{H^1}^2 + \|v_t\|_{H^2}^2 + \|v\|_{H^2}^2\ ds. \qquad (3.3.37)
\end{aligned}
$$

Combining (3.3.33) and (3.3.37) with (A.2.14)-(A.2.15) yields

$$
\begin{aligned}
\int_0^t |(\tilde{R}^{(2)}, v_{tt})|\ ds \quad \leq\ & C\epsilon X(t) + C\epsilon(\|f_f\circ\eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 + \|\partial_t(f_f\circ\eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2) \\
&+C\epsilon X(0) + C\epsilon(\|(f_f\circ\eta)(0)\|_{L^2}^2 + \|f_e(0)\|_{L^2}^2 + \|\partial_t(f_f\circ\eta)(0)\|_{L^2}^2 \\
&+\|\partial_t f_e(0)\|_{L^2}^2) + C\epsilon\int_0^t (X(s) + \|f_f\circ\eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 \\
&+\|\partial_t(f_f\circ\eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2).
\end{aligned}
$$

$\square$

Combining (3.2.66), (3.3.24), and (3.3.30), we have the following:

$$
\begin{aligned}
&E^{(2)}(t) + \int_0^t E^{(2)}(s)\ ds + \int_0^t D^{(2)}(s)\ ds \\
&\leq CX(0) + C\epsilon X(t) + C\epsilon(\|f_f\circ\eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 + \|\partial_t(f_f\circ\eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2) \\
&\quad +C\epsilon(\|(f_f\circ\eta)(0)\|_{L^2}^2 + \|f_e(0)\|_{L^2}^2 + \|\partial_t(f_f\circ\eta)(0)\|_{L^2}^2 + \|\partial_t f_e(0)\|_{L^2}^2) \\
&\quad +C\epsilon\int_0^t X(s)\ ds + C\epsilon\int_0^t (\|f_f\circ\eta\|_{L^2}^2 + \|f_e\|_{L^2}^2) + \|\partial_t(f_f\circ\eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2)\ ds \\
&\quad +C\epsilon\int_0^t D^{(2)}(s)\ ds + \epsilon\int_0^t \|v_t\|_{L^2}^2\ ds + C_\epsilon\|\partial_{tt}(f_f\circ\eta)\|_{L^2(\Omega_f\times[0,T])}^2 \\
&\quad +C_\epsilon\|\partial_{tt} f_e\|_{L^2(\Omega_e\times[0,T])}^2. \qquad (3.3.38)
\end{aligned}
$$

Lemma 3.3.6. *With R_m defined in* (3.2.70)*, we have*

$$\int_0^t |(R_m, \partial_m v)| \; ds \;\; \leq \;\; C\epsilon \int_0^t X(s) \; ds + C\epsilon \int_0^t \|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 \; ds \quad (3.3.39)$$

for $t \in [0, T]$.

Proof. We estimate using the Holder, Sobolev, and Young's inequalities similarly to the previous arguments to obtain

$$\begin{aligned}
\int_0^t |(R_m, \partial_m v)| \; ds \;\; &\leq \;\; C \int_0^t \|\partial_m (a_\ell^j a_\ell^k - \delta_{ik})\|_{L^6} \|\partial_k v^i\|_{L^3} \|\partial_j \partial_m v^i\|_{L^2} \\
&\quad + \int_0^t \|\partial_m (a_i^k - \delta_{jk})\|_{L^3} \|q\|_{L^3} \|\partial_j \partial_k v^i\|_{L^2} \\
&\quad + \int_0^t \|\partial_m (a_i^k - \delta_{jk})\|_{L^3} \|q\|_{L^3} \|\partial_j \partial_m v^i\|_{L^2} \\
&\leq \;\; C \int_0^t \|\partial_m (a_\ell^j a_\ell^k - \delta_{ik})\|_{H^1} \|\partial_k v^i\|_{H^1} \|\partial_k \partial_m v^i\|_{L^2} \\
&\quad + \int_0^t \|\partial_m (a_i^k - \delta_{jk})\|_{H^1} \|q\|_{H^1} \|\partial_j \partial_k v^i\|_{L^2} \\
&\quad + \int_0^t \|\partial_m (a_i^k - \delta_{jk})\|_{H^1} \|q\|_{H^1} \|\partial_j \partial_m v^i\|_{L^2} \\
&\leq \;\; C \int_0^t \|aa^T - I\|_{H^1} \|v\|_{H^2}^2 + \int_0^t \|a - I\|_{H^2} \|q\|_{H^1} \|v\|_{H^2} \\
&\quad + \int_0^t \|a - I\|_{H^2} \|q\|_{H^1} \|v\|_{H^2} \\
&\leq \;\; C\epsilon \int_0^t \|v\|_{H^2}^2 + \|q\|_{H^1}^2, \qquad (3.3.40)
\end{aligned}$$

where (3.3.7) was used. We conclude by applying (A.2.14), i.e.

$$\int_0^t |(R_m, \partial_m(\eta - x))| \, ds \quad \leq \quad C\epsilon \int_0^t X(s) \, ds + C\epsilon \int_0^t \|f_f \circ \eta\|_{L^2}^2 + \|f_e\|_{L^2}^2 \, ds. \tag{3.3.41}$$

$\square$

Lemma 3.3.7. *With* $\tilde{R}_m$ *defined in* (3.2.72)*, we have*

$$\begin{aligned} \int_0^t |(\tilde{R}_m, \partial_m v)| \, ds \quad &\leq \quad (C + Ct^2)\epsilon \int_0^t X(s) \, ds \\ &\quad + (C + Ct^2)\epsilon \int_0^t \|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 \, ds \end{aligned} \tag{3.3.42}$$

for $t \in [0, T]$.

Proof. We estimate (3.2.72) similarly as (3.2.70), that is

$$\begin{aligned} &\int_0^t (\tilde{R}_m(s), \partial_m(\eta(x,s) - x)) \, ds \\ &\leq C \int_0^t \|\partial_m(a_\ell^j a_\ell^k - \delta_{jk})\|_{L^6} \|\partial_k v^i\|_{L^3} \|\partial_j \partial_m(\eta^i - x^i)\|_{L^2} \, ds \\ &\quad + C \int_0^t \|\partial_t(a_\ell^j a_\ell^k)\|_{L^\infty} \|\partial_k \partial_m(\eta^i - x^i)\|_{L^2} \|\partial_j \partial_m(\eta^i - x^i)\|_{L^2} \, ds \\ &\quad + C \int_0^t \|\partial_m(a_i^k - \delta_{jk})\|_{L^6} \|q\|_{L^3} \|\partial_k \partial_m(\eta^i - x^i)\|_{L^2} \, ds \\ &\quad + \int_0^t \int_{\Omega_f} a_i^k \partial_m q \partial_k \partial_m(\eta^i - x^i) \, d\Omega_f \, ds \\ &\leq C \int_0^t \|\partial_m(a_\ell^j a_\ell^k - \delta_{jk})\|_{H^1} \|\nabla v\|_{H^1} \|\nabla \partial_m(\eta - x)\|_{L^2} \, ds \end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t \|\partial_t(aa^T)\|_{H^1}\|\nabla\partial_m(\eta-x)\|_{L^2}^2\;ds \\
&+C\int_0^t \|\partial_m(a_i^k-\delta_{jk})\|_{H^1}\|q\|_{H^1}\|\nabla\partial_m(\eta-x)\|_{L^2}\;ds \\
&+\int_0^t\int_{\Omega_f} a_i^k\partial_m q\partial_k\partial_m(\eta^i-x^i)\;d\Omega_f\;ds \\
\le\; & C\int_0^t \|aa^T-I\|_{H^2}\|v\|_{H^2}\|\nabla\partial_m(\eta-x)\|_{L^2}\;ds \\
&+C\int_0^t \|\partial_t(aa^T)\|_{H^1}\|\nabla\partial_m(\eta-x)\|_{L^2}^2\;ds \\
&+C\int_0^t \|a-I\|_{H^2}\|q\|_{H^1}\|\nabla\partial_m(\eta-x)\|_{L^2}\;ds \\
&+\int_0^t\int_{\Omega_f} a_i^k\partial_m q\partial_k\partial_m(\eta^i-x^i)\;d\Omega_f\;ds \\
\le\; & C\epsilon\int_0^t \|v\|_{H^2}^2+\|q\|_{H^1}^2+\|\nabla\partial_m(\eta-x)\|_{L^2}^2\;ds \\
&+\int_0^t\int_{\Omega_f} a_i^k\partial_m q\partial_k\partial_m(\eta^i-x^i)\;d\Omega_f\;ds. \qquad (3.3.43)
\end{aligned}
$$

We apply (3.3.10) to the $\nabla\partial_m(\eta-x)$ term in (3.3.43) to obtain

$$
C\epsilon\int_0^t \|\nabla\partial_m(\eta-x)\|_{L^2}^2\;ds \;\le\; C\epsilon t^2\int_0^t \|\nabla v\|_{H^2}^2\;d\tau \le C\epsilon t^2\int_0^t \|v\|_{H^3}^2\;d\tau. \quad (3.3.44)
$$

Next consider the last term in (3.3.43). We rewrite this term using the divergence-free condition

$$
\partial_m a_i^k\partial_k v^i + a_i^k\partial_k\partial_m v^i = 0. \quad (3.3.45)
$$

This implies that

$$a_i^k \partial_k \partial_m v^i = -\partial_m a_i^k \partial_k v^i. \tag{3.3.46}$$

Integrating in time yields

$$\int_0^t a_i^k \partial_k \partial_m v^i \; ds = -\int_0^t \partial_m a_i^k \partial_k v^i \; ds. \tag{3.3.47}$$

We now integrate by parts:

$$a_i^k \partial_k \partial_m \int_0^t v^i \; ds - \int_0^t \partial_t a_i^k \partial_k \partial_m (\eta - x) \; ds = -\int_0^t \partial_m a_i^k \partial_k v^i \; ds. \tag{3.3.48}$$

Since $\eta - x = \int_0^t v \; ds$, we conclude

$$a_i^k \partial_k \partial_m (\eta^i - x^i) - \int_0^t \partial_t a_i^k \partial_k \partial_m (\eta - x) \; ds = -\int_0^t \partial_m a_i^k \partial_k v^i \; ds. \tag{3.3.49}$$

Using (3.3.49) in the last term on the right side of (3.3.43) we have

$$\begin{aligned} &\int_0^t \int_{\Omega_f} a_i^k \partial_m q \partial_k \partial_m (\eta^i - x^i) \; d\Omega_f \; ds \\ &\quad = \int_0^t \left[\int_{\Omega_f} \partial_m q \left(\int_0^s \partial_t a_i^k \partial_k \partial_m (\eta^i - x^i) - \partial_m a_i^k \partial_k v^i \; d\tau \right) dx \right] ds. \end{aligned} \tag{3.3.50}$$

We estimate the terms inside the time integral using the Holder and Sobolev inequalities.

$$\begin{aligned} \int_{\Omega_f} a_i^k \partial_m q \partial_k \partial_m (\eta^i - x^i) \; d\Omega_f \;\; &= \;\; \int_0^t \int_{\Omega_f} (\partial_m q(t) \partial_t a_i^k \partial_k \partial_m (\eta^i - x^i) \\ &\qquad -\partial_m q(t) \partial_m a_i^k \partial_k v^i) \; ds \; dx \end{aligned}$$

$$
\begin{aligned}
&\leq C\int_0^t \|\partial_m q(t)\|_{L^6}\|\partial_t a_i^k\|_{L^3}\|\partial_k\partial_m(\eta^i - x^i)\|_{L^2}\ ds \\
&\quad + C\int_0^t \|\partial_m q(t)\|_{L^3}\|\partial_m a_i^k\|_{L^3}\|\partial_k v^i\|_{L^3}\ ds \\
&\leq C\int_0^t \|q(t)\|_{H^2}\|a_t\|_{H^1}\|\nabla(\eta - x)\|_{H^1}\ ds \\
&\quad + C\int_0^t \|q(t)\|_{H^2}\|\partial_m a\|_{H^1}\|v\|_{H^2}\ ds \\
&\leq C\epsilon\int_0^t \|q(t)\|_{H^2}\|\nabla(\eta - x)\|_{H^1}\ ds \\
&\quad + C\epsilon\int_0^t \|q(t)\|_{H^2}\|v\|_{H^2}\ ds \\
&\leq C\epsilon\int_0^t \|q(t)\|_{H^2} s^{\frac{1}{2}}\left(\int_0^s \|v\|_{H^1}^2\ d\tau\right)^{\frac{1}{2}}\ ds \\
&\quad + C\epsilon\int_0^t \|q(t)\|_{H^2}\|v\|_{H^2}\ ds, \qquad (3.3.51)
\end{aligned}
$$

where (3.3.10) was used. Integrating in time results in the following:

$$
\begin{aligned}
\int_0^t \int_{\Omega_f} a_i^k \partial_m q \partial_k \partial_m (\eta^i - x^i)\ d\Omega_f\ ds &\leq C\epsilon \int_0^t \int_0^s \|q(s)\|_{H^2} \tau^{\frac{1}{2}} \left(\int_0^\tau \|v\|_{H^1}^2\ d\xi\right)^{\frac{1}{2}}\ d\tau\ ds \\
&\quad + C\epsilon \int_0^t \int_0^s \|q(s)\|_{H^2}\|v\|_{H^2}\ d\tau\ ds \\
&\leq C\epsilon \int_0^t s^{\frac{1}{2}}\|q(s)\|_{H^2} \int_0^s \left(\int_0^s \|v\|_{H^1}^2\ d\xi\right)^{\frac{1}{2}}\ d\tau\ ds
\end{aligned}
$$

$$
\begin{aligned}
& +C\epsilon\int_0^t \|q(s)\|_{H^2}\int_0^s \|v\|_{H^2}\ d\tau\ ds \\
\leq\ & C\epsilon\int_0^t s^{\frac{1}{2}}\|q(s)\|_{H^2}\int_0^s \left(\int_0^t \|v\|^2_{H^1}\ d\xi\right)^{\frac{1}{2}}\ d\tau\ ds \\
& +C\epsilon\int_0^t \|q(s)\|_{H^2}s^{\frac{1}{2}}\left(\int_0^s \|v\|^2_{H^2}\ d\tau\right)^{\frac{1}{2}}\ ds. \\
\leq\ & C\epsilon\int_0^t s^{\frac{3}{2}}\|q(s)\|_{H^2}\left(\int_0^t \|v\|^2_{H^1}\ d\xi\right)^{\frac{1}{2}}\ ds \\
& +C\epsilon\int_0^t \|q(s)\|_{H^2}s^{\frac{1}{2}}\left(\int_0^s \|v\|^2_{H^2}\ d\tau\right)^{\frac{1}{2}}\ ds. \\
\leq\ & C\epsilon\int_0^t s\|q(s)\|_{H^2}s^{\frac{1}{2}}\left(\int_0^t \|v\|^2_{H^1}\ d\xi\right)^{\frac{1}{2}}\ ds \\
& +C\epsilon\int_0^t \|q(s)\|_{H^2}s^{\frac{1}{2}}\left(\int_0^s \|v\|^2_{H^2}\ d\tau\right)^{\frac{1}{2}}\ ds.
\end{aligned}
\tag{3.3.52}
$$

Next, we apply Young's inequality to obtain

$$
\begin{aligned}
\int_0^t\int_{\Omega_f} a_i^k\partial_m q\partial_k\partial_m(\eta^i - x^i)\ d\Omega_f\ ds\ \leq\ & C\epsilon\int_0^t s^2\|q(s)\|^2_{H^2} + C\epsilon\int_0^t s\left(\int_0^s \|v\|^2_{H^1}\ d\xi\right)\ ds \\
& +C\epsilon\int_0^t \|q(s)\|^2_{H^2}\ ds + C\epsilon\int_0^t s\int_0^s \|v\|^2_{H^2}\ d\tau\ ds. \\
\leq\ & C\epsilon t^2\int_0^t \|q(s)\|^2_{H^2} + C\epsilon t^2\int_0^t \|v\|^2_{H^1}\ d\xi \\
& +C\epsilon\int_0^t \|q(s)\|^2_{H^2}\ ds + C\epsilon\int_0^t s\int_0^t \|v\|^2_{H^2}\ d\tau\ ds.
\end{aligned}
$$

$$\leq C\epsilon t^2 \int_0^t \|q(s)\|_{H^2}^2 \, ds + C\epsilon t^2 \int_0^s \|v\|_{H^1}^2 \, d\xi$$
$$+C\epsilon \int_0^t \|q(s)\|_{H^2}^2 \, ds + C\epsilon t^2 \int_0^t \|v\|_{H^2}^2 \, d\tau. \tag{3.3.53}$$

Combining (3.3.43), (3.3.44), and (3.3.53), we arrive at the following result:

$$\begin{aligned}\int_0^t |(\tilde{R}_m, \partial_m v)| \, ds &\leq C\epsilon \int_0^t \|v\|_{H^2}^2 + \|q\|_{H^2}^2 \, ds + C\epsilon t^2 \int_0^t \|v\|_{H^3}^2 + \|q\|_{H^2}^2 \, ds \\ &\leq (C + Ct^2)\epsilon \int_0^t X(s) \, ds \\ &\quad +(C + Ct^2)\epsilon \int_0^t \|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 \, ds. \end{aligned} \tag{3.3.54}$$

□

Combining (3.2.73), (3.3.39), and (3.3.54), we arrive at the following inequality:

$$\begin{aligned}&E_m(t) + \int_0^t E_m(s) \, ds + \int_0^t D_m(s) \, ds \\ &\leq CE_m(0) + (C + Ct^2)\epsilon \int_0^t X(s) \, ds + (C + Ct^2)\epsilon \int_0^t \|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 \, ds \\ &\quad +C_\epsilon \|\partial_m (f_f \circ \eta)\|_{L^2(\Omega_e \times [0,T])}^2 + C_\epsilon \|\partial_m f_e\|_{L^2(\Omega_e \times [0,T])}^2. \end{aligned} \tag{3.3.55}$$

Lemma 3.3.8. *With R_{mm} defined in* (3.2.78), *we have*

$$\int_0^t (R_{mm}(s), \partial_{mm} v(s)) \, ds \leq C\epsilon \int_0^t X(s) \, ds + C\epsilon \int_0^t \|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 \, ds. \tag{3.3.56}$$

for $t \in [0,T]$.

Proof. We estimate (3.2.78) using the Holder, Sobolev, and Young inequalities to obtain

$$\begin{aligned}
&\int_0^t (R_{mm}(s), \partial_{mm} v(s))\ ds \\
&\quad \leq C \int_0^t \|\partial_m (a_\ell^j a_\ell^k - \delta_{jk})\|_{L^6} \|\partial_k \partial_m v^i\|_{L^3} \|\partial_j \partial_{mm} v^i\|_{L^2}\ ds \\
&\qquad + C \int_0^t \|\partial_{mm} (a_\ell^j a_\ell^k - \delta_{jk})\|_{L^2} \|\partial_k v^i\|_{L^\infty} \|\partial_j \partial_{mm} v^i\|_{L^2}\ ds \\
&\qquad + C \int_0^t \|\partial_{mm} (a_i^k - \delta_{ik})\|_{L^2} \|q\|_{L^\infty} \|\partial_k \partial_{mm} v^i\|_{L^2}\ ds \\
&\qquad + C \int_0^t \|\partial_m (a_i^k - \delta_{ik})\|_{L^6} \|\partial_m q \partial_k\|_{L^3} \|\partial_{mm} v^i\|_{L^2}\ ds \\
&\qquad + \int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm} v^i\ dx\ ds \\
&\quad \leq C \int_0^t \|\partial_m (a_\ell^j a_\ell^k - \delta_{jk}\|_{H^1} \|\partial_k \partial_m v^i)\|_{H^1} \|\partial_j \partial_{mm} v^i\|_{L^2}\ ds \\
&\qquad + C \int_0^t \|\partial_{mm} (a_\ell^j a_\ell^k - \delta_{jk})\|_{L^2} \|\partial_k v^i\|_{H^2} \|\partial_j \partial_{mm} v^i\|_{L^2}\ ds \\
&\qquad + C \int_0^t \|\partial_{mm} (a_i^k - \delta_{ik})\|_{L^2} \|q\|_{H^2} \|\partial_k \partial_{mm} v^i\|_{L^2}\ ds \\
&\qquad + C \int_0^t \|\partial_m (a_i^k - \delta_{ik})\|_{H^1} \|\partial_m q\|_{H^1} \|\partial_k \partial_{mm} v^i\|_{L^2}\ ds \\
&\qquad + \int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm} v^i\ dx\ ds
\end{aligned}$$

$$
\begin{aligned}
&\leq C \int_0^t \|aa^T - I\|_{H^2} \|v\|_{H^3}^2 \; ds + C \int_0^t \|aa^T - I\|_{H^2} \|v\|_{H^3}^2 \; ds \\
&\quad + C \int_0^t \|a - I\|_{H^2} \|q\|_{H^2} \|v\|_{H^3} \; ds + C \int_0^t \|a - I\|_{H^2} \|q\|_{H^2} \|v\|_{H^3} \; ds \\
&\quad + \int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm} v^i \; dx \; ds \\
&\leq C\epsilon \int_0^t \|v\|_{H^3}^2 + \|q\|_{H^2}^2 \; ds + \int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm} v^i \; dx \; ds. \qquad (3.3.57)
\end{aligned}
$$

Consider the last term on the right side of (3.3.57). We use the divergence-free condition

$$
\partial_{mm} a_i^k \partial_k v^i + 2\partial_m a_i^k \partial_k \partial_m v^i + a_i^k \partial_k \partial_{mm} v^i = 0
$$

to write

$$
\begin{aligned}
\int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm} v^i \; dx \; ds &= -\int_0^t \int_{\Omega_f} \partial_{mm} a_i^k \partial_{mm} q \partial_k v^i \; dx \; ds \\
&\quad -2 \int_0^t \int_{\Omega_f} \partial_m a_i^k \partial_{mm} q \partial_k \partial_m v^i \; dx \; ds \\
&\leq C \int_0^t \|\partial_{mm} a_i^k\|_{L^2} \|\partial_{mm} q\| L^2 \|\partial_k v^i\|_{L^\infty} \; ds \\
&\quad + C \int_0^t \|\partial_m a_i^k\|_{L^6} \|\partial_{mm} q\|_{L^2} \|\partial_k \partial_m v^i\|_{L^3} \; ds \\
&\leq C \int_0^t \|\partial_{mm} a\|_{L^2} \|q\|_{H^2} \|v\|_{H^3} \; ds \\
&\quad + C \int_0^t \|\partial_m a\|_{H^1} \|q\|_{H^2} \|v\|_{H^3} \; ds
\end{aligned}
$$

$$\leq \quad C\epsilon \int_0^t \|v\|_{H^3}^2 + \|q\|_{H^2}^2 \; ds. \tag{3.3.58}$$

(3.3.57) and (3.3.58) together with (A.2.14) imply

$$\begin{aligned}
\int_0^t (R_{mm}(s), \partial_{mm} v(s)) \; ds \quad &\leq \quad C\epsilon \int_0^t \|v\|_{H^3}^2 + \|q\|_{H^2}^2 \; ds \\
&\leq \quad C\epsilon \int_0^t X(s) \; ds \\
&\qquad + C\epsilon \int_0^t \|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 \; ds. \qquad (3.3.59)
\end{aligned}$$

□

Lemma 3.3.9. *With $\tilde{R}_{mm}$ defined in* (3.2.80), *we have*

$$\begin{aligned}
\int_0^t (\tilde{R}_{mm}(s), \partial_{mm} v(s)) \; ds \quad &\leq \quad (C + Ct^2)\epsilon \int_0^t X(s) \; ds \\
&\qquad + (C + Ct^2)\epsilon \int_0^t \|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 \; ds, \quad (3.3.60)
\end{aligned}$$

for $t \in [0, T]$.

Proof. We estimate (3.2.80) using the Holder, Sobolev, and Young inequalities.

$$\begin{aligned}
&\int_0^t (\tilde{R}_{mm}(s), \partial_{mm}(\eta(x,s) - x)) \; ds \\
&\leq C \int_0^t \|\partial_{mm}(a_\ell^j a_\ell^k - \delta_{jk})\|_{L^2} \|\partial_k v^i\|_{L^\infty} \|\partial_j \partial_{mm}(\eta^i - x^i)\|_{L^2} \; ds \\
&\quad + C \int_0^t \|\partial_m(a_\ell^j a_\ell^k - \delta_{jk})\|_{L^6} \|\partial_k v^i\|_{L^3} \|\partial_j \partial_{mm}(\eta^i - x^i)\|_{L^2} \; ds
\end{aligned}$$

$$
\begin{aligned}
&\quad + C\int_0^t \|\partial_t(a^j_\ell a^k_\ell)\|_{L^\infty}\|\partial_k\partial_{mm}(\eta^i - x^i)\partial_j\|_{L^2}\|\partial_{mm}(\eta^i - x^i)\|_{L^2}\ ds \\
&\quad + C\int_0^t \|\partial_{mm}(a^k_i - \delta_{jk})\|_{L^2}\|q\|_{L^\infty}\|\partial_k\partial_{mm}(\eta^i - x^i)\|_{L^2}\ ds \\
&\quad + C\int_0^t \|\partial_m(a^k_i - \delta_{ik})\|_{L^6}\|\partial_m q\|_{L^3}\|\partial_k\partial_{mm}(\eta^i - x^i)\|_{L^2}\ ds \\
&\quad + \int_0^t\int_{\Omega_f} a^k_i\partial_{mm}q\partial_k\partial_{mm}(\eta^i - x^i)\ d\Omega_f\ ds \\
&\leq C\int_0^t \|aa^T - I\|_{H^2}\|v\|_{H^3}\|\nabla(\eta - x)\|_{H^2}\ ds \\
&\quad + C\int_0^t \|aa^T - I\|_{H^2}\|v\|_{H^2}\|\nabla(\eta - x)\|_{H^2}\ ds \\
&\quad + C\int_0^t \|\partial_t(aa^T)\|_{H^2}\|\nabla(\eta - x)\|^2_{H^2}\ ds \\
&\quad + C\int_0^t \|a - I\|_{H^2}\|q\|_{H^2}\|\nabla(\eta - x)\|_{H^2}\ ds \\
&\quad + C\int_0^t \|a - I\|_{H^2}\|q\|_{H^2}\|\nabla(\eta - x)\|_{H^2}\ ds \\
&\quad + \int_0^t\int_{\Omega_f} a^k_i\partial_{mm}q\partial_k\partial_{mm}(\eta^i - x^i)\ d\Omega_f\ ds \\
&\leq C\epsilon\int_0^t \|v\|^2_{H^3} + \|q\|^2_{H^2} + \|\nabla(\eta - x)\|^2_{H^2}\ ds \\
&\quad + \int_0^t\int_{\Omega_f} a^k_i\partial_{mm}q\partial_k\partial_{mm}(\eta^i - x^i)\ d\Omega_f\ ds. \qquad (3.3.61)
\end{aligned}
$$

We use the divergence-free condition similarly as in (3.3.57)-(3.3.58) in order to conclude

$$
\begin{aligned}
&\int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm}(\eta^i - x^i)\ d\Omega_f\ ds \\
&\quad = \int_0^t \int_{\Omega_f} \partial_{mm} q \int_0^s \Big(\partial_t a_i^k \partial_k \partial_{mm}(\eta^i - x^i) \\
&\quad\quad - \partial_{mm} a_i^k \partial_k v^i - 2\partial_m a_i^k \partial_k \partial_m v^i\ d\tau \Big)\ dx\ ds \\
&\quad = \int_0^t \int_0^s \int_{\Omega_f} \partial_{mm} q(s) \partial_t a_i^k \partial_k \partial_{mm}(\eta^i - x^i)\ dx\ d\tau\ ds \\
&\quad\quad - \int_0^t \int_0^s \int_{\Omega_f} \partial_{mm} q(s) \partial_{mm} a_i^k \partial_k v^i\ dx\ d\tau\ ds \\
&\quad\quad - 2\int_0^t \int_0^s \int_{\Omega_f} \partial_{mm} q(s) \partial_m a_i^k \partial_k \partial_m v^i\ dx\ d\tau\ ds \\
&\quad \leq C \int_0^t \int_0^s \|\partial_{mm} q(s)\|_{L^2} \|\partial_t a_i^k\|_{L^\infty} \|\partial_k \partial_{mm}(\eta^i - x^i)\|_{L^2}\ d\tau\ ds \\
&\quad\quad + C \int_0^t \int_0^s \|\partial_{mm} q(s)\|_{L^2} \|\partial_{mm} a_i^k\|_{L^2} \|\partial_k v^i\|_{L^\infty}\ d\tau\ ds \\
&\quad\quad + C \int_0^t \int_0^s \int_{\Omega_f} \|\partial_{mm} q(s)\|_{L^2} \|\partial_m a_i^k\|_{L^6} \|\partial_k \partial_m v^i\|_{L^3}\ d\tau\ ds \\
&\quad \leq C \int_0^t \int_0^s \|q(s)\|_{H^2} \|a_t\|_{H^2} \|\nabla(\eta - x)\|_{H^2}\ d\tau\ ds \\
&\quad\quad + C \int_0^t \int_0^s \|q(s)\|_{H^2} \|\partial_{mm} a\|_{L^2} \|v\|_{H^3}\ d\tau\ ds \\
&\quad\quad + C \int_0^t \int_0^s \|q(s)\|_{H^2} \|\partial_m a\|_{H^1} \|v\|_{H^3}\ d\tau\ ds. \qquad (3.3.62)
\end{aligned}
$$

We use Holders inequality with (3.3.10) to write

$$
\begin{aligned}
&\int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm}(\eta^i - x^i)\ d\Omega_f\ ds \\
&\quad = \int_0^t \int_{\Omega_f} \partial_{mm} q \int_0^s \Big(\partial_t a_i^k \partial_k \partial_{mm}(\eta^i - x^i) \\
&\qquad -\partial_{mm} a_i^k \partial_k v^i - 2\partial_m a_i^k \partial_k \partial_m v^i\ d\tau \Big)\ dx\ ds \\
&\quad = \int_0^t \int_0^s \int_{\Omega_f} \partial_{mm} q(s) \partial_t a_i^k \partial_k \partial_{mm}(\eta^i - x^i)\ dx\ d\tau\ ds \\
&\qquad - \int_0^t \int_0^s \int_{\Omega_f} \partial_{mm} q(s) \partial_{mm} a_i^k \partial_k v^i\ dx\ d\tau\ ds \\
&\qquad -2 \int_0^t \int_0^s \int_{\Omega_f} \partial_{mm} q(s) \partial_m a_i^k \partial_k \partial_m v^i\ dx\ d\tau\ ds \\
&\quad \le C \int_0^t \int_0^s \|\partial_{mm} q(s)\|_{L^2} \|\partial_t a_i^k\|_{L^\infty} \|\partial_k \partial_{mm}(\eta^i - x^i)\|_{L^2}\ d\tau\ ds \\
&\qquad + C \int_0^t \int_0^s \|\partial_{mm} q(s)\|_{L^2} \|\partial_{mm} a_i^k\|_{L^2} \|\partial_k v^i\|_{L^\infty}\ d\tau\ ds \\
&\qquad + C \int_0^t \int_0^s \int_{\Omega_f} \|\partial_{mm} q(s)\|_{L^2} \|\partial_m a_i^k\|_{L^6} \|\partial_k \partial_m v^i\|_{L^3}\ d\tau\ ds \\
&\quad \le C \int_0^t \int_0^s \|q(s)\|_{H^2} \|a_t\|_{H^2} \|\nabla(\eta - x)\|_{H^2}\ d\tau\ ds \\
&\qquad + C \int_0^t \int_0^s \|q(s)\|_{H^2} \|\partial_{mm} a\|_{L^2} \|v\|_{H^3}\ d\tau\ ds \\
&\qquad + C \int_0^t \int_0^s \|q(s)\|_{H^2} \|\partial_m a\|_{H^1} \|v\|_{H^3}\ d\tau\ ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\epsilon \int_0^t \int_0^s \|q(s)\|_{H^2} \|\nabla(\eta - x)\|_{H^2} \; d\tau \; ds \\
&\quad + C\epsilon \int_0^t \|q(s)\|_{H^2} \int_0^s \|v\|_{H^3} \; d\tau \; ds \\
&\leq C\epsilon \int_0^t \int_0^s \|q(s)\|_{H^2} \tau^{\frac{1}{2}} \left(\int_0^\tau \|v\|_{H^3}^2 \; d\xi \right)^{\frac{1}{2}} \; d\tau \; ds \\
&\quad + C\epsilon \int_0^t \|q(s)\|_{H^2} s^{\frac{1}{2}} \left(\int_0^s \|v\|_{H^3}^2 \; d\tau \right)^{\frac{1}{2}} \; ds. \qquad (3.3.63)
\end{aligned}
$$

The Cauchy-Schwarz inequality implies that

$$
\begin{aligned}
&\int_0^t \int_{\Omega_f} a_i^k \partial_{mm} q \partial_k \partial_{mm} (\eta^i - x^i) \; d\Omega_f \; ds \\
&\quad \leq C\epsilon \int_0^t \|q(s)\|_{H^2} s^{\frac{1}{2}} \int_0^s \left(\int_0^s \|v\|_{H^3}^2 \; d\xi \right)^{\frac{1}{2}} \; d\tau \; ds \\
&\qquad + C\epsilon \int_0^t \|q(s)\|_{H^2}^2 \; ds + C\epsilon \int_0^t s \int_0^s \|v\|_{H^3}^2 \; d\tau \; ds \\
&\quad \leq C\epsilon \int_0^t \|q(s)\|_{H^2} s^{\frac{3}{2}} \left(\int_0^s \|v\|_{H^3}^2 \; d\xi \right)^{\frac{1}{2}} \; ds \\
&\qquad + C\epsilon \int_0^t \|q(s)\|_{H^2}^2 \; ds + C\epsilon \int_0^t s \int_0^t \|v\|_{H^3}^2 \; d\tau \; ds \\
&\quad \leq C\epsilon \int_0^t s \|q(s)\|_{H^2} s^{\frac{1}{2}} \left(\int_0^s \|v\|_{H^3}^2 \; d\xi \right)^{\frac{1}{2}} \; ds \\
&\qquad + C\epsilon \int_0^t \|q(s)\|_{H^2}^2 \; ds + C\epsilon t^2 \int_0^t \|v\|_{H^3}^2 \; d\tau \\
&\quad \leq C\epsilon \int_0^t s^2 \|q(s)\|_{H^2}^2 + C\epsilon \int_0^t s \int_0^s \|v\|_{H^3}^2 \; d\xi \; ds
\end{aligned}
$$

$$+C\epsilon\int_0^t \|q(s)\|_{H^2}^2\ ds + C\epsilon t^2\int_0^t \|v\|_{H^3}^2\ d\tau$$

$$\leq C\epsilon t^2\int_0^t \|q(s)\|_{H^2}^2 + C\epsilon\int_0^t s\int_0^t \|v\|_{H^3}^2\ d\xi\ ds$$

$$+C\epsilon\int_0^t \|q(s)\|_{H^2}^2\ ds + C\epsilon t^2\int_0^t \|v\|_{H^3}^2\ d\tau$$

$$\leq C\epsilon t^2\int_0^t \|q(s)\|_{H^2}^2 + C\epsilon t^2\int_0^t \|v\|_{H^3}^2\ d\xi$$

$$+C\epsilon\int_0^t \|q(s)\|_{H^2}^2\ ds + C\epsilon t^2\int_0^t \|v\|_{H^3}^2\ d\tau$$

$$\leq C\epsilon t^2\int_0^t \|v\|_{H^3}^2 + \|q\|_{H^2}^2 + C\epsilon\int_0^t \|q(s)\|_{H^2}^2\ ds. \tag{3.3.51}$$

Using (3.3.61) and (3.3.51) together with (A.2.14) implies

$$\begin{aligned}\int_0^t (\tilde{R}_{mm}(s), \partial_{mm}v(s))\ ds \quad &\leq \quad (C+Ct^2)\epsilon\int_0^t X(s)\ ds \\ &\qquad +(C+Ct^2)\epsilon\int_0^t \|f_f\circ\eta\|_{H^1}^2 + \|f_e\|_{H^1}^2\ ds. \quad (3.3.51)\end{aligned}$$

□

Combining (3.2.82), (3.3.56) and (3.3.60) yields

$$\begin{aligned}&E_{mm}(t) + \int_0^t E_{mm}(s)\ ds + \int_0^t D_{mm}(s)\ ds \\ &\quad \leq CE_{mm}(0) + (C+Ct^2)\epsilon\int_0^t (X(s) + \|f_f\circ\eta\|_{H^1}^2 + \|f_e\|_{H^1}^2)\ ds \\ &\qquad + C_\epsilon\|\partial_{mm}(f_f\circ\eta)\|_{L^2(\Omega_f\times[0,T])}^2 + \ C_\epsilon\|\partial_{mm}f_e\|_{L^2(\Omega_e\times[0,T])}^2. \end{aligned}\tag{3.3.52}$$

Lemma 3.3.10. *With R_{tm} defined in* (3.2.86)*, we have*

$$\int_0^t |(R_{tm}, \partial_t \partial_m v)| \; ds \quad \leq \quad C\epsilon \int_0^t X(s) \; ds + C\epsilon \int_0^t (\|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 + \|\partial_t (f_f \circ \eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2) \; ds \tag{3.3.53}$$

for $t \in [0, T]$.

Proof. This follows from similar arguments above combined with the divergence-free condition

$$\partial_t \partial_m a_i^k \partial_k v^i + \partial_m a_i^k \partial_k v_t^i + \partial_t a_i^k \partial_k \partial_m v^i + a_i^k \partial_k \partial_m v_t^i = 0.$$

$\square$

Lemma 3.3.11. *With $\tilde{R}_{tm}$ defined in* (3.2.88)*, we have*

$$\int_0^t |(\tilde{R}_{tm}, \partial_m v)| \; ds \quad \leq \quad C\epsilon \int_0^t X(s) \; ds + C\epsilon \int_0^t (\|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 + \|\partial_t (f_f \circ \eta)\|_{L^2}^2 + \|\partial_t f_e\|_{L^2}^2) \; ds \tag{3.3.54}$$

for $t \in [0, T]$.

Proof. This follows from similar arguments above combined with the divergence-free condition

$$\partial_m a_i^k \partial_k v^i + a_i^k \partial_k \partial_m v^i = 0.$$

$\square$

Combining (3.3.55) with (3.3.39) and (3.3.54), we have

$$E_{tm}(t) + \int_0^t E_{tm}(s) \; ds + \int_0^t D_{tm}(s) \; ds$$
$$\leq C E_{tm}(0) + C\|\nabla \partial_m v(0)\|_{L^2}^2 + C\epsilon \int_0^t X(s) \; ds + C\epsilon \int_0^t \|f_f \circ \eta\|_{H^1}^2 + \|f_e\|_{H^1}^2 \; ds$$

$$+C\epsilon\int_0^t \|\partial_t(f_f\circ\eta)\|_{L^2}^2+\|\partial_t f_e\|_{L^2}^2\ ds+C_\epsilon\|\partial_t\partial_m(f_f\circ\eta)\|_{L^2(\Omega_f\times[0,T])}^2$$

$$+C_\epsilon\|\partial_t\partial_m f_e\|_{L^2(\Omega_e\times[0,T])}^2+C\epsilon\int_0^t \|\partial_m v\|_{L^2}^2\ ds. \tag{3.3.55}$$

Summing the energy bounds obtained in Section 3.2, using estimates (3.3.13)-(3.3.55), and absorbing the small terms that appear on the right-hand side yields

$$\begin{aligned}
&X(t)+\int_0^t X(s)\ ds+\int_0^t \hat{D}(s)\ ds\\
&\le CX(0)+C\epsilon(1+t^2)\int_0^t (X(s)+\|f_f\circ\eta\|_{L^2}^2+\|f_e\|_{L^2}^2)\ ds+C\epsilon\Big(\|f_f\circ\eta\|_{H^1}^2\\
&\quad+\|f_e\|_{H^1}^2+\|\partial_t(f_f\circ\eta)\|_{L^2}^2+\|\partial_t f_e\|_{L^2}^2\Big)+C\epsilon\Big(\|(f_f\circ\eta)(0)\|_{H^1}^2+\|f_e(0)\|_{H^1}^2\\
&\quad+\|\partial_t(f_f\circ\eta)(0)\|_{L^2}^2+\|\partial_t f_e(0)\|_{L^2}^2\Big)+C\int_0^t (\|f_f\circ\eta\|_{L^2}^2+\|f_e\|_{L^2}^2+\|\partial_t(f_f\circ\eta)\|_{L^2}^2\\
&\quad+\|\partial_t f_e\|_{L^2}^2+\|\partial_{tt}(f_f\circ\eta)\|_{L^2}^2+\|\partial_{tt} f_e\|_{L^2}^2)\ ds+C\sum_{m=1}^2\Big(\int_0^t \|\partial_m(f_f\circ\eta)\|_{L^2}^2\\
&\quad+\|\partial_m f_e\|_{L^2}^2+\|\partial_{mm}(f_f\circ\eta)\|_{L^2}^2+\|\partial_{mm} f_e\|_{L^2}^2+\|\partial_t\partial_m(f_f\circ\eta)\|_{L^2}^2\\
&\quad+\|\partial_t\partial_m f_e\|_{L^2}^2\ ds\Big).
\end{aligned} \tag{3.3.56}$$

3.4 Decay of the Energy Norm $X(t)$

We denote $\mathcal{F}(t)$ by

$$\begin{aligned}
\mathcal{F}(t) &= \|(f_f\circ\eta)(t)\|_{H^1(\Omega_f)}^2+\|f_e(t)\|_{H^1(\Omega_e)}^2+\|\partial_t(f_f\circ\eta)(t)\|_{L^2(\Omega_f)}^2+\|\partial_t f_e(t)\|_{L^2(\Omega_e)}^2\\
&\quad+\|\partial_{tt}(f_f\circ\eta)(t)\|_{L^2(\Omega_f)}^2+\|\partial_{tt} f_e(t)\|_{L^2(\Omega_e)}^2+\sum_{m=1}^2\|\partial_{mm}(f_f\circ\eta)(t)\|_{L^2(\Omega_f)}^2\\
&\quad+\|\partial_{mm} f_e(t)\|_{L^2(\Omega_e)}^2+\|\partial_t\partial_m(f_f\circ\eta)(t)\|_{L^2(\Omega_f)}^2+\|\partial_t\partial_m f_e(t)\|_{L^2(\Omega_e)}^2,
\end{aligned} \tag{3.4.1}$$

and re-express (3.3.56) in terms of $\mathcal{F}(t)$ as follows:

$$\begin{aligned} X(t) + \int_0^t X(s)\ ds + \int_\tau^t \mathcal{F}(s)\ ds \ \ &\leq \ \ CX(0) + C\epsilon(1+t^2)\int_0^t (X(s)+\mathcal{F}(s))\ ds \\ &\quad + C\int_0^t \mathcal{F}(s)\ ds + C\mathcal{F}(t) + C\mathcal{F}(0). \end{aligned} \tag{3.4.2}$$

Shifting in time, we obtain

$$\begin{aligned} X(t) + \int_\tau^t X(s)\ ds + \int_0^t \mathcal{F}(s)\ ds \ \ &\leq \ \ C_0X(\tau) + \epsilon(C_0 + C_0(t-\tau)^2)\int_\tau^t (X(s)+\mathcal{F}(s))\ ds \\ &\quad + C_0\int_\tau^t \mathcal{F}(s)\ ds + C_0\mathcal{F}(t) + C_0\mathcal{F}(\tau) \end{aligned} \tag{3.4.3}$$

for $0 \leq \tau \leq t$. We now prove the decay of the norm $X(t)$.

Lemma 3.4.1. *Suppose that $X : [0,\infty) \to [0,\infty]$ is continuous for all t such that $X(t)$ is finite and assume it satisfies (3.4.3) for $0 \leq \tau \leq t$ where $C \geq 1$. Assume $X(0) + \mathcal{F}(0) \leq M$, $M > 0$, and $\mathcal{F} \in W^{1,1}(\mathbb{R}^+)$. Then there exists $C > 0$, $\omega > 0$, and $T > 0$ all depending on C_0 such that*

$$X(t) \leq CMe^{-\frac{t}{C}} + C\int_0^t e^{-\omega(t-s)}\big(\mathcal{F}(s) + \mathcal{F}'(s)\big)\ ds + C\mathcal{F}(t). \tag{3.4.4}$$

Proof. Let $T > 0$ such that

$$2C_0 + T > 4C_0^2 \tag{3.4.5}$$

and $\epsilon > 0$ such that

$$\epsilon < \frac{1}{4C_0T^2}. \tag{3.4.6}$$

Then we can write

$$C_0 \epsilon T^2 < \frac{1}{4}. \tag{3.4.7}$$

Then, when $t-\tau \leq T$, we have that $C_0\epsilon(t-\tau)^2 \leq \frac{1}{4}$. We can thus rewrite (3.4.3) as

$$\begin{aligned} X(t) + \int_\tau^t X(s)\ ds + \int_\tau^t \mathcal{F}(s)\ ds \ \ &\leq\ \ C_0 X(\tau) + \frac{1}{2}\int_\tau^t X(s)\ ds \\ &\quad + \left(C_0 + \frac{1}{2}\right)\int_\tau^t \mathcal{F}(s)\ ds \\ &\quad + C_0\mathcal{F}(t) + C_0\mathcal{F}(\tau), \end{aligned} \tag{3.4.8}$$

which implies

$$\begin{aligned} X(t) + \frac{1}{2}\int_\tau^t X(s)\ ds + \frac{1}{2}\int_\tau^t \mathcal{F}(s)\ ds \ \ &\leq\ \ C_0 X(\tau) + C_0\int_\tau^t \mathcal{F}(s)\ ds \\ &\quad + C_0\mathcal{F}(t) + C_0\mathcal{F}(\tau). \end{aligned} \tag{3.4.9}$$

Multiplying by 2 and bounding below, we have

$$X(t) + \int_\tau^t X(s)\ ds + \int_\tau^t \mathcal{F}(s)\ ds \ \ \leq\ \ 2C_0[X(\tau) + \mathcal{F}(\tau)] + 2C_0\int_\tau^t \mathcal{F}(s)\ ds + 2C_0\mathcal{F}(t). \tag{3.4.10}$$

For $\tau \in [0,T]$, we have

$$X(T) + \int_\tau^T X(s)\ ds + \int_\tau^T \mathcal{F}(s)\ ds \ \ \leq\ \ 2C_0[X(\tau) + \mathcal{F}(\tau)] + 2C_0\int_\tau^T \mathcal{F}(s)\ ds + 2C_0\mathcal{F}(T). \tag{3.4.11}$$

We move all but the first term on the right-hand side to the left of the inequality, i.e.

$$\frac{1}{2C_0}X(T) - \int_\tau^T \mathcal{F}(s)\ ds - \mathcal{F}(T) \ \ \leq \ \ X(\tau) + \mathcal{F}(\tau). \tag{3.4.12}$$

Then on $[0, T]$, we have from (3.4.11)

$$\begin{aligned} X(T) + \int_0^T \left(\frac{1}{2C_0}X(T) - \int_\tau^T \mathcal{F}(s)\ ds - \mathcal{F}(T)\right) d\tau \ \ \leq \ \ & 2C_0[X(0) + \mathcal{F}(0)] \\ & +2C_0 \int_0^T \mathcal{F}(s)\ ds \\ & +2C_0\mathcal{F}(T). \end{aligned} \tag{3.4.13}$$

This implies that

$$\begin{aligned} X(T) + \frac{T}{2C_0}X(T) \ \ \leq \ \ & 2C_0[X(0) + \mathcal{F}(0)] + 2C_0 \int_0^T \mathcal{F}(s)\ ds + 2C_0 \int_0^T \int_\tau^T \mathcal{F}(s)\ ds\ d\tau \\ & +(2C_0 + T)\mathcal{F}(T) \\ \leq \ \ & 2C_0[X(0) + \mathcal{F}(0)] + 2C_0 \int_0^T \mathcal{F}(s)\ ds + (2C_0 + T) \int_0^T \mathcal{F}(s)\ ds \\ & +(2C_0 + T)\mathcal{F}(T). \end{aligned} \tag{3.4.14}$$

Combining like terms yields

$$\left(\frac{2C_0 + T}{2C_0}\right)X(T) \ \ \leq \ \ 2C_0[X(0) + \mathcal{F}(0)] + (2C_0 + T) \int_0^T \mathcal{F}(s)\ ds + (2C_0 + T)\mathcal{F}(T). \tag{3.4.15}$$

Thus,

$$X(T) \leq \left(\frac{4C_0^2}{2C_0+T}\right)[X(0)+\mathcal{F}(0)] + 2C_0\int_0^T \mathcal{F}(s)\ ds + 2C_0\mathcal{F}(T). \quad (3.4.16)$$

By (3.4.5), we have that

$$\frac{4C_0^2}{2C_0+T} < 1.$$

Choose $\kappa = \frac{4C_0^2}{2C_0+T}$. Then,

$$X(T) \leq \kappa[X(0)+\mathcal{F}(0)] + 2C_0\int_0^T \mathcal{F}(s)\ ds + 2C_0\mathcal{F}(T). \quad (3.4.17)$$

We now apply these steps on intervals of the form $[T, 2T],\ [2T, 3T], \ldots$. We obtain the following bound on $X(nT)$, $n = 1, 2, \ldots$:

$$\begin{aligned} X(nT) \leq\ & \kappa^n[X(0)+\mathcal{F}(0)] + 2C_0\sum_{j=0}^{n-1}\kappa^j \int_{(n-j-1)T}^{(n-j)T} \mathcal{F}(s)\ ds \\ & +(2C_0+1)\sum_{j=0}^{n-1}\kappa^j\mathcal{F}\big((n-j)T\big). \end{aligned} \quad (3.4.18)$$

Consider the second term above. We have that

$$\begin{aligned} \sum_{j=0}^{n-1}\kappa^j \int_{(n-1-j)T}^{(n-j)T} \mathcal{F}(s)\ ds &= \sum_{\ell=0}^{n-1}\kappa^{n-1-\ell}\int_{\ell T}^{(\ell+1)T} \mathcal{F}(s)\ ds \\ &= \sum_{\ell=0}^{n-1}\kappa^{n-1}\int_{\ell T}^{(\ell+1)T} \kappa^{-\ell}\mathcal{F}(s)\ ds. \end{aligned}$$

Since $\kappa < 1$, there is an $\omega > 0$ such that

$$\kappa = e^{-\omega T}. \quad (3.4.19)$$

This implies that $\kappa^{-\ell} = e^{\omega \ell T}$ and $\kappa^{n-1} = e^{-\omega(n-1)T} = e^{\omega T} e^{-\omega n T}$. This implies that

$$
\begin{aligned}
\sum_{j=0}^{n-1} \kappa^j \int\limits_{(n-1-j)T}^{(n-j)T} \mathcal{F}(s)\ ds &= \sum_{\ell=0}^{n-1} e^{\omega T} e^{-\omega n T} \int\limits_{\ell T}^{(\ell+1)T} e^{\omega \ell T} \mathcal{F}(s)\ ds. \\
&\leq \sum_{\ell=0}^{n-1} e^{\omega T} e^{-\omega n T} \int\limits_{\ell T}^{(\ell+1)T} e^{\omega s} \mathcal{F}(s)\ ds \text{ [since } s > \ell T] \\
&\leq e^{\omega T} e^{-\omega n T} \int\limits_{0}^{nT} e^{\omega s} \mathcal{F}(s)\ ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
X(nT) &\leq \kappa^n [X(0) + \mathcal{F}(0)] + 2C_0 e^{\omega T} e^{-\omega n T} \int\limits_0^{nT} e^{\omega s} \mathcal{F}(s)\ ds \\
&\qquad + (2C_0 + 1) \sum_{j=0}^{n-1} \kappa^j \mathcal{F}\big((n-j)T\big). \\
&\leq \kappa^n [X(0) + \mathcal{F}(0)] + 2C_0 e^{\omega T} \int\limits_0^{nT} e^{-\omega(nT-s)} \mathcal{F}(s)\ ds \\
&\qquad + (2C_0 + 1) \sum_{j=0}^{n-1} \kappa^j \mathcal{F}\big((n-j)T\big). \qquad (3.4.20)
\end{aligned}
$$

Next let $t > 0$. Then there exists an integer n such that $t \in [nT, (n+1)T]$. By (3.4.10), we have

$$
\begin{aligned}
X(t) &\leq 2C_0 [X(nT) + \mathcal{F}(nT)] + 2C_0 \int\limits_{nT}^{t} \mathcal{F}(s)\ ds + 2C_0 \mathcal{F}(t) \\
&\leq 2C_0 \kappa^n [X(0) + \mathcal{F}(0)] + 4C_0^2 e^{\omega T} \int\limits_0^{nT} e^{-\omega(nT-s)} \mathcal{F}(s)\ ds + 2C_0 \int\limits_{nT}^{t} \mathcal{F}(s)\ ds
\end{aligned}
$$

$$+2C_0(2C_0+2)\sum_{j=0}^{n-1}\kappa^j\mathcal{F}\big((n-j)T\big)+2C_0\mathcal{F}(t). \tag{3.4.21}$$

In the last integral on the right hand side of (3.4.21), we have

$$\begin{aligned} nT \le s \quad &\Longrightarrow \quad -nT \ge -s \\ &\Longrightarrow \quad 0 \ge nT - s \\ &\Longrightarrow \quad 0 \le -\omega(nT-s) \\ &\Longrightarrow \quad 1 \le e^{-\omega(nT-s)} \\ &\Longrightarrow \quad \mathcal{F}(s) \le \mathcal{F}(s)e^{-\omega(nT-s)}. \end{aligned}$$

We also make the observation that $2C_0 \le 4C_0^2e^{\omega T}$. These inequalities together allow us to express (3.4.21) as

$$\begin{aligned} X(t) \quad \le \quad & 2C_0\kappa^n[X(0)+\mathcal{F}(0)] + 4C_0^2e^{\omega T}\int_0^{nT} e^{-\omega(nT-s)}\mathcal{F}(s)\ ds + 4C_0^2e^{\omega T}\int_{nT}^{t}\mathcal{F}(s)\ ds \\ & +2C_0(2C_0+2)\sum_{j=0}^{n-1}\kappa^j\mathcal{F}\big((n-j)T\big)+2C_0\mathcal{F}(t) \\ \le \quad & 2C_0\kappa^n[X(0)+\mathcal{F}(0)] + 4C_0^2e^{\omega T}\int_0^{t}\mathcal{F}(s)\ ds \\ & +2C_0(2C_0+2)\sum_{j=0}^{n-1}\kappa^j\mathcal{F}\big((n-j)T\big)+2C_0\mathcal{F}(t). \end{aligned} \tag{3.4.22}$$

We now make the following observations:

$$\begin{aligned} \left\lceil \frac{t-T}{T} \right\rceil = n \quad &\Longrightarrow \quad n \ge \frac{t-T}{T} \text{ for } t \in \big(nT,(n+1)T\big) \\ &\Longrightarrow \quad \kappa^n \le \kappa^{\frac{t-T}{T}} \text{ since } \kappa < 1, \end{aligned}$$

which implies

$$\kappa^n \le \kappa^{\frac{t-T}{T}} = e^{-\frac{|\ln(\kappa)|}{T}(t-T)} = e^{|\ln(\kappa)|} e^{-\frac{|\ln(\kappa)|}{T} t}. \tag{3.4.23}$$

Furthermore,

$$\begin{aligned} n \ge \frac{t-T}{T} \text{ for } t \in \big(nT, (n+1)T\big) &\implies nT - s \ge t - T - s \\ &\implies -\omega(nT-s) \le -\omega(t-T-s) \\ &\implies e^{-\omega(nT-s)} \le e^{-\omega(t-T-s)}. \end{aligned} \tag{3.4.24}$$

Using (3.4.23) and (3.4.24) in (3.4.22), we obtain

$$\begin{aligned} X(t) \;\le\;& 2C_0 e^{|\ln(\kappa)|}[X(0) + \mathcal{F}(0)] e^{-\frac{|\ln(\kappa)|}{T} t} + 4C_0^2 e^{\omega T} \int_0^t e^{-\omega(t-T-s)} \mathcal{F}(s)\, ds \\ & + 2C_0(2C_0+2) \sum_{j=0}^{n-1} \kappa^j \mathcal{F}\big((n-j)T\big) + 2C_0 \mathcal{F}(t) \\ \le\;& 2C_0 e^{|\ln(\kappa)|}[X(0) + \mathcal{F}(0)] e^{-\frac{|\ln(\kappa)|}{T} t} + 4C_0^2 e^{2\omega T} \int_0^t e^{-\omega(t-s)} \mathcal{F}(s)\, ds \\ & + 2C_0(2C_0+2) \sum_{j=0}^{n-1} \kappa^j \mathcal{F}\big((n-j)T\big) + 2C_0 \mathcal{F}(t). \end{aligned} \tag{3.4.25}$$

Now we consider the $\sum_{j=0}^{n-1} \kappa^j \mathcal{F}\big((n-j)T\big)$ term in (3.4.25). First, we express κ as an exponential and re-index the sum by setting $r = j+1$.

$$\begin{aligned} \sum_{j=0}^{n-1} \kappa^j \mathcal{F}\big((n-j)T\big) &= \sum_{j=0}^{n-1} e^{-\omega j T} \mathcal{F}\big((n-j)T\big) \\ &= \sum_{r=1}^{n} e^{-\omega(r-1)T} \mathcal{F}\big((n-(r-1))T\big). \end{aligned} \tag{3.4.26}$$

We set $f(r) = e^{-\omega(r-1)T} \mathcal{F}\big((n-(r-1))T\big)$, and we denote

$$s_n = \sum_{j=0}^{n-1} \kappa^j \mathcal{F}\big((n-j)T\big) = \sum_{r=1}^{n} f(r). \tag{3.4.27}$$

Following the strategy of [64], we apply Abel's partial summation formula to (3.4.27) to obtain

$$s_n = f(1) + \int_1^n f(r)\ dr + \int_1^n f'(r)\ dr. \tag{3.4.28}$$

We now seek to bound each term on the right side of (3.4.28). We observe that $f(1) = \mathcal{F}(nT)$. Since $\mathcal{F} \in W^{1,1}(\mathbb{R}^+)$, we know that $\mathcal{F}$ is absolutely continuous. This implies that

$$\mathcal{F}(t) = \mathcal{F}(nT) + \int_{nT}^{t} \mathcal{F}'(s)\ ds \tag{3.4.29}$$

on $[nT, (n+1)T]$. Therefore,

$$\begin{aligned} \mathcal{F}(nT) &= \mathcal{F}(t) - \int_{nT}^{t} \mathcal{F}'(s)\ ds \\ &\leq \mathcal{F}(t) + \int_{nT}^{t} \mathcal{F}'(s)\ ds. \end{aligned} \tag{3.4.30}$$

We also make the following observation: for $s \in [nT, (n+1)T]$,

$$\begin{aligned} nT \leq s \leq (n+1)T \quad &\implies \quad 0 \leq -\omega(nT - s) \\ &\implies \quad 1 \leq e^{-\omega(nT-s)}. \end{aligned} \tag{3.4.31}$$

Using (3.4.31) and (3.4.24) in yields

$$
\begin{aligned}
\mathcal{F}(nT) &\leq \mathcal{F}(t) + \int_{nT}^{t} e^{-\omega(nT-s)} \mathcal{F}'(s)\, ds. \\
&\leq \mathcal{F}(t) + C_T \int_{nT}^{t} e^{-\omega(t-s)} \mathcal{F}'(s)\, ds. \qquad (3.4.32)
\end{aligned}
$$

Next, we bound the $\int_1^n f(r)\, dr$ term in (3.4.28). Using the change of variable $s = \big(n - (r-1)\big)T$, we obtain

$$
\begin{aligned}
\int_1^n f(r)\, dr &= \int_1^n e^{-\omega(r-1)T} \mathcal{F}\big((n-(r-1))T\big)\, dr \\
&= C_T \int_T^{nT} e^{-\omega(t-s)} \mathcal{F}(s)\, ds. \qquad (3.4.33)
\end{aligned}
$$

We follow a similar approach with the $\int_1^n f'(r)\, dr$ term in (3.4.28) to obtain

$$
\int_1^n f'(r)\, dr = C_T \int_T^{nT} e^{-\omega(t-s)} \mathcal{F}(s)\, ds + C_T \int_T^{nT} e^{-\omega(t-s)} \mathcal{F}'(s)\, ds. \qquad (3.4.34)
$$

Using (3.4.32), (3.4.33), and (3.4.34) in (3.4.28), we obtain

$$
\begin{aligned}
s_n &\leq \mathcal{F}(t) + C_T \int_{nT}^{t} e^{-\omega(t-s)} \mathcal{F}'(s)\, ds + C_T \int_T^{nT} e^{-\omega(t-s)} \mathcal{F}(s)\, ds + C_T \int_T^{nT} e^{-\omega(t-s)} \mathcal{F}'(s)\, ds \\
&\leq \mathcal{F}(t) + C_T \int_0^{t} e^{-\omega(t-s)} \mathcal{F}'(s)\, ds + C_T \int_0^{t} e^{-\omega(t-s)} \mathcal{F}(s)\, ds. \qquad (3.4.35)
\end{aligned}
$$

Using (3.4.35) in (3.4.25), we obtain the following bound on $X(t)$, $t > 0$:

$$X(t) \leq 2C_0 e^{|\ln(\kappa)|}[X(0) + \mathcal{F}(0)]e^{-\frac{|\ln(\kappa)|}{T}t} + (4C_0^2 e^{2\omega T} + C_T)\int_0^t e^{-\omega(t-s)}\mathcal{F}(s)\ ds$$
$$+C_T \int_0^t e^{-\omega(t-s)}\mathcal{F}'(s)\ ds + (2C_0 + 1)\mathcal{F}(t). \tag{3.4.36}$$

Let $C_1 = \max\{2C_0 e^{|\ln(\kappa)|}, \frac{T}{|\ln(\kappa)|},\ 4C_0^2 e^{2\omega T} + C_T,\ C_T\}$. Then,

$$X(t) \leq C_1[X(0) + \mathcal{F}(0)]e^{-\frac{t}{C_1}} + C_1 \int_0^t e^{-\omega(t-s)}\mathcal{F}(s)\ ds$$
$$+C_1 \int_0^t e^{-\omega(t-s)}\mathcal{F}'(s)\ ds + C_1\mathcal{F}(t). \tag{3.4.37}$$

□

Remark 3.4.2. *Lemma 3.4.1 also holds in the case where $\mathcal{F}(t)$ decays exponentially, i.e.*

$$\mathcal{F}(t) \leq C_{\mathcal{F}}\mathcal{F}(0)e^{-\frac{t}{C_{\mathcal{F}}}},\ t \geq 0. \tag{3.4.38}$$

In this case, we do not require $\mathcal{F}'(t) \in L^1(\mathbb{R}^+)$. We adjust the proof of the lemma by applying (3.4.38) *to* (3.4.22) *to obtain a bound of*

$$X(t) \leq CMe^{-\frac{t}{C}}, \tag{3.4.39}$$

where $\mathcal{X}(0) + \mathcal{F}(0) < M$.

Chapter 4

Well-posedness Theory for FSI

The goal for this section is to investigate and build upon the current well-posedness theory for free-boundary fluid structure interactions. As previously mentioned, the main challenges associated with studying the existence of solutions for FSI models include the nonlinear nature of the equations, the time-dependence of the domains, and the coupling of parabolic and hyperbolic phases. These difficulties have left the question of existence and uniqueness of solutions open until recently. At this stage, we divide our attention between two cases: local and global theory. In both cases, the first step in obtaining a solution for the coupled system is to write the equations on a fixed reference configuration using the flow mapping η. This configuration is known as the Lagrangian framework.

4.1 Local Theory: Quasilinear Elasticity

The analysis of an FSI model from the perspective of optimal control heavily relies on the existing well-posedness results in the literature. We thus consider the following simplified model proposed in [18] to address the existence of a local-in-time solution for a fluid-structure problem:

$$\begin{cases} u_t - \nu\Delta u + Du\cdot u + \nabla p = f_f, & \Omega_f(t), \\ \mathrm{div} u = 0, & \Omega_f(t), \\ \eta^e_{tt} - \mathrm{Div}\Sigma = f_e, & \Omega_e \times (0,T), \\ u \circ \eta^f = \eta^e_t, & \Gamma_c \times (0,T), \\ \Sigma\cdot N = [\sigma(u,p)\circ\eta^f][(D\eta^f)^{-T}N], & \Gamma_c\times(0,T), \\ u = 0, & \Gamma_f \times (0,T). \end{cases} \tag{4.1.1}$$

Here, $\eta^e(x,t) = \eta(x,t)|_{\Omega_e}$ and $\eta^f(x,t) = \eta(x,t)|_{\Omega_f}$ as the positions at time $t \geq 0$ of the material point x in Ω_e and Ω_f, respectively. That is

$$\eta^e : \Omega_e \to \Omega_e(t),\ \eta^f : \Omega_e \to \Omega_f(t).$$

Let $F = (F_f, F_e)$ denote the external forcing in the material frame on the coupled system (note: in the context of (3.1.2), $F^f = f_f \circ \eta$ and F^e is the external force restricted to the solid domain Ω_e). In this framework, the full coupled system can be written componentwise in Lagrangian coordinates as follows:

$$\begin{cases} v^i_t - \partial_j(a^j_\ell a^k_\ell \partial_k v^i) + \partial_k(a^k_i q) = F^i_f, & \Omega_f\times(0,T), \\ a^k_i\partial_k v^i = 0, & \Omega_f\times(0,T), \\ v_t - c^{mjk\ell}\partial_\ell[(\partial_m\eta\cdot\partial_j\eta - \delta_{mj})\partial_k\eta] = F_e, & \Omega_e\times(0,T), \\ c^{mjk\ell}(\partial_m\eta\cdot\partial_j\eta-\delta_{mj})\partial_k\eta^i N_\ell = \nu a^k_\ell a^j_\ell\partial_k v^i N_j - a^j_i q N_j, & \Gamma_c\times(0,T), \\ v = 0, & \Gamma_f\times(0,T), \\ v(x,0) = u_0, & \mathcal{D}, \\ \eta(x,0) = x, & \mathcal{D}, \end{cases} \tag{4.1.2}$$

where N is the normal field pointing inward to the fluid domain and $c^{ijk\ell}$ is the fourth order tensor (2.5.25) expressed componentwise, i.e.

$$c^{ijk\ell}\lambda\delta_{jk}\delta_{k\ell} + \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}).$$

Local Well-posedness

To obtain the existence and uniqueness of a local solution for (4.1.2), we consider the following functional framework: define

$$\begin{aligned} V_f^4(T) &:= \left\{\phi \in L^2(0,T;\mathbf{H}^4(\Omega_f)) \mid \partial_t^n \phi \in L^2(0,T;\mathbf{H}^{4-n}(\Omega_f)), n=1,2,3\right\}, \\ V_e^4(T) &:= \left\{\psi \in L^2(0,T;\mathbf{H}^4(\Omega_e)) \mid \partial_t^n \psi \in L^2(0,T;\mathbf{H}^{4-n}(\Omega_e)), n=1,2,3\right\}, \end{aligned}$$

on the fluid and solid domains, respectively. On the entire domain $\mathcal{D}$, we introduce the space

$$\mathbf{L}^2_{\text{div},f}(\mathcal{D}) := \{\psi \in \mathbf{L}^2(\mathbf{D}) \mid \text{div}\psi = 0 \text{ in } \Omega_f,\ \psi \cdot n = 0 \text{ on } \partial\mathcal{D}\}.$$

Next, we define the set of fluid and flow velocities:

$$X_T := \left\{v \in L^2(0,T;\mathbf{H}^1_0(\mathcal{D})) \mid \left(v^f, \int_0^{(\cdot)} v^e\right) \in V_f^4(T) \times V_e^4(T)\right\}.$$

We note that X_T is a separable Hilbert space, and it is endowed with its natural norm

$$\|v\|^2_{X_T} = \|v\|^2_{L^2(0,T;\mathbf{H}^1_0(\mathcal{D}))} + \sum_{n=0}^{3}\left[\|\partial_t^n v^f\|^2_{L^2(0,T;\mathbf{H}^{4-n}(\Omega_f))} + \left\|\partial_t^n\left(\int_0^{(\cdot)} v^e\right)\right\|^2_{L^2(0,T;\mathbf{H}^{4-n}(\Omega_e))}\right].$$

In order to specify additional regularity on the solid, we define the following subspace of X_T:

$$\begin{aligned} W_T := &\left\{v \in X_T \mid v_{ttt} \in L^\infty(0,T;\mathbf{L}^2(\mathcal{D})),\ \partial_t^n \int_0^{(\cdot)} v^e \in L^\infty(0,T;\mathbf{H}^{4-n}(\Omega_e)),\right. \\ &\left. n = 0,1,2,3\right\}, \end{aligned} \tag{4.1.3}$$

with the norm given by

$$\|v\|^2_{W_T} = \|v\|^2_{X_T} + \|v_{ttt}\|^2_{L^\infty(0,T;\mathbf{L}^2(\mathcal{D}))} + \sum_{n=0}^{3}\left\|\partial_t^n \int_0^{(\cdot)} v^e\right\|^2_{L^\infty(0,T;\mathbf{H}^{4-n}(\Omega_e))}.$$

For the pressure regularity on the fluid domain, let

$$Y_T \ := \ \big\{q \in L^2\big(0,T;H^3(\Omega_f)\big) \mid \partial_t^n q \in L^2\big(0,T;H^{3-n}(\Omega_f)\big),\ n=1,2\big\}\,, \quad (4.1.4)$$

endowed with its natural Hilbert norm

$$\|q\|_{Y_T}^2 \ = \ \sum_{n=0}^{2} \|\partial_t^n q\|_{L^2(0,T;H^{3-n}(\Omega_f))}^2.$$

Furthermore, we consider its subspace

$$Z_T \ := \ \big\{q \in Y_T \mid q_{tt} \in L^\infty\big(0,T;\mathbf{L}^2(\Omega_f)\big)\big\} = Y_T \cap W^{2,\infty}(0,T;\mathbf{L}^2(\Omega_f)), \quad (4.1.5)$$

with norm given by

$$\|q\|_{Z_T}^2 \ = \ \|q\|_{Y_T}^2 + \|\partial_t^2 q\|_{L^\infty(0,T;L^2(\Omega_f))}^2.$$

We now state the assumptions imposed on the source f and its initial data:

Assumption 4.1.1. *Suppose that source term $f(t)$ is defined on $[0,\overline{T}]$. Assume*

$$f \in E_{\overline{T}} := \{\phi \in L^2(0,\overline{T};\boldsymbol{H}^3(\mathcal{D})) \mid \partial_t^n \phi \in L^2(0,\overline{T};\boldsymbol{H}^{3-n}(\mathcal{D}) \text{ for } n=1,2,3\}, \quad (4.1.6)$$

$$f(0) \in \boldsymbol{H}^4(\mathcal{D}), \quad f_t(0) \in \boldsymbol{H}^4(\mathcal{D}).$$

The following existence and uniqueness result is provided in [18]:

Theorem 4.1.2. *Let $\mathcal{D} \subset \mathbb{R}^3$ be an open bounded domain of class H^4, and let $\Omega_e \subset \mathcal{D}$ be an open set of class H^4 such that $\overline{\Omega}_e \subset \mathcal{D}$. Suppose $u_0 \in \boldsymbol{H}^6(\Omega_f) \cap \boldsymbol{H}^6(\Omega_e) \cap \boldsymbol{H}^1_0(\mathcal{D}) \cap \boldsymbol{L}^2_{div,f}$. Let $\Omega_f = \mathcal{D} \cap (\overline{\Omega}_e)^c$, and let $\nu > 0$, $\lambda > 0$, $\mu > 0$ be given. Under Assumption 4.1.1 and necessary compatibility conditions on the initial data (provided in [18, Thm. 1]), there exists $T \in (0,\overline{T})$ depending on u_0, f, and Ω_f, such that there is a unique solution $(v,q) \in W_T \times Z_T$ of the problem (4.1.2). Furthermore,*

$$\eta \in C([0,T];\boldsymbol{H}^4(\Omega_f) \cap \boldsymbol{H}^4(\Omega_e) \cap \boldsymbol{H}^1(\mathcal{D}))\,. \quad (4.1.7)$$

4.2 Lemmas for the Construction of a Global Solution

Our goal for this section is to establish the existence of a unique, global in time solution for the coupled system (3.2.4). We follow the strategy outlined in [33] for the proof of well-posedness. First, we obtain the existence and uniqueness of solutions for an associated linear problem in which the coefficient matrix a is given and close to the identity matrix. It is at this stage that we address the mismatch in regularity between the parabolic and hyperbolic phases in the fluid-structure system. Using the maximal parabolic regularity of the Stokes system [51] in conjunction with the sharp trace regularity for the wave equation given in [41], we can transfer the regularity properties of the fluid across the common interface without losing derivatives.

The construction of the solution for the nonlinear problem arises from an iteration scheme based on the well-posedness results for the linear problem, energy estimates that yield a Gronwall-type inequality for lower order norms of the iterate solutions, and the contraction mapping theorem.

We now state the main result, which will be proven in Section 4.3:

Theorem 4.2.1. *Let $\alpha, \beta > 0$. Assume that $v_0 \in V \cap H^{\frac{7}{2}}(\Omega_f)$, $v_t(0) \in V \cap H^{\frac{5}{2}}(\Omega_f)$, $v_{tt}(0) \in V$, $w_0 \in H^{\frac{15}{4}-\delta}(\Omega_e)$, and $w_1 \in H^{\frac{11}{4}-\delta}(\Omega_e)$ for $\delta \in (0, \frac{1}{4})$ satisfying the compatibility conditions*

$$
\begin{aligned}
&w_1 = v_0,\\
&\Delta w_0 - \alpha w_1 - \beta w_0 + f_e(0) = \Delta v_0 - \nabla q_0 + f_f(0),\\
&\Delta w_1 - \alpha w_{tt}(0) - \beta w_1 + \partial_t f_e(0) = \Delta v_t(0) - \nabla q_t(0) + \partial_j(\partial_t a_j^k(0)\partial_k v^i(0))\\
&\qquad\qquad -\partial_t a_i^k(0)\partial_k q(0) + Df_f(0)v_0 + \partial_t f_f(0),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial w_0}{\partial N}\cdot\tau = \frac{\partial v_0}{\partial N}\cdot\tau,\\
&\frac{\partial w_1}{\partial N}\cdot\tau = \frac{\partial}{\partial N}\left(\Delta v_0 - \nabla q_0 + f_f(0)\right)\cdot\tau,\\
&\frac{\partial w_{tt}(0)}{\partial N}\cdot\tau = \frac{\partial}{\partial N}\big(\Delta v_t(0) - \nabla q_t(0) + \partial_j(\partial_t a_j^k(0)\partial_k v^i(0)) - \partial_t a_i^k(0)\partial_k q(0)
\end{aligned}
$$

$$+Df_f(0)v_0 + \partial_t f_f(0)\big) \cdot \tau,$$

on Γ_c, where τ is a unit tangential vector. Furthermore,

$$\begin{aligned}
&v_0 = 0,\\
&\Delta v_0 - \nabla q_0 + f_f(0) = 0,\\
&-\partial_j(\partial_t a_j^k(0)\partial_k v^i(0)) - \Delta\partial_t v^i(0) + \partial_t a_i^k(0)\partial_k q(0) + \partial_{it} q(0) + Df_f(0)v_0 + \partial_t f_f(0) = 0,
\end{aligned}$$

on Γ_f. Assume, in addition, that

$$\|v_0\|_{H^3} + \|v_t(0)\|_{H^1} + \|v_{tt}(0)\|_{L^2} + \|w_0\|_{H^3} + \|w_1\|_{H^2} \leq \epsilon,$$

where $\epsilon > 0$ is a sufficiently small constant. Let $f = (f_f, f_e)$ satisfy

$$\begin{cases}
f_f \in L^2([0,T], H^3(\Omega));\ \partial_t f_f \in L^2([0,T], H^2(\Omega)),\ \partial_{tt} f_f \in L^2([0,T], H^1(\Omega)),\\
f_e, \in W^{1,1}([0,T], H^{\frac{3}{2}}(\Omega_e)) \cap W^{\frac{3}{2},1}([0,T], L^2(\Omega_e)),\\
\partial_{tt} f_e, \in L^1([0,T], H^{\frac{1}{2}}(\Omega_e)),\ \partial_{ttt} f_e \in L^1([0,T], H^{-\frac{1}{4}-\delta}(\Omega_e)),\\
\partial_m f_e, \in L^1([0,T], H^{\frac{3}{2}}(\Omega_e)) \cap W^{\frac{1}{2},1}([0,T], L^2(\Omega_e)),\\
\partial_{tt}\partial_m f_e \in L^1([0,T], H^{-\frac{1}{4}-\delta}(\Omega_e)),
\end{cases} \tag{4.2.1}$$

for $m = 1, 2$ and lie in the ball

$$\begin{aligned}
\Bigg\{ f = (f_f, f_e) \quad | \quad & \|f_f\|_{L^2([0,T];H^3(\Omega_f))} + \|\partial_t f_f\|_{L^2([0,T];H^2(\Omega_f))} + \|\partial_{tt} f_f\|_{L^2([0,T];H^1(\Omega_f))}\\
&+ \|f_e\|_{W^{1,1}([0,T];H^{\frac{3}{2}}(\Omega_e))} + \|f_e\|_{W^{\frac{3}{2},1}([0,T];L^2(\Omega_e))} + \|\partial_{tt} f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))}\\
&+ \|\partial_{ttt} f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} + \sum_{m=1}^{2}\Bigg(\|\partial_m f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))}\\
&+ \|\partial_m f_e\|_{W^{\frac{1}{2},1}([0,T];L^2(\Omega_e))} + \|\partial_{tt}\partial_m f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}\Bigg) \leq \|y_0\|_{\tilde{Y}(0)} \Bigg\}.
\end{aligned} \tag{4.2.2}$$

If the norm

$$\|f_f(t)\|_{H^3(\Omega)} + \|\partial_t f_f(t)\|_{H^2(\Omega)} + \|\partial_{tt} f_f(t)\|_{H^1(\Omega)} + \|f_e(t)\|_{H^1(\Omega)} + \|\partial_t f_e(t)\|_{L^2(\Omega)}$$

$$+\|\partial_{tt} f_e(t)\|_{L^2(\Omega)} + \sum_{m=0}^{2} \|\partial_{mm} f_e(t)\|_{L^2(\Omega)} + \|\partial_t \partial_m f_e(t)\|_{L^2(\Omega_f)} \tag{4.2.3}$$

is sufficiently small in $H^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, then, for any $T > 0$, there exists a unique global solution (v, w, q) to (3.2.4) *in the space*

$$\begin{aligned} \tilde{Y} \ &= \ \{(v, q, w) \mid v \in L^2([0,T]; H^4(\Omega_f)) \cap H^1([0,T]; H^3(\Omega_f)), \\ &\quad v_t \in H^1([0,T]; H^2(\Omega_f)),\ v_{tt} \in H^1([0,T]; L^2(\Omega_f)), \\ &\quad q \in L^2([0,T]; H^3(\Omega_f)) \cap H^1([0,T]; H^2(\Omega_f)),\ q_t \in H^1([0,T]; H^1(\Omega_f)), \\ &\quad \partial_t^j w \in L^\infty([0,T]; H^{\frac{15}{4}-\delta-j}(\Omega_e)),\ j = 0,1,2,3\}. \end{aligned} \tag{4.2.4}$$

First, we construct a local solution to a linear problem with given coefficient matrix a that is close to identity in some sense. It's at this stage that we address the mismatch in regularity between the parabolic and hyperbolic phases using the sharp trace regularity of the Dirichlet-Neumann map given by [41] and the maximal parabolic regularity of [51]. Unfortunately, the solution obtained will not be sufficiently regular to take advantage of the decay obtained via Lemma 3.4.1. Thus we construct higher regularity solutions that allow us to take advantage of the decay mechanism of Lemma 3.4.1 in extending local solutions for all times $t > 0$.

4.2.1 Linear Stokes

We begin this analysis by looking at a linear Stokes system with Neumann boundary condition. The following result, given in [51], provides the maximal parabolic regularity used to pass regularity information from the fluid to the solid across the boundary Γ_c.

Lemma 4.2.2. *Consider the system*

$$\begin{cases} \partial_t v^k - \Delta v^k + \nabla q^k = f^k, & \Omega_f \times (0,T), \\ \partial_i v^i = g, & \Omega_f \times (0,T), \end{cases} \tag{4.2.5}$$

subject to the mixed boundary conditions

$$\begin{cases} \partial_j v^k N^j - qN^k = h^k, & \Gamma_c \times (0,T), \\ v^k = 0, & \Gamma_f \times (0,T), \end{cases} \tag{4.2.6}$$

for $k = 1, 2, 3$ *and subject to the compatibility conditions*

$$\int\limits_{\Omega_f} g(0)\ dx \quad = \quad \int\limits_{\Gamma_c} v_0 \cdot N\ d\Gamma_c. \tag{4.2.7}$$

and the assumption

$$g_t \quad = \quad divA + B, \tag{4.2.8}$$

where $A, B \in L^2(\Omega_f \times [0,T])$. *Let* $v_0 \in H^1(\Omega_f)$. *If the terms obey* $f \in L^2(\Omega_f \times [0,T])$, $g \in L^2\big([0,T]; H^1(\Omega_f)\big)$, *and* $h \in L^2\big([0,T]; H^{\frac{1}{2}}(\Omega_f)\big) \cap H^{\frac{1}{4}}\big([0,T]; L^2(\Gamma_c)\big)$, *then there is a unique solutions* $y = (v,q)$ *on* $[0,T]$ *to the non-homogeneous system which satisfies*

$$\begin{aligned} \|y\| \quad \le \quad C\Bigg(& \|v_0\|_{H^1(\Omega_f)} + \|f\|_{L^2(\Omega_f \times [0,T]} + \|g\|_{L^2([0,T];H^1(\Omega_f))} + \|A\|_{L^2([0,T];L^2(\Omega_f))} \\ & + \|B\|_{L^2([0,T];L^2(\Omega_f))} + \|h\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|h\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Bigg), \end{aligned} \tag{4.2.9}$$

where $C > 0$ *is a constant and the norm* $\|y\|$ *is given by*

$$\begin{aligned} \|y\| \quad = \quad & \|v\|_{L^2([0,T];H^2(\Omega_f))} + \|v\|_{C([0,T];H^2(\Omega_f))} + \|q\|_{L^2([0,T];H^1(\Omega_f))} \\ & + \|q\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|v_t\|_{L^2([0,T];L^2(\Omega_f))}. \end{aligned} \tag{4.2.10}$$

We now apply Lemma 4.2.2 to the following system that couples the linear Stokes equations (4.2.5)-(4.2.7) with the damped wave equation (2.5.18).

$$\begin{cases} \partial_t v - \Delta v + \nabla q = f_f, & \Omega_f \times (0,T), \\ \text{div} v = g, & \Omega_f \times (0,T), \\ w_{tt} - \Delta w + \alpha w_t + \beta w = f_e, & \Omega_e \times (0,T), \end{cases} \tag{4.2.11}$$

with boundary conditions

$$\begin{cases} v = w_t, & \Gamma_c \times (0,T), \\ v = 0, & \Gamma_f \times (0,T), \\ \frac{\partial v}{\partial N} - qN = \frac{\partial w}{\partial N} + h, & \Gamma_c \times (0,T). \end{cases} \tag{4.2.12}$$

Here, the functions f_f, f_e, g, and h are given data.

The following lemma provides the existence and uniqueness of solutions (v, q, w) satisfying (4.2.11)-(4.2.12).

Lemma 4.2.3. *Consider* (4.2.11) - (4.2.12). *Suppose that the initial data satisfies* $(v_0, w_0, w_1) \in (V \cap H^{\frac{5}{2}}(\Omega_f)) \times H^{\frac{11}{4}-\delta}(\Omega_e) \times H^{\frac{7}{4}-\delta}(\Omega_e)$ *for* $\delta \in (0, \frac{1}{4})$, *and*

$$\Delta v(0) - \nabla q(0) + f_f(0) \in H^1(\Omega_f) \tag{4.2.13}$$

with $q(0)$ *determined from the elliptic system* (4.2.18). *Additionally, let* f_f, f_e, g, *and* h *satisfy*

$$\begin{aligned} & f_f \in L^2([0,T]; H^1(\Omega_f)), \\ & \partial_t f_f \in L^2([0,T]; L^2(\Omega_f)), \\ & f_e \in L^1([0,T]; H^{\frac{3}{2}}(\Omega_e)) \cap W^{\frac{1}{2},1}([0,T]; L^2(\Omega_e)), \\ & \partial_t f_e \in L^1([0,T]; H^{\frac{1}{2}}(\Omega_e)), \\ & \partial_{tt} f_e \in L^1([0,T]; H^{-\frac{1}{4}-\delta}(\Omega_e)), \\ & g \in L^2([0,T]; H^2(\Omega_f)), \\ & g_t \in L^2([0,T]; H^1(\Omega_f)), \\ & h \in L^2([0,T]; H^{\frac{3}{2}}(\Gamma_c)), \\ & h_t \in H^{\frac{1}{4}}([0,T]; L^2(\Gamma_c)) \cap L^2([0,T]; H^{\frac{1}{2}}(\Gamma_c)), \\ & A, B \in L^2([0,T]; L^2(\Omega_f)), \end{aligned} \tag{4.2.14}$$

and

$$g_t \quad = \quad divA + B, \tag{4.2.15}$$

for some time $T > 0$. Assume the following compatibility conditions

$$\begin{cases} w_1 = v_0, & \Gamma_c, \\ \frac{\partial w_0}{\partial N} \cdot \tau = \frac{\partial v_0}{\partial N} \cdot \tau + h(0) \cdot \tau, & \Gamma_c, \\ \Delta w_0 - \alpha w_1 - \beta w_0 + f_e(0) = \Delta v_0 - \nabla q_0 + f_f(0), & \Gamma_c, \end{cases} \tag{4.2.16}$$

and

$$\begin{cases} v_0 = 0, & \Gamma_f, \\ \Delta v_0 - \nabla q_0 + f_f(0) = 0, & \Gamma_f, \end{cases} \tag{4.2.17}$$

hold where q_0 solves

$$\Delta q_0 \quad = -g_t(0) + \Delta g(0) + div f_f(0), \qquad \Omega_f, \tag{4.2.18}$$

with boundary conditions

$$\begin{cases} \frac{\partial q_0}{\partial N} = \Delta v_0 \cdot N + f_f(0) \cdot N, & \Gamma_f, \\ q_0 = \frac{\partial v_0}{\partial N} \cdot N - \frac{\partial w_0}{\partial N} \cdot N - h(0) \cdot N, & \Gamma_c. \end{cases} \tag{4.2.19}$$

Assume also $w_0 = 0$ on Γ_c. Then there exists a solution (v, q, w) on $(0, T)$ which belongs to Y where

$$\begin{aligned} Y \quad = \quad & \{(v, q, w) \mid v \in L^2([0,T]; H^3(\Omega_f)),\ v_t \in L^2([0,T]; H^2(\Omega_f)), \\ & v_{tt} \in L^2([0,T]; L^2(\Omega_f)),\ q \in L^2([0,T]; H^2(\Omega_f)),\ q_t \in L^2([0,T]; H^1(\Omega_f)), \\ & q_t|_{\Gamma_c} \in H^{\frac{1}{4}}([0,T]; L^2(\Gamma_c)),\ \partial_t^j w \in L^\infty([0,T]; H^{\frac{11}{4}-\delta-j}(\Omega_e)),\ j = 0, 1, 2\}. \end{aligned} \tag{4.2.20}$$

Proof. **Step 1: a priori Bounds on the Solution**

We apply Lemma 4.2.2 to obtain estimate (4.2.9) for the time derivative of the coupled system (4.2.11) - (4.2.12). We set $y = (v_t, q_t)$ as in Lemma (4.2.2). Then we have

$$\|y\| \quad = \quad \|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^2(\Omega_f))} + \|q_t\|_{L^2([0,T];H^1(\Omega_f))}$$

$$+\|q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))}.$$

which satisfies

$$\begin{aligned}\|y\| \;\leq\; & C\Big(\|v_t(0)\|_{H^1(\Omega_f)} + \|\partial_t f_f\|_{L^2(\Omega_f\times[0,T])} + \|g_t\|_{L^2([0,T];H^1(\Omega_f))} + \|A\|_{L^2([0,T];L^2(\Omega_f))} \\ & +\|B\|_{L^2([0,T];L^2(\Omega_f))} + \|h_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|h_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & +\left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Big). \end{aligned} \tag{4.2.21}$$

Our objective is to bound the terms on the left-hand side of (4.2.21) by the initial data, boundary data, and source terms. To estimate the normal derivative terms, we apply the sharp trace regularity result of [41] to obtain

$$\begin{aligned}\left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \;\leq\; & C\Big(\|w_1\|_{H^{\frac{3}{2}}(\Omega_e)} + \|w_{tt}(0,\cdot)\|_{H^{\frac{1}{2}}(\Omega_e)} \\ & +\|v\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} + \|v\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} \\ & +\|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} \\ & +\|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Big), \end{aligned} \tag{4.2.22}$$

where the velocity matching condition $w_t = v$ on Γ_c was used. Next, we estimate the trace of v on the right-hand side of (4.2.22). Trace, interpolation, and Young's inequalities yield the following estimates:

$$\|v\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \;\leq\; \epsilon\|v\|_{L^2([0,T];H^3(\Omega_f))} + C_\epsilon\|v\|_{L^2([0,T];H^1(\Omega_f))},\; \epsilon \in (0,1] \tag{4.2.23}$$

and

$$\|v\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} \;\leq\; \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|v\|_{L^2([0,T];H^2(\Omega_f))},\; \epsilon \in (0,1]. \tag{4.2.24}$$

Furthermore, if we use (4.2.23) and (4.2.24) in (4.2.22), we obtain

$$\begin{aligned}&\left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ &\quad \leq C\Big(\|w_1\|_{H^{\frac{3}{2}}(\Omega_e)} + \|w_{tt}(0,\cdot)\|_{H^{\frac{1}{2}}(\Omega_e)} + \epsilon\|v\|_{L^2([0,T];H^3(\Omega_f))}\end{aligned}$$

$$+\epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} + \|v\|_{L^2([0,T];H^2(\Omega_f))} + \|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))}$$
$$+\|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Big). \tag{4.2.25}$$

Applying (4.2.25) to (4.2.21) yields the following estimate.

$$\begin{aligned}\|y\| \leq\ & C\Big(\|v_t(0)\|_{H^1(\Omega_f)} + \|f_t\|_{L^2(\Omega_f\times[0,T])} + \|g_t\|_{L^2([0,T];H^1(\Omega_f))} + \|A\|_{L^2([0,T];L^2(\Omega_f))} \\ & +\|B\|_{L^2([0,T];L^2(\Omega_f))} + \|h_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|h_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|w_1\|_{H^{\frac{3}{2}}(\Omega_e)} \\ & +\|w_{tt}(0,\cdot)\|_{H^{\frac{1}{2}}(\Omega_e)} + \epsilon\|v\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} \\ & +C_\epsilon\|v\|_{L^2([0,T];H^2(\Omega_f))} + \|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} + \|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Big).\end{aligned} \tag{4.2.26}$$

We also require the elliptic Stokes estimate (see [3]) in order to obtain the full regularity for the fluid velocity and pressure, i.e.

$$\begin{aligned}&\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} \\ &\leq C\Big(\|f\|_{L^2([0,T];H^1(\Omega_f))} + \|v_t\|_{L^2([0,T];H^1(\Omega_f))} + \|g\|_{L^2([0,T];H^2(\Omega_f))} + \|h\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \\ &\quad +\left\|\frac{\partial w}{\partial N}\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))}\Big).\end{aligned} \tag{4.2.27}$$

A second application of the sharp trace regularity for the wave equation provides an estimate on the normal derivative of w on the right side of (4.2.114), i.e.

$$\begin{aligned}\left\|\frac{\partial w}{\partial N}\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} &\leq \left\|\frac{\partial w}{\partial N}\right\|_{H^{\frac{3}{2}}(\Sigma_c)} \\ &\leq C\Big(\|w\|_{H^{\frac{5}{2}}(\Sigma_c)} + \|w_0\|_{H^{\frac{5}{2}}(\Omega_e)} + \|w_1\|_{H^{\frac{3}{2}}(\Omega_e)} \\ &\qquad +\|f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))}\Big).\end{aligned} \tag{4.2.28}$$

Here we take $\Sigma_c = \Gamma_c \times [0,T]$ and $H^s(\Sigma_c) := L^2([0,T];H^s(\Gamma_c)) \cap H^s([0,T];L^2(\Gamma_c))$. Using the norm on $H^s(\Sigma_c)$ as defined in as in [46], we have

$$\begin{aligned}
\|w\|_{H^{\frac{5}{2}}(\Gamma_c\times[0,T])} &= \|\eta - x\|_{L^2([0,T];H^{\frac{5}{2}}(\Gamma_c))} + \|w\|_{H^{\frac{5}{2}}([0,T];L^2(\Gamma_c))} \\
&\leq \left\| \int_0^t v\ ds \right\|_{L^2([0,T];H^{\frac{5}{2}}(\Gamma_c))} + \|w\|_{H^{\frac{5}{2}}([0,T];L^2(\Gamma_c))}. \qquad (4.2.29)
\end{aligned}$$

For the first term on the right hand side of the inequality above, we write

$$\left\| \int_0^t v\ ds \right\|^2_{L^2([0,T];H^{\frac{5}{2}}(\Gamma_c))} = \int_0^T \left\| \int_0^t v\ ds \right\|^2_{H^{\frac{5}{2}}(\Gamma_c)} dt. \qquad (4.2.30)$$

We apply the Bochner inequality (given in [55]) and the Cauchy-Schwarz inequality to the integral term on the right side of (4.2.30) as follows:

$$\begin{aligned}
\int_0^T \left\| \int_0^t v\ ds \right\|^2_{H^{\frac{5}{2}}(\Gamma_c)} dt &\leq \int_0^T \left(\int_0^t \|v\|_{H^{\frac{5}{2}}(\Gamma_c)}\ ds \right)^2 dt \\
&\leq \int_0^T \int_0^t ds \int_0^t \|v\|^2_{H^{\frac{5}{2}}(\Gamma_c)}\ ds\ dt \\
&\leq \int_0^T t \int_0^T \|v\|^2_{H^{\frac{5}{2}}(\Gamma_c)}\ ds\ dt \\
&\leq \|v\|^2_{L^2([0,T];H^{\frac{5}{2}}(\Gamma_c))} \int_0^T t\ dt \\
&\leq CT^2 \|v\|^2_{L^2([0,T];H^{\frac{5}{2}}(\Gamma_c))}.
\end{aligned}$$

Next we apply the trace inequality to obtain

$$\left\| \int_0^t v\ ds \right\|^2_{H^{\frac{5}{2}}(\Gamma_c)} dt \leq CT^2 \|v\|^2_{L^2([0,T];H^3(\Omega_f))}.$$

Therefore,

$$\Big\| \int_0^t v \; ds \Big\|_{L^2([0,T];H^{\frac{5}{2}}(\Gamma_c))} \leq CT \|v\|_{L^2([0,T];H^3(\Omega_f))}.$$

Next, we consider the $\|w\|_{H^{\frac{5}{2}}([0,T];L^2(\Gamma_c))}$ term. By the definition of the fractional order Sobolev norm, we may write

$$\|w\|_{H^{\frac{5}{2}}([0,T];L^2(\Gamma_c))} = \|w_t\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} + \|w_{tt}\|_{H^{\frac{1}{2}}([0,T];L^2(\Gamma_c))}.$$

We focus on the second term. Recall that $w_0 = 0$ on Γ_c. Then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|w\|_{L^2(\Gamma_c)} &\leq \int_0^t \|v\|_{L^2(\Gamma_c)} \\ &\leq C_T \|v\|_{L^2(\Gamma_c)}. \end{aligned}$$

This implies that

$$\|w\|_{L^2([0,T];L^2(\Gamma_c))} \leq C_T \|v\|_{L^2([0,T];L^2(\Gamma_c))} \leq C_T \|v\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))}.$$

Furthermore, the boundary condition $w_t = v$ on Γ_c implies

$$\|w_t\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} = \|v\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))}.$$

Thus,

$$\|w_t\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} = \|v\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))}.$$

To estimate $\|v\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))}$, we first estimate $\|v\|_{L^2(\Gamma_c)}$ using the following trace inequality [23]:

$$\|v\|_{L^2(\Gamma_c)} \leq C\|v\|_{H^1(\Omega_f)} \leq C\|v\|_{H^{\frac{3}{2}}(\Omega_f)}.$$

Thus,

$$\|v\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} \leq C\|v\|_{H^{\frac{3}{2}}([0,T];H^{\frac{3}{2}}(\Omega_f))}.$$

We also utilize the following interpolation inequality given in [45]:

$$\|v\|_{H^{\frac{3}{2}}([0,T];H^{\frac{3}{2}}(\Omega_f))} \leq C\|v\|^{\frac{3}{4}}_{H^2([0,T];L^2(\Omega_f))}\|v\|^{\frac{1}{4}}_{L^2([0,T];H^2(\Omega_f))}. \tag{4.2.31}$$

We estimate the right hand side of (4.2.31) using Young's Inequality with $p = \frac{4}{3}$ and $q = 4$ to obtain

$$\|v\|^{\frac{3}{4}}_{H^2([0,T];L^2(\Omega_f))}\|v\|^{\frac{1}{4}}_{L^2([0,T];H^2(\Omega_f))} \leq \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|v\|_{L^2([0,T];H^2(\Omega_f))}.$$

Substituting back into (4.2.29), we have

$$\|w\|_{H^{\frac{5}{2}}(\Gamma_c\times[0,T])} \leq CT\|v\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|v\|_{L^2([0,T];H^2(\Omega_f))}. \tag{4.2.32}$$

We now use (4.2.28) and (4.2.32) in (4.2.114) to obtain

$$\begin{aligned}\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} \leq\ & C\Big(\|f_f\|_{L^2([0,T];H^1(\Omega_f))} + \|v_t\|_{L^2([0,T];H^1(\Omega_f))} \\ & + \|g\|_{L^2([0,T];H^2(\Omega_f))} + \|h\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \\ & + \|w_0\|_{H^{\frac{5}{2}}(\Gamma_c\times[0,T])} + \|w_1\|_{H^{\frac{3}{2}}(\Omega_e)} \\ & + T\|v\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} \\ & + T\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))}\Big).\end{aligned} \tag{4.2.33}$$

Summing (4.2.26) with (4.2.33) and using the inequality

$$\|v\|_{L^2([0,T];H^2(\Omega_f))} \leq CT\|v_0\|_{H^2(\Omega_f)} + CT\|v_t\|_{L^2([0,T];H^2(\Omega_f))},$$

we obtain

$$\begin{aligned}&\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^2(\Omega_f))} + \|q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ &\quad + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} + \|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} \\ &\quad \leq (C_\epsilon T + C)\|v_t(0)\|_{H^1(\Omega_f)} + C\Big(\|f_f\|_{L^2([0,T];H^1(\Omega_f))} + \|\partial_t f_f\|_{L^2(\Omega_f\times[0,T])}\end{aligned}$$

$$
\begin{aligned}
&+\|g\|_{L^2([0,T];H^2(\Omega_f))} + \|g_t\|_{L^2([0,T];H^1(\Omega_f))} + \|A\|_{L^2([0,T];L^2(\Omega_f))} + \|B\|_{L^2([0,T];L^2(\Omega_f))} \\
&+\|h\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} + \|h_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|h_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\
&+\|w_0\|_{H^{\frac{5}{2}}(\Omega_e)} + \|w_1\|_{H^{\frac{3}{2}}(\Omega_e)} + \|w_{tt}(0,\cdot)\|_{H^{\frac{1}{2}}(\Omega_e)} + \epsilon\|v\|_{L^2([0,T];H^3(\Omega_f))} \\
&+T\|v\|_{L^2([0,T];H^3(\Omega_f))} + T\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} \\
&+C\epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))} + \|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} \\
&+\|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Bigg). \qquad (4.2.34)
\end{aligned}
$$

Taking ϵ and T sufficiently small, we absorb the last four terms on the left-hand side, i.e.

$$
\begin{aligned}
&\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^2(\Omega_f))} + \|q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\
&\quad+\|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} + \|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} \\
&\quad\leq (C_\epsilon T + C)\|v_t(0)\|_{H^1(\Omega_f)} + C\Bigg(\|f_f\|_{L^2([0,T];H^1(\Omega_f))} + \|\partial_t f_f\|_{L^2(\Omega_f\times[0,T])} \\
&\qquad+\|g\|_{L^2([0,T];H^2(\Omega_f))} + \|g_t\|_{L^2([0,T];H^1(\Omega_f))} + \|A\|_{L^2([0,T];L^2(\Omega_f))} \\
&\qquad+\|B\|_{L^2([0,T];L^2(\Omega_f))} + \|h_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|h_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|w_0\|_{H^{\frac{5}{2}}(\Omega_e)} \\
&\qquad+\|w_1\|_{H^{\frac{3}{2}}(\Omega_e)} + \|w_{tt}(0,\cdot)\|_{H^{\frac{1}{2}}(\Omega_e)} + \|h\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))} \\
&\qquad+\|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} + \|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Bigg). \qquad (4.2.35)
\end{aligned}
$$

We also require the interior regularity of the wave equation [41], which gives

$$
\begin{aligned}
&\|w_{tt}\|_{L^\infty([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} + \|w_{ttt}\|_{L^\infty([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} \\
&\quad\leq C\|v_t\|_{H^{\frac{3}{4}-\delta}(\Gamma_c\times[0,T])} + \|w_{tt}(0)\|_{H^{\frac{3}{4}-\delta}(\Omega_e)} + \|w_{ttt}(0)\|_{H^{-\frac{1}{4}-\delta}(\Omega_e)} \\
&\qquad+ \|\partial_{tt} f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} \\
&\quad\leq C\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + C\|v_t\|_{H^{\frac{3}{4}-\delta}([0,T];L^2(\Gamma_c)} + \|w_{tt}(0)\|_{H^{\frac{3}{4}-\delta}(\Omega_e)} \\
&\qquad+\|w_{ttt}(0)\|_{H^{-\frac{1}{4}-\delta}(\Omega_e)} + \|\partial_{tt} f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}, \qquad (4.2.36)
\end{aligned}
$$

where $T \leq \frac{1}{C}$ was used (without loss of generality). We also have the following interpolation trace inequality

$$
\|v_t\|_{H^{\frac{3}{4}}([0,T];L^2(\Gamma_c)} \leq C\|v_t\|^{\frac{3}{4}}_{H^1([0,T];L^2(\Omega_f)}\|v_t\|^{\frac{1}{4}}_{L^2([0,T];H^2(\Omega_f)}. \qquad (4.2.37)
$$

Using (4.2.37) and Young's inequality in (4.2.36), we have

$$\begin{aligned}
&\|w_{tt}\|_{L^\infty([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} + \|w_{ttt}\|_{L^\infty([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} \\
&\quad \le \|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{H^1([0,T];L^2(\Omega_f)} + \|w_{tt}(0)\|_{H^{\frac{3}{4}-\delta}(\Omega_e)} + \|w_{ttt}(0)\|_{H^{-\frac{1}{4}-\delta}(\Omega_e)} \\
&\qquad + \|\partial_{tt} f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}. \qquad (4.2.38)
\end{aligned}$$

By elliptic regularity (using $T \le \frac{1}{C}$) for w and w_t, respectively, we obtain

$$\begin{aligned}
\|w(t)\|_{L^\infty([0,T];H^{\frac{11}{4}-\delta}(\Omega_e))} &\le \|w_{tt}(t)\|_{L^\infty([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} + C\alpha \|w_t(t)\|_{L^\infty([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} \\
&\quad + C\Big\| \int_0^t v \; ds \Big\|_{L^\infty([0,T];H^{\frac{9}{4}-\delta}(\Gamma_c))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} \\
&\le \|w_{tt}(t)\|_{L^\infty([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} + C\alpha \|w_t(t)\|_{L^\infty([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} \\
&\quad + C\|v\|_{L^2([0,T];H^{\frac{11}{4}-\delta}(\Omega_f))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} \qquad (4.2.39)
\end{aligned}$$

and

$$\begin{aligned}
\|w_t(t)\|_{L^\infty([0,T];H^{\frac{7}{4}-\delta}(\Omega_e))} &\le \|w_{ttt}(t)\|_{L^\infty([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} + C\alpha \|w_{tt}(t)\|_{L^\infty([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} \\
&\quad + C\Big\| \int_0^t v_t \; ds \Big\|_{L^\infty([0,T];H^{\frac{5}{4}-\delta}(\Gamma_c))} + C\|w_1\|_{H^{\frac{5}{4}-\delta}(\Gamma_c)} \\
&\quad + \|\partial_t f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} \\
&\le \|w_{ttt}(t)\|_{L^\infty(0,T);H^{-\frac{1}{4}-\delta}(\Omega_e)} + C\alpha \|w_t(t)\|_{L^\infty(0,T);H^{-\frac{1}{4}-\delta}(\Omega_e)} \\
&\quad + C\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + C\|w_1\|_{H^{\frac{7}{4}-\delta}(\Omega_e)} \\
&\quad + \|\partial_t f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}. \qquad (4.2.40)
\end{aligned}$$

We set

$$\begin{aligned}
\|w\| &= \|w(t)\|_{L^\infty([0,T];H^{\frac{11}{4}-\delta}(\Omega_e))} + \|w_t(t)\|_{L^\infty([0,T];H^{\frac{7}{4}-\delta}(\Omega_e))} + \|w_{tt}\|_{L^\infty([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} \\
&\quad + \|w_{ttt}\|_{L^\infty([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}
\end{aligned}$$

and sum estimates (4.2.36), (4.2.39), and (4.2.40) to obtain

$$\|w\| \le C\|v\|_{L^2([0,T];H^{\frac{11}{4}-\delta}(\Omega_f))} + C\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + C\|v_t\|_{H^1([0,T];L^2(\Omega_f))}$$

$$+C\|w_1\|_{H^{\frac{7}{4}-\delta}(\Omega_e)} + \|w_{tt}(0)\|_{H^{\frac{3}{4}-\delta}(\Omega_e)} + \|w_{ttt}(0)\|_{H^{-\frac{1}{4}-\delta}(\Omega_e)}$$
$$+\|f_e\|_{L^1([0,T];H^{\frac{3}{4}-\delta}(\Omega_e))} + \|\partial_{tt} f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}. \tag{4.2.41}$$

Step 2: The Fixed Point

Next we describe the construction of the solution using a contraction mapping approach. Let Y_v be the space defined by

$$Y_v \;=\; \{v \mid v \in L^2([0,T];H^3(\Omega_f)),\ v_t \in L^2([0,T];H^2(\Omega_f)),\ v_{tt} \in L^2([0,T];L^2(\Omega_f))\}. \tag{4.2.42}$$

Define the mapping $\mathcal{T} : Y_v \to Y_v$ as follows: given $\hat{v} \in Y_v$,

$$\hat{v} \mapsto \hat{v}|_{\Gamma_c} \mapsto \frac{\partial w}{\partial N} \mapsto v, \tag{4.2.43}$$

i.e. $\mathcal{T}(\hat{v}) = v$. Given $\hat{v} \in Y_v$, consider the following modified coupled system (4.2.11) – (4.2.12):

$$\begin{cases} \partial_t v - \Delta v + \nabla q = f_f, & \Omega_f(t), \\ \nabla \cdot v = g, & \Omega_f \times (0,T), \\ \nabla v \cdot N - qN = \nabla w \cdot N + h, & \Gamma_c \times (0,T), \\ v = 0, & \Gamma_f \times (0,T), \end{cases} \tag{4.2.44}$$

and

$$\begin{cases} w_{tt} - \Delta w + \alpha w_t + \beta w = f_e, & \Omega_e \times (0,T), \\ w_t = \hat{v}, & \Gamma_c. \end{cases} \tag{4.2.45}$$

We follow the following scheme: given $\hat{v}$, we solve the wave equation with Dirichlet data $w_t = \hat{v}$ on Γ_c, i.e. we solve (4.2.45). We first estimate the trace of $\hat{v}$ on Γ_c by $\hat{v}$ on the interior of Ω_f. Using (4.2.23) – (4.2.24), we have

$$\|\hat{v}\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \;\leq\; \epsilon\|\hat{v}\|_{L^2([0,T];H^3(\Omega_f))} + C_\epsilon\|\hat{v}\|_{L^2([0,T];H^1(\Omega_f))},\ \epsilon \in (0,1] \tag{4.2.46}$$

and

$$\|\hat{v}\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} \leq \epsilon\|\hat{v}\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|\hat{v}\|_{L^2([0,T];H^2(\Omega_f))}, \ \epsilon \in (0,1]. \quad (4.2.47)$$

The sharp trace regularity estimates on the normal derivatives of the displacement of the solid yield bounds on w and w_t in terms of $\hat{v}$ on the common interface. In the following, the notation (data) will stand for the contribution of the initial conditions and given source terms to the a-priori estimates. $(4.2.28)-(4.2.29)$ yield

$$\begin{aligned} \left\|\frac{\partial w}{\partial N}\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \leq \ & C\Big((\text{data}) + T\|\hat{v}\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|\hat{v}\|_{H^2([0,T];L^2(\Omega_f))} \\ & + T\|\hat{v}_t\|_{L^2([0,T];H^2(\Omega_f))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))}\Big). \end{aligned} \quad (4.2.48)$$

Furthermore, $(4.2.22)-(4.2.25)$ with $(4.2.23)-(4.2.24)$ give the following bound in terms of $\hat{v}$ on the interior (through the trace inequality)

$$\begin{aligned} & \left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & \quad \leq C\Big((\text{data}) + \epsilon\|\hat{v}\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|\hat{v}\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|\hat{v}\|_{L^2([0,T];H^2(\Omega_f))} \\ & \qquad + \|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} + \|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Big). \end{aligned} \quad (4.2.49)$$

We now solve (4.2.44) with Neumann data. Using the standard Stokes estimate (see [3]) on (4.2.44) yields

$$\begin{aligned} \|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} \leq \ & C\Big((\text{data}) + \|v_t\|_{L^2([0,T];H^1(\Omega_f))} \\ & + \left\|\frac{\partial w}{\partial N}\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))}\Big), \end{aligned} \quad (4.2.50)$$

which, when combined with Lemma (4.2.2) applied to the time-differentiated system, provides estimates on v and v_t in terms of $\frac{\partial w}{\partial N}$ and $\frac{\partial w_t}{\partial N}$, i.e.

$$\begin{aligned} & \|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^2(\Omega_f))} + \|q_t\|_{L^2([0,T];H^1(\Omega_f))} \\ & \quad + \|q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \end{aligned}$$

$$\leq C\Bigg((\text{data}) + \left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Bigg). \quad (4.2.51)$$

We now sum (4.2.50) and (4.2.51) to obtain

$$\begin{aligned}
&\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^2(\Omega_f))} \\
&\quad + \|q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \leq C\Bigg((\text{data}) + \left\|\frac{\partial w}{\partial N}\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Bigg),
\end{aligned} \tag{4.2.52}$$

where we estimated $\|v_t\|_{L^2([0,T];H^1(\Omega_f))}$ by (4.2.51). Using (4.2.48) – (4.2.49) in (4.2.52), we obtain

$$\begin{aligned}
&\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^2(\Omega_f))} \\
&\quad + \|q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \leq C\Big((\text{data}) + T\|\hat{v}\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|\hat{v}\|_{H^2([0,T];L^2(\Omega_f))} \\
&\qquad + T\|\hat{v}_t\|_{L^2([0,T];H^2(\Omega_f))} + \|F_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))} + \epsilon\|\hat{v}\|_{L^2([0,T];H^3(\Omega_f))} \\
&\qquad + \epsilon\|\hat{v}\|_{H^2([0,T];L^2(\Omega_f))} + \|\hat{v}\|_{L^2([0,T];H^2(\Omega_f))} + \|\partial_t F_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} \\
&\qquad + \|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Big) \\
&\quad \leq C\Big((\text{data}) + (CT+\epsilon)\|\hat{v}\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|\hat{v}\|_{H^2([0,T];L^2(\Omega_f))} \\
&\qquad + C_\epsilon T^{\frac{1}{2}}\|\hat{v}_t\|_{L^2([0,T];H^2(\Omega_f))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))} + \|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} \\
&\qquad + \|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Big).
\end{aligned} \tag{4.2.53}$$

We concern ourselves with the bounds on v only and add $\epsilon\|\hat{v}_t\|_{C([0,T];H^2(\Omega_f))}$ to the right hand side, i.e.

$$\begin{aligned}
&\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^2(\Omega_f))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\leq C\Big((\text{data}) + (CT+\epsilon)\|\hat{v}\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|\hat{v}\|_{H^2([0,T];L^2(\Omega_f))} + T\|\hat{v}_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&\quad + \epsilon\|\hat{v}_t\|_{C([0,T];H^2(\Omega_f))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))} + \|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))}
\end{aligned}$$

$$+\|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Big). \tag{4.2.54}$$

We also absorb $\|\hat{v}\|_{H^2([0,T];L^2(\Omega_f))}$ into the $\hat{v}$, $\hat{v}_t$, and $\hat{v}_{tt}$ on the right side of (4.2.55).

$$\begin{aligned}
&\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^2(\Omega_f))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \leq C\Big((\text{data}) + (CT+\epsilon)\|\hat{v}\|_{L^2([0,T];H^3(\Omega_f))} + C_\epsilon T^{\frac{1}{2}}\|\hat{v}_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&\qquad + \epsilon\|\hat{v}_t\|_{C([0,T];H^2(\Omega_f))} + \epsilon\|\hat{v}_{tt}\|_{L^2([0,T];L^2(\Omega_f))} + \|f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))} \\
&\qquad + \|\partial_t f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} + \|\partial_t^{\frac{1}{2}} f_e\|_{L^1([0,T];L^2(\Omega_e))}\Big).
\end{aligned} \tag{4.2.55}$$

We now show that $\mathcal{T}$ is a contraction from Y_v into Y_v. Let $\hat{v}_1$ and $\hat{v}_2$ be two elements in Y_v. Then $v_1 = \mathcal{T}\hat{v}_1$ and $v_2 = \mathcal{T}\hat{v}_2$. We consider the system satisfied by $V = v_1 - v_2$:

$$\begin{cases}
\partial_t V - \Delta V + \nabla Q = 0, & \Omega_f \times (0,T), \\
\nabla \cdot V = 0, & \Omega_f \times (0,T), \\
\nabla V \cdot N - QN = \nabla W \cdot N, & \Gamma_c \times (0,T), \\
v = 0, & \Gamma_f \times (0,T),
\end{cases} \tag{4.2.56}$$

and

$$\begin{cases}
W_{tt} - \Delta W + \alpha W_t + \beta W = 0, & \Omega_e \times (0,T), \\
W_t = \hat{V}, & \Gamma_c \times (0,T).
\end{cases} \tag{4.2.57}$$

Here $Q = q_1 - q_2$ and $W = w_1 - w_2$. Applying (4.2.55), we obtain

$$\begin{aligned}
&\|V\|_{L^2([0,T];H^3(\Omega_f))} + \|V_t\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{C([0,T];H^2(\Omega_f))} + \|V_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \leq C\Big((CT+\epsilon)\|\hat{V}\|_{L^2([0,T];H^3(\Omega_f))} + C_\epsilon T^{\frac{1}{2}}\|\hat{V}_t\|_{L^2([0,T];H^2(\Omega_f))} + \epsilon\|\hat{V}_t\|_{C([0,T];H^2(\Omega_f))} \\
&\qquad + \epsilon\|\hat{V}_{tt}\|_{L^2([0,T];L^2(\Omega_f))}\Big),
\end{aligned} \tag{4.2.58}$$

i.e.,

$$\|v_1 - v_2\|_{L^2([0,T];H^3(\Omega_f))} + \|\partial_t v_1 - \partial_t v_2\|_{L^2([0,T];H^2(\Omega_f))}$$

$$
\begin{aligned}
&+\|\partial_t v_1 - \partial_t v_2\|_{C([0,T];H^2(\Omega_f))} + \|\partial_{tt} v_1 - \partial_{tt} v_2\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \le C\bigg((CT+\epsilon)\|\hat v_1 - \hat v_2\|_{L^2([0,T];H^3(\Omega_f))} + C_\epsilon T^{\frac{1}{2}}\|\partial_t \hat v_1 - \partial_t \hat v_2\|_{L^2([0,T];H^2(\Omega_f))} \\
&\qquad +\epsilon\|\partial_t \hat v_1 - \partial_t \hat v_2\|_{C([0,T];H^2(\Omega_f))} + \epsilon\|\partial_{tt} \hat v_1 - \partial_{tt} \hat v_2\|_{L^2([0,T];L^2(\Omega_f))}\bigg). \qquad (4.2.59)
\end{aligned}
$$

Thus we can write

$$
\|\mathcal{T}\hat v_1 - \mathcal{T}\hat v_2\|_{Y_v} \;\; \le \;\; C(T,\epsilon)\|\hat v_1 - \hat v_2\|_{Y_v}, \qquad (4.2.60)
$$

where we can take ϵ and T sufficiently small to insure that $\mathcal{T}$ is a contraction from Y_v onto itself. Thus there exists a unique fixed point $\mathcal{T}v = v$, and (4.2.11) – (4.2.12) has a unique solution (given v, we can solve the wave equation for w).

□

Define the space of initial data

$$
\begin{aligned}
Y(0) \;\; = \;\; &\big\{(v_0, w_0, w_1) \in (V \cap H^{\frac{5}{2}}(\Omega_f)) \times H^{\frac{11}{4}-\delta}(\Omega_e) \times H^{\frac{7}{4}-\delta}(\Omega_e), \\
&\Delta v(0) - \nabla q(0) + f(0) \in H^1(\Omega_f) \text{ with compatibility} \\
&\text{conditions } (4.2.16) - (4.2.17)\big\}, \qquad (4.2.61)
\end{aligned}
$$

where $q(0)$ satisfies (4.2.18).

Remark 4.2.4. *We note that the space of regularity $Y(0)$ is invariant under the dynamics, that is given initial data $(v_0, w_0, w_1) \in Y(0)$, the associated solution $(v, w, w_1) \in Y$ satisfies $v \in C([0,T]; H^{\frac{5}{2}}(\Omega_f))$ and $v_t \in C([0,T]; H^1(\Omega_f))$. This fact allows us to continue the solution for all times $T > 0$.*

We also define D to be the space of regularity requirements given by (4.2.14):

$$
\begin{aligned}
D \;\; = \;\; &\{(f_f, f_e, g, h) \mid f_f \in L^2([0,T]; H^1(\Omega_f)),\; \partial_t f_f \in L^2([0,T]; L^2(\Omega_f)), \\
&f_e \in L^1([0,T]; H^{\frac{3}{2}}(\Omega_e)) \cap W^{\frac{1}{2},1}([0,T]; L^2(\Omega_e)),\; \partial_t f_e \in L^1([0,T]; H^{\frac{1}{2}}(\Omega_e)), \\
&\partial_{tt} f_e \in L^1([0,T]; H^{-\frac{1}{4}-\delta}(\Omega_e)),\; g \in L^2([0,T]; H^2(\Omega_f)),\; g_t \in L^2([0,T]; H^1(\Omega_f)), \\
&h \in L^2([0,T]; H^{\frac{3}{2}}(\Gamma_c)),\; h_t \in H^{\frac{1}{4}}([0,T]; L^2(\Gamma_c)) \cap L^2([0,T]; H^{\frac{1}{2}}(\Gamma_c)), \\
&A, B \in L^2([0,T]; L^2(\Omega_f))\}. \qquad (4.2.62)
\end{aligned}
$$

Lemma 4.2.3 states that given initial data $y_0 = (v_0, w_0, w_1) \in Y(0)$, for any $T > 0$, and for $d = (f_f, f_e, g, h) \in D$, there exists a unique solution $y(t) = (v(t), q(t), w(t))$ such that

$$\|y\|_Y \leq C_T \|y_0\|_{Y(0)} + C_T \|d\|_D. \tag{4.2.63}$$

4.2.2 Linear Stokes with Given Coefficients a Coupled with a Damped Linear Wave Equation

Next, we prove the existence of a global solution for (3.2.4) in the case of given coefficients a close to identity. Note that the PDE remains linear under these assumptions.

Lemma 4.2.5. *Consider the linear system (written component-wise)*

$$\begin{cases} \partial_t v^i - \partial_j (a_\ell^j a_\ell^k \partial_k v^i) + \partial_k (a_i^k q) = F^i, & \Omega_f \times (0, T), \\ a_i^k \partial_k v^i = G, & \Omega_f \times (0, T), \\ w_{tt}^i - \Delta w^i + \alpha w_t^i + \beta w^i = f_e^i, & \Omega_e \times (0, T), \end{cases} \tag{4.2.64}$$

with boundary conditions

$$\begin{cases} v^i = w_t^i, & \Gamma_c \times (0, T), \\ v^i = 0, & \Gamma_f \times (0, T), \\ a_\ell^j a_\ell^k \partial_k v^i N_j - a_i^k q^i N_k = \partial_j w^i N_j + H^i, & \Gamma_c \times (0, T). \end{cases} \tag{4.2.65}$$

where the coefficient matrix $a = a(x, t)$ is given such that

$$a(0) = I, \ \partial_k a_j^k = 0, \ k, j = 1, 2, 3, \tag{4.2.66}$$

in addition to the conditions

$$\begin{aligned} &\|a - I\|_{L^\infty([0,T];H^2(\Omega_f))}, \ \|aa^T - I\|_{L^\infty([0,T];H^2(\Omega_f))}, \ \|\partial_t (aa^T)\|_{L^\infty([0,T];H^2(\Omega_f))}, \\ &\quad \|\partial_t a\|_{L^\infty([0,T];H^2(\Omega_f))}, \ \|\partial_{tt} a\|_{L^\infty([0,T];H^1(\Omega_f))} \leq \epsilon \end{aligned} \tag{4.2.67}$$

for some sufficiently small ϵ, where $\epsilon < \epsilon_0 \in (0, 1)$, and $T > 0$. Assume that the initial data (v_0, w_0, w_1) satisfies the assumptions and compatibility conditions in Lemma 4.2.3.

We also assume that $(F, f_e, G, H) \in D$*, where* D *is the topology given by* (4.2.62)*. Then there exists a unique solution* (v, q, w, w_t) *to the system* (4.2.64) *with boundary conditions* (4.2.65)-(4.2.65) *on* $(0, T)$ *belonging to the space* Y*. More precisely, for any* $T > 0$*,* $\epsilon < \frac{1}{2}C_T$ *(where* C_T *is determined by* (4.2.63)*), data* $y_0 = (v_0, w_0, w_1) \in Y(0)$*, and* $d = (F, G, H, f_e)$*, the solution* $y \in Y$ *satisfies the estimate*

$$\|y\|_Y \leq 2C_T(\|y_0\|_{Y(0)} + \|d\|_D). \tag{4.2.68}$$

Proof. The proof is based on a fixed point argument for the system

$$\begin{cases} \partial_t v - \Delta v + \nabla q = f, & \Omega_f \times (0, T), \\ \nabla \cdot v = g, & \Omega_f \times (0, T), \\ w_{tt} - \Delta w + \alpha w_t + \beta w = f_e, & \Omega_e \times (0, T), \end{cases} \tag{4.2.69}$$

with boundary conditions

$$\begin{cases} w_t = u, & \Gamma_c \times (0, T), \\ v = 0, & \Gamma_f \times (0, T), \\ \nabla v \cdot N - qN = \frac{\partial w}{\partial N} + h, & \Gamma_c \times (0, T), \end{cases} \tag{4.2.70}$$

where f, g, h are given by

$$\begin{cases} f^i = \partial_j((\delta_{jk} - a^j_\ell a^k_\ell)\partial_k u^i) + (\delta_{ki} - a^k_i)\partial_k p + F^i, \\ g = (\delta_{kj} - a^k_j)\partial_k u^j + G, \\ h^i = (\delta_{jk} - a^j_\ell a^k_\ell)\partial_k u^i N_j - (\delta_{ki} - a^k_i)pN_k + H^i, \end{cases} \tag{4.2.71}$$

with $(u, p) \in Y_{vq}$, where

$$\begin{aligned} Y_{vq} \;\; = \;\; & \{(v, q) \mid v \in L^2([0, T]; H^3(\Omega_f)),\ v_t \in L^2([0, T]; H^2(\Omega_f)), \\ & v_{tt} \in L^2([0, T]; L^2(\Omega_f)), q \in L^2([0, T]; H^2(\Omega_f)),\ q_t \in L^2([0, T]; H^1(\Omega_f)), \\ & q_t|_{\Gamma_c} \in H^{\frac{1}{4}}([0, T]; L^2(\Gamma_c))\}. \end{aligned} \tag{4.2.72}$$

Step 1: Estimates on the given data

For simplicity, take $G = H = 0$. We now show that f, g, and h satisfy the regularity requirements of Lemma 4.2.3. Since $a(0) = I$, we have

$$\Delta v(0) - \nabla q(0) + f(0) = \Delta v(0) - \nabla q(0) + F(0) \in H^1(\Omega_f).$$

First, we show $f \in L^2([0,T]; H^1(\Omega_f))$. To this end, we rewrite the pressure component in divergence form (making use of the divergence-free condition $\partial_k a_j^k = 0$ for $j = 1, 2, 3$):

$$(\delta_{ki} - a_i^k)\partial_k p = \partial_k((\delta_{ki} - a_i^k)p).$$

Estimating f in $H^1(\Omega_f)$ yields,

$$\begin{aligned}
\|f\|_{H^1(\Omega_f)} &\leq \|\partial_j((\delta_{jk} - a_\ell^j a_\ell^k)\partial_k u^i)\|_{H^1(\Omega_f)} + \|\partial_k((\delta_{ki} - a_i^k)p)\|_{H^1(\Omega_f)} + \|F\|_{H^1(\Omega_f)} \\
&\leq \|\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)\partial_k u^i\|_{H^1(\Omega_f)} + \|(\delta_{jk} - a_\ell^j a_\ell^k)\partial_j\partial_k u^i\|_{H^1(\Omega_f)} \\
&\quad + \|\partial_k((\delta_{ki} - a_i^k)p)\|_{H^1(\Omega_f)} + \|F\|_{H^1(\Omega_f)}. \qquad (4.2.73)
\end{aligned}$$

For the first term in (4.2.73), we write

$$\begin{aligned}
&\|\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)\partial_k u^i\|_{H^1(\Omega_f)} \\
&\quad = \|\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)\partial_k u^i\|_{L^2(\Omega_f)} + \|D(\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)\partial_k u^i)\|_{L^2(\Omega_f)} \\
&\quad \leq \|\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)\partial_k u^i\|_{L^2(\Omega_f)} + \|D(\partial_j(\delta_{jk} - a_\ell^j a_\ell^k))\partial_k u^i\|_{L^2(\Omega_f)} \\
&\qquad + \|\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)D(\partial_k u^i)\|_{L^2(\Omega_f)}. \qquad (4.2.74)
\end{aligned}$$

We apply Holder's inequality and Sobolev's imedding theorem to all three terms in (4.2.74) and obtain

$$\begin{aligned}
\|\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)\partial_k u^i\|_{H^1(\Omega_f)} &\leq \|\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)\|_{L^3(\Omega_f)}\|\partial_k u^i\|_{L^6(\Omega_f)} \\
&\quad + \|D(\partial_j(\delta_{jk} - a_\ell^j a_\ell^k))\|_{L^2(\Omega_f)}\|\partial_k u^i\|_{L^\infty(\Omega_f)} \\
&\quad + \|\partial_j(\delta_{jk} - a_\ell^j a_\ell^k)\|_{L^3(\Omega_f)}\|D(\partial_k u^i)\|_{L^6(\Omega_f)} \\
&\leq C\|I - aa^T\|_{H^2(\Omega_f)}\|u\|_{H^3(\Omega_f)}. \qquad (4.2.75)
\end{aligned}$$

We follow this procedure for the second and third terms in (4.2.73), i.e.

$$\begin{aligned}&\|(\delta_{jk}-a^j_\ell a^k_\ell)\partial_j\partial_k u^i\|_{H^1(\Omega_f)}+\|\partial_k((\delta_{ki}-a^k_i)p)\|_{H^1(\Omega_f)}\\&\quad\le C\|I-aa^T\|_{H^2(\Omega_f)}\|u\|_{H^3(\Omega_f)}+C\|I-a\|_{H^2(\Omega_f)}\|p\|_{H^2(\Omega_f)}.\end{aligned}\tag{4.2.76}$$

Applying (4.2.75)-(4.2.76) to (4.2.73) and using Holder's inequality (in time), we have that

$$\begin{aligned}\|f\|_{L^2([0,T];H^1(\Omega_f))}\ \le\ & C\|I-aa^T\|_{L^\infty([0,T];H^2(\Omega_f))}\|u\|_{L^2([0,T];H^3(\Omega_f))}\\&+C\|I-a\|_{L^\infty([0,T];H^2(\Omega_f))}\|p\|_{L^2([0,T];H^2(\Omega_f))}\\&+C\|F\|_{L^2([0,T];H^1(\Omega_f))}.\end{aligned}\tag{4.2.77}$$

Using (4.2.67) in (4.2.77), we conclude

$$\|f\|_{L^2([0,T];H^1(\Omega_f))}\ \le\ C\epsilon\|u\|_{L^2([0,T];H^3(\Omega_f))}+C\epsilon\|p\|_{L^2([0,T];H^2(\Omega_f))}+C\|F\|_{L^2([0,T];H^1(\Omega_f))}.\tag{4.2.78}$$

Following a similar argument, we can conclude that $f_t\in L^2([0,T]\times\Omega_f)$, $g\in L^2([0,T];H^2(\Omega_f))$, and $g_t\in L^2([0,T];H^1(\Omega_f))$ via the following estimates:

$$\begin{aligned}\|f_t\|_{L^2([0,T]\times\Omega_f)}\ \le\ & C\epsilon\|u_t\|_{L^2([0,T];H^2(\Omega_f))}+C\epsilon\|p_t\|_{L^2([0,T];H^1(\Omega_f))},\\&+\|F_t\|_{L^2([0,T]\times\Omega_f)}\end{aligned}\tag{4.2.79}$$

$$\|g\|_{L^2([0,T];H^2(\Omega_f))}\ \le\ \epsilon\|u\|_{L^2([0,T];H^3(\Omega_f))},\tag{4.2.80}$$

$$\|g_t\|_{L^2([0,T];H^1(\Omega_f))}\ \le\ \epsilon\|u_t\|_{L^2([0,T];H^2(\Omega_f))}+C\epsilon\|u\|_{L^2([0,T];H^3(\Omega_f))}.\tag{4.2.81}$$

Next we show $A\in L^2([0,T];L^2(\Omega_f))$. We write

$$g_{tt}=\partial_{tt}((\partial_{jk}-a^k_j)\partial_k u^j)=\partial_{tt}(\partial_k(\delta_{kj}-a^k_j)u^j)=\partial_k(\partial_{tt}(\delta_{kj}-a^k_j)u^j)=\mathrm{div}A.\tag{4.2.82}$$

By setting $A_k=\partial_{tt}((\delta_{kj}-a^k_j)u^j)$ for $k=1,2,3$ and $B=0$, we can use a similar approach as in (4.2.78)-(4.2.81) to conclude

$$\|A\|_{L^2([0,T];L^2(\Omega_f))}\ \le\ C\epsilon\|u\|_{L^\infty([0,T];H^2(\Omega_f))}+C\epsilon\|u_t\|_{L^2([0,T]\times\Omega_f)}+C\epsilon\|u_{tt}\|_{L^2([0,T]\times\Omega_f)}.\tag{4.2.83}$$

Next we show $h \in L^2([0,T]; H^{\frac{3}{2}}(\Gamma_c)$ and $h_t \in H^{\frac{1}{4}}([0,T]; L^2(\Gamma_c) \cap L^2([0,T]; H^{\frac{1}{2}}(\Gamma_c)$. By the trace inequality, we have that

$$\begin{aligned} &\|h^i\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \\ &\quad \leq C\|(\delta_{jk} - a^j_\ell a^k_\ell)\partial_k u^i\|_{L^2([0,T];H^2(\Omega_f))} + C\|(\delta_{ki} - a^i_k)p\|_{L^2([0,T];H^2(\Omega_f))}, \end{aligned} \tag{4.2.84}$$

for $i = 1, 2, 3$. Thus, if we follow the procedure used in (4.2.77) – (4.2.81), we obtain

$$\|h\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \leq \epsilon\|u\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|p\|_{L^2([0,T];H^2(\Omega_f))} \tag{4.2.85}$$

and

$$\begin{aligned} \|h_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} \leq \; & C\epsilon\|u_t\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|u\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|p_t\|_{L^2([0,T];H^1(\Omega_f))} \\ & + C\epsilon\|p\|_{L^2([0,T];H^1(\Omega_f))}. \end{aligned} \tag{4.2.86}$$

Also, we use Kato-Ponce type estimates as in [33] to write

$$\begin{aligned} \|h_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \leq \; & C\|aa^T - I\|_{L^\infty([0,T]\times\Omega_f)}\|\nabla u_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & + C\|aa^T - I\|_{W^{\frac{1}{4},4}([0,T],L^\infty(\Omega_f))}\|\nabla u_t\|_{L^4([0,T];L^2(\Gamma_c))} \\ & + C\|\partial_t(aa^T)\|_{L^\infty([0,T]\times\Omega_f)}\|\nabla u\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & + C\|\partial_t(aa^T)\|_{H^{\frac{1}{4}}([0,T],L^4(\Gamma_c))}\|\nabla u\|_{L^\infty([0,T];L^4(\Gamma_c))} \\ & + C\|a - I\|_{L^\infty([0,T]\times\Omega_f)}\|p_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & + C\|a - I\|_{H^{\frac{1}{4}}([0,T],L^\infty(\Gamma_c))}\|p_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & + C\|\partial_t a\|_{L^\infty([0,T]\times\Omega_f)}\|p\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & + C\|\partial_t a\|_{H^{\frac{1}{4}}([0,T],L^4(\Gamma_c))}\|p\|_{L^\infty([0,T];L^4(\Gamma_c))}. \end{aligned} \tag{4.2.87}$$

We also use the space-time interpolation inequalities for v (and similar inequalities for p):

$$\|\nabla u_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \leq \epsilon_0\|u_t\|_{H^1([0,T];L^2(\Gamma_c))} + C_{\epsilon_0}\|u_t\|_{L^2([0,T];H^2(\Omega_f))} \tag{4.2.88}$$

and

$$\|\nabla u_t\|_{L^4([0,T];L^2(\Gamma_c))} \leq \epsilon_0 \|u_t\|_{H^1([0,T];L^2(\Gamma_c))} + C_{\epsilon_0}\|u_t\|_{L^2([0,T];H^2(\Omega_f))} \tag{4.2.89}$$

as well as

$$\|\nabla u\|_{L^\infty([0,T];L^4(\Gamma_c))} \leq \epsilon_0 \|u_t\|_{L^2([0,T];H^2(\Gamma_c))} + C_{\epsilon_0}\|u\|_{L^2([0,T];H^2(\Omega_f))} \tag{4.2.90}$$

for $\epsilon_0 \in (0,1]$. We thus obtain

$$\begin{aligned}\|h_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \leq\ & C\epsilon\|u_t\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|u_t\|_{L^2([0,T];H^2(\Omega_f))}\\ &+C\epsilon\|u\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|u\|_{L^2([0,T];H^2(\Omega_f))}\\ &+C\epsilon\|p_t\|_{L^2([0,T];H^1(\Omega_f))} + C\epsilon\|p\|_{L^2([0,T];H^1(\Omega_f))}.\end{aligned} \tag{4.2.91}$$

These bounds show that f, g, h and their time derivatives satisfy the conditions of Lemma 4.2.3. Since $\epsilon \in (0,1]$ is small, the estimates (4.2.35) and (4.2.41) lead to the existence and uniqueness of the solution to the system (4.2.69) – (4.2.71).

Step 2: The Fixed Point

Consider the map $\mathcal{T} : Y_{vq} \to Y_{vq}$ following the diagram of transfer (4.2.43). Given $(u,p), (\tilde{u},\tilde{p}) \in Y_{vq}$ with $(v,q) = \mathcal{T}(u,p)$ and $(\tilde{v},\tilde{q}) = \mathcal{T}(\tilde{u},\tilde{p})$, we define

$$U = u - \tilde{u},\ P = p - \tilde{p},\ V = v - \tilde{v},\ Q = q - \tilde{q}.$$

The variables U, P, V, and Q satisfy the following PDE:

$$\begin{cases} V_t - \Delta V + \nabla Q = \hat{F}, & \Omega_f \times (0,T),\\ \nabla \cdot V = \hat{G}, & \Omega_f \times (0,T),\\ \frac{\partial V}{\partial N} - QN = \frac{\partial W}{\partial N} + \hat{H}, & \Gamma_c \times (0,T),\\ V = 0, & \Gamma_f \times (0,T), \end{cases} \tag{4.2.92}$$

and

$$\begin{cases} W_{tt} - \Delta W + \alpha W_t + \beta W = 0, & \Omega_e \times (0,T), \\ W_t = U, & \Gamma_c \times (0,T), \end{cases} \tag{4.2.93}$$

where,

$$\begin{cases} \hat{F}^i = \partial_j((\delta_{jk} - a^j_\ell a^k_\ell)\partial_k U^i) + (\delta_{ki} - a^k_i)\partial_k P, \\ \hat{G} = (\delta_{kj} - a^k_j)\partial_k U^j, \\ \hat{H}^i = (\delta_{jk} - a^j_\ell a^k_\ell)\partial_k U^i N_j - (\delta_{ki} - a^k_i) P N_k, \end{cases}$$

for $i = 1,2,3$. We apply estimates (4.2.46) – (4.2.55), noting that we need to include the specific terms generated right-hand side (that appear as "(data)"). We begin by solving (4.2.93). To this end, we require the analogues to (4.2.46) – (4.2.49). First, we use (4.2.23) – (4.2.24), to write

$$\|U\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \leq \epsilon\|U\|_{L^2([0,T];H^3(\Omega_f))} + C_\epsilon\|U\|_{L^2([0,T];H^1(\Omega_f))}, \ \epsilon \in (0,1],$$

$$\|U\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} \leq \epsilon\|U\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|U\|_{L^2([0,T];H^2(\Omega_f))}, \ \epsilon \in (0,1].$$

Next we use the sharp trace regularity estimates (4.2.28) – (4.2.29) on the normal derivatives of the solid displacement

$$\begin{aligned} \left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \leq\ & C\Big(T\|U\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|U\|_{H^2([0,T];L^2(\Omega_f))} \\ & + T\|U_t\|_{L^2([0,T];H^2(\Omega_f))}\Big). \end{aligned} \tag{4.2.94}$$

Furthermore, (4.2.22) – (4.2.25) with (4.2.23) – (4.2.24) give the following bound on the normal derivative of W_t in terms of U on the interior:

$$\begin{aligned} & \left\|\frac{\partial W_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial W_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & \quad \leq C\Big(\epsilon\|U\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|U\|_{H^2([0,T];L^2(\Omega_f))} + \|U\|_{L^2([0,T];H^2(\Omega_f))}\Big). \end{aligned}$$

$$(4.2.95)$$

Next, we solve (4.2.92) with Neumann data, using (4.2.114) to obtain

$$\begin{aligned} \|V\|_{L^2([0,T];H^3(\Omega_f))} + \|Q\|_{L^2([0,T];H^2(\Omega_f))} \;\; \le \;\; & C\bigg(\|V_t\|_{L^2([0,T];H^1(\Omega_f))} + \left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \\ & + \|\hat{F}\|_{L^2([0,T];H^1(\Omega_f))} + \|\hat{G}\|_{L^2([0,T];H^2(\Omega_f))} \\ & + \|\hat{H}\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))}\bigg). \end{aligned} \tag{4.2.96}$$

We combine (4.2.96) with Lemma 4.2.3 applied to the time derivative of (4.2.92) to obtain an estimate on V and V_t in terms of the normal derivatives of W and W_t.

$$\begin{aligned} & \|V_t\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{C([0,T];H^2(\Omega_f))} + \|Q_t\|_{L^2([0,T];H^1(\Omega_f))} \\ & \quad + \|Q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|V_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\ & \quad \le C\bigg(\left\|\frac{\partial W_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial W_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|\hat{F}_t\|_{L^2(\Omega_f\times[0,T])} \\ & \qquad + \|\hat{G}_t\|_{L^2([0,T];H^1(\Omega_f))} + \|\hat{A}\|_{L^2([0,T];L^2(\Omega_f))} + \|\hat{H}_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} \\ & \qquad + \|\hat{H}_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\bigg), \end{aligned} \tag{4.2.97}$$

where $\hat{A} = A - \hat{A}$. Combining these estimates gives the following on (v, q) in terms of (u, p) in Y_{vq}. Sum (4.2.96) and (4.2.97) to obtain

$$\begin{aligned} & \|V\|_{L^2([0,T];H^3(\Omega_f))} + \|Q\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{C([0,T];H^2(\Omega_f))} \\ & \quad + \|Q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|Q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\ & \quad \le C\bigg(\left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} + \left\|\frac{\partial W_t}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial W_t}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\ & \qquad + \|\hat{F}\|_{L^2([0,T];H^1(\Omega_f))} + \|\hat{G}\|_{L^2([0,T];H^2(\Omega_f))} + \|\hat{F}_t\|_{L^2(\Omega_f\times[0,T])} + \|\hat{G}_t\|_{L^2([0,T];H^1(\Omega_f))} \\ & \qquad + \|\hat{A}\|_{L^2([0,T];L^2(\Omega_f))} + \|\hat{H}_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|\hat{H}_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\bigg). \end{aligned} \tag{4.2.98}$$

Using (4.2.94) – (4.2.95) in (4.2.98), we obtain

$$\|V\|_{L^2([0,T];H^3(\Omega_f))} + \|Q\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{C([0,T];H^2(\Omega_f))}$$

$$
\begin{aligned}
&+\|Q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|Q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
\leq\ & C\Big(T\|U\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|U\|_{H^2([0,T];L^2(\Omega_f))} + T\|U_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&+T\|U_t\|_{L^2([0,T];H^2(\Omega_f))} + \epsilon\|U\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|U\|_{H^2([0,T];L^2(\Omega_f))} \\
&+\|U\|_{L^2([0,T];H^2(\Omega_f))} + \|\hat{F}\|_{L^2([0,T];H^1(\Omega_f))} + \|\hat{G}\|_{L^2([0,T];H^2(\Omega_f))} + \|\hat{F}_t\|_{L^2(\Omega_f\times[0,T])} \\
&+\|\hat{G}_t\|_{L^2([0,T];H^1(\Omega_f))} + \|\hat{A}\|_{L^2([0,T];L^2(\Omega_f))} + \|\hat{H}_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} \\
&+\|\hat{H}_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Big) \\
\leq\ & C\Big((T+\epsilon)\|U\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|U\|_{H^2([0,T];L^2(\Omega_f))} + T\|U_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&+\|\hat{F}\|_{L^2([0,T];H^1(\Omega_f))} + \|\hat{G}\|_{L^2([0,T];H^2(\Omega_f))} + \|\hat{F}_t\|_{L^2(\Omega_f\times[0,T])} + \|\hat{G}_t\|_{L^2([0,T];H^1(\Omega_f))} \\
&+\|\hat{A}\|_{L^2([0,T];L^2(\Omega_f))} + \|\hat{H}_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|\hat{H}_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Big) \\
\leq\ & C\Big((T+\epsilon)\|U\|_{L^2([0,T];H^3(\Omega_f))} + T\|U_t\|_{L^2([0,T];H^2(\Omega_f))} + \|\hat{F}\|_{L^2([0,T];H^1(\Omega_f))} \\
&+\|\hat{G}\|_{L^2([0,T];H^2(\Omega_f))} + \|\hat{F}_t\|_{L^2(\Omega_f\times[0,T])} + \|\hat{G}_t\|_{L^2([0,T];H^1(\Omega_f))} + \|\hat{A}\|_{L^2([0,T];L^2(\Omega_f))} \\
&+\|\hat{H}_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|\hat{H}_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Big). \qquad (4.2.99)
\end{aligned}
$$

Add $\epsilon\|U_t\|_{C([0,T];H^2(\Omega_f))}$ to the right hand side to obtain

$$
\begin{aligned}
&\|V\|_{L^2([0,T];H^3(\Omega_f))} + \|Q\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{C([0,T];H^2(\Omega_f))} \\
&+\|Q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|Q_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
\leq\ & C\Big((T+\epsilon)\|U\|_{L^2([0,T];H^3(\Omega_f))} + T\|U_t\|_{L^2([0,T];H^2(\Omega_f))} + \epsilon\|U_t\|_{C([0,T];H^2(\Omega_f))} \\
&+\|\hat{F}\|_{L^2([0,T];H^1(\Omega_f))} + \|\hat{G}\|_{L^2([0,T];H^2(\Omega_f))} + \|\hat{F}_t\|_{L^2(\Omega_f\times[0,T])} + \|\hat{G}_t\|_{L^2([0,T];H^1(\Omega_f))} \\
&+\|\hat{A}\|_{L^2([0,T];L^2(\Omega_f))} + \|\hat{H}_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|\hat{H}_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Big). \qquad (4.2.100)
\end{aligned}
$$

Next, we estimate $\hat{F}, \hat{G}$, and $\hat{H}$ using (4.2.73) – (4.2.91) to obtain

$$\|\hat{F}\|_{L^2([0,T];H^1(\Omega_f))} \leq C\epsilon\|U\|_{L^2([0,T];H^3(\Omega_f))} + C\epsilon\|P\|_{L^2([0,T];H^2(\Omega_f))}, \qquad (4.2.101)$$

$$\|\hat{F}_t\|_{L^2([0,T]\times\Omega_f)} \leq C\epsilon\|U_t\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|P_t\|_{L^2([0,T];H^1(\Omega_f))}, \qquad (4.2.102)$$

$$\|\hat{G}\|_{L^2([0,T];H^2(\Omega_f))} \leq \epsilon\|U\|_{L^2([0,T];H^3(\Omega_f))}, \tag{4.2.103}$$

$$\|\hat{G}_t\|_{L^2([0,T];H^1(\Omega_f))} \leq \epsilon\|U_t\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|U\|_{L^2([0,T];H^3(\Omega_f))}, \tag{4.2.104}$$

$$\|\hat{H}\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \leq \epsilon\|U\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|P\|_{L^2([0,T];H^2(\Omega_f))}, \tag{4.2.105}$$

$$\begin{aligned}\|\hat{H}_t\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} \leq\ & C\epsilon\|U_t\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|U\|_{L^2([0,T];H^2(\Omega_f))} \\ & + C\epsilon\|P_t\|_{L^2([0,T];H^1(\Omega_f))} + C\epsilon\|P\|_{L^2([0,T];H^1(\Omega_f))},\end{aligned} \tag{4.2.106}$$

$$\begin{aligned}\|\hat{H}_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \leq\ & C\epsilon\|U_t\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|U_t\|_{L^2([0,T];H^2(\Omega_f))} \\ & + C\epsilon\|U\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|U\|_{L^2([0,T];H^2(\Omega_f))} \\ & + C\epsilon\|P_t\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + C\epsilon\|P\|_{L^2([0,T];H^1(\Omega_f))} \\ & + C\epsilon\|P_t\|_{L^2([0,T];H^1(\Omega_f))}.\end{aligned} \tag{4.2.107}$$

We also estimate $\hat{A}$ using (4.2.83).

$$\|\hat{A}\|_{L^2([0,T];L^2(\Omega_f))} \leq C\epsilon\|U\|_{L^\infty([0,T];H^2(\Omega_f))} + C\epsilon\|U_t\|_{L^\infty([0,T]\times\Omega_f)} + C\epsilon\|U_{tt}\|_{L^\infty([0,T]\times\Omega_f)}. \tag{4.2.108}$$

Using (4.2.101) – (4.2.107) in (4.2.100), we conclue that

$$\|\mathcal{T}(u,p) - \mathcal{T}(\tilde{u},\tilde{p})\|_{Y_{vq}} \leq C(T,\epsilon)\|(u,p) - (\tilde{u},\tilde{p})\|_{Y_{vq}},$$

where we can take ϵ and T sufficiently small to insure that $\mathcal{T}$ is a contraction from Y_{vq} onto itself. Thus there exists a unique fixed point $\mathcal{T}(v,q) = (v,q)$, and (4.2.64) has a unique solution (given v, we solve the wave for w).

□

Due to the invariance of the regularity of the initial data under the dynamics, the solution obtained above can be continued indefinitely.

4.2.3 Higher Regularity Solutions

The space Y (see (4.2.20)) provides sufficient regularity to accommodate the topological restrictions (4.2.67) for the matrix $a(x,t)$. Furthermore, we will use the decay estimate (3.4.4) for the norm X(t) to control the regularity of $a(x,t)$. Unfortunately, the topology of the space Y is insufficient in using this decay to construct solutions for the full nonlinear coupled system. We must therefore construct solutions for the linear problem with higher regularity.

Lemma 4.2.6. *Let the coefficient matrix $a(x,t)$ be given such that (4.2.66) and (4.2.67) hold with $\epsilon < \frac{1}{C}$ for sufficiently large C. Assume*

$$\|a\|_{L^2([0,T];H^3(\Omega_f))} < \infty, \ \|a_{ttt}\|_{L^2([0,T];L^2(\Omega_f))} < \infty.$$

Consider the linear system (4.2.64) with boundary conditions (4.2.65) – (4.2.65) where $F = f_f$ and $G = H = 0$. Assume that the initial data satisfy $v_0 \in V \cap H^{\frac{7}{2}}(\Omega_f)$, $\partial_t v_0 \in V \cap H^{\frac{5}{2}}(\Omega_f)$, $\Delta v_t(0) - \nabla q_t(0) \in V$, $w_0 \in H^{\frac{15}{4}-\delta}(\Omega_e)$, $w_1 \in H^{\frac{11}{4}-\delta}(\Omega_e)$ for some $\delta \in (0, \frac{1}{4})$, with compatibility conditions (4.2.16) – (4.2.17). In addition,

$$\begin{cases} \Delta w_1 - \alpha w_{tt}(0) - \beta w_1(0) + \ \partial_t f_e(0) = \Delta v_t(0) - \nabla q_t(0) + \partial_t f_f(0), & \Gamma_c, \\ \dfrac{\partial w_1}{\partial N} \cdot \tau = \dfrac{\partial}{\partial N}(\Delta v_0 - \nabla q_0 + f_f(0)) \cdot \tau, & \Gamma_c, \\ -\Delta v_t(0) + \nabla q_t(0) = f_f(0), & \Gamma_f. \end{cases} \tag{4.2.109}$$

Furthermore, we assume the forcing data satisfies (4.2.14) as well as

$$\begin{aligned} &\partial_t f_f, \ \partial_m f_f \in L^2([0,T]; H^1(\Omega_f)), \\ &\partial_{tt} f_f, \ \partial_m \partial_t f_f \in L^2([0,T]; L^2(\Omega_f)), \\ &\partial_t f_e, \ \partial_m f_e \in L^1([0,T]; H^{\frac{3}{2}}(\Omega_e)) \cap W^{\frac{1}{2},1}([0,T]; L^2(\Omega_e)), \\ &\partial_{tt} f_e, \ \partial_m \partial_t f_e \in L^1([0,T]; H^{\frac{1}{2}}(\Omega_e)), \\ &\partial_{ttt} f_e, \ \partial_m \partial_{tt} f_e \in L^1([0,T]; H^{-\frac{1}{4}-\delta}(\Omega_e)), \end{aligned} \tag{4.2.110}$$

for $m = 1, 2$. Then the unique solution $(v, q, w) \in Y$ from Lemma 4.2.5 obeys $(v_t, q_t, w_t) \in Y$ and $(\partial_m v, \partial_m q, \partial_m w) \in Y$ for $m = 1, 2$. Here ∂_m denotes the tangential derivative. The obtained solution can be extended to arbitrary $T > 0$ with the estimates given at the end *of the proof.*

Proof. Consider the system

$$\begin{cases} \partial_t \phi^i - \partial_j(a^j_\ell a^k_\ell \partial_k \phi^i) + \partial_k(a^k_i \chi) = f^i, & \Omega_f(t), \\ a^k_i \partial_k \phi^i = g, & \Omega_f \times (0,T), \\ \psi^i_{tt} - \Delta \psi^i + \alpha \psi^i_t + \beta \psi^i = \partial_t f^i_e, & \Omega_e \times (0,T), \end{cases} \tag{4.2.111}$$

with boundary conditions

$$\begin{cases} \phi^i = \psi^i_t, & \Gamma_c, \\ \phi^i = 0, & \Gamma_f \times (0,T), \\ a^j_\ell a^k_\ell \partial_k \phi^i N_j - a^k_i \chi^i N_k = \partial_j \psi^i N_j + h^i, & \Gamma_c \times (0,T), \end{cases} \tag{4.2.112}$$

with f, g, and h are defined by

$$\begin{cases} f^i_1 = \partial_j(\partial_t(a^j_\ell a^k_\ell)\partial_k v^i) - \partial_t a^k_i \partial_k q + \partial_t f^i_f, \\ g_1 = -\partial_t a^k_i \partial_k v^i, \\ h^i_1 = \partial_t(a^j_\ell a^k_\ell)\partial_k v^i N_j - \partial_t a^k_i a N_k, \end{cases}$$

where $(v, q, w) \in Y$ is the unique solution to the system (4.2.64) constructed in Lemma 4.2.5. Note that the PDE system above is obtained by differentiating (4.2.64) with respect to t and moving all time-differentiated a terms to the right-hand side of the equations. We take initial data $\phi_0 = v_t(0) \in V \cap H^{\frac{5}{2}}(\Omega_f)$, $\phi_t(0) = v_{tt}(0) \in V$, $\psi_0 = w_t(0) \in H^{\frac{11}{4}-\epsilon}(\Omega_e)$, and $\psi_t(0) = w_{tt}(0) \in H^{\frac{7}{4}-\epsilon}(\Omega_e)$ and observe that f, g, and h satisfy the requirements of Lemma 4.2.5x. This allows us to conclude that (4.2.111) has a unique solution satisfying

$$\begin{aligned} \phi &\in L^2([0,T]; H^3(\Omega_f)), \\ \phi_t &\in L^2([0,T]; H^2(\Omega_f)), \\ \chi &\in L^2([0,T]; H^2(\Omega_f)), \\ \chi_t &\in L^2([0,T]; H^1(\Omega_f)), \\ \psi &\in L^\infty([0,T]; H^{\frac{11}{4}-\delta}(\Omega_e)), \\ \psi_t &\in L^\infty([0,T]; H^{\frac{7}{4}-\delta}(\Omega_e)). \end{aligned}$$

By uniqueness, $\phi = v_t$, $\chi = q_t$, and $\psi = w_t$. Thus $(v_t, q_t, w_t) \in Y$. For tangential regularity, we apply the same procedure with forcing terms f, g, and h given by

$$\begin{cases} f_2^i = \partial_j(\partial_m(a_\ell^j a_\ell^k)\partial_k v^i) - \partial_m a_i^k \partial_k q + \ \partial_m f_f^i, \\ g_2 = -\partial_m a_i^k \partial_k v^i, \\ h_2^i = \partial_m(a_\ell^j a_\ell^k)\partial_k v^i N_j - \partial_m a_i^k a N_k, \end{cases}$$

for $m = 1, 2$ and initial data $\phi_0 = \partial_m v(0) \in V \cap H^{\frac{5}{2}}(\Omega_f)$, $(\Delta\phi_0 - \nabla\chi_0) \in V$, $\psi_0 = \partial_m w(0) \in H^{\frac{11}{4}-\epsilon}(\Omega_e)$, and $\psi_t(0) = \partial_m w_t(0) \in H^{\frac{7}{4}-\epsilon}(\Omega_e)$. Using Lemma 4.2.5 and uniqueness of solutions, we obtain $(\partial_m v, \partial_m q, \partial_m w) \in Y$ for $m = 1, 2$. As a consequence, we obtain higher regularity for w through the elliptic estimate

$$\begin{aligned} \|w\|_{H^{\frac{15}{4}-\delta}(\Omega_e)} \;\; \leq \;\; & C\|w_{tt}\|_{H^{\frac{7}{4}-\delta}(\Omega_e)} + C\|w_t\|_{H^{\frac{7}{4}-\delta}(\Omega_e)} + C\|w\|_{H^{\frac{7}{4}-\delta}(\Omega_e)} + C\|D'w\|_{H^{\frac{11}{4}-\delta}(\Omega_e)} \\ & + C\|f_e\|_{H^{\frac{7}{4}-\delta}(\Omega_e)}, \end{aligned} \tag{4.2.113}$$

for $t \in (0, T)$, where D' represents the tangential derivative. Using similar elliptic estimates for the stationary Stokes operator, i.e.

$$\begin{aligned} \|v\|_{L^2([0,T];H^4(\Omega_f))} + \|q\|_{L^2([0,T];H^3(\Omega_f))} \;\; \leq \;\; & C\bigg(\|f_f\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{L^2([0,T];H^2(\Omega_f))} \\ & + \|g\|_{L^2([0,T];H^3(\Omega_f))} + \|h\|_{L^2([0,T];H^{\frac{5}{2}}(\Gamma_c))} \\ & + \left\| \frac{\partial w}{\partial N} \right\|_{L^2([0,T];H^{\frac{5}{2}}(\Gamma_c))} \bigg), \end{aligned} \tag{4.2.114}$$

we can conclude the higher regularity for v and q, i.e.

$$\begin{aligned} v &\in L^2([0,T]; H^4(\Omega_f)), \\ q &\in L^2([0,T]; H^3(\Omega_f)). \end{aligned}$$

$\square$

Note that the space $\tilde{Y}$ given by (4.2.4) is the solution space obtained in Lemma 4.2.6. Futhermore, we define the space of initial data $\tilde{Y}(0)$ as

$$\tilde{Y}(0) \;\; = \;\; \{(v_0, w_0, w_1) \in (V \cap H^{\frac{7}{2}}(\Omega_f)) \times H^{\frac{15}{4}-\delta}(\Omega_e) \times H^{\frac{11}{4}-\delta}(\Omega_e),$$

$$\partial_t v_0 \in V \cap H^{\frac{5}{2}}(\Omega_f),\ \Delta v_t(0) - \nabla q_t(0) \in H^1(\Omega_f),\ \text{with compatibility conditions } (4.2.16) - (4.2.17)\}. \tag{4.2.115}$$

and the space of regularity for the source terms $\tilde{D}$ as

$$\begin{aligned}\tilde{D} \ = \ & \{(f_f, f_e) \in D \mid f_f \in H^1([0,T]; H^1(\Omega_f)),\ \partial_{tt} f_f \in L^2([0,T]; L^2(\Omega_f)), \\ & \partial_m f_f \in L^2([0,T]; H^1(\Omega_f)),\ f_e \in W^{1,1}([0,T]; H^{\frac{3}{2}}(\Omega_e)) \cap W^{\frac{3}{2},1}([0,T]; L^2(\Omega_e)), \\ & \partial_{tt} f_e \in L^1([0,T]; H^{\frac{1}{2}}(\Omega_e)), \partial_{ttt} f_e \in L^1([0,T]; H^{-\frac{1}{4}-\delta}(\Omega_e)), \\ & \partial_m f_e \in L^1([0,T]; H^{\frac{3}{2}}(\Omega_e)) \cap W^{\frac{1}{2},1}([0,T]; L^2(\Omega_e)), \\ & \partial_m \partial_{tt} f_e \in L^1([0,T]; H^{-\frac{1}{4}-\delta}(\Omega_e)),\ m = 1, 2\}. \end{aligned} \tag{4.2.116}$$

4.3 Construction of a Solution for the Nonlinear Coupled System

We now construct the solution to the coupled system (3.2.4). We assume that for $\delta \in (0, \frac{1}{4})$, the initial data $y_0 \in (v_0, w_0, w_1) \in \tilde{Y}(0)$ is small, i.e. $\|y_0\|_{X(0)} \leq \epsilon$, where

$$\|y_0\|_{X(0)} = \|v_0\|_{H^3} + \|v_t(0)\|_{H^1} + \|v_{tt}(0)\|_{L^2} + \|w_0\|_{H^3} + \|w_1\|_{H^2}. \tag{4.3.1}$$

Further, we assume y_0 satisfies the compatibility conditions of Lemma 4.2.3. We now prove Theorem 4.2.1.

proof of Theorem 4.2.1. Let $(u^{(0)}, p^{(0)}, \psi^{(0)})$ be the unique solution to the linear, homogeneous problem (4.2.64) with boundary conditions (4.2.65) – (4.2.65), $a^{(0)} = a^{(0)}(x,t)$, forcing terms $F = f_f$ and $G = H = 0$, and initial data (v_0, w_0, w_1). Assume that the coefficient matrix $a^{(0)}$ satisfies

$$a^{(0)}(x,0) = I,\ \partial_t a^{(0)}(x,0) = -\nabla u^{(0)}(x,0) \tag{4.3.2}$$

and (3.3.6) – (3.3.7), i.e. $a^{(0)}$ is close to identity. We also assume that $\mathrm{Div}((a^{(0)})^T) = 0$. Given iterates $y^{(j)} = (u^{(j)}, p^{(j)}, \psi^{(j)})$ for $j = 0, 1, \ldots, n-1$, we have that

$$\eta_t^{(j)} = u^{(j)},\ \eta^{(j)}(x,0) = x. \tag{4.3.3}$$

We construct $a^{(n)}$ to be

$$(a^{(n)})^T = \mathrm{cof}(D\eta^{(n-1)}), \tag{4.3.4}$$

which guarantees that $\mathrm{Div}((a^{(n)})^T) = 0$. This is equivalent to

$$a^{(n)} = \det(D\eta^{(n-1)})(D\eta^{(n-1)})^{-1}.$$

Let $J_n = \det(D\eta^{(n-1)})$. Then $a^{(n)}$ satisfies the following ODE:

$$a_t^{(n)} J_n^{-1}(D\eta^{(n-1)}) + a^{(n)} \frac{d}{dt}[J_n^{-1}(D\eta^{(n-1)})] = 0$$

which can be equivalently expressed as

$$a_t^{(n)}(a^{(n)})^{-1} + a^{(n)} \frac{d}{dt}[J_n^{-1}(D\eta^{(n-1)})] = 0.$$

We then obtain

$$J_n^{-1} a_t^{(n)} = J_n^{-2} \frac{dJ_n}{dt} a^{(n)} - J_n^{-2} a^{(n)} D u^{(n-1)} a^{(n)}.$$

By Jacobi's identity, we have

$$\frac{dJ_n}{dt} = J_n \mathrm{Tr}\left((D\eta^{(n-1)})^{-1} \frac{d}{dt} D\eta^{(n-1)} \right) = \mathrm{Tr}(a^{(n)} D u^{(n-1)}).$$

Thus, the ODE satisfied by $a^{(n)}$ becomes

$$a_t^{(n)} = J_n^{-1} \mathrm{Tr}(a^{(n)} D u^{(n-1)}) a^{(n)} - J_n^{-1} a^{(n)} D u^{(n-1)} a^{(n)}.$$

Furthermore, we define $H_n = J_n^{-1}$. Then $a^{(n)}$ and H_n satisfy the following IVPs:

$$\begin{cases} a_t^{(n)} = -H_n a^{(n)} D u^{(n-1)} a^{(n)} + H_n \mathrm{Tr}(a^{(n)} D u^{(n-1)}) a^{(n)}, \\ a^{(n)}(0) = I, \end{cases} \tag{4.3.5}$$

$$\begin{cases} \partial_t H_n = -H_n^2 \mathrm{Tr}(a^{(n)} D u^{(n-1)}), \\ H_n(0) = 1. \end{cases} . \tag{4.3.6}$$

Given $a^{(n)}$, we solve the linear system for the new iterate

$$y^{(n)} = (u^{(n)}, p^{(n)}, \psi^{(n)}).$$

Denote $(v, q, w) = (u^{(n)}, p^{(n)}, \psi^{(n)})$. Then,

$$\begin{cases} \partial_t v^i - \partial_j((a^{(n)})^j_\ell (a^{(n)})^k_\ell \partial_k v^i) + \partial_k((a^{(n)})^k_i q) = f^i_f \circ \eta^{(n-1)}, & \Omega_f \times (0,T), \\ (a^{(n)})^k_i \partial_k v^i = 0, & \Omega_f \times (0,T), \\ w^i_{tt} - \Delta w^i + \alpha w^i_t + \beta w^i = f^i_e, & \Omega_e \times (0,T), \end{cases} \tag{4.3.7}$$

with boundary conditions

$$\begin{cases} v^i = w^i_t \text{ on } \Gamma_c \times (0,T), \\ v^i = 0 \text{ on } \Gamma_f \times (0,T), \\ (a^{(n)})^j_\ell (a^{(n)})^k_\ell \partial_k v^i N_j - (a^{(n)})^k_i q^i N_k = \partial_j w^i N_j \text{ on } \Gamma_c \times (0,T). \end{cases} \tag{4.3.8}$$

$\eta^{(0)}$ is Close to x

Using definition (4.3.3), we first show that $\eta^{(0)}(x,t)$ remains close to x for all time. We have

$$\begin{aligned} \eta^{(0)}_t = u^{(0)} \quad &\implies \quad \eta^{(0)}(x,t) - \eta^{(0)}(x,0) = \int_0^t u^{(0)} \, ds \\ &\implies \quad \eta^{(0)} - x = \int_0^t u^{(0)} \, ds. \end{aligned}$$

Estimating in H^3, we obtain

$$\|\eta^{(0)} - x\|_{H^3(\Omega_f)} \leq \int_0^t \|u^{(0)}\|_{H^3(\Omega_f)} \, ds.$$

Using (4.2.3) and Lemma 3.4.1 with η taken as identity yields

$$
\begin{aligned}
\|\eta^{(0)} - x\|_{H^3} &\leq \int_0^t \Big(C[X(0) + \mathcal{F}(0)]e^{-\frac{s}{C}} + C\int_0^s e^{-\omega(s-\tau)}\big(\mathcal{F}(\tau) + \mathcal{F}'(\tau)\big)\ d\tau \\
&\qquad + C\mathcal{F}(s)\Big)^{\frac{1}{2}}\ ds \\
&\leq C[X(0) + \mathcal{F}(0)]^{\frac{1}{2}}(1 - e^{-\frac{t}{2C}}) \\
&\qquad + C\int_0^t \Big(\int_0^s e^{-\omega(s-\tau)}\big(\mathcal{F}(\tau) + \mathcal{F}'(\tau)\big)\ d\tau \Big)^{\frac{1}{2}}\ ds + \int_0^t \mathcal{F}(s)^{\frac{1}{2}}\ ds.
\end{aligned}
\tag{4.3.9}
$$

Thus for sufficiently small initial data and source terms satisfying (4.2.2)-(4.2.3), we conclude

$$
\|\eta^{(0)} - x\|_{H^3} \leq \epsilon. \tag{4.3.10}
$$

Composition Estimates for $f_f \circ \eta^{(0)}$

In order to apply Lemma 4.2.6 to the iterated system (4.3.7), we require estimates on the composite function $f_f \circ \eta^{(n-1)}$ in the spaces given by (4.2.14)-(4.2.110). We focus our attention first on the first iterate $f_f \circ \eta^{(0)}$. To estimate $f_f \circ \eta^{(0)}$ in $L^2([0,T]; H^1(\Omega_f))$, we first obtain a bound on $f_f \circ \eta^{(0)} - f_f$ in $L^2(\Omega)$ following the strategy in [18, p. 329]:

$$
\begin{aligned}
&\|f_f^i \circ \eta^{(0)} - f_f^i\|_{L^2(\Omega)} \\
&\quad = \Big\| \int_0^1 \partial_{t'} f_f^i(\eta^{(0)}(x,t) + t'(\eta^{(0)}(x,t) - x), t)\ dt'((\eta^{(0)})^i(x,t) - x) \Big\|_{L^2(\Omega)} \\
&\quad \leq \Big\| \int_0^1 \partial_{t'} f_f^i(\eta^{(0)}(x,t) + t'(\phi(x,t) - x), t)\ dt' \Big\|_{L^2(\Omega)} \|(\eta^{(0)})^i(x,t) - x^i\|_{L^\infty(\Omega)} \\
&\quad \leq C\|(\eta^{(0)})^i - x^i\|_{H^2(\Omega)} \|f_f^i\|_{H^1(\Omega)},
\end{aligned}
\tag{4.3.11}
$$

where Holder's inequality and the Sobolev embedding theorem were used. The reverse triangle inequality combined with (4.3.50) implies

$$\begin{aligned} \|f_f \circ \eta^{(0)}\|_{L^2(\Omega_f)} &\leq C\|\eta^{(0)} - x\|_{H^2(\Omega)}\|f_f\|_{H^1(\Omega)} + \|f_f\|_{L^2(\Omega)} \\ &\leq C\epsilon\|f_f\|_{H^1(\Omega)} + \|f_f\|_{L^2(\Omega)}. \end{aligned} \tag{4.3.12}$$

Next, we measure $D(f_f \circ \eta^{(0)}) \in L^2(\Omega)$. We begin by estimating the difference $D(f_f \circ \eta^{(0)}) - Df_f \in L^2(\Omega)$. Using the chain rule, Holder and Sobolev inequalities, and (4.3.50), we obtain

$$\begin{aligned} \|D(f_f \circ \eta^{(0)}) - Df_f\|_{L^2(\Omega)} &= \|(Df_f \circ \eta^{(0)})D\eta^{(0)} - Df_f\|_{L^2(\Omega)} \\ &= \|(Df_f \circ \eta^{(0)} - Df_f)D\eta^{(0)} \\ &\quad + Df_f(D\eta^{(0)} - I)\|_{L^2(\Omega)} \\ &\leq \|(Df_f \circ \eta^{(0)} - Df_f)D\eta^{(0)}\|_{L^2(\Omega)} \\ &\quad + \|Df_f(D\eta^{(0)} - I)\|_{L^2(\Omega)} \\ &\leq \|Df_f \circ \eta^{(0)} - Df_f\|_{L^2(\Omega)}\|D\eta^{(0)}\|_{L^\infty(\Omega)} \\ &\quad + \|Df_f\|_{L^2(\Omega)}\|(D\eta^{(0)} - I)\|_{L^\infty(\Omega)} \\ &\leq \|Df_f \circ \eta^{(0)} - Df_f\|_{L^2(\Omega)}\|D\eta^{(0)}\|_{H^2(\Omega)} \\ &\quad + \|f_f\|_{H^1(\Omega)}\|D(\eta^{(0)} - x)\|_{H^2(\Omega)} \\ &\leq \|Df_f \circ \eta^{(0)} - Df_f\|_{L^2(\Omega)}\|D\eta^{(0)}\|_{H^2(\Omega)} \\ &\quad + \|f_f\|_{H^1(\Omega)}\|D(\eta^{(0)} - x)\|_{H^2(\Omega)} \\ &\leq C\|Df_f \circ \eta^{(0)} - Df_f\|_{L^2(\Omega)} + C\epsilon\|f_f\|_{H^1}. \end{aligned}$$

Next we write $\|Df_f \circ \eta^{(0)} - Df_f\|_{L^2(\Omega)}$ component-wise, i.e.

$$\begin{aligned} \|Df_f \circ \eta^{(0)} - Df_f\|_{L^2(\Omega)} &= \int_\Omega \sum_{i,j} (\partial_j f_f^i \circ \eta^{(0)} - \partial_j f_f^i)^2 \, d\Omega \\ &= \sum_{i,j} \int_\Omega (\partial_j f_f^i \circ \eta^{(0)} - \partial_j f_f^i)^2 \, d\Omega \\ &= \sum_{i,j} \|\partial_j f_f^i \circ \eta^{(0)} - \partial_j f_f^i\|_{L^2(\Omega)}^2. \end{aligned}$$

We again apply the strategy used in [18, p. 150] to obtain

$$\begin{aligned}\|\partial_j f_f^i \circ \eta^{(0)} - \partial_j f_f^i\|^2_{L^2(\Omega)} &\leq C\|\eta^{(0)} - x\|^2_{H^2(\Omega)}\|\partial_j f_f\|^2_{H^1(\Omega)} \\ &\leq C\epsilon\|f_f\|^2_{H^2(\Omega)}.\end{aligned}$$

Thus,

$$\|Df_f \circ \eta^{(0)} - Df_f\|_{L^2(\Omega)} \leq C\epsilon\|f_f\|_{H^2(\Omega)},$$

and

$$\|D(f_f \circ \eta^{(0)}) - Df_f\|_{L^2(\Omega)} \leq C\epsilon\|f_f\|_{H^2(\Omega)} + C\epsilon\|f_f\|_{H^1(\Omega)}.$$

The reverse triangle inequality implies

$$\|D(f_f \circ \eta^{(0)})\|_{L^2(\Omega)} \leq C\epsilon\|f_f\|_{H^2(\Omega)} + C\epsilon\|f_f\|_{H^1(\Omega)}. \tag{4.3.13}$$

Summing the squares of (4.3.12) and (4.3.13), integrating in time, and applying Young's inequality yields

$$\|f_f \circ \eta^{(0)}\|_{L^2([0,T];H^1(\Omega))} \leq C\epsilon\|f_f\|_{L^2([0,T];H^2(\Omega))} + C\|f_f\|_{L^2([0,T];H^1(\Omega))}. \tag{4.3.14}$$

We follow the same procedure for the time and tangential derivatives of $f_f \circ \eta^{(0)}$. In the derivation of the estimates, we make use of the decay of $\|u^{(0)}\|_X$ given by (4.3.25). We obtain

$$\begin{aligned}\|\partial_t(f_f \circ \eta^{(0)})\|_{L^2([0,T];H^1(\Omega))} &\leq C\epsilon\|f_f\|_{L^2([0,T];H^3(\Omega_f))} + C\epsilon\|f_f\|_{L^2([0,T];H^2(\Omega_f))} \\ &\quad + C\epsilon\|\partial_t f_f\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|\partial_t f_f\|_{L^2([0,T];H^1(\Omega_f))} \\ &\quad + C\|\partial_t f_f\|_{L^2([0,T];H^1(\Omega_f))} + C\|\partial_t f_f\|_{L^2([0,T];L^2(\Omega_f))},\end{aligned} \tag{4.3.15}$$

$$\begin{aligned}\|\partial_{tt}(f_f \circ \eta^{(0)})\|_{L^2([0,T];L^2(\Omega))} &\leq C\epsilon\|f_f\|_{L^2([0,T];H^3(\Omega))} + C\epsilon\|f_f\|_{L^2([0,T];H^2(\Omega))} \\ &\quad + C\epsilon\|f_f\|_{L^2([0,T];H^1(\Omega))} + C\epsilon\|\partial_t f_f\|_{L^2([0,T];H^1(\Omega_f))} \\ &\quad + C\epsilon\|\partial_t f_f\|_{L^2([0,T];H^2(\Omega))} + C\epsilon\|\partial_{tt} f_f\|_{L^2([0,T];H^1(\Omega_f))}\end{aligned}$$

$$+C\|\partial_{tt}f_f\|_{L^2([0,T];L^2(\Omega_f))}, \tag{4.3.16}$$

$$\begin{aligned} \|\partial_m(f_f\circ\eta^{(0)})\|_{L^2([0,T];H^1(\Omega))} &\leq C\epsilon\|f_f\|_{L^2([0,T];H^3(\Omega))} + C\|f_f\|_{L^2([0,T];H^2(\Omega))} \\ &\quad +C\|f_f\|_{L^2([0,T];H^1(\Omega))}. \end{aligned} \tag{4.3.17}$$

Inequalities (4.3.14)-(4.3.17) imply that $f_f\circ\eta^{(0)}$ is sufficiently regular for the existence of solutions to (4.3.7) provided $f_f \in L^2([0,T];H^3(\Omega))$, $\partial_t f_f \in L^2([0,T];H^2(\Omega))$, and $\partial_{tt}f_f \in L^2([0,T];H^1(\Omega))$. Furthermore, the norm $\mathcal{F}(t)$ given (3.4.1) with $\eta=\eta^{(0)}$ is small in $W^{1,1}(\mathbb{R}^+)$. In fact, this norm is bounded in terms of the square of the norm (4.2.3) (which is small in $L^2(\mathbb{R}^+)$), and therefore satisfies the requirements of Lemma 3.4.1.

4.3.1 $a^{(n)}$ is Close to Identity

We want to show that the iterate $a^{(n)}$ given by $(4.3.4)-(4.3.8)$ satisfy $(3.3.6)-(3.3.7)$, which places us within the framework of Lemma (4.2.6). We proceed by induction. Assume that we can construct iterates $y^{(j)}$, $a^{(j)}$, and $\eta^{(j-1)}$, $j=0,1,...,n-1$ satisfying $(3.3.6)-(3.3.7)$ with $\|\eta^{(j-1)}-x\|_{H^3(\Omega_f)}<\epsilon$ for $\epsilon>0$.

$a^{(n)}$ Close to Identity in H^2

We now show that $\|I-a^{(n)}(t)\|_{H^2(\Omega_f)}<\epsilon_a$, where $\epsilon_a>0$ is sufficiently small. We have by the ODE (4.3.5) that

$$\begin{aligned} \partial_t(a^{(n)}-I) &= \partial_t a^{(n)} \\ &= -H_n a^{(n)}Du^{(n-1)}a^{(n)} + H_n\mathrm{Tr}(a^{(n)}Du^{(n-1)})a^{(n)} \\ &= -(H_n-1)a^{(n)}Du^{(n-1)}a^{(n)} - a^{(n)}Du^{(n-1)}a^{(n)} \\ &\quad +(H_n-1)\mathrm{Tr}(a^{(n)}Du^{(n-1)})a^{(n)} + \mathrm{Tr}(a^{(n)}Du^{(n-1)})a^{(n)} \\ &= -(H_n-1)(a^{(n)}-I)Du^{(n-1)}a^{(n)} - (H_n-1)Du^{(n-1)}a^{(n)} \\ &\quad -(a^{(n)}-I)Du^{(n-1)}a^{(n)} - Du^{(n-1)}a^{(n)} \\ &\quad +(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a^{(n)} + (H_n-1)\mathrm{Tr}(Du^{(n-1)})a^{(n)} \\ &\quad +\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a^{(n)} + \mathrm{Tr}(Du^{(n-1)})a^{(n)} \end{aligned}$$

$$
\begin{aligned}
= \; & -(H_n - 1)(a^{(n)} - I)Du^{(n-1)}(a^{(n)} - I) - (H_n - 1)(a^{(n)} - I)Du^{(n-1)} \\
& -(H_n - 1)Du^{(n-1)}(a^{(n)} - I) - (H_n - 1)Du^{(n-1)} \\
& -(a^{(n)} - I)Du^{(n-1)}(a^{(n)} - I) - (a^{(n)} - I)Du^{(n-1)} \\
& -Du^{(n-1)}(a^{(n)} - I) - Du^{(n-1)} \\
& +(H_n - 1)\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)})(a^{(n)} - I) \\
& +(H_n - 1)\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)}) \\
& +(H_n - 1)\mathrm{Tr}(Du^{(n-1)})(a^{(n)} - I) + (H_n - 1)\mathrm{Tr}(Du^{(n-1)}) \\
& +\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)})(a^{(n)} - I) + \mathrm{Tr}((a^{(n)} - I)Du^{(n-1)}) \\
& +\mathrm{Tr}(Du^{(n-1)})(a^{(n)} - I) + \mathrm{Tr}(Du^{(n-1)}). \qquad (4.3.18)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
a^{(n)} - I \;\; = \;\; & -\int_0^t (H_n - 1)(a^{(n)} - I)Du^{(n-1)}(a^{(n)} - I)\; ds \\
& -\int_0^t (H_n - 1)(a^{(n)} - I)Du^{(n-1)}\; ds \\
& -\int_0^t (H_n - 1)Du^{(n-1)}(a^{(n)} - I)\; ds - \int_0^t (H_n - 1)Du^{(n-1)}\; ds \\
& -\int_0^t (a^{(n)} - I)Du^{(n-1)}(a^{(n)} - I)\; ds - \int_0^t (a^{(n)} - I)Du^{(n-1)}\; ds \\
& -\int_0^t Du^{(n-1)}(a^{(n)} - I)\; ds - \int_0^t Du^{(n-1)}\; ds \\
& +\int_0^t (H_n - 1)\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)})(a^{(n)} - I)\; ds \\
& +\int_0^t (H_n - 1)\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)})\; ds \\
& +\int_0^t (H_n - 1)\mathrm{Tr}(Du^{(n-1)})(a^{(n)} - I)\; ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t (H_n-1)\mathrm{Tr}(Du^{(n-1)})\ ds+\int_0^t \mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})(a^{(n)}-I)\ ds\\
&+\int_0^t \mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})\ ds+\int_0^t \mathrm{Tr}(Du^{(n-1)})(a^{(n)}-I)\ ds\\
&+\int_0^t \mathrm{Tr}(Du^{(n-1)})\ ds.
\end{aligned}
$$

Using the Sobolev product estimate $\|uv\|_{H^2} \leq \|u\|_{H^2}\|v\|_{H^2}$, we have

$$
\begin{aligned}
\|(a^{(n)}-I)(t)\|_{H^2} \ &\leq\ C\int_0^t \|H_n-1\|_{H^2}\|a^{(n)}-I\|_{H^2}^2\|Du^{(n-1)}\|_{H^2}\ ds\\
&\quad +C\int_0^t \|H_n-1\|_{H^2}\|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\ ds\\
&\quad +C\int_0^t \|H_n-1\|_{H^2}\|Du^{(n-1)}\|_{H^2}\ ds\\
&\quad +C\int_0^t \|a^{(n)}-I\|_{H^2}^2\|Du^{(n-1)}\|_{H^2}\ ds\\
&\quad +C\int_0^t \|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\ ds+C\int_0^t \|Du^{(n-1)}\|_{H^2}\ ds\\
&\leq\ C\int_0^t Q(\|a^{(n)}-I\|_{H^2},\ \|H_n-1\|_{H^2})\|Du^{(n-1)}\|_{H^2}\ ds, \qquad (4.3.19)
\end{aligned}
$$

where Q is a cubic function in the indicated norms. Similarly, we have from the ODE (4.3.6) that

$$
\begin{aligned}
\partial_t(H_n-1)\ &=\ -H_n^2\mathrm{Tr}(a^{(n)}Du^{(n-1)})\\
&=\ -(H_n-I)^2\mathrm{Tr}(a^{(n)}Du^{(n-1)})-2H_n\mathrm{Tr}(a^{(n)}Du^{(n-1)})+\mathrm{Tr}(a^{(n)}Du^{(n-1)})\\
&=\ -(H_n-I)^2\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})-(H_n-I)^2\mathrm{Tr}(Du^{(n-1)})\\
&\quad -2(H_n-1)\mathrm{Tr}(a^{(n)}Du^{(n-1)})-\mathrm{Tr}(a^{(n)}Du^{(n-1)})
\end{aligned}
$$

$$
\begin{aligned}
&= -(H_n - I)^2\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)}) - (H_n - I)^2\mathrm{Tr}(Du^{(n-1)}) \\
&\quad -2(H_n - 1)\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)}) - 2(H_n - 1)\mathrm{Tr}(Du^{(n-1)}) \\
&\quad -\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)}) - \mathrm{Tr}(Du^{(n-1)}). \qquad (4.3.20)
\end{aligned}
$$

Integrating in time, we have

$$
\begin{aligned}
H_n - 1 &= -\int_0^t (H_n - I)^2\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)})\ ds - \int_0^t (H_n - I)^2\mathrm{Tr}(Du^{(n-1)})\ ds \\
&\quad -2\int_0^t (H_n - 1)\mathrm{Tr}((a^{(n)} - I)Du^{(n-1)})\ ds - 2\int_0^t (H_n - 1)\mathrm{Tr}(Du^{(n-1)})\ ds \\
&\quad -\int_0^t \mathrm{Tr}((a^{(n)} - I)Du^{(n-1)})\ ds - \int_0^t \mathrm{Tr}(Du^{(n-1)})\ ds. \qquad (4.3.21)
\end{aligned}
$$

Estimating in $H^2(\Omega_f)$, we have

$$
\begin{aligned}
\|H_n - 1\|_{H^2} &\le C\int_0^t \|H_n - I\|_{H^2}^2\|a^{(n)} - I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\ ds \\
&\quad +C\int_0^t \|H_n - I\|_{H^2}^2\|Du^{(n-1)}\|_{H^2}\ ds \\
&\quad +C\int_0^t \|H_n - I\|_{H^2}\|a^{(n)} - I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\ ds \\
&\quad +C\int_0^t \|H_n - I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\ ds \\
&\quad +C\int_0^t \|a^{(n)} - I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\ ds + C\int_0^t \|Du^{(n-1)}\|_{H^2}\ ds \\
&\le C\int_0^t Q(\|a^{(n)} - I\|_{H^2},\ \|H_n - I\|_{H^2})\|Du^{(n-1)}\|_{H^2}.\ ds \qquad (4.3.22)
\end{aligned}
$$

Combining (4.3.19) and (4.3.23), we have

$$\begin{aligned}&\|a^{(n)}(t)-I\|_{H^2}+\|H_n(t)-1\|_{H^2}\\&\leq C\int\limits_0^t Q(\|a^{(n)}-I\|_{H^2},\ \|H_n-I\|_{H^2})\|Du^{(n-1)}\|_{H^2}\ ds,\end{aligned}\tag{4.3.23}$$

where Q is a cubic polynomial in $\|a^{(n)}-I\|_{H^2}$ and $\|H_n-I\|_{H^2}$. Denote $\tilde{X}(t)$ and $\tilde{\mathcal{F}}(t)$ to be the norms (3.3.1) and (3.4.1) derived from (4.3.7) for the $(n-1)$-st iterate $y^{(n-1)}$. Using the smallness of (4.2.3) in $L^1(\mathbb{R}^+)$, we can show that $\tilde{\mathcal{F}}(t)$ satisfies the requirements of Lemma 3.4.1 (following a similar argument for $f_f\circ\eta^{(n-2)}$ as that used for $f_f\circ\eta^{(0)}$ in (4.3.14)-(4.3.17). since $\|\eta^{(n-2)}-x\|_{H^3(\Omega_f)}<\epsilon$). Then the Stoke's estimate (A.2.14) combined with (3.4.37) implies

$$\begin{aligned}\|Du^{(n-1)}\|^2_{H^2}\ &\leq\ C\tilde{X}(t)+\|f_f\circ\eta^{(n-2)}\|^2_{H^1}+\|f_e\|^2_{H^1}\\&\leq\ C\tilde{X}(t)+\tilde{\mathcal{F}}(t)\\&\leq\ C[\tilde{X}(0)+\tilde{\mathcal{F}}(0)]e^{-\frac{t}{C}}+C\int\limits_0^t e^{-\omega(t-s)}(\tilde{\mathcal{F}}(s)+(\tilde{\mathcal{F}})'(s))\ ds\\&\qquad+C\tilde{\mathcal{F}}(t).\end{aligned}\tag{4.3.24}$$

Recall that $\tilde{X}(0)+\tilde{\mathcal{F}}(0)=X(0)+\mathcal{F}(0)$. Assuming $X(0)+\mathcal{F}(0)<\epsilon$, $\epsilon>0$ sufficiently small, we conclude that

$$\|Du^{(n-1)}\|^2_{H^2}\ \leq\ C\epsilon e^{-\frac{t}{C}}+C\int\limits_0^t e^{-\omega(t-s)}(\tilde{\mathcal{F}}(s)+\tilde{\mathcal{F}}'(s))\ ds+C\tilde{\mathcal{F}}(t).\tag{4.3.25}$$

Using (4.3.25) in (4.3.23), we have

$$\begin{aligned}&\|a^{(n)}(t)-I\|_{H^2}+\|H_n(t)-1\|_{H^2}\\&\leq C\int\limits_0^t Q(\|a^{(n)}-I\|_{H^2},\ \|H_n-I\|_{H^2})\Bigg(C\epsilon e^{-\frac{s}{C}}+C\int\limits_0^s e^{-\omega(s-\tau)}\tilde{\mathcal{F}}(\tau)\\&\quad+\tilde{\mathcal{F}}'(\tau))\ d\tau+C\tilde{\mathcal{F}}(s)\Bigg)^{\frac{1}{2}}\ ds\end{aligned}$$

$$\leq C\int_0^t Q(\|a^{(n)}-I\|_{H^2},\ \|H_n-I\|_{H^2})\Bigg(\Big(\epsilon e^{-\frac{s}{C}}\Big)^{\frac{1}{2}}+\Big(\int_0^s e^{-\omega(s-\tau)}(\tilde{\mathcal{F}}(\tau)$$

$$+\tilde{\mathcal{F}}'(\tau))\ d\tau\Big)^{\frac{1}{2}}+\tilde{\mathcal{F}}(s)^{\frac{1}{2}}\Bigg)\ ds. \qquad (4.3.26)$$

Let $x(t)=\max\{\|a^{(n)}(t)-I\|_{H^2},\ \|H_n(t)-I\|_{H^2}\}$ and set $z(t)=\sup\limits_{\tau\leq t}x(\tau)$. Then,

$$\begin{aligned} z &\leq C\tilde{Q}(z)\Bigg(\int_0^t \epsilon e^{-\frac{s}{2C}}\ ds+\int_0^t\Big(\int_0^s e^{-\omega(s-\tau)}(\tilde{\mathcal{F}}(\tau)+\tilde{\mathcal{F}}'(\tau))\ d\tau\Big)^{\frac{1}{2}}\ ds+\int_0^t\tilde{\mathcal{F}}(s)^{\frac{1}{2}}\ ds\Bigg)\\ &\leq C\tilde{Q}(z)\Bigg(\epsilon(1-e^{-\frac{t}{C}})+\int_0^t\Big(\int_0^s e^{-\omega(s-\tau)}(\tilde{\mathcal{F}}(\tau)+\tilde{\mathcal{F}}'(\tau))\ d\tau\Big)^{\frac{1}{2}}\ ds+\int_0^t\tilde{\mathcal{F}}(s)^{\frac{1}{2}}\ ds\Bigg)\\ &\leq C\tilde{Q}(z)\Bigg(\epsilon+\int_0^t\Big(\int_0^s e^{-\omega(s-\tau)}(\tilde{\mathcal{F}}(\tau)+\tilde{\mathcal{F}}'(\tau))\ d\tau\Big)^{\frac{1}{2}}\ ds+\int_0^t\tilde{\mathcal{F}}(s)^{\frac{1}{2}}\ ds\Bigg). \qquad (4.3.27)\end{aligned}$$

We now consider the integral

$$\int_0^t\Big(\int_0^s e^{-\omega(s-\tau)}(\tilde{\mathcal{F}}(\tau)+\tilde{\mathcal{F}}'(\tau))\ d\tau\Big)^{\frac{1}{2}}\ ds. \qquad (4.3.28)$$

Consider the innermost integral in (4.3.28). We integrate by parts as follows:

$$\begin{aligned} 0 &\leq \int_0^s e^{-\omega(s-\tau)}(\tilde{\mathcal{F}}(\tau)+\tilde{\mathcal{F}}'(\tau))\ d\tau\\ &\leq e^{-\omega s}\Bigg(\int_0^s e^{\omega\tau}\tilde{\mathcal{F}}(\tau)\ d\tau+\int_0^s e^{\omega\tau}\tilde{\mathcal{F}}'(\tau)\Bigg)\\ &\leq e^{-\omega s}\Bigg(\frac{1}{\omega}e^{\omega s}\tilde{\mathcal{F}}(s)-\frac{1}{\omega}\tilde{\mathcal{F}}(0)-\frac{1}{\omega}\int_0^s e^{\omega\tau}\tilde{\mathcal{F}}(\tau)\ d\tau+\int_0^s e^{\omega\tau}\tilde{\mathcal{F}}'(\tau)\Bigg)\\ &\leq e^{-\omega s}\Bigg(\frac{1}{\omega}e^{\omega s}\tilde{\mathcal{F}}(s)-\frac{1}{\omega}\tilde{\mathcal{F}}(0)-\Big(\frac{1}{\omega}-1\Big)\int_0^s e^{\omega\tau}\tilde{\mathcal{F}}(\tau)\ d\tau\Bigg). \qquad (4.3.29)\end{aligned}$$

In addition to (3.4.5), we assume that T is taken sufficiently large such that $0 < \omega < 1$ (see (3.4.19)). Then $\frac{1}{\omega} - 1 > 0$, and

$$e^{-\omega s}\left(\frac{1}{\omega}e^{\omega s}\tilde{\mathcal{F}}(s) - \frac{1}{\omega}\tilde{\mathcal{F}}(0) - \left(\frac{1}{\omega} - 1\right)\int_0^s e^{\omega\tau}\tilde{\mathcal{F}}(\tau)\ d\tau\right) \leq \frac{1}{\omega}\tilde{\mathcal{F}}(s). \tag{4.3.30}$$

Using (4.3.30) in (4.3.28), we obtain

$$\int_0^t \left(\int_0^s e^{-\omega(s-\tau)}(\tilde{\mathcal{F}}(\tau) + \tilde{\mathcal{F}}'(\tau))\ d\tau\right)^{\frac{1}{2}} ds \ \leq \ \frac{1}{\sqrt{w}}\int_0^t \tilde{\mathcal{F}}(s)^{\frac{1}{2}}\ ds. \tag{4.3.31}$$

Using (4.3.31) in (4.3.33), we have

$$z \ \leq \ C\tilde{Q}(z)\left(\epsilon + \int_0^t \tilde{\mathcal{F}}(s)^{\frac{1}{2}}\ ds\right). \tag{4.3.32}$$

Using analogous composition estimates to (4.3.14)-(4.3.17) for $f_f \circ \eta^{(n-2)}$ and the fact that $\|\eta^{(n-2)} - x\|_{H^3(\Omega_f)} < \epsilon$, we see that $\tilde{\mathcal{F}}(t)^{\frac{1}{2}}$ is bounded by the norm (4.2.3) and therefore is small in $L^1(\mathbb{R}^+)$. We thus conclude

$$z \ \leq \ C\epsilon\tilde{Q}(z). \tag{4.3.33}$$

Then, referring back to (4.3.18) to more specifically characterize $\tilde{Q}$, we can write down the following:

$$\begin{aligned} z \ &\leq \ C\epsilon(z^3 + 3z^2 + 3z + 1) \\ &\leq \ C\epsilon(z+1)^3 \\ &\leq \ C\epsilon(z^3 + 1) \\ &\leq \ C\epsilon z^3 + C\epsilon, \end{aligned} \tag{4.3.34}$$

where the inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ was used in the third line. Furthermore, we observe that $z(0) = 0$. Then, by Proposition A.3.1, we conclude

$$\|a^{(n)}(t) - I\|_{H^2} + \|H_n(t) - 1\|_{H^2} < \epsilon, \ t \geq 0. \tag{4.3.35}$$

$a^{(n)}(a^{(n)})^T$ Close to Identity in H^2

We now show $a^{(n)}(a^{(n)})^T$ is close to identity, i.e. $\|a^{(n)}(a^{(n)})^T - I\|_{H^2} < \epsilon_a$. Using the ODEs (4.3.5), we obtain

$$
\begin{aligned}
(a^{(n)}(a^{(n)})^T - I)_t &= a_t^{(n)}(a^{(n)})^T + a^{(n)}(a_t^{(n)})^T \\
&= (-H_n a^{(n)} Du^{(n-1)} a^{(n)} + H_n \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a^{(n)})(a^{(n)})^T \\
&\quad + a^{(n)}(-H_n a^{(n)} Du^{(n-1)} a^{(n)} + H_n \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a^{(n)})(a^{(n)})^T \\
&= -H_n a^{(n)} Du^{(n-1)} a^{(n)} (a^{(n)})^T + 2H_n \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a^{(n)} (a^{(n)})^T \\
&\quad - H_n a^{(n)} (a^{(n)})^T (Du^{(n-1)})^T (a^{(n)})^T \\
&= -(H_n - 1) a^{(n)} Du^{(n-1)} a^{(n)} (a^{(n)})^T - a^{(n)} Du^{(n-1)} a^{(n)} (a^{(n)})^T \\
&\quad + 2(H_n - 1)\mathrm{Tr}(a^{(n)} Du^{(n-1)}) a^{(n)} (a^{(n)})^T \\
&\quad + 2\mathrm{Tr}(a^{(n)} Du^{(n-1)}) a^{(n)} (a^{(n)})^T \\
&\quad - (H_n - 1) a^{(n)} (a^{(n)})^T (Du^{(n-1)})^T (a^{(n)})^T \\
&\quad - a^{(n)} (a^{(n)})^T (Du^{(n-1)})^T (a^{(n)})^T \\
&= -(H_n - 1)(a^{(n)} - I) Du^{(n-1)} a^{(n)} (a^{(n)})^T \\
&\quad - (H_n - 1) Du^{(n-1)} a^{(n)} (a^{(n)})^T - (a^{(n)} - I) Du^{(n-1)} a^{(n)} (a^{(n)})^T \\
&\quad - Du^{(n-1)} a^{(n)} (a^{(n)})^T \\
&\quad + 2(H_n - 1)\mathrm{Tr}((a^{(n)} - I) Du^{(n-1)}) a^{(n)} (a^{(n)})^T \\
&\quad + 2(H_n - 1)\mathrm{Tr}(Du^{(n-1)}) a^{(n)} (a^{(n)})^T \\
&\quad + 2\mathrm{Tr}((a^{(n)} - I) Du^{(n-1)}) a^{(n)} (a^{(n)})^T \\
&\quad + 2\mathrm{Tr}(Du^{(n-1)}) a^{(n)} (a^{(n)})^T \\
&\quad - (H_n - 1)[a^{(n)} (a^{(n)})^T - I](Du^{(n-1)})^T (a^{(n)})^T \\
&\quad - (H_n - 1)(Du^{(n-1)})^T (a^{(n)})^T \\
&\quad - [a^{(n)} (a^{(n)})^T - I](Du^{(n-1)})^T (a^{(n)})^T - (Du^{(n-1)})^T (a^{(n)})^T \\
&= -(H_n - 1)(a^{(n)} - I) Du^{(n-1)} [a^{(n)} (a^{(n)})^T - I] \\
&\quad - (H_n - 1)(a^{(n)} - I) Du^{(n-1)} \\
&\quad - (H_n - 1) Du^{(n-1)} [a^{(n)} (a^{(n)})^T - I] - (H_n - 1) Du^{(n-1)} \\
&\quad - (a^{(n)} - I) Du^{(n-1)} [a^{(n)} (a^{(n)})^T - I] - (a^{(n)} - I) Du^{(n-1)} \\
&\quad - Du^{(n-1)} [a^{(n)} (a^{(n)})^T - I] - Du^{(n-1)}
\end{aligned}
$$

$$
\begin{aligned}
&+2(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})[a^{(n)}(a^{(n)})^T-I]\\
&+2(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})\\
&+2(H_n-1)\mathrm{Tr}(Du^{(n-1)})[a^{(n)}(a^{(n)})^T-I]\\
&+2(H_n-1)\mathrm{Tr}(Du^{(n-1)})\\
&+2\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})[a^{(n)}(a^{(n)})^T-I]\\
&+2\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})+2\mathrm{Tr}(Du^{(n-1)})[a^{(n)}(a^{(n)})^T-I]\\
&+2\mathrm{Tr}(Du^{(n-1)})-(H_n-1)[a^{(n)}(a^{(n)})^T-I](Du^{(n-1)})^T(a^{(n)}-I)^T\\
&-(H_n-1)[a^{(n)}(a^{(n)})^T-I](Du^{(n-1)})^T\\
&-(H_n-1)(Du^{(n-1)})^T(a^{(n)}-I)^T\\
&-(H_n-1)(Du^{(n-1)})^T-[a^{(n)}(a^{(n)})^T-I](Du^{(n-1)})^T(a^{(n)}-I)^T\\
&-[a^{(n)}(a^{(n)})^T-I](Du^{(n-1)})^T-(Du^{(n-1)})^T(a^{(n)}-I)^T\\
&-(Du^{(n-1)})^T. \qquad (4.3.36)
\end{aligned}
$$

Integrating from 0 to t and estimating in H^2, we obtain

$$
\begin{aligned}
&\|a^{(n)}(a^{(n)})^T-I\|_{H^2}\\
&\quad= C\int_0^t \|H_n-1\|_{H^2}\|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\|a^{(n)}(a^{(n)})^T-I\|_{H^2}\,ds\\
&\qquad+C\int_0^t \|H_n-1\|_{H^2}\|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\,ds\\
&\qquad+C\int_0^t \|H_n-1\|_{H^2}\|Du^{(n-1)}\|_{H^2}\|a^{(n)}(a^{(n)})^T-I\|_{H^2}\,ds\\
&\qquad+C\int_0^t \|H_n-1\|_{H^2}\|Du^{(n-1)}\|_{H^2}\,ds\\
&\qquad+C\int_0^t \|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\|a^{(n)}(a^{(n)})^T-I\|_{H^2}\,ds\\
&\qquad+C\int_0^t \|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\,ds
\end{aligned}
$$

$$+C\int_0^t \|Du^{(n-1)}\|_{H^2}\|a^{(n)}(a^{(n)})^T - I\|_{H^2}\ ds + C\int_0^t \|Du^{(n-1)}\|_{H^2}\ ds$$

$$+C\int_0^t \|H_n - 1\|_{H^2}\|a^{(n)} - I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\|a^{(n)}(a^{(n)})^T - I\|_{H^2}\ ds.$$

$$= C\int_0^t \Big((C\epsilon+1)\|a^{(n)}(a^{(n)})^T - I\|_{H^2} + (C\epsilon+1)\Big)\|Du^{(n-1)}\|_{H^2}\ ds, \qquad (4.3.37)$$

where (4.3.35) was used. Let $z = \sup_{\tau\le t}\|a^{(n)}(a^{(n)})^T(\tau) - I\|_{H^2}$. Then,

$$z \le C((C\epsilon+1)z + (C\epsilon+1))\int_0^t \|Du^{(n-1)}\|_{H^2}\ ds. \qquad (4.3.38)$$

Following the procedure used in (4.3.24) -(4.3.33), we obtain

$$z \le C\epsilon z + C\epsilon. \qquad (4.3.39)$$

Recall that $z(0) = 0$ and $z \ge 0$. Then,

$$0 \le z \le \frac{C\epsilon}{1 - C\epsilon}. \qquad (4.3.40)$$

which is small for ϵ sufficiently small. Therefore,

$$\|a^{(n)}(a^{(n)})^T(t) - I\|_{H^2} < \epsilon_a,\ t \ge 0. \qquad (4.3.41)$$

$a_t^{(n)}$ is Small in H^2

We now show $a_t^{(n)}$ is small. Recall from (4.3.18) that

$$\begin{aligned} a_t^{(n)} = & -(H_n-1)(a^{(n)}-I)Du^{(n-1)}(a^{(n)}-I) - (H_n-1)(a^{(n)}-I)Du^{(n-1)} \\ & -(H_n-1)Du^{(n-1)}(a^{(n)}-I) - (H_n-1)Du^{(n-1)} \\ & -(a^{(n)}-I)Du^{(n-1)}(a^{(n)}-I) \\ & -(a^{(n)}-I)Du^{(n-1)} - Du^{(n-1)}(a^q w(n) - I) - Du^{(n-1)} \end{aligned}$$

$$
\begin{aligned}
&+(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})(a^{(n)}-I)+(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})\\
&+(H_n-1)\mathrm{Tr}(Du^{(n-1)})(a^{(n)}-I)+(H_n-1)\mathrm{Tr}(Du^{(n-1)})\\
&+\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})(a^{(n)}-I)+\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})I\\
&+\mathrm{Tr}(Du^{(n-1)})(a^{(n)}-I)+\mathrm{Tr}(Du^{(n-1)})I.
\end{aligned}
$$

Estimating in H^2, we have

$$
\begin{aligned}
\|a_t^{(n)}\|_{H^2} \leq\ & C\|H_n-1\|_{H^2}\|a^{(n)}-I\|_{H^2}^2\|Du^{(n-1)}\|_{H^2}\\
&+C\|H_n-1\|_{H^2}\|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\\
&+C\|H_n-1\|_{H^2}\|Du^{(n-1)}\|_{H^2}+C\|a^{(n)}-I\|_{H^2}^2\|Du^{(n-1)}\|_{H^2}\\
&+C\|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}+C\|Du^{(n-1)}\|_{H^2}\\
&+C\|Du^{(n-1)}\|_{H^2}.
\end{aligned}
\tag{4.3.42}
$$

Recall that the source term $\mathcal{F}$ in (4.3.25) lies in $W^{1,1}(\mathbb{R}^+)$. Then $\mathcal{F}$ is absolutely continuous, and for any $t \geq 0$ we can write

$$
\mathcal{F}(t)=\mathcal{F}(0)+\int_0^t \mathcal{F}'(s)\ ds \leq \mathcal{F}(0)+\int_0^\infty \mathcal{F}'(s)\ ds. \tag{4.3.43}
$$

Since $\mathcal{F}(0)$ and $\|\mathcal{F}'\|_{L^1(\mathbb{R}^+}$ are both small, we may conclude that $a_t^{(n)}$ is small in H^2 by using (4.3.35), (4.3.25), and (A.2.14) in (4.3.42).

$(a^{(n)}(a^{(n)})^T)_t$ is Small in H^2

We now show $(a^{(n)}(a^{(n)})^T)_t$ is small. We have that

$$
\begin{aligned}
(a^{(n)}(a^{(n)})^T)_t =\ & a_t^{(n)}(a^{(n)})^T+a^{(n)}(a_t^{(n)})^T\\
&=a_t^{(n)}(a^{(n)}-I)^T+a_t^{(n)}+(a^{(n)}-I)(a_t^{(n)})^T+(a_t^{(n)})^T
\end{aligned}
$$

Estimating in H^2, we have

$$
\|(a^{(n)}(a^{(n)})^T)_t\|_{H^2} \leq C\|a_t^{(n)}\|_{H^2}\|a^{(n)}-I\|_{H^2}+C\|a_t^{(n)}\|_{H^2}.
$$

We now use the fact that a is close to identity and a_t is small to write

$$\|(a^{(n)}(a^{(n)})^T)_t\|_{H^2} \quad \leq \quad \epsilon_a.$$

a_{tt} is Small in H^1

We now show a_{tt} is small. Recall from (4.3.18) that

$$a_t^{(n)} \quad = \quad -H_n a^{(n)} Du^{(n-1)} a^{(n)} + H_n \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a^{(n)}.$$

We have that

$$\begin{aligned}
a_{tt}^{(n)} \quad = \quad & -\partial_t H_n a^{(n)} Du^{(n-1)} a^{(n)} - H_n a_t^{(n)} Du^{(n-1)} a^{(n)} - H_n a^{(n)} Du_t^{(n-1)} a^{(n)} \\
& -H_n a^{(n)} Du^{(n-1)} a_t^{(n)} + \partial_t H_n \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a^{(n)} \\
& +H_n \mathrm{Tr}(a_t^{(n)} Du^{(n-1)}) a^{(n)} + H_n \mathrm{Tr}(a^{(n)} Du_t^{(n-1)}) a^{(n)} + H_n \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a_t^{(n)} \\
= \quad & -\partial_t H_n (a^{(n)} - I) Du^{(n-1)} a^{(n)} - \partial_t H_n Du^{(n-1)} a^{(n)} - H_n a_t^{(n)} Du^{(n-1)} (a^{(n)} - I) \\
& -H_n a_t^{(n)} Du^{(n-1)} - (H_n - 1) a^{(n)} Du_t^{(n-1)} a^{(n)} - a^{(n)} Du_t^{(n-1)} a^{(n)} \\
& -(H_n - 1) a^{(n)} Du^{(n-1)} a_t^{(n)} - a^{(n)} Du^{(n-1)} a_t^{(n)} \\
& +\partial_t H_n \mathrm{Tr}((a^{(n)} - I) Du^{(n-1)}) a^{(n)} + \partial_t H_n \mathrm{Tr}(Du^{(n-1)}) a^{(n)} \\
& +(H_n - 1) \mathrm{Tr}(a_t^{(n)} Du^{(n-1)}) a^{(n)} + \mathrm{Tr}(a_t^{(n)} Du^{(n-1)}) a^{(n)} \\
& +(H_n - 1) \mathrm{Tr}(a^{(n)} Du_t^{(n-1)}) a^{(n)} + \mathrm{Tr}(a^{(n)} Du_t^{(n-1)}) a^{(n)} \\
& +(H_n - 1) \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a_t^{(n)} + \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a_t^{(n)} \\
& +(H_n - 1) \mathrm{Tr}(a_t^{(n)} Du^{(n-1)}) a^{(n)} + \mathrm{Tr}(a_t^{(n)} Du^{(n-1)}) a^{(n)} \\
& +(H_n - 1) \mathrm{Tr}(a^{(n)} Du_t^{(n-1)}) a^{(n)} + \mathrm{Tr}(a^{(n)} Du_t^{(n-1)}) a^{(n)} \\
& +(H_n - 1) \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a_t^{(n)} + \mathrm{Tr}(a^{(n)} Du^{(n-1)}) a_t^{(n)} \\
= \quad & -\partial_t H_n (a^{(n)} - I) Du^{(n-1)} (a^{(n)} - I) - \partial_t H_n (a^{(n)} - I) Du^{(n-1)} \\
& -\partial_t H_n Du^{(n-1)} (a^{(n)} - I) - \partial_t H_n Du^{(n-1)} - (H_n - 1) a_t^{(n)} Du^{(n-1)} (a^{(n)} - I) \\
& -a_t^{(n)} Du^{(n-1)} (a^{(n)} - I) - (H_n - 1) a_t^{(n)} Du^{(n-1)} - a_t^{(n)} Du^{(n-1)} \\
& -(H_n - 1)(a^{(n)} - I) Du_t^{(n-1)} a^{(n)} - (H_n - 1) Du_t^{(n-1)} a^{(n)} \\
& -(a^{(n)} - I) Du_t^{(n-1)} a^{(n)} - Du_t^{(n-1)} a^{(n)} - (H_n - 1)(a^{(n)} - 1) Du^{(n-1)} a_t^{(n)} \\
& -(H_n - 1)(a^{(n)} - 1) Du^{(n-1)} a_t^{(n)} - (a^{(n)} - I) Du^{(n-1)} a_t^{(n)} - Du^{(n-1)} a_t^{(n)}
\end{aligned}$$

$$
\begin{aligned}
& +\partial_t H_n \mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})(a^{(n)}-I) + \partial_t H_n \mathrm{Tr}((a^{(n)}-I)Du^{(n-1)}) \\
& +\partial_t H_n \mathrm{Tr}(Du^{(n-1)})(a^{(n)}-I) + \partial_t H_n \mathrm{Tr}(Du^{(n-1)}) \\
& +(H_n-1)\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})(a^{(n)}-I) + (H_n-1)\mathrm{Tr}(a_t^{(n)}Du^{(n-1)}) \\
& +\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})(a^{(n)}-I) + \mathrm{Tr}(a_t^{(n)}Du^{(n-1)}) \\
& +(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})a^{(n)} + (H_n-1)\mathrm{Tr}(Du_t^{(n-1)})a^{(n)} \\
& +\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})a^{(n)} + \mathrm{Tr}(Du_t^{(n-1)})a^{(n)} \\
& +(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)} + (H_n-1)\mathrm{Tr}(Du^{(n-1)})a_t^{(n)} \\
& +\mathrm{Tr}(a^{(n)}Du^{(n-1)})a_t^{(n)} + \mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)} \\
& +(H_n-1)\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})(a^{(n)}-I) + (H_n-1)\mathrm{Tr}(a_t^{(n)}Du^{(n-1)}) \\
& +\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})(a^{(n)}-I) + \mathrm{Tr}(a_t^{(n)}Du^{(n-1)}) \\
& +(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})a^{(n)} + (H_n-1)\mathrm{Tr}(Du_t^{(n-1)})a^{(n)} \\
& +\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})a^{(n)} + \mathrm{Tr}(Du_t^{(n-1)})a^{(n)} \\
& +(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)} + (H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)} \\
& +\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)} + \mathrm{Tr}(Du^{(n-1)})a_t^{(n)} \\
= \; & -\partial_t H_n(a^{(n)}-I)Du^{(n-1)}(a^{(n)}-I) - \partial_t H_n(a^{(n)}-I)Du^{(n-1)} \\
& -\partial_t H_n Du^{(n-1)}(a^{(n)}-I) - \partial_t H_n Du^{(n-1)} \\
& -(H_n-1)a_t^{(n)}Du^{(n-1)}(a^{(n)}-I) - a_t^{(n)}Du^{(n-1)}(a^{(n)}-I) \\
& -(H_n-1)a_t^{(n)}Du^{(n-1)} - a_t^{(n)}Du^{(n-1)} - (H_n-1)(a^{(n)}-I)Du_t^{(n-1)}(a^{(n)}-I) \\
& -(H_n-1)(a^{(n)}-I)Du_t^{(n-1)} - (H_n-1)Du_t^{(n-1)}(a^{(n)}-I) - (H_n-1)Du_t^{(n-1)} \\
& -(a^{(n)}-I)Du_t^{(n-1)}(a^{(n)}-I) - (a^{(n)}-I)Du_t^{(n-1)} - Du_t^{(n-1)}(a^{(n)}-I) \\
& -Du_t^{(n-1)} - (H_n-1)(a^{(n)}-1)Du^{(n-1)}a_t^{(n)} - (H_n-1)(a^{(n)}-1)Du^{(n-1)}a_t^{(n)} \\
& -(a^{(n)}-I)Du^{(n-1)}a_t^{(n)} - Du^{(n-1)}a_t^{(n)} + \partial_t H_n \mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})(a^{(n)}-I) \\
& +\partial_t H_n \mathrm{Tr}((a^{(n)}-I)Du^{(n-1)}) + \partial_t H_n \mathrm{Tr}(Du^{(n-1)})(a^{(n)}-I) + \partial_t H_n \mathrm{Tr}(Du^{(n-1)}) \\
& +(H_n-1)\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})(a^{(n)}-I) + (H_n-1)\mathrm{Tr}(a_t^{(n)}Du^{(n-1)}) \\
& +\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})(a^{(n)}-I) + \mathrm{Tr}(a_t^{(n)}Du^{(n-1)}) \\
& +(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})(a^{(n)}-I) + (H_n-1)\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)}) \\
& +(H_n-1)\mathrm{Tr}(Du_t^{(n-1)})(a^{(n)}-I) + (H_n-1)\mathrm{Tr}(Du_t^{(n-1)}) \\
& +\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})(a^{(n)}-I) + \mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)}) \\
& +\mathrm{Tr}(Du_t^{(n-1)})(a^{(n)}-I) + \mathrm{Tr}(Du_t^{(n-1)})
\end{aligned}
$$

$$
\begin{aligned}
&+(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)}+(H_n-1)\mathrm{Tr}(Du^{(n-1)})a_t^{(n)}\\
&+\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)}+\mathrm{Tr}(Du^{(n-1)})a_t^{(n)}+\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)}\\
&+(H_n-1)\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})(a^{(n)}-I)+(H_n-1)\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})\\
&+\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})(a^{(n)}-I)+\mathrm{Tr}(a_t^{(n)}Du^{(n-1)})\\
&+(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})(a^{(n)}-I)+(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})\\
&+(H_n-1)\mathrm{Tr}(Du_t^{(n-1)})(a^{(n)}-I)+(H_n-1)\mathrm{Tr}(Du_t^{(n-1)})\\
&+\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})(a^{(n)}-I)+\mathrm{Tr}((a^{(n)}-I)Du_t^{(n-1)})\\
&+\mathrm{Tr}(Du_t^{(n-1)})(a^{(n)}-I)+\mathrm{Tr}(Du_t^{(n-1)})+(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)}\\
&+(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)}+\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})a_t^{(n)}\\
&+\mathrm{Tr}(Du^{(n-1)})a_t^{(n)}. \qquad (4.3.44)
\end{aligned}
$$

We observe from (4.3.20) that

$$
\begin{aligned}
\partial_t H_n &= -(H_n-I)^2\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})-(H_n-I)^2\mathrm{Tr}(Du^{(n-1)})\\
&\quad -2(H_n-1)\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})-2(H_n-1)\mathrm{Tr}(Du^{(n-1)})\\
&\quad -\mathrm{Tr}((a^{(n)}-I)Du^{(n-1)})-\mathrm{Tr}(Du^{(n-1)}). \qquad (4.3.45)
\end{aligned}
$$

Estimating in H^2, we have

$$
\begin{aligned}
\|\partial_t H_n\|_{H^2} &= C\|H_n-I\|_{H^2}^2\|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}+C\|H_n-I\|_{H^2}^2\|Du^{(n-1)}\|_{H^2}\\
&\quad +C\|H_n-1\|_{H^2}\|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}+C\|a^{(n)}-I\|_{H^2}\|Du^{(n-1)}\|_{H^2}\\
&\quad +C\|Du^{(n-1)}\|_{H^2}. \qquad (4.3.46)
\end{aligned}
$$

Using (4.3.35), (A.2.14), and (4.3.25) in (4.3.46), we have

$$
\|\partial_t H_n\|_{H^2} \leq \epsilon. \qquad (4.3.47)
$$

Estimating (4.3.44) in H^1 and using (4.3.35), (A.2.14), (A.2.15), (4.3.25), and (4.3.47) in (4.3.44), we conclude that

$$
\|a_{tt}^{(n)}\|_{H^1} \leq \epsilon. \qquad (4.3.48)
$$

$\eta^{(n-1)}$ is close to x

We now show that $\eta^{(n-1)}$ with $\eta^{(n-1)}(x,0) = x$ is close to x for all t. We have

$$
\begin{aligned}
\eta_t^{(n-1)} = u \quad \implies \quad & \eta^{(n-1)}(x,t) - \eta^{(n-1)}(x,0) = \int_0^t u^{(n-1)}\, ds \\
\implies \quad & \eta^{(n-1)} - x = \int_0^t u^{(n-1)}\, ds.
\end{aligned}
$$

Estimating in H^3, we obtain

$$
\|\eta^{(n-1)} - x\|_{H^3} \leq \int_0^t \|u^{(n-1)}\|_{H^3}\, ds.
$$

We have by Lemma 3.4.1

$$
\begin{aligned}
\|\eta^{(n-1)} - x\|_{H^3} \;\leq\; & \int_0^t \Bigg(C[X(0) + \mathcal{F}(0)]e^{-\frac{s}{C}} \\
& \qquad + C\int_0^s e^{-\omega(s-\tau)}\big(\mathcal{F}(\tau) + \mathcal{F}'(\tau)\big)\, d\tau + C\mathcal{F}(s)\Bigg)\, ds \\
\leq\; & C[X(0) + \mathcal{F}(0)](1 - e^{-\frac{t}{C}}) + C\int_0^\infty \mathcal{F}(\tau) + \mathcal{F}'(\tau)\, d\tau. \qquad (4.3.49)
\end{aligned}
$$

Thus for sufficiently small initial data, we conclude

$$
\|\eta^{(n-1)} - x\|_{H^3} \;\leq\; \epsilon. \qquad (4.3.50)
$$

4.3.2 Iterate Solutions Lie in a Ball in $\tilde{Y}$

Next, we demonstrate that all iterates lie in a closed ball $B_M \subset \tilde{Y}$ that depends on the initial data. Using the 1D Agmon inequality in time (see [35, Appedix A]), we may conclude that $\tilde{Y} \subset X$, where X is given by

$$\begin{aligned} X &= \{(v,q,w) \mid v \in L^\infty([0,T];H^3(\Omega_f)),\ v_t \in L^\infty([0,T];H^2(\Omega_f)), \\ &\quad v_{tt} \in L^\infty([0,T];L^2(\Omega_f)),\ \nabla v_{tt} \in L^2([0,T];L^2(\Omega_f)), \\ &\quad q \in L^\infty([0,T];H^2(\Omega_f)),\ q_t \in L^\infty([0,T];H^1(\Omega_f)), \\ &\quad \partial_t^j w \in C([0,T];H^{3-j}(\Omega_e)),\ j=0,1,2,3\}. \end{aligned} \tag{4.3.51}$$

Thus from the conclusion of Lemma 4.2.6, we obtain the following a-priori bound on the solution $y \in \tilde{Y}$ to the system (4.2.64):

$$\begin{aligned} \|y\|_{\tilde{Y}} &\leq C_T(\|y\|_{\tilde{Y}(0)} + \|f\|_{\tilde{D}}) + C_T(\|a\|_{L^2([0,T];H^3(\Omega_f))} + \|a_t\|_{L^2([0,T];H^2(\Omega_f))} \\ &\quad + \|a_{tt}\|_{L^2([0,T];H^1(\Omega_f))} + \|a_{ttt}\|_{L^2([0,T];L^2(\Omega_f))})\|y\|_X, \end{aligned} \tag{4.3.52}$$

which arises from applying the estimate (4.2.63) to the solution of the systems given in (4.2.6).

We now apply (4.3.52) to the iterates $y^{(n)}$

$$\begin{aligned} \|y^{(n)}\|_{\tilde{Y}} &\leq C\|y_0\|_{\tilde{Y}(0)} + C\big(\|a^{(n)}\|_{L^2([0,T];H^3(\Omega_f))} + \|a_t^{(n)}\|_{L^2([0,T];H^2(\Omega_f))} \\ &\quad + C\|a_{tt}^{(n)}\|_{L^2([0,T];H^2(\Omega_f))} + C\|a_{ttt}^{(n)}\|_{L^2([0,T];L^2(\Omega_f))}\big)\|y^{(n)}\|_X + \|f\|_{\tilde{D}} \\ &\leq C\|y_0\|_{\tilde{Y}(0)} + C\epsilon\|a^{(n)}\|_{L^2([0,T];H^3(\Omega_f))} + C\epsilon\|a_t^{(n)}\|_{L^2([0,T];H^2(\Omega_f))} \\ &\quad + C\epsilon\|a_{tt}^{(n)}\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|a_{ttt}^{(n)}\|_{L^2([0,T];L^2(\Omega_f))} + \|f\|_{\tilde{D}}. \end{aligned} \tag{4.3.53}$$

on $[0,T]$ where C depends on T. The space X is given by (4.3.51).

Consider the term $\|a^{(n)}\|_{L^2([0,T];H^4(\Omega_f)}$ on the right side of (4.3.53). Using (4.3.5), we may write

$$\|a^{(n)}\|_{H^3(\Omega_f)} \leq C\int_0^t \|H_n\|_{H^3(\Omega_f)}\|a^{(n)}\|^2_{H^3(\Omega_f)}\|Du^{(n-1)}\|_{H^3(\Omega_f)} + C.$$

Similarly, we have the following for $\|H_n\|_{H^3(\Omega_f)}$:

$$\|H_n\|_{H^3(\Omega_f)} \leq C\int_0^t \|H_n\|^2_{H^3(\Omega_f)}\|a^{(n)}\|_{H^3(\Omega_f)}\|Du^{(n-1)}\|_{H^3(\Omega_f)} + C.$$

Summing $\|a^{(n)}\|_{H^3(\Omega_f)}$ and $\|H_n\|_{H^3(\Omega_f)}$, we obtain

$$\begin{aligned}
\|a^{(n)}\|_{H^3(\Omega_f)} + \|H_n\|_{H^3(\Omega_f)} &\leq C\int_0^t (\|H_n\|_{H^3(\Omega_f)}\|a^{(n)}\|^2_{H^3(\Omega_f)} \\
&\quad +\|H_n\|^2_{H^3(\Omega_f)}\|a^{(n)}\|_{H^3(\Omega_f)})\|Du^{(n-1)}\|_{H^3(\Omega_f)} + C \\
&\leq C\int_0^t (\|H_n\|_{H^3(\Omega_f)} + \|a^{(n)}\|_{H^3(\Omega_f)})^3\|Du^{(n-1)}\|_{H^3(\Omega_f)} + C.
\end{aligned}$$

We apply a Gronwall-type inequality (see [22, Theorem 25]), the Cauchy-Schwarz inequality, and the fact that $Du^{(n-1)}$ lies in $L^2([0,T];H^3(\Omega_f))$ to obtain

$$\begin{aligned}
\|a^{(n)}\|_{H^3(\Omega_f)} + \|H_n\|_{H^3(\Omega_f)} &\leq \frac{C}{1 - C^2\int_0^t \|Du^{(n-1)}\|_{H^3(\Omega_f)}\, ds} \\
&\leq \frac{C}{1 - C^2\int_0^T \|Du^{(n-1)}\|_{H^3(\Omega_f)}\, ds} \\
&\leq \frac{C}{1 - C^2T^{\frac{1}{2}}\|Du^{(n-1)}\|_{L^2([0,T];H^3(\Omega_f))}},
\end{aligned}$$

for $t \in [0,T]$ with $T < C^4\|Du^{(n-1)}\|_{L^2([0,T];H^3(\Omega_f))}$. Here we take C is sufficiently large to accommodate T. Then we may write

$$\|a^{(n)}\|_{H^3(\Omega_f)} + \|H_n\|_{H^3(\Omega_f)} \leq \tilde{C}$$

for $t \in [0,T]$, and

$$\|a^{(n)}\|_{L^2([0,T];H^3(\Omega_f))} \leq \tilde{C} \leq C_T\|y_0\|_{\tilde{Y}(0)}. \tag{4.3.54}$$

For $\|a_t^{(n)}\|_{L^2([0,T];H^2(\Omega_f))}$, we have (from (4.3.5) and (4.3.6))

$$\|a_t^{(n)}\|_{H^2(\Omega_f)} + \|\partial_t H^{(n)}\|_{H^2(\Omega_f)} \leq C(\|H_n\|_{H^2(\Omega_f)}\|a^{(n)}\|^2_{H^2(\Omega_f)}$$

$$+\|H_n\|^2_{H^2(\Omega_f)}\|a^{(n)}\|_{H^2(\Omega_f)})\|Du^{(n-1)}\|_{H^2(\Omega_f)}.$$

Using (4.3.54), we have

$$\|a_t^{(n)}\|_{H^2(\Omega_f)} \;\; \leq \;\; C\|Du^{(n-1)}\|_{H^2(\Omega_f)},$$

which implies

$$\|a_t^{(n)}\|_{L^2([0,T];H^2(\Omega_f))} \;\; \leq \;\; C\|Du^{(n-1)}\|_{L^2([0,T];H^2(\Omega_f))} \leq C\|y^{(n-1)}\|_{\tilde{Y}}.$$

The remaining terms on the right side of (4.3.53) are obtained by differentiating (4.3.5) and (4.3.6) the appropriate number of times and applying a similar argument as above. Using the decay bound (3.4.37) with Proposition A.1.1, (A.2.14)-(A.2.15), and reducing ϵ (if necessary), we obtain

$$\|y^{(n)}\|_{\tilde{Y}} \;\; \leq \;\; C\|y_0\|_{\tilde{Y}(0)} + \frac{1}{2}\|y^{(n-1)}\|_{\tilde{Y}} + \|f\|_{\tilde{D}}. \tag{4.3.55}$$

Here, $\|f\|_{\tilde{D}}$ is given by

$$\begin{aligned}\|f\|_{\tilde{D}} \;\; = \;\; & \|f_f\circ\eta^{(n-1)}\|_{H^1([0,T];H^1(\Omega))} + \|\partial_{tt}(f_f\circ\eta^{(n-1)})\|_{L^2([0,T];L^2(\Omega))} \\ & +\|f_e\|_{W^{1,1}([0,T];H^{\frac{3}{2}}(\Omega_e))} + \|f_e\|_{W^{\frac{3}{2},1}([0,T];L^2(\Omega_e))} + \|\partial_{tt}f_e\|_{L^1([0,T];H^{\frac{1}{2}}(\Omega_e))} \\ & +\|\partial_{ttt}f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} + \sum_{m=1}^{2}\Big(\|\partial_m(f_f\circ\eta^{(n-1)})\|_{L^2([0,T];H^1(\Omega))} \\ & +\|\partial_m f_e\|_{L^1([0,T];H^{\frac{3}{2}}(\Omega_e))} + \|\partial_m f_e\|_{W^{\frac{1}{2},1}([0,T];L^2(\Omega_e))} \\ & +\|\partial_m\partial_{tt}f_e\|_{L^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}\Big).\end{aligned} \tag{4.3.56}$$

Using (4.3.14)-(4.3.17), we obtain the following estimate for $y^{(n)}$ in $\tilde{Y}$.

$$\begin{aligned}\|y^{(n)}\|_{\tilde{Y}} \;\; \leq \;\; & C\|y_0\|_{\tilde{Y}(0)} + \frac{1}{2}\|y^{(n-1)}\|_{\tilde{Y}} + C\epsilon\|f_f\|_{L^2([0,T];H^3(\Omega_f))} + C\epsilon\|\partial_t f_f\|_{L^2([0,T];H^2(\Omega_f))} \\ & +C\epsilon\|\partial_{tt}f_f\|_{L^2([0,T];H^1(\Omega_f))} + \|f_e\|_{W^{\frac{5}{2},1}([0,T];H^1(\Omega_e))} + \|\partial_t f_e\|_{L^2([0,T];H^{\frac{3}{2}}(\Omega_e))} \\ & +\|\partial_{tt}f_e\|_{H^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} + \sum_{m=1}^{2}\Big(\|\partial_m f_e\|_{W^{\frac{3}{2},1}([0,T];H^1(\Omega_e))}\end{aligned}$$

$$+\|\partial_m\partial_t f_e\|_{L^2([0,T];H^{\frac{3}{2}}(\Omega_e))} + \|\partial_m\partial_{tt} f_e\|_{H^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}\Big). \tag{4.3.57}$$

Repeating this process $n-1$ more times, we obtain

$$\begin{aligned}
\|y^{(n)}\|_{\tilde{Y}} \leq\ & C\|y_0\|_{\tilde{Y}(0)} + C\|f_f\|_{L^2([0,T];H^3(\Omega_f))} + C\|\partial_t f_f\|_{L^2([0,T];H^2(\Omega_f))} \\
& + C\|\partial_{tt} f_f\|_{L^2([0,T];H^1(\Omega_f))} + C\|f_e\|_{W^{\frac{5}{2},1}([0,T];H^1(\Omega_e))} + C\|\partial_t f_e\|_{L^2([0,T];H^{\frac{3}{2}}(\Omega_e))} \\
& + C\|\partial_{tt} f_e\|_{H^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))} + C\sum_{m=1}^{2}\Big(\|\partial_m f_e\|_{W^{\frac{3}{2},1}([0,T];H^1(\Omega_e))} \\
& + \|\partial_m\partial_t f_e\|_{L^2([0,T];H^{\frac{3}{2}}(\Omega_e))} + \|\partial_m\partial_{tt} f_e\|_{H^1([0,T];H^{-\frac{1}{4}-\delta}(\Omega_e))}\Big).
\end{aligned} \tag{4.3.58}$$

Therefore, if f belongs to the ball (4.2.2), then all the iterates belong to a ball B_M in $\tilde{Y}$ on $[0, T_0]$, where T_0 is a fixed constant (i.e. we can take $T_0 = 1$). That is,

$$\|y^{(n)}\|_{\tilde{Y}} \leq M\|y_0\|_{\tilde{Y}(0)}.$$

4.3.3 Fixed Point Iteration

We now construct a solution on the interval $[0, T_0]$. Consider the map $\Lambda : \tilde{Y} \to \tilde{Y}$ between the successive iterates, i.e. $\Lambda(u^{(j)}, p^{(j)}, \psi^{(j)}) = (u^{(j+1)}, p^{(j+1)}, \psi^{(j+1)})$. Denote

$$(u, p, \psi) = (u^{(n-1)}, p^{(n-1)}, \psi^{(n-1)}),$$

with $\eta_t = \eta_t^{n-1} = u$ on Ω_f, and

$$(v, q, w) = (u^{(n)}, p^{(n)}, \psi^{(n)}).$$

Also, let $\eta = \eta^{(n-1)} = \int_0^t u^{(n-1)}\, ds + x = \int_0^t u\, ds + x$ on Ω_f. Thus $(v, q, w) = \Lambda(u, p, \psi)$ is the solution to the system

$$\begin{cases}
v_t - \Delta v + \nabla q = f + f_f \circ \eta, & \Omega_f \times (0, T), \\
\nabla \cdot v = g, & \Omega_f \times (0, T), \\
w_{tt} - \Delta w = f_e - \alpha w_t - \beta w, & \Omega_e \times (0, T),
\end{cases}$$

with boundary conditions

$$\begin{cases} w_t = v, & \Gamma_c \times (0,T), \\ v = 0, & \Gamma_f \times (0,T), \\ \frac{\partial v}{\partial N} - qN = \frac{\partial w}{\partial N} + h, & \Gamma_c \times (0,T), \end{cases}$$

where,

$$\begin{cases} f^i = \partial_j((\delta_{jk} - a^j_\ell a^k_\ell)\partial_k v^i) + (\delta_{ki} - a^k_i)\partial_k q, \\ g = (\delta_{kj} - a^k_j)\partial_k v^j, \\ h^i = (\delta_{jk} - a^j_\ell a^k_\ell)\partial_k v^i N_j + (\delta_{ki} - a^k_i) q N_k, \end{cases}$$

for $i = 1, 2, 3$. The matrix $a(x,t)$ corresponds to u, and is found by solving

$$\begin{cases} a_t = -HaDua + H\mathrm{Tr}(aDu)a, \\ a(x,0) = I, \end{cases} \tag{4.3.59}$$

where the scalar-valued function $H(x,t)$ is obtained from

$$\begin{cases} \partial_t H = -H^2 \mathrm{Tr}(aDu), \\ H(x,0) = 1. \end{cases} \tag{4.3.60}$$

We now show that Λ is a contraction on the lower topological space

$$\begin{aligned} Y_0 \;=\; & \big\{(v,q,w) : v \in L^2([0,T]; H^2(\Omega_f)),\ v_t \in L^2([0,T]; L^2(\Omega_f)), \\ & q \in L^2([0,T]; H^1(\Omega_f)),\ q \in H^{\frac{1}{4}}([0,T]; L^2(\Gamma_c)), \\ & w \in L^\infty([0,T]; H^1(\Omega_e)),\ w_t \in L^\infty([0,T]; L^2(\Omega_e))\big\}. \end{aligned} \tag{4.3.61}$$

with $T = T_0$ and norm

$$\begin{aligned} \|(v,q,w)\|_{Y_0} \;=\; & \|v\|_{L^2([0,T],H^2(\Omega_f))} + \|v_t\|_{L^2([0,T],L^2(\Omega_f))} + \|q\|_{L^2([0,T],H^1(\Omega_f))} \\ & + \|q\|_{H^{\frac{1}{4}}([0,T],L^2(\Gamma_c))} + \epsilon_0 \|w\|_{L^\infty([0,T],H^1(\Omega_e))} + \epsilon_0 \|w_t\|_{L^\infty([0,T],L^2(\Omega_e))}. \end{aligned} \tag{4.3.62}$$

Here ϵ_0 is a sufficiently small constant determined below.

Let (u, p, ψ) and $(\tilde{u}, \tilde{p}, \tilde{\psi})$ be two elements in the ball B_M in $\tilde{Y}$. We estimate the difference between two solutions (v, q, w) and $(\tilde{v}, \tilde{q}, \tilde{w})$ arising from (u, p, ψ) and $(\tilde{u}, \tilde{p}, \tilde{\psi})$ in the topology of Y_0. We define the differences of the old variables

$$U = u - \tilde{u},\ P = p - \tilde{p},\ \Psi = \psi - \tilde{\psi},$$

and the differences of the new variables

$$V = v - \tilde{v},\ Q = q - \tilde{q},\ W = w - \tilde{w},$$

respectively. These variables obey the following equations:

$$\begin{cases} V_t - \Delta V + \nabla Q = F, & \Omega_f \times (0,T), \\ \nabla \cdot V = G, & \Omega_f \times (0,T), \\ \frac{\partial V}{\partial N} - QN = \frac{\partial W}{\partial N} + H, & \Gamma_c \times (0,T), \\ V = 0, & \Gamma_f \times (0,T), \end{cases} \tag{4.3.63}$$

and

$$\begin{cases} W_{tt} - \Delta W + \alpha W_t + \beta W = 0, & \Omega_e \times (0,T), \\ W_t = V, & \Gamma_c \times (0,T), \end{cases} \tag{4.3.64}$$

where,

$$\begin{cases} F^i & = \partial_j((\delta_{jk} - \tilde{a}^j_\ell \tilde{a}^k_\ell)\partial_k \tilde{v}^i) - \partial_j((\delta_{jk} - a^j_\ell a^k_\ell)\partial_k v^i) + (\delta_{ki} - \tilde{a}^k_i)\partial_k \tilde{q} - (\delta_{ki} - a^k_i)\partial_k q \\ & \quad + (f^i_f \circ \phi - f^i_f \circ \tilde{\phi}), \\ G & = (\delta_{kj} - \tilde{a}^k_j)\partial_k \tilde{v}^j - (\delta_{kj} - a^k_j)\partial_k v^j = \partial_k((\delta_{kj} - \tilde{a}^k_j)\tilde{v}^j - (\delta_{kj} - a^k_j)v^j), \\ H^i & = (\delta_{jk} - \tilde{a}^j_\ell \tilde{a}^k_\ell)\partial_k \tilde{v}^i N_j - (\delta_{jk} - a^j_\ell a^k_\ell)\partial_k v^i N_j + (\delta_{ki} - \tilde{a}^k_i)\tilde{q} N_k - (\delta_{ki} - a^k_i) q N_k, \end{cases}$$

for $i = 1, 2, 3$, and

$$G_t = \mathrm{div}\tilde{A} + \tilde{B},$$

with $\tilde{B} = 0$ and

$$\tilde{A}_k = \partial_t((\partial_{ki} - \tilde{a}^k_j)\tilde{v}^j - (\delta_{kj} - a^k_j)v^j), \ k = 1, 2, 3.$$

Here a is the matrix of coefficients associated with the flow map induced by u and $\tilde{a}$ is the matrix of coefficients associated with the flow map induced by $\tilde{u}$. Note that the forcing term f_e cancelled when we subtracted iterates.

We apply Lemma 4.2.2 to the linear system (4.3.63) – (4.3.63) to obtain

$$\begin{aligned}
&\|V\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{C([0,T];H^2(\Omega_f))} + \|Q\|_{L^2([0,T];H^1(\Omega_f))} + \|Q\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\
&\quad + \|V_t\|_{L^2([0,T];L^2(\Omega_f))} \\
&\qquad \le C\Bigg(\|F\|_{L^2(\Omega_f \times [0,T])} + \|G\|_{L^2([0,T];H^1(\Omega_f))} + \|\tilde{A}\|_{L^2([0,T];L^2(\Omega_f))} + \|\tilde{B}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\qquad\quad + \|H\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \|H\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + \left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} \\
&\qquad\quad + \left\|\frac{\partial W}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\Bigg).
\end{aligned} \tag{4.3.65}$$

We now estimate the terms on the right side of (4.3.65). For F, which we write as

$$\begin{aligned}
F^i &= \partial_j((\delta_{jk} - \tilde{a}^j_\ell \tilde{a}^k_\ell)\partial_k \tilde{v}^i) - \partial_j((\delta_{jk} - a^j_\ell a^k_\ell)\partial_k v^i) + (\delta_{ki} - \tilde{a}^k_i)\partial_k \tilde{q} - (\delta_{ki} - a^k_i)\partial_k q \\
&\quad + (f^i_f \circ \phi - f^i_f \circ \tilde{\phi}), \\
&= -\partial_j((\tilde{a}^j_\ell \tilde{a}^k_\ell - \delta_{jk})\partial_k \tilde{v}^i) + \partial_j((a^j_\ell a^k_\ell - \delta_{jk})\partial_k v^i) - (a^k_i - \delta_{ki})\partial_k q + (a^k_i - \delta_{ki})\partial_k q \\
&\quad + (f^i_f \circ \phi - f^i_f \circ \tilde{\phi}), \\
&= \partial_j((\tilde{a}^j_\ell \tilde{a}^k_\ell - \delta_{jk})\partial_k V^i) - \partial_j((a^j_\ell a^k_\ell - \tilde{a}^j_\ell \tilde{a}^k_\ell)\partial_k v^i) + (\tilde{a}^k_i - \delta_{ki})\partial_k Q - (a^k_i - \tilde{a}^k_i)\partial_k q \\
&\quad + (f^i_f \circ \phi - f^i_f \circ \tilde{\phi}).
\end{aligned}$$

Estimating, we have

$$\begin{aligned}
&\|F\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \le C \sum_{i,j} \|(\tilde{a}^j_\ell \tilde{a}^k_\ell - \delta_{jk})\partial_k V^i\|_{L^2([0,T];H^1(\Omega_f))} + \|(a^j_\ell a^k_\ell - \tilde{a}^j_\ell \tilde{a}^k_\ell)\partial_k v^i\|_{L^2([0,T];H^1(\Omega_f))} \\
&\qquad + \|(a^k_i - \delta_{ki})\partial_k Q\|_{L^2([0,T];L^2(\Omega_f))} + \|(a^k_i - \tilde{a}^k_i)\partial_k q\|_{L^2([0,T];L^2(\Omega_f))} \\
&\qquad + \|f^i_f \circ \phi - f^i_f \circ \tilde{\phi}\|_{L^2([0,T];L^2(\Omega_f))}.
\end{aligned}$$

Applying Holder and Sobolev inequalities, we have

$$\begin{aligned}\|F\|_{L^2(\Omega_f)} &\leq C(\|\tilde{a}^j_\ell \tilde{a}^k_\ell - \delta_{jk}\|_{H^{1.5+\epsilon}(\Omega_f)}\|V\|_{H^2(\Omega_f)} + \|\tilde{a}^j_\ell \tilde{a}^k_\ell - a^j_\ell a^k_\ell\|_{H^1(\Omega_f)}\|\nabla v\|_{L^\infty(\Omega_f)} \\ &\quad + \|\tilde{a}^k_\ell - \delta_{ki}\|_{H^{1.5+\epsilon}(\Omega_f)}\|Q\|_{H^1(\Omega_f)} + \|a^j_i - \tilde{a}^k_i\|_{L^6(\Omega_f)}\|\nabla q\|_{L^3(\Omega_f)}). \\ &\quad + \|f^i_f \circ \phi - f^i_f \circ \tilde{\phi}\|_{L^2(\Omega_f)}. \end{aligned} \tag{4.3.66}$$

Here we use that the coefficient matrices a and $\tilde{a}$ obey (3.3.6) and (3.3.7). Furthermore, we use a similar approach used in Subsection 4.3.1 with Gronwall's inequality to obtain

$$\|a - \tilde{a}\|_{H^1} \leq C \int_0^t \|U\|_{H^2}\ ds. \tag{4.3.67}$$

Similarly,

$$\|a^j_\ell a^k_\ell - \tilde{a}^j_\ell \tilde{a}^k_\ell\|_{H^1} \leq C \int_0^t \|U\|_{H^2}\ ds. \tag{4.3.68}$$

Using (4.3.67)-(4.3.68) in (4.3.66), we have

$$\begin{aligned}\|F\|_{L^2([0,T];L^2(\Omega_f))} &\leq C\epsilon\|U\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|P\|_{L^2([0,T];H^1(\Omega_f))} \\ &\quad + C\epsilon\|V\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|Q\|_{L^2([0,T];H^1(\Omega_f))} \\ &\quad + \|f_f \circ \phi - f_f \circ \tilde{\phi}\|_{L^2([0,T];L^2(\Omega_f))}, \end{aligned} \tag{4.3.69}$$

where $T \leq T_0$ is assumed, and we use that the coefficient matrices a and $\tilde{a}$ obey (3.3.6) and (3.3.7).

We require an estimate on $\|f_f \circ \phi - f_f \circ \tilde{\phi}\|_{L^2([0,T];L^2(\Omega_f))}$. Following the strategy outlined in Subsection 4.3, we have

$$\|f_f \circ \phi - f_f \circ \tilde{\phi}\|_{L^2([0,T];L^2(\Omega_f))} \leq C\|f_f\|_{L^2([0,T];H^1(\Omega_f))}\|\phi - \tilde{\phi}\|_{L^2([0,T];H^2(\Omega_f))}. \tag{4.3.70}$$

Recall that $\mathcal{F}(t)$ is assumed to be small in $W^{1,1}(\mathbb{R}^+)$. This implies that $\|f_f(t)\|_{H^1(\Omega_f)}$ is small for all $t \geq 0$. Therefore,

$$\|f_f \circ \phi - f_f \circ \tilde{\phi}\|_{L^2([0,T];L^2(\Omega_f))} \leq C\epsilon \|U\|_{L^2([0,T];H^2(\Omega_f))}.$$

For G, we use a similar strategy as with estimating F to obtain

$$\|G\|_{H^1(\Omega_f)} \leq \|\tilde{a}_j^k - \delta_{jk}\|_{L^\infty(\Omega_f)} \|\nabla V\|_{H^1(\Omega_f)} + C\|\tilde{a}_j^k - \tilde{a}_j^k\|_{H^1(\Omega_f)} \|\nabla v\|_{L^\infty(\Omega_f)}.$$

Thus,

$$\|G\|_{L^2([0,T];H^1(\Omega_f))} \leq C\epsilon \|U\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon \|V\|_{L^2([0,T];H^2(\Omega_f))}. \tag{4.3.71}$$

For H, we write

$$\begin{aligned} H^i &= (\delta_{jk} - \tilde{a}_\ell^j \tilde{a}_\ell^k)\partial_k \tilde{v}^i N_j - (\delta_{jk} - a_\ell^j a_\ell^k)\partial_k v^i N_j + (\delta_{ki} - \tilde{a}_i^k)\tilde{q} N_k - (\delta_{ki} - a_i^k) q N_k, \\ &= (\delta_{jk} - \tilde{a}_\ell^j \tilde{a}_\ell^k)\partial_k \tilde{v}^i N_j - (\delta_{jk} - \tilde{a}_\ell^j \tilde{a}_\ell^k)\partial_k v^i N_j + (\delta_{jk} - \tilde{a}_\ell^j \tilde{a}_\ell^k)\partial_k v^i N_j \\ &\quad -(\delta_{jk} - a_\ell^j a_\ell^k)\partial_k v^i N_j + (\delta_{ki} - \tilde{a}_i^k)\tilde{q} N_k + (\delta_{ki} - \tilde{a}_i^k) q N_k - (\delta_{ki} - \tilde{a}_i^k) q N_k \\ &\quad -(\delta_{ki} - a_i^k) q N_k, \\ &= (\delta_{jk} - \tilde{a}_\ell^j \tilde{a}_\ell^k)(\partial_k \tilde{v}^i - \partial_k v^i) N_j + (a_\ell^j a_\ell^k - \tilde{a}_\ell^j \tilde{a}_\ell^k)\partial_k v^i N_j + (\delta_{ki} - \tilde{a}_i^k)(\tilde{q} - q) N_k \\ &\quad +(\tilde{a}_i^k - a_i^k) q N_k. \end{aligned}$$

Estimating, we obtain

$$\|H\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} \leq \|H\|_{L^2([0,T];H^1(\Omega_f))}.$$

Then, following the same process for estimating F we obtain

$$\begin{aligned} \|H\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} &\leq C\epsilon \|U\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon \|P\|_{L^2([0,T];H^1(\Omega_f))} \\ &\qquad + C\epsilon \|V\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon \|Q\|_{L^2([0,T];H^1(\Omega_f))}. \end{aligned} \tag{4.3.72}$$

Using Kato-Ponce type estimes (see [33]), we have

$$\|H\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \leq C\|\tilde{a}^T : \tilde{a} - I\|_{L^\infty([0,T]\times\Omega_f)} \|\nabla V\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}$$

$$
\begin{aligned}
&+C\|\tilde{a}^T:\tilde{a}-I\|_{W^{\frac{1}{4},4}([0,T],L^\infty(\Omega_f))}\|\nabla V\|_{L^4([0,T];L^2(\Gamma_c))}\\
&+C\|a^T:a-\tilde{a}^T:\tilde{a}\|_{L^\infty([0,T];L^4(\Gamma_c))}\|\nabla v\|_{H^{\frac{1}{4}}([0,T];L^4(\Gamma_c))}\\
&+C\|a^T:a-\tilde{a}^T:\tilde{a}\|_{H^{\frac{1}{4}}([0,T],L^2(\Gamma_c))}\|\nabla v\|_{L^\infty([0,T]\times\Omega_f}\\
&+C\|\tilde{a}-I\|_{L^\infty([0,T]\times\Omega_f)}\|Q\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\\
&+C\|\tilde{a}-I\|_{W^{\frac{1}{4},4}([0,T],L^\infty(\Omega_f))}\|Q\|_{L^4([0,T];L^2(\Gamma_c))}\\
&+C\|a-\tilde{a}\|_{L^\infty([0,T];L^4(\Omega_f))}\|q\|_{H^{\frac{1}{4}}([0,T];L^4(\Gamma_c))}\\
&+C\|a-\tilde{a}\|_{H^{\frac{1}{4}}([0,T],L^2(\Gamma_c))}\|q\|_{L^\infty([0,T];L^\infty(\Gamma_c))}.
\end{aligned}
\tag{4.3.73}
$$

For the $\|\nabla V\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}$ term, we have by the trace inequality [45, p. 41-42]

$$
\|\nabla V\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\le\|V\|_{H^{\frac{1}{4}}([0,T];H^1(\Gamma_c))}\le\|V\|_{H^{\frac{1}{4}}([0,T];H^{\frac{3}{2}}(\Omega_f))}.
$$

Next we use the space-time interpolation inequality [45, p. 47] with $X=L^2(\Omega_f)$, $Y=H^2(\Omega_f)$, $s_1=1$, $s_2=0$, and $\theta=\frac{3}{4}$, in combination with the interpolation inequality [45, p. 43] with $s_1=0$, $s_2=2$, and $\theta=\frac{3}{4}$:

$$
\|\nabla V\|_{H^{\frac{1}{4}}([0,T];H^{\frac{3}{2}}(\Omega_f))}\le C\|V\|^{\frac{1}{4}}_{H^1([0,T];L^2(\Omega_f))}\|V\|^{\frac{3}{4}}_{L^2([0,T];H^2(\Omega_f))}. \tag{4.3.74}
$$

Apply Young's inequality with $p=4$ and $q=\frac{4}{3}$ to obtain

$$
\|\nabla V\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\le\epsilon_0\|V\|_{H^1([0,T];L^2(\Omega_f))}+C_{\epsilon_0}\|V\|_{L^2([0,T];H^2(\Omega_f))},\ \epsilon_0\in(0,1]. \tag{4.3.75}
$$

The Sobolev imbedding $H^{\frac{1}{4}}\to L^4$ implies

$$
\|Q\|_{L^4([0,T];L^2(\Gamma_c))}\le C\|Q\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}. \tag{4.3.76}
$$

We have for $a^T:a-\tilde{a}^T\tilde{a}$ and $a-\tilde{a}$ that

$$
\|a-\tilde{a}\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\le C\|\nabla U\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \tag{4.3.77}
$$

and

$$
\|a^T:a-\tilde{a}^T\tilde{a}\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}\le C\|\nabla U\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}, \tag{4.3.78}
$$

for $T \le T_0$. Thus,

$$\begin{aligned}\|H\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} &\le C\epsilon\|U\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|U\|_{L^2([0,T];H^2(\Omega_f))} \\ &\quad + C\epsilon\|P\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} + C\epsilon\|V\|_{H^1([0,T];L^2(\Omega_f))} \\ &\quad + C\epsilon\|V\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|Q\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}. \quad (4.3.79)\end{aligned}$$

Optimal trace regularity [41] for the wave equation yields

$$\begin{aligned}\left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial W}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];H^2(\Gamma_c))} &\le C\left\|\int_0^t V\right\|_{L^2([0,T];H^{\frac{3}{2}}(\Gamma_c))} \\ &\quad + C\|W\|_{H^{\frac{3}{2}}([0,T];L^2(\Gamma_c))} \\ &\le CT^{\frac{1}{2}}\|V\|_{L^2([0,T];H^2(\Omega_f))} \\ &\quad + C\|V\|_{H^{\frac{1}{2}}([0,T];L^2(\Gamma_c))}, \quad (4.3.80)\end{aligned}$$

where $W = \int_0^t V \, ds$ on Γ_c was used. For the last term on the right side we use the interpolation inequality

$$\|V\|_{H^{\frac{1}{2}}([0,T];L^2(\Gamma_c))} \le \epsilon_0\|V_t\|_{L^2([0,T];L^2(\Omega_f))} + C_{\epsilon_0}\|V\|_{L^2([0,T];H^1(\Omega_f))}.$$

Thus,

$$\begin{aligned}\left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{\frac{1}{2}}(\Gamma_c))} + \left\|\frac{\partial W}{\partial N}\right\|_{H^{\frac{1}{4}}([0,T];H^2(\Gamma_c))} &\le C\|V\|_{L^2([0,T];H^2(\Omega_f))} \\ &\quad + C\|V_t\|_{L^2([0,T];L^2(\Omega_f))}. \quad (4.3.81)\end{aligned}$$

To estimate $G_t = \mathrm{div}\tilde{A}$, we write

$$\tilde{A}_k = (\tilde{a}^k_j - \delta_{kj})\partial_t V^j - (\tilde{a}^k_j - a^k_j)\partial_t v^j + \partial_t\tilde{a}^k_j V^j - (\partial_t\tilde{a}^k_j - \partial_t a^k_j)v^j.$$

Then using Cauchy-Scwarz with $p = 3$ and $q = \frac{3}{2}$ (on the second and third terms), we have

$$\|\tilde{A}(t)\|_{L^2(\Omega_f)} \le C\|a - I\|_{L^\infty(\Omega_f)}\|V_t\|_{L^2(\Omega_f)} + C\|a - \tilde{a}\|_{L^6(\Omega_f)}\|\partial_t v\|_{L^3(\Omega_f)}$$

$$+C\|\partial_t\tilde{a}\|_{L^3(\Omega_f)}\|V\|_{L^6(\Omega_f)} + C\|\partial_t(a-\tilde{a})\|_{L^2(\Omega_f)}\|v\|_{L^\infty(\Omega_f)}. \tag{4.3.82}$$

Using (4.3.67) and $\|\partial_t(a-\tilde{a})\|_{L^2([0,T];L^2(\Omega_f))} \le C\|\nabla V\|_{L^2([0,T];L^2(\Omega_f))}$, we have

$$\|\tilde{A}(t)\|_{L^\infty([0,T];L^2(\Omega_f))} \;\le\; C\epsilon\|V\|_{L^2([0,T];H^1(\Omega_f))} + C\epsilon\|V_t\|_{L^2([0,T];L^2(\Omega_f))}. \tag{4.3.83}$$

We also have (by the interior regularity of the wave equation [41])

$$\|W\|_{L^\infty([0,T];H^1(\Omega_e))} + \|W_t\|_{L^\infty([0,T];L^2(\Omega_e))} \;\le\; C\|W\|_{H^1(\Sigma_c)}. \tag{4.3.84}$$

Using the definition of the $H^1(\Sigma_c)$ norm, we have

$$\begin{aligned}
\|W\|_{H^1(\Sigma_c)} &= \left\|\int_0^t V\right\|_{L^2([0,T];H^1(\Gamma_c))} + \|W\|_{H^1([0,T];L^2(\Gamma_c))} \\
&\le CT^{\frac{1}{2}}\|V\|_{L^2([0,T];H^{\frac{3}{2}}(\Omega_f))} + \|V\|_{L^2([0,T];L^2(\Gamma_c))} \\
&\le C\|V\|_{L^2([0,T];H^2(\Omega_f))},
\end{aligned} \tag{4.3.85}$$

for $T \le T_0$. From the above estimates, we have

$$\begin{aligned}
&\|V\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{L^2([0,T];L^2(\Omega_f))} + \|Q\|_{L^2([0,T];H^1(\Omega_f))} + \|Q\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))} \\
&\quad \le C\epsilon(\|U\|_{L^2([0,T];H^2(\Omega_f))} + \|U_t\|_{L^2([0,T];L^2(\Omega_f))} + \|P\|_{L^2([0,T];H^1(\Omega_f))} \\
&\qquad + \|P\|_{H^{\frac{1}{4}}([0,T];L^2(\Gamma_c))}),
\end{aligned} \tag{4.3.86}$$

and

$$\|W\|_{L^\infty([0,T];H^1(\Omega_f))} + \|W_t\|_{L^\infty([0,T];L^2(\Omega_f))} \;\le\; C\|V\|_{L^2([0,T];H^2(\Omega_f))}. \tag{4.3.87}$$

If $\epsilon > 0$ and $\epsilon_0 > 0$ sufficiently small, we obtain

$$\|(V,Q,W)\|_{Y_0} \;\le\; \frac{1}{4}\|(U,P,\Psi)\|_{Y_0}. \tag{4.3.88}$$

Therefore, Λ is a contraction on the space Y_0, and there exists a unique fixed point solution (v,q,w) in Y_0 that belongs to $\tilde{Y}$. The solution satisfies the bound

$$\|v(t)\|_X^2 \;\leq\; C\epsilon e^{-\frac{t}{C}} + C\int_0^t e^{-\omega(t-s)}(\mathcal{F}(s)+\mathcal{F}'(s))\;ds + \mathcal{F}(t),\;\; 0\leq t\leq T_0.$$

We now continue the argument on intervals $[kT_0,(k+1)T_0]$, $k=1,2,\ldots$. Define $Y_0[T_1,T_2]$ to be the space (4.3.61) with $[0,T]$ replaced by $[T_1,T_2]$ with the norm $\|\cdot\|_{Y_0[T_1,T_2]}$ given by (4.3.62) also with $[0,T]$ replaced by $[T_1,T_2]$. When $k=0$, we have

$$\|(V,Q,W)\|_{Y_0[0,T_0]} \;\leq\; \frac{1}{4}\|(U,P,\Psi)\|_{Y_0[0,T_0]}. \tag{4.3.89}$$

Let $k\in\mathbb{N}$. Then, for $t\in[kT_0,(k+1)T_0]$, (4.3.67) is replaced by

$$\begin{aligned}\|a-\tilde{a}\|_{H^1} &\leq C\int_0^{kT_0}\|U\|_{H^2}\;ds + C\int_{kT_0}^{t}\|U\|_{H^2}\;ds\\ &\leq C_k\|U\|_{L^2([0,kT_0];H^2(\Omega_f))} + C\|U\|_{L^2([kT_0,(k+1)T_0];H^2(\Omega_f))}.\end{aligned} \tag{4.3.90}$$

Similarly, we split the integrals in the terms with the coefficient matrix a in (4.3.69) to obtain

$$\begin{aligned}\|F\|_{L^2([kT_0,(k+1)T_0];L^2(\Omega_f))} &\leq C_k\epsilon\|U\|_{L^2([0,kT_0];H^2(\Omega_f))} + C_k\epsilon\|P\|_{L^2([0,kT_0];H^1(\Omega_f))}\\ &\quad +C\epsilon\|U\|_{L^2([kT_0,(k+1)T_0];H^2(\Omega_f))}\\ &\quad +C\epsilon\|P\|_{L^2([kT_0,(k+1)T_0];H^1(\Omega_f))}\\ &\quad +\|f_f\circ\phi - f_f\circ\tilde{\phi}\|_{L^2([kT_0,(k+1)T_0];L^2(\Omega_f))}\\ &\leq C_k\epsilon\|U\|_{L^2([0,kT_0];H^2(\Omega_f))} + C_k\epsilon\|P\|_{L^2([0,kT_0];H^1(\Omega_f))}\\ &\quad +C\epsilon\|U\|_{L^2([kT_0,(k+1)T_0];H^2(\Omega_f))}\\ &\quad +C\epsilon\|P\|_{L^2([kT_0,(k+1)T_0];H^1(\Omega_f))}\\ &\quad +\|f_f\circ\phi - f_f\circ\tilde{\phi}\|_{L^2([kT_0,(k+1)T_0];L^2(\Omega_f))}.\end{aligned} \tag{4.3.91}$$

We have analogous replacements for the inequalities (4.3.69)-(4.3.85). Using these inequalities, we replace (4.3.88) with

$$\|(V,Q,W)\|_{Y_0(kT_0,(k+1)T_0)} \;\leq\; C_k\|(U,P,\Psi)\|_{Y_0(0,kT_0)} + \frac{1}{4}\|(U,P,\Psi)\|_{Y_0(kT_0,(k+1)T_0)}.$$

$$(4.3.92)$$

We thus can write

$$\|y^{n+1} - y^n\|_{Y_0(kT_0,(k+1)T_0)} \leq C_k\|y^n - y^{n-1}\|_{Y_0(0,kT_0)} + \frac{1}{4}\|y^n - y^{n-1}\|_{Y_0(kT_0,(k+1)T_0)}, \tag{4.3.93}$$

for all n. By induction, the first term on the right side of (4.3.93) converges to zero, which allows us to conclude that the iterates converge to zero on the interval $[kT_0, (k+1)T_0]$ as n goes to infinity. We also have a generalization of (4.3.55), i.e.

$$\|y^{(n)}\|_{\tilde{Y}(kT_0,(k+1)T_0)} \leq C\|y^{(n)}(kT_0)\|_{\tilde{Y}(0)} + \frac{1}{2}\|y^{(n-1)}\|_{\tilde{Y}(kT_0,(k+1)T_0)} + \|f\|_{\tilde{D}}.$$

Here, $\tilde{Y}(T_1, T_2)$ denotes the analogue of $\tilde{Y}$ on the interval $[T_1, T_2]$. Thus the iterates lie in a ball B_M in $\tilde{Y}(kT_0, (k+1)T_0)$ for large enough M, allowing us to conclude the proof. $\square$

Chapter 5

Optimal Control

Using a body force, our objective is to minimize the vorticity inside a fluid in the case of an elastic solid moving and deforming within it. In the stationary case, the drag force in the direction of the leading velocity w is represented by a distributed integral of the classical term $Du..Du = \sum_{i,j}(\partial_i u_j)^2$. Since $Du..Du = |\text{curl}(u)|^2 + D(u)..D(u)$ (for the symmetrized gradient $D(u)$), the dissipated energy will comprise the energy captured by the wake (dependent on the curl), which measures the rate of turbulence in the flow. We seek to reduce this turbulence using an external force acting upon the body of the system.

Let u represent the velocity of the flow on the time-dependent domain $\Omega^f(t)$ within the control volume Ω which is comprised of both the fluid and the solid. Denote by $\mathcal{U}_{ad}$ some set of admissible controls f acting on the system. Due to the movement of the domains within Ω, we proceed as in [18] to quantify the control as acting on all Ω.

Remark 5.0.1 (Control from the outer layer of the fluid)**.** *We can restrict the set of admissible controls to a bounded closed convex subset, comprised of functions supported on a collar of $\mathcal{D}$ with an H^1-extension by 0 into the rest of the domain. Then, at least locally in time, the control affects only the outer layer of the fluid. The same minimization analysis carries through in this context; though, of course, potentially resulting in a larger minimal attainable cost.*

The cost functional can be expressed as

$$j(f,w) := \int_0^T \int_{\Omega^f(t)} c_1 |\nabla \times u|^2 \; dx \; dt + c_2 \|f\|^2_{\mathcal{E}(0,T;\Omega)}, \tag{5.0.1}$$

with $\alpha, \beta > 0$. Denoting by $u[f]$ the flow-field corresponding to the control f, the constrained drag minimization problem can be formulated in as $\min\limits_{f \in \mathcal{U}_{ad}} j(f, u[f])$. The control norm $\mathcal{E}(0,T;\Omega)$ will be derived from the known well-posedness results for the control-to-state map.

Our goal is to prove the existence of an optimal control for the above cost functional. This result shows that the turbulence in the flow can be reduced by applying a force on the domain, potentially only on a layer of the fluid as described in Remark 5.0.1. One of the main challenges to optimal controllability is the limited well-posedness theory for fluid-structure interaction problems. The optimality proof relies on sufficient regularity and/or smallness conditions in order to take advantage of the available existence results. Furthermore, the cost functional (5.0.1) must be reformulated to accommodate the evolution of the domain using the deformation mapping.

Because $\nabla \times u$ is the image of Du via a linear map $\mathcal{L} : \mathbb{R}^9 \to \mathbb{R}^3$, we can work the following cost functional defined on v, η in the reference configuration, and the control f on the domain Ω

$$J(f, v, \eta) := \int_0^T \int_{\Omega_f} \left| \mathcal{L}\big[Dv(x) \cdot \mathrm{cof}(D\eta(x))\big]^* \right|^2 dxdt + \|f\|^2_{\mathcal{E}(0,T;\Omega)}, \tag{5.0.2}$$

where, without loss of generality, we take the positive weights $c_1, c_2 = 1$.

5.1 Quasilinear Elasticity

In the case of a fluid interacting with a solid modeled by quasilinear elasticity (as in (4.1.2)), we recall the well-posedness result of [18] given in section 4.1. Provided that the control f satisfies Assumption 4.1.1, we can conclude the existence and uniqueness of a local in time solution via Theorem 4.1.2.

Consider the Hilbert space $E_{\overline{T}}$ in (4.1.6) with norm given by

$$\|f\|^2_{E_{\overline{T}}} = \|f\|^2_{L^2(0,\overline{T};\mathbf{H}^3(\Omega))} + \sum_{n=1}^{3} \|\partial_t^n f\|^2_{L^2(0,\overline{T};\mathbf{H}^{3-n}(\Omega))}.$$

According to Theorem 4.1.2, for each $u_0 \in \mathbf{H}^6(\Omega_f) \cap \mathbf{H}^6(\Omega_e) \cap \mathbf{H}^1_0(\Omega) \cap \mathbf{L}^2_{\mathrm{div},f}$ (satisfying the necessary compatibility conditions) and $f \in E_{\overline{T}}$, there exist a time $T(u_0, f, \Omega_f) > 0$

with $T \le \overline{T}$ such that (4.1.2) has a unique solution $(v, q) \in W_T \times Z_T$.

Define the set of admissible controls in the Lagrangian formulation to be

$$Q_{\mathrm{ad}}(\overline{T}, R) := \overline{B_{E_{\overline{T}}(0,R)}} = \{f \in E_{\overline{T}} \mid \|f\|_{E_{\overline{T}}} \le R\}, \text{ for some fixed } R > 0.$$

Then for each $f \in Q_{\mathrm{ad}}(\overline{T}, R)$, (4.1.2) has unique solution on $[0, T]$ with $T = T(R, u_0) \le \overline{T}$. From a priori estimates in [18] it follows that for a fixed R, the existence time T is at worst non-decreasing as $\overline{T}$ decreases provided $T < \overline{T}$. Thus, we may fix R and choose $\overline{T}$ such that

$$T = \overline{T}, \quad \text{and henceforth abbreviate} \quad E := E_{\overline{T}}, \quad Q_{\mathrm{ad}} = Q_{\mathrm{ad}}(\overline{T}, R).$$

Let $\mathcal{A} : E \to W_T \times Z_T$ map an admissible control function to the solution of the transported (to Lagrangian coordinates) state problem (4.1.2). Then for a given initial condition u_0, the optimization problem in Lagrangian framework reads: *find $\overline{f} \in Q_{ad}$ such that for the corresponding velocity $\overline{v} = v[\overline{f}]$ and deformation $\overline{\eta} = \eta[\overline{f}]$, the functional J in* (5.0.2) satisfies

$$J(\overline{f}, \overline{v}, \overline{\eta}) = \min_{f \in Q_{\mathrm{ad}}} J(f, v[f], \eta[f]) \,. \tag{5.1.1}$$

Theorem 5.1.1. *Let $u_0 \in \boldsymbol{H}^6(\Omega_0^f) \cap \boldsymbol{H}^6(\Omega_0^s) \cap \boldsymbol{H}_0^1(\Omega) \cap \boldsymbol{L}^2_{div,f}$ satisfying the compatibility conditions as in Theorem 4.1.2. Given $R > 0$, there is $T > 0$ such that the minimization problem* (5.1.1) *has a solution, i.e. there is $\overline{f} \in Q_{ad}(T, R)$ and a solution $(\overline{v}, \overline{q}) \in W_T \times Z_T$ with the associated deformation map $\overline{\eta}$ as in* (4.1.7)*, so that the functional $f \mapsto J(f, v[f], \eta[f])$ attains its minimum on Q_{ad} at $\overline{f}$, and $(\overline{v}, \overline{q})$ is the solution of* (4.1.2) *with Lagrangian field $\overline{\eta}$ and forcing term $\overline{f}$.*

The strategy of the proof for Theorem 5.1.1 is as follows: using a (weakly convergent) minimizing sequence of controls f_n for J and the associated sequence of solutions for the coupled system associated with f_n, we reconstruct the cost functional evaluated at its minimizer. We then use Lions-Aubin compactness arguments to obtain strong convergence in the nonlinear terms in the PDE system, which implies that the weak limits of these sequences satisfy the FSI coupled system.

proof of Theorem 5.1.1. **The minimizing sequence:** Let $\{f_n\} \in Q_{\mathrm{ad}}$ be a minimizing sequence for J, and set $(v_n, q_n, \eta_n) = (v[f_n], q[f_n], \eta[f_n])$ to be the associated solution of (4.1.2) with right hand side f_n. By the coercivity of J, we know that $\{f_n\}$ is a bounded sequence in E with weak subsequential limits residing in the closed convex subset $Q_{\rm ad}$. Now we use the following estimate given in [18, p. 328]:

$$\|(v_n, q_n)\|_{W_T\times Z_T} \leq C\left[\mathcal{N}_0(u_0, (w_i)_{i=1}^3) + \mathcal{M}_0(f_n) + \mathcal{N}\big((q_j)_{j=0}^2\big)\right] \text{ for all } n, \tag{5.1.2}$$

where $\mathcal{N}_0(u_0, (w_i)_{i=1}^3)$ and $\mathcal{N}((q_i)_{i=0}^2)$ are smooth functions depending only on the norms $\|w_{3-i}\|_{\mathbf{H}^i(\Omega_e)}$, $\|w_{3-i}\|_{\mathbf{H}^i(\Omega_e)}$ and $\|q_{2-j}\|_{H^j(\Omega_f)}$, and $\mathcal{M}_0(f_n)$ is a smooth function depending on $\|f_n\|_E$. Hence

$$(v_n, q_n) \text{ are bounded in } W_T \times Z_T\,.$$

Identifying weak and weak* limits: The space W_T (4.1.3) embeds continuously into

$$V(0,T) := \{u \in L^2(0,T;\mathbf{H}_0^1(\Omega)) \mid \partial_t^3 u \in L^2(0,T;\mathbf{L}^2(\Omega))\}, \tag{5.1.3}$$

equipped with the Hilbert norm:

$$\|u\|_{V(0,T)}^2 = \|u\|^2_{L^2(0,T;\mathbf{H}_0^1(\Omega))} + \|\partial_t^3 u\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))}.$$

Set $\tilde{X}_T = X_T \cap V(0,T)$. The space $W_T \times Z_T$ continuously embeds into $\tilde{X}_T \times Y_T$, and thus we have the following convergence for a reindexed subsequence:

$$(v_n, q_n) \quad \text{converge weakly to} \quad (\overline{v}, \overline{q}) \text{ in } \tilde{X}_T \times Y_T.$$

Next we show that $\overline{v} \in W_T$ and $\overline{q} \in Z_T$. Since v_n converge weakly to v in $V(0,T)$, we have that on a subsequence

$$\partial_t^3 v_n \to \partial_t^3 \overline{v} \quad \text{weakly in} \quad L^2(0,T;\mathbf{L}^2(\Omega))\,.$$

Moreover, $\partial_t^3(v_n)$ is bounded in W_T, so on a subsequence

$$\partial_t^3(v_n) \to \tilde{v} \quad \text{weakly* in} \quad L^\infty(0,T;\mathbf{L}^2(\Omega)).$$

Therefore, for a test function $\phi \in \mathbf{C}_c^\infty((0,T)\times\Omega_f)$, we have

$$\int_0^T\int_\Omega \partial_t^3 v_n\cdot\phi \to \int_0^T\int_\Omega \partial_t^3\overline{v}\cdot\phi \text{ and } \int_0^T\int_\Omega \partial_t^3 v_n\cdot\phi \to \int_0^T\int_\Omega \tilde{v}\cdot\phi.$$

This convergence holds for any smooth compactly supported ϕ on $(0,T)\times\Omega_f$, so $\tilde{v}=\partial_t^3\overline{v}$ almost everywhere and hence

$$\partial_t^3\overline{v}\in L^\infty(0,T;\mathbf{L}^2(\Omega)), \tag{5.1.4}$$

which is the requirement for elements of W_T (4.1.3) on the entire domain Ω.

It remains to verify the rest of the conditions for W_T-membership, which amount to L^∞ regularity of time-derivatives. We have that $v_n\to\overline{v}$ weakly in X_T as well. Consequently,

$$\partial_t^k\int_0^{(\cdot)} v_n^s \to \partial_t^k\int_0^{(\cdot)}(\overline{v})^s \quad \text{weakly in} \quad L^2(0,T;\mathbf{H}^{4-k}(\Omega_e)),\quad k=0,1,2,3.$$

From v_n being in W_T we may assume

$$\partial_t^k\int_0^{(\cdot)} v_n^s \to \hat{v}^k \quad \text{weakly* in} \quad L^\infty(0,T;\mathbf{H}^{4-k}(\Omega_e)),\quad k=0,1,2,3.$$

As above, using test functions $\phi\in\mathbf{C}_c^\infty((0,T)\times\Omega_e)$, we obtain that $\partial_t^k\int_0^{(\cdot)}(\bar{v})^s=\hat{v}_k$ a.e. in $(0,T)\times\Omega_e$, for $0\le k\le 3$. In combination with (5.1.4) and $\overline{v}\in X_T$ it gives that $\overline{v}\in W_T$.

Next we focus on the pressure term q_n. Since $q_n\to\overline{q}$ weakly in Y_T then

$$\partial_{tt}q_n\to\partial_{tt}\overline{q}\quad\text{weakly in}\quad L^2(0,T;H^1(\Omega_f))\,.$$

In addition, $\{\partial_{tt}q_n\}$ is bounded in $L^\infty(0,T;\mathbf{L}^2(\Omega_f))$, hence

$$\partial_{tt}q_n\to\hat{q}\quad\text{weakly* in}\quad L^\infty(0,T;L^2(\Omega_f)).$$

The duality argument with test functions $\phi\in\mathbf{C}_c^\infty((0,T)\times\Omega_f)$ implies that $\hat{q}=\partial_{tt}\overline{q}$ almost everywhere in $(0,T)\times\Omega_f$. Consequently the weak limit $\overline{q}$ is in Z_T.

Convergence of $\{q_n\}$, $\{v_n\}$ and $\{\eta_n\}$ in various topologies: The following observations will be useful to establish convergence of the minimizing sequence to the solution of the problem (4.1.2) which also minimizes the cost functional.

For the pressure term we have $\{q_n\}$ a priori bounded in Z_T which continuously embeds into $L^2(0,T;H^3(\Omega_f))$; moreover, $\partial_t q_n$ are bounded in $L^2(0,T;H^2(\Omega_f))$. Hence Aubin's compactness result (see, e.g., [57, p. 67]) implies that for any $\epsilon > 0$ we can assume (up to the extraction of a subsequence) that $\{q_n\}$ converges strongly in $L^2(0,T;H^{3-\epsilon}(\Omega_f))$.

Likewise, with $\{\partial_t q_n\}$ bounded in $L^2(0,T;H^2(\Omega_f))$ and $\{\partial_{tt} q_n\}$ bounded in $L^2(0,T;H^1(\Omega_f))$, we can for any fixed $\epsilon > 0$ assume that $\partial_t q_n$ converge strongly in $L^2(0,T;H^{2-\epsilon}(\Omega_f))$. Consequently

$$q_n \text{ converge strongly in } H^1(0,T;H^{2-\epsilon}(\Omega_f)) \hookrightarrow C([0,T];H^{2-\epsilon}(\Omega_f))\,. \tag{5.1.5}$$

On the solid region Ω_e, the bound on v_n in W_T gives that $\{\partial_{tt}\eta_n\}$ is bounded in $L^\infty(0,T;\mathbf{H}^2(\Omega_e))$ and $\partial_{ttt}\eta_n$ are bounded in $L^\infty(0,T;\mathbf{H}^1(\Omega_e))$. Hence for any fixed $\epsilon > 0$ we can assume without loss of generality that

$$\partial_{tt}\eta_n \text{ converge strongly in } L^\infty(0,T;\mathbf{H}^{2-\epsilon}(\Omega_e)).$$

Using the definition of W_T, the same argument applied to $\partial_t\eta_n$ and $\partial_{tt}\eta_n$ gives

$$\partial_t\eta_n \text{ converge strongly in } L^\infty(0,T;\mathbf{H}^{3-\epsilon}(\Omega_e)),$$

and likewise, for η_n and $\partial_t\eta_n$, obtaining that η_n converge strongly in $L^\infty(0,T;\mathbf{H}^{4-\epsilon}(\Omega_e))$. In particular, we can assume

$$\eta_n \text{ converge strongly in } W^{1,\infty}(0,T;\mathbf{H}^{3-\epsilon}(\Omega_e)). \tag{5.1.6}$$

On the fluid region Ω_f, the fact that $\{v_n\}$ is bounded in $\tilde{X}_T \subset X_T$ implies $\partial_t^k v_n$ are bounded in $L^2(0,T;\mathbf{H}^{4-k}(\Omega_f))$, for $k = 0,1,2,3$. These bounds imply that for any fixed $\epsilon > 0$ we may assume that

$$v_n \text{ converge strongly in } L^2(0,T;\mathbf{H}^{4-\epsilon}(\Omega_f)).$$

Hence

$$\eta_n \text{ converge strongly in } H^1(0,T;\mathbf{H}^{4-\epsilon}(\Omega_f))\,.$$

From the 3D embedding $H^{\delta+3/2} \to L^\infty$, for any $\delta > 0$, we conclude that (up to a subsequence)

$$\eta_n \text{ converge strongly in } C([0,T];\mathbf{W}^{2,\infty}(\Omega))\,. \tag{5.1.7}$$

We also get

$$v_n \text{ converge strongly in } H^1(0,T;\mathbf{H}^{3-\epsilon}(\Omega_f)). \tag{5.1.8}$$

$$v_n \text{ converge strongly in } H^2(0,T;\mathbf{H}^{2-\epsilon}(\Omega_f))$$

On the entire domain Ω, the uniform bound on v_n in W_T gives that $\{v_n\}$ is bounded in $L^2(0,T;\mathbf{H}^1(\Omega))$ and $\{\partial_{tt}v_n\}$ is bounded in $L^\infty(0,T;\mathbf{L}^2(\Omega))$. Then $\{\partial_t v_n\}$ is bounded in $L^\infty(0,T;\mathbf{L}^2(\Omega))$ and therefore, for any fixed $\epsilon > 0$, we may assume

$$v_n \text{ converge strongly in } L^\infty(0,T;\mathbf{H}^{1-\epsilon}(\Omega))\,.$$

Convergence in the system (4.1.2)**:** It suffices to focus on the convergence of nonlinear terms in the interior and on the boundary.

On the fluid region Ω_f we require the terms to converge in $L^2(\Omega_f \times (0,T))$. Consider the individual summands from the corresponding equation in (4.1.2). For the terms $\partial_k[(a_n)_i^k q_n]$ it suffices to have the convergence of $(a_n)_i^k q_n$ in $L^2(0,T;H^1(\Omega_f))$. In turn, it is sufficient to have $(a_n)_i^k$ convergent in $L^\infty(0,T;W^{1,\infty}(\Omega_f))$ and q_n convergent in $L^\infty(0,T;H^1(\Omega_f))$. Recalling that each $(a_n)_i^k$ is a polynomial in first derivatives of η_n we can use convergence (5.1.7) for the first requirement and (5.1.5) for the pressure.

Next, consider $\partial_j[(a_n)_l^j \cdot (a_n)_l^k \cdot \partial_k v_n^i]$ in $L^2(0,T;L^2(\Omega_f))$. It is sufficient to have convergence of (a_n) components in $L^\infty(0,T;W^{1,\infty}(\Omega_f))$ which has already been established, and of

$$\partial_k v_n^i \text{ in } L^2\big(0,T;H^1(\Omega_f)\big)\,.$$

From (5.1.8) we have that $(\partial_k v_n^i)$ converge strongly in $H^1(0,T;H^{2-\epsilon}(\Omega_f))$ which is sufficient.

On the solid region Ω_e we focus on the terms $\partial_l\big[(\partial_m\eta_n)(\partial_j\eta_n)(\partial_k\eta_n)\big]$ in the topology $L^2(0,T;\mathbf{L}^2(\Omega_e))$. For that it certainly suffices to have convergence of $(\partial_m\eta_n)(\partial_j\eta_n)(\partial_k\eta_n)$ in $L^\infty(0,T;\mathbf{H}^1(\Omega_e))$, or that of η_n in $L^\infty(0,T;\mathbf{W}^{2,\infty}(\Omega_e))$ which holds by (5.1.7).

Interface: Next, consider the trace identity on the reference interface Γ_c. For the solutions of considered regularity, the traces of the limiting problem can be identified in $L^2(\Gamma_c \times (0,T))$.

For the $(a_n)_l^k(a_n)_l^j(\partial_k v_n^i)$ terms, the desired convergence result follows from the strong convergence of these products in $L^2(0,T;H^1(\Omega_f))$ and the continuity of the trace map

$H^1(\Omega_f) \hookrightarrow H^{1/2}(\Gamma) \subset L^2(\Gamma)$. The terms $\partial_m \eta_n \cdot \partial_j \eta_n \cdot \partial_k \eta_n^i$ likewise feature convergence in $L^\infty(0,T;\mathbf{H}^1(\Omega_e))$ which suffices for their traces to converge in $L^\infty(0,T;\mathbf{L}^2(\Omega_e))$. The analogous conclusion holds for the boundary product $(a_n)_i^j q_n$ using the interior convergence in $L^2(0,T;H^1(\Omega_f))$.

Finally, since the reference configuration has Γ_c of class C^1 (in fact, of class H^4 by assumption), then components N_j of the normal vector do not affect the convergence in $L^2(\Gamma_c \times (0,T))$.

Attaining minimal cost: Some subsequence of controls $\{f_n\}$ goes weakly to $\overline{f}$ in E (4.1.6), with the limit confined to the closed convex subset $Q_{\text{ad}} \subset E$. By the weak lower-semicontinuity of the norm

$$\|\bar{f}\|_E \leq \liminf_{n\to\infty} \|f_n\|_E^2 .$$

Furthermore, (5.1.7) and (5.1.8) together imply that subsequences $\{D\eta_n\}$ and $\{Dv_n\}$ converge strongly in $\mathbf{L}^\infty((0,T)\times\Omega_f)$. Thus,

$$\int_0^T \int_{\Omega_f} |\mathcal{L}[Dv_n(x)\cdot \operatorname{cof}(D\eta_n(x))]^*|^2 \, dxdt \to \int_0^T \int_{\Omega_f} |\mathcal{L}[D\overline{v}(x)\cdot \operatorname{cof}(D\overline{\eta}(x))]^*|^2 \, dxdt.$$

Via the definition of the cost functional (5.0.2) it follows

$$J(\overline{f},\overline{v},\overline{\eta}) \leq J(f, v[f], \eta[f]) \text{ for all } f \in Q_{\text{ad}}$$

completing the proof of Theorem 5.1.1.

$\square$

5.2 Damped Linear Wave Equation

We now focus our attention on the case where the solid is governed by a damped linear wave equation as in (3.2.4). Define the space of admissible controls $\mathcal{E}(0,T;\Omega)$ to coincide with the space of source terms satisfying the regularity, compatibility, and smallness conditions given in Theorem 4.2.1 (note that this is a closed, convex space). Then for $f \in \mathcal{E}(0,T;\Omega)$, we obtain a unique solution to the coupled system (3.2.4) associated with the control f.

Theorem 5.2.1. *Let $v_0 \in V \cap H^{\frac{7}{2}}(\Omega_f)$, $v_t(0) \in V \cap H^{\frac{5}{2}}(\Omega_f)$, $v_{tt}(0) \in V$, $w_0 \in H^{\frac{15}{4}-\delta}(\Omega_e)$, and $w_1 \in H^{\frac{11}{4}-\delta}(\Omega_e)$ for $\delta \in (0, \frac{1}{4})$ satisfying the compatibility conditions given in Theorem 4.2.1. Then there exists $\overline{f} \in \mathcal{E}(0,T;\Omega)$ and a solution $(\overline{v}, \overline{q}, \overline{w}) \in \tilde{Y}$ and associated deformation map $\overline{\eta}$ such that the functional $f \mapsto J(f, v[f], \eta[f])$, given by (5.1.1), attains its minimum on $\mathcal{E}(0,T;\Omega)$ at $\overline{f}$, and $(\overline{v}, \overline{q}, \overline{w})$ is the solution of (3.2.4) with Lagrangian field $\overline{\eta}$ and forcing term $\overline{f}$.*

Proof. **The minimizing sequence:** Let $\{f_n\} \in \mathcal{E}(0,T;\Omega)$ be a minimizing sequence for J, and set $(v_n, q_n, \eta_n) = (v[f_n], q[f_n], \eta[f_n])$ to be the associated solution of (3.2.4) with right hand side f_n. Recall from Theorem 4.2.1 that each $f_n \in \mathcal{E}(0,T,\Omega)$ lies in the ball (4.2.2). Thus, $\{f_n\}$ is bounded in $\mathcal{E}(0,T,\Omega)$ with weak subsequential limit $\overline{f}$ residing in $\mathcal{E}(0,T,\Omega)$. Furthermore, the bound (4.3.58) implies

$$\|y_n\|_{\tilde{Y}} \leq C\|y_0\| + \|f_n\|_{\mathcal{E}(0,T;\Omega)}, \tag{5.2.1}$$

implying that $\{y_n\}$ is bounded in the Hilbert space $\tilde{Y}$. Therefore,

$$\begin{aligned}
&v_n \rightharpoonup \overline{v}_1 \in L^2([0,T];H^4(\Omega_f)) \cap H^1([0,T];H^3(\Omega_f))\\
&(v_n)_t \rightharpoonup \overline{v}_2 \in H^1([0,T];H^2(\Omega_f))\\
&(v_n)_{tt} \rightharpoonup \overline{v}_3 \in H^1([0,T];L^2(\Omega_f))\\
&q_n \rightharpoonup \overline{q}_1 \in L^2([0,T];H^3(\Omega_f)) \cap H^1([0,T];H^2(\Omega_f))\\
&(q_n)_t \rightharpoonup \overline{q}_2 \in H^1([0,T];H^1(\Omega_f)).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\partial_t^j w_n \overset{*}{\rightharpoonup} \tilde{w}_j \in L^\infty([0,T];H^{\frac{15}{4}-\delta-j}(\Omega_e))\\
&\partial_t^j w_n \rightharpoonup \overline{w}_j \in L^2([0,T];H^{\frac{15}{4}-\delta-j}(\Omega_e)),
\end{aligned}$$

since $L^\infty([0,T];H^{\frac{15}{4}-\delta-j}(\Omega_e))$ embeds continuously into $L^2([0,T];H^{\frac{15}{4}-\delta-j}(\Omega_e))$. We may again identify the weak limit $\overline{w}$ and weak* limit $\tilde{w}$ using a duality argument to conclude that $\partial_t^j \overline{w} \in L^\infty([0,T];H^{\frac{15}{4}-\delta-j}(\Omega_e))$.

Convergence of $\{q_n\}$, $\{v_n\}$, $\{w_n\}$, and $\{\eta_n\}$ in various topologies: We will use the following observations to establish convergence of the minimizing sequence to the solution of the problem (3.2.4) associated with forcing term $\overline{f}$.

Recall that the pressure $\{q_n\}$ is bounded in $L^2(0,T;H^3(\Omega_f))$ and its derivative $\{\partial_t q_n\}$ is bounded in $L^2(0,T;H^2(\Omega_f))$. Aubin's compactness result (see, e.g., [57, p. 67]) therefore implies that, for any $\epsilon > 0$, we can assume (up to the extraction of a subsequence) that $\{q_n\}$ converges strongly in $L^2(0,T;H^{3-\epsilon}(\Omega_f))$.

Similarly, $\{\partial_t q_n\}$ is bounded in $L^2(0,T;H^2(\Omega_f))$ and $\{\partial_{tt} q_n\}$ is bounded in $L^2(0,T;$ $H^1(\Omega_f))$. Then for any fixed $\epsilon > 0$, we may conclude that $\partial_t q_n$ converge strongly in $L^2(0,T;H^{2-\epsilon}(\Omega_f))$. Consequently

$$q_n \text{ converge strongly in } H^1(0,T;H^{2-\epsilon}(\Omega_f)) \hookrightarrow C([0,T];H^{2-\epsilon}(\Omega_f))\,. \tag{5.2.2}$$

On the solid region Ω_e, $\{\partial_{tt} w_n\}$ is bounded in $L^\infty(0,T;\mathbf{H}^{\frac{7}{4}-\delta}(\Omega_e))$ and $\partial_{ttt} w_n$ are bounded in $L^\infty(0,T;\mathbf{H}^{\frac{3}{4}-\delta}(\Omega_e))$ for $\delta \in (0,\frac{1}{4})$. Hence for any fixed $\epsilon > 0$ we can assume without loss of generality that

$$\partial_{tt} w_n \text{ converge strongly in } L^\infty(0,T;\mathbf{H}^{\frac{7}{4}-\delta-\epsilon}(\Omega_e)).$$

The same argument applied to $\partial_t w_n$ and $\partial_{tt} w_n$ yields

$$\partial_t w_n \text{ converge strongly in } L^\infty(0,T;\mathbf{H}^{\frac{11}{4}-\delta-\epsilon}(\Omega_e)),$$

and likewise, for w_n and $\partial_t w_n$, obtaining that w_n converge strongly in $L^\infty(0,T;\mathbf{H}^{\frac{15}{4}-\delta-\epsilon}$ $(\Omega_e))$. In particular, we can assume

$$w_n \text{ converge strongly in } W^{1,\infty}(0,T;\mathbf{H}^{\frac{11}{4}-\delta-\epsilon}(\Omega_e)). \tag{5.2.3}$$

On the fluid region Ω_f, $\partial_t^k v_n$ are bounded in $L^2(0,T;\mathbf{H}^{4-k}(\Omega_f))$, for $k = 0,1,2,3$. These bounds imply that for any fixed $\epsilon > 0$ we may assume that

$$v_n \text{ converge strongly in } L^2(0,T;\mathbf{H}^{4-\epsilon}(\Omega_f)).$$

Hence

$$\eta_n \text{ converge strongly in } H^1(0,T;\mathbf{H}^{4-\epsilon}(\Omega_f))\,.$$

From the 3D embedding $H^{\delta+3/2} \to L^\infty$, for any $\delta > 0$, we conclude that (up to a subsequence)

$$\eta_n \text{ converge strongly in } C([0,T];\mathbf{W}^{2,\infty}(\Omega))\,. \tag{5.2.4}$$

We also get

$$v_n \text{ converge strongly in } H^1(0,T;\mathbf{H}^{3-\epsilon}(\Omega_f)). \tag{5.2.5}$$

$$v_n \text{ converge strongly in } H^2(0,T;\mathbf{H}^{2-\epsilon}(\Omega_f)).$$

Convergence in the system (3.2.4)**:** Again it's enough to focus on the convergence of nonlinear terms in the interior and on the boundary of the system. Note that the fluid component of (3.2.4) is identical to that of (4.1.2). Furthermore, the fluid velocity and pressure sequences and their derivatives are bounded in the same spaces of regularity given in the proof of Theorem 5.1.1. We conclude with a similar argument that $\partial_k[(a_n)^k_i q_n]$ and $\partial_j[(a_n)^j_l(a_n)^k_l\partial_k v^i_n]$ converge strongly in $L^2(0,T;L^2(\Omega_f))$.

For the source term $f_n\circ\eta_n$, we have

$$\begin{aligned}
\|f_n\circ\eta_n-\overline{f}\circ\overline{\eta}\|_{L^2(\Omega_f)} &\leq \|f_n\circ\eta_n-f_n\circ\overline{\eta}\|_{L^2(\Omega_f)}+\|f_n\circ\overline{\eta}-\overline{f}\circ\overline{\eta}\|_{L^2(\Omega_f)}\\
&\leq \|f_n\circ\eta_n-f_n\circ\overline{\eta}\|_{L^2(\Omega_f)}+\|f_n\circ\overline{\eta}-\overline{f}\circ\overline{\eta}\|_{L^2(\Omega_f)}\\
&\leq \|f_n\circ\eta_n-f_n\circ\overline{\eta}\|_{L^2(\Omega_f)}+\|(f_n-\overline{f})\circ\overline{\eta}\|_{L^2(\Omega_f)}\\
&\leq \|f_n\circ\eta_n-f_n\circ\overline{\eta}\|_{L^2(\Omega_f)}+\|(f_n-\overline{f})\circ\overline{\eta}-(f_n-\overline{f})\|_{L^2(\Omega)}\\
&\qquad +\|f_n-\overline{f}\|_{L^2(\Omega)}.
\end{aligned} \tag{5.2.6}$$

We follow the strategy used in [18, p. 329] (similar to the derivation of estimate (4.3.11)) to estimate (5.2.6), i.e.

$$\begin{aligned}
\|f_n\circ\eta_n-\overline{f}\circ\overline{\eta}\|_{L^2(\Omega_f)} &\leq C\|\eta_n-\overline{\eta}\|_{L^\infty(\Omega_f)}\|f_n\|_{H^1(\Omega)}+\|(f_n-\overline{f})\circ\overline{\eta}\|_{L^2(\Omega_f)}\\
&\leq C\|\eta_n-\overline{\eta}\|_{L^\infty(\Omega_f)}\|f_n\|_{H^1(\Omega)}+C\|\overline{\eta}-I\|_{L^\infty(\Omega_f)}\|f_n-\overline{f}\|_{H^1(\Omega)}\\
&\qquad +\|f_n-\overline{f}\|_{L^2(\Omega)},
\end{aligned} \tag{5.2.7}$$

which implies

$$\begin{aligned}
\|f_n\circ\eta_n-\overline{f}\circ\overline{\eta}\|_{L^2([0,T];L^2(\Omega_f))} &\leq C\|\eta_n-\overline{\eta}\|_{L^\infty([0,T];L^\infty(\Omega_f))}\|f_n\|_{L^2([0,T];H^1(\Omega))}\\
&\qquad +C\|\overline{\eta}-I\|_{L^\infty([0,T];L^\infty(\Omega_f))}\|f_n-\overline{f}\|_{L^2([0,T];H^1(\Omega))}\\
&\qquad +\|f_n-\overline{f}\|_{L^2([0,T];L^2(\Omega))},
\end{aligned}$$

$$
\begin{aligned}
\leq\ & C\|\eta_n - \overline{\eta}\|_{L^\infty([0,T];L^\infty(\Omega_f))}\|f_n\|_{L^2([0,T];H^1(\Omega))} \\
& +C\|\overline{\eta} - I\|_{L^\infty([0,T];H^2(\Omega_f))}\|f_n - \overline{f}\|_{L^2([0,T];H^1(\Omega))} \\
& +\|f_n - \overline{f}\|_{L^2([0,T];L^2(\Omega))},
\end{aligned} \tag{5.2.8}
$$

Recall that $\{f_n\}$ is bounded in $L^2([0,T];H^3(\Omega))$ and $\{\partial_t f_n\}$ is bounded in $L^2([0,T];H^2(\Omega))$. By Aubin's compactness result, we may conclude (for any $\epsilon > 0$) that (up to a subsequence)

$$
f_n \text{ converge strongly in } L^2([0,T];\mathbf{H}^{3-\epsilon}(\Omega)). \tag{5.2.9}
$$

Using (5.2.9) and (5.2.4) in (5.2.8) allows us to obtain the desired convergence of $f_n \circ \eta_n \to \overline{f} \circ \overline{\eta}$ in $L^2([0,T];L^2(\Omega_f))$.

Interface and Attaining Minimal Cost: The strong convergence nonlinear terms on the interface follows a similar argument as that presented in Theorem 5.1.1. Furthermore, we observe that a subsequence of controls $\{f_n\}$ goes weakly to $\overline{f}$ in $\mathcal{E}(0,T;\Omega)$. Using the lower-semicontinuity of norm, we conclude

$$
\|\bar{f}\|_{\mathcal{E}(0,T;\Omega)} \leq \liminf_{n\to\infty} \|f_n\|^2_{\mathcal{E}(0,T;\Omega)}.
$$

In addition, (5.2.4) and (5.2.5) imply that subsequences $\{D\eta_n\}$ and $\{Dv_n\}$ converge strongly in $\mathbf{L}^\infty((0,T)\times\Omega_f)$. Thus,

$$
\int_0^T \int_{\Omega_f} |\mathcal{L}[Dv_n(x)\cdot \mathrm{cof}(D\eta_n(x))]^*|^2\, dxdt \to \int_0^T \int_{\Omega_f} |\mathcal{L}[D\overline{v}(x)\cdot \mathrm{cof}(D\overline{\eta}(x))]^*|^2\, dxdt.
$$

Via the definition of the cost functional (5.0.2) it follows

$$
J(\overline{f},\overline{v},\overline{\eta}) \leq J(f, v[f], \eta[f]) \text{ for all } f \in Q_{\mathrm{ad}}
$$

completing the proof of Theorem 5.2.1.

$\square$

Chapter 6

Conclusion

6.1 Summary

In this section, we briefly summarize the conclusions of this thesis regarding the well-posedness and optimal control theory for fluid-structure interactions.

1. Local theory: Using the local well-posedness theory developed in [17, 18], we proved the existence of a body force that minimizes the turbulence inside a fluid in the case of an elastic body moving and deforming within it. Here, the elastic body is modeled by the nonlinear equations of elasticity, and the existence of the control depends upon the existence of a local solution to the model and an a-priori bound on the control-to-state map established in [18]. This result necessitates high regularity for the initial data and body forces, which were inherited from [18].

2. Global theory: The limitation of the existence and uniqueness result for an FSI given in [33], from the viewpoint of optimization or control problems, is the fact that they focus on homogeneous equations for both fluid and elastic structure. Therefore, the first step towards control is developing a well-posedness theory that accommodates sources, either distributed or located on the boundary. We proved global existence of solutions for the moving boundary interactions in the case of small distributed sources and small initial data. This is the first result on global well-posedness of solutions and optimal control for moving boundary fluid-structure interactions with distributed forces (i.e. controls). Furthermore, we demonstrated the existence of an optimal distributed control for the problem of minimizing turbulence in the fluid

flow.

6.2 Future Work

In this this, we demonstrated the existence of a distributed control in the case of a solid immersed in a fluid. This framework ensures the no-slip condition on the fixed outer boundary, which permits the use of the Poincaré inequality in our calculations. Using a generalized Poincaré, we plan to adapt this work to fit the more physically relevant cases of boundary control and of a solid surrounding the fluid flow. The latter is relevant in the study of blood flow in a stenosed or stented artery [16, 56, 58], which is of interest in the medical community. Furthermore, we worked with initially flat subdomains in order to simplify the exposition. This assumption, which is not essential, bypasses the use of commutators in the a-priori tangential estimates. We seek to generalize these results to obtain global well-posedness and existence of an optimal control in the case of a smooth domain with a general initial configuration.

The next step is to characterize the optimal control through deriving the first order necessary conditions of optimality associated with the problems, paving the way for the numerical study of the problems. In order to achieve this goal, a suitable adjoint problem must be derived and used to explicitly compute the gradient of the cost functional we seek to optimize. Due to the nonlinearity of the state equation and the moving domains, the cost function gradient cannot reduce to the derivative with respect to the control of the Lagrangian function at its saddle point. Optimality conditions must therefore be derived from differentiability arguments on the cost functional with respect to the control. The main challenge stems from the dependence of cost integrals on the unknown domain, which also depends on the control.

From the application of the chain rule, the derivative with respect to the control of the state variables will appear in the computation of the gradient of the cost functional. To describe the the derivative of the state variables, we need to take the Gâteaux derivative of the original coupled system, which was formally computed in [5, 6, 7] (in the case of quasilinear elasticity) using a pseudo-shape derivative method. The sensitivity system revealed the influence of the geometry of the common interface in the linearization through new terms that appeared on the boundary. In particular, the curvature of the boundary plays a key role in the analysis of the linearized system. We seek to use the resulting adjoint problem obtained from the linearization to derive the first order necessary optimality conditions that characterize the optimal control.

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APPENDIX

Appendix A

Auxillary Lemmas

A.1 Bounds on η and the Coefficient Matrix a

Proposition A.1.1. *Let the coefficient matrix $a = a(x,t)$, the scalar function $H = H(x,t)$, and the deformation $\eta = \eta(x,t)$ satisfy*

$$\begin{cases} a_t = -HaDva + H\,Tr(aDv)a \\ a(x,0) = I \end{cases}, \tag{A.1.1}$$

$$\begin{cases} \partial_t H = -H^2\,Tr(aDv) \\ H(x,0) = 1 \end{cases}, \tag{A.1.2}$$

$$\begin{cases} \eta_t = v \\ \eta(x,0) = x \end{cases}, \tag{A.1.3}$$

where $v = v(x,t)$ is a vector function. Assume that $\|Dv\|_{L^\infty([0,T];\ H^2(\Omega_f))}$. With $T \in [0, \frac{1}{2C^3M}]$, where C is a sufficiently large constant, the following statements hold:

(i) $\|\nabla\eta\|_{H^2(\Omega_f)} \leq C$ for $t \in [0,T]$.

(ii) $\|a\|_{H^2(\Omega_f)} + \|H\|_{H^2(\Omega_f)} \leq C$ for $t \in [0,T]$.

(iii) $\|a_t\|_{L^p(\Omega_f)} + \|H_t\|_{L^p(\Omega_f)} \leq C\|Dv\|_{L^p(\Omega_f)}$ for $t \in [0,T]$.

(iv) $\|\partial_i a_t\|_{L^2(\Omega_f)} + \|\partial_i H_t\|_{L^2(\Omega_f)} \le C\|v\|_{H^2(\Omega_f)}$ for $t \in [0,T]$.

(v) $\|\partial_{ij} a_t\|_{L^2(\Omega_f)} + \|\partial_{ij} H_t\|_{L^2(\Omega_f)} \le C\|v\|_{H^3}$ for $t \in [0,T]$ for $i,j = 1,2,3$ and $t \in [0,T]$.

(vi) $\|a_{tt}\|_{L^2(\Omega_f)} + \|H_{tt}\|_{L^2(\Omega_f)} \le C\|Dv\|_{H^2(\Omega_f)}\|v\|_{H^1(\Omega_f)} + C(\|Dv\|_{H^2(\Omega_f)} + 1)\|v_t\|_{H^1(\Omega_f)}$ for $t \in [0,T]$.

(vii) $\|\partial_i a_{tt}\|_{L^2(\Omega_f)} + \|\partial_i H_{tt}\|_{L^2(\Omega_f)} \le C(\|Dv\|_{H^2(\Omega_f)} + 1)(\|v\|_{H^2(\Omega_f)} + \|v_t\|_{H^2(\Omega_f)})$ for $t \in [0,T]$.

(viii) $\|a_{ttt}\|_{H^1(\Omega_f)} \le C\|Dv\|^2_{H^2(\Omega_f)}\|v\|_{H^1(\Omega_f)} + C\|Dv\|_{H^2(\Omega_f)}\|v_t\|_{H^1(\Omega_f)} + C(\|Dv\|_{H^2(\Omega_f)} + 1)\|v_{tt}\|_{H^2(\Omega_f)}$ for $t \in [0,T]$.

(ix) for every $\epsilon \in [0, \frac{1}{2}]$ and all $t \le T^* = \min\{\frac{\epsilon}{CM^2}, T\}$, we have

$$\|\delta_{jk} - a^j_\ell a^k_\ell\|^2_{H^2(\Omega_f)} \le \epsilon,\ j,k = 1,2,3 \tag{A.1.4}$$

and

$$\|\delta_{jk} - a^j_k\|^2_{H^2(\Omega_f)} \le \epsilon,\ j,k = 1,2,3. \tag{A.1.5}$$

In particular, the form $a^j_\ell a^k_\ell \xi^k_j \xi^i_k$ satisfies the ellipticity estimate

$$a^j_\ell a^k_\ell \xi^k_j \xi^i_k \ge \frac{1}{C}|\xi|^2,\ \xi \in \mathbb{R}^{n^2}, \tag{A.1.6}$$

for all $t \in [0,T^*]$ and $x \in \Omega_f$, provided $\epsilon < \frac{1}{C}$ with C sufficiently large.

Proof. (i) Relation (2.2.1) implies that

$$\nabla\eta = \int_0^t Dv\ ds + I$$

for $t \in [0,T]$. Estimating in $H^2(\Omega_f)$ and using $\|Dv\|_{L^\infty([0,T];\ H^2(\Omega_f))}$, we obtain

$$\|\nabla\eta\|_{H^2(\Omega_f)} \le \int_0^t \|Dv\|_{H^2(\Omega_f)}\ ds + \|I\|_{H^2(\Omega_f)}$$

$$\begin{aligned} &\leq M \int_0^T ds + \|I\|_{H^2(\Omega_f)} \\ &\leq C. \end{aligned}$$

(ii) Integrating (A.1.1) and (A.1.2) in time yield

$$a = \int_0^t -HaDva + H\mathrm{Tr}(aDv)a \; ds + I$$

and

$$H = -\int_0^t H^2 \mathrm{Tr}(aDv) \; ds + 1$$

Thus we have

$$\begin{aligned} \|a\|_{H^2(\Omega_f)} + \|H\|_{H^2(\Omega_f)} &\leq \int_0^t (\|HaDva\|_{H^2} + \|H\mathrm{Tr}(aDv)a\|_{H^2} \\ &\quad + \|H^2\mathrm{Tr}(aDv)\|_{H^2(\Omega_f)}) ds + C \\ &\leq C \int_0^t (\|H\|_{H^2(\Omega_f)} \|a\|^2_{H^2(\Omega_f)} \\ &\quad + \|H\|^2_{H^2(\Omega_f)} \|a\|_{H^2(\Omega_f)}) \|Dv\|_{H^2(\Omega_f)} \; ds + C \\ &\leq C \int_0^t (\|a\|_{H^2(\Omega_f)} + \|H\|_{H^2(\Omega_f)})^3 \|Dv\|_{H^2(\Omega_f)} \; ds + C \end{aligned} \tag{A.1.7}$$

By the Gronwall lemma [22, Theorem 25], we conclude for $T < \frac{1}{2C^3M}$ that

$$\begin{aligned} \|a\|_{H^2(\Omega_f)} + \|H\|_{H^2(\Omega_f)} &\leq \frac{C}{1 - 2C^3TM} \\ &\leq \tilde{C}, \end{aligned} \tag{A.1.8}$$

where $\tilde{C}$ is sufficiently large.

(iii) We apply Holder's inequality, Sobolev's inequality, and (ii), i.e.

$$\begin{aligned}
\|a_t\|_{L^p(\Omega_f)} + \|H_t\|_{L^p(\Omega_f)} &\leq C(\|H\|_{L^\infty(\Omega_f)}\|a\|^2_{L^\infty(\Omega_f)} \\
&\quad +\|H\|^2_{L^\infty(\Omega_f)}\|a\|_{L^\infty(\Omega_f)})\|Dv\|_{L^p(\Omega_f)} \\
&\leq C(\|H\|_{H^2(\Omega_f)}\|a\|^2_{H^2(\Omega_f)} \\
&\quad +\|H\|^2_{H^2(\Omega_f)}\|a\|_{H^2(\Omega_f)})\|Dv\|_{L^p(\Omega_f)} \\
&\leq C\|Dv\|_{L^p(\Omega_f)}.
\end{aligned}$$

(iv) We differentiate (A.1.1) and (A.1.2) with respect to the i-th spatial coordinate, i.e.

$$\begin{aligned}
\partial_i a_t &= -\partial_i H a D v a - H\partial_i a D v a - H a D(\partial_i v)a - H a D v \partial_i v a + \partial_i H \mathrm{Tr}(aDv)a \\
&\quad +H\mathrm{Tr}(\partial_i a D v)a + H\mathrm{Tr}(aD(\partial_i v))a + H\mathrm{Tr}(aDv)\partial_i a
\end{aligned}$$

and

$$\partial_t\partial_i H = -2H\partial_i H\mathrm{Tr}(aDv) - H^2\mathrm{Tr}(\partial_i a D v) - H^2\mathrm{Tr}(aD(\partial_i v)).$$

We apply Holder's inequality, Sobolev's inequality, and (ii) to obtain

$$\begin{aligned}
\|\partial_i a_t\|_{L^2(\Omega_f)} + \|\partial_i H_t\|_{L^2(\Omega_f)} &= C(\|\partial_i H\|_{L^4(\Omega_f)}\|a\|^2_{L^\infty}\|Dv\|_{L^4(\Omega_f)} \\
&\quad +\|H\|_{L^\infty(\Omega_f)}\|\partial_i a\|_{L^4(\Omega_f)}\|Dv\|_{L^4(\Omega_f)}\|a\|_{L^\infty(\Omega_f)} \\
&\quad +\|H\|_{L^\infty(\Omega_f)}\|a\|^2_{L^\infty(\Omega_f)}\|D(\partial_i v)\|_{L^2(\Omega_f)}) \\
&\quad +2\|H\|_{L^\infty(\Omega_f)}\|\partial_i H\|_{L^4(\Omega_f)}\|a\|_{L^\infty(\Omega_f)}\|Dv\|_{L^4(\Omega_f)} \\
&\quad +\|H\|^2_{L^\infty(\Omega_f)}\|\partial_i a\|_{L^4(\Omega_f)}\|Dv\|_{L^4(\Omega_f)} \\
&\quad +\|H\|^2_{L^\infty(\Omega_f)}\|a\|_{L^\infty(\Omega_f)}\|D(\partial_i v)\|_{L^2(\Omega_f)} \\
&\leq C\|v\|_{H^2(\Omega_f)}.
\end{aligned}$$

(v) Differentiate (A.1.1) and (A.1.2) once with respect to the i-th component and once with respect to the j-th component. This result then follows from a similar argument as (iv).

(vi) Differentiate (A.1.1) and (A.1.2) with respect to time. We obtain

$$\begin{aligned} a_{tt} &= -H_t aDva - Ha_t Dva - HaDv_t a - HaDva_t + H_t \mathrm{Tr}(aDv)a + H\mathrm{Tr}(a_t Dv)a \\ &\quad + H\mathrm{Tr}(aDv_t)a + H\mathrm{Tr}(aDv)a_t \end{aligned} \tag{A.1.9}$$

and

$$H_{tt} = -2HH_t \mathrm{Tr}(aDv) - 2H^2 \mathrm{Tr}(a_t Dv) - 2H^2 \mathrm{Tr}(aDv_t). \tag{A.1.10}$$

We substitute (A.1.1) and (A.1.2) into (A.1.9) and (A.1.10), respectively. We obtain

$$\begin{aligned} a_{tt} &= -2H^2 \mathrm{Tr}(aDv)aDva + 2H^2 aDvaDva - HaDv_t a \\ &\quad + H^2 \mathrm{Tr}(aDv)^2 a - H^2 \mathrm{Tr}(aDvaDv)a + H\mathrm{Tr}(aDv_t)a \end{aligned} \tag{A.1.11}$$

and

$$H_{tt} = H^3 \mathrm{Tr}(aDv)^2 + H^3 \mathrm{Tr}(aDvaDv) - 2H^2 \mathrm{Tr}(aDv_t). \tag{A.1.12}$$

Finally, we estimate using Holder's inequality, Sobolev's inequality, and (ii) to obtain

$$\begin{aligned} \|a_{tt}\|_{L^2(\Omega_f)} + \|H_{tt}\|_{L^2(\Omega_f)} &= C\|Dv\|_{H^2(\Omega_f)}\|v\|_{H^1(\Omega_f)} \\ &\quad + C(\|Dv\|_{H^2(\Omega_f)} + 1)\|v_t\|_{H^1(\Omega_f)}. \end{aligned}$$

(vii) Differentiate (A.1.11) and (A.1.12) with respect to the i-th component. The result follows from a similar argument as (vi).

(viii) Differentiate (A.1.11) and (A.1.12) with respect to time. The result follows from a similar argument as (vi)-(vii).

(ix) The proof is identical to that of Lemma 3.2.2 given in [31].

□

A.2 Stoke's and Elliptic Type Estimates

Proposition A.2.1. *Assume v and q are solutions to the system*

$$\begin{cases} v_t^i - \partial_j(a_\ell^j a_\ell^k \partial_k v^i) + \partial_k(a_i^k q) = f_f^i \circ \eta, & \Omega_f \times (0,T), \\ a_i^k \partial_k v^i = 0, & \Omega_f, \\ \partial_j w^i N_j = a_\ell^j a_\ell^k \partial_k v^i N_j - a_i^k q N_k, & \Gamma_c, \\ v^i = 0, & \Gamma_f, \end{cases} \tag{A.2.1}$$

for given coefficients a_j^i with $i, j = 1, 2, 3$ satisfying (3.3.6)-(3.3.7) *for sufficiently small $\epsilon > 0$. Then the estimate*

$$\|v\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)} \leq C\|v_t\|_{H^s(\Omega_f)} + C\|f_f \circ \eta\|_{H^s(\Omega_f)} + \left\|\frac{\partial w}{\partial N}\right\|_{H^{s+\frac{1}{2}}(\Gamma_c)} \tag{A.2.2}$$

holds for $s = 0, 1$ and for all $t \in (0,T)$. Moreover, the time derivatives v_t and q_t satisfy

$$\begin{aligned} \|v_t\|_{H^2(\Omega_f)} + \|q_t\|_{H^1(\Omega_f)} \leq\ & C\|v_{tt}\|_{L^2(\Omega_f)} + C\|f_f \circ \eta\|_{L^2(\Omega_f)} + C\|\partial_t(f_f \circ \eta)\|_{L^2(\Omega_f)} \\ & + \left\|\frac{\partial w}{\partial N}\right\|_{H^{\frac{1}{2}}(\Gamma_c)} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{\frac{1}{2}}(\Gamma_c)}. \end{aligned} \tag{A.2.3}$$

Proof. Let ϕ be the solution to the elliptic problem

$$\Delta\phi = -(\delta_{jk} - a_j^k)\partial_k v^j \text{ in } \Omega_f$$

with Dirichlet data $\phi = 0$ on $\Gamma_c \cup \Gamma_f$. Then the function $u = v + \nabla\phi$ satisfies the stationary Stokes problem

$$\begin{cases} -\Delta u^i + \partial_i q = -\Delta\partial_i\phi - \partial_j((\delta_{jk} - a_\ell^j a_\ell^k)\partial_k v^i) + \partial_k((\delta_{ik} - a_i^k)q) - v_t^i + f_f^i \circ \eta, & \Omega_f, \\ \partial_j u^j = 0, & \Omega_f, \\ u = \nabla\phi, & \Gamma_f, \\ \partial_j u^i N_j - q N_i = \partial_j w^i N_j + \partial_{ij}\phi N_j + (\delta_{jk} - a_\ell^j a_\ell^k)\partial_k v^i N_j - (\delta_{ik} - a_i^k) q N_k, & \Gamma_c. \end{cases}$$

$$(A.2.4)$$

Thus we have (see [31], for instance)

$$\begin{aligned}
&\|u\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)} \\
&\quad \le C\|\Delta\nabla\phi\|_{H^s(\Omega_f)} + C\|\partial_j((\delta_{jk} - a^j_\ell a^k_\ell)\partial_k v^i)\|_{H^s(\Omega_f)} + C\sum_{i=1}^3 \|(\delta_{ik} - a^k_i)q\|_{H^s(\Omega_f)} \\
&\quad\quad + C\|v_t\|_{H^s(\Omega_f)} + C\|f_f \circ \eta\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+\frac12}(\Gamma_c)} + C\left\|\frac{\partial(\nabla\phi)}{\partial N}\right\|_{H^{s+\frac12}(\Gamma_c)} \\
&\quad\quad + C\|(\delta_{jk} - a^j_\ell a^k_\ell)\partial_k v N_j\|_{H^{s+\frac12}(\Gamma_c)} + C\sum_{i=1}^3 \|(\delta_{ik} - a^k_i)qN_k\|_{H^{s+\frac12}(\Gamma_c)} \\
&\quad\quad + C\|\nabla\phi\|_{H^{s+\frac32}(\Gamma_f)}. \qquad (A.2.5)
\end{aligned}$$

By the trace theorem, $\|\nabla\Delta\phi\|_{H^s} \le C\|(\delta_{jk} - a^k_j)\partial_k v^j\|_{H^{s+1}(\Omega_f)}$, the multiplicative Sobolev inequalities, and (3.3.7), we have

$$\begin{aligned}
&\|v\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)} \\
&\quad \le C\|(\delta_{jk} - a^k_j)\partial_k v^j\|_{H^{s+1}(\Omega_f)} + C\sum_{i=1}^3 \|(\delta_{jk} - a^j_\ell a^k_\ell)\partial_k v^i\|_{H^{s+1}(\Omega_f)} \\
&\quad\quad + C\sum_{i,k=1}^3 \|(\delta_{ik} - a^k_i)q\|_{H^{s+1}(\Omega_f)} + C\|v_t\|_{H^s(\Omega_f)} + C\|f_f \circ \eta\|_{H^s(\Omega_f)} \\
&\quad\quad + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+\frac12}(\Gamma_c)} \\
&\quad \le C\|\delta_{jk} - a^k_j\|_{H^2}\|\partial_k v^j\|_{H^{s+1}(\Omega_f)} + C\sum_{i=1}^3 \|\delta_{jk} - a^j_\ell a^k_\ell\|_{H^2}\|\partial_k v^i\|_{H^{s+1}(\Omega_f)} \\
&\quad\quad + C\sum_{i,k=1}^3 \|\delta_{ik} - a^k_i\|_{H^2}\|q\|_{H^{s+1}(\Omega_f)} + C\|v_t\|_{H^s(\Omega_f)} + C\|f_f \circ \eta\|_{H^s(\Omega_f)} \\
&\quad\quad + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+\frac12}(\Gamma_c)} \\
&\quad \le C\epsilon\|v\|_{H^{s+2}(\Omega_f)} + C\epsilon\|q\|_{H^{s+1}(\Omega_f)} + C\|v_t\|_{H^s(\Omega_f)} + C\|f_f \circ \eta\|_{H^s(\Omega_f)} \\
&\quad\quad + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+\frac12}(\Gamma_c)}. \qquad (A.2.6)
\end{aligned}$$

Absorbing terms on the left-hand side, we obtain the desired result

$$\|v\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)} \leq C\|v_t\|_{H^s(\Omega_f)} + C\|f_f \circ \eta\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+\frac{1}{2}}(\Gamma_c)}. \tag{A.2.7}$$

For (A.2.3), we differentiate (A.2.4) with respect to time and repeat the analysis above (making use of (A.2.2)) to obtain the desired result. □

Note that the normal derivative terms in (A.2.2) and (A.2.3) can be bounded using the trace theorem, i.e.

$$\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+\frac{1}{2}}(\Gamma_c)} \leq \|w\|_{H^{s+2}(\Omega_e)} \tag{A.2.8}$$

and

$$\left\|\frac{\partial w_t}{\partial N}\right\|_{H^{\frac{1}{2}}(\Gamma_c)} \leq \|w_t\|_{H^2(\Omega_e)}. \tag{A.2.9}$$

To obtain bounds on the right side of (A.2.8)-(A.2.9), we apply the elliptic regularity result of [3] to the damped wave equation and its time derivative, i.e.

$$\|w\|_{H^{s+2}(\Omega_e)} \leq C\|w_{tt}\|_{H^s(\Omega_e)} + C\|w_t\|_{H^s(\Omega_e)} + \|(D')^2 w\|_{H^s(\Omega_f)} + C\|f_e\|_{H^s(\Omega_f)} \tag{A.2.10}$$

and

$$\|w_t\|_{H^2(\Omega_e)} \leq C\|w_{ttt}\|_{L^2(\Omega_e)} + C\|w_{tt}\|_{L^2(\Omega_e)} + C\|D' w_t\|_{H^1(\Omega_f)} + C\|\partial_t f_e\|_{L^2(\Omega_f)}. \tag{A.2.11}$$

Here, $D'w = (\partial_1 w, \partial_2 w)$ denotes the tangential derivative of w and $(D')^2$ is the matrix whose ij-entry is $\partial_{ij} w$ for $i, j = 1, 2$.

Applying (A.2.10)-(A.2.11) to (A.2.2)-(A.2.3) yields

$$\begin{aligned}\|v\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)} \leq\ & C\|v_t\|_{H^s(\Omega_f)} + C\|w_{tt}\|_{H^s(\Omega_e)} + C\|w_t\|_{H^s(\Omega_e)} \\ & + \|(D')^2 w\|_{H^s(\Omega_f)} + C\|f_f \circ \eta\|_{H^s(\Omega_f)} + C\|f_e\|_{H^s(\Omega_f)}\end{aligned}$$

$$(A.2.12)$$

and

$$\begin{aligned}\|v_t\|_{H^2(\Omega_f)} + \|q_t\|_{H^1(\Omega_f)} &\leq C\|v_{tt}\|_{L^2(\Omega_f)} + C\|w_{tt}\|_{L^2(\Omega_e)} + C\|w_t\|_{L^2(\Omega_e)} \\ &\quad + C\|(D')^2 w\|_{L^2(\Omega_f)} + C\|w_{ttt}\|_{L^2(\Omega_e)} + C\|w_{tt}\|_{L^2(\Omega_e)} \\ &\quad + \|D' w_t\|_{H^1(\Omega_f)} + C\|f_f \circ \eta\|_{L^2(\Omega_f)} + C\|f_e\|_{L^2(\Omega_f)} \\ &\quad + C\|\partial_t (f_f \circ \eta)\|_{L^2(\Omega_f)} + C\|\partial_t f_e\|_{L^2(\Omega_f)}. \qquad (A.2.13)\end{aligned}$$

Estimates (A.2.2)-(A.2.3) combined with the definition of the $X(t)$ norm given by (3.3.1) yield the following estimates in the case where the coefficient matrix a is close to identity:

$$\|v\|^2_{H^{s+2}(\Omega_f)} + \|q\|^2_{H^{s+1}(\Omega_f)} \leq CX(t) + C\|f_f \circ \eta\|^2_{H^s(\Omega_f)} + C\|f_e\|^2_{H^s(\Omega_f)}, \ s = 0, 1 \qquad (A.2.14)$$

and

$$\begin{aligned}\|v_t\|^2_{H^2(\Omega_f)} + \|q_t\|^2_{H^1(\Omega_f)} &\leq CX(t) + C\|f_f \circ \eta\|^2_{L^2(\Omega_f)} + C\|f_e\|^2_{L^2(\Omega_f)} \\ &\quad + C\|\partial_t (f_f \circ \eta)\|^2_{L^2(\Omega_f)} + C\|\partial_t f_e\|^2_{L^2(\Omega_f)}. \qquad (A.2.15)\end{aligned}$$

A.3 Barrier Argument with Cubic Equation

Proposition A.3.1. *Let $C > 0$ and $\epsilon > 0$, and let $z(t) \geq 0$ such such that*

$$z(t) \leq C\epsilon z(t)^3 + C\epsilon, \ z(0) = 0.$$

Then there exists $\tilde{\epsilon} > 0$ small such that $z(t) \leq \tilde{\epsilon}$.

Proof. We have

$$z^3 - \frac{1}{C\epsilon} z + 1 \geq 0.$$

Choose ϵ sufficiently small such that

$$\frac{(1)^2}{4} - \frac{1}{27(C\epsilon)^3} < 0$$

Then the equation $z^3 - \frac{1}{C\epsilon}z + 1$ admits 3 real roots given by

$$\begin{aligned}
z_0 &= \frac{2}{\sqrt{3C\epsilon}} \cos\left[\frac{1}{3}\arccos\left(\frac{-(3C\epsilon)^{\frac{3}{2}}}{2}\right)\right] \\
z_1 &= \frac{2}{\sqrt{3C\epsilon}} \cos\left[\frac{1}{3}\arccos\left(\frac{-(3C\epsilon)^{\frac{3}{2}}}{2}\right) - \frac{2\pi}{3}\right] \\
z_2 &= \frac{2}{\sqrt{3C\epsilon}} \cos\left[\frac{1}{3}\arccos\left(\frac{-(3C\epsilon)^{\frac{3}{2}}}{2}\right) - \frac{4\pi}{3}\right].
\end{aligned}$$

We use the identity $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$ and the Taylor expansion of $\arcsin(x)$ to obtain

$$\begin{aligned}
\arccos(x) &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2(2n+1)} x^{2n+1} \\
&= \frac{\pi}{2} - x - \sum_{n=1}^{\infty} \frac{(2n)!}{4^n(n!)^2(2n+1)} x^{2n+1}
\end{aligned}$$

Consider the tail of the Taylor series $\sum\limits_{n=1}^{\infty} \frac{(2n)!}{4^n(n!)^2(2n+1)} x^{2n+1}$. We can bound this term using the Taylor Remainder theorem, that is if $T_1(x)$ is the first order Taylor polynomial for $\arcsin(x)$, we have

$$\begin{aligned}
\left| \sum_{n=1}^{\infty} \frac{(2n)!}{4^n(n!)^2(2n+1)} x^{2n+1} \right| &= |\arcsin(x) - T_1(x)| \\
&\leq \left| \frac{M}{2!} x^2 \right| \qquad \text{(A.3.1)}
\end{aligned}$$

provided $|\frac{d}{dx}(arcsin(x))| = |\frac{1}{\sqrt{1-x^2}}| \leq M$ for x close to zero. Let $\epsilon_1 > 0$ be small. Then for $|x| < \sqrt{\frac{\sqrt{1-\epsilon_1}-1}{\sqrt{1+\epsilon_1}}}$, we have that

$$\left| \frac{1}{\sqrt{1-x^2}} - 1 \right| < \epsilon_1 \implies \frac{1}{\sqrt{1-x^2}} < \epsilon_1 + 1.$$

Furthermore, $|x| < \sqrt{\frac{\sqrt{1-\epsilon_1}-1}{\sqrt{1+\epsilon_1}}} < \sqrt{\epsilon_1}$ implies that $x^2 < \epsilon_1$. Using these facts in (A.3.1), we have

$$\left| \sum_{n=1}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \right| \leq \frac{1}{2}\epsilon_1(\epsilon_1 + 1). \qquad \text{(A.3.2)}$$

Thus if we let $\epsilon_1 > 0$ small and choose ϵ small enough such that

$$\left| \frac{(3C\epsilon)^{\frac{3}{2}}}{2} \right| < \sqrt{\frac{\sqrt{1-\epsilon_1}-1}{\sqrt{1+\epsilon_1}}},$$

we can write

$$\arccos\left(\frac{-(3C\epsilon)^{\frac{3}{2}}}{2} \right) \sim \frac{\pi}{2} + \frac{(3C\epsilon)^{\frac{3}{2}}}{2} + \frac{1}{2}\epsilon_1(\epsilon_1 + 1), \qquad \text{(A.3.3)}$$

which implies

$$\frac{1}{3}\arccos\left(\frac{-(3C\epsilon)^{\frac{3}{2}}}{2} \right) \sim \frac{\pi}{6} + \frac{(3C\epsilon)^{\frac{3}{2}}}{6} + \frac{1}{6}\epsilon_1(\epsilon_1 + 1), \qquad \text{(A.3.4)}$$

Then

$$z_i \sim \frac{2}{\sqrt{3C\epsilon}} \cos\left[\frac{\pi}{6} + \frac{(3C\epsilon)^{\frac{3}{2}}}{6} + \frac{1}{6}\epsilon_1(\epsilon_1 + 1) - \frac{2\pi i}{3} \right], \; i = 0, 1, 2. \qquad \text{(A.3.5)}$$

For ϵ sufficiently small, we have

$$z_2 < 0 < z_1 < z_0.$$

To satisfy the conditions $z(t) \geq 0$ and $z(0) = 0$, we conclude that

$$0 \leq z(t) \leq z_1.$$

Furthermore,

$$z_1 \sim \frac{2}{\sqrt{3C\epsilon}} \cos\left[\frac{\pi}{6} + \frac{(3C\epsilon)^{\frac{3}{2}}}{6} + \frac{1}{6}\epsilon_1(\epsilon_1 + 1) - \frac{2\pi}{3} \right]$$

$$
\begin{aligned}
&\sim \quad \frac{2}{\sqrt{3C\epsilon}} \cos\left[\frac{(3C\epsilon)^{\frac{3}{2}}}{6} + \frac{1}{6}\epsilon_1(\epsilon_1+1) - \frac{\pi}{2}\right] \\
&\sim \quad \frac{2}{\sqrt{3C\epsilon}} \cos\left[\frac{\pi}{2} - \left(\frac{(3C\epsilon)^{\frac{3}{2}}}{6} + \frac{1}{6}\epsilon_1(\epsilon_1+1)\right)\right] \\
&\sim \quad \frac{2}{\sqrt{3C\epsilon}} \sin\left[\frac{(3C\epsilon)^{\frac{3}{2}}}{6} + \frac{1}{6}\epsilon_1(\epsilon_1+1)\right].
\end{aligned}
$$

Using the small angle approximation for sin, we conclude that 5

$$
z(t) \quad \sim \quad \frac{(3C\epsilon)^{\frac{3}{2}}}{6} + \frac{1}{6}\epsilon_1(\epsilon_1+1) \sim \tilde{\epsilon}, \tag{A.3.6}
$$

where $\tilde{\epsilon} = \max\left\{\frac{(3C\epsilon)^{\frac{3}{2}}}{6},\ \frac{1}{6}\epsilon_1(\epsilon_1+1)\right\}$. □