ABSTRACT

YANG, YUE. Neutrino Flavor Transformations in Supernovae: the Role of Turbulence, Gravity, and Nonstandard Neutrino Self-interactions. (Under the direction of Dr. James Kneller.)

In recent years neutrino physics has been booming with numerous experimental and theoretical achievements. It has been increasingly realized that neutrinos are one of the most important keys to many unsolved mysteries in both particle physics and astrophysics and could be the gateway to the next major breakthrough for physics beyond the Standard Model. Among many natural and artificial neutrino sources, dying stars are one of the most intriguing and important due to their extreme intensities and the connection to some basic questions of the universe like the origin of chemical elements. In February 1987, detectors around the world collected 25 neutrinos which are believed to come from the supernova 1987A in the Large Magellanic Cloud, marking the beginning of neutrino astronomy. Since then a lot of studies have looked at supernova neutrinos and revealed how neutrinos play a major role in the dynamics of supernova explosion and the following nucleosynthesis. The picture that has emerged is somewhat complicated because of the complex interactions of neutrinos with the surrounding matter and more annoyingly, with themselves. One important question many want to answer is how the flavor composition of the neutrinos generated in the dense core of a supernova changes as the neutrinos propagate through the mantle of the star. This dissertation addresses this problem in three different cases, which are: neutrino flavor transformation in the presence of (a) a strong turbulence, (b) a strong gravitational field, and (c) a nonstandard neutrino self-interaction. First we study the change to neutrino flavor transformations due to turbulence found inside supernovae. We develop a general formalism using the “Rotating Wave Approximation” to analytically predict the evolution of a N-flavor neutrino under a arbitrary Fourier-decomposed potential, and apply it to a 3-flavor neutrino propagating through turbulence. The results show a good agreement with numerical calculations and an effect similar to “induced transparency” in quantum optics has been discovered. As an extension of this study, we also apply the analytical approach to deal with the neutrino self-interaction problem, and the preliminary results show a good match with numeric calculations. Then we study the influence of strong gravity on neutrino flavor transformations. We treat neutrinos as massless particles and point out 3 major general relativistic effects that should be considered: energy redshift, time dilation, and trajectory bending. We find the overall effect is a significant enhancement of the neutrino self-interaction potential. When we do the flavor transformation calculations using density profiles and neutrino spectra appropriate for the accretion phase and cooling phase of supernova explosion we find that while the GR effect is limited in the former case it could be more significant in the latter case and it may also impact the nucleosynthesis. Finally, we investigate the scenario of a nonstandard neutrino-neutrino interaction via a scalar/pseudoscalar field and how it would change the picture of neutrino
flavor transformations. It is shown that if (and only if) neutrinos are Majorana fermions there will be a nonstandard effective self-interaction potential in addition to the standard V-A potential that can give rise to a variety of new effects, notably a shut-down of the ordinary neutrino collective oscillation caused by the standard V-A interactions between neutrinos. This can provide a new way of probing interactions beyond the standard model as well as the nature of neutrinos.
Neutrino Flavor Transformations in Supernovae: the Role of Turbulence, Gravity, and Nonstandard Neutrino Self-interactions

by
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DEDICATION

To my beloved parents Ping Yang and Shuping Gu,
who have always been supportive of my endeavor to understand the mysteries of the universe.

And to Keyu Xiang,
who makes my Ph.D. life colorful and unforgettable.
BIOGRAPHY

The author was born in Beijing, the capital of the People's Republic of China. Yue has been receiving quality education in the middle/high school attached Tsinghua University until being accepted by the famous Fudan University located in Shanghai. Yue began with studying Applied Mechanics for his first year of college and then shifted to Physics major at the second year. Yue obtained his Bachelor of Science in physics in 2010. After that Yue entered Prof. Changqin Wu's group and began his scientific research in condensed matter physics as a graduate student at Fudan University. He graduated with Master of Science in theoretical physics in 2013.

Yue continued his study of physics at North Carolina State University from the fall of 2013. After passing the qualifier exam he joined Prof. Keith Weninger's group for a short period of time, and then joined Prof. James Kneller's group, which is focused on neutrino astrophysics. The projects Yue has been working on at NCSU are related to neutrino flavor transformations in supernova environments, which feature turbulences, strong gravity, and possible nonstandard neutrino interactions.
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I would like to thank my advisor Dr. James Kneller for his support and help, without which it is impossible for me to get this far. I also would like to thank Dr. Keith Weninger, who was very understanding and supportive about my decision to change direction and let me devote myself to what I am really interested in.
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1.1 Short history of neutrino physics

1.1.1 Discovery of the neutrino

Like many other profound scientific advances in the 20th century, the discovery of neutrino is an fascinating story by no means less appealing than the mysterious particles themselves. The concept of the “neutrino” traces back to the Pauli’s proposal in 1930 that an invisible particle generated in neutron beta decay is responsible for the apparent violation of energy conservation. At first this particle was called the “neutron” but shortly after Pauli’s conjecture, Chadwick discovered another neutral particle inside the nucleus which he too named the “neutron”. To distinguish the two terms, Fermi coined the name “neutrino” for Pauli’s mysterious particle - “neutrino” means “little neutral one” in Italian - because Chadwick’s neutron had a mass similar to the proton and Pauli’s “neutron” was much lighter. However, even after Pauli’s proposal for a new particle had gained greater acceptance, the general opinion was that it would be impossible to design experiments that were able to detect these elusive particles.

In 1942 G.Wang proposed using beta capture as the avenue to detect neutrinos, which was later employed by Reines and Cowan in their famous 1956 experiment which confirmed the existence of neutrino. The Cowan-Reines neutrino experiment used the capture of an antineutrino by proton

\[ \bar{\nu}_e + p \rightarrow n + e^+ \]  

(1.1)
as the primary way of detection. The released antielectrons will annihilate with ambient electrons, producing a $\gamma$ ray of 511keV. The neutron will be captured by another nearby nucleus, producing another $\gamma$ ray after a short period of time. This signature coincidence is the key to identifying antineutrino capture events, and is still used in modern experiments such as KamLAND. The neutrino species discovered in 1956 is now known as the electron antineutrino. Back then the standard model of particle physics was yet to take its current form and it was not realized there are more than one generation or 'family' of leptons. The existence of distinct flavors of neutrino was shown by Lederman, Schwartz and Steinberger in 1962 who found that the neutrinos produced in association with the muon when a pion decays were distinct from the neutrinos produced in association with the electron because the muon-associated neutrinos can only go on to produce muons in the reaction
\[ \nu_\mu + n \rightarrow p + \mu^-, \] (1.2)
or anti-muons in the reaction
\[ \bar{\nu}_\mu + p \rightarrow n + \mu^+. \] (1.3)
After the discovery of the second generation of matter particles, the multi-generation structure of the model of particle physics became more clear. So when the third generation lepton $\tau$ was discovered in a series of experiments between 1974 and 1977, the existence of its associated neutrino was immediately indicated. Being one of the most elusive particles, it was not until 2000 that the DONUT experiment finally obtained the direct evidence of tau neutrino events, making it the second latest discovered elementary particle next to the Higgs boson.

1.1.2 Early theories about neutrino oscillations

The concept of neutrino oscillation came into being as early as 1957, when Pontecorvo postulated the neutrino might be able to oscillate into its own anti-particle, if these two were distinguishable [1]. After the discovery of muon neutrino in 1962, it was suggested that neutrino oscillations may not be happening between neutrinos and antineutrinos, but rather between different types of neutrinos. In their 1962 seminal work Maki, Nakagawa, and Sakata proposed the 2-component model of the neutrino in which the “true neutrino” is related to the “weak neutrino” through a mixing matrix [2]
\[ \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos \delta & \sin \delta \\ -\sin \delta & \cos \delta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \] (1.4)
This idea was further developed by Pontecorvo in the 1968 paper, followed by another paper in 1969 where the idea of neutrino oscillations was used for the first time to explain the newly discovered solar neutrino deficit [3, 4].
1.1.3 Solar neutrino problem

The sun is a major natural source of neutrinos that can be used for measuring key neutrino properties. Solar neutrinos are a by-product of the energy-producing process in the core of the sun called the “pp-chain”. The majority of solar neutrinos are produced in the following reaction

\[ p + p \rightarrow ^2H + e^+ + \nu_e \quad (86.6\% \text{ of the total flux}) \] (1.5)

with a maximum energy less than 0.5MeV. There are also some reaction branches of the “pp-chain” that produce higher energy neutrinos with much smaller flux such as

\[ ^8B \rightarrow ^7Be^* + e^+ + \nu_e \quad (0.017\% \text{ of the total flux}) \] (1.6)

which were very important for early experiments since they often have high energy thresholds.

In the 1960s, Ray Davis began the first solar neutrino experiment with a large chlorine-based detector in Homestake, planning to compare his detection with the prediction of standard solar model [5, 6]. The reaction used for the neutrino detection was

\[ \nu_e + ^{37}\text{Cl} \rightarrow ^{37}\text{Ar} + e^- . \] (1.7)

Soon after the experiment began Davis realized there was an apparent deviation of the number of events from the prediction of the standard solar model [7, 8]. After 25 years of data accumulation the final result shows a factor of one third between the observed events and theoretical prediction [9]. The result was astonishing and it was no surprise that the initial response of the scientific community was skeptical. Independent verifications of Davis’ discovery was carried out by the Kamiokande-II project (water Cherenkov) in 1985, GALLEX/GNO project (radiochemical) in 1991 and the SNO project (heavy water/water Cherenkov) in 1999 respectively. While many solutions for the deficit were put forward, the suggestion that the neutrinos were changing flavor became the preferred solution because all the other solutions involved changes to the solar model which were incompatible with helioseismology. The most thorough and convincing evidence of the oscillation of solar neutrinos came from the SNO experiment, which is capable of measuring both the flux of electron neutrinos and the flux of muon/tau neutrinos. In the SNO experiment, three types of interaction were employed which are: the charged current reaction

\[ \nu_e + ^2\text{H} \rightarrow p + p + e^- , \] (1.8)

the neutral current reaction

\[ \nu_\alpha + ^2\text{H} \rightarrow p + n + \nu_\alpha , \] (1.9)
and the elastic scattering reaction
\[ \nu_\alpha + e^- \rightarrow \nu_\alpha + e^- . \] (1.10)

Theoretically the relationship between the neutrino fluxes measured from the three interactions are
\[ \Phi_{CC} = \Phi_{\nu_e}, \]
\[ \Phi_{NC} = \Phi_{\nu_e} + \Phi_{\nu_x}, \]
\[ \Phi_{ES} = \Phi_{\nu_e} + 0.1553 \Phi_{\nu_x} \] (1.11)

where \( \nu_x \) represents the combination of mu and tau neutrinos. The first two equations can be used for obtaining the fluxes of \( \nu_e \) and \( \nu_x \), while the third equation can be used as a verification of the result. The final measured ratio of electron neutrino flux to the total neutrino flux is
\[ \frac{\Phi_{\nu_e}}{\Phi_{\nu_e} + \Phi_{\nu_x}} = 0.34 \pm 0.023, \] (1.12)

which is in good agreement with the Homestake result. Even though the detection of non-zero flux of mu and tau neutrinos from the Sun are a good evidence of neutrino oscillation, the results of the experiments cannot be explained by simple theory of neutrino vacuum oscillation. As we are going to discuss in section 1.3, the matter effect is the key to correctly interpret the quantitative results of solar neutrino experiments.

**1.1.4 Atmospheric neutrino anomaly**

Cosmic rays are another important natural neutrino source besides the Sun. The collisions between the particles in cosmic rays and the molecules in the Earth’s atmosphere produce pions, whose decay generates copious neutrinos of both electron type and muon type through
\[ \pi^+ \rightarrow \mu^+ + \nu_\mu \]
\[ \pi^- \rightarrow \mu^- + \bar{\nu}_\mu \] (1.13)

and
\[ \mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu \]
\[ \mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \] (1.14)

Due to the fact that the high energy cosmic rays are barely influenced by the geomagnetic field, the high energy neutrinos they produce are uniformly distributed around the globe. It can be shown that at any given location on earth, the atmospheric neutrino fluxes of each flavor coming from opposite directions should be equal in the absence of flavor transformation [10]. Any asymmetry of neutrino fluxes in opposite directions is therefore an indication of neutrino flavor transformation.
For each neutrino flavor $\alpha$, we can define the up/down ratio as

$$R_{\alpha}^{\text{up-down}} = \frac{\Phi_{\alpha}^{\text{up}}}{\Phi_{\alpha}^{\text{down}}},$$

(1.15)

or equivalently we can define the asymmetry

$$A_{\alpha}^{\text{up-down}} = \frac{\Phi_{\alpha}^{\text{up}} - \Phi_{\alpha}^{\text{down}}}{\Phi_{\alpha}^{\text{up}} + \Phi_{\alpha}^{\text{down}}}.$$  

(1.16)

Since neutrino detectors are all located near the surface of the earth, the neutrinos coming from below a detector must have traveled a long distance inside the earth to reach the detector. Therefore, if there is any flavor transformation for neutrinos of flavor $\alpha$, the up-down asymmetry $A_{\alpha}^{\text{up-down}}$ should be negative. In 1998 the Super-Kamiokande collaboration presented the result

$$A_{\mu}^{\text{up-down}} = -0.296 \pm 0.048,$$

(1.18)

which clearly indicates a flavor transformation in the muon neutrino sector [11]. The remaining problem was to determine the oscillation channel. There are two options, which are $\nu_{\mu} \leftrightarrow \nu_{e}$ and $\nu_{\mu} \leftrightarrow \nu_{\tau}$. However, if the first one is true, we should expect asymmetries in both $\nu_{\mu}$ flux and $\nu_{e}$ flux, which contradicts the observations. Therefore the $\nu_{\mu} \leftrightarrow \nu_{e}$ channel is disfavored [12]. If the $\nu_{\mu} \leftrightarrow \nu_{\tau}$ channel is assumed, the flavor transformation can be calculated by the formula of vacuum oscillation since there is no matter effect between $\mu$ and $\tau$ flavors. Combined with the measured asymmetry for $\nu_{\mu}$ flux, the SK was able to determine the allowed parameter region for $|\Delta m_{32}^{2}|$ and $\sin^{2}(2\theta_{23})$ [12].

### 1.2 Vacuum oscillation

In this section we review the theory of neutrino oscillation in vacuum. It is well-known that the charged current of the weak interaction mixes quarks from different generations through the CKM matrix. In the lepton sector though, the situation is different. Because neutrinos do not have mass in the standard model, it is more convenient to define the flavor states of the neutrino so that the weak charged current term is diagonal. This makes a lot of sense since the existence of neutrinos has been primarily felt through their weak interactions with other particles. The limitation of the convention was revealed only after the discovery of neutrino oscillation which strongly indicated neutrinos are massive, though their masses are likely to be incredibly small compared with other leptons. The mismatch between the mass eigenstates and flavor eigenstates of neutrinos can be
incorporated into a transformation as

\[ |\nu_k\rangle = \sum_a U_{ak} |\nu_a\rangle, \]  

(1.19)

or in terms of state vector we have

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}
= 
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
\]  

(1.20)

where \( k = 1, 2, 3 \) is the mass index, \( \alpha = e, \mu, \tau \) is the flavor index, and the transformation matrix \( U \) is called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. Similar to the CKM matrix, the PMNS matrix can be parameterized as

\[
U = U_{23} U_{13} U_{12}
\]

\[
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13} e^{-i\delta_{\text{CP}}} \\
0 & 1 & 0 \\
-s_{13} e^{i\delta_{\text{CP}}} & 0 & c_{13}
\end{pmatrix}
\begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 0
\end{pmatrix},
\]  

(1.21)

where \( c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij} \), and the \( \theta_{ij} \)'s are the mixing angles. If neutrinos were Majorana fermions, there would be two additional Majorana phases applied to the \( U \) matrix. In vacuum the equation of motion for the mass eigenstates is

\[
i \frac{d}{dt} |\nu_k\rangle = E_k |\nu_k\rangle,
\]  

(1.22)

where the energy is

\[ E_k = \sqrt{p^2 + m_k^2}. \]  

(1.23)

Here we have assumed every neutrino produced through weak interaction has a well-defined momentum. The solution is

\[ |\nu_k(t)\rangle = |\nu_k\rangle e^{-iE_k t}. \]  

(1.24)

Thus the time dependence of the flavor eigenstates can be obtained as

\[ |\nu_\alpha(t)\rangle = \sum_k U_{ak}^* |\nu_k(t)\rangle = \sum_k U_{ak}^* |\nu_k\rangle e^{-iE_k t}. \]  

(1.25)
Now suppose we have a neutrino with flavor $\alpha$ at $t = 0$, we want to know the probability that it is detected to be in flavor $\beta$ at time $t$. The transition probability can be expressed as

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j} e^{-i(E_k - E_j)t}. \quad (1.26)$$

Since neutrino masses are supposed to be tiny, we can safely expand the energy as

$$E_k \approx E + \frac{m_k^2}{2E}, \quad (1.27)$$

in which we used $c \equiv 1$ and $E \equiv p$. Then we obtain

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j} e^{-\frac{\Delta m^2_{kj}}{4E}t}. \quad (1.28)$$

Sometimes it is also useful to define the neutrino survival probability as

$$P_{\nu_\alpha \rightarrow \nu_\alpha}(t) = 1 - 4\sum_{k>j} |U_{\alpha k}|^2 |U_{\alpha j}|^2 \sin^2\left(\frac{\Delta m^2_{kj}}{4E}t\right). \quad (1.29)$$

These formulae for vacuum oscillations are very useful for the determination of neutrino squared mass differences and the mixing angles in terrestrial neutrino experiments. The current best-fit values of the mixing angles and the squared-mass differences are [13]: $\theta_{12} = 34.5^\circ$, $\theta_{23} = 41.0^\circ$, $\theta_{13} = 8.44^\circ$ and $\Delta m^2_{21} = 7.56 \times 10^{-5} \text{eV}^2$, $|\Delta m^2_{31}| = 2.55 \times 10^{-3} \text{eV}^2$ (normal hierarchy), $|\Delta m^2_{31}| = 2.49 \times 10^{-3} \text{eV}^2$ (inverted hierarchy). Even though experiments have been able to measure most oscillation parameters to good precision, they have yet to tell the order of the three masses. This is the so-called “mass hierarchy” problem. If $m_3 > m_1$ it is called “normal hierarchy” and otherwise it is called “inverted hierarchy”. The schematic of the two possible mass hierarchies is shown in figure 1.1. In addition, the CP-violating phase $\delta_{CP}$ is also unknown.

### 1.3 Matter effect

Neutrinos interact with ordinary matter through the charged current and neutral current weak interactions. When propagating through dense matter, neutrinos may undergo both coherent and incoherent scattering which can alter their quantum states. The full calculation, taking all types of interactions into account, is a difficult task due to the need to solve the quantum kinetic equations. However, due to the weakness of neutrino interactions, incoherent scattering is a rare occurrence in any environment with a mass density lower than $\sim 10^{12} \text{g/cm}^3$, such as the interior of the sun. In these environments, the neutrino flavor transformations are only due to coherent forward scattering.
from the matter particles, which keeps the momentum of the neutrino unchanged while altering their quantum phases. As we shall see, under some conditions this will lead to significant flavor transitions that cannot be predicted by vacuum oscillations alone. The first person to recognize the role of matter in neutrino flavor evolution was Wolfenstein. In his seminal paper in 1978, Wolfenstein demonstrated the presence of matter can alter the survival probability of solar neutrinos, while the influence on neutrinos passing through rocks inside the earth is negligible \[14\]. But unfortunately, the matter potential used in this paper, along with another paper in the early studies, has the wrong sign. The first correct treatment appeared in the paper \[15\]. Later in 1985 Mikheyev and Smirnov further pointed out that a slow decrease of the matter density can resonantly enhance the neutrino flavor transformation. Combining these works the neutrino matter effect was later named as the Mikheyev-Smirnov-Wolfenstein effect (MSW effect), and was employed to finally give a satisfactory explanation of the long-standing solar neutrino problem \[16, 17\].

As the first step of deriving the matter effect, we write down the Hamiltonian densities of the charged current and neutral current interactions between neutrino and matter made up of electrons, protons and neutrons. The charged current term is (Fierz transformed)

\[
\mathcal{H}_{\text{CC}}^{\text{eff}} = \frac{1}{\sqrt{2}} G_F \left[ \bar{\nu}_e \gamma^\rho \left( 1 - \gamma_5 \right) \nu_e \right] \left[ \bar{e} \gamma_\rho \left( 1 - \gamma_5 \right) e \right].
\]  

(1.30)
while the neutral current term is

\[
\mathcal{H}_{\text{eff}}^{\text{NC}} = \frac{1}{\sqrt{2}} G_F \sum_{a=e,\mu,\tau} \left[ \bar{\nu}_a \gamma^\rho (1 - \gamma_5) \nu_a \right] \sum_{f=e,p,n} \left[ f \gamma_\rho \left( g_V^f - g_A^f \gamma_5 \right) f \right].
\] (1.31)

where \( G_F \) is the Fermi constant, and \( g_V^f \) and \( g_A^f \) are the vector and pseudo-vector coupling constants for the corresponding fermion \( f \). Here we notice that the contributions from the neutral current interaction to all neutrino flavors are equal. This leads to three equal diagonal elements in the effective single particle Hamiltonian that only gives rise to a common phase factor. Thus, we are free to ignore the contribution from the neutral current term. By averaging over the single neutrino quantum state in the whole space, we end up with an effective Hamiltonian in the flavor basis as [10]

\[
H_{\text{matter}}^{(f)} = \sqrt{2} G_F N_e \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\] (1.32)

where \( N_e \) is the electron density in the environment.

Now let us look at the flavor evolution when taking the additional matter potential into account in the evolution equation. The total Hamiltonian is

\[
H^{(f)} = H^{(f)}_{\text{vac}} + H_{\text{matter}}^{(f)} = \frac{1}{2E} M^2 U^\dagger + H_{\text{matter}}^{(f)},
\] (1.33)

where

\[
M^2 = \begin{pmatrix} m_1^2 \\ m_2^2 \\ m_3^2 \end{pmatrix},
\] (1.34)

and \( U \) is the vacuum mixing matrix relating the flavor basis and the mass basis. In a 2-flavor model, the Hamiltonian can be simplified into the following form

\[
H^{(f)} = \begin{pmatrix} -\frac{\Delta m^2}{4E} \cos 2\theta_V + \frac{1}{\sqrt{2}} G_F N_e & \frac{\Delta m^2}{4E} \sin 2\theta_V \\ -\frac{1}{\sqrt{2}} G_F N_e & \frac{\Delta m^2}{4E} \cos 2\theta_V - \frac{1}{\sqrt{2}} G_F N_e \end{pmatrix},
\] (1.35)

where \( \Delta m^2 \) is the chosen squared-mass difference and \( \theta_V \) is the corresponding vacuum mixing angle. To solve the evolution equation, it is convenient to transform into a basis where the Hamiltonian is diagonal. This can be done by solving the eigenvalue equation and the resulting effective mixing matrix is

\[
U_M = \begin{pmatrix} \cos \theta_M & \sin \theta_M \\ -\sin \theta_M & \cos \theta_M \end{pmatrix},
\] (1.36)
where

\[ \theta_M = \frac{1}{2} \tan^{-1} \left( \frac{\Delta m^2 \sin 2\theta_V}{\Delta m^2 \cos 2\theta_V - 2\sqrt{2} G_F N_e E} \right) \]  

(1.37)

and the diagonalized matrix is

\[ K^{(m)} \equiv U_M^\dagger H(f) U_M = \begin{pmatrix} -\Delta m^2_M & 0 \\ 0 & \Delta m^2_M \end{pmatrix} \]  

(1.38)

with

\[ \Delta m^2_M = \sqrt{\left( \Delta m^2 \cos 2\theta - 2\sqrt{2} G_F N_e E \right)^2 + (\Delta m^2 \sin 2\theta_V)^2}. \]  

(1.39)

With these quantities defined, we can rewrite the evolution equation in the “matter basis” which is related to the flavor basis through \( U_M \) as

\[ \psi(f) \equiv \begin{pmatrix} \psi_e \\ \psi_\mu \end{pmatrix} = U_M \phi^{(m)} \equiv U_M \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \]  

(1.40)

After substituting this into the evolution equation in the flavor basis and replacing the time variable \( t \) with the displacement variable \( x = c t \equiv t \), we find

\[ i \frac{d \phi^{(m)}}{d x} = \left( K^{(m)} - i U_M^\dagger \frac{d U_M}{d x} \right) \phi^{(m)}, \]  

(1.41)

or explicitly

\[ i \frac{d}{d x} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{4E} \begin{pmatrix} -\Delta m^2_M & -4i E d \theta_M / d x \\ 4i E d \theta_M / d x & \Delta m^2_M \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \]  

(1.42)

The solution is simple when the matter density is independent of position, which indicates \( d \theta_M / d t = 0 \). This is equivalent to the vacuum oscillation with \( \theta \) and \( \Delta m^2 \) replaced by the effective values \( \theta_M \) and \( \Delta m^2_M \). So the transition probability can be easily calculated as

\[ P_{\nu_e \rightarrow \nu_\mu} = \sin^2 2\theta_M \sin^2 \left( \frac{\Delta m^2_M x}{4E} \right). \]  

(1.43)

It can be seen there exists a particular electron density

\[ N_e^R = \frac{\Delta m^2 \cos 2\theta_V}{2\sqrt{2} G_F E} \]  

(1.44)

that makes \( \theta_M = 45^\circ \). From equation (1.43) we can see when \( \theta_M = 45^\circ \) the transition probability reaches its maximum value \( P_{\nu_e \rightarrow \nu_\mu} = 1 \), this is called the “MSW resonance” and it is an example of matter enhanced neutrino flavor transformation.
However, if neutrinos are passing through matter with a varying density, the calculation of the survival (or transition) probability becomes more difficult. Taking the sun as an example, electron neutrinos are produced in the center of the sun and then they propagate through the interior of the sun before escaping from the surface into vacuum. To compute the neutrino survival probability at the surface of the sun, we can numerically solve equation (1.42), but in order to do that we need the details about the matter density profile inside the sun. But there is another way to solve the equation if we assume the length scale for the change of matter density inside the sun is much longer than the length scale for the neutrino oscillations. In this case, the off-diagonal terms on the right-hand side of equation (1.42) satisfies
\[
\left| \frac{d\theta_M}{dx} \right| \ll \frac{\Delta m^2_M}{4E}, \tag{1.45}
\]
and therefore the neutrino evolution can be deemed as “adiabatic”. This allows us to assume the transition between the two eigenstates in the matter basis can be neglected and the initial state is related with the final state through

\[
\begin{pmatrix}
\phi_1(x) \\
\phi_2(x)
\end{pmatrix} = \begin{pmatrix}
\cos \theta_M & -\sin \theta_M \\
\sin \theta_M & \cos \theta_M
\end{pmatrix} \begin{pmatrix}
\phi_1(0) \\
\phi_2(0)
\end{pmatrix}. \tag{1.46}
\]

In the solar neutrino case, \(x = 0\) corresponds to the center of the sun, and \(x = x\) corresponds to the surface of the sun. Initially the neutrino is in a pure electron flavor state, so the initial state in the matter basis is

\[
\begin{pmatrix}
\phi_1(0) \\
\phi_2(0)
\end{pmatrix} = \begin{pmatrix}
\cos \theta_M^{(i)} & -\sin \theta_M^{(i)} \\
\sin \theta_M^{(i)} & \cos \theta_M^{(i)}
\end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix}
\cos \theta_M^{(i)} \\
\sin \theta_M^{(i)}
\end{pmatrix}, \tag{1.47}
\]

and therefore we have

\[
\begin{pmatrix}
\phi_1(x) \\
\phi_2(x)
\end{pmatrix} = \begin{pmatrix}
\cos \theta_M^{(i)} e^{i \int_0^x \frac{\Delta m^2_M(x') dx'}{4E}} \\
\sin \theta_M^{(i)} e^{-i \int_0^x \frac{\Delta m^2_M(x') dx'}{4E}}
\end{pmatrix}. \tag{1.48}
\]

At the surface of the sun the matter density drops to zero, so the final state in flavor basis is related to the final state in matter basis through

\[
\begin{pmatrix}
\psi_e(x) \\
\psi_\mu(x)
\end{pmatrix} = \begin{pmatrix}
\cos \theta_V & \sin \theta_V \\
-\sin \theta_V & \cos \theta_V
\end{pmatrix} \begin{pmatrix}
\phi_1(x) \\
\phi_2(x)
\end{pmatrix}. \tag{1.49}
\]

When we insert 1.48 into 1.49 and calculate the survival probability for electron neutrinos we obtain

\[
P_{\nu_e \to \nu_e} = |\psi_e(x)|^2 = \frac{1}{2} \left[ 1 + \cos 2\theta_M^{(i)} \cos 2\theta_V + \sin 2\theta_M^{(i)} \sin 2\theta_V \cos \left( \int_0^x \frac{\Delta m^2_M(x')}{2E} dx' \right) \right]. \tag{1.50}
\]

Note that since the core of the sun is much larger than the oscillation length for the neutrinos,
neutrinos generated from different parts of the core will have a range of different phases in the cosine function, so that on average this term is zero. If we consider high energy neutrinos such as those from the $^8\text{B}$ reaction, the density in the core is much higher than the MSW resonance density. Thus, the initial effective mixing angle $\theta_{M}^{(i)}$ is very close to $\pi/2$, and we can simplify equation 1.50 further into
\[ \langle P_{\nu_e \rightarrow \nu_e} \rangle = \frac{1}{2} (1 - \cos 2\theta_V). \] (1.51)

Using the vacuum mixing angle $\theta_{12}$ for $\theta_V$, we obtain $\langle P_{\nu_e \rightarrow \nu_e} \rangle = 0.32$, which is in good agreement with the deficit of electron flavor neutrinos at the high energy region of the solar neutrino spectrum, as measured by Davis in the Homestake experiment [10]. On the other hand, if we consider low energy neutrinos like those from the $pp$ reaction, then the density in the core is much lower than the MSW resonance density. In this case the initial effective mixing angle $\theta_{M}^{(i)}$ is very close to the vacuum mixing angle, and equation 1.50 can be simplified into
\[ \langle P_{\nu_e \rightarrow \nu_e} \rangle = \frac{1}{2} (1 + \cos^2 2\theta_V). \] (1.52)

When we insert $\theta_{12}$ for the value of $\theta_V$ we obtain $\langle P_{\nu_e \rightarrow \nu_e} \rangle = 0.56$, which is in good agreement with the deficit of electron flavor neutrinos at the low energy region of the solar neutrino spectrum, as those measured by GALLEX/GNO and SAGE [10].

In addition to the MSW effect, there are also other types of matter effect that can cause large flavor transformations even though the matter density does not fall in the region of MSW resonance. In chapter 2 we will introduce the “stimulated neutrino flavor transformation”, in which a neutrino propagating through a fluctuating density profile undergoes flavor transformation when the frequency of one or several Fourier modes in the Fourier-decomposed density match one of the splits between the neutrino’s eigenstates in the matter basis.

1.4 Supernova neutrinos

On February 23,1987, a supernova was observed in the Large Magellanic Cloud, by both optical and neutrino signals, for the first time. About three hours before the light of the supernova reached Earth, Kamiokande-II, IMB, and Baksan, the three neutrino detectors around the world which were operating at that time, received a burst of neutrino events. In total, 25 neutrino signals were recorded within a period of about 13 seconds, in agreement with the typical time scale of a core-collapse supernova (CCSN) [18, 19, 20]. Even though the small statistics mean the uncertainties of these neutrino signals are large, they provided us with the first-hand data that not only qualitatively confirmed the dynamical model of core-collapse supernova, but also allow us to put on important limits on some neutrino properties, such as neutrino masses, neutrino mixing and neutrino magnetic
moment.

Core-collapse supernovae are the endpoint of stellar evolution for stars with mass larger than \(8 M_\odot\). Once the conditions in the core of the star allow for photodisintegration of the iron nuclei and electron capture by free protons, the catastrophic loss of both thermal and electron degeneracy pressure means the star begins to collapse toward its gravitational center, releasing tremendous amount of gravitational energy. This energy is emitted in the form of neutrino emission (~99%), kinetic energy of the envelope (~1%) and electromagnetic radiation (~0.01%). In the remainder of this section we will describe the dynamics of CCSN and the neutrino output.

1.4.1 Supernova dynamics

As shown in figure 1.2, the process of exploding a star via core-collapse has several stages. They can be summarized as follows:\(^1\):

(a) For a star with a mass larger than \(\sim 10 M_\odot\), the temperature in the core is so high that it can burn nuclei through the periodic table until iron and nickel begin to form. This leads to a onion-like structure with progressively heavier atomic nuclei build up in each layer.

(b) As an inner core of iron and nickel is formed, the energy output comes to a halt, and the only force to prevent the core from collapse is the degeneracy pressure of electrons. Due to the high temperature of the core, processes such as photo-dissociation of iron nuclei and electron capture by protons begins to occur. Catastrophic collapse occurs as the electron degeneracy pressure is reduced and there is nothing left to support the core. As a result, the core will contract from a radius of \(\sim 1000\) km to \(\sim 20\) km within a period of \(\sim 200\) milliseconds.

(c) The contraction will cease when the density of core surpasses the density of nuclear matter, which is about \(10^{14}\) g/cm\(^3\). At this density the repulsion between nucleons can withstand the gravitational force and prevent the core from further contraction. At this point the core becomes incompressible and a protoneutron star is formed.

(d) Although a protoneutron star has formed in the inner core, the matter around it is unaware of the abrupt termination of the contraction and continues falling inwards. Thus the matter of the outer core hits and rebounds from the roto-neutrons star and creates a outgoing shock. As nuclei pass through the shock they are photo-dissociated into free nucleons. The electrons will be captured by protons behind the shock front and produce a huge quantity of electron neutrinos in a period of tens of milliseconds, releasing about \(10^{51}\) erg of energy. This short burst of neutrinos is called the **neutronization burst**, and it is very important for neutrino observations due to its sensitivity to neutrino mass hierarchy.

(e) The energy of the shock is dissipated as it photo-dissociates the nuclei and emits neutrinos along the way. At \(t \sim 100\) ms the shock reaches \(r \sim 200\) km and stalls. At this point the system will

---

\(^1\)Here we only describe the typical iron-core core-collapse supernova with \(M \sim 10 M_\odot\) and larger, not the O-Mg-Ne supernova which typically have \(M \sim 8 - 10 M_\odot\).
Figure 1.2 The process of core-collapse supernova explosion. Adopted from the Wikipedia. (a) Onion-like structure formed. (b) Core-collapse begins. (c) Protoneutron star formed. (d) Outer core rebounds from the protoneutron star, forming an outward shock. (e) The shock stalls. (f) The shock is revived by neutrino heating.

Enter a state of instabilities, and in multi-dimensional simulations one may observe significant asphericity in the form of convection and the standing accretion shock instability (SASI) [21]. As matter from the outer layers of the star continues to fall into the shocked region, the protoneutron star keeps growing, so this stage is also called the “accretion phase”.

(f) To give rise to a successful supernova explosion, the stalled shock must gain energy from other sources and get revived. The mechanisms capable of reviving the shock is an important field of study, and the heating by the thermal neutrinos emitted from the core is widely considered to be the leading contender among the possible candidates [22]. After the revival of the shock, the core will continue to contract as the emission of thermal neutrinos effectively cools it. This stage of supernova explosion is thus called the “cooling phase”.

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1.4.2 Neutrino emission

In this part we discuss the features of neutrino emission during the supernova explosion. It is important to note that much of the neutrino emission data has been inferred from simulations which do not include flavor transformations. However, neutrino flavor transformations play important roles in both supernova dynamics and the prediction of observations. So a further discussion of how the supernova environment can distort the neutrino flavor contents will be given. The focus will be on the matter effect on supernova neutrinos, and the collective neutrino transformation effect will be discussed in the next section.

As mentioned, during the process of core collapse, the gravitational energy released is of the order of $\sim 3 \times 10^{53}$ erg, the majority of which is carried away by the neutrinos, and the rest of which is converted to kinetic energy and electromagnetic radiation resulting in the optical supernova we can see directly. The reactions that are responsible for the production of neutrinos include:

1. **The electron/positron captures**
   
   $$e^- + p \rightarrow n + \nu_e$$  \hspace{1cm} (1.53)
   $$e^+ + n \rightarrow p + \bar{\nu}_e.$$ \hspace{1cm} (1.54)

2. **The electron-nucleon and nucleon-nucleon bremsstrahlung**
   
   $$e^\pm + N \rightarrow e^\pm + N + \nu_\alpha + \bar{\nu}_\alpha$$ \hspace{1cm} (1.55)
   $$N + N \rightarrow N + N + \nu_\alpha + \bar{\nu}_\alpha.$$ \hspace{1cm} (1.56)

3. **The plasmon decay and photoannihilation**
   
   $$\gamma \rightarrow \nu_\alpha + \bar{\nu}_\alpha$$ \hspace{1cm} (1.57)
   $$\gamma + e^\pm \rightarrow e^\pm + \nu_\alpha + \bar{\nu}_\alpha.$$ \hspace{1cm} (1.58)

4. **The electron/positron pair annihilation**
   
   $$e^- + e^+ \rightarrow \nu_\alpha + \bar{\nu}_\alpha.$$ \hspace{1cm} (1.59)

While the reactions in (1) are charged current processes that only produce electron flavor neutrinos and antineutrinos, the reactions in (2)-(4) are neutral current processes that equally produce all flavors of neutrinos with thermal spectra.

Processes (1)-(5) contribute to the neutrino fluxes in different phases of the supernova explosion. In figure 1.3 the neutrino luminosities, fluxes and mean energies for all flavors are shown as a function of time after bounce $t_{\text{ph}}$. In the first few milliseconds after the formation of the shock, a
large amount of electron neutrinos are produced via electron captures by free nucleons in the mantle of the core, resulting in the neutronization burst that corresponds to the spike in of luminosity/flux in the figure. Even though the instantaneous flux can be as large as \( \sim 2 \times 10^{58} / s \), the duration is too short to carry away a significant portion of the total lepton number of the core. Most electron neutrinos produced in this stage are still trapped in the core region where the density is higher than \( \sim 10^{11} \) g/cm³. After the neutronization burst, the supernova enters the accretion phase and the matter around the core continues to fall into the shock and heat the medium behind it. The fluxes of electron antineutrinos and muon and tau neutrino/antineutrinos will grow rapidly and approach the flux of electron neutrinos, since the medium behind the shock is so hot that neutrinos of all flavors are produced equally and thermally via the reactions in (2)-(5), and the density is low enough for the neutrinos to escape. In addition to the thermal processes, electron neutrino and antineutrinos are also produced via the reactions in (1) because both positron and electron exist copiously in the hot medium. Thus, the fluxes of electron flavor neutrino and antineutrino will be larger than heavy lepton neutrinos during the accretion phase. At the same time, neutrino captures by the neutrons and protons in the supernova medium are also happening frequently:

\[
\begin{align*}
    n + \nu_e & \rightarrow e^- + p \\
    p + \bar{\nu}_e & \rightarrow e^+ + n,
\end{align*}
\]  

These processes are the major avenue by which the energy and lepton number are transferred from the neutrino fluxes to the supernova medium, which plays a critical role in the revival of stalled shock. Furthermore, as we can see from this pair of reactions the relative strength of the fluxes of neutrino and antineutrino will have a significant influence on the equilibrium ratio of neutrons to protons which is a very important factor for the calculation of nucleosynthesis. The reactions in 1.60 also explain the hierarchy of mean energies for neutrinos of different flavors. Since the mantle of protoneutron star is neutron-rich, electron neutrinos are more likely to be captured by the medium, so they have a larger opacity than electron antineutrinos and therefore a larger neutrinosphere. The heavy lepton neutrinos \( \nu_\mu \) and \( \nu_\tau \) do not interact with nucleons via charged current process so they have even lower opacities, and therefore have the smallest neutrinospheres. The neutrinos coming from a deeper layer of PNS are hotter so they have a larger mean energy. Thus we expect the hierarchy of neutrino mean energies is

\[
\langle E_{\nu_e} \rangle < \langle E_{\bar{\nu}_e} \rangle < \langle E_{\nu_\mu} \rangle, \quad \langle E_{\nu_\tau} \rangle,
\]  

which is in agreement with figure 1.3. Finally, after \( t_{pb} \sim 0.35s \), the shock has been revived and the accretion phase ends. At this phase neutrinos are produced through the cooling of the protoneutron star, which means all flavors have the same luminosity, just as shown in the figure 1.3.
Figure 1.3 The neutrino luminosity, number flux, and mean energy, as well as the matter density profiles of 3 different time slices for the Basel 10.8$M_\odot$ simulation. The two black dashed lines in the density plot indicate the region where collective oscillations usually happen. Data is adopted from [23].
1.5 Neutrino self-interaction and collective flavor transformations

During violent astronomical processes such as supernova explosions and neutron star mergers, the neutrino emission is so intense that the interactions between neutrinos begin to play a significant role in their flavor evolutions. A typical core-collapse supernova explosion releases $\sim 10^{53}$ erg energy through neutrinos within 10 seconds, which have mean energy of 10 MeV, that is equivalent to $\sim 10^{57}$ neutrino/s. In the region a few tens of kilometers above the protoneutron star surface, the neutrino density is comparable to that of other leptons (mainly electrons), so the neutrino self-interaction Hamiltonian is in the same order of magnitude as the matter potential. This means we need to include a self-interaction contribution to the Hamiltonian when calculating neutrino flavor evolution.

Even though the weak interaction between neutrinos is similar to that between neutrino and other fermions, they are different in that neutrino-neutrino interactions contain only a neutral current component and it has both diagonal and off-diagonal contributions in the mean field approximation. The off-diagonal terms were often missing in early works on neutrino self-interaction \cite{24, 25}. The study by Pantaleone in 1992 was the first to correctly derive the neutrino forward coherent scattering self-interaction Hamiltonian \cite{26, 27}. After the correct form of neutrino self-interaction was derived, it was first applied to the study of neutrino flavor transformations in the early universe, where the concept of “neutrino collective oscillation” was first proposed. \cite{28, 29, 30}. It was later found that this kind of neutrino flavor transformation can also be significant in the supernova environments and it potentially has large influence on the electron fraction in the neutrino driven wind \cite{31}. This is going to be the major topic in the section of “the bulb model”, and a comprehensive review can be found in \cite{32}.

1.5.1 The neutrino self-interaction potential

Unlike the interaction between neutrino and other fermions, neutrinos only interact with other neutrinos through the neutral current channel. The interaction Hamiltonian can be written as

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = \frac{1}{\sqrt{2}} G_F \left( \frac{1}{2} \sum_{\alpha = e, \mu, \tau} \bar{\nu}_\alpha \gamma^\lambda (1 - \gamma_5) \nu_\alpha \right) \left( \frac{1}{2} \sum_{\beta = e, \mu, \tau} \bar{\nu}_\beta \gamma^\lambda (1 - \gamma_5) \nu_\beta \right)$$

$$= \frac{1}{\sqrt{2}} G_F \left( \sum_{\alpha = e, \mu, \tau} \bar{\nu}_{\alpha L} \gamma^\lambda \nu_{\alpha L} \right) \left( \sum_{\beta = e, \mu, \tau} \bar{\nu}_{\beta L} \gamma^\lambda \nu_{\beta L} \right)$$

In order to derive the effective single-particle Hamiltonian, the mean field approximation can be used. The first step is to transform the 4-fermion operator into an effective 2-fermion operator using
the “Hartree-Fock” procedure [33]. The result is

\[ \mathcal{H}_{\text{int}} = \frac{1}{\sqrt{2}} G_F \sum_{a, \beta = e, \mu, \tau} \tilde{\nu}_{\alpha L} \gamma^\lambda v_{\alpha L} \bar{\nu}_{\beta L} \gamma^\lambda v_{\beta L} \]

\[ \approx \frac{1}{\sqrt{2}} G_F \sum_{a, \beta = e, \mu, \tau} \left[ \left( \bar{\nu}_{\alpha L} \gamma^\lambda v_{\alpha L} \right) \bar{\nu}_{\beta L} \gamma^\lambda v_{\beta L} + \left( \bar{\nu}_{\beta L} \gamma^\lambda v_{\beta L} \right) \bar{\nu}_{\alpha L} \gamma^\lambda v_{\alpha L} + 2 \left( \bar{\nu}_{\alpha L} \gamma^\lambda v_{\beta L} \right) \bar{\nu}_{\beta L} \gamma^\lambda v_{\alpha L} \right], \]

where the last 2 numeric terms in the summation were dropped because they only give rise to an overall phase factor in the neutrino evolution.

Now let us be more specific and consider the interaction between a test neutrino/antineutrino and a background of neutrinos and antineutrinos whose quantum states can be specified by their momentum \( \mathbf{p} \). In such a system, the test neutrino/antineutrino can be seen as an individual from the ensemble of the background neutrinos and antineutrinos and they share the same set of quantum states \(| \nu_e (\mathbf{p}) \rangle \) and \(| \bar{\nu}_e (\mathbf{p}) \rangle \) for all \( \mathbf{p} \). The differential contribution to the interaction Hamiltonian from neutrinos and antineutrinos with momentum \( \mathbf{p} \) from the background can be written as

\[ d \mathcal{H}_{\nu\nu} (\mathbf{p}) \]

\[ = \frac{1}{\sqrt{2}} G_F \sum_{a, \beta = e, \mu, \tau} dN_v (\mathbf{p}) \left[ \left( \nu (\mathbf{p}) \right| \bar{\nu}_{\alpha L} \gamma^\lambda v_{\alpha L} \left| \nu (\mathbf{p}) \right) \bar{\nu}_{\beta L} \gamma^\lambda v_{\beta L} \left| \nu (\mathbf{p}) \right) \bar{\nu}_{\alpha L} \gamma^\lambda v_{\alpha L} + \left( \nu (\mathbf{p}) \right| \bar{\nu}_{\beta L} \gamma^\lambda v_{\beta L} \left| \nu (\mathbf{p}) \right) \bar{\nu}_{\alpha L} \gamma^\lambda v_{\alpha L} + 2 \left( \nu (\mathbf{p}) \right| \bar{\nu}_{\alpha L} \gamma^\lambda v_{\beta L} \left| \nu (\mathbf{p}) \right) \bar{\nu}_{\beta L} \gamma^\lambda v_{\alpha L} \right], \]

where \( N_v (\mathbf{p}) \) is the number of neutrinos(antineutrinos) with momentum \( \mathbf{p} \). As the second step, the interaction Hamiltonian needs to be evaluated under the free-propagating single-particle state of the test neutrino/antineutrino to give the single-particle Hamiltonian that can be used to calculate the flavor evolution of the test neutrino in the neutrino background. Without loss of generality we take the 2-flavor neutrino model as a simple example. Here the single-particle state for neutrino and antineutrino can be expressed as

\[ \left| \nu (\mathbf{p}) \right) = a_e \left| \nu_e (\mathbf{p}) \right) + a_x \left| \nu_x (\mathbf{p}) \right), \]

\[ \left| \bar{\nu} (\mathbf{p}) \right) = \bar{a}_e \left| \bar{\nu}_e (\mathbf{p}) \right) + \bar{a}_x \left| \bar{\nu}_x (\mathbf{p}) \right), \]

and the corresponding mean values can be evaluated to be

\[ \left( \nu (\mathbf{p}) \right| \bar{\nu}_{\alpha L} \gamma^\lambda v_{\beta L} \left| \nu (\mathbf{p}) \right) = \frac{p^\lambda}{E V} a^*_\alpha a_\beta, \left( \bar{\nu} (\mathbf{p}) \right| \bar{\nu}_{\alpha L} \gamma^\lambda v_{\beta L} \left| \bar{\nu} (\mathbf{p}) \right) = -\frac{p^\lambda}{E V} \bar{a}^*_\alpha \bar{a}_\beta, \]

where \( p^\lambda = (E, \mathbf{p}) \) is the 4-momentum of the neutrino and \( V \) is the volume of the space containing the neutrino ensemble.

Now we can evaluate the interaction potential between a test neutrino/antineutrino with momentum \( \mathbf{q} \) and background neutrino and antineutrinos with momentum \( \mathbf{p} \) by taking mean values
on \( \mathcal{H}_{\nu\nu}(p) \) as follows

\[
[d H_{\nu\nu}(q,p)]_{\bar{\xi}\eta} = \int \left\langle \nu_{\xi}(q) \left| d \mathcal{H}_{\nu\nu}(p) \right| \nu_{\eta}(q) \right\rangle d^3 x \tag{1.68}
\]

\[
[d \bar{H}_{\nu\nu}(q,p)]_{\bar{\xi}\eta} = \int \left\langle \bar{\nu}_{\xi}(q) \left| d \mathcal{H}_{\nu\nu}(p) \right| \bar{\nu}_{\eta}(q) \right\rangle d^3 x, \tag{1.69}
\]

where the first expression is for a test neutrino and the second is for a test antineutrino. If we define the single-particle density matrices by

\[
\rho_p = \begin{pmatrix} |a_e|^2 & a_e a_x^* \\ a_x^* a_e & |a_x|^2 \end{pmatrix}, \quad \bar{\rho}_p = \begin{pmatrix} |\bar{a}_e|^2 & \bar{a}_e \bar{a}_x^* \\ \bar{a}_x^* \bar{a}_e & |\bar{a}_x|^2 \end{pmatrix}, \tag{1.70}
\]

then we end up with the expressions for \( d H_{\nu\nu}(q,p) \) and \( d \bar{H}_{\nu\nu}(q,p) \) as

\[
d H_{\nu\nu}(q,p) = \sqrt{2} G_F (1 - \hat{q} \cdot \hat{p}) \left[ \rho_p d n_\nu(p) - \bar{\rho}_p^* d n_{\bar{\nu}}(p) \right] \tag{1.71}
\]

\[
d \bar{H}_{\nu\nu}(q,p) = \sqrt{2} G_F (1 - \hat{q} \cdot \hat{p}) \left[ -\rho_p^* d n_\nu(p) + \bar{\rho}_p d n_{\bar{\nu}}(p) \right], \tag{1.72}
\]

here \( d n_\nu(p) = d N_\nu(p) / V \) and \( d n_{\bar{\nu}}(p) = d N_{\bar{\nu}}(p) / V \) are the corresponding neutrino and antineutrino densities in the background. One important characteristic of the neutrino self-interaction potential with the form of equation (1.71) is the angular dependence coming from \( 1 - \hat{q} \cdot \hat{p} \). A test neutrino interacts most strongly with other neutrinos propagating in the opposite direction while barely interacting with those propagating in the same direction. This angular dependence will have important indications when neutrino fluxes are not isotropic as we will see in the next section about supernova environments.

### 1.5.2 The neutrino bulb model

While it became more clear that the neutrino self-interaction plays an important role in the flavor evolution of supernova neutrinos, handling the problem is by no means an easy task. Unlike the homogeneous environment such as the early universe, where neutrino background is uniform and isotropic in all directions and the angular dependence in the self-interaction potential can be dropped, the majority of the background neutrinos in supernova environments come from the supernova core and are propagating outward, with the intensity of the fluxes decreasing as \( 1 / r^2 \). Thus the self-interaction potential between a test neutrino and the background would be strongly dependent on the position and direction of propagation of the test neutrino. In the most general case, to determine the self-interaction potential one would need (1) The spatial coordinates and the momentum vector of the test neutrino; (2) The flavor compositions and the fluxes of all the background neutrino beams coming from the part of the neutrinosphere surface that can be seen
by the test neutrino. Due to the requirement of self-consistency, the evolution history of all neutrino beams must be recorded and evolved simultaneously, which is an impossible task with limited computational resources. In order to formulate a tractable problem, a dramatic simplification is needed. In their seminal 2006 paper [34] Duan et al. proposed a practical model that can be used to handle the problem of neutrino flavor evolution in supernova environments self-consistently. The model is comprised of a spherical “neutrino bulb” as the source of neutrino emission and a background formed by the neutrino beams originated from the “bulb”. The bulb model possess the following properties:

- **The neutrinosphere and the neutrino field above it are spherically symmetric.**
- **The system is cylindrically symmetric along any radial direction.**
- **The neutrino flavor states above the neutrinosphere are stationary.**

With these assumptions, we can move forward to derive the neutrino self-interaction Hamiltonian in the bulb model, whose configuration is shown in figure (1.4). We continue the derivation with equation (1.71). In the context of the bulb model, the meaning of \( d n_\nu(p) \) is the differential contribution to the neutrino number density at \( r \) from those neutrinos with energy between \( |p| \) and \( |p| + dp \), and propagating in the directions between \( \hat{p} \) and \( \hat{p} + d\hat{p} \) (the hats indicate unit vectors). At a distance \( r \) from the center of the neutrinosphere this can be expressed in terms of the neutrino radiance as

\[
 d n_\nu(r, p) = j_\nu(p) d(\cos \theta') d\phi, \tag{1.73}
\]

where \( \theta' \) is the angle between \( p \) and the radial direction and we have assumed the neutrino emission is half-isotropic for simplicity\(^2\). At the neutrinosphere, we have

\[
 \frac{L_\nu}{\langle E_\nu \rangle} f_\nu(p) dp = 4\pi R_\nu^2 \int_0^1 2\pi j_\nu(p) \cos \theta_R d(\cos \theta_R) = 4\pi^2 R_\nu^2 j_\nu(p), \tag{1.74}
\]

so that

\[
 j_\nu(p) = \frac{L_\nu}{4\pi^2 R_\nu^2 \langle E_\nu \rangle} f_\nu(p) dp, \tag{1.75}
\]

On the other hand, due to the cylindrical symmetry along any radial direction, we have

\[
 \int (1 - \hat{q} \cdot \hat{p}) F(\theta') d(\cos \theta') d\phi = 2\pi \int (1 - \cos \theta \cos \theta') F(\theta') d(\cos \theta'), \tag{1.76}
\]

where \( \theta \) is the angle between \( q \) and the radial direction and \( F(\theta') \) is an arbitrary function of \( \theta' \). With this we obtain from equation (1.71) the expression for the neutrino self-interaction potential

\(^2\)This is not a requirement by the bulb model, so this condition can be relaxed if necessary.
between the test neutrino \( q \) and the background beam \( p \) as

\[
dH_{\nu\nu}(q, p) = \frac{\sqrt{2} G_F}{2 \pi R^2} (1 - \cos \theta \cos \theta^\prime) \left[ \frac{T_{\nu\nu}}{\langle E_{\nu\nu} \rangle} f_{\nu}(p) \rho_p(r, \theta^\prime) - \frac{T_{\bar{\nu}\nu}}{\langle E_{\bar{\nu}\nu} \rangle} f_{\bar{\nu}}(p) \bar{\rho}_p^*(r, \theta^\prime) \right] d(\cos \theta^\prime) d\theta.
\]

Since what we need is the overall potential between the test neutrino and all other neutrino beams in the background, we need to integrate over all possible intersection angles \( \theta^\prime \). Direct integration over \( \theta^\prime \) is not convenient in numeric computations since the range of \( \theta^\prime \) is dependent on the distance \( r \). But we can change the variable of integration from the intersection angle \( \theta^\prime \) to the corresponding emission angle \( \theta_R^\prime \) by the following relationship

\[
\cos \theta^\prime = \sqrt{1 - \left( \frac{R_{\nu}}{r} \right)^2 \sin^2 \theta_R^\prime},
\]

so that \( d(\cos \theta^\prime) \) becomes

\[
d(\cos \theta^\prime) = \left( \frac{R_{\nu}}{r} \right)^2 \frac{d\left( \sin^2 \theta_R^\prime \right)}{\cos \theta^\prime}.
\]

Therefore the overall interaction potential between the test neutrino and the background can be obtained as

\[
H_{\nu\nu}(r, q) = \int dH_{\nu\nu}(q, p) = \frac{\sqrt{2} G_F}{2 \pi r^2} \int (1/ \cos \theta - \cos \theta) \left[ \frac{T_{\nu\nu}}{\langle E_{\nu\nu} \rangle} f_{\nu}(p) \rho_p(r, \theta^\prime) - \frac{T_{\bar{\nu}\nu}}{\langle E_{\bar{\nu}\nu} \rangle} f_{\bar{\nu}}(p) \bar{\rho}_p^*(r, \theta^\prime) \right] d\left( \sin^2 \theta_R^\prime \right) d\theta.
\]

Thus, we see that by enforcing artificial symmetries on the supernova environment, we have reduced the necessary parameters describing the neutrino field to the emission angle \( \theta_R \), the energy \( q \) and the distance \( r \). In terms of numeric computation, this requires solving \( N_A \times N_E \) differential equation about \( r \) in the same time, where \( N_A \) is the number of angular modes and \( N_E \) is the number of energy modes. This approach is called the “multi-angle” method, and it can be very computationally expensive depending on the particular settings of the supernova model. For example, when the matter density is high, the “multi-angle” method requires better angular resolution since insufficient number of angular bins has been found to cause artificial flavor instabilities [35]. It is also possible to further simplify the calculation by applying further approximations on the “multi-angle” method. A very popular approximation often found in literature is the so called “single-angle” approximation, where it is assumed all angular modes have exactly the same evolution history. That means we can represent the flavor states of neutrinos with different emission angle by one with a particular emission angle, which is often chosen to be either the neutrino emitted with 0° or 45° relative to the radial direction. Thus the number of angular modes is reduced to 1 and the calculation is greatly simplified. Under the “single-angle” approach the self-interaction Hamiltonian 1.80 needs to be averaged over all angular modes. If we take \( \theta_R = 0^\circ (\theta = 0^\circ) \) as the representative mode, then the

\[3\]This also applies to the test neutrino.
The schematic of neutrino bulb model. \( R_\nu \) is the radius of the neutrinosphere, \( r \) is the distance of observational point from the center. \( q \) represents the momentum for the test neutrino and \( \theta_R \) is the neutrino emission angle, while \( \theta \) is the intersection angle between the neutrino beam and the radial direction. \( p \) represents the momentum for a background neutrino beam, and its other quantities are marked by a prime.

The averaging procedure is

\[
\langle 1 - \cos \theta \cos \theta' \rangle = \int_{\theta_{\text{max}}}^{\theta} (1 - \cos \theta \cos \theta') d \cos \theta' = \frac{1}{2} (1 - \cos \theta_{\text{max}})^2 = \frac{1}{2} \left( 1 - \sqrt{1 - \left( \frac{R_\nu}{r} \right)^2} \right)^2.
\]

Therefore equation 1.80 becomes

\[
H_{\nu\nu}(r) = \frac{\sqrt{2} G_F}{2 \pi R_\nu^2} D\left( r / R_\nu \right) \int \left[ \frac{L_{\nu}}{\langle E_{\nu} \rangle} f_{\nu}(p) \rho_{\nu}(r) - \frac{L_{\bar{\nu}}}{\langle E_{\bar{\nu}} \rangle} f_{\bar{\nu}}(p) \tilde{\rho}_{\nu}^*(r) \right] dp,
\]

where \( D\left( r / R_\nu \right) = \frac{1}{2} \left( 1 - \sqrt{1 - \left( \frac{R_\nu}{r} \right)^2} \right)^2 \) is called the geometric factor.

In chapter 3 and 4, we will generalize the neutrino bulb model to include the effect of strong gravitational field and non-standard neutrino self-interaction (NSSI). Our results will show that even though the neutrino bulb model has its own limitation on describing more asymmetric, realistic supernova environments, it is still the only practical model that is able to calculate neutrino flavor transformation self-consistently. And by generalizing the bulb model in different aspects, we are able to get more insight into what is going on deep inside the an exploding supernova in various scenarios.
1.5.3 Major achievements of the neutrino bulb model

In the last decade the neutrino bulb model has been widely used in various supernova problems and the results provided important insights on how the paradigms of supernova neutrinos must be changed. One of the prominent effects of collective flavor oscillations found in the bulb model is the “spectral swap”, in which neutrinos of different flavors swap their spectra in certain energy intervals bounded by sharp spectral splits [36, 37, 38, 39]. In figure (1.5) an example of spectral swap in supernova cooling phase is shown. In the region of $E \lesssim 10$ MeV no flavor conversion between electron neutrinos and heavy-lepton neutrinos occurs, while in the region of $E \gtrsim 10$ MeV electron neutrinos almost completely converted to heavy-lepton neutrinos. An obvious consequence is that the neutrino signals from the next Galactic supernova should contain the feature of swapped spectra. A discussion on this issue can be found in [40, 41].

Apart from the observational consequences, collective oscillations also have significant implications on supernova dynamics. In the 1985 paper by Bethe and Wilson which proposed neutrino heating mechanism as a possible avenue to the revival of stalled shock [22] it was estimated if...
only 0.1% of the neutrinos emitted from the core were absorbed by the outer layers, it would be enough for a successful supernova explosion. Later investigations of multi-dimensional supernova model found the neutrino heating can be more efficient thanks to the existence of asymmetrical hydrodynamical instabilities [42]. The influence of neutrino flavor transformation on supernova dynamics comes from the fact that the only electron neutrino and antineutrinos can be captured by the nucleons and thus effectively transfer their energy to the material in the outer layers of the supernova. And since collective neutrino oscillations are able to cause flavor conversions between $\nu_e$, $\bar{\nu}_e$ and $\nu_x$, in order to correctly calculate the neutrino heating rate it is necessary to figure out the change of neutrino flavor composition throughout the whole energy spectrum as the neutrinos propagate outward in the supernova environment, and the neutrino bulb model is readily useful for this task. The effect of “spectral swap” on neutrino heating process has been discussed in details under realistic supernova settings in literature such as [43, 44, 45, 46]. Generally speaking it was found that the neutrino heating would be enhanced by spectral swap because the supernova could obtain some “free” electron neutrinos with higher energies by a swap between electron neutrinos and heavy-flavor neutrinos in the high energy region (the region of $E > 15$ MeV in figure (1.5), for example), but how significant the enhancement is is still controversial, with some of the works concluding that collective oscillations only take place close to, or outside, of the shock, therefore the enhancement is not every effective. Nevertheless we note that these earlier works lack the self-consistency since neutrino flavor evolution and supernova dynamical simulation were carried out separately and independently. More recent efforts in this subject are trying to combine neutrino flavor transformation with supernova hydrodynamics and classical neutrino transports to construct a complete and self-consistent picture of the dynamical evolution of core-collapse supernovae.

Another big concern is how collective neutrino oscillation would change the conditions for nucleosynthesis in the supernova environment. The major factor to be considered here is the electron fraction $Y_e$ as a function of position in the supernova environment, which is directly used in the calculations of nucleosynthesis. The speculation on the potential influences of neutrino flavor transformation on nucleosynthesis has been around for nearly two decades, but earlier works suffered from many limitations, which includes

- Lack of precise neutrino parameters, especially accurate mixing angles.
- Primitive treatment of neutrino collective oscillations like single-angle approximation that enforces artificial synchronizations among different neutrino modes.
- Absence of sophisticated multi-dimensional supernova simulations as the provider of both neutrino fluxes and physical conditions for the nuclear reactions.

Over the years these aspects have been greatly improved. An incomplete list of some more recent and representative works on this subject including [47, 48, 49]. However, debates about how collective
oscillation changes the yields of elements are still ongoing since collective oscillations are as sensitive to many astrophysical factors as the nucleosynthesis and the interplay between them further complicates the situation. A major progress on understanding the effects of collective oscillation on nucleosynthesis cannot be achieved without improvements on both the model of neutrino flavor evolution and supernova simulations.

Apart from the calculations of nucleosynthesis in supernovae, it is also worth noting that some of the techniques used in the bulb model has been tentatively applied to the investigations of other potential sites of r-process nucleosynthesis such as neutron star mergers [50, 51, 52, 52]. But due to the complex geometry of these neutrino sources it is even harder to self-consistently calculate neutrino flavor transformations than in the bulb model. This again highlights the need to establish a completely new numerical scheme that, if combined with sufficient computational power, is able to handle neutrino evolution problems in any environment with an arbitrary degree of symmetry.
REFERENCES


2

NEUTRINO FLAVOUR EVOLUTION THROUGH FLUCTUATING MATTER

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2.1 Abstract

A neutrino propagating through fluctuating matter can experience large amplitude transitions between its states. Such transitions occur in supernovae and compact object mergers due to turbulent matter profiles and neutrino self-interactions. In this paper we study, both numerically and analytically, three-flavour neutrino transformation through fluctuating matter built from two and three Fourier modes. We find flavor transformation effects which cannot occur with just two flavours. For the case of two Fourier modes we observe the equivalent of “induced transparency” from quantum optics whereby transitions between a given pair of states are suppressed due to the presence of a resonant mode between another pair. When we add a third Fourier mode we find a new effect whereby the third mode can manipulate the transition probabilities of the two mode case so as to force complete transparency or, alternatively, restore “opacity” meaning the perturbative
Hamiltonian regains its ability to induce neutrino flavour transitions. In both applications we find analytic solutions are able to match the amplitude and wavenumber of the numerical results to within a few percent. We then consider a case of turbulence and show how the theory can be used to understand the very different response of a neutrino to what appears to be two, almost identical, instances of turbulence.

2.2 Introduction

Perhaps one of the more unusual cases of a driven quantum mechanical system is the flavour evolution in space/time of a neutrino as it propagates through inhomogeneously distributed matter and/or through a field of other neutrinos. The phenomenology of neutrino flavour evolution in environments such as core-collapse supernovae, the merger of two neutron stars or a neutron star and a black hole, has been found to be very rich: for recent reviews of supernova neutrinos see Mirizzi et al. [1] and Horiuchi & Kneller [2]. An equivalence of a neutrino to other driven quantum systems can be made because the neutrino flavour evolution in these environments is calculated from a Schrödinger equation governed by a Hamiltonian. Due to the difference in the neutrino masses, a $N_f$ flavour neutrino has $N_f$ distinct eigenstates of the Hamiltonian which we can treat just like the eigenstates of an atom or molecule. Some of the causes of neutrino flavour transformation in supernovae and compact-object merger environments are well understood - e.g. the Mikheyev-Smirnov-Wolfenstein effect [3, 4] - however at the present time the flavour transformation induced by turbulence in the matter is only understood for the case of two neutrino flavours, and similarly the phenomenology of neutrino flavour transformation due to ‘self-interactions’ [5] have been solved only in simplified scenarios [6]. In order to understand neutrino flavour evolution in such environments, we need analytical tools which are able to predict the response of a neutrino to such stimulations.

The effect of matter fluctuations upon neutrinos has received a lot of attention from an analytic perspective. Both Floquet theory and the Rotating Wave Approximation approaches have been used to calculate the effect upon a two flavor neutrino of matter fluctuations described by a single Fourier mode (FM), and through periodic layers of constant density [7, 8, 11, 9, 12, 13, 10, 14]. From these studies it has been found that for a given matter structure, at some neutrino energies the probability for the neutrino to transition between its two states can be enhanced via parametric resonance. Recently the case of two flavour neutrino evolution through non-constant, non-periodic matter fluctuations, as one would find in a turbulent medium, was considered by Patton, Kneller and McLaughlin (PKM) [15]. Note an alternative analysis of the similar problem of a two-level atom interacting with a stochastic electromagnetic field is found in Cummings [19]. PKM based their theoretical description of the evolution also upon the Rotating Wave Approximation (RWA) and found it gave predictions which were in remarkably good agreement with numerical calculations on
a case-by-case basis even though the turbulence is aperiodic. They named their model Stimulated Transitions and found there is a direct correspondence with the predicted response of an irradiated polar molecule [16, 17, 18]. Like the other studies of matter effects upon neutrinos, PKM also saw the effect of parametric resonance but, in addition, they also observed a suppression effect when low frequency / long wavelength modes were present in the turbulence.

However neutrinos have (at least) three flavours and it is well known that one finds richer phenomenology when a quantum system possesses three or more eigenstates. Perhaps the best known examples are in the field of quantum optics where one observes the phenomenon of electromagnetic induced transparency [22, 23, 24, 25] and coherent population trapping into dark states [26] which has also been seen in quantum dots and solid-state systems [27, 28]. Whether similar effects occur for three flavour neutrinos is presently unknown.

In our paper we study the effect of matter fluctuations upon a three flavour neutrinos passing through matter fluctuations to a) observe three flavour oscillation phenomena and b) examine the utility of a analytical tool for predicting the response of neutrinos to Hamiltonians which can be decomposed into a Fourier series. In section §2.3 we study the case of two anharmonic FMs and find the equivalent of electromagnetic induced transparency. In section §2.3.2 we add another FM and find a new effect we call Restored Opacity. In both studies we find the analytic solutions and numerical calculations are in excellent agreement. We then finish with a case of turbulent matter fluctuations and show how the insight gained from the two and three FM cases can be used to understand why a neutrino can respond so differently to two cases of turbulence which appear, at first glance, to be almost identical. Our conclusions and directions for further study are presented in section §2.5.

2.3 Neutrino Propagation Through Fluctuating Matter

The problem we wish to solve is the case of a three-flavour neutrino propagating through fluctuating matter. While this is an example of the evolution of a $N$-level quantum system subject to a time-dependent Hamiltonian note that, as commonly found in the literature on neutrino flavour transformation, we switch the variable from time $t$ to position along the neutrino trajectory $r$ where $r = c t$ since the neutrino wavepacket is localized in space and typically the energy of neutrinos is much larger than their rest mass hence they move at a speed close to $c$. If the neutrino is initially in state $\phi^{(f)}(0)$ in the flavor basis then at position $r$ the neutrino is in the state $\phi^{(f)}(r)$ related to $\phi^{(f)}(0)$ via the evolution matrix $S$ i.e. $\phi^{(f)}(r) = S \phi^{(f)}(0)$. This matrix can be found by solving the Schrödinger equation

$$\frac{dS}{dr} = H^{(f)} S$$

(2.1)
given the initial condition \( S(0) = 1 \). The Hamiltonian, \( H \), governing the neutrino flavour evolution through the fluctuating matter is the sum of a constant vacuum term \( H_V \) and a term coming from the effect of matter \( H_M \) \cite{3, 4}, that is \( H = H_V + H_M \). The vacuum Hamiltonian in the flavour basis is

\[
H_V^{(f)} = \frac{1}{2E} U_V \begin{pmatrix}
    m_1^2 - m_2^2 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & m_3^2 - m_2^2
\end{pmatrix} U_V^\dagger
\]  

(2.2)

where \( U_V \) is the vacuum mixing matrix and \( m_i \) the three neutrino masses. We set the squared mass differences \( m_1^2 - m_2^2 = -7.5 \times 10^{-5} \text{ eV}^2 \) and \( m_3^2 - m_2^2 = 2.32 \times 10^{-3} \text{ eV}^2 \) which are compatible with the mass-squared differences as given by the Particle Data Group \cite{29}. Throughout this paper we choose the neutrino energy \( E \) to be 5 MeV. \( U_V \) is parameterized by three mixing angles \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \) - we set all possible phases to zero \cite{30} - and given by

\[
U_V = \begin{pmatrix}
    c_{12} c_{13} & s_{12} c_{13} & s_{13} \\
    -s_{12} c_{23} - c_{12} s_{13} s_{23} & c_{12} c_{23} - s_{12} s_{13} s_{23} & c_{13} s_{23} \\
    s_{12} s_{23} - c_{12} s_{13} c_{23} & -c_{12} s_{23} - s_{12} s_{13} c_{23} & c_{13} c_{23}
\end{pmatrix}
\]  

(2.3)

where the notation is that \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \). We take the angles to be \( \theta_{12} = 34^\circ \), \( \theta_{13} = 9^\circ \) and \( \theta_{23} = 45^\circ \) \cite{29}.

### 2.3.1 Two Fourier Modes

We first consider the case where the matter Hamiltonian \( H_M \) is taken to be a constant upon which are superposed two FMs with wavenumbers \( q_1 \) and \( q_2 \) not in a rational ratio. The matter is regarded as affecting only the electron flavour type, not the other two flavours. The form of the Hamiltonian in the flavour basis, with the first row/column indicating the electron flavour, is thus

\[
H_{M}^{(f)}(r) = V_\ast \left[ 1 + A_1 \cos(q_1 r + \phi_1) + A_2 \cos(q_2 r + \phi_2) \right] \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}
\]  

(2.4)

with \( V_\ast \) the potential from the constant background, \( A_1 \) and \( A_2 \) the amplitudes of the fluctuations. In what follows we set \( V_\ast \) to \( V_\ast = 6 \times 10^{-25} \text{ erg} \) and \( \phi_1, \phi_2 \) to zero.

The vacuum Hamiltonian and the constant potential \( V_\ast \) form the 'unperturbed' Hamiltonian \( \tilde{H} \). In the flavour basis \( \tilde{H} \) is not diagonal. We can diagonalize \( \tilde{H} \) by first finding its matrix of eigenvalues, denoted by \( K = \text{diag}(k_1, k_2, k_3) \), and then the unitary matrix \( \tilde{U} \) which satisfies \( \tilde{H}^{\dagger} = \tilde{U} K \tilde{U}^{\dagger} \). Since this is standard textbook quantum mechanics, we leave this as an exercise for the reader. For reference, the differences between the eigenvalues are found to be \( k_3 - k_1 = 3.835 \times 10^{-22} \text{ erg} \) and
Figure 2.1 The three unperturbed eigenstates and the transitions between them in the two and three FM problems. The modes $q_1$ and $q_2$ are the FMs that drive transitions between the indicates states. The mode $q_3$ is the "switch mode" which switches on and off the effect of transitions induced by modes $q_1$ and $q_2$.

$k_3 - k_2 = 3.715 \times 10^{-22}$ erg. Note that since $\tilde{H}$ is a function of $V_\star$, the eigenvalues and unperturbed mixing matrix, $\tilde{U}$ are also functions of $V_\star$. The level scheme we end up with is shown in figure (2.1) with the three eigenstates of the unperturbed system denoted by $|k_1\rangle$, $|k_2\rangle$ and $|k_3\rangle$.

The two FMs in the matter Hamiltonian are a Fourier-decomposed perturbation and so we can apply the analytic solution for this kind of perturbation derived in 2.6. The evolution matrix in the basis of the eigenstates of the unperturbed Hamiltonian, denoted as $S(\tilde{u})$, is written as the product $S(\tilde{u}) = \tilde{S} W B$ where $\tilde{S}$ is the evolution matrix for the unperturbed states, $W$ is a diagonal matrix designed so as to remove the diagonal elements of the perturbing Hamiltonian in this basis, and $B$ is the evolution matrix which describes the transitions. While this problem may be solved in general, let us consider the case where we set the wavenumbers for the two modes so that $q_1 \approx k_3 - k_1$ and $q_2 \approx k_3 - k_2$ as shown in figure (2.1). According to the solution found in 2.6, we must find a set of integers - the RWA integers - for every element $i j$ of the perturbing Hamiltonian for each FM $a$. These integers are labeled $n_{a;i;j}$. Given our choice for $q_1$ and $q_2$, the RWA integers we select for the 1, 3 element are $\{n_{1;13}, n_{2;13}\} = \{+1, 0\}$ and for the 2, 3 element we pick $\{n_{1;23}, n_{2;23}\} = \{0, +1\}$. The integer set for the 1, 2 element must therefore be $\{n_{1;12}, n_{2;12}\} = \{+1, -1\}$ in order that $n_{a;12} + n_{a;23} = n_{a;13}$. The evolution of $B$ is determined by (see equation 3.9 in appendix A)

$$\frac{dB}{dt} = H^{(B)}B$$

(2.5)
where the Hamiltonian $H^{(B)}$ is

$$H^{(B)} = \begin{pmatrix}
0 & -i\kappa_{12} e^{-i(\delta k_{12} + q_1 - q_2)r} & -i\kappa_{13} e^{i(\delta k_{13} + q_1)r} \\
-i\kappa_{12} e^{-i(\delta k_{12} + q_1 - q_2)r} & 0 & -i\kappa_{23} e^{-i(\delta k_{23} + q_2)r} \\
i\kappa_{13} e^{-i(\delta k_{13} + q_1)r} & -i\kappa_{23} e^{-i(\delta k_{23} + q_2)r} & 0
\end{pmatrix} \tag{2.6}$$

with

$$\kappa_{12} = \frac{2i G_{1:12}}{z_{1:12}} j_1(z_{1:12}) j_1(z_{2:12}) + \frac{2i G_{2:12}}{z_{2:12}} j_1(z_{1:12}) j_1(z_{2:12}) \tag{2.7}$$

$$\kappa_{13} = \frac{2i G_{1:13}}{z_{1:13}} j_1(z_{1:13}) j_0(z_{2:13}) \tag{2.8}$$

$$\kappa_{23} = \frac{2i G_{2:23}}{z_{2:23}} j_0(z_{1:23}) j_1(z_{2:23}), \tag{2.9}$$

and the quantities $z_{a:i}$ are defined to be

$$z_{a:i} = \frac{A_a V_a (|\tilde{U}_{e:i}|^2 - |U_{e:j}|^2)}{2 q_a} \tag{2.10}$$

We notice that both terms in $\kappa_{12}$ are proportional to the product of two Bessel functions $J_1$ so once we recall that the Bessel function $j_n(z) \sim z^{|n|}$ for small $z$, we see that the element $\kappa_{12}$ is smaller in magnitude than $\kappa_{13}$ and $\kappa_{23}$ since the values of $z_{a:i}$ are very small. That is confirmed when we compute the numerical values and find $\kappa_{12} = 6.419 \times 10^{-32}$ i erg, $\kappa_{13} = -3.888 \times 10^{-27}$ i erg and $\kappa_{23} = -1.311 \times 10^{-26}$ i erg.

We now proceed to solve for $B$ following the steps found in the appendix A. If we make the approximation that $\kappa_{12}$ is negligibly small compared to $\kappa_{13}$ and $\kappa_{23}$ and that the two wavenumbers are exactly on resonance, $q_1 = k_3 - k_1$, $q_2 = k_3 - k_2$, then we find the analytical expression for the $B$ matrix is

$$B = \exp(\im \left[ \Lambda - k_3 \right] r) \begin{pmatrix}
\frac{|\kappa_{13}|^2}{Q^2} + \frac{|\kappa_{12}|^2}{Q^2} \cos(Q r) & \frac{|\kappa_{13}|^2}{Q^2} + \frac{|\kappa_{23}|^2}{Q^2} \cos(Q r - 1) & -\frac{\kappa_{13}}{Q} \sin(Q r) \\
\frac{|\kappa_{12}|^2}{Q^2} \cos(Q r - 1) & \frac{|\kappa_{12}|^2}{Q^2} \cos(Q r) & -\frac{\kappa_{23}}{Q} \sin(Q r) \\
-\frac{\kappa_{13}}{Q} \sin(Q r) & -\frac{\kappa_{23}}{Q} \sin(Q r) & \cos(Q r)
\end{pmatrix} , \tag{2.11}$$

where $Q^2 = |\kappa_{13}|^2 + |\kappa_{23}|^2$. From this result we can extract the transition probability from unperturbed eigenstate 1 to unperturbed eigenstate 3 by taking the squared magnitude of $B_{13}$

$$P_{13} = |B_{13}|^2 = \frac{|\kappa_{13}|^2}{Q^2} \sin^2(Q r) = \left(1 - \frac{|\kappa_{23}|^2}{Q^2}\right) \sin^2(Q r). \tag{2.12}$$

This result is interesting because it indicates the transition probability $P_{13}$ depends upon the
wavenumber \( q_2 \) which is driving transitions from unperturbed eigenstate 2 to unperturbed eigenstate 3. In the extreme case when \( \kappa_{23} \) is significantly larger than \( \kappa_{13} \), the transition from states 1 to 3 is strongly suppressed. This is an analog of the Electromagnetically Induced Transparency (EIT) - see, for example, [22, 23, 24, 25] - in atomic physics where the presence of a second possible transition between atomic levels 2 and level 3 will inhibit the primary transition from atomic level 1 to level 3 leading to little absorption, and thus transparency, for the light frequency corresponding to the energy splitting of level 1 and 3.

To illustrate this neutrino version of induced transparency, in figure (2.2) we plot the transition probability as a function of \( r \) when the system is at perfect resonance, namely when \( q_1 = k_3 - k_1 \) and \( q_2 = k_3 - k_2 \). The reader will observe that indeed, even though the wavenumber \( q_1 \) is exactly on resonance with the transition between neutrino states 1 and 3, the probability of being in state 3 has a maximum of only 10% when \( A_2 \neq 0 \). When we remove the second FM \( q_2 \) the transition probability \( P_{13} \) increases to 100%. Note also a) that the solution is periodic even though the two wavenumbers \( q_1 \) and \( q_2 \) do not form rational ratio, and b) how well the numerical solution to the problem agrees with the analytic solution. The predicted amplitude and the wavenumber match the amplitude and wavenumber of the numerical solution to within a few percent.

To see the effect of induced transparency more clearly, we fix \( q_2 \) at the resonance between states 2 and 3 and scan in \( q_1 \). The solution for \( B \) can be found by evaluating the formal solution and from the element \( B_{13} \) we extract the transition probability \( P_{13} \). In figure (2.3) we plot the amplitude of the oscillations in \( P_{13} \) as a function of \( q_1 \). We see that in the presence of mode \( q_2 \), the transition probability has a peculiar shape with peaks off-resonance and local minimum at the resonance. If we turn off the second perturbing mode by setting \( A_2 \) to zero we recover the expected shape for a resonance at \( q_1 \). Again, we find the analytic solution is able to reproduce the shape of \( P_{13} \) versus \( q_1 \) very well at all the values of \( A_2 \) used.

### 2.3.2 Three Fourier modes

Now we add a third FM to the perturbing Hamiltonian which we give an amplitude \( A_3 \) and wavenumber \( q_3 \). Thus the perturbing Hamiltonian in the flavour basis becomes

\[
\delta H^{(f)}(r) = V z \sum_{j=1}^{3} A_j \cos(q_j r + \phi_j) \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]  

We shall leave \( V_z \) unchanged so that the unperturbed Hamiltonian is the same as the previous case of two FMs with the same eigenvalues.

Let us again set the wavenumbers for the first two modes so that \( q_1 = k_3 - k_1 \) and \( q_2 = k_3 - k_2 \) for the level diagram shown in figure (2.1). Neither \( A_1 \) nor \( A_2 \) are zero and \( A_2 > A_1 \). For a two FM case this
choice for the wavenumbers $q_1$ and $q_2$ and ratio of amplitudes would put the system exactly at the midpoint of figure (2.3) so the transition probability $P_{13}$ is suppressed even though the wavenumber $q_1$ is exactly on resonance. We shall not set the mode $q_3$ to a particular value yet but we shall only consider wavenumbers such that $q_3$ is much smaller than $k_3 - k_1$, $k_3 - k_2$ and $k_2 - k_1$, i.e. $q_3$ is not on resonance with any pair of eigenvalue splittings. Thus the sets of RWA integers are very similar to the sets for the two FM case: for the 1, 3 element they are $\{n_{1;13}, n_{2;13}, n_{3;13}\} = \{+1, 0, 0\}$ and for the 2, 3 element we pick $\{n_{1;23}, n_{2;23}, n_{3;23}\} = \{0, +1, 0\}$. Again, these choices mean the integer set for the 1, 2 element is determined and must therefore be $\{n_{1;12}, n_{2;12}, n_{3;12}\} = \{+1, -1, 0\}$ in order that $n_{a;12} + n_{a;23} = n_{a;13}$. The structure of the Hamiltonian for $H^{(B)}$ is exactly the same as in equation (3.25). The expressions for the $\kappa$’s are:

\begin{align*}
\kappa_{12} &= -\frac{2i G_{1;12}}{z_{1;12}} J_1(z_{1;12}) J_1(z_{2;12}) J_0(z_{3;12}) + \frac{2i G_{2;12}}{z_{2;12}} J_1(z_{1;12}) J_1(z_{2;12}) J_0(z_{3;12}) \quad (2.14) \\
\kappa_{13} &= \frac{2i G_{1;13}}{z_{1;13}} J_1(z_{1;13}) J_0(z_{2;13}) J_0(z_{3;13}) \quad (2.15) \\
\kappa_{23} &= \frac{2i G_{2;23}}{z_{2;23}} J_0(z_{1;23}) J_1(z_{2;23}) J_0(z_{3;23}) \quad (2.16)
\end{align*}

where $z_{a;ij}$ has the same meaning as for the two FM case, which is defined by equation (2.10). The expressions again show $\kappa_{12}$ is much smaller than $\kappa_{13}$ and $\kappa_{23}$ when $z_{1;ij}$ and $z_{2;ij}$ are small. These expressions look very similar to those given in equations (2.7) - (2.9) for the two FM case, in fact the only difference is the presence of $J_0(z_{3;ij})$. But the presence of this new term permits new phenomena because as we vary the wavenumber $q_3$ and/or its amplitude $A_3$ it becomes possible for either $z_{3;13}$ or $z_{3;23}$ to become equal to a zero of the Bessel function $J_0$. The effect will be to either switch off $\kappa_{13}$ or $\kappa_{23}$. If we switch off $\kappa_{13}$ then no transitions between states 1 and 3 can occur thus $P_{13} = 0$ even though mode $q_1$ is on resonance. At this value of $q_3$ the induced transparency effect, which was only partial for the two FM case, will become complete for three FMs. If we switch off $\kappa_{23}$ then the effect of the third FM is to cancel the induced transparency effect and so restore amplitude of the oscillations of $P_{13}$ to 100%. We call this effect Restored Opacity. In summary, by scanning in the non-resonant mode $q_3$ we can tune the opacity of the system from zero to 100% even though this mode is nowhere close to being resonant.

To test these predictions we solve the for the transition probability $P_{13}$ numerically making no approximation. As for the two FM case, we set the potential $V_c$ to $V_c = 6 \times 10^{-23}$ erg and the wavenumbers $q_1$ and $q_2$ are set to $q_1 = k_3 - k_1$ and $q_2 = k_3 - k_2$ with amplitudes $A_1 = 0.02$ and $A_2 = 0.1$. The third wavenumber $q_3$ is set to $q_3 = 5.24 \times 10^{-10}$ cm$^{-1}$ and we consider two cases: $A_3 = 0$ and $A_3 = 0.2$. The comparison between the numerical and analytic solutions is shown in figure (2.4). In the $A_3 = 0$ case we expect induced transparency and indeed the figures shows that is correct with very small amplitude oscillations in $P_{13}$ even though the wavenumber $q_1$ is exactly on resonance.
between those pair of states. When we switch on the third mode we find $z_{3;23}$ is equal to a root of $J_0$ which means $\kappa_{23} = 0$. This should return the amplitude of the oscillations of the transition probability $P_{13}$ back to unity and the figure indicates that does indeed occur: the presence of the third FM with this amplitude and wavenumber leads to a restoration of the opacity.

To further illustrate the power of the third FM, in figure (2.5) we fix the amplitudes at $A_1 = 0.002, A_2 = 0.01$ and $A_3 = 0.02$, and scan in the wavenumber $q_3$. The purpose of using smaller amplitudes for the FMs is to suppress the fluctuations of the transition probability seen in the numerical results which make it hard to determine the transition amplitude. Note this choice also makes the corresponding value of $q_3$ which cause the Bessel functions to hit their roots smaller than in the example shown in figure (2.4). From every numerical solution we fit two sinusoids with amplitudes that enclose the oscillations of $P_{13}$ as seen in figure (2.4). The spread in amplitudes forms the width of the band for the numerical results shown in figure (2.5). The comparison of the theory and numerical solutions in figure (2.5) indicate the theory does a very good job of reproducing the numerical results. At $q_3 = 5.24 \times 10^{-11}$ cm$^{-1}$, $\kappa_{23}$ is zero and therefore opacity is restored. When $q_3 = 4.22 \times 10^{-11}$ cm$^{-1}$ we find $z_{3;13}$ is a root of $J_0$ which forces $\kappa_{13}$ to be zero and thus we have complete transparency.

2.4 Turbulence

We finish by considering the case of neutrino evolution through a turbulent medium. We assume the medium has an average density $\rho_\star$ and in what follows we have set the mean density of the medium to be $\rho_\star = 100$ g/cm$^3$ which is a typical matter density at $r \sim 10^5$ km above the proto-neutron star for a supernova in the late cooling phase. This density sets the scale $V_\star$ for the matter Hamiltonian $H_M$ to be $V_\star = \sqrt{2} G_F Y_e \rho_\star / m_p$ where $G_F$ is the Fermi constant, $m_p$ is the proton mass, and $Y_e$ is the electron fraction. For our calculations we have adopted $Y_e = 0.5$ which is also consistent with the electron fraction found at $r \sim 10^5$ km in supernova simulations at late times. The matter Hamiltonian $H_M$ in the flavour basis can be written as

$$H_M^{(f)}(r) = V_\star \left( 1 + \sum_{a=1}^{N_\mathcal{Q}} \left\{ A_a \cos(q_a r) + B_a \sin(q_a r) \right\} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (2.17)

where the $A_a$'s and $B_a$'s are the amplitudes of the FMs, the $q_a$'s are the wavenumbers, and $N_\mathcal{Q}$ is the number of FMs. We assume the turbulence is a Gaussian random field with a power spectrum $E(q)$ which is an inverse power law i.e.

$$E(q) = \frac{\alpha - 1}{2 q_\star} \left( \frac{q}{q_\star} \right)^\alpha \Theta(|q| - q_\star).$$ (2.18)
In this equation $q_\star$ is a cut-off scale for the turbulence wavenumbers $q$ and $\alpha$ is the power spectral index. The spectral index we use is the Kolmogorov value of $\alpha = 5/3$. The number of Fourier modes $N_q$ is determined by the dynamic range i.e. the ratio of the largest spatial scale to the smallest. It is found in practice that for every decade of dynamic range, one needs at least 3 wavenumbers in order to reproduce the statistical properties of the field satisfactorily [31, 32]. In our case, we determined $q_\star$ and $N_q$ by first finding the eigenvalues $k_1$, $k_2$ and $k_3$ of the unperturbed Hamiltonian. We then picked a value for $q_\star$ and dynamic range of the turbulence so as to cover the wavenumbers corresponding to the differences of eigenvalues $k_3 - k_1$ and $k_3 - k_2$. We used $q_\star = 1.0 \times 10^{-6}$ cm$^{-1}$, the dynamic range was 2.5 orders of magnitude and we use $N_q = 40$ FMs. An instance of a random field may be constructed by selecting the amplitudes $A_a$, $B_a$ and the wavenumbers $q_a$ from probability distributions chosen so as to satisfy the chosen statistical properties. For this work the Gaussian random field for the turbulence is generated by the ‘Algorithm C’ from Kramer, Kurbanmuradov and Sabelfeld [31] after setting the rms amplitude of the random field to be 0.25.

With all the parameters set, we generated an instance of the turbulence and the wavenumbers and amplitudes of the FMs we obtained are given in §2.7. Before using them we examined the wavenumbers generated by the algorithm and adjusted the two FMs which were closest to the splittings $k_3 - k_1$ and $k_3 - k_2$ in order that these two FMs were exactly equal to the resonant values. The amplitudes were left unchanged. This was done so that we had an instance of turbulence which, according to the two FM case discussed earlier, should be a case of Induced Transparency. The two wavenumbers which were changed are indicated in table (2.1). With this tweaked instance of turbulence we then proceeded to construct $H_M^{(f)}$ and numerically solved the Schrödinger equation. The ‘e-e’ element of the matter Hamiltonian and the resulting transition probability $P_{13}$ are shown in figure (2.6) where we observe that the transition probability $P_{13}$ is suppressed even though the stimulative FM in the turbulence is resonant. Knowing the amplitudes and wavenumbers of the turbulence also allows us to use the analytical theory to make a prediction for this transition probability. The prediction is also shown in figure (2.6) and we see the theory gives the amplitude qualitatively well and wavenumber to within $\sim 20\%$.

In order to verify this was an instance of Induced Transparency, we set to zero the amplitude of the mode which matched the splitting between eigenvalues $k_3 - k_2$. We then reconstructed the matter Hamiltonian $H_M$ and solved the Schrödinger equation. The transition probability is shown in figure (2.7). Note how the matter Hamiltonian in the left panel is almost identical to that in the left panel of figure (2.6). Nevertheless, by setting to zero the amplitude of the mode whose wavenumber matches $k_3 - k_2$ we find the amplitude of the oscillations of $P_{13}$ are unity as one would expect for the case of a resonant FM. Again, we are able to use the analytical theory to predict the amplitude and wavenumber of the oscillations of $P_{13}$ and this prediction is also shown in figure (2.7). The theory predicts the amplitude to be 100% and the wavenumber is correct to within $\sim 20\%$. 
2.5 Conclusions

In this paper we have analyzed how a three-flavour neutrino evolves when subject to density fluctuations composed anharmonic FMs. We found effects in these calculations which are not possible with just two neutrino flavours. Using theory based upon the Rotating Wave Approximation, we are able to predict the amplitude and wavenumber of the neutrino transition probabilities between pairs of states to within a few percent. As we varied the wavenumbers of the FMs we discovered the equivalent of electromagnetic induced transparency when the expected maximal oscillations between a given pair of neutrino states could be switched off by the presence of a second resonant FM between another pair of states. When we added a third FM we found the neutrino flavour evolution could be further controlled. At one value for the wavenumber the third FM was able to complete the transparency induced by the second but, at another value, we found the opacity could be restored.

While these cases are non-trivial and produced interesting results, ultimately our goal is to show how more complex cases such as neutrino evolution through turbulence and neutrino self-interaction can be understood in a similar framework. We showed how case of turbulence can be approached by presenting two calculations which were chosen to show the presence of Induced Transparency. We found the analytical theory predicted amplitudes for the transition probabilities through turbulence which were a good match to the numerical solutions and one could therefore understand why the neutrino could respond so differently to what appear to be two, almost identical, instances of turbulence.

The transition to neutrino self-interaction is not as simple because the Hamiltonian is itself a function of the evolution [5, 6]. Thus the approach one would need to take is to take the general solution, insert the solution into the actual self-interaction Hamiltonian and demand self-consistency. It is our intention to undertake this application in future work.

2.6 Appendix A: Stimulated Transformation

The problem of the flavour oscillations of a neutrino propagating through a fluctuating medium is a specific case of the more general quantum mechanical problem, namely the time evolution of a multi-level quantum system given a time-dependent perturbation. Determining the response of a quantum mechanical system to a time-dependent perturbation is a frequent endeavor of both experiment and theory in a large number of subfields of physics. A number of phenomena have been found to occur in quantum optics, in electronic spin and nuclear magnetic resonance, and in ultracold atoms and molecules to name just a few. Many reviews of driven quantum systems can be found e.g. Cohen-Tannoudji [33]. There are several analytic approaches to the calculation of the transition probability between the states of the system in textbook literature with the various techniques
We now write the evolution matrix in the unperturbed eigenbasis as the product

\[ H(X) \]

Thus, in order to facilitate applications to other quantum systems, the theory is in the most general circumstances where knowing the evolution from one basis to a different basis is useful. Note also that throughout this paper we set \( \hbar = c = 1 \).

At some initial time \( t_1 \) we prepare the system in some arbitrary state - represented by a column vector - which we decompose in terms of the \( N \) states of some basis \( (X) \). The system then evolves to a time \( t_2 \) at which we decompose the state in terms of the \( N \) states of a possibly different basis \( (Y) \). The evolution is described by a matrix \( S^{(XY)}(t_2, t_1) \) and the transition probabilities are the set of probabilities that the system in a given initial state \( x \) of \( (X) \) at \( t_1 \) is detected in the state \( y \) of \( (Y) \) at \( t_2 \). These transition probabilities are denoted by \( P^{(XY)}_{y x}(t_2, t_1) \) and are related to the elements of \( S^{(XY)} \) by

\[ P^{(XY)}_{y x} = |S^{(XY)}_{x y}|^2. \]

Since \( S^{(XY)} \) must be unitary, one needs \( N^2 \) independent real parameters in order to describe the matrix \( S^{(XY)} \) but note only \((N - 1)^2\) of the elements of \( P^{(XY)} \) are independent. Hereafter we shall work with the case where the bases \( (X) \) and \( (Y) \) are the same although there are certainly circumstances where knowing the evolution from one basis to a different basis is useful. Note also that throughout this paper we set \( \hbar = c = 1 \).

In the generic basis \( (X) \) the evolution matrix can be found by solving the Schrödinger equation

\[ \frac{dS^{(XX)}}{dt} = H^{(X)} S^{(XX)} \]  

(2.19)

where \( H^{(X)} \) is the Hamiltonian in the basis \( (X) \). The initial condition is \( S^{(XX)}(t_1, t_1) = 1 \). We make no assumption about the structure of \( H^{(X)} \) except that it be possible to separate the Hamiltonian into an unperturbed piece \( \tilde{H}^{(X)}(t) \) and a position dependent perturbation \( \delta H^{(X)}(t) \) i.e. \( H^{(X)}(t) = \tilde{H}^{(X)}(t) + \delta H^{(X)}(t) \).

If \( \tilde{H}^{(X)}(t) \) is not diagonal then we introduce an instantaneous unperturbed eigenbasis \( (u) \) by finding the unitary matrix \( \tilde{U}(t) \) defined by \( \tilde{H}^{(X)} = \tilde{U} K \tilde{U}^\dagger \) where \( K \) is the diagonal matrix of the eigenvalues of \( \tilde{H} \), that is \( K = \text{diag}(k_1, k_2, \ldots) \). The evolution matrix in the instantaneous unperturbed eigenbasis is related to the evolution \( S^{(XX)} \) by \( S^{(uu)}(t_2, t_1) = \tilde{U}^\dagger(t_2) S^{(XX)}(t_2, t_1) \tilde{U}(t_1) \). In this unperturbed eigenbasis

\[ H^{(u)} = K - i \tilde{U}^\dagger \frac{d\tilde{U}}{dt} + \tilde{U}^\dagger \delta H^{(X)} \tilde{U} \]

(2.20)

We now write the evolution matrix in the unperturbed eigenbasis as the product \( S^{(uu)} = \tilde{S} A \) where \( \tilde{S} \) having strengths and weaknesses depending upon the form of the perturbation. Even when we restrict our attention to harmonic perturbations, one may compute the transition probability (or transition rate) between states using Floquet theory [34], the Rotating Wave Approximation (RWA) [35], or Fermi’s Golden Rule. A comparison between these techniques for a two-level system can be found in Dion & Hirschfelder [36]. The approach we adopt to solve this problem is to generalize the method found in Patton, Kneller & McLaughlin [15] to an arbitrary number of neutrino flavours and arbitrary, but Fourier decomposed, density fluctuations. While the applications of our solution found in this paper are neutrino related, we recognize the generality of this problem to other fields.

At some initial time \( t_1 \) we prepare the system in some arbitrary state - represented by a column vector - which we decompose in terms of the \( N \) states of some basis \( (X) \). The system then evolves to a time \( t_2 \) at which we decompose the state in terms of the \( N \) states of a possibly different basis \( (Y) \). The evolution is described by a matrix \( S^{(XY)}(t_2, t_1) \) and the transition probabilities are the set of probabilities that the system in a given initial state \( x \) of \( (X) \) at \( t_1 \) is detected in the state \( y \) of \( (Y) \) at \( t_2 \). These transition probabilities are denoted by \( P^{(XY)}_{y x}(t_2, t_1) \) and are related to the elements of \( S^{(XY)} \) by \( P^{(XY)}_{y x} = |S^{(XY)}_{x y}|^2. \) Since \( S^{(XY)} \) must be unitary, one needs \( N^2 \) independent real parameters in order to describe the matrix \( S^{(XY)} \) but note only \((N - 1)^2\) of the elements of \( P^{(XY)} \) are independent. Hereafter we shall work with the case where the bases \( (X) \) and \( (Y) \) are the same although there are certainly circumstances where knowing the evolution from one basis to a different basis is useful. Note also that throughout this paper we set \( \hbar = c = 1 \).

In the generic basis \( (X) \) the evolution matrix can be found by solving the Schrödinger equation

\[ \frac{dS^{(XX)}}{dt} = H^{(X)} S^{(XX)} \]  

(2.19)

where \( H^{(X)} \) is the Hamiltonian in the basis \( (X) \). The initial condition is \( S^{(XX)}(t_1, t_1) = 1 \). We make no assumption about the structure of \( H^{(X)} \) except that it be possible to separate the Hamiltonian into an unperturbed piece \( \tilde{H}^{(X)}(t) \) and a position dependent perturbation \( \delta H^{(X)}(t) \) i.e. \( H^{(X)}(t) = \tilde{H}^{(X)}(t) + \delta H^{(X)}(t) \).

If \( \tilde{H}^{(X)}(t) \) is not diagonal then we introduce an instantaneous unperturbed eigenbasis \( (u) \) by finding the unitary matrix \( \tilde{U}(t) \) defined by \( \tilde{H}^{(X)} = \tilde{U} K \tilde{U}^\dagger \) where \( K \) is the diagonal matrix of the eigenvalues of \( \tilde{H} \), that is \( K = \text{diag}(k_1, k_2, \ldots) \). The evolution matrix in the instantaneous unperturbed eigenbasis is related to the evolution \( S^{(XX)} \) by \( S^{(uu)}(t_2, t_1) = \tilde{U}^\dagger(t_2) S^{(XX)}(t_2, t_1) \tilde{U}(t_1) \). In this unperturbed eigenbasis

\[ H^{(u)} = K - i \tilde{U}^\dagger \frac{d\tilde{U}}{dt} + \tilde{U}^\dagger \delta H^{(X)} \tilde{U} \]  

(2.20)

We now write the evolution matrix in the unperturbed eigenbasis as the product \( S^{(uu)} = \tilde{S} A \) where \( \tilde{S} \)
is defined to be the solution of
\[
\frac{d\tilde{S}}{dt} = \left[ K - i\tilde{U}^\dagger \frac{d\tilde{U}}{dt} \right] \tilde{S}.
\] (2.21)

If we know the solution to the unperturbed problem, \(\tilde{S}\), we can solve for the effect of the perturbation by finding the solution to the differential equation for \(A\):
\[
\frac{dA}{dt} = \tilde{S}^\dagger \hat{U}^\dagger \delta H^{(X)} \hat{U} \tilde{S} A.
\] (2.22)

In general the term \(\hat{U}^\dagger \delta H^{(X)} \hat{U}\) which appears in this equation possesses both diagonal and off-diagonal elements. The diagonal elements are easily removed by writing the matrix \(A\) as \(A = W B\) where \(W = \exp(-i\Xi)\) and \(\Xi\) a diagonal matrix \(\Xi = \text{diag}(\xi_1, \xi_2, \ldots)\). Substitution into (3.8) gives a differential equation for \(B\)
\[
\frac{dB}{dt} = W^\dagger \left[ \tilde{S}^\dagger \hat{U}^\dagger \delta H^{(X)} \hat{U} \tilde{S} - \frac{d\Xi}{dt} \right] WB \equiv H^{(B)} B
\] (2.23)

and \(\Xi\) is chosen so that \(d\Xi/dt\) removes the diagonal elements of \(\tilde{S}^\dagger \hat{U}^\dagger \delta H^{(X)} \hat{U} \tilde{S}\). Once \(\Xi\) has been found, determining transition probabilities is reduced to solving for the \(B\) matrix.

### 2.6.1 Fourier-decomposed Perturbations

We now consider the specific case of a constant potential for \(\tilde{H}^{(X)}\). This form for \(\tilde{H}\) means \(\tilde{S}\) is a diagonal matrix \(\tilde{S} = \exp(-iK t)\). The perturbation \(\delta \tilde{H}\) is taken to be a Fourier-like series of the form
\[
\delta \tilde{H}^{(X)} = \sum_a \left( C_a e^{i q_a t} + C_a^\dagger e^{-i q_a t} \right)
\] (2.24)

where \(C_a\) is an arbitrary complex matrix and \(q_a\) the frequency of the \(a^{th}\) FM of the perturbation. We make no restriction on the number of FMs, the frequencies \(q_a\) nor the size or structure of the matrices \(C_a\). This generalization to arbitrary structure for the \(C_a\)’s is where we depart from previous analyses by PKM [15]. We also refer the reader to Brown, Meath & Tran [20] and Avetissian, Avchyan & Mkrtchian [21] who considered the related but simpler problem of the effect of two lasers of different colors, i.e. two FMs, upon a two-level dipolar molecule.

Given this form for the perturbation, equation (3.9) indicates we need to consider the combination \(\hat{U}^\dagger C_a \hat{U}\). If we write the diagonal elements of \(\hat{U}^\dagger C_a \hat{U}\) as
\[
\text{diag}(\hat{U}^\dagger C_a \hat{U}) = \frac{F_a}{2 \lambda} \exp(i\Phi_a)
\] (2.25)

where \(F_a\) is a diagonal matrix of amplitudes \(F_a = \text{diag}(f_{a;1}, f_{a;2}, \ldots)\) and \(\Phi_a\) the diagonal matrix of
phases \( \Phi_a = \text{diag}(\Phi_{a1}, \Phi_{a2}, \ldots) \), then the matrix \( \Xi \) is found to be \( \Xi(t) = \sum_a \Xi_a(t) \) with
\[
\Xi_a(t) = \frac{F_a}{q_a} \left[ \cos \Phi_a - \cos(\Phi_a + q_a t) \right].
\] (2.26)

We denote the diagonal elements of \( \Xi_a \) as \( \xi_{a1}, \xi_{a2}, \ldots \). Next we rewrite the off-diagonal elements of \( \tilde{U}^\dagger C_a \tilde{U} \) as a matrix \( G_a \) i.e \( G_a = \text{offdiag}(\tilde{U}^\dagger C_a \tilde{U}) \). Putting together the solution for \( \Xi \) and \( \tilde{S} \) and inserting the new matrix \( G_a \), we find the Hamiltonian for \( B \) is
\[
H(B) = \exp(i\Xi)\exp(iKt) \left( \sum_a [G_a e^{i\eta_a t} + G_a^\dagger e^{-i\eta_a t}] \right) \exp(-iKt)\exp(-i\Xi)
\] (2.27)

Written explicitly the element \( i\ j \) of the Hamiltonian is
\[
H(B)_{ij} = e^{i\delta k_{ij} t + i\delta \xi_{ij}} \sum_a \left[ G_{a\;ij} e^{i\eta_a t} + G_{a\;ji}^\dagger e^{-i\eta_a t} \right]
\] (2.28)

where \( \delta k_{ij} = k_i - k_j \) and \( \delta \xi_{ij} = \xi_i - \xi_j \).

If we define
\[
\begin{align*}
x_{a\;ij} &= \frac{f_{a\;i}}{q_a} \cos \Phi_{a\;i} - \frac{f_{a\;j}}{q_a} \cos \Phi_{a\;j} \\
y_{a\;ij} &= \frac{f_{a\;i}}{q_a} \sin \Phi_{a\;i} - \frac{f_{a\;j}}{q_a} \sin \Phi_{a\;j} \\
z_{a\;ij} &= \sqrt{x_{a\;ij}^2 + y_{a\;ij}^2} \\
\psi_{a\;ij} &= \arctan \left( \frac{y_{a\;ij}}{x_{a\;ij}} \right)
\end{align*}
\] (2.29-2.32)

then the term \( \delta \xi_{ij} \) is equal to
\[
\delta \xi_{ij} = \sum_a \left[ x_{a\;ij} - z_{a\;ij} \cos(q_a t + \psi_{a\;ij}) \right].
\] (2.33)

The presence of \( y_{a\;ij} \) in these equations is a new feature of the more general perturbing Hamiltonian we are considering. We now make use of the Jacobi-Anger expansion for \( e^{i\delta \xi_{ij}} \)
\[
e^{i\delta \xi_{ij}} = \prod_a \left\{ e^{ix_{a\;ij}} \sum_{m_a = -\infty}^{\infty} (-i)^{m_a} J_{m_a}(z_{a\;ij}) \exp[i m_a(q_a t + \psi_{a\;ij})] \right\}.
\] (2.34)

If we substitute this expansion into the expression for the elements of \( H(B) \) and define \( \mu_{a\;m_a\;ij} \) and \( \lambda_{a\;m_a\;ij} \) to be
\[
\lambda_{a\;m_a\;ij} = (-i)^{m_a} e^{ix_{a\;ij}} J_{m_a}(z_{a\;ij}) e^{im_a \psi_{a\;ij}}
\] (2.35)
\[ \mu_{a,m_i;i} = (-i)^{m_a} e^{i\zeta_{a;i}} \left[ G_{a;i}^* J_{m_a+1}(z_{a;i}) e^{i(m_a+1)\psi_{a;i}} - G_{a;i} J_{m_a-1}(z_{a;i}) e^{i(m_a-1)\psi_{a;i}} \right] \] (2.36)

then we find the element \( i j \) of the Hamiltonian is given by

\[ H_{ij}^{(B)} = i \sum_a \left\{ \sum_{m_a} \mu_{a,m_i;ij} e^{i(m_a q_a + \delta k_{ij}) t \prod_{b \neq a} \sum_{m_b} \lambda_{b,m_b;ij} e^{im_b q_b t}} \right\} \]. (2.37)

### 2.6.2 Rotating Wave Approximation

Even though we started with a very general perturbing Hamiltonian, we have found a form for \( H^{(B)} \) which has the same structure as that found by PKM. From here on, we follow the same procedure to solve for the matrix \( B \). First we adopt the Rotating Wave Approximation. The RWA amounts to selecting a particular value for the integers \( m_a \) and \( m_b \) in equation (3.22) and dropping all others.

We do not specify a procedure for selecting those integers though algorithms exist. We expect there is not one procedure that can be adopted universally for all situations. There are some restrictions to be placed on the selection of the integers. In order that the resulting Hamiltonian be solvable we cannot make choices for \( m_a \) and \( m_b \) for every element \( i j \) independently. Only \( N - 1 \) elements are to be regarded as independent and a suitable set could be either those on the sub/superdiagonal or the off-diagonal elements in a particular row or column. The values of \( m_a \) and \( m_b \) we select will be different for each independent element. We denote these integers by \( n_{a;ij} \) since they are specific both to the frequency \( a \) and the element of the Hamiltonian \( i j \), and define

\[ \kappa_{ij} = \sum_a \mu_{a,n_{a;ij};ij} \prod_{b \neq a} \lambda_{b,n_{b;ij};ij} \] (2.38)

then \( H_{ij}^{(B)} \) is simplified to

\[ H_{ij}^{(B)} = -i \kappa_{ij} \exp \left[ i \left( \sum_a n_{a;ij} q_a + \delta k_{ij} \right) t \right], \] (2.39)

or, in full matrix form

\[
H^{(B)} = \begin{pmatrix}
0 & -i \kappa_{12} e^{-i \delta k_{12} + i \sum_a n_{a,12} q_a t} & -i \kappa_{13} e^{-i \delta k_{13} + i \sum_a n_{a,13} q_a t} & \cdots \\
-i \kappa_{12} e^{i \delta k_{12} + i \sum_a n_{a,12} q_a t} & 0 & -i \kappa_{23} e^{i \delta k_{23} + i \sum_a n_{a,23} q_a t} & \cdots \\
-i \kappa_{13} e^{i \delta k_{13} + i \sum_a n_{a,13} q_a t} & -i \kappa_{23} e^{i \delta k_{23} + i \sum_a n_{a,23} q_a t} & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \] (2.40)
Again, we remind the reader that, for example, only $n_{a;12}$, $n_{a;23}$, $n_{a;34}$ etc. are independent: in all other cases the integer $n_{a;ij} = n_{a;il} + n_{a;jl}$. As shown by PKM, with this simplified Hamiltonian, equation (3.9), can be solved for the evolution matrix $B$ and we reproduce their solution here for completeness. Since both $n_{a;ij} = n_{a;il} + n_{a;jl}$ and $\delta k_{ij} = \delta k_{il} + \delta k_{lj}$, we can factorize $H^{(B)}(t)$ into the form $H^{(B)}(t) = T(t)M T^\dagger(t)$ where the matrix $M$ is a constant, i.e. it contains the couplings $\kappa_{ij}$ only. The matrix $T$ is of the form $T(t) = \exp(i \Lambda t)$, where $\Lambda$ is also a constant matrix that depends only on $\delta k_{ij}$, the integer sets $\{n_{a;ij}\}$ and the frequencies $q_a$. Explicitly we can write

$$M = \begin{pmatrix} 0 & -i\kappa_{12} & -i\kappa_{13} & \cdots \\ i\kappa_{12}^* & 0 & -i\kappa_{23} & \cdots \\ i\kappa_{13}^* & i\kappa_{23}^* & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2.41)$$

and one possible choice for the matrix $\Lambda$ is

$$\Lambda = \begin{pmatrix} k_1 + \sum_a n_{a;1} q_a & 0 & 0 & \cdots \\ 0 & k_2 + \sum_a n_{a;2} q_a & 0 & \cdots \\ 0 & 0 & k_3 + \sum_a n_{a;3} q_a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2.42)$$

where $n_{a;i}$ are integers chosen so that $n_{a;i} - n_{a;ij} = n_{a;ij}$. Using this factorization of $H^{(B)}(t)$ we find equation (3.9) can be rewritten as

$$i T^\dagger(t) \frac{dB}{dt} = M T^\dagger(t) B \quad (2.43)$$

Instead of solving for $B$ we solve for the combination $\Omega = T^\dagger B$. The differential equation for $\Omega$ is found to be

$$i \frac{d\Omega}{dt} = (M + \Lambda) \Omega = H^{(\Omega)} \Omega. \quad (2.44)$$

Since the both $M$ and $\Lambda$ are constant matrices, the matrix $H^{(\Omega)}$ is also independent of $t$ meaning $\Omega$ has the formal solution $\Omega(t) = \exp(-iH^{(\Omega)}t)\Omega(0)$. The solution for $B$ is thus

$$B(t) = T(t) \exp(-iH^{(\Omega)}t) T^\dagger(0) B(0). \quad (2.45)$$

Now that we have the solution for $B$, the full evolution matrix in the basis $(u)$ is $S = \tilde{S} W B$ but given that both $\tilde{S}$ and $W$ are diagonal matrices, the transition probability between certain unperturbed eigenstates is simply the square magnitude of the corresponding off-diagonal element of $B$. 

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2.7 Appendix B: Fourier modes in turbulence

In this appendix we list in table (2.1) the wavenumbers and amplitudes of the 40 Fourier modes used to generate the turbulence in section 2.4. To plot figure 2.6 the 2 boxed wavenumbers in the list are replaced by the resonant frequencies $q_2 = k_3 - k_2 = 1.1681 \times 10^{-5}$ and $q_1 = k_3 - k_1 = 1.2036 \times 10^{-5}$ respectively, with their amplitudes unchanged. To plot figure 2.7 we further set the amplitudes of mode $q_2$ to zero.
Figure 2.2 The transition probabilities from unperturbed eigenstate 1 to unperturbed eigenstate 3. The blue dashed line is the numerical result for the case $A_1 = 0.1, A_2 = 0$ and the red dashed line for $A_1 = 0.1, A_2 = 0.5$. The symbols represent the corresponding analytic results.
Figure 2.3 The amplitude of $P_{13}$ as a function of $q_1$. The parameters used are $V_\star = 6 \times 10^{-25} \text{ erg}$, $A_1 = 0.1$, and $A_2 = 0/0.1/0.2/0.5$ for the black/blue/green/red dashed line and symbols, $A_2 = 0$ for the blue dashed line and symbols. The symbols represent numerical results, while the dashed lines are from analytic evaluation.
Figure 2.4 The probability $P_{13}$ as a function of $r$ for a three flavour neutrino model. The wavenumbers $q_1$ and $q_2$ are set to $q_1 = k_3 - k_1$ and $q_2 = k_3 - k_2$ with amplitudes $A_1 = 0.02$ and $A_2 = 0.1$. The third wavenumber is $q_3 = 5.24 \times 10^{-10} \text{cm}^{-1}$. The dashed lines are the numerical solutions, the triangle symbols are the analytic prediction. The blue curve is for the case $A_3 = 0$ and produces an example of Induced Transparency. The red curve is for $A_3 = 0.2$ and produces an example of Restored Opacity.
Figure 2.5 The amplitude of $P_{13}$ as a function of the wavenumber $q_3$. The potential $V^* = 6 \times 10^{-25}$ erg and the wavenumbers $q_1$ and $q_2$ are set to $q_1 = k_3 - k_1$ and $q_2 = k_3 - k_2$ with amplitudes $A_1 = 0.002$ and $A_2 = 0.01$. The amplitude of the third FM $q_3$ is $A_3 = 0.02$. The black solid line is from analytic evaluation and the thick red line is from the numerical solutions with the thickness of the band indicating the width of the fluctuations, an example of which is shown in figure (2.4). The values of $q_3$ which give Complete Transparency and Restored Opacity are indicated.

Figure 2.6 Left panel: The turbulence profile with 40 Fourier modes. Right panel: Numerical (red solid) and analytical (black dotted) transition probability with induced transparency.
Figure 2.7 Left panel: The same turbulence profile with the suppressive mode removed. Right panel: Numerical (red solid) and analytical (black dotted) transition probability without induced transparency.
Table 2.1 The 40 Fourier modes used to generate the turbulence.

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3.1 Abstract

The neutrino self-interaction problem has been recently re-evaluated using an effective Hamiltonian. In this treatment of the problem, the background neutrinos are considered separately and it is assumed that the flavor evolution of a particular neutrino will not alter its own evolution. In previous work we have established a method that can take any Fourier-decomposed Hamiltonian as input and analytically calculate the flavor evolution of a neutrino under such a Hamiltonian by using Rotating Wave Approximation (RWA). In this paper we combine the two approaches by constructing an effective Hamiltonian from a numerically calculated self-interaction and show how our method can give analytical results which match the amplitude and wavenumber of the numerical results to within a few percent.
3.2 Introduction

The flavor evolution of a free-streaming neutrino can be treated using a Schrödinger equation governed by a Hamiltonian. In environments with sufficiently large neutrino densities, such as core-collapse supernovae and the early Universe, the Hamiltonian is composed of three terms: the vacuum $H_V$, a matter term $H_M$, and a self-interaction $H_{SI}$. The effect of the first two contributions alone is well understood. The effect of neutrino self-interactions is much more complex because $H_{SI}$ depends upon the evolution matrix one is trying to compute. This dependency makes the problem non-linear. Furthermore, in cases such as supernovae where there is a net neutrino current, one also has to include the effect that the weak interaction between two neutrinos depends upon the factor $1 - \cos \theta$ where $\theta$ is the angle between the neutrino trajectories. The numerical solutions for the flavor evolution including self-interaction have revealed an array of interesting behavior - see [20, 21] for reviews. Although no analytic solutions for the self-interaction problem in the early Universe nor supernovae are known, some important features of the system which determine its fate have been found. For example, the essential difference between neutrino evolution with self-interactions and neutrino evolution through matter alone were pointed out in [23]. Also much recent effort has gone into ‘stability analysis’ of the neutrino self-interaction problem which has revealed the conditions that lead to large neutrino flavor oscillations, if not exactly what those oscillations look like [24, 25, 26].

Analytical solutions for the self-interaction problem only exist for simplified models. Though missing many important facets of the problem found in supernovae and the early Universe, these models have proved useful in understanding the phenomenology of the flavor evolution. For example, the models considered in Hannestad et al. [27] are able to describe the neutrino flavor evolutions by analogy with motions of a pendulum in the neutrino flavor isospin space. A recent study by Hansen and Smirnov [28] has approached the problem of neutrino self-interaction by studying the case of a single probe neutrino passing through a beam of colinear neutrinos. They showed how this model also has an analytical solution and used the solution to explain how the time/space dependence of the flavor oscillations in the beam can induce large flavor transitions in evolution of the probe neutrino.

In our previous work, a technique for solving the flavor evolution of a neutrino given an arbitrary, Fourier-decomposed Hamiltonian has been developed [29]. The technique is based upon time-dependent perturbation theory and can be applied to the self-interaction problem if we treat the self-interaction as the perturbation. In this paper we combine the approach of an effective Hamiltonian for the self-interaction and the general solution of a Fourier-decomposed perturbation in order to take the first step of providing another avenue for solving neutrino self-interaction problems analytically. We find the analytical solutions of the evolution turn out to give good matches to the amplitude and wavelength of the neutrino oscillations giving us confidence for further developing
the effective Hamiltonian approach so that it can be applied to realistic problems such as neutrino flavor evolution in supernova environments.

3.2.1 Fourier Decomposition of the Neutrino Self-Interaction Problem

Let us describe the neutrino self-interaction problem we wish to study. At some initial position \( r_1 \) we prepare the neutrino in some arbitrary state - represented by a column vector. The neutrino then propagates to a position \( r_2 \). The evolution is described by a matrix \( S(r_2, r_1) \) and the transition probabilities are the set of probabilities that the system in a given initial state \( x \) at \( r_1 \) is detected in the state \( y \) at \( r_2 \). These transition probabilities are denoted by \( P_y x = |S_{yx}|^2 \). The evolution matrix can be found by solving the Schrödinger equation

\[
\frac{dS}{dr} = HS
\]  

where \( H \) is the Hamiltonian. In most case one wishes to study the Hamiltonian \( H \) governing the neutrino flavor evolution through matter is the sum of a constant vacuum term \( H_V \) and a term coming from the effect of matter \( H_M \) [13, 14], and the self interaction \( H_{SI} \). However for the purposes of this paper we shall ignore the matter contribution which can be included if necessary by adjusting the parameters which are used to define the vacuum Hamiltonian. In what follows we treat the vacuum as an unperturbed Hamiltonian and the self-interaction as the perturbation. The vacuum Hamiltonian for two neutrino flavors can be written as

\[
H_V = \frac{m_1^2 - m_2^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}
\]  

(3.2)

The ‘mass splitting’ is take to be \( m_1^2 - m_2^2 = -2.43 \times 10^{-3} \text{ eV}^2 \) and we adopt a vacuum mixing angle of \( \theta = 1^\circ \). The energy of the neutrino and antineutrino are set to \( E = 20 \text{ MeV} \).

The perturbation is the neutrino self-interactions \( H_{SI} \) [22]. To avoid the complicating factors of supernovae and early Universe neutrinos, we choose to study the simplified self-interaction problem presented in Hannestad et al. [27] which considered a monoenergetic neutrino and antineutrino in vacuum only with a constant self-interaction strength \( \mu \). This is a well studied model that also forms the basis for the models used to study the matter-neutrino resonance [30, 31, 32, 33]. The self-interaction Hamiltonian for the neutrinos in the flavor basis for this model is

\[
H_{SI} = \mu [\rho(r) - \alpha \tilde{\rho}^*(r)]
\]

\[
= \mu [S\rho(0)S^\dagger - \alpha (\tilde{S}\tilde{\rho}(0)\tilde{S}^\dagger)^*)]
\]  

(3.3)

where \( \rho(r) \) and \( \tilde{\rho}(r) \) are the density matrices of the neutrinos and antineutrinos respectively at position \( r \), \( S \) and \( \tilde{S} \) the evolution matrices. For the antineutrinos the self-interaction Hamiltonian,
\( \hat{H}_{SI} \) is related to that of the neutrinos by \( \hat{H}_{SI} = -\hat{H}_{SI}^* \). The parameter \( \alpha \) sets the asymmetry between the neutrinos and antineutrinos and we shall assume the density matrices at the initial point are 100% electron neutrino and electron antineutrino flavor. For definiteness, in what follows we set \( \mu = 1.682 \times 10^{-21} \) erg.

### 3.2.2 The symmetric case

Let us first look at the symmetric case, \( \alpha = 1 \), in which neutrinos and antineutrinos appear in equal quantities. The self-interaction potential we find numerically is shown by the dashed lines in Fig. (3.1). Only the imaginary parts of the off-diagonal entries of \( H_{SI} \) are non-zero which has the consequence that the antineutrino self-interaction Hamiltonian is equal to the neutrino self-interaction Hamiltonian. Thus the antineutrinos evolve exactly the same way as the neutrinos in this particular case. The figure also shows the self-interaction Hamiltonian is clearly periodic though not sinusoidal. We compute the Fourier decomposition of the Hamiltonian to find the modes present and their amplitudes and find the fundamental frequency is \( q_1 = 3.28 \times 10^{-7} \) cm\(^{-1} \) with an amplitude of \( A_1 = 2.97 \times 10^{-22} \) erg for the imaginary component of the off-diagonal element. The relative amplitudes of the other modes are shown in Fig. (3.2) and it is interesting to note that only the odd modes are present in the spectrum.

From the numerical solution we find that the neutrino self-interaction is of the form

\[
H_{SI} = \sum_a \left( C_a e^{i q_a r} + C_a^* e^{-i q_a r} \right) \tag{3.4}
\]

where \( C_a \) is an arbitrary complex matrix and \( q_a \) the wavenumber of the \( a \)'th Fourier mode. The sum in this series extends to infinity so let us determine how many Fourier modes need to be kept. We can do this by directly trying to reproduce the potential with different numbers of modes and checking how close we are able to recover the Hamiltonian obtained numerically. Fig. (3.1) shows the comparison of the reproduced potentials with different number of modes. We see that for the symmetric case, \( \alpha = 1 \), just five modes are needed, i.e. the modes with 1, 3, 5, 7 and 9 times the fundamental wavenumber \( q_1 \), are sufficient. We have verified this conclusion further by rewriting our self-interaction code to use the Fourier decomposed \( H_{SI} \) as an external potential and found the evolution matrix we obtain is the same as from the self-interaction code.

### 3.2.3 The asymmetric case

Let us now switch to a more demanding example of an asymmetric case with \( \alpha = 0.8 \). Our procedure is the same as in the symmetric case above but for \( \alpha \neq 1 \) we find the self-interaction Hamiltonian is now much more complicated with oscillatory behavior in every element of the matrix, as shown in Fig. (3.4). Nevertheless, the Hamiltonian is again seen to be periodic with a fundamental wavenumber of
$q_i = 5.19 \times 10^{-6}$ cm$^{-1}$. An analysis finds that all entries have essentially the same Fourier spectrum. The relative amplitudes of the modes for the real part of the off-diagonal element of the Hamiltonian are shown in Fig. (3.5). The amplitude of the modes have been normalized to the amplitude of the largest which has a wavenumber 25 times the fundamental. The seven modes with the largest amplitudes correspond to wavenumbers which are 4, 25, 33, 54, 62, 83, and 91 times the fundamental frequency. It is interesting to notice that the fundamental is not among these seven important modes and also to notice the pattern that the splitting between adjacent modes is 21, 8, 21, 8, . . . times the fundamental wavenumber. As before we can determine the number of modes we need to keep by comparing the Fourier decomposed Hamiltonian with the numerical solution. The comparison is shown in Fig. (3.6) where we see we need the seven largest amplitude modes shown in Fig. (3.4) in order to reproduce the real part of the off-diagonal element of the Hamiltonian.

### 3.3 Analytical Solutions

Given the periodic nature of the self-interaction Hamiltonian we can derive approximate analytical solutions if we consider an effective Hamiltonian of the form

$$H_{SI} = \sum_a (C_a e^{i q a} + \bar{C}_a e^{-i q a})$$

(3.5)

These solutions are shown in figures (3.3), (3.7) and (3.8) compared to the numerical solutions. Let us explain how the analytical solutions were found. In the mass basis, the full Hamiltonian is

$$H^{(m)} = K + U^\dagger H_{SI} U$$

(3.6)

We now write the evolution matrix in the mass basis as the product $S^{(m)} = \tilde{S} A$ where $\tilde{S}$ is defined to be the solution of

$$i \frac{d \tilde{S}}{dr} = K \tilde{S}.$$  

(3.7)

which has the solution $\tilde{S} = \exp(-i K r)$. Given this solution, we can solve for the effect of the self-interaction by finding the solution to the differential equation for $A$:

$$i \frac{d A}{dr} = \tilde{S}^\dagger U^\dagger H_{SI} U \tilde{S} A.$$  

(3.8)

In general the term $U^\dagger H_{SI} U$ which appears in this equation possesses both diagonal and off-diagonal elements. The diagonal elements are easily removed by writing the matrix $A$ as $A = W B$ where $W = \exp(-i \Xi)$ and $\Xi$ a diagonal matrix $\Xi = \text{diag}(\xi_1, \xi_2)$. Substitution into (3.8) gives a differ-
ential equation for $B$

$$\frac{dB}{dr} = W^\dagger \left[ \tilde{S}^\dagger U H_{SI} U \tilde{S} - \frac{d\Xi}{dr} \right] W B \equiv H^{(B)} B \tag{3.9}$$

and $\Xi$ is chosen so that $d\Xi/d\, r$ removes the diagonal elements of $\tilde{S}^\dagger U H_{SI} U \tilde{S}$. Given this form for the self-interaction, equation (3.9) indicates that in order to solve for $\Xi$ we need to consider the combination $U^\dagger C_a U$. If we write the diagonal elements of $U^\dagger C_a U$ as

$$\text{diag}(U^\dagger C_a U) = \frac{F_a}{2i} \exp(i\Phi_a), \tag{3.10}$$

where $F_a$ is a diagonal matrix of amplitudes $F_a = \text{diag}(f_{a;1}, f_{a;2})$ and $\Phi_a$ the diagonal matrix of phases $\Phi_a = \text{diag}(\phi_{a;1}, \phi_{a;2})$, then the matrix $\Xi$ is found to be $\Xi(r) = \sum_a \Xi_a(r)$ with

$$\Xi_a(r) = \frac{F_a}{q_a} \left[ \cos\Phi_a - \cos(\Phi_a + q_a \, r) \right]. \tag{3.11}$$

We denote the diagonal elements of $\Xi_a$ as $\xi_{a;1}, \xi_{a;2}$. Next we rewrite the off-diagonal elements of $U^\dagger C_a U$ as a matrix $G_a$ i.e $G_a = \text{offdiag}(U^\dagger C_a U)$. Putting together the solution for $\Xi$ and $\tilde{S}$ and inserting the new matrix $G_a$, we find the Hamiltonian for $B$ is

$$H^{(R)} = \exp(t\ Xi) \exp(t\ K \, r) \left( \sum_a \left[ G_a e^{i\xi_{a;1} r} + G_a^* e^{-i\xi_{a;2} r} \right] \right) \exp(-t\ K \, r) \exp(-t\ Xi) \tag{3.12}$$

Written explicitly the element 12 of the Hamiltonian is

$$H^{(R)}_{12} = e^{i(\delta k_{12} r + \delta \xi_{12})} \sum_a \left[ G_{a;12} e^{i\xi_{a;1} r} + G_{a;21}^* e^{-i\xi_{a;2} r} \right] \tag{3.13}$$

where $\delta k_{12} = k_1 - k_2$ and $\delta \xi_{12} = \xi_1 - \xi_2$. If we define

$$x_a = \frac{f_{a;1}}{q_a} \cos\phi_{a;1} - \frac{f_{a;2}}{q_a} \cos\phi_{a;2} \tag{3.14}$$

$$y_a = \frac{f_{a;1}}{q_a} \sin\phi_{a;1} - \frac{f_{a;2}}{q_a} \sin\phi_{a;2} \tag{3.15}$$

$$z_a = \sqrt{x_a^2 + y_a^2} \tag{3.16}$$

$$\tan\psi_a = y_a/x_a \tag{3.17}$$

then the term $\delta \xi_{12}$ is equal to

$$\delta \xi_{12} = \sum_a \left[ x_a - z_a \cos(q_a \, r + \psi_a) \right] \tag{3.18}$$
We now make use of the Jacobi-Anger expansion for \( e^{i\delta \xi_{12}} \)

\[
e^{i\delta \xi_{12}} = \prod_a \left( e^{i x_a} \sum_{m_a=-\infty}^{\infty} (-1)^m J_m(z_a) \exp \left[ i m_a \left( q_a + \psi_a \right) \right] \right).
\]

(3.19)

If we substitute this expansion into the expression for the off-diagonal element of \( H^{(B)} \) and define \( \kappa_{a,m_a} \) and \( \lambda_{a,m_a} \) to be

\[
\lambda_{a,m_a} = (-1)^m e^{ix_a} J_m(z_a) e^{im_a \psi_a}
\]

(3.20)

\[
\kappa_{a,m_a} = (-1)^m e^{ix_a} \left[ G_{a;2i}^{*} J_{m_a+1}(z_a) e^{i(m_a+1)\psi_a} - G_{a} J_{m_a-1}(z_a) e^{i(m_a-1)\psi_a} \right]
\]

(3.21)

then the we find the off-diagonal element of the Hamiltonian \( H^{(B)} \) is given by

\[
H_{12}^{(B)} = \sum_a \left\{ \sum_{m_a} \kappa_{a,m_a} e^{i(m_a q_a + \delta \kappa_{12}) r} \prod_{b \neq a} \sum_{m_b} \lambda_{b,m_b} e^{i(m_b q_b r)} \right\}.
\]

(3.22)

### 3.3.1 Rotating Wave Approximation

Even though we started with a very general perturbing Hamiltonian, we have found a form for \( H^{(B)} \) which has the same structure as that found by Patton, Kneller & McLaughlin. From here on, we follow the same procedure to solve for the matrix \( B \). First we adopt the Rotating Wave Approximation. The RWA amounts to selecting a particular value for the integers \( m_a \) and \( m_b \) in Eq.(3.22) and dropping all others. We do not specify a procedure for selecting those integers though algorithms exist. We expect there is not one procedure that can be adopted universally for all situations. We call these chosen integers \( n_a \) and define

\[
\kappa = \sum_a \kappa_{n_a} \prod_{b \neq a} \lambda_{n_b}
\]

(3.23)

However one complication in our particular case occurs when the Fourier modes are such that the ratio between any pair of frequencies, \( q_a \) and \( q_b \), is a rational fraction. When this occurs there are multiple sets of the integers \( \{n\} \) which all have the same detuning frequency. One can account for these degenerate RWAs by simply adding them together so that \( \kappa \) becomes

\[
\kappa = \sum_{\{n\}} \left( \sum_a \kappa_{n_a} \prod_{b \neq a} \lambda_{n_b} \right).
\]

(3.24)

If all ratios between the frequencies are rational then accounting for the degeneracy in the RWA becomes an exercise in combinatorics. Even so, typically one finds the sum is dominated by one set - the one where the sum of the absolute values of the integers \( \{n\} \) is smallest.
After making the RWA we find that, in full matrix form,

$$H^{(B)} = \begin{pmatrix} 0 & -i \kappa e^{i \delta k_{12} + \sum_{a} n_a q_a} r \\ i \kappa^* e^{-i \delta k_{12} + \sum_{a} n_a q_a} r & 0 \end{pmatrix}$$  \quad (3.25)$$

Since both $n_a = n_a;1 + n_a;2$ and $\delta k_{12} = \delta k_{1\ell} + \delta k_{ij}$, we can factorize $H^{(B)}(r)$ into the form $H^{(B)}(r) = \Upsilon(r) M \Upsilon^\dagger(r)$ where the matrix $M$ is a constant, i.e. it contains the couplings $\kappa$ only. The matrix $\Upsilon$ is of the form $\Upsilon(r) = \exp(i \Lambda r)$, where $\Lambda$ is also a constant matrix that depends only on $\delta k_{12}$, the integer sets $\{n_a\}$ and the frequencies $q_a$. Explicitly we can write

$$M = \begin{pmatrix} 0 & -i \kappa \\ i \kappa^* & 0 \end{pmatrix},$$  \quad (3.26)$$

and one possible choice for the matrix $\Lambda$ is

$$\Lambda = \begin{pmatrix} k_1 + \sum_a n_a;1 q_a & 0 \\ 0 & k_2 + \sum_a n_a;2 q_a \end{pmatrix},$$  \quad (3.27)$$

where $n_a;1$ are integers chosen so that $n_a;1 - n_a;2 = n_a$. Using this factorization of $H^{(B)}(r)$ we find equation (3.9) can be rewritten as

$$i \Upsilon^\dagger \frac{dB}{dr} = M \Upsilon^\dagger B$$  \quad (3.28)$$

Instead of solving for $B$ we solve for the combination $\Omega = \Upsilon^\dagger B$. The differential equation for $\Omega$ is found to be

$$i \frac{d\Omega}{dr} = (M + \Lambda) \Omega = H^{(B)} \Omega.$$  \quad (3.29)$$

Since both the matrix $M$ and $\Lambda$ are constants, the matrix $H^{(B)}$ is also independent of $r$ meaning $\Omega$ has the formal solution $\Omega(r) = \exp(-i H^{(B)} r) \Omega(0)$. The solution for $B$ is thus

$$B(t) = \Upsilon(r) \exp(-i H^{(B)} r) \Upsilon^\dagger(0) B(0).$$  \quad (3.30)$$

Now that we have the solution for $B$, the full evolution matrix in the mass basis is $S = \tilde{S} W B$ but given that both $\tilde{S}$ and $W$ are diagonal matrices, the transition probability between the mass eigenstates is simply the square magnitude of the off-diagonal element of $B$. Note that we have made no restrictions on the number of Fourier modes, the frequencies nor the size and structure of the amplitude matrices. For the particular case of two flavors we can write out the solution succinctly by introducing the detuning frequency $p$ via $2p = \delta k_{12} + \sum_a n_a q_a$ and the Rabi flopping frequency $Q$
by $Q^2 = p^2 + \kappa^2$. Using these quantities, $B(r)$ for two flavors is found to be

$$
B = \begin{pmatrix}
        e^{i p r} \left[ \cos(Qr) - i \frac{p}{Q} \sin(Qr) \right] & -e^{i p r} \frac{\kappa}{Q} \sin(Qr) \\
        e^{-i p r} \frac{\kappa}{Q} \sin(Qr) & e^{-i p r} \left[ \cos(Qr) + i \frac{p}{Q} \sin(Qr) \right]
\end{pmatrix}.
$$

(3.31)

and so we see the transition probability between the mass states 1 and 2 is again

$$
P_{12} = |B_{12}|^2 = \frac{\kappa^2}{Q^2} \sin^2(Qr).
$$

(3.32)

### 3.3.2 The symmetric case

Let us apply the theory from the previous section to the two cases studied earlier. First we consider the symmetric case. We have found the effective Hamiltonian for the self-interaction could be written in the form

$$
H_{SI} = \sum_{a=1}^{5} \left( C_a e^{i q_a r} + C_a^\dagger e^{-i q_a r} \right)
$$

$$
= \begin{pmatrix}
    0 & -i \sum_{a=1}^{5} A_a \sin(q_a r) \\
    i \sum_{a=1}^{5} A_a \sin(q_a r) & 0
\end{pmatrix},
$$

(3.33)

where $A_a$’s are the amplitudes of the largest five Fourier modes. Only sinusoidal terms appear because of the symmetry of the potential. Thus the $C_a$ matrices are

$$
C_a = \frac{1}{2} A_a \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

(3.34)

The next step is to convert $C_a$ matrices into $G_a$ by using the matrices $U$ which diagonalize the unperturbed Hamiltonian which, in our case, is simply the vacuum Hamiltonian. The $U$ matrix is

$$
U = \begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix}.
$$

(3.35)

With this matrix we find the $G_a$’s for the symmetric case are also purely off-diagonal and equal to the $C_a$’s. Usually the next step in the RWA procedure would be to extract from the $G_a$’s the diagonal elements in order to compute the $F_a$’s and $\Phi_a$’s which are then inserted into Eqs. (3.14,3.15,3.16,3.17) to get $x_a, y_a, z_a, \psi_a$. However this is not necessary in this test problem because $G_a$ matrices are all purely off-diagonal so we know immediately that $z_a = 0$ for all modes.

We now have to determine the set - or sets - of integers \{n_1, n_2, n_3, n_4, n_5\} to use for the RWA equations. The approach we adopt is to scan the grid points of the five dimensional hypercube from $-2 \leq n_a \leq +2$ and select the combination that maximizes the amplitude of the RWA solution i.e.
\( \kappa^2 / Q^2 \). We find this set to be \( \{ n_1, n_2, n_3, n_4, n_5 \} = \{-1, 0, 0, 0, 0\} \). Normally the next step is to recognize that since the wavenumbers for the Fourier modes are all harmonics, their ratios are rational and thus we should be in a situation where we have degenerate RWA. There are other sets of integers which have the same detuning frequency as the set \( \{-1, 0, 0, 0, 0\} \) and if we scan through the grid points of the hypercube defined by \(-2 \leq n_a \leq +2\) we find there are sixty four other degenerate integer sets. However, as noted, the \( z_a \)'s for all modes are zero so the value of \( \lambda_{a,m_a} \), equation (3.20), will also be zero except when \( m_a = 0 \). Since \( \kappa \) is a product of four \( \lambda_{a,m_a} \)'s, only one of the RWA integers can be non-zero; if two or more integers in the set are non-zero the contribution to the amplitude vanishes. Thus the set \( \{-1, 0, 0, 0, 0\} \) is, in fact, the only set that contributes to the amplitude and it is not necessary to compute the contributions from the degenerate solutions.

Now that we know which set of RWA integers to use we simply compute the quantity \( \kappa \) and \( Q \) in order determine the RWA solution. The comparison between the RWA solution and the transition probability as calculated numerically is shown in Fig. (3.3). Though the shape of the transition probability from the numerical solution and RWA do not match - the numerical solution varies approximately as \( P_{12} \sim \sin^{12}(Q' r) \) whereas the solution from the RWA is that \( P_{12} \propto \sin^2(Q r) \) - what we observe is that the amplitude of the two curves are in good agreement and that the predicted wavelength matches the wavelength of the numerical solutions to within \( \sim 5\% \).

### 3.3.3 The asymmetric case

Given the decomposition of the self-interaction for the asymmetric case, we follow the same procedure as the symmetric case. This test problem is more demanding in several aspects. In this particular problem the Hamiltonian for the antineutrinos is not equal to the neutrino Hamiltonian so we must compute the antineutrino evolution separately. Another difference is that the \( C_a \) and \( G_a \) matrices contain non-zero diagonal elements so the \( z_a \)'s are also non-zero. For the seven modes we are using we find \( z_a \)'s are all small, typically \( z_a \sim 10^{-3} \). We scan through the \( 5^7 = 78,125 \) grid points of the seven dimensional hypercube defined by \(-2 \leq n_a \leq +2\) and find that the set integers which give the largest amplitude RWA solution are \( \{ n_1, n_2, n_3, n_4, n_5, n_6, n_7 \} = \{-1, 0, 0, 0, 0, 0, 0\} \). Unlike in the symmetric case of \( \alpha = 1 \), the contribution to the solution from the degenerate RWA integer sets can be non-zero. Within the seven dimensional hypercube we find 379 other sets with the same detuning frequency as the \( \{-1, 0, 0, 0, 0, 0\} \) set ranging from the simple such as \( \{1,1,-1,0,0,0,0\} \) to the exotic such as \( \{2,-2,-1,2,1,1,-2\} \). We must add the contribution from these degenerate sets to our value for \( \kappa \).

The numerical result and RWA prediction for the neutrinos are shown in Fig.(3.7) and for the antineutrinos in Fig.(3.8). Again the numerical solution is seen to be periodic function that resembles a ‘pinched’ sinusoid - a fit indicates the numerical solution evolves as \( P_{12} \sim \sin^{18}(Q' r) \) - which the RWA is not able to reproduce but, as before, the amplitude is calculated to with \( \sim 10\% \) for
the neutrinos, within \( \sim 1\% \) for the antineutrinos and the wavelength is accurate to \( \sim 5\% \). It is interesting to note that the transition wavelength for the neutrinos is overestimated i.e. too large, but underestimated for the antineutrinos. We also see that the amplitude for the oscillations in the antineutrinos is 100\% i.e. they are ‘on resonance’ but the neutrinos are not. Note this resonance is different from that described by Raffelt [34]. In figures (3.7) and (3.8) we show the RWA solution using just the dominant integer set and then the solution when we include all the degenerate solutions with the same detuning frequency. There is no discernible difference between the two indicating RWA degeneracy does not greatly modify the solution. This is due to the smallness of the \( z_a \)’s: for small arguments the Bessel function \( J_n \) scales as \( J_n \propto z^n \) so the \( \lambda_{a,n_a} \)’s, equation (3.20), are suppressed for non-zero \( n_a \).

### 3.4 Conclusion

In this paper we have tackled the neutrino self-interaction problem using an effective Hamiltonian obtained by a numeric calculation, which is composed of a harmonic Fourier series with just a few large amplitude modes. We constructed an analytic solution using the RWA and found the solution is able to predict the amplitude of the transition probability and the transition wavelength to better than 10\% in two test cases we considered. The applications to more realistic problems are left to future investigations.
Figure 3.1 The match between the Fourier reconstructed $H_{SI}$ and the numerical $H_{SI}$. The top panel uses one Fourier mode, the middle panel uses two modes, and the bottom panel uses five modes. In all panels the dashed lines are the numerical result.
Figure 3.2 The relative amplitudes of the Fourier components of the self-interaction potential.
Figure 3.3 The transition probabilities of electron neutrinos in the flavor basis in the symmetric case. Top panel, the numerical solution: bottom panel the RWA solution calculated using the five largest amplitude modes.
Figure 3.4 The diagonal and off-diagonal entries of $H_{SI}$. Top panel: the imaginary (solid line) and real (dashed line) part of the off-diagonal entries. Middle panel: the real part of the first diagonal element. Bottom panel: the real parts of the second diagonal element.
Figure 3.5 The relative amplitudes of the Fourier components of the real part of the off-diagonal element of the Hamiltonian shown in Fig.3.4.
Figure 3.6 The comparison between the real component of the off-diagonal element of the Fourier reconstructed $H_{SI}$ and the numerical $H_{SI}$. In the top panel we use one Fourier mode, the middle panel is with two modes, and the bottom panel uses seven modes. In each panel the dashed line is the numerical result.
Figure 3.7 The transition probabilities of electron neutrinos in the flavor basis in the asymmetric, $\alpha = 0.8$ case. Top panel, the numerical solution. Bottom panel the RWA solution using the seven largest amplitude modes. In the lower panel the solid line is the RWA solution using just the dominant RWA integer set, the dashed line is the solution computed with all the integer sets that have a degenerate detuning frequency.
Figure 3.8 The transition probabilities of electron antineutrino in the flavor basis for the same calculation as in figure (3.7).
REFERENCES

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4.1 Abstract

The strong gravitational field around a proto-neutron star can modify the neutrino flavor transformations that occur above the neutrinosphere via three General Relativistic (GR) effects: time dilation, energy redshift, and trajectory bending. Depending on the compactness of the central object, the neutrino self-interaction potential is up to three times as large as that without GR principally due to trajectory bending which increases the intersection angles between different neutrino trajectories, and time dilation which changes the fluxes. We determine whether GR effects are important for flavor transformation during the different epochs of a supernova by using multi-angle flavor transformation calculations and consider a density profile and neutrino spectra representative of both the accretion and cooling phases. We find the GR effects are smaller during the accretion phase due to low compactness of the proto-neutron star and merely delay the decoherence; the neutrino
bipolar oscillations during the cooling phase are also delayed due to the GR effects but the delay may be more important because the delay occurs at radii where it might alter the nucleosynthesis in the neutrino driven wind.

4.2 Introduction

The collapse of the core of a massive star at the end of its life forms a hot and dense object known as a proto-neutron star which cools via the emission of neutrinos over a period of ∼ 10 s [1, 2]. The spectra and flavor distribution of the neutrinos that emerge from the supernova are not the same as those emitted from the proto-neutron star: for a recent review see Mirizzi et al. [3]. At the present time the most sophisticated calculations of the neutrino flavor transformation adopt the so-called ‘bulb’ model: the neutrino source is a spherically symmetric, hard neutrinosphere, the calculation assumes a steady state, and neutrinos are followed along multiple trajectories characterized by their angle of emission relative to the radial direction - the ‘multi-angle’ approach [4, 5]. The Hamiltonian governing the flavor evolution for a single neutrino depends on the local density profile plus a contribution from all the other neutrinos which are escaping the proto-neutron star - the neutrino self-interaction. The neutrino self-interaction depends upon the neutrino luminosity, mean energy and a term proportional to 1 – cos Θ due to the current-current nature of the weak interaction where Θ is the angle between two neutrino trajectories. Curiously, while the density profile and the neutrino spectra are sometimes taken from hydrodynamical simulations of supernova which include General Relativistic (GR) effects either exactly or approximately, e.g. from the simulations by Fischer et al. [6], the calculations of the neutrino flavor transformation ignore them.

The flavor transformation that occurs in a supernova will alter the expected signal from the next Galactic supernova [7, 8, 9, 10], as well as modify the Diffuse Supernova Neutrino Background [11, 12, 13, 14, 15, 16, 17], and the nucleosynthesis that occurs in the neutrino driven wind [18, 19, 20, 21, 22]. Neutrino heating in the region behind the shock is thought to be the mechanism by which the star explodes and such heating depends upon the neutrino spectra of each flavor which depends upon the flavor transformation [23, 24]. With so many different consequences of flavor transformation, one wonders how including GR in the flavor transformation calculations might alter our expectations.

GR effects upon neutrino oscillations in vacuum have been considered on several occasions e.g. [25, 26, 27, 28, 29, 30, 31, 32]. The inclusion of matter is occasionally considered [33, 34, 35, 36] and the effect of GR usually limited to a shift in location and adiabaticity of the Mikheyev-Smirnov-Wolfenstein (MSW) resonance [37, 38] via the redshift of the neutrino energy. The effects of GR upon neutrino self-interactions have not been considered. The effect of GR has also been studied for the neutrinos emitted from the accretion disk surrounding a black hole formed in the merger of two neutron stars, a black hole and a neutron star, or in a collapsar. For example, Caballero,
McLaughlin and Surman [39] studied the GR effects for accretion disk neutrinos (but without neutrino transformation) and found the effects upon the nucleosynthesis were large because of the significant changes to the neutrino flux.

The aim of this paper is to explore the GR effects upon flavor transformation in supernovae including neutrino self-interactions and determine whether they might be important in different phases of the explosion. Our paper is organized as follows. In section §4.3 we describe our calculation and how the GR effects are included. Section §4.4 contains our results for the two representative cases we study: luminosities, mean and rms energies, density profiles and source compactness characteristic of the accretion phase, and a different set representative of the cooling phase. In section §4.5 we discuss the conditions that lead to the formation of a neutrino halo - neutrinos that were emitted but which later turned-around and returned to the proto-neutron star. We present a summary and our conclusions in section §4.6.

4.3 Calculation Description

4.3.1 GR Effects Upon Neutrinos

Before describing the formulation of neutrino oscillations in a curved spacetime, we first describe the three general relativistic effects that will be important. For this paper we adopt an exterior Schwarzschild metric for the space beyond the neutrinosphere\(^1\) which is given by

\[
d\tau^2 = B(r) dt^2 - \frac{dr^2}{B(r)} - r^2 d\psi^2 - r^2 \sin^2 \psi \, d\phi^2, \tag{4.1}
\]

where the function \(B(r)\) is \(B(r) = 1 - \frac{r_s}{r}\) and \(r_s\) is the Schwarzschild radius given by \(r_s = 2GM\) with \(M\) the gravitational mass. Throughout our paper we set \(\hbar = c = 1\). Since the rest mass of all neutrino species are much smaller than the typical energies of supernova neutrinos, we can comfortably take the ultra-relativistic limit and assume neutrinos follow null geodesics just like photons. The Schwarzschild metric is isotropic so all geodesics are planar. By setting \(d\tau^2 = 0\) and \(d\phi = 0\) so that the geodesic lies in the plane perpendicular to the equatorial plane, we obtain

\[
B(r) \, dt^2 = \frac{dr^2}{B(r)} + r^2 d\psi^2. \tag{4.2}
\]

The energy of a neutrino \(E\) decreases as it climbs out of the gravitational well such that its energy at a given radial coordinate \(r\) relative to its energy at \(r \to \infty\), \(E_\infty\), is

\[
\frac{E}{E_\infty} = \frac{1}{\sqrt{B(r)}}. \tag{4.3}
\]

\(^1\)For simplicity we ignore the gravitational effect of the matter outside the neutrinosphere.
The angular momentum $\ell$ of the neutrino also decreases as it climbs out of the potential well by the same scaling. This means the ratio of the neutrino’s angular momentum to its energy is constant and in our chosen plane is given by

$$\frac{\ell}{E} = \frac{r^2}{B(r)} \left| \frac{d\psi}{dt} \right| = b$$

(4.4)

where $b$ is a constant called the impact parameter. The impact parameter can be evaluated at the neutrinosphere $r = R_\nu$ where we find it is given by

$$b = \frac{R_\nu \sin \theta_R}{\sqrt{1 - \frac{r_s}{R_\nu}}}$$

(4.5)

where $\theta_R$ is the the emission angle of the neutrino with respect to the radial direction at the neutrinosphere. Using Eq. (4.4) to eliminate $dt$ from Eq. (4.2) we find\(^2\)

$$d\psi = \pm \left[ \frac{1}{b^2} - \frac{1}{r^2} B(r) \right]^{1/2} \frac{dr}{r^2}$$

(4.6)

This equation can be used to describe the neutrino trajectory associated with a certain emission angle $\theta_R$. Or using Eq. (4.4) to eliminate $d\psi$ from Eq. (4.2) gives

$$dt = \pm \frac{1}{B(r)} \frac{dr}{\sqrt{1 - \frac{b^2}{r^2} B(r)}}$$

(4.7)

For an observer at position $r$ the relation between the coordinate time $t$ and the local proper time\(^3\) $\tau$ is

$$d\tau^2 = B(r) dt^2$$

(4.8)

so using the result from Eq. (4.7) we find

$$d\tau = \pm \frac{1}{\sqrt{B(r)}} \frac{dr}{\sqrt{1 - \frac{b^2}{r^2} B(r)}}$$

(4.9)

This collection of equations will be useful when we describe flavor oscillations in a curved spacetime.
4.3.2 Neutrino Oscillations In A Curved Spacetime

Our calculations of the effects of GR on neutrino flavor transformation are based upon the neutrino bulb model established by Duan et al. [4, 5]. In this model, neutrinos are emitted from a hard neutrinosphere with radius $R_\nu$ and for simplicity we assume the angular distribution of emission is half-isotropic. The setup is illustrated in Fig. 4.1 which shows the trajectory of a neutrino emitted at the neutrinosphere $R_\nu$ with angle $\theta_R$ relative to the radial direction. After propagating to radial coordinate $r$ with angle $\psi$ relative to the radial direction at the point of emission, it makes an angle $\theta$ relative to the radial direction at $(r, \psi)$. The formulation of neutrino flavor transformation in a curved spacetime has been considered on multiple occasions [28, 29, 30, 31, 32, 33]. The flavor state at some local proper time $\tau$ of a neutrino with momentum $q$ is related to the flavor state at the local proper time of emission $\tau_0$ with momentum $q_0$ via an evolution matrix $S(\tau, q; \tau_0, q_0)$ which evolves according to the Schrödinger equation.

$$i \frac{dS}{d\tau} = H(\tau)S.$$  \hfill (4.10)

Here $H$ is the Hamiltonian which is also a function of the local proper time for the case of neutrinos in a non-uniform medium. The local proper time $\tau$ may be replaced with the radial coordinate $r$ by using Eq. (4.9) once the impact parameter/emission angle is given. Similarly, the evolution of the antineutrinos is given by an evolution matrix $\bar{S}$ which evolves according to a Hamiltonian $\bar{H}$. Once the evolution matrix has been found, the probability that a neutrino in some generic initial state $\nu_j$ with momentum $q_0$ at $\tau_0$ is later detected as state $\nu_i$ at proper time $\tau$ and momentum $q$ is

$$P(\nu_j \rightarrow \nu_i) = P_{ij} = |S_{ij}(\tau, q; \tau_0, q_0)|^2.$$  

The Hamiltonian $H$ is the sum of three terms: $H = H_V + H_M + H_{SI}$, where $H_V$ is the vacuum term, $H_M$ is the matter term to describe the effect of passing through matter, and $H_{SI}$ is a term due to neutrino self-interactions. For the antineutrinos the Hamiltonian is also a sum of three terms with $\bar{H} = \bar{H}_V + \bar{H}_M + \bar{H}_{SI}$, which are related to the corresponding terms in the neutrino Hamiltonian via $\bar{H}_V = H_V^*, \bar{H}_M = -H_M^*, \bar{H}_{SI} = -H_{SI}^*$. In a flat spacetime the vacuum term for a neutrino with energy $E$ takes the form of

$$H_V^{(f)} = \frac{1}{2E} U_V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U_V^\dagger.$$  \hfill (4.11)

where $m_i$ are the neutrino masses and $U_V$ is the unitary matrix relating the ‘mass’ and flavor bases. The flavor basis is denoted by the superscript $(f)$ upon relevant quantities and we order the

\footnote{Here the plus sign is for outgoing neutrinos, the minus sign is for ingoing neutrinos, this is true for all following equations.}

\footnote{The “local proper time” is defined as the clock time of an observer sitting at a particular point along the neutrino trajectory.}
rows/columns as $e, \mu, \tau$ (here $\tau$ is the neutrino flavor, not local proper time). We adopt the Particle Data Group parameterization of the matrix $U_V$ which is in terms of three mixing angles $\theta_{12}, \theta_{13}$ and $\theta_{23}$ plus a CP violating phase $\delta_{CP}$ [40]. In a curved spacetime the energy of a neutrino is dependent on position due to the gravitational redshift so the vacuum term will change accordingly and is

$$H^{(f)}_V = \frac{\sqrt{B(r)}}{2E_\infty} U_V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger_V.$$  \hspace{1cm} (4.12)$$

The matter Hamiltonian $H^M_m$ in the flavor basis depends upon the electron density $n_e(r)$ and is simply

$$H^{(f)}_M = \sqrt{2} G_F n_e(r) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \hspace{1cm} (4.13)$$

### 4.3.3 The GR correction to neutrino self-interactions

In addition to the vacuum and matter terms, in a neutrino dense environment such as a supernova we must add to the Hamiltonian a term due to neutrino self-interactions. The form of the self-interaction is

$$H_{SI}(r, q) = \sqrt{2} G_F \sum_{\alpha = e, \mu, \tau} \int \left[ 1 - \hat{q} \cdot \hat{q}' \right] \left[ \rho_\alpha(r, q') d n_\alpha(r, q') - \rho^*_\alpha(r, q') d n_\alpha(r, q) \right] dq'$$  \hspace{1cm} (4.14)$$

where $\rho_\alpha(r, q)$ is the density matrix of the neutrinos at position $r$ with momentum $q$ and initial flavor $\alpha$ defined as $\rho_\alpha(r, q) = \psi_\alpha(r, q) \psi^\dagger_\alpha(r, q)$, with $\psi_\alpha(r, q)$ being the corresponding normalized neutrino wave function, $d n_\alpha(r, q)$ is the differential neutrino number density [4], which is the differential contribution to the neutrino number density at $r$ from those neutrinos with initial flavor $\alpha$ and energy $|q|$ propagating in the directions between $\hat{q}$ and $\hat{q} + d\hat{q}$, per unit energy (the hats on $q$ and $q'$ indicate unit vectors). Note that here we have replaced the local proper time $\tau$ with the radial coordinate $r$ to denote the location along a given neutrino trajectory.

In order to use Eq. (4.14) we have to first specify the expression for $d n_\alpha(r, q)$. This requires relating the neutrino momenta $q$ at radial coordinate $r$ back to their values $q_0$ at the neutrinosphere where they are initialized. After this relationship is obtained we can substitute $d n_\alpha(r, q)$ with $d n_\alpha(R, q_0)$ and integrate over the neutrino momentum distributions at the neutrinosphere. While the magnitude of $q$ is related to the magnitude of $q_0$ via an energy redshift $q = q_0 \sqrt{B(R)/B(r)}$, relating $\hat{q}$ to $\hat{q}_0$ means finding the relation between the emission angle $\theta_R$ and the angle $\theta$ shown in Fig. 4.1 since the neutrino trajectory is planar. In flat spacetime, the relation between $\theta_R$ and $\theta$
can be found through geometric arguments [4]. In a curved spacetime, however, $\theta$ and $\theta_R$ might be expected to be related only after solving for the neutrino trajectory. But fortunately, for the Schwarzschild metric the relation between $\theta$ and $\theta_R$ can also be found simply by making use of the fact that the impact parameter $b$ is a conserved quantity along each neutrino trajectory [39]. It makes no difference whether the impact parameter is evaluated at $R_\nu$ or at $r$, therefore $b(r) = b(R_\nu)$.

Using this conserved quantity we must have

$$\frac{r \sin \theta}{\sqrt{1 - r_s/r}} = \frac{R_\nu \sin \theta_R}{\sqrt{1 - r_s/R_\nu}},$$

from which we find

$$\cos \theta = \sqrt{1 - \left(\frac{R_\nu \sin \theta_R}{r}\right)^2 \left(\frac{1 - r_s/r}{1 - r_s/R_\nu}\right)}.$$

In Fig. 4.2 we plot the angle $\theta$ as a function of emission angle $\theta_R$ for three different ratios of $r_s$ to $R_\nu$ at $r = 10 R_\nu$. The figure shows that for each particular emission angle $\theta_R$, the trajectory bending effect always makes the angle $\theta$ larger than without GR. In the bulb model $(1 - \hat{q} \cdot \hat{q}')$ is found to be equivalent to $(1 - \cos \theta \cos \theta')$ after averaging over the angles in the plane perpendicular to the radial direction. Thus the correction to $\cos \theta$ by GR increases the magnitude of $H_{SI}$ by increasing the value of $1 - \hat{q} \cdot \hat{q}'$ for every neutrino.

Now we have the expression relating $\theta$ to $\theta_R$, we can write the expression for the differential number density, after taking time dilation into account, as

$$d n_\alpha(r, q) = d n_\alpha(R_\nu, q_0, \theta_R) = \frac{1}{2\pi r^2 \sqrt{B(r)}} \left[ \frac{L_{\alpha,\infty}}{E_{\alpha,\infty}} \right] f_\alpha(q_0) \left( \frac{\cos \theta_R}{\cos \theta} \right) \left( \frac{dq_0}{dq} \right) d \cos \theta_R,$$

where $f_\alpha(q_0)$ is the normalized distribution function for flavor $\alpha$ with momentum $q_0$ that redshifts to $q$ at $r$, $L_{\alpha,\infty}$ is the luminosity of flavor $\alpha$ at infinity if no flavor transformation had occurred, and similarly $\langle E_{\alpha,\infty} \rangle$ is the mean energy of neutrinos of flavor $\alpha$ at infinity again assuming no flavor transformation had occurred. The expression for the antineutrinos is similar. The derivation of Eq. (4.17) can be found in the Appendix.

The density matrix $\rho_\alpha(r, q)$ for neutrinos at $r$ with momentum $q$ is related to the corresponding density matrix at the neutrinosphere via $\rho_\alpha(r, q) = S(r, q; R_\nu, q_0) \rho_\alpha(R_\nu, q_0) S^\dagger(r, q; R_\nu, q_0)$ and the same for the antineutrinos using the evolution matrix $\bar{S}(r, q; R_\nu, q_0)$.

Combining these equations together, we obtain the GR corrected expression of neutrino self-
interaction in curved spacetime as

\[
H_{SI}(r, q) = \frac{\sqrt{2} G_F}{2\pi r^2 \sqrt{B(r)}} \sum_{\alpha=e,\mu,\tau} \int \left(1 - \cos \theta \cos \theta' \right) \left[ \frac{L_{\alpha,\infty}}{E_{\alpha,\infty}} \right] \rho_{\alpha}(r, q') f_{\alpha}(q_0') - \left[ \frac{L_{\bar{\alpha},\infty}}{E_{\bar{\alpha},\infty}} \right] \bar{\rho}^*_{\bar{\alpha}}(r, q') \bar{f}_{\bar{\alpha}}(q_0')
\times \left( \frac{\cos \theta'_R}{\cos \theta'} \right) d \cos \theta'_R d q_0'.
\] (4.18)

When we take the weak gravity limit \( r_s \ll r \) and \( r_s \ll R_\nu \) we find this expression reduces to the same equation found in Duan et al. [4, 5]. This equation includes two GR effects: trajectory bending and time dilation (the energy redshift of the luminosity cancels with the energy redshift of the mean energy). In order to appreciate how significant the GR effects can be for the self-interaction Hamiltonian we show in Fig. 4.3 the neutrino trajectories which converge at a certain point above the surface of the central proto-neutron star. From the perspective of an observer at this point, the neutrinos seem to be coming from an expanded source whose radius is increased by a factor of

\[
\frac{1 - r_s / r}{1 - r_s / R_\nu},
\]

which can be seen from Eq. (4.16). As noted earlier, the effect of trajectory bending causes the neutrino trajectories to cross at larger angles than in the case without GR. Time dilation also enhances the self-interaction because it leads to a larger effective neutrino flow rate. Close to the neutrinosphere time dilation is the larger effect because the effect of trajectory bending is small. At larger radii the situation is reversed with trajectory bending more important than time dilation.

To quantify the magnitude of the GR effects upon the self-interaction we show in the top panel of Fig. 4.4 the enhancement of the self-interaction due to GR, which is defined to be the ratio of the magnitude of the self-interaction potential with GR effects to that without, as a function of the coordinate \( r \) and assuming no flavor oscillation occurs, for different values of \( r_s / R_\nu \). The striking feature of the GR effects is that, even though the spacetime curvature is only pronounced near the proto-neutron star, the enhancement of the neutrino self-coupling turns out to be a long-range effect that is asymptotic to a value greater than unity which depends upon the ratio \( r_s / R_\nu \). Since the influence of GR on neutrino flavor transformation is not just a local effect, it can have repercussions upon processes at larger radii such as neutrino heating in the accretion phase and nucleosynthesis in the cooling phase.

As we have seen, the magnitude of the GR effect is governed by ratio of the radius of the neutrinosphere relative to the Schwarzschild radius of the proto-neutron star which itself is just proportional to the mass of the proto-neutron star. This suggests we define a neutrino 'compactness' -
similar to the definition of compactness found in O’Connor & Ott [41] - as

$$\xi_\nu = \frac{M/M_\odot}{R_\nu/10 \text{ km}} = \frac{r_s/2.95 \text{ km}}{R_\nu/10 \text{ km}} = 3.39 \frac{r_s}{R_\nu}. \quad (4.19)$$

In the bottom panel of Fig. 4.4 we plot the enhancement factor as a function of compactness at different distances from the center of the proto-neutron star. For a very compact neutrino source we find the enhancement of the self-interaction can be as large as a factor of 300% if $$\xi_\nu \sim 2.26$$ which corresponds to $$r_s/R_\nu = 2/3$$. We shall explain the significance of this compactness in section §4.5. The blue line in this figures shows the enhancement factor at the neutrinosphere, where the trajectory bending effect is minimal. Here the enhancement is purely due to time dilation.

### 4.4 Numerical Calculations

With the formulation complete and with the insights gained from the computation of the enhancement as a function of compactness, we proceed to compute numerically the multi-angle neutrino flavor evolution for two representative cases. These are a density profile, neutrino spectra and compactness typical of the accretion phase of a supernova, and one representative of the cooling phase. The neutrino mixing angles and square mass differences we adopt are $$m_2^2 - m_1^2 = 7.5 \times 10^{-5} \text{ eV}^2$$, $$m_3^2 - m_2^2 = -2.32 \times 10^{-3} \text{ eV}^2$$, $$\theta_{12} = 33.9^\circ$$, $$\theta_{13} = 9^\circ$$ and $$\theta_{23} = 45^\circ$$. The CP phase $$\delta_{CP}$$ is set to zero. We do not consider a normal mass ordering on the basis of the results by Chakraborty et al. [42] and Wu et al. [22].

#### 4.4.1 Application to SN accretion phase

For the accretion phase we use the density profile at $$t_{pb} = 0.3 \text{ s}$$ postbounce from Fischer et al. [6] for the 10.8 $$M_\odot$$ progenitor. As previously stated, this simulation includes GR effects in both the hydrodynamics and evolution of the neutrino phase space density (see Liebendörfer et al. [43] for further details about the code). The density profile at this snapshot time is shown by the red line in Fig. 4.5. We set the neutrinosphere radius to be $$R_\nu = 25 \text{ km}$$ which corresponds to the minimum of the electron fraction for this model at this time. This working definition for the neutrinosphere

<table>
<thead>
<tr>
<th>Flavor</th>
<th>Luminosity $$L_{\alpha,\infty}$$</th>
<th>Mean Energy $$\langle E_{\alpha,\infty} \rangle$$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$$\nu$$</td>
<td>$$41.52 \times 10^{51} \text{ erg/s}$$</td>
<td>10.39 MeV</td>
</tr>
<tr>
<td>$$\mu$$, $$\tau$$</td>
<td>$$14.23 \times 10^{51} \text{ erg/s}$$</td>
<td>16.19 MeV</td>
</tr>
<tr>
<td>$$\bar{\nu}$$</td>
<td>$$42.35 \times 10^{51} \text{ erg/s}$$</td>
<td>12.67 MeV</td>
</tr>
<tr>
<td>$$\bar{\mu}$$, $$\bar{\tau}$$</td>
<td>$$14.39 \times 10^{51} \text{ erg/s}$$</td>
<td>16.40 MeV</td>
</tr>
</tbody>
</table>

Table 4.1 The luminosities and mean energies used for the accretion phase calculation.
radius comes from noting the coincidence of the electron fraction minimum and the neutrinosphere radii shown in figures (7) and (8) in Fischer et al. and produces a curve which is similar to figure (15) found in their paper. We note that the value of $R_\nu$ we adopt is different from the value estimated by others, e.g. [44, 42], which tend to use relatively larger values for $R_\nu$ during the accretion phase. From the simulation we find the mass enclosed within the $R_\nu = 25$ km radius is $M = 1.33 \, M_\odot$, giving a compactness of $\xi_\nu = 0.53$. The neutrino luminosities and mean energies we use are also taken from the same simulation and are listed in table (4.1). To save computational resources we use a source distribution $f_\alpha(q_0)$ which is a delta-function at a single energy taken to be 15 MeV. Single-energy calculations were also undertaken by Chakraborty et al. [45] when they also studied the self-interaction effects during the accretion phase. As previously stated, the angular distribution is assumed to be half-isotropic which is the same distribution used in Duan et al. [4, 5].

Our results are shown in Fig. 4.6 which is a plot of the electron flavor survival probability averaged over all angular bins as a function of distance. In the figure we also include three vertical dashed lines to indicate the start of the bipolar oscillation region, the position of the shockwave, and the end of the bipolar oscillation region. The predictions for the beginning and end of the bipolar oscillation region come from equations given in Chakraborty et al. [45]. The change in the angle-averaged survival probability $P_{ee}$ which occurs at $r \sim 475$ km is simply decoherence [42]. Comparing the results with and without GR effects we see the decoherence is slightly delayed when GR is included but the difference is only of order $\sim 20$ km and the final result is identical to the case without GR. Thus it appears GR has little effect upon flavor transformation during the accretion phase and where little change occurs is in a region where it has little consequence.

4.4.2 Application to SN cooling phase

As the proto-neutron star cools it contracts which increases the compactness. The sensitivity of the neutrino self-interaction to the compactness means we might expect a larger effect from GR during the cooling phase. To test whether this is the case we use the density profile at $t_{pb} = 2.8$ s postbounce from the Fischer et al. [6] simulation for the same 10.8 $M_\odot$ progenitor and which is shown by the blue line in Fig. 4.5. We set the neutrinosphere radius to be $R_\nu = 17$ km which, again, is close to the minimum of the electron fraction for this model at this time and consistent with figure

<table>
<thead>
<tr>
<th>Flavor</th>
<th>Luminosity $L_{\alpha,\infty}$</th>
<th>Mean Energy $\langle E_{\alpha,\infty} \rangle$</th>
</tr>
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<tbody>
<tr>
<td>$e$</td>
<td>$2.504 \times 10^{51}$ erg/s</td>
<td>9.891 MeV</td>
</tr>
<tr>
<td>$\mu$, $\tau$</td>
<td>$2.864 \times 10^{51}$ erg/s</td>
<td>12.66 MeV</td>
</tr>
<tr>
<td>$\bar{e}$</td>
<td>$2.277 \times 10^{51}$ erg/s</td>
<td>11.83 MeV</td>
</tr>
<tr>
<td>$\bar{\mu}$, $\bar{\tau}$</td>
<td>$2.875 \times 10^{51}$ erg/s</td>
<td>12.70 MeV</td>
</tr>
</tbody>
</table>

Table 4.2 The luminosities, mean energies, and rms energies used for the cooling phase calculation.
The mass enclosed within this radius is $M \approx 1.44 M_\odot$, giving a compactness of $\xi_v = 0.85$. For this cooling epoch calculation we use multi-energy as well as multi-angle. The neutrino energy range is chosen to be $E_\infty = 1 \text{ MeV}$ to $E_\infty = 60 \text{ MeV}$, and is divided into 300, equally spaced, energy bins. To generate the neutrino spectra for flavor $\alpha$ at the neutrinosphere we use the luminosities, mean energies and rms energies at this snapshot of the simulation - listed in table (4.2) - and insert them into the pinched thermal spectrum of Keil, Raffelt and Janka [46] which has the form

$$f_\alpha(q_0) = \frac{(A_\alpha + 1)q_0^{A_\alpha}}{(E_{\alpha, R_v})^{A_\alpha+1} \Gamma(A_\alpha + 1)} \exp\left(\frac{(A_\alpha + 1)q_0}{\langle E_{\alpha, R_v} \rangle}\right),$$

with $\langle E_{\alpha, R_v} \rangle = \langle E_{\alpha, \infty} \rangle / \sqrt{B(R_v)}$ and the pinch parameter $A_\alpha$ for flavor $\alpha$ is given by

$$A_\alpha = \frac{2\langle E_{\alpha, \infty} \rangle^2 - \langle E_{\alpha, \infty}^2 \rangle}{\langle E_{\alpha, \infty}^2 \rangle - \langle E_{\alpha, \infty} \rangle^2}.$$ (4.21)

The result of this calculation is shown in Fig. 4.7 where we plot the electron neutrino flavor survival probability averaged over all angular bins and energy bins (using the emitted neutrino spectrum as the weighting function) as a function of distance. At this epoch self-interaction effects occur much closer to the proto-neutron star and the effect of GR is more important. The net result of adding GR is to delay the onset of bipolar oscillations by around 25 km and once more we find the probability at large radii are almost identical to that without GR. But while this shift in the onset of bipolar oscillations may seem small, we note the neutrino flavor evolution in the region from $50 \text{ km} \lesssim r \lesssim 500 \text{ km}$ was found to be crucial for determining the nucleosynthesis yields in the calculations by Duan et al. [20] and Wu et al. [22] so even a relatively small delay of flavor transformation caused by GR might have a consequence.

## 4.5 The GR Neutrino Halo

So far we have considered only cases where all neutrinos propagate to $r \to \infty$. However if the compactness of the source becomes too large the neutrinosphere becomes smaller than the “photon sphere”, whose radius is $3r_s/2$. When this occurs there will be a critical emission angle for neutrinos beyond which they cannot escape to infinity. Following the argument in Hartle [47], one can obtain a condition for the neutrinos to escape to infinity to be

$$\frac{2}{3\sqrt{3}} \frac{R_v}{r_s} \frac{1}{\sqrt{1 - r_s/R_v}} \sin \theta_R < 1.$$ (4.22)

We show three example neutrino trajectories for the case where $R_v/r_s < 3/2$ in Fig. 4.8. Trajectories 1 and 2 are open and a neutrino emitted along these trajectories will propagate to infinity: the
trajectories of neutrinos emitted at sufficiently large angles - such as trajectory 3 - will turn around and return to the proto-neutron star. Note that the farthest place where a neutrino can turn around is the photon sphere. The consequence of such trajectories are included in simulations which include GR. In principle there is a substantial change to the flavor evolution calculations when neutrinos start to follow trajectories such as the trajectory 3 in Fig. 4.8 because they lead to the formation of a neutrino 'halo' around the proto-neutron star, similar to the neutrino halos produced by scattering on matter [48, 49].

From Eq. (4.22) we can evaluate the critical angle as a function of $R_{\nu}/r_s$. The relation between the critical angle as a function of $R_{\nu}/r_s$ is shown in Fig. 4.9. If $R_{\nu}/r_s > 3/2$, clearly neutrinos with all emission angles can escape and no neutrino halo is formed. We define a critical compactness $\xi_{\nu}$ to be the case where $R_{\nu}/r_s = 3/2$ and find it equal to $\xi_{\nu} = 2.26$ - the value discussed earlier. The compactness of the sources we have considered for our previous numerical calculations did not approach this value because the mass of the proto-neutrons star is not sufficiently large and the neutrinospheres lay beyond the photon sphere. To reach the critical compactness for formation of the halo we require a more massive proto-neutron star with a smaller neutrinosphere. Whether a proto-neutron star surpasses the critical compactness while the proto-neutron star is still cooling via neutrino emission will depend upon the Equation of State of dense matter and the neutrino opacity [50, 51]. Note that from causality, the radius of a neutron star is required to be greater than $R_{NS} \gtrsim 2.823M$ [50] which, if we set $R_{\nu} = R_{NS}$, corresponds to a compactness of $\xi_{\nu} = 2.4$, which is beyond the critical value $\xi_{\nu}^{\star}$. A halo will certainly form immediately preceding the collapse of a proto-neutron star to a black hole.

The formation of a neutrino halo has consequences for the cooling of the proto-neutron star as well as the flavor transformation due to neutrino self-interaction. One can find a presentation of the changes that occur to the emitted neutrino spectra as the mass of the proto-neutron star approaches its maximum mass in Liebendörfer et al. [43]. In their simulations, as the maximum mass is approached (but before the black holes forms) the luminosity of the $\mu$ and $\tau$ flavors increases due to contraction of the proto-neutron star while the luminosities of electron neutrino and electron antineutrinos drop. The mean energies of all flavors increases.

When a halo forms, in principle, one would have to completely change how the flavor calculations are undertaken in the halo region - the zone between the neutrinosphere and the photon sphere. In such cases the flavor evolution up to the photon sphere cannot be treated as an initial value problem - as we have done in this paper - because the flavor evolution up to the photon sphere of outward moving neutrinos is affected by neutrinos that were also emitted in an outward direction but which turned around and are now moving inwards. Thus in the halo region a paradigm beyond the bulb model would be needed to correctly deal with the flavor evolution. Prevailing understanding from the extant literature would indicate that in the case of three active flavors of neutrino emitted spherically symmetrically, one should not expect flavor transformation within the halo: if this is true then the
only effect of the formation of a halo would be to alter the luminosity and angular distribution of
the neutrinos beyond the photon sphere (which now becomes the effective neutrinosphere). But
in other circumstances - such as calculations that include sterile neutrinos [52, 53, 54, 55, 56] or
calculations with non-standard neutrino interactions [57, 58, 59, 44] - flavor transformation can
occur much closer to the neutrinosphere in which case the formation of a halo may have greater
consequences.

4.6 Conclusions

In this paper we have considered the effects of General Relativity upon neutrino flavor transforma-
tion in a core-collapse supernova. We adopted a Schwarzschild metric to describe the spacetime
and included three GR effects - trajectory bending, time dilation, and energy redshift. Of the three,
time dilation is the major effect close to the proto-neutron star, whose role is replaced by trajec-
tory bending at larger radii. The size of the GR effects were found to scale with a single parameter
which is the compactness of the source: the relative ratio of the Schwarzschild radius to the neu-
trinosphere radius. For large compactness with $R_\nu$ close to the radius of the photon sphere, the
neutrino self-interaction Hamiltonian can be up to approximately three times larger than without
GR. We calculated the flavor evolution in two representative cases to determine whether the GR
effects led to significant differences compared to calculations without GR. These cases were a density
profile and neutrino spectra typical of the accretion phase, and a density profile and neutrino spectra
typical of the cooling phase. In both cases we found the effect of GR was to delay the onset of flavor
transformation but for the accretion phase the flavor transformation occurred due to decoherence
at large radii where the change would have little consequence. In contrast, the change to the onset
of bipolar oscillations during the cooling phase may be more important because it is much closer to
the proto-neutron star and may impact the nucleosynthesis in the neutrino driven wind. Finally,
we showed that GR effects can produce a halo of neutrinos surrounding the proto-neutron star for
very compact neutrino sources. If a halo forms then, in principle, one would have to treat flavor
transformation in the halo region using a different technique than the usual approach of treating it
as an initial-value problem.

4.7 Appendix: The GR corrected expression for the neutrino
self-interaction

In order to get the correct expression for $d n_\alpha(r, q, \theta)$, we start from the conservation of neutrino
flow through an enclosing spherical surface after taking time dilation into account but ignoring
flavor transformation. This allows us to write

\[ r^2 \sqrt{B(r)} F_\alpha(r, q) \, dq = R_v^2 \sqrt{B(R_v)} F_\alpha(R_v, q_0) \, dq_0, \]

(4.23)

where \( F_\alpha(r, q) \) is the flux of neutrinos with energy \( q \) at \( r \) per unit energy that were emitted with energy \( q_0 \) at the neutrinosphere. Integrated over all momenta, both sides of this equation must evaluate to \( \frac{1}{4\pi} \frac{L_{a,\infty}}{\langle E_{a,\infty} \rangle} \) where \( L_{a,\infty} \) is the luminosity of flavor \( a \) at infinity assuming no oscillations, and similarly \( \langle E_{a,\infty} \rangle \) is the mean energy at infinity again assuming no oscillations. At the neutrinosphere \( R_v \) we have

\[ F_\alpha(R_v, q_0) = \int_0^{\theta_{\max}} 2\pi j_\alpha(q_0, \theta'_R) \cos \theta'_R \, d \theta'_R, \]

(4.24)

where \( j_\alpha(q_0, \theta_R) \) is the emitted intensity of flavor \( \alpha \) with energy \( q_0 \) at angle \( \theta_R \) with respect to the radial direction. At radial coordinate \( r \) the flux is

\[ F_\alpha(r, q) = \int_0^{\theta_{\max}} \cos \theta' \, d n_\alpha(r, q, \theta'), \]

(4.25)

where \( \theta_{\max} \) is the angle with respect to the radial direction of neutrinos that were emitted at the neutrinosphere with angle \( \theta_R = \pi/2 \). Combining Eq. (4.23), (4.24) and (4.25) we obtain the result that

\[ d n_\alpha(r, q, \theta) = \frac{2\pi R_v^2}{r^2} \sqrt{B(r)} \frac{B(R_v)}{B(r)} j_\alpha(q_0, \theta_R) \left( \frac{\cos \theta_R}{\cos \theta} \right) \left( \frac{dq_0}{dq} \right) \, d \cos \theta_R, \]

(4.26)

In the case of half-isotropic emission the intensity \( j_\alpha \) is independent of \( \theta_R \) and can be written as

\[ j_\alpha(q_0) = \frac{1}{4\pi^2 R_v^2 \sqrt{B(R_v)}} \left[ \frac{L_{a,\infty}}{\langle E_{a,\infty} \rangle} \right] f_\alpha(q_0), \]

(4.27)

where \( f_\alpha(q_0) \) is the normalized spectral distribution for flavor \( \alpha \) at \( R_v \). The final expression for \( d n_\alpha(r, q, \theta) \) is thus

\[ d n_\alpha(r, q, \theta) = \frac{1}{2\pi r^2 \sqrt{B(r)}} \left[ \frac{L_{a,\infty}}{\langle E_{a,\infty} \rangle} \right] f_\alpha(q_0) \left( \frac{\cos \theta_R}{\cos \theta} \right) \left( \frac{dq_0}{dq} \right) \, d \cos \theta_R. \]

(4.28)
**Figure 4.1** The schematic of a neutrino trajectory in strong gravitational field. Here $R_{\nu}$ is the radius of neutrinosphere, $r$ is the distance from the center, $\theta_R$ is the emission angle, $\psi$ is the polar angle, and $\theta$ is the angle of intersection.

**Figure 4.2** The relationship between $\theta$ and $\theta_R$ for $r_s/R_{\nu} = 0, 0.2$ and $0.5$ evaluated at $r = 10 R_{\nu}$. 
Figure 4.3 The neutrino trajectories converging at $r = 3R_\nu$ for (a) $r_s / R_\nu = 0$ and (b) $r_s / R_\nu = 0.6$. 

\[ \theta \]
Figure 4.4 *Top:* The enhancement factor as a function of distance for three different ratios of the Schwarzschild radius relative to the neutrinosphere radius. *Bottom:* The enhancement factor as a function of compactness, at three different distances. The two vertical dashed lines indicate the compactness of the sources we use in our calculations for the accretion phase and cooling phase, respectively.
Figure 4.5 The matter density profiles of the 10.8 M$_\odot$ simulation by Fischer et al. [6] at postbounce times $t_{pb} = 0.3$ s (red solid line) and $t_{pb} = 2.8$ s (blue solid line).
Figure 4.6 The survival probability of electron neutrinos as a function of distance in the SN accretion phase, when \( t_{\nu b} = 0.3 \) s. The result is averaged over all angular bins. \( R_\nu \) is set to 25 km, red solid line and blue dotted line are the results with and without GR effect, respectively. The vertical dashed lines labeled \( r_{\text{sync}} \) and \( r_{\text{end}} \) are the predicted beginning and ending locations of bipolar oscillations as given by the equations given in [45]. The position of the shock wave is also indicated and labeled as \( r_{\text{shock}} \).
Figure 4.7 The survival probability of electron neutrinos as a function of distance using neutrino spectra and a density profile taken from the cooling phase of a simulation of a 10.8 $M_\odot$ progenitor by Fischer et al. [6]. The electron flavor survival probability is averaged over all angular bins and energy bins. The red solid line and blue dotted line are the results with and without GR effects respectively.
Figure 4.8 Typical neutrino trajectories near a ultra-compact source. The inner dashed lines and the outer dashed lines represent the Schwarzschild radius and the photon sphere respectively. The three trajectories correspond to three different emission angles.
Figure 4.9 The maximum emission angle of neutrinos that can escape the source, for different values of $R_\nu/r_s$. The vertical dashed line indicates the position of the photon sphere.
REFERENCES


5

NEUTRINO FLAVOR TRANSFORMATION IN SUPERNOVAE AS A PROBE FOR NONSTANDARD NEUTRINO-SCALAR INTERACTIONS

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5.1 Abstract

We explore the possibility of probing the nonstandard interactions between the neutrino and a hypothetical massive scalar or pseudoscalar via neutrino flavor transformation in supernovae. We find that in the ultrarelativistic limit, the effective interaction between the neutrinos vanishes if neutrinos are Dirac fermions but not if they are Majorana fermions. The impact of the new neutrino interaction upon the flavor transformation above the neutrinosphere is calculated in the context of
the multi-angle “neutrino bulb model”. We find that the addition of the nonstandard neutrino self-
interaction (NSSI) to the ordinary V-A self-interaction between neutrinos is capable of dramatically
altering the collective oscillations when its strength is comparable to the standard, V-A, interaction.
The effect of flavor-preserving (FP) NSSI is generally to suppress flavor transformation, while the
flavor-violating (FV) interactions are found to promote flavor transformations. If the neutrino
signal from a Galactic supernova can be sufficiently well understood, supernova neutrinos can
provide complimentary constraints on scalar/pseudoscalar interactions of neutrinos as well as
distinguishing whether the neutrino is a Majorana or Dirac fermion.

5.2 Introduction

The physical conditions found in the core of a core-collapse supernova (CCSN) provide us with an
alternative and complimentary laboratory for probing the properties of the neutrino. In addition to
the extreme matter density, the neutrino density in the vicinity of the proto-neutron star (PNS) is so
high that neutrinos can experience coherent forward-scattering from the other neutrinos emitted
from the PNS. Indeed, during some epochs of the explosion, this neutrino-neutrino self-interaction
can dominate the flavor evolution. The complete description of the flavor transformation in CCSN is
given in terms of Quantum Kinetic Equations [1, 2, 3, 4] which are found to reduce to a Schrödinger-
like equation in the limit where the exchange of energy and momentum between neutrinos and the
medium vanishes. Using Standard Model physics, the Hamiltonian $H$ that enters this equation is
built out of a vacuum contribution $H_V$, a matter contribution $H_M$, and a self-interaction $H_{SI}$. The
self-interaction makes the flavor evolution of one neutrino dependent upon the flavor evolution of
every other neutrino it encounters. The full problem is currently beyond the scope of computing
platforms. The current state-of-the-art model for the calculations of neutrino flavor transformation
in supernovae is known as the “neutrino bulb model” which imposes both spherical symmetry for
neutrino emissions from the neutrinosphere, and axial symmetry around every radial ray, in order to
reduce the number of independent variables needed to describe the neutrino field to just three. The
three degrees of freedom are typically chosen to be: the radial coordinate along a ray, the neutrino
energy, and the angle of emission relative to the normal at the neutrinosphere [5]. Multiple studies
of the neutrino flavor transformation in CCSN using the bulb model have found the addition of $H_{SI}$
can leave distinct features in the neutrino spectra which vary with time and which one would hope
to observe in the signal from a future Galactic supernova: for recent reviews we refer the reader to
Mirizzi et al. [6] and Horiuchi & Kneller [7]

The conditions found in a CCSN mean that any change to the properties of the neutrino often
modify the outcome of the flavor transformation. For example, new - sterile - flavors of neutrinos
have been considered on several occasions [8, 9, 10, 11, 12, 13, 14, 15]. Authors have found that
active-sterile mass-splittings of order $\sim 0.1 \text{ eV}^2$ or greater, and mixing angles larger than $\sim 0.01^\circ$
can introduce new adiabatic Mikheyev-Smirnov-Wolfenstein (MSW) \[16, 17, 18\] resonances close to the PNS whose effect upon the neutrino flavor composition of the flux changes the dynamics of the explosion \[12, 14\] as well as the flavor evolution at larger radii and the neutrino signal \[15, 12\]. Similarly one can also consider new interactions of neutrinos coupled via some new field to either matter (electrons and quarks) or to other neutrinos. There are several studies of the effect of nonstandard interactions of neutrinos with charged fermions and a pair of recent reviews can be found in Miranda and Nunokawa \[19\] and Ohlsson \[20\]. Again, these scenarios often lead to new resonances and flavor evolution which differs substantially from the Standard Model, V-A, case \[21, 22, 23, 24, 25, 26, 27, 28, 29\]. For example, it has been shown one can observe neutrino self-interaction effects in the normal mass ordering when nonstandard interactions are included that cannot occur with just Standard Model physics \[26, 27, 28, 29\]. Alternatively one can also consider non-standard interactions of neutrinos among themselves - so-called non-standard self-interactions (NSSI). Compared with nonstandard interactions of neutrinos with charged fermions, the parameters of NSSI are much less constrained by terrestrial experiments \[30, 31, 32, 33\] and current constraints show that NSSI can be as large as the standard neutrino self-interaction. This provides an unique opportunity for us to take advantage of the CCSN environment as a neutrino laboratory and place complimentary constraints upon unknown interactions among neutrinos.

The form of the NSSI is not unique. Blennow et al. \[27\] and Das et al. explored NSSI for supernova neutrino originating from a non-standard model gauge boson. This form of interaction leads to an effective neutrino-neutrino interaction Hamiltonian similar to the standard V-A except for a flavor-dependent coupling strength and flavor-violating terms \[34\]. Dighe and Sen later applied instability analysis to study the “fast conversion” in the presence of such a NSSI \[35\]. These works show clearly that the presence of NSSI can have significant influence on neutrino flavor transformation in supernovae. For example, it is pointed out the presence of NSSI can lead to flavor equilibration in both mass hierarchies \[27\], and it can also cause collective oscillation in normal mass hierarchy if NSSI is stronger than standard V-A \[34\].

While the gauge boson model is well-motivated, it represents just one category of possible NSSI candidates. Another strong candidate for NSSI is a Yukawa coupling between neutrinos and nonstandard scalar or pseudoscalar fields. This type of interaction has a long history and is used in several models to explain the origin of neutrino mass. One prominent example is the “majoron model” by Gelmini \[36, 37\]. Indeed, constraints on the neutrino-majoron coupling by using the neutrino signal from SN1987A have been made \[38, 39, 40, 41, 42, 43\] although these previous works did not link the neutrino-scalar coupling to neutrino flavor transformation.

Our goal in this paper is to explore the consequence of a neutrino-scalar/pseudoscalar interaction upon the flavor transformation. Our paper is organized in the following way. In section §5.3 we write out the neutrino evolution equation and derive the single-particle effective Hamiltonian of NSSI under the mean field framework, showing the difference between the case of a Dirac neutrino.
and a Majorana neutrino. In section §5.4 we solve the neutrino flavor evolution equations numerically with the NSSI term added to the standard Hamiltonian, using realistic supernova profiles and spectra, and show its impact on neutrino collective oscillations at two different snapshots of a CCSN. We also make a comparison of the results by “single-angle” approach and “multi-angle” approach. In §5.5 we summaize our results and conclude.

5.3 The flavor evolution of supernova neutrinos

In this section we describe the formulism of neutrino flavor transformations in the supernova environment. During a supernova explosion, the ambient region around the contracting core is an environment featuring dense matter, violent turbulence, and an intense flux of neutrinos. What we want to compute is the flavor evolution history of the \( \sim 10^{58} \) neutrinos emitted as the PNS cools. As mentioned earlier, a full treatment of neutrino flavor evolution requires solving the quantum-kinetic equations taking all refraction and scattering effects into account. This is a gigantic task in terms of computational expense. Fortunately it has been demonstrated that neutrino flavor transformations usually happens in regions relatively far from the core due to the dense matter and multiangle suppression effect \([44, 45]\), thus only the refraction effect is relevant and the Schrödinger-like flavor evolution equation for streaming neutrinos can be applied\(^1\).

5.3.1 The equations of flavor evolution

The flavor evolution equation of a test neutrino propagating with momentum \( q \) in the supernova environment takes the following form:

\[
\frac{i}{\Delta} \frac{d S_q}{d \tau} = H(\tau, q) S_q, \tag{5.1}
\]

where \( \tau \) is the “local proper time” \([53]\) and \( S_q \) is the matrix encoding the evolution history of the test neutrino. In ultrarelativistic and weak gravity limit, we can replace \( \tau \) with the distance \( r \) from the center of the neutrinosphere\(^2\). The probability that a neutrino in some generic initial state \( \nu_j \) with momentum \( q \) at distance \( r_0 \) is later detected as state \( \nu_i \) at distance \( r \) is \( P(\nu_j \rightarrow \nu_i) = P_{ij} = |<S_{q,ij}(r; r_0)>|^2 \).

Similarly, the evolution of the antineutrinos is given by an evolution matrix \( \bar{S} \) which evolves according to a Hamiltonian \( \bar{H} \). The total Hamiltonian can be divided into three parts as

\[
H(r, q) = H_V(E) + H_M(r) + H_{SI}(r, \hat{q}) \tag{5.2}
\]

\(^1\)We also note that more recent works on “neutrino fast conversion” \([46, 47, 48, 49, 50, 51, 52, 35]\) indicate flavor transformations may occur close to the PNS potentially upsetting this paradigm.

\(^2\)Throughout the paper we set \( \hbar = c \equiv 1 \).
with \( \hat{q} \) indicating a unit vector in the direction of the neutrino’s momentum. Note that the vacuum term \( H_V \) is only a function of neutrino energy \( E = |q| \), while the matter term \( H_M \) is only dependent on position \( r \). The vacuum term and matter term are straightforward to write out in the flavor basis for a relativistic three flavor neutrino with energy \( E \):

\[
H_V = \frac{1}{2E} \mathbf{U}_V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} \mathbf{U}_V^\dagger, \quad H_M = \sqrt{2} G_F n_\nu(r) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (5.3)

where \( \mathbf{U}_V \) is the vacuum mixing matrix. In the standard model the self-interaction term in the Hamiltonian, \( H_{SI} \), has a form which arises from the V-A interaction and is dependent on both the position and direction of the neutrino’s momentum. The expression for the self-interaction from the V-A interaction is

\[
H_{V-A}(r, \hat{q}) = \sqrt{2} G_F \int \left(1 - \hat{p} \cdot \hat{q}\right) \left[ \rho(r, p) d n_\nu(r, p) - \bar{\rho}^*(r, p) d n_\bar{\nu}(r, p) \right] d E_p.
\] (5.4)

where \( \rho(r, p) \) is the density matrix of the ambient neutrinos at position \( r \) with momentum \( p \) and \( d n_\nu(r, p) \) is the differential neutrino number density [5], which is the differential contribution to the neutrino number density at \( r \) from those neutrinos with energy \( E_p = |p| \) propagating in the directions between \( \hat{p} \) and \( \hat{p} + d\hat{p} \), per unit energy (the hat indicates a unit vector). The quantities \( \bar{\rho}(r, p) \) and \( d n_{\bar{\nu}}(r, p) \) are similar in meaning but for antineutrinos. The differential contribution \( \rho(r, p) d n_\nu(r, p) \) can be further decomposed into \( \rho(r, p) d n_\nu(r, p) = \sum_{\alpha=e,\mu,\tau} \rho_\alpha(r, p) d n_\nu\alpha(r, p) \) by summing over the original flavor states of the neutrinos at the neutrinosphere.

### 5.3.2 The effective Hamiltonian of NSSI

Let us consider the form of the additional contribution to \( H_{SI} \) from a hypothetical coupling between neutrinos via a scalar or pseudoscalar interaction, as shown by Fig. 5.1. Instead of asking the nature of the hypothetical scalar fields, we focus on the phenomenological consequences if such a Yukawa coupling between neutrinos and some scalar fields exists. Generally the coupling can be written as

\[
-\mathcal{L}_{\text{int}} = \frac{1}{2} \mathbf{g}_{a\beta} \bar{\nu}_a \gamma_\beta \phi + \frac{i}{2} \mathbf{h}_{a\beta} \bar{\nu}_a \gamma^a \gamma_\beta \chi.
\] (5.5)

where the \( \phi/\chi \) is the hypothetical scalar/pseudoscalar field, and \( \mathbf{g} \) and \( \mathbf{h} \) are the hermitian coupling matrices\(^3\). In many models the scalar fields are taken to be massless leading to new long range interactions, while in other models the scalar fields are massive leading to a shortening of the range of the interaction considerably. The assumed mass of the scalar/pseudoscalar field and the

\(^3\)For simplicity we assume \( \mathbf{g} \) and \( \mathbf{h} \) are real and symmetric in the following without loss of generality.
typical energy of the neutrinos have considerable impact upon the neutrino phenomenology. In this paper we assume the scalar/pseudoscalar field has a mass larger than the GeV scale, which is well beyond the typical energies of supernova neutrinos. This excludes many scenarios in which the neutrino-scalar field coupling could change the CCSN dynamics through “cooling effects” [43]. This also makes it possible to adopt the “4-fermion” approximation, which is the basis of discussing neutrino-neutrino coherent forward scattering effect in the supernova environments. With this assumption, we can derive an effective neutrino NSSI Hamiltonian in addition to the regular V-A type neutrino self-interaction.

Under the assumption that the mediating particles $\phi$ and $\chi$ are sufficiently massive, the effective interaction Hamiltonian can be written in a 4-fermion form

$$H_{int} = -\mathcal{L}_{int} \approx \frac{1}{8m_\phi} g_{\alpha\beta} g_{\xi\eta} (\bar{\nu}_\alpha \nu_\beta) (\bar{\nu}_\xi \nu_\eta) - \frac{1}{8m_\chi} h_{\alpha\beta} h_{\xi\eta} (\bar{\nu}_\alpha \gamma^5 \nu_\beta) (\bar{\nu}_\xi \gamma^5 \nu_\eta), \quad (5.6)$$

where $m_\phi$ and $m_\chi$ are the rest mass of $\phi$ and $\chi$, respectively. Note that a factor of 1/2 has been introduced to avoid double counting. Just as with the V-A self-interaction, by applying the mean field approximation we can transform the 4-neutrino operators into an effective 2-neutrino operator (see appendix 5.6). Interestingly, the resulting effective Hamiltonian holds different implications for Dirac neutrino and Majorana neutrino. For the Dirac neutrino we find

$$\left(\bar{\nu}_\alpha \nu_\beta\right)\left(\bar{\nu}_\xi \nu_\eta\right) \approx \frac{1}{2} \left(\bar{\nu}_\alpha L \gamma^\mu \nu_\beta\right)\left(\bar{\nu}_\xi R \gamma^\mu \nu_\eta\right) - \frac{1}{2} \left(\bar{\nu}_\alpha R \gamma^\mu \nu_\beta\right)\left(\bar{\nu}_\xi L \gamma^\mu \nu_\eta\right) + (\alpha \eta \leftrightarrow \xi \beta) \quad (5.7)$$

and

$$\left(\bar{\nu}_\alpha \gamma^5 \nu_\beta\right)\left(\bar{\nu}_\xi \gamma^5 \nu_\eta\right) \approx \frac{1}{2} \left(\bar{\nu}_\alpha L \gamma^\mu \nu_\beta\right)\left(\bar{\nu}_\xi R \gamma^\mu \nu_\eta\right) + \frac{1}{2} \left(\bar{\nu}_\alpha R \gamma^\mu \nu_\beta\right)\left(\bar{\nu}_\xi L \gamma^\mu \nu_\eta\right) + (\alpha \eta \leftrightarrow \xi \beta) \quad (5.8)$$

where we have used $(\alpha \eta \leftrightarrow \xi \beta)$ to denote the same terms as the earlier part of the equation but with

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subscripts exchanged. Thus we have decomposed the scalar/pseudoscalar coupling of neutrino fields into products of left-left coupling and right-right coupling of the vector-vector type. However, in the ultrarelativistic limit the right-handed component of neutrino fields vanishes, resulting in a zero contribution to these equations from right-handed neutrino current. So in the Dirac neutrino case, neither scalar nor pseudoscalar interactions can give observable effects in the limit of vanishing neutrino mass.

But if neutrinos are Majorana fermions we find instead

\[
(\bar{\nu}_a \gamma^\mu \nu_\beta)(\bar{\nu}_\xi \gamma_\eta \nu) \approx -\frac{1}{2} \left( \bar{\nu}_{aL} \gamma^\mu \nu_\eta \gamma L \right) \left( \bar{\nu}_\xi \gamma_\eta \nu \right) + \frac{1}{2} \left( \bar{\nu}_\xi \gamma_\eta \nu \gamma L \right) \left( \bar{\nu}_{aL} \gamma^\mu \nu_\eta \right) + (\alpha \eta \leftrightarrow \xi \beta) \quad (5.9)
\]

and

\[
(\bar{\nu}_a \gamma^5 \nu_\beta)(\bar{\nu}_\xi \gamma^5 \nu) \approx \frac{1}{2} \left( \bar{\nu}_{aL} \gamma^\mu \nu_\eta \gamma L \right) \left( \bar{\nu}_\xi \gamma^5 \nu \gamma L \right) + \frac{1}{2} \left( \bar{\nu}_\xi \gamma^5 \nu \gamma L \right) \left( \bar{\nu}_{aL} \gamma^\mu \nu_\eta \right) + (\alpha \eta \leftrightarrow \xi \beta) \quad (5.10)
\]

Unlike the Dirac neutrino, the charge conjugate currents of Majorana neutrino do not vanish even in the limit of zero neutrino mass. From the effective Hamiltonian operators (5.9) and (5.10) we can derive the single-particle Hamiltonian that can be used in neutrino flavor evolution equations by evaluating the average value of neutrino currents under single-particle states. In the following derivation we consider a 2-flavor neutrino but from our result the generalization to neutrinos with more then 2 flavors is straightforward. The single-particle states for neutrino and antineutrino with momentum \( \mathbf{p} \) are

\[
\begin{align*}
| \nu(\mathbf{p}) \rangle &= a_e | \nu_e(\mathbf{p}) \rangle + a_x | \nu_x(\mathbf{p}) \rangle, \\
| \bar{\nu}(\mathbf{p}) \rangle &= \bar{a}_e | \bar{\nu}_e(\mathbf{p}) \rangle + \bar{a}_x | \bar{\nu}_x(\mathbf{p}) \rangle.
\end{align*}
\]

(5.11)

Evaluating the average values on the single-particle states we obtain (see appendix 5.6)

\[
\langle \nu(\mathbf{p}) | \bar{\nu}_{aL} \gamma^\mu \nu_\beta \gamma L | \nu(\mathbf{p}) \rangle = \frac{p^\mu}{E_p V} a^*_a a_\beta, \\
\langle \bar{\nu}(\mathbf{p}) | \bar{\nu}_{aL} \gamma^\mu \nu_\beta \gamma L | \bar{\nu}(\mathbf{p}) \rangle = -\frac{p^\mu}{E_p V} \bar{a}^*_a \bar{a}_\beta
\]

(5.12)

for normal currents and

\[
\langle \nu(\mathbf{p}) | \bar{\nu}_{aL} \gamma^\mu \nu_\beta \gamma L | \nu(\mathbf{p}) \rangle = -\frac{p^\mu}{E_p V} a^*_a a_\beta, \\
\langle \bar{\nu}(\mathbf{p}) | \bar{\nu}_{aL} \gamma^\mu \nu_\beta \gamma L | \bar{\nu}(\mathbf{p}) \rangle = \frac{p^\mu}{E_p V} \bar{a}^*_a \bar{a}_\beta
\]

(5.13)

for charge conjugate currents, respectively. Here \( p^\mu \equiv (E_p, \mathbf{p}) \) is the 4-momentum. If we define the single-particle density matrices as [5]

\[
\rho(\mathbf{p}) = \begin{pmatrix} |a_e|^2 & a_e a^*_x \\ a^*_e a_x & |a_x|^2 \end{pmatrix}, \\
\tilde{\rho}(\mathbf{p}) = \begin{pmatrix} |\tilde{a}_e|^2 & \tilde{a}_e \tilde{a}^*_x \\ \tilde{a}^*_e \tilde{a}_x & |\tilde{a}_x|^2 \end{pmatrix}
\]

(5.14)

for neutrinos and antineutrinos respectively, then the final single-particle effective Hamiltonian of

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the nonstandard neutrino self-interaction can be obtained as (see the appendix 5.7 for details)

\[ H_S(r, \hat{q}) = 4 \int (1 - \hat{p} \cdot \hat{q}) \left\{ \tilde{g} \left[ \rho^*(r, p) d n_\nu(r, p) - \check{\rho}(r, p) d n_\bar{\nu}(r, p) \right] \right \} d E_p \] (5.15)

for neutrino-neutrino interaction via a scalar field and similarly,

\[ H_P(r, \hat{q}) = 4 \int (1 - \hat{p} \cdot \hat{q}) \left\{ \tilde{h} \left[ \rho^*(r, p) d n_\nu(r, p) - \check{\rho}(r, p) d n_\bar{\nu}(r, p) \right] \right \} d E_p \] (5.16)

for neutrino-neutrino interaction through a pseudoscalar field. Here \( E_p \) is the energy of the background neutrinos with momentum \( p \), and the elements of \( \tilde{g} \) and \( \tilde{h} \) are \((\tilde{g})_{\alpha\beta} \equiv \tilde{g}_{\alpha\beta} = \frac{1}{4m_\nu} g_{\alpha\beta} \) and \((\tilde{h})_{\alpha\beta} \equiv \tilde{h}_{\alpha\beta} = \frac{1}{4m_\chi} h_{\alpha\beta} \). Note that Eqs. (5.15) and (5.16) are valid for a neutrino model with arbitrary number of flavors.

Thus we can add to the standard V-A self-interaction a new term given in Eqs. (5.15) and/or (5.16) so that

\[ H_{SI} = H_{V-A} + H_{S/P}. \] (5.17)

At first glance the expressions for the NSSI looks very similar to the NSSI Hamiltonian due to gauge bosons [34], as both of them have a current-current nature and are modulated by the coupling matrix \( \tilde{g} \). However, they are distinct in that wherever the gauge boson Hamiltonian uses the density matrix \( \rho_p (\check{\rho}_p^*) \) the NSSI uses \( \rho^*_p (\check{\rho}_p) \). In addition, the NSSI mediated by a scalar or pseudoscalar field emerges only from the “exchange terms” of the interaction so we do not find the term \( \tilde{g} \text{ Tr} \left[ (\rho_p - \check{\rho}_p^*) \tilde{g} \right] \) which appears in the gauge boson case [34]. We shall see that these subtle nuances between the form of the self-interaction with the standard V-A or, indeed, any gauge-mediated boson interaction, and a scalar/pseudoscalar interaction are key for the NSSI to have distinct observable effects.

### 5.4 The effects of NSSI on neutrino flavor transformation in supernovae

Since the NSSI from scalar and pseudoscalar interactions have the same form we treat them as indistinguishable and focus on the phenomenological consequences of the scalar part of the NSSI. We define two parameters \( \alpha_1 \) and \( \alpha_2 \) so that the \( \tilde{g} \) matrix is parameterized as

\[ \tilde{g} = \left[ \frac{\sqrt{2}}{4} G_F \right]^{1/2} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 \\ \alpha_2 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_2 & \alpha_1 \end{pmatrix}. \] (5.18)

The parameter \( \alpha_1 \) indicates the strength of flavor-preserving (FP) NSSI while \( \alpha_2 \) indicates the strength of flavor-violating (FV) NSSI. When \( \alpha_1 \) or \( \alpha_2 \) is equal to unity it means the corresponding NSSI has an
Figure 5.2 The matter density profiles being used for the calculations of neutrino flavor transformation. The two dashed lines in each plot indicate the beginning and end of the calculation.

Table 5.1 The luminosities, mean energies, and rms energies used for the $t_{pb} = 1.0$ s calculation.

<table>
<thead>
<tr>
<th>Flavor</th>
<th>Luminosity $L_{\nu,\infty}$</th>
<th>Mean Energy $\langle E_{\nu,\infty}\rangle$</th>
<th>rms Energy $\sqrt{\langle E_{\nu,\infty}^2\rangle}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$4.606 \times 10^{51}$ erg/s</td>
<td>10.24 MeV</td>
<td>11.44 MeV</td>
</tr>
<tr>
<td>$\mu, \tau$</td>
<td>$5.473 \times 10^{51}$ erg/s</td>
<td>14.32 MeV</td>
<td>16.78 MeV</td>
</tr>
<tr>
<td>$\bar{e}$</td>
<td>$4.572 \times 10^{51}$ erg/s</td>
<td>12.88 MeV</td>
<td>14.51 MeV</td>
</tr>
<tr>
<td>$\bar{\mu}, \bar{\tau}$</td>
<td>$5.522 \times 10^{51}$ erg/s</td>
<td>14.42 MeV</td>
<td>16.93 MeV</td>
</tr>
</tbody>
</table>

strength equal to the standard V-A interaction. For simplicity we have assumed the flavor-preserving and flavor-violating parameters are identical for all flavors but note this is a restriction that can be relaxed.

The neutrino mixing angles and square mass differences we adopt throughout the rest of the paper are $m_2^2 - m_1^2 = 7.59 \times 10^{-5}$ eV$^2$, $|m_3^2 - m_2^2| = 2.32 \times 10^{-3}$ eV$^2$ $\theta_{12} = 33.9^\circ$ $\theta_{13} = 9^\circ$ and $\theta_{23} = 45^\circ$ which are consistent with the Particle Data Group evaluations [54]. The CP phase $\delta_{CP}$ is set to zero. In the following calculations we will generally work with the inverted mass ordering (IMO) but will show some results using the normal mass ordering (NMO) and will indicate when this occurs.

The density profiles and neutrino spectra for our calculations comes from the 1-D GR-compatible CCSN simulation for the 10.8 M$_{\odot}$ progenitor calculated by Fischer et al. [55]. The matter density profiles are shown in figure (5.2). The neutrino emission is assumed to be half-isotropic and the neutrino spectra at $r$ are given by the pinched thermal spectra found by Keil et al. [56]. Therefore we have

$$dn_{\nu}(r,p) = \frac{L_{\nu,\infty}}{4\pi^2 R_{\nu,\infty}^2} f_\nu(E_p) d(cos \theta) d\phi$$  \hspace{1cm} (5.19)

with

$$f_\nu(E_p) = \frac{(\gamma + 1)^{\gamma+1}}{\Gamma(\gamma + 1)} \frac{E_p^{\gamma+1}}{\langle E_{\nu,\infty}\rangle^{\gamma+1}} \exp\left(-\frac{(\gamma + 1)E_p}{\langle E_{\nu,\infty}\rangle}\right), \hspace{1cm} (5.20)$$
where $\theta$ is the angle between the neutrino beams and the radial direction at $r$, $\phi$ the azimuthal angle of the beam, $L_{\nu,\infty}$ the neutrino luminosity, $\langle E_{\nu,\infty}\rangle$ the mean energy and $\gamma_\nu$ the pinch parameter which can be derived from the mean energy $\langle E_{\nu,\infty}\rangle$ and the mean square energy $\langle E_{\nu,\infty}^2\rangle$ via

$$\gamma_\nu = \frac{2\langle E_{\nu,\infty}\rangle^2 - \langle E_{\nu,\infty}^2\rangle}{\langle E_{\nu,\infty}^2\rangle - \langle E_{\nu,\infty}\rangle^2}. \quad (5.21)$$

The numerical values for the neutrino luminosities, mean and rms energies for post-bounce times of $t_{pb} = 1.0$ s and $t_{pb} = 2.8$ s are shown in tables (5.1) and (5.2). These two snapshots are representative of the early to intermediate cooling phase of CCSN explosion and were chosen based on the results from Wu et al. [57] which showed flavor transformations at these two epochs for the 18.0 $M_\odot$ simulation by Fischer et al. [55] and the similarity of the neutrino spectra in this model with the 10.8 $M_\odot$ simulation also by Fischer et al. The neutrinosphere radius is set to $R_\nu = 19$ km for the $t_{pb} = 1.0$ s profile and $R_\nu = 17$ km for the $t_{pb} = 2.8$ s. For both time slices we compute the evolution starting from $r = 100$ km. Our calculation adopts the multi-angle, multi-energy bulb model framework for energies ranging from 1 MeV to 60 MeV in 200 bins, and the neutrino emission angles ranging from 0° to 90° in 200 bins\(^4\). We have also verified our results have converged with the number of energy and angular bins.

5.4.1 Flavor transformation at $t_{pb} = 1.0$ s

Figure (5.3) shows the numerical results of the survival probabilities of electron neutrino and antineutrino as a function of distance $r$ from the neutrinosphere, for $t_{pb} = 1.0$ s and different values of $\alpha_1$ when $\alpha_2 = 0$. In the left panels the probabilities are averaged over the energy and angular bins used in the calculation; in the right panels the survival probabilities are shown at $r = 400$ km as a function of neutrino energy averaged over the angular distribution only. We see that when there is no NSSI there is a noticeable amount of electron neutrinos transformation into muon and tau neutrinos, and that there are also flavor transformations in the electron antineutrino sector. This is in agreement with the results from Wu et al. [57]. When we add NSSI we can see the flavor transformation in the neutrino sector is delayed although the average survival probability at $r = 400$ km is essentially unchanged. The spectra of the electron neutrinos at $r = 400$ km also look similar for the three values of $\alpha_1$ shown though larger NSSI seems to suppress the transformation of the higher energy neutrinos.

The flavor transformation in the antineutrino sector, however, is more affected by NSSI. As the NSSI is turned on, the transformation is immediately suppressed, with the final survival probability

\(^4\)Determination of the number of angle bins needed in multi-angle calculations can be difficult. Insufficient angular resolution has been found to cause spurious flavor instabilities[58]. However, for the CCSN cooling phase, the matter density is generally not high enough for such artifacts to develop so the required number of angular bins can be reduced. Convergence has been checked to make sure 200 bins are sufficient for both $t_{pb} = 1.0$ s and $t_{pb} = 2.8$ s.
Table 5.2 The luminosities, mean energies, and rms energies used for the $t_{pb} = 2.8$ s calculation.

<table>
<thead>
<tr>
<th>Flavor</th>
<th>Luminosity $L_{\nu,\infty}$</th>
<th>Mean Energy $\langle E_{\nu,\infty} \rangle$</th>
<th>rms Energy $\sqrt{\langle E_{\nu,\infty}^2 \rangle}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$2.504 \times 10^{51}$ erg/s</td>
<td>9.891 MeV</td>
<td>11.12 MeV</td>
</tr>
<tr>
<td>$\mu, \tau$</td>
<td>$2.864 \times 10^{51}$ erg/s</td>
<td>12.66 MeV</td>
<td>14.99 MeV</td>
</tr>
<tr>
<td>$\bar{\nu}$</td>
<td>$2.277 \times 10^{51}$ erg/s</td>
<td>11.83 MeV</td>
<td>13.65 MeV</td>
</tr>
<tr>
<td>$\bar{\mu}, \bar{\tau}$</td>
<td>$2.875 \times 10^{51}$ erg/s</td>
<td>12.70 MeV</td>
<td>15.07 MeV</td>
</tr>
</tbody>
</table>

going back to $P_{\nu_e, \nu_e} = 1$. This suppression effect can be seen more clearly in the sequence of 2-D plots shown figure (5.4), where we can see the region of flavor transformation keeps shrinking with an increasing NSSI in both neutrino and antineutrino sectors.

The effect of the NSSI becomes even more interesting when the flavor-violating NSSI parameter $\alpha_2$ is non-zero. Figure 5.5 shows that the flavor-violating NSSI have the effect of undoing the suppression of the flavor-preserving NSSI. As we can see from the blue curve in the figure, the flavor transformation is restored to the original level (i.e. no NSSI) for the combination $\alpha_1 = 1.5$, $\alpha_2 = 0.6$. At smaller $\alpha_2$, the transformation is only partially restored across the spectrum, as shown by the red curve in the figure. The sequence of 2-D plots shown in figure 5.6 also show the pattern of transformed regions can be largely restored when flavor-violating NSSI is significant.

5.4.2 Flavor transformation at $t_{pb} = 2.8$ s

In order to make sure the “shut-down” effect of NSSI is not specific to some certain settings of the supernova environment, we perform the same kind of calculations for the $t_{pb} = 2.8$ s time slice of 10.8 M⊙ progenitor. In figure (5.7) we plot the results with flavor-preserving NSSI only. It shows a similar “shut-down” effect in the neutrino sector as at $t_{pb} = 1.0$ s. However, flavor transformation does not take place in the antineutrino sector with just the V-A term - this result is consistent with the Wu et al. results [57] - so there is no difference when NSSI is added. From the spectrum at $r = 400$ km we can see the dip in the survival probability becomes shallower as NSSI increased, but the range of flavor transformation remains the same. The sequence of 2-D plots shown in figure (5.8) also show a shrinking of the transformed regions due to NSSI, similar to the shrinking seen in the $t_{pb} = 1.0$ s case. And also as before, the effect of the flavor-violating NSSI is a restoration of flavor transformation to a state as if NSSI does not exist, as shown by figure (5.9) and (5.10).

Finally, it is also interesting to look at the effects of a pure flavor-violating NSSI. As seen in figure (5.11), the pure flavor-violating NSSI is capable of enforcing flavor transformation in the antineutrino sector for the IMO at the post-bounce time of $t_{pb} = 2.8$ s, and the flavor transformation in the neutrino sector is also enhanced for this ordering. When the mass ordering is normal the NSSI can also lead to some flavor oscillations for both neutrino and antineutrinos, especially in the region close to the neutrinosphere, although the final survival probabilities are not very different.
from the result without NSSI even for the case where the flavor-violating parameter $\alpha_2 = 2$. These results with non-zero pure flavor-violating NSSI are qualitatively similar to that found by Das, Dighe and Sen with the gauge boson NSSI [34, 35]. This flavor transformation with pure flavor-violating NSSI can be also compared to the results with only the standard V-A interaction found in Wu et al. [57]. Using the 18.0 $M_\odot$ simulation by Fischer et al. [55], Wu et al. observed no transformation in the antineutrinos and only a small amount of transformation in the neutrinos at these late times.

5.4.3 “Single-angle” vs “multi-angle” approach

In the previous sections we have demonstrated the suppression effect by flavor-preserving NSSI and the effect of undoing the suppression effect by the flavor-violating terms in the NSSI under the “multi-angle” framework. One often sees in the literature on supernova neutrinos reference to a “single-angle” approximation. This approximation assumes the evolution history of a neutrino is independent of its emission direction and is identical with that of the neutrinos propagating in a chosen direction\(^5\). This approximation has been used in previous works about NSSI and supernova neutrinos such as [34, 27]. The “single-angle” approximation greatly reduces runtimes but its drawback is that it has been known to produce collective flavor transformation which is not seen in “multi-angle” calculation due to its artificial synchronization of different angular modes. While in some cases the “single-angle” approach gives qualitatively similar results as “multi-angle” approach, it also lacks the decoherence effect and can often result in flavor transformation occurring at much smaller radii than seen in multi-angle calculations [45]. In this section we compare the “multi-angle” results with “single-angle” counterparts to see whether the effects caused by NSSI can be reproduced more expediently in the single-angle calculations. In the “single-angle” approximation all neutrinos with the same energy share the same evolution history regardless of their direction of propagation, so the NSSI Hamiltonian (5.15) and (5.16) can be simplified to be [5]

\[
\begin{align*}
H_S(r) &= \frac{D(r/R_\nu)}{2\pi R_\nu^2} \int \left\{ \mathbf{g} \left[ \rho^*(r, E) \frac{L_{\nu,\infty}}{E_{\nu,\infty}} f_\nu(E) - \bar{\rho}(r, E) \frac{L_{\bar{\nu},\infty}}{E_{\bar{\nu},\infty}} f_{\bar{\nu}}(E) \right] \mathbf{g} \right\} dE \\
H_P(r) &= \frac{D(r/R_\nu)}{2\pi R_\nu^2} \int \left\{ \mathbf{h} \left[ \rho^*(r, E) \frac{L_{\nu,\infty}}{E_{\nu,\infty}} f_\nu(E) - \bar{\rho}(r, E) \frac{L_{\bar{\nu},\infty}}{E_{\bar{\nu},\infty}} f_{\bar{\nu}}(E) \right] \mathbf{h} \right\} dE
\end{align*}
\]

(5.22) (5.23)

where

\[
D(r/R_\nu) = \frac{1}{2} \left\{ 1 - \sqrt{1 - (R_\nu/r)^2} \right\}^2
\]

(5.24)

is the geometric factor obtained after averaging over all the angular modes. $E \equiv E_\nu$ is the energy of the background neutrinos. The expression for the single-angle version of the V-A interactions can be found in Duan et al. [5].

\(^5\)The chosen direction is often set to be either the radial direction or 45° relative to the radial direction at the neutrinosphere. Here we adopted the radial direction.
In figure (5.12) we plot the survival probabilities for $t_{pb} = 2.8$ s in neutrino sector computed with “single-angle” approach. In the upper panels, we only include the flavor-preserving NSSI. Here we can see that unlike in the “multi-angle” case, the NSSI do not suppress the flavor transformation. Instead, in the final spectrum we notice that the flavor-preserving NSSI actually enhances flavor transformation of the neutrinos in the high energy tail. In the lower panels we again add the flavor-violating terms, and just as “multi-angle” case the effect of the flavor-preserving NSSI is largely wiped out, since the enhanced transformation in the high energy tail disappears. Thus it appears the presence of flavor-preserving NSSI has different effects in “single-angle” and “multi-angle” cases but that single-angle does reproduce the correct trend that the flavor-violating terms always tends to undo any effect caused by flavor-preserving NSSI. The mechanism through which flavor-preserving NSSI shuts down collective oscillations in multi-angle calculation is still a point of interest that needs further investigation.

5.5 Summary and Discussion

In this paper we have derived the effective neutrino-neutrino Hamiltonian due to a NSSI with a scalar/pseudoscalar field and applied it to the case of neutrino flavor transformations at two epochs of a core-collapse supernova. We find that, as in the case of NSSI due to a new neutrino interaction via a gauge boson, there is a suppression effect of the flavor-preserving NSSI which is capable of delaying or shutting down entirely collective flavor oscillation when the strength of the NSSI is comparable to the standard V-A interaction. The presence of flavor-violating terms in the NSSI has the effect of reducing the suppression effect of the flavor-preserving interactions and can even restore the collective flavor oscillations to more-or-less the Standard Model behavior when sufficiently large. When only flavor-violating interactions occur, NSSI can increase the flavor transformation beyond those of V-A alone and even induce oscillations in circumstances where the standard V-A does not. Finally, we find that while the single-angle approximation can give qualitatively similar results to multi-angle calculations as we vary the NSSI parameters, there are large quantitative differences between the two.

In order to exploit our findings we must successfully identify the signatures of collective flavor oscillation in the signal from a Galactic CCSN. If that can be done, our results indicate that supernova neutrinos can provide several complimentary methods for the determination of neutrino properties should the neutrino be a Majorana fermion and the neutrino-scalar interaction be comparable to the standard V-A interaction (but with small flavor violation). First, the effects of observation of scalar or pseudoscalar NSSI could be used as a complimentary method for identifying the Majorana or Dirac nature of the neutrino. If the NSSI is of the order of the weak interaction, NSSI effects have nothing to do with the neutrino mass so appear even if the mass ordering is normal and the Majorana phases conspire to give an neutrinoless double beta decay effective Majorana mass $m_{\beta\beta}$.
which is exactly zero. At the same time, the presence or absence of NSSI signatures in the neutrino signal from a Galactic supernova neutrino burst provides a complimentary tool for measuring, or placing upper limits upon, the coupling strength of NSSI. Current bounds on neutrino-scalar coupling strength are found by a variety of analyses to be $|g|^2 < 10^{-7} \sim 10^{-6}$ for scalar masses below 100 MeV, but there are presently no bounds for scalar masses above 300 MeV [59, 60]. The effective neutrino-neutrino self-interaction we derived is valid only for scalar fields with large masses so NSSI of supernova neutrinos are able to provide constraints in what is currently a blank area in the neutrino-scalar coupling exclusion plot. Finally, the NSSI we have considered in this paper are flavor symmetric even though they may be flavor-violating. Other than simplicity, there is no reason to expect this property to be true. The interaction strength might be unequal for different neutrino flavors or between different pairs of neutrino flavors. Such flavor asymmetry would introduce new phenomenology, as indicated by the results from Das, Dighe & Sen and Dighe & Sen [34, 35] for NSSI due to gauge bosons.

5.6 Appendix A: The mean field approximation

In this section we first derive the mean field expressions of the 4-neutrino operators that appear in the NSSI mediated by scalar fields, namely eq. (5.7) (5.8) (5.9) and (5.10). For generality we start by defining a generic 4-fermion operator as follows

$$M_{1234}^{ab} = \langle \bar{\psi}_1 \Gamma^a \psi_2 \rangle \langle \bar{\psi}_3 \Gamma^b \psi_4 \rangle,$$

(5.25)

here $\Gamma^a$ can be anyone of the 16 $\Gamma$-matrices forming the basis of the vectorial space of all 4 $\times$ 4 matrices. Applying the mean field approximation on the 4-fermion operator results in the following expression

$$M_{1234}^{ab} \approx \langle \bar{\psi}_1 \Gamma^a \psi_2 \rangle \langle \bar{\psi}_3 \Gamma^b \psi_4 \rangle + \langle \bar{\psi}_1 \Gamma^a \psi_2 \rangle \langle \bar{\psi}_3 \Gamma^b \psi_4 \rangle - \sum_{c,d=S,P,V,A,T} C_{ab,cd} \langle \bar{\psi}_1 \Gamma^c \psi_2 \rangle \langle \bar{\psi}_3 \Gamma^d \psi_4 \rangle + \langle \bar{\psi}_1 \Gamma^c \psi_2 \rangle \langle \bar{\psi}_3 \Gamma^d \psi_4 \rangle,$$

(5.26)

The first two terms of Eq. (5.26) represent the regular “Hartree terms”, while the following terms inside the summation are the “exchange terms” arising from the mean field treatment [61]. Note that: a Fierz transformation has been performed to the “exchange terms” since the fermion operators contain spinors, we have dropped the constant term that is present in the mean field expression because it does not have any effect in the evolution equations. In the case of scalar-scalar interaction,
we have $a, b = S$. Replacing the generic fermion fields $\psi$ with neutrino fields, we have

$$\langle \bar{\nu}_1 v_2 \rangle \langle \nu_3 v_4 \rangle \approx \langle \bar{\nu}_1 v_2 \rangle \langle \nu_3 v_4 \rangle + \langle \bar{\nu}_3 v_4 \rangle \langle \nu_1 v_2 \rangle - \sum_{c,d=\text{SS}, PV, AA, T} C_{SS, cd} \left[ \langle \bar{\nu}_3 \Gamma^d v_2 \rangle \langle \nu_1 \Gamma^c v_4 \rangle + \langle \bar{\nu}_1 \Gamma^c v_4 \rangle \langle \nu_3 \Gamma^d v_2 \rangle \right].$$

(5.27)

In the relativistic limit only vector and pseudovector terms can survive the averaging in the single-particle state [62] so we can drop all terms in the right-hand side of equation (5.27) except for the terms with $V \times V$ or $A \times A$ form. Interestingly, the “Hartree terms” are among those who do not survive, which is not the case in the NSSI mediated by gauge bosons where the “Hartree terms” are vector-vector type. Therefore we are left with

$$\langle \bar{\nu}_1 v_2 \rangle \langle \nu_3 v_4 \rangle \approx -\frac{1}{4} \left\langle \bar{\nu}_1 \Gamma^V \nu_4 \right\rangle \left\langle \nu_3 \Gamma^V v_2 \right\rangle + \frac{1}{4} \left\langle \bar{\nu}_1 \Gamma^A \nu_4 \right\rangle \left\langle \nu_3 \Gamma^A v_2 \right\rangle + (14 \leftrightarrow 32)
$$

(5.28)

$$= -\frac{1}{2} \left\langle \bar{\nu}_1 \gamma^\mu P_R v_4 \right\rangle \left\langle \nu_3 \gamma^\mu P_L v_2 \right\rangle - \frac{1}{2} \left\langle \bar{\nu}_1 \gamma^\mu P_L v_4 \right\rangle \left\langle \nu_3 \gamma^\mu P_R v_2 \right\rangle + (14 \leftrightarrow 32),$$

where $\Gamma^V \equiv \gamma^\mu$, $\Gamma^A \equiv \gamma^\mu \gamma^5$ and $P_{L/R} = \frac{1}{2}(1 \pm \gamma^5)$ are the projection operators. Decomposing the neutrino into $\nu = \left( \begin{array}{c} \nu_L
\nu_R \end{array} \right)^T$ for Dirac neutrinos, and $\nu = \left( \begin{array}{c} \nu_L
\nu^C_L \end{array} \right)^T$ for Majorana neutrino, we eventually obtain equations (5.7) and (5.9). The derivation for the equations (5.8) and (5.10) follows a similar path.

Next we derive the expressions for equations (5.12) and (5.13). First we write down the quantized field operator for Majorana neutrino

$$\nu(x) = \sum_{h=\pm 1} \sum_p \frac{1}{2E_p V} \left[ a^{(h)}(p) u^{(h)}(p) e^{-i p \cdot x} + a^{(h)\dagger}(p) \nu^{(h)}(p) e^{i p \cdot x} \right] \equiv \nu^C(x),$$

(5.29)

where $x \equiv x^\mu \equiv (t, \mathbf{x})$ is the 4-position and $p \equiv p^\mu \equiv (E_p, \mathbf{p})$ is the 4-momentum. Then we can decompose the neutrino field into its 2 chirality components $\nu_L(x) = P_L \nu(x)$ and $\nu^C_L(x) = P_R \nu(x)$. If neutrino has mass then both helicity states are present for each of the 2 chirality fields. But in the relativistic limit, for each helicity state, one of the 2 chirality components will be suppressed, resulting in the following equations

$$\nu_L(x) = \sum_p \frac{1}{2E_p V} \left[ a^{(-)}(p) u^{(-)}(p) e^{-i p \cdot x} + a^{(+)}(p) \nu^{(+)}(p) e^{i p \cdot x} \right],$$

(5.30)

and

$$\nu^C_L(x) = \sum_p \frac{1}{2E_p V} \left[ a^{(+)}(p) u^{(+)}(p) e^{-i p \cdot x} + a^{(-)}(p) \nu^{(-)}(p) e^{i p \cdot x} \right].$$

(5.31)

Since Majorana particles are their own antiparticles, we cannot distinguish a Majorana neutrino.
from an antineutrino by their creation and annihilation operators. Nevertheless it is customary to call Majorana neutrino with negative(positive) helicity neutrino(antineutrino), therefore we have (flavor subscripts omitted)

\[ |\nu(p)\rangle \equiv |\nu(p)\rangle = \frac{1}{\sqrt{2E_pV}} a^{(-)\dagger}(p)|0\rangle, \quad |\bar{\nu}(p)\rangle \equiv |\bar{\nu}(p)\rangle = \frac{1}{\sqrt{2E_pV}} a^{(+)\dagger}(p)|0\rangle, \quad (5.32) \]

Note we adopt the finite volume normalization convention from [63] so that the 4-momentum is summed instead of integrated. The corresponding commutation relations for the creation and annihilation operators are

\[ \{a^{(h)}_\alpha(p), a^{(h')\dagger}(p')\} = (2E_pV) \delta_{\alpha\beta} \delta_{hh'} \delta_{pp'}, \quad (5.33) \]

with \(\alpha, \beta\) denoting the neutrino flavor. Combining equations (5.30), (5.31), (5.32) and (5.33), we can obtain the current equations (5.12) and (5.13) with the flavor-superposition states (5.11).

### 5.7 Appendix B: The effective Hamiltonian

In this section we derive the effective single-particle Hamiltonian for the nonstandard neutrino self-interaction, which is to be used in the flavor evolution equation. For simplicity we consider the case in which there are only neutrinos with momentum \(p\) in the background, and the momentum of the test neutrino is \(q\). We start with combining Eqs. (5.6) and (5.9). If we only consider the scalar part, the mean field Hamiltonian operator becomes

\[ H_{\text{MF}}^{\text{S}} = -\bar{g}_{\alpha\beta} \left[ \left( \bar{\nu}_{aL} \gamma^\mu \bar{\nu}_{\eta L} \right) \bar{\nu}_{\xi L} \gamma_\mu \nu_{\beta L} + \left( \bar{\nu}_{\xi L} \gamma_\mu \nu_{\beta L} \right) \bar{\nu}_{\eta L} + \left( a\eta \leftrightarrow \xi\beta \right) \right] \bar{g}_{\bar{\eta}\bar{\xi}} N_\nu, \quad (5.34) \]

where \(N_\nu\) is the number of neutrinos in the background, and \(\bar{g}_{\alpha\beta} = \frac{1}{4m_\nu} g_{\alpha\beta}\). Here we note that the absence of the “Hartree terms” such as \(\left( \bar{\nu}_{aL} \gamma^\mu \bar{\nu}_{\eta L} \right) \bar{\nu}_{\xi L} \gamma_\mu \nu_{\beta L}\) in the Eq. (5.34) is the one of the major differences between a scalar/pseudoscalar NSSI and the NSSI mediated by gauge bosons. Using Eqs. (5.12) and (5.13) we obtain

\[ H_{\text{MF}}^{\text{S}} = \bar{g}_{\alpha\beta} \left( \frac{p^\mu}{E_p} \right) \left[ c_\eta^* c_\alpha \bar{\nu}_{\xi L} \gamma_\mu \nu_{\beta L} - c_\xi^* c_\beta \bar{\nu}_{\eta L} \gamma_\mu \nu_{\alpha L} + \left( a\eta \leftrightarrow \xi\beta \right) \right] \bar{g}_{\bar{\eta}\bar{\xi}} \left( \frac{N_\nu}{V} \right) \quad (5.35) \]

The next step is to evaluate the matrix elements by averaging over the single-particle states of the test neutrino with four momentum \(q^\mu \equiv (E_q, q)\). The \(i, j\) element of the Hamiltonian matrix is

\[ H_{S,ij} = \int_V d^3 x \langle \nu_i(q) | H_{\text{MF}}^{\text{S}} | \nu_j(q) \rangle = 2\left(1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}\right) \left( \bar{g}_{\alpha j} \bar{g}_{\bar{\eta} i} c_\eta^* s_\alpha + \bar{g}_{\alpha i} \bar{g}_{\bar{\eta} j} c_\eta s_\alpha \right) n_\nu, \quad (5.36) \]

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where \( i, j \) are the flavor indices and also representing the corresponding element of \( H_S \). \( n_\nu = N_\nu / V \) is the neutrino density. The angular factor \( 1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} \) comes from the inner product of \( \left( \frac{p^\mu}{E_p} \right) \) and \( \left( \frac{q^\mu}{E_q} \right) \). Since in this paper we assume the coupling matrices are real and symmetric, the result can be simplified to be

\[
H_S = 2 \left( 1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} \right) \left( \tilde{\mathbf{g}} \rho^*(\mathbf{p}) \tilde{\mathbf{g}} + \tilde{\mathbf{g}}^T \rho^*(\mathbf{p}) \tilde{\mathbf{g}}^T \right) n_\nu = 4 \left( 1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} \right) \left( \tilde{\mathbf{g}} \rho^*(\mathbf{p}) \tilde{\mathbf{g}} \right) n_\nu, \tag{5.37}
\]

where the density matrix \( \rho(\mathbf{p}) \) is defined according to Eq. (5.14). Due to the absence of the Hartree terms, we notice there is no term such as \( \tilde{\mathbf{g}} \text{Tr}(\rho \tilde{\mathbf{g}}) \) that appears in the Hamiltonian of the gauge boson case. Finally, the addition of antineutrinos into the background results in an extra term in the Hamiltonian

\[
H_S = 4 \left( 1 - \hat{\mathbf{p}} \cdot \hat{\mathbf{q}} \right) \tilde{\mathbf{g}} \left( \rho^*(\mathbf{p}) n_\nu - \tilde{\rho}(\mathbf{p}) n_\nu \right) \tilde{\mathbf{g}}. \tag{5.38}
\]

In the context of the bulb model we have a collection of neutrino and antineutrino states of different energies and emission angles. To obtain the effective Hamiltonian in the bulb model we need to perform integrations over these distributions which means we must replace \( n_\nu \rightarrow \int d n_\nu d E_p \) and \( n_{\bar{\nu}} \rightarrow \int d n_{\bar{\nu}} d E_p \) thus leading to Eq. (5.15). The derivation of Eq. (5.16) is similar.
Figure 5.3 Survival probability of electron neutrinos (top panels) and antineutrinos (bottom panels) with flavor-preserving NSSI at $t_{pb} = 1.0\, s$. The left panels are the flux averaged probabilities as a function of distance $r$ while the right panels are plotted as function of energy at $r = 400\, km$. The combinations of the NSSI parameters are given in the legends.
Figure 5.4 Top panels: The heatmaps of survival probability of electron neutrinos at $t_{pb} = 1.0\, s$ and $r = 400\, km$ as a function of energy and emission angle when there is only flavor-preserving NSSI. Bottom panels: The same but for electron antineutrinos.
Figure 5.5 Top panels: Survival probability of electron neutrinos at $t_{\mu\nu} = 1.0$ s as a function of distance (left panel) and energy (right panel) at $r = 400$ km with flavor-violating NSSI. The bottom panels are the same but for electron antineutrinos.
Figure 5.6 Top panels: The heatmaps of survival probability of electron neutrinos at $t_{pb} = 1.0\, s$ and $r = 400\, km$ as a function of energy and emission angle when there is flavor-violating NSSI. Bottom panels: The same but for electron antineutrinos.
Figure 5.7 Top panels: Survival probability of electron neutrinos at $t_{\mu\nu} = 2.8 s$ as a function of distance (left panel) and energy (right panel) at $r = 400$ km with flavor-preserving NSSI. The bottom panels are the same but for electron antineutrinos.
Figure 5.8 Top panels: The heatmaps of survival probability of electron neutrinos at $t_{pb} = 2.8\, s$ and $r = 400\, km$ as a function of energy and emission angle when there is only flavor-preserving NSSI. Bottom panels: The same but for electron antineutrinos.
Figure 5.9 Top panels: Survival probability of electron neutrinos at $t_{\mu,\nu} = 2.8$ s as a function of distance (left panel) and energy (right panel) at $r = 400$ km with flavor-violating NSSI. Bottom panels: The same but for electron antineutrinos.
Figure 5.10 Top panels: The heatmaps of survival probability of electron neutrinos at $t_{pb} = 2.8 \text{s}$ and $r = 400 \text{km}$ as a function of energy and emission angle when there is flavor-violating NSSI. Bottom panels: The same but for electron antineutrinos.
Figure 5.11 Top panels: Survival probability of electron neutrinos (left) and antineutrinos (right) at $t_{pb} = 2.8$ s as a function of distance with pure flavor violating NSSI for IMO. Bottom panels: The same as top panels but for NMO.
Figure 5.12 Top panels: “Single-angle” survival probability of electron neutrinos at $t_{\mu,b} = 2.8$ s as a function of distance (left panel) and energy (right panel) at $r = 400$ km with flavor-preserving NSSI. Bottom panels: The same but with flavor-violating terms.
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In this dissertation we have addressed three major issues in the subject of neutrino flavor transformations in supernovae, which are turbulence, gravity and nonstandard neutrino self-interactions. Although these three problems seem independent of each other, they are all important pieces of the jigsaw of a complete and realistic picture of how neutrinos evolve in supernova environments. To conclude the dissertation, in the following part of this chapter we will summarize what we have achieved, and what can possibly be achieved in the future.

**Neutrino Flavor Evolution Through Fluctuating Matter**

This work is a continuation and generalization of a previous trilogy on this subject by K.Patton, J.P.Kneller and G.McLaughlin [1, 2, 3]. In these papers, neutrino flavor evolution under the influence of a Fourier-decomposed matter potential has been investigated in a 2-flavor neutrino model. The investigations mainly showed that the flavor transformation can be greatly enhanced by the matter potential if one of the Fourier modes matches exactly the characteristic frequency of the neutrino model, namely the splitting between the two eigenvalues of the unperturbed Hamiltonian. This effect is called “stimulated flavor transformation”. Additionally, a strong suppression of flavor transformation can also be achieved when there is a low frequency Fourier mode in the matter potential that hits one of the roots of the Bessel functions comprising the transition amplitude [2]. Our new work generalized the previous works in two aspects. First, we generalized the 2-flavor
model to a 3-flavor model. Second, we relaxed the limitation on the form of the Fourier-decomposed potential from one being real and diagonal to one being complex and possibly also containing off-diagonal elements.

By the first generalization, we increased the number of eigenstates of the system from two to three, therefore there are three possible transitions between the eigenstates, corresponding to three resonance frequencies that can cause large flavor transformations. If it is the only difference from the 2-flavor model, then it falls short of an interesting subject since we can always factorize a 3-flavor model into three separate 2-flavor models. However, a real 3-flavor model is distinct from three separate 2-flavor models in that the transitions between different pairs of eigenstates can interfere with each other and give rise to new phenomena such as “induced transparency” and “restored opacity” that cannot be found in 2-flavor models, as described in chapter 2.

A natural next step in this direction is to apply the analytical equations to realistic matter profiles from multi-dimensional supernova simulations. Unlike the plots of Fig. (12) in [3] which has only one stimulating region and one suppression region, there could be several stimulating regions and suppression regions in the 3-flavor case. Once the regions are determined, any Fourier-decomposed matter profile can be put into the diagram to predict if it is possible for a large flavor transformation to happen. And of course, a generalization to incorporate a varying background matter density (which is added to the fluctuating turbulence) is also highly desirable.

The second generalization also has great potential in applications, especially when we want to take neutrino self-interactions into consideration. The V-A neutrino self-interaction potential is known to give rise to off-diagonal elements in the Hamiltonian, which can only be handled in our generalized version of RWA approach. We noticed that some recent works such as [4] are trying to establish an effective theory for the neutrino self-interaction problems. We are convinced that the RWA method can be another avenue of achieving this goal. This part of our work has been collected in the chapter 3 of the dissertation, and we are going to summary our results in the following section.

**Effective Hamiltonian approach for neutrino self-interaction problems**

In this work we made use of the full power of the “Rotating Wave Approximation” method we developed in the study of neutrino flavor evolutions in turbulent environments, and we were trying to establish an effective Hamiltonian approach to handle the neutrino self-interaction problems.

One important motivation of this project is that the neutrino self-interactions are hard to deal with numerically due to its dependence on the solution of the neutrino evolution as an input, which makes the problem non-linear. In numerical calculations this requires a feedback of the solution in each step to the calculation of the Hamiltonian that is to be used in the next step, resulting in major efficiency issues. For the realistic calculations in supernova environments, this often becomes painful when a large number of angular bins and energy bins are used. In the future, if we want
to go beyond the spherically symmetric “bulb model” and explore the richness of the directional dependence of neutrino flavor evolution, the problem will become even more formidable. By using the effective Hamiltonian approach, it is possible to mitigate this issue. And hopefully, in some particular cases the determination of whether a collective flavor oscillation would happen could be done without numerically solving the evolution equations.

For this purpose, an attempt to reproduce the numeric result of the evolution of a monochromatic neutrino in a homogeneous neutrino background was made. The match between the numeric result and the RWA result was found to be within a few percent. In the analytic calculation we used the Fourier-decomposed self-interaction potential from numerical calculations as the input. Even though this approach is still some distance away from being practical since the calculation should be self-contained and self-consistent, it is still a promising first step on verifying the RWA approach when complex potentials with off-diagonal elements are involved. We expect further progress can be made in the future.

**GR effect in neutrino flavor transformations**

In this project we investigated how a strong gravitational field influences the neutrino flavor transformations in supernova environments. This work was initially inspired by a well known astrophysical effect called “gravitational lensing” that makes parts of an invisible rear surface of a neutron star become visible. If supernova neutrinos are ultrarelativistic and can be effectively treated as photons, then there must be a “gravitational lensing” effect for neutrinos, too. With this intuitive picture in mind, we went on to determine how significant the effect could be in realistic supernova settings. To quantitatively determine the enhancement of the neutrino self-interaction, we need to take all generally relativistic effects into account to make the calculation self-consistent. We have identified three GR effects that are important in supernova environment, which are (a) Energy redshift, which makes the effective energy of neutrinos larger when close to the neutrinosphere; (b) Time dilation, which makes the effective neutrino flux larger when close to the neutrinosphere; (c) Trajectory bending, which makes the intersection angle between neutrinos larger. All the three effects contribute to increase the effective neutrino self-interaction potential. The first two dominate in the region close to the neutrinosphere whereas the third one is found to be a long range effect that dominates the regions far away.

Our numerical calculations show the enhancement factor of the neutrino self-interaction is strongly dependent on the compactness of the neutrino source. A typical calculation gives an enhancement factor of $\sim 120\%$ in the supernova accretion phase and $\sim 130\%$ in the supernova cooling phase. However, even though the enhancements of self-interaction potential are considerable, numerical calculations of neutrino flavor evolution show that the enhancements fall short of significantly influencing the survival probability of electron flavor neutrinos. But it is also found that the
collective oscillations are delayed by the enlarged self-interactions, especially in the cooling phase. This may have significant impacts on late time nucleosynthesis since it is very sensitive to the flavor composition of the neutrino fluxes. We also speculate that in the cases where the neutrino source has a larger compactness, such as failed supernovae that eventually form black holes, the GR effect will play a bigger role or even be decisive on how neutrinos evolve.

In our future work, we intend to investigate the influence of GR effects on nucleosynthesis quantitatively. Our work could be built upon existing works on how neutrino flavor transformations could change nucleosynthesis such as [5], and take the GR effect into account. In [5] it is shown that the “multi-angle” calculation, which results in a much later collective oscillation than the gross “single-angle” method, makes a better prediction for the yields of nucleosynthesis. Since the GR effect will generally delay the collective flavor oscillations, we expect the inclusion of GR effects will push the prediction further in the right direction.

We also look forward to generalizing the analysis to other types of neutrino sources such as neutron star mergers. Even though there have already been papers discussing GR effects on neutrino fluxes and spectral properties in sources with disk-like geometry such as black hole accretion disk [6], and papers about neutrino flavor transformations in neutron star mergers using a simplified “single-trajectory” approximation [7, 8, 9], combining different aspects of the GR effect to obtain a complete and self-consistent picture of neutrino evolution near compact celestial bodies remains to be a big challenge.

**Nonstandard neutrino self-interactions**

In this project, we investigated the effect of nonstandard neutrino self-interactions (NSSI) on neutrino flavor transformations in supernovae. This work was inspired by a recent study on how NSSI influences neutrino flavor transformation using a neutrino-neutrino coupling mediated by a hypothetical gauge boson [10], which was followed by a paper putting NSSI in the context of neutrino fast conversions [11]. This kind of non-standard neutrino self-interaction can be seen as a minimal extension of the standard V-A interaction, since they have the same type of mediating particle (massive vector boson) and only involve the left-handed component of the neutrino field. All of the effects of such a nonstandard self-interaction come from its flavor-dependent coupling strengths and flavor-violating components. The resulting alteration to neutrino flavor transformations is mainly dependent on manipulating the corresponding coupling parameters. Moreover, such a NSSI has little to do with whether the neutrinos are Dirac fermions or Majorana fermions. In [12] that studies the possibility of distinguishing Dirac neutrino from Majorana neutrino, the point is made that with interactions only involving the left-hand component of neutrino field there are no observable effects that can be used to distinguish Dirac neutrino from Majorana neutrino when neutrino masses approach zero. This is informally known as the “Practical Dirac-Majorana Confusion Theorem”, and
it is manifested in the neutrinoless double $\beta$—decay \(^{1}\).

Besides nonstandard gauge bosons, scalar particles are another interesting candidate for mediating the NSSI. This type of interaction has a long history in neutrino physics as a possible origin of the neutrino mass. Similar with NSSI via a gauge boson, current experiments have yet to put strong constraints on the coupling strength of neutrino-scalar interactions, especially when the scalar particle is massive \([13]\). The strongest constraints so far come from the measurement of meson decays and the search for heavy neutrinos, but the exclusion diagram is still blank in the region where the mass of the scalar particle is greater than a few hundred MeV \([13]\). Fortunately, in this sector the 4-fermion approximation becomes valid and an effective Hamiltonian can be derived for the coherent forward scattering between neutrinos that occurs in supernova environments. This is why our work provides a complementary way of further constraining the neutrino couplings to nonstandard scalar fields.

In deriving the effective Hamiltonian of the nonstandard neutrino self-interactions we found that the potential would vanish in the limit of zero neutrino mass if neutrinos were Dirac fermions, but it is not the case if neutrinos were Majorana fermions. This kind of distinction is only possible when there are cross terms of the left-handed and the right-handed components of the neutrino field, and coupling the neutrino field to a scalar field is one way of producing the needed terms \([12]\).

Since even the most recent experiments of neutrinoless double $\beta$—decay \([14, 15]\) are not conclusive, it is helpful to find some alternative avenues of determining the nature of neutrino. Our supernova scheme is a convenient one in that if such a dramatic alteration to flavor transformation were to be discovered, we would confirm that neutrinos are Majorana fermions \(\text{and} \) they interact via nonstandard scalar fields in the same time. Though a conspiracy between nonstandard interactions and neutrino being a Majorana fermion is needed for this scheme to succeed, we note that the absence of the alternation to flavor transformations in supernovae is also a valuable negation of either of the two hypotheses. The moment of truth will eventually come when the neutrinos from the next Galactic supernova arrive.

As an extension to our current work, we can add flavor dependence to the neutrino-scalar NSSI and explore the full diversity of the phenomenology. Moreover, as neutrino “fast flavor conversion” (see \([11]\), for example) is becoming a hot topic in the research field of neutrino flavor transformations, a study of the overlap could also be inspiring. Additionally, applying our analysis in the current work to other intense neutrino sources such as neutron star mergers is yet another interesting course to pursue.

\(^{1}\)Neutrinoless double $\beta$—decay cannot occur if the neutrino is massless.
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