

## ABSTRACT

BOULWARE, NAOMI GRACE. Hyperplane Arrangements and  $q$ -Varchenko Matrices. (Under the direction of Kailash C. Misra and Naihuan Jing.)

A hyperplane is an  $(n - 1)$ -dimensional subspace of an  $n$ -dimensional space. Each hyperplane divides the underlying space into two half-spaces. Any real arrangement of hyperplanes partitions the underlying space into disjoint open sets, called regions. The study of hyperplane arrangements has a long history with early origins in questions regarding how many partitions of space a certain number of hyperplanes can form. Arrangements have emerged independently as important objects in various fields of mathematics such as combinatorics, geometry, representation theory, reflection groups, and computer science.

If  $R$  and  $R'$  are two regions in a hyperplane arrangement, then  $\#sep(R, R')$  counts the number of hyperplanes separating the two regions. This combinatorial information about the hyperplane arrangement is recorded in a square matrix called the  $q$ -Varchenko matrix. The Smith normal form of the  $q$ -Varchenko matrix is the unique matrix that reveals the combinatorial structure of the hyperplane arrangement: if the  $q$ -Varchenko matrices for two hyperplane arrangements have the same Smith normal form, then the hyperplane arrangements have the same combinatorial structure.

It is well known that the Smith normal form of a  $q$ -Varchenko matrix for a hyperplane arrangement exists over a principal ideal domain such as  $\mathbb{Q}[q]$ . However, very little is known about when the  $q$ -Varchenko matrix associated with a hyperplane arrangement has a Smith normal form over the ring  $\mathbb{Z}[q]$ . Previous work on this topic focused on hyperplane arrangements in two dimensions. Cai, Chen, and Mu considered several special cases of hyperplane arrangements in the plane, and showed that the  $q$ -Varchenko matrices associated with them had a Smith normal form over  $\mathbb{Z}[q]$ .

In this thesis, we investigate the  $q$ -Varchenko matrices for some hyperplane arrangements with symmetry in two and three dimensions, and prove that they have a Smith normal form over  $\mathbb{Z}[q]$ . In particular, we examine the hyperplane arrangement for a regular  $n$ -gon in the plane, which we call the cyclic model. In  $\mathbb{R}^3$ , we examine the hyperplane arrangement which we define to be the dihedral model; also we investigate the hyperplane arrangements for several polytopes. In each case, we prove that the  $q$ -Varchenko matrix associated with the hyperplane arrangement has a Smith normal form over  $\mathbb{Z}[q]$ , give the transition matrices, and determine the Smith normal form which reveals the combinatorial structure of the hyperplane arrangement.

Hyperplane Arrangements and  $q$ -Varchenko Matrices

by

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## **DEDICATION**

To my parents.

## **ACKNOWLEDGEMENTS**

I would like to thank my advisors for all of their help, inspiration, mentoring, patience, support, and advice.

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## CHAPTER

# 1

## INTRODUCTION

A hyperplane is an  $(n - 1)$ -dimensional subspace in an  $n$ -dimensional space. For example, in the plane, a hyperplane is a line. Every hyperplane divides the underlying space into two half-spaces. Any real arrangement of hyperplanes partitions the underlying space into open sets, called regions. If  $\mathcal{A} = \{h_1, h_2, \dots, h_k\}$  is a real hyperplane arrangement, then for two regions  $R, R'$ ,  $\text{sep}(R, R') = \{h \in \mathcal{A} : h \text{ separates } R \text{ and } R'\}$ . Then it follows that  $\#\text{sep}(R, R') = \#\{h \in \mathcal{A} : h \text{ separates } R \text{ and } R'\}$ , and  $\#\text{sep}(R, R')$  forms a metric on the set of regions of  $\mathcal{A}$ . Hyperplane arrangements have a rich and beautiful variety of structures; they have attracted the attention of researchers in geometry, combinatorics, and algebra.

The study of hyperplane arrangements has a long history with many applications. One of the early papers on hyperplane arrangements was published in 1826 by J. Steiner (see [PS09]); the topic focused on obtaining bounds on the number of cells in arrangements of hyperplanes and circles in the plane, and hyperplanes and spheres in  $\mathbb{R}^3$ . Later, in [OT92], one of the earliest comprehensive books on the topic, Orlik and Terao presented the origins of the study of hyperplane arrangements as a problem in a 1943 monthly journal: "Show that  $n$  cuts can divide a cheese into as many as  $(n + 1)(n^2 - n + 6)/6$  pieces." The solution to this particular problem demonstrated that the maximum number of pieces could be obtained only if the arrangement of planes was in *general position*: every pair of planes intersects in a line, and only one pair of planes intersects in each line; every set of three planes intersect in a point, and only one set of three planes intersect in each point. In 1971, B. Grünbaum wrote a couple of papers summarizing much of what was known at the time (see [OT92]). Here we quote from the introduction to his paper [Grü71]: "...I would like to survey the somewhat related field of *arrangements of hyperplanes*, which I expect to become increasingly popular during the next few years...the theory of arrangements may be developed, much like topology, in rectilinear or curved versions as well as in discrete and continuous variants, and that in these developments it impinges upon many aspects of convexity, topology, and geometry which seemed to be quite unrelated."

In 1993, Alexandre Varchenko defined the Varchenko matrix associated with a hyperplane arrangement in [Var93]. The Varchenko matrix has rows and columns indexed by the regions of the hyperplane arrangement. Each hyperplane in the arrangement is assigned an indeterminate,  $a_H$ . Then the  $(i, j)$  entry of the Varchenko matrix is obtained by taking the product over all  $a_H$  such that  $H \in \text{sep}(R_i, R_j)$ .

More recently, in [GZ18], Gao and Zhang determined the necessary and sufficient conditions for the Varchenko matrix associated with a hyperplane arrangement to have a diagonal form. The Varchenko matrix  $V(\mathcal{A})$  has a diagonal form if and only if  $\mathcal{A}$  is in *semigeneral form*: for any  $k$  hyperplanes in  $\mathcal{A}$  that intersect in one place  $x$ , either  $\text{codim}(x) = k$  or  $x = \emptyset$ . In [Sta16], Stanley noted that Gao and Zhang proved this result for pseudosphere arrangements, which are a generalization of hyperplane arrangements.

$q$ -Varchenko matrices come from these Varchenko matrices. The  $(i, j)$  entry of the  $q$ -Varchenko matrix is  $q^{\#\text{sep}(R_i, R_j)}$ . In [Shi04], Shiu discussed how the invariant factors of these matrices are of interest to many researchers. The invariant factors of these  $q$ -Varchenko matrices are the diagonal entries of their Smith normal form. A square matrix  $A$  over a ring  $R$  has a Smith normal form if there exist transition matrices  $P$  and  $Q$  with determinants that are units in  $R$  such that  $PAQ = D$  where  $D = \text{diag}(d_1, d_2, \dots, d_n)$  is a diagonal matrix whose entries satisfy the condition that  $d_i$  divides  $d_{i+1}$  for all  $1 \leq i < n$ . Significantly, the Smith normal form provides a factorization of the determinant of the matrix  $A$ . While the Smith normal form of a matrix is guaranteed to exist over a principal ideal domain such as  $\mathbb{Q}[q]$  and a standard algorithm for finding it exists, the algorithm is not always computationally practical. In [DH97], Denham and Hanlon illustrate the two problems with directly applying the algorithm: one is that the size of the  $q$ -Varchenko matrix becomes extremely large very quickly in proportion to the number of hyperplanes in the arrangement; the second is that the degrees of the polynomial matrix entries "blow up during the intermediate stages in the computation".

There is very little known about the existence of a Smith normal form of a matrix over rings that are not principal ideal domains. Since  $\mathbb{Z}[q]$  is not a principal ideal domain, a Smith normal form for a  $q$ -Varchenko matrix may or may not exist over  $\mathbb{Z}[q]$ . Most recently, in [CCM16], Cai, Chen, and Mu studied the  $q$ -Varchenko matrices for several specific types of hyperplane arrangements in  $\mathbb{R}^2$ , and showed that they had a Smith normal form over  $\mathbb{Z}[q]$ . In particular, they defined a peelable hyperplane arrangement, and showed that it has a Smith normal form over  $\mathbb{Z}[q]$ ; they also examined the case of a hyperplane arrangement in  $\mathbb{R}^2$  where all of the lines went through the same point, proved that the  $q$ -Varchenko matrix had a Smith normal form over  $\mathbb{Z}[q]$ , and gave the Smith normal form. Then they presented their results in the case of a regular  $n$ -gon hyperplane arrangement in  $\mathbb{R}^2$ : the form of the  $q$ -Varchenko matrix, the steps to transform it into its Smith normal form, and the Smith normal form over  $\mathbb{Z}[q]$ .

The Smith normal form of the  $q$ -Varchenko matrix for a hyperplane arrangement uniquely represents the combinatorial structure of the hyperplane arrangement. Therefore if the Smith normal form of the  $q$ -Varchenko matrices for two hyperplane arrangements is the same, we know that the arrangements have the same combinatorial structure regardless of their distinguishing features. In this thesis, we focus on determining the Smith normal form over  $\mathbb{Z}[q]$  of  $q$ -Varchenko matrices associated with hyperplane arrangements in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  with symmetry. First we revisit the hyperplane arrangement of a regular  $n$ -gon in  $\mathbb{R}^2$  which we call the cyclic model; we formulate an algorithm that allows us to use the symmetry of the arrangement to obtain its  $q$ -Varchenko matrix and determine the Smith normal form over  $\mathbb{Z}[q]$  for an arbitrary  $n$ . Then we move

into  $\mathbb{R}^3$  and define the dihedral model hyperplane arrangement; we obtain its  $q$ -Varchenko matrix using the symmetry of the arrangement and determine the Smith normal form over  $\mathbb{Z}[q]$  for an arbitrary  $n$ . We also give the transition matrices for the Smith normal form. Next, in chapters four and five, we consider the hyperplane arrangements in  $\mathbb{R}^3$  corresponding to a tetrahedron and a cube. Using two different methods, we determine the Smith normal form over  $\mathbb{Z}[q]$  for each of these arrangements and give the transition matrices. Then in chapter six we investigate the  $q$ -Varchenko matrix for the hyperplane arrangement corresponding to an octahedron. Using the symmetry of the arrangement, we give the transition matrices and determine its Smith normal form over  $\mathbb{Z}[q]$ . In chapter seven, the methods used in the study of the  $q$ -Varchenko matrix for the octahedron arrangement are then modified and applied to show that the  $q$ -Varchenko matrix for a hyperplane arrangement corresponding to a pyramid with a square base also has a Smith normal form over  $\mathbb{Z}[q]$ . Lastly, we consider the hyperplane arrangement corresponding to a pyramid with a regular pentagonal base and show that its  $q$ -Varchenko matrix has a Smith normal form over  $\mathbb{Z}[q]$ . In each case, we give the transition matrices and determine the Smith normal form.

## CHAPTER

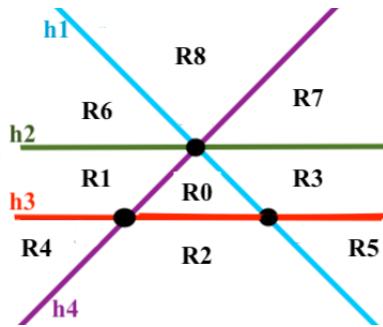
# 2

## PRELIMINARIES

If  $\mathcal{A} = \{h_1, h_2, \dots, h_k\}$  is a real hyperplane arrangement, the set of regions of  $\mathcal{A}$  is denoted by  $\mathcal{R}(\mathcal{A})$ .  $\#sep(R, R')$  forms a metric on  $\mathcal{R}(\mathcal{A})$ .

**Definition 2.0.1.** [Sta04] If we fix a base region  $R_0$ , then the distance enumerator of  $\mathcal{A}$  with respect to  $R_0$  is

$$D_{\mathcal{A}, R_0}(t) = \sum_{R \in \mathcal{R}(\mathcal{A})} t^{d(R_0, R)}.$$



**Figure 2.1** Regions of a Hyperplane Arrangement in  $\mathbb{R}^2$

**Example 2.0.2.** The distance enumerator for  $\mathcal{A}$  shown in Fig. 2.1 is  $D_{\mathcal{A}, R_0}(t) = 1 + 3t + 4t^2 + t^3$ .

**Definition 2.0.3.** [Sta04] The **weak order** (with respect to  $R_0$ ) of  $\mathcal{A}$  is the partial order  $W_{\mathcal{A}}$  on the set of

regions,  $\mathcal{R}(\mathcal{A})$ , given by  $R \leq R'$  if  $\text{sep}(R_0, R) \subseteq \text{sep}(R_0, R')$ . It is a partial ordering of  $\mathcal{R}(\mathcal{A})$  and it is graded by distance from  $R_0$ .

**Example 2.0.4.**  $R_1 \leq R_6 \leq R_8$  in  $W_{\mathcal{A}}$  for the arrangement in Fig. 2.1. The weak order for the arrangement  $\mathcal{A}$  shown in Fig. 2.1 is illustrated below in Fig. 2.2.

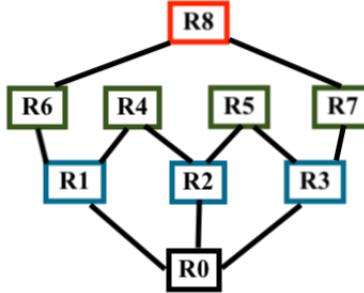


Figure 2.2 The Weak Order for  $\mathcal{A}$  in Fig. 2.1.

**Definition 2.0.5.** [Sta04] The **intersection poset**  $L(\mathcal{A})$  is the set of all nonempty intersections of hyperplanes in  $\mathcal{A}$ , including the underlying space itself as the intersection over the empty set.  $x \leq y$  in the intersection poset if  $y \subseteq x$ . Therefore the underlying space is the minimal element,  $\hat{0}$ .

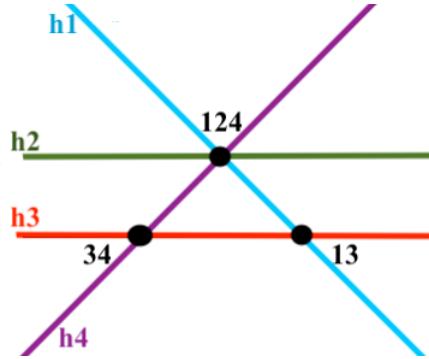


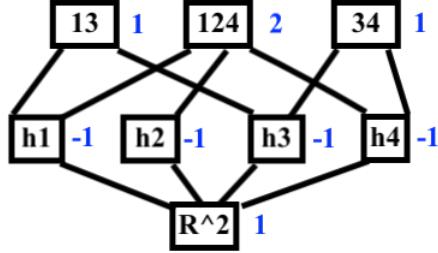
Figure 2.3 Hyperplane Arrangement in  $\mathbb{R}^2$ :  $\mathcal{A} = \{h_1, h_2, h_3, h_4\}$ .

**Example 2.0.6.** Let  $y = h_3 \cap h_4$  which is labelled as 34 in Fig. 2.3. Then  $h_3 \leq y$  and  $h_4 \leq y$  in  $L(\mathcal{A})$ .

**Definition 2.0.7.** The **Möbius function**  $\mu(x, y)$  is defined recursively on the interval  $[x, y] = \{z : x \leq z \leq y\}$ :

1.  $\mu(x, x) = 1$ .
2. For  $x < y$ ,  $\sum_{z \in [x,y]} \mu(x, z) = 0$ .

**Example 2.0.8.** The Möbius function values on the intersection poset  $L(\mathcal{A})$  for the arrangement  $\mathcal{A}$  in Fig. 2.3 are shown in Fig. 2.4.



**Figure 2.4**  $L(\mathcal{A})$  with Möbius function values for the arrangement in Fig. 2.3.

The intersection poset  $L(\mathcal{A})$  and the Möbius function provide a way of counting the number of regions in a hyperplane arrangement.

**Definition 2.0.9.** The **characteristic polynomial** for an arrangement  $\mathcal{A}$  is defined to be

$$\chi(\mathcal{A}, q) = \sum_{x \in L(\mathcal{A})} \mu(\hat{0}, x) q^{\dim(x)}.$$

**Example 2.0.10.** The characteristic polynomial for the arrangement shown in Fig. 2.3 with the intersection poset  $L(\mathcal{A})$  and Möbius function values shown in Fig. 2.4 is

$$\chi(\mathcal{A}, q) = q^2 - 4q + 4.$$

**Theorem 2.0.11.** [Sta04] Let  $\mathcal{A}$  be an arrangement in an  $n$ -dimensional real vector space. Then the number of regions of  $\mathcal{A}$  is given by  $(-1)^n \chi(\mathcal{A}, -1)$ .

**Example 2.0.12.** For the arrangement shown in Fig. 2.3 with the characteristic polynomial  $\chi(\mathcal{A}, q) = q^2 - 4q + 4$ , we can see that  $(-1)^n \chi(\mathcal{A}, -1) = (-1)^2((-1)^2 - 4(-1) + 4) = 1 + 4 + 4 = 9$  which indeed equals the number of regions of  $\mathcal{A}$ .

**Definition 2.0.13.** [Sta04]  $\mathcal{A}_x$  is a subarrangement of  $\mathcal{A}$  defined as  $\mathcal{A}_x = \{h \in \mathcal{A} : x \subseteq h\}$ .

**Example 2.0.14.** Take  $x$  to be the element in  $L(\mathcal{A})$  in Fig. 2.3 labelled as 124:  $x = (h_1 \cap h_2 \cap h_4)$ . Then the subarrangement  $\mathcal{A}_{124}$  is shown in Fig. 2.5.

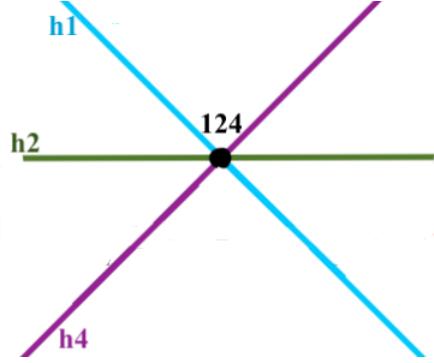


Figure 2.5  $\mathcal{A}_x$

**Definition 2.0.15.** [Sta04] An arrangement  $\mathcal{A}^x$  is a hyperplane arrangement in the affine space  $x \in L(\mathcal{A})$  defined by  $\mathcal{A}^x = \{x \cap h \neq \emptyset : h \in \mathcal{A} - \mathcal{A}_x\}$ .

Let  $\mathcal{A} = \{h_1, h_2, \dots, h_n\}$  be a hyperplane arrangement. If we take  $x = h_k$ , then  $\mathcal{A}^{h_k}$  is an arrangement in the affine subspace  $h_k$ :

$$\mathcal{A}^{h_k} = \{h_k \cap h_i \neq \emptyset : h_i \in \mathcal{A} - \{h_k\}\}. \quad (2.0.1)$$

**Example 2.0.16.** Take  $x$  to be the element  $h_3$  in  $L(\mathcal{A})$  where  $\mathcal{A}$  is the arrangement in Fig. 2.3. Then the arrangement  $\mathcal{A}^{h_3}$  is the hyperplane arrangement of two points in the line  $h_3$  shown in Fig. 2.6.



Figure 2.6  $\mathcal{A}^x$

**Definition 2.0.17.** [CCM16] Let  $\mathcal{A}$  be a finite hyperplane arrangement and let  $h$  be a hyperplane in  $\mathcal{A}$ . Then  $h$  is **peelable** (from  $\mathcal{A}$ ) if there is one half-space  $h_f$  of  $h$  such that if  $R$  is a region of  $\mathcal{A}$  and  $R \in h_f$ , then  $\overline{R} \cap h$  is the closure of a region of  $\mathcal{A}_h$ .

**Definition 2.0.18.** [CCM16] Let  $\mathcal{A} = \{h_1, h_2, \dots, h_m\}$  be a finite hyperplane arrangement.  $\mathcal{A}$  is inductively defined to be a **peelable arrangement** as follows:

1. If  $m = 1$  then  $\mathcal{A} = \{h_1\}$  is peelable.
2. If there is one peelable hyperplane  $h \in \mathcal{A}$  such that both  $\mathcal{A} - \{h\}$  and  $\mathcal{A}_h$  are peelable, then  $\mathcal{A}$  is peelable.

**Example 2.0.19.** The hyperplane  $h_3$  in Fig. 2.1 is peelable from  $\mathcal{A}$  because the intersection of  $h_3$  with the closure of each of the regions  $R_4, R_2, R_5$  is the closure of a region of  $\mathcal{A}^{h_3}$  shown in Fig. 2.6. However,  $\mathcal{A}$  is

not a peelable arrangement since  $\mathcal{A} - \{h_3\}$  (which is the same as  $\mathcal{A}_{124}$  in this case) shown in Fig. 2.5 has no peelable hyperplanes.

Let  $h_k$  be a hyperplane in  $\mathcal{A} = \{h_1, h_2, \dots, h_m\}$ .  $\mathcal{A}' = \mathcal{A} - \{h_k\}$  is called the **deleted arrangement**. The arrangement in  $h_k$  defined by  $\mathcal{A}'' = \{h_i \cap h_k : h_i \in \mathcal{A}'\}$  is called the **restricted arrangement**. Then the triple  $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$  can be used to recursively solve the problem of counting the number of regions of  $\mathcal{A}$ : the **number of regions** of  $\mathcal{A}$  equals the number of regions of  $\mathcal{A}'$  plus the number of regions of  $\mathcal{A}''$  [OT92].

**Example 2.0.20.** Take  $\mathcal{A}$  to be the arrangement shown in Fig. 2.3, and choose  $h_k$  to be  $h_3$ . Then  $\mathcal{A}' = \mathcal{A} - \{h_3\}$  is shown in Fig. 2.5 and  $\mathcal{A}''$  is shown in Fig. 2.6. Observe that the number of regions of  $\mathcal{A} = 9 = 6 + 3 =$  the number of regions of  $\mathcal{A}'$  plus the number of regions of  $\mathcal{A}''$ .

The **q-Varchenko matrix** for a hyperplane arrangement was defined in the introduction.

**Example 2.0.21.** The  $q$ -Varchenko matrix for the hyperplane arrangement shown in Fig. 2.1 is

$$\begin{pmatrix} 1 & q & q & q & q^2 & q^2 & q^2 & q^2 & q^3 \\ q & 1 & q^2 & q^2 & q & q^3 & q & q^3 & q^2 \\ q & q^2 & 1 & q^2 & q & q & q^3 & q^3 & q^4 \\ q & q^2 & q^2 & 1 & q^3 & q & q^3 & q & q^2 \\ q^2 & q & q & q^3 & 1 & q^2 & q^2 & q^4 & q^3 \\ q^2 & q^3 & q & q & q^2 & 1 & q^4 & q^2 & q^3 \\ q^2 & q & q^3 & q^3 & q^2 & q^4 & 1 & q^2 & q \\ q^2 & q^3 & q^3 & q & q^4 & q^2 & q^2 & 1 & q \\ q^3 & q^2 & q^4 & q^2 & q^3 & q^3 & q & q & 1 \end{pmatrix}.$$

## CHAPTER

# 3

## SYMMETRY MODELS

### 3.1 The Cyclic Model

The Smith normal form for the  $q$ -Varchenko matrix associated with the hyperplane arrangement of a regular  $n$ -gon,  $\mathcal{A} = \{h_1, h_2, \dots, h_n\}$ , was presented in [CCM16]. Here we refer to their method as the cyclic model.

**The Cyclic Model,  $C_n$ :** Take the base region  $R_0$  to be the center  $n$ -gon region.  $\#sep(R_0, R_0) = 0$ .

By Definition 2.0.1, when  $\mathcal{A}$  is arrangement of a regular  $n$ -gon and  $R_0$  is the  $n$ -gon's central region,

$$D_{\mathcal{A}, R_0}(t) = 1 + \sum_{k=1}^p nt^k, \quad \text{where } p = \lfloor \frac{n+1}{2} \rfloor. \quad (3.1.1)$$

For  $1 \leq k \leq p$ , there are  $n$  regions  $R$  such that  $\#sep(R_0, R) = k$ .

The form of the  $q$ -Varchenko matrix, the steps to transform the  $q$ -Varchenko matrix into Smith normal form, and the Smith normal form were given in [CCM16]. Here we prove their results. To do so, we present the following algorithm to label the  $n$  hyperplanes and index the regions in such a way that their separating sets can be specified.

The algorithm is based on the existence of  $n$  disjoint saturated chains of length  $p$  in the weak partial ordering of the set of regions,  $\mathcal{R}(\mathcal{A})$ , given by  $W_{\mathcal{A}}$  as it was defined in 2.0.3. Recall that  $R \leq R'$  if  $sep(R_0, R) \subseteq sep(R_0, R')$ .

#### Algorithm 3.1.1.

1. Label the  $n$  hyperplanes  $h_1, \dots, h_n$  so that each  $h_x$  forms an edge of the  $n$ -gon that shares vertices with the edges formed by  $h_{x \pm 1 \pmod n}$ , and so that  $h_2$  follows  $h_1$  moving along the edges in a counter-clockwise direction.

2. Label the central region  $R_0$ .
3. For  $1 \leq x \leq n$ , label the region  $R$  such that  $\text{sep}(R_0, R) = h_x$  as  $R_x$ .
4. Then, label the remaining regions according the following pseudo-code:

Set  $k = 1$ .

While  $k \leq n\{$

Set  $m = 0$ .

While  $(m + 1) \leq p - 1\{$

Choose  $R'$  such that  $\text{sep}(R_0, R_{m+((k+1) \bmod(n))}) = \text{sep}(R_{mn+k}, R')$ .

Label it as  $R_{(m+1)n+k}$ .

$m = m + 1.\}$

$k = k + 1.\}$

For  $1 \leq x \leq n$ :  $R_0 \leq R_x \leq R_{n+x} \leq R_{2n+x} \leq \dots \leq R_{(p-1)n+x}$ . These saturated chains provide a partition of the poset of regions. The  $k = 1$  loop of the algorithm uses:

$$\text{sep}(R_0, R_0) \subseteq \text{sep}(R_0, R_1) \subseteq \text{sep}(R_0, R_{n+1}) \subseteq \text{sep}(R_0, R_{2n+1}) \subseteq \dots \subseteq \text{sep}(R_0, R_{(p-1)n+1}).$$

To illustrate the algorithm, the steps are as follows. Once the  $n$  regions  $R'$  with  $\#\text{sep}(R_0, R') = 1$  have been labelled, choose the region  $R'$  such that  $\text{sep}(R_0, R_2) = \text{sep}(R_1, R')$  and label it as  $R_{n+1}$ .

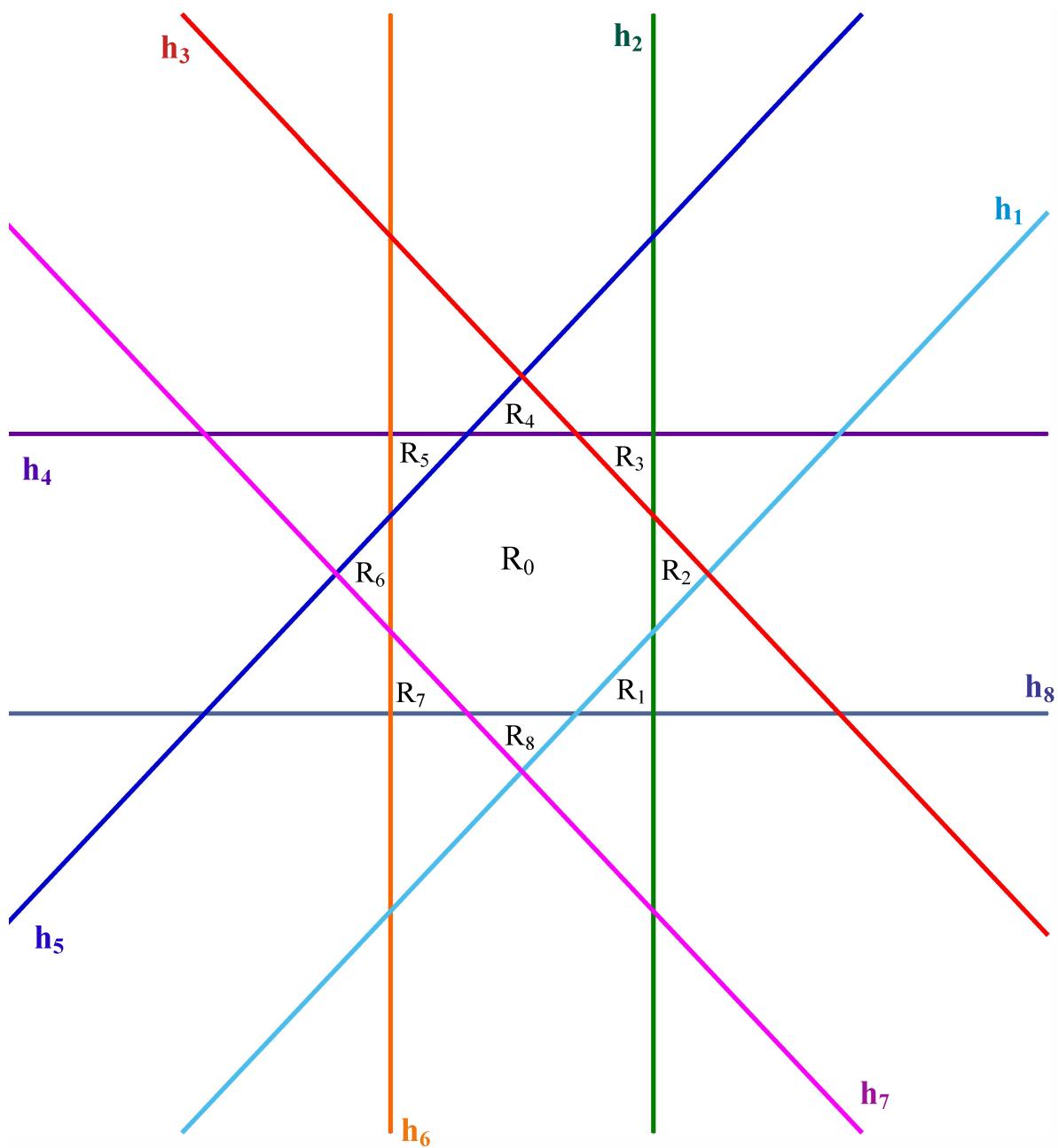
Next choose the region  $R'$  such that  $\text{sep}(R_0, R_3) = \text{sep}(R_{n+1}, R')$  and label it as  $R_{2n+1}$ . Continue until the region  $R_{(p-1)n+1}$  has been labelled.

Then, choose the region  $R'$  such that  $\text{sep}(R_0, R_3) = \text{sep}(R_2, R')$  and label it as  $R_{n+2}$ . Next choose the region  $R'$  such that  $\text{sep}(R_0, R_4) = \text{sep}(R_{n+2}, R')$  and label it as  $R_{2n+2}$ . Continue until the region  $R_{(p-1)n+2}$  has been labelled.

Repeat the process until regions  $R_1, R_{n+1}, \dots, R_{(p-1)n+1}, R_2, R_{n+2}, R_{(p-1)n+2}, \dots, R_n$  have been labelled.

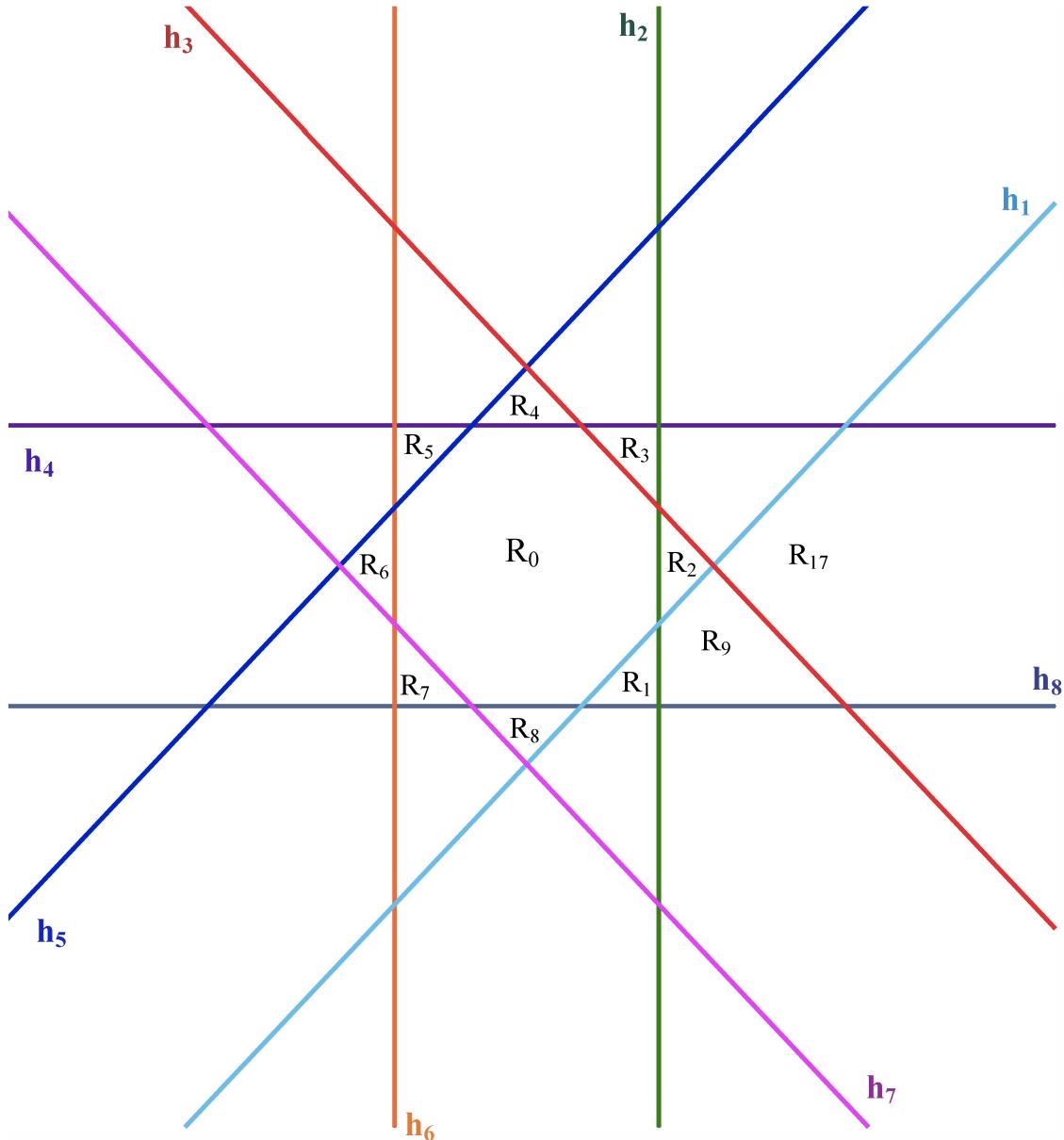
Then choose the region  $R'$  such that  $\text{sep}(R_0, R_1) = \text{sep}(R_n, R')$  and label it as  $R_{2n}$ . Next choose the region  $R'$  such that  $\text{sep}(R_0, R_2) = \text{sep}(R_{2n}, R')$  and label it as  $R_{3n}$ . Continue until the region  $R_{(p-1)n+n}$  has been labelled.

**Illustration after step 3 of Algorithm 3.1.1**



**Figure 3.1** The  $C_8$  hyperplane arrangement after step 3 of Algorithm 3.1.1.

**Illustration of a loop in step 4 of Algorithm 3.1.1:**  $k = 1, m + 1 = p - 1$ .



Note: Before the following steps:

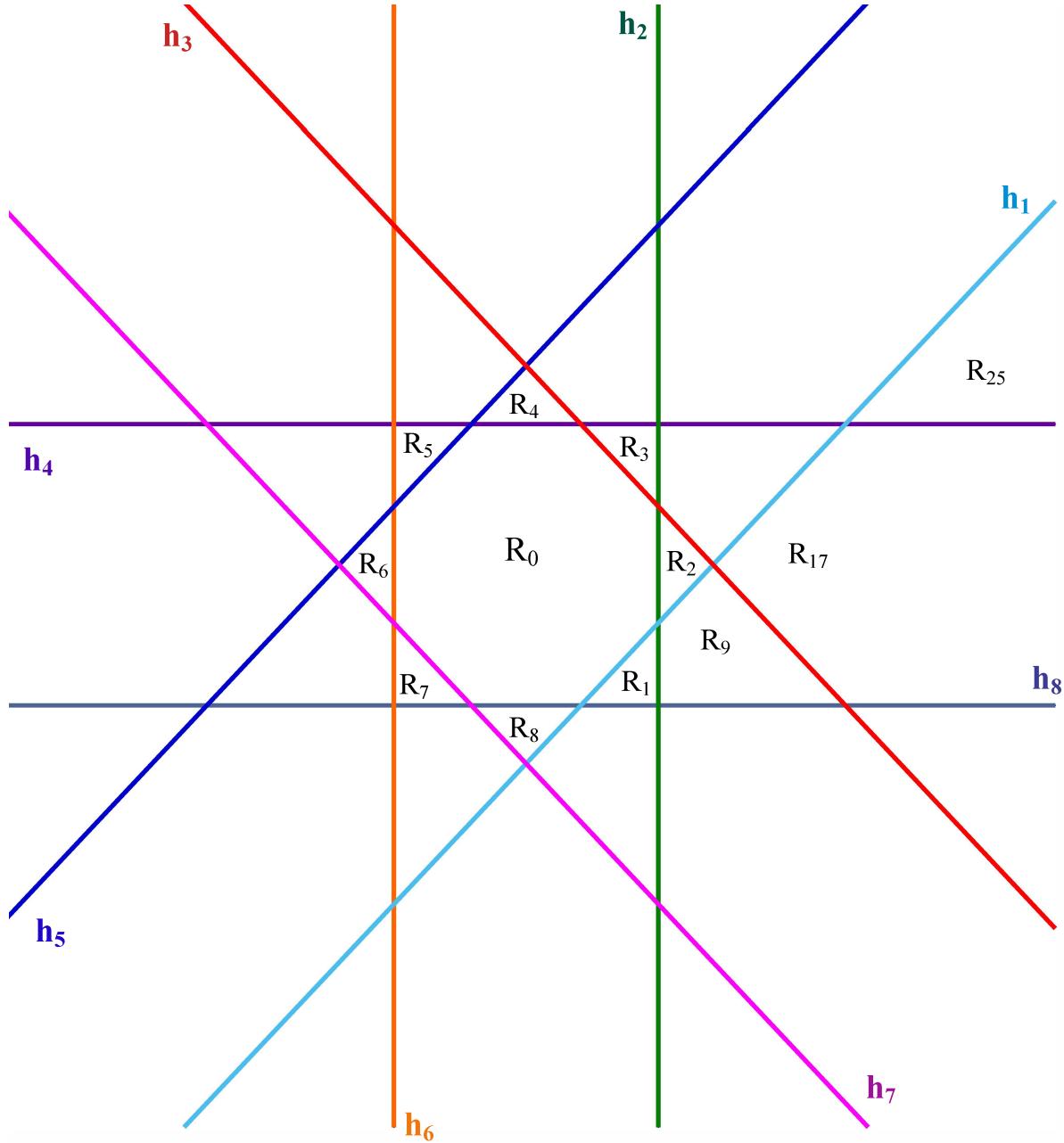
Choose  $R'$  such that  $\text{sep}(R_0, 2 + ((1+1) \bmod n)) = \text{sep}(R_{(3)8+1}, R')$ .

Label it as  $R_{(3)8+1}$ .

$m = m + 1$ .

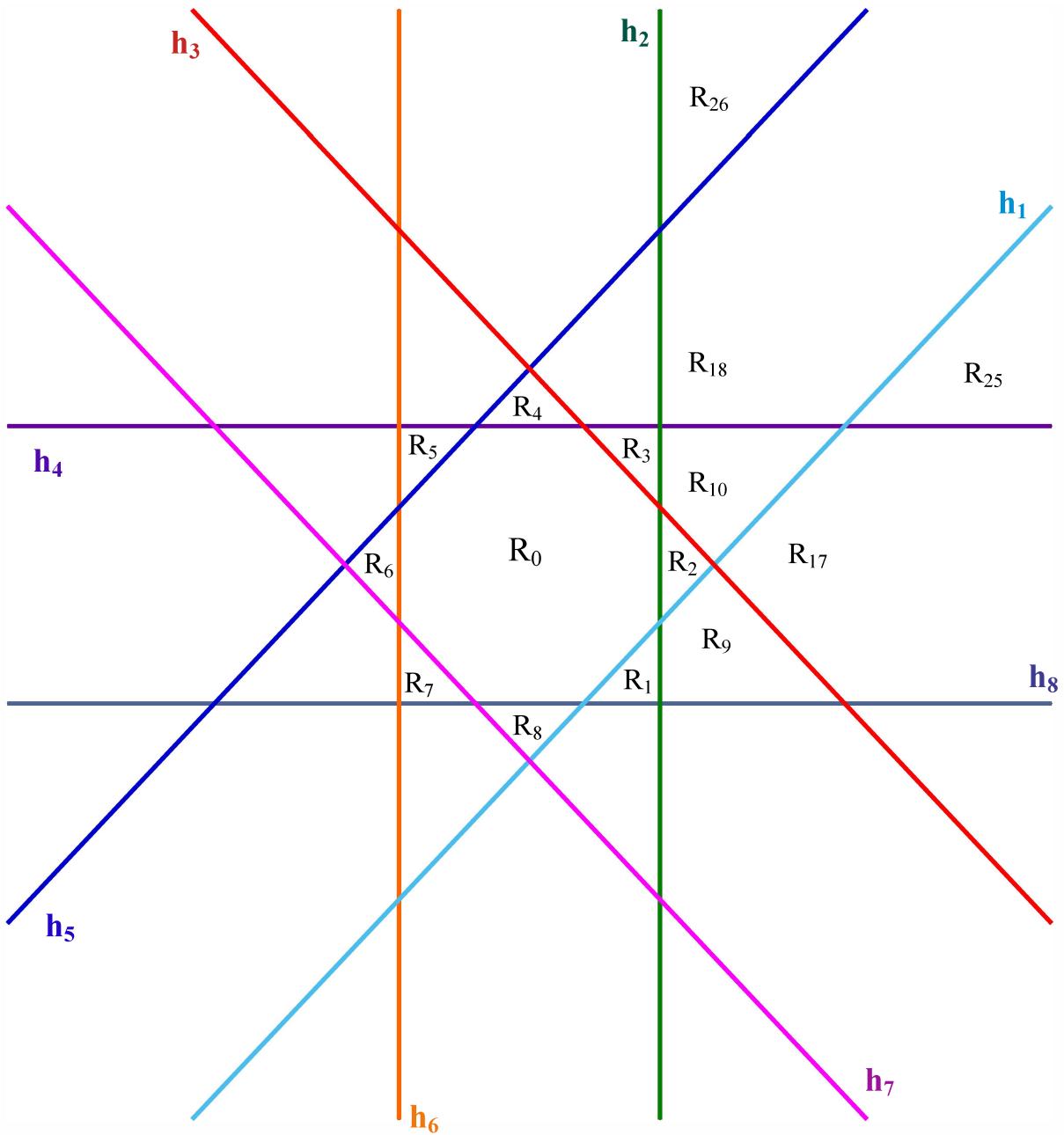
**Figure 3.2** The  $C_8$  hyperplane arrangement in step 4 of Algorithm 3.1.1, where  $k = 1$ , at the beginning of the loop when  $m = 2$ .

**Illustration of a loop in step 4 of Algorithm 3.1.1:  $k = 1, m + 1 = p$ .**



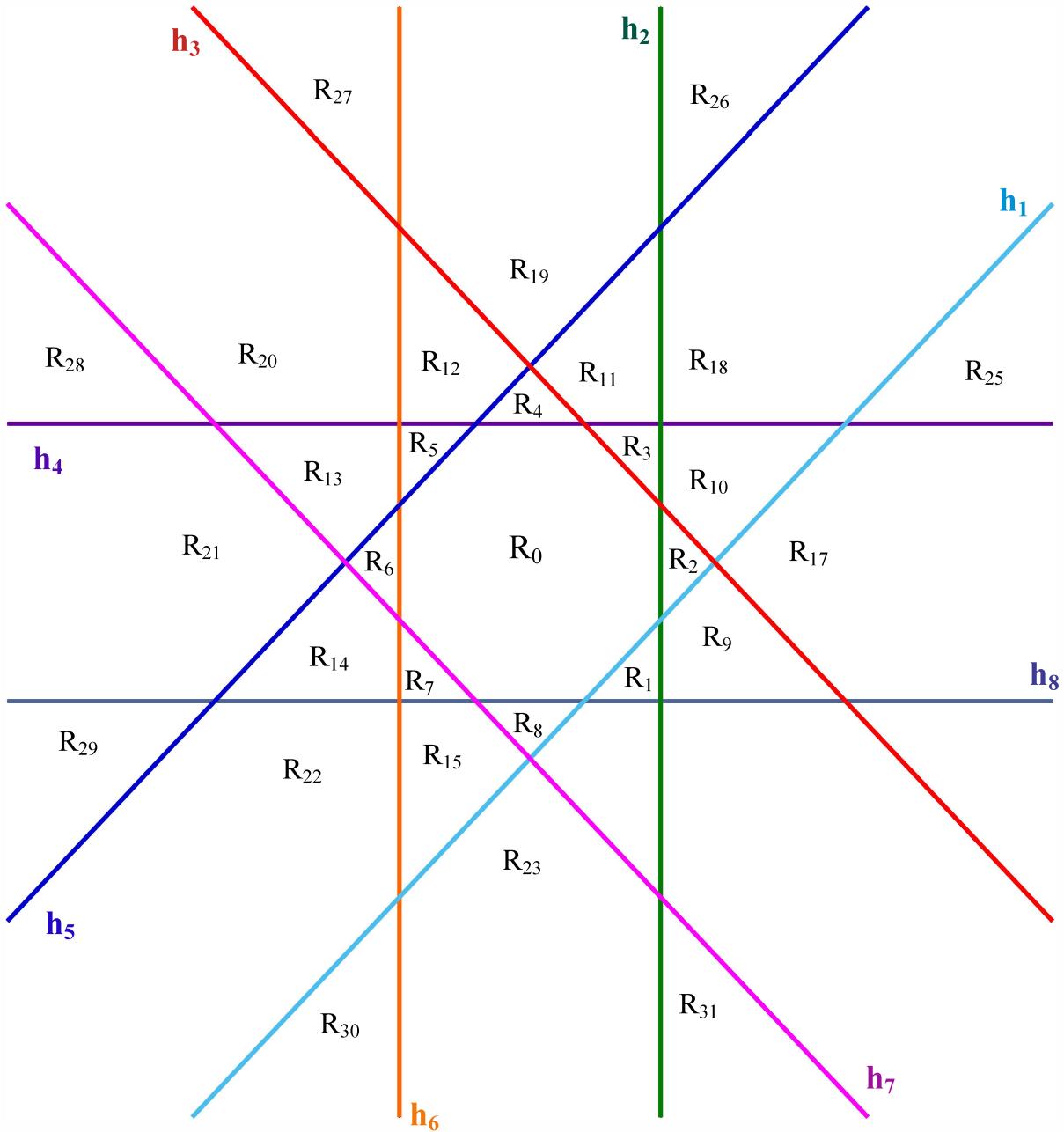
**Figure 3.3** The  $C_8$  hyperplane arrangement in step 4 of Algorithm 3.1.1, where  $k = 1$ , at the end of the loop when  $m = 2$  is re-assigned to  $m = 3$ .

**Illustration of a loop in step 4 of Algorithm 3.1.1:  $k = 2, m + 1 = p$ .**



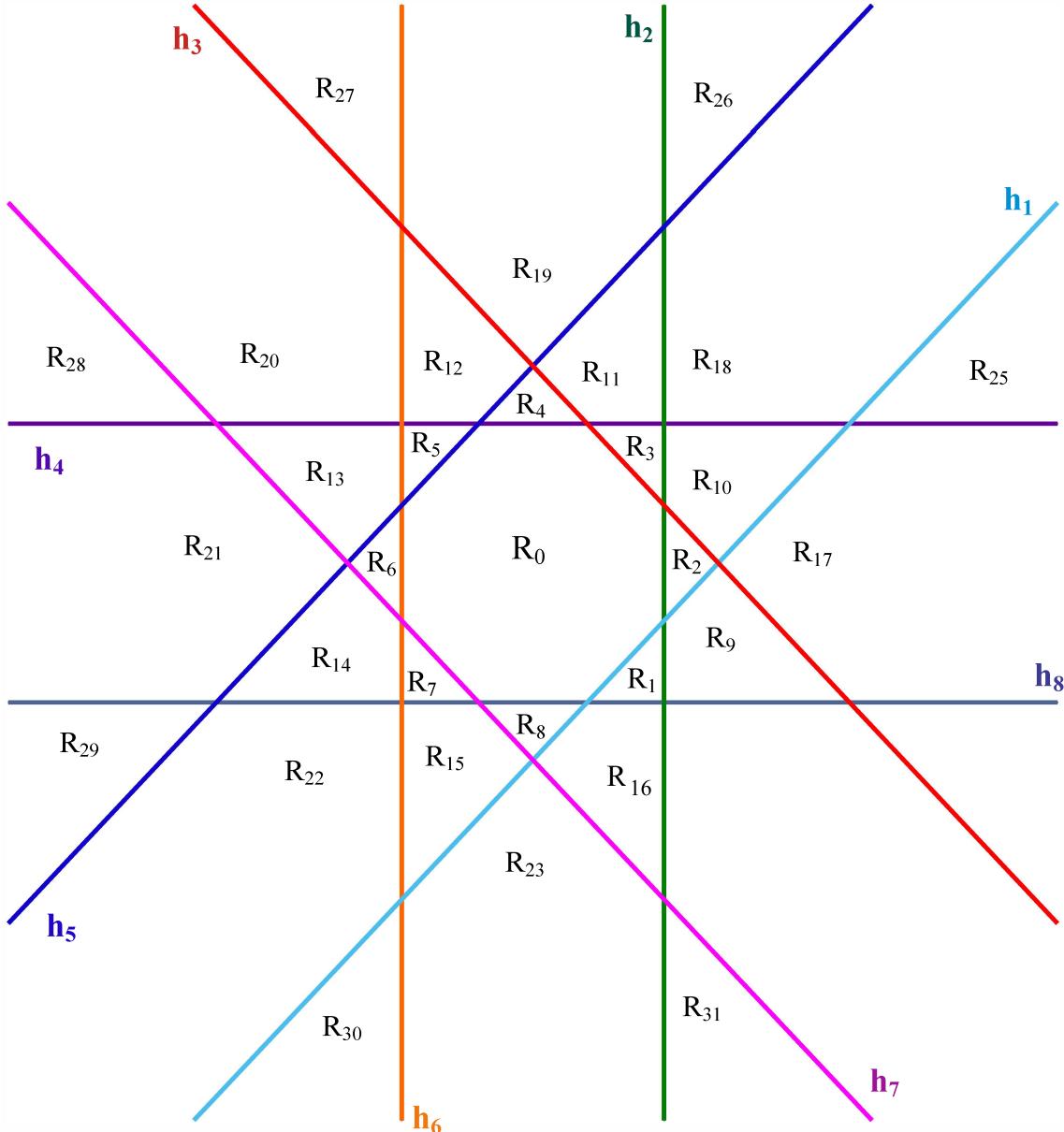
**Figure 3.4** The  $C_8$  hyperplane arrangement in step 4 of Algorithm 3.1.1, where  $k = 2$  and  $m = 3$ .

**Illustration of a loop in step 4 of Algorithm 3.1.1:  $k = n - 1, m + 1 = p$ .**



**Figure 3.5** The  $C_8$  hyperplane arrangement in step 4 of Algorithm 3.1.1, where  $k = n - 1$  and  $m = 3$ .

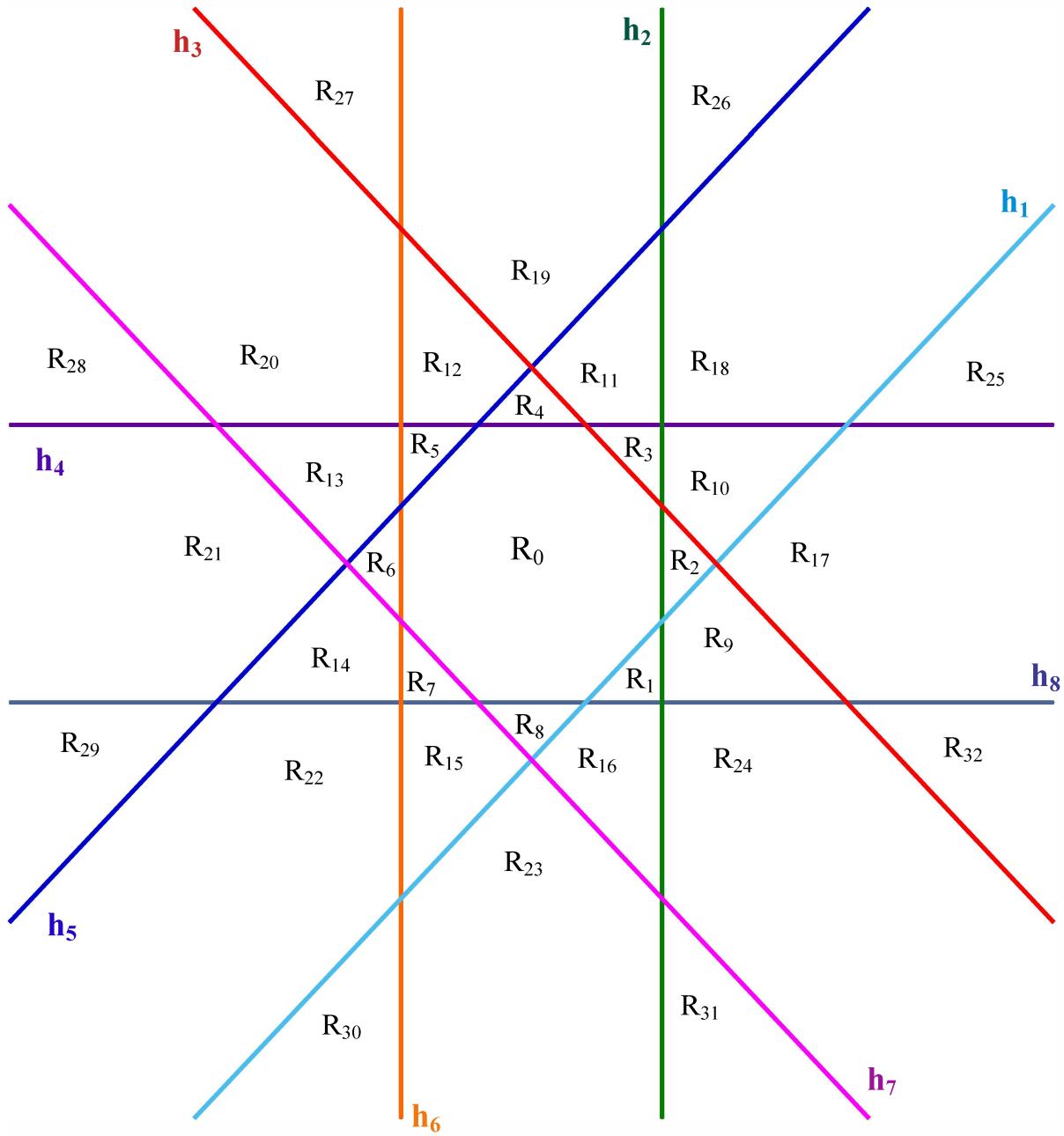
**Illustration of a loop in step 4 of Algorithm 3.1.1:  $k = n, m = 1$ .**



Note: After the  $m = 0$  loop which performed the following steps:  
 Choose  $R'$  such that  $\text{sep}(R_0, 0 + ((8+1) \bmod(n))) = \text{sep}(R_{0(8)+8}, R')$ .  
 Label it as  $R_{(1)8+8}$ .  
 $m = 1$ .

**Figure 3.6** The  $C_8$  hyperplane arrangement in step 4 of Algorithm 3.1.1, where  $k = 8$ , at the beginning of the loop when  $m = 1$ .

**Illustration of Algorithm 3.1.1 at the end of step 4:  $k = n, m + 1 = p$ .**



**Figure 3.7** The  $C_8$  hyperplane arrangement in step 4 of Algorithm 3.1.1, where  $k = 8$  and  $m = 3$ .

**Proposition 3.1.2.**

$$\text{sep}(R, R') = \text{sep}(R_0, R) \cup \text{sep}(R_0, R') - (\text{sep}(R_0, R') \cap \text{sep}(R_0, R)).$$

*Proof.* ( $\subseteq$ ): Let  $h \in \text{sep}(R, R')$ . Then either  $R$  is on the  $-$  side of  $h$  while  $R'$  is on the  $+$  side of  $h$ , or vice versa.  $R_0$  must be in one of the two half-spaces defined by  $h$ . Without loss of generality, suppose  $R_0$  is on the  $+$  side of  $h$ . Then  $h \in \text{sep}(R_0, R)$  and  $h \notin \text{sep}(R_0, R')$ . Therefore,  $h \in \text{sep}(R_0, R) \cup \text{sep}(R_0, R') - (\text{sep}(R_0, R') \cap \text{sep}(R_0, R))$ .

( $\supseteq$ ): Let  $h \in \text{sep}(R_0, R) \cup \text{sep}(R_0, R') - (\text{sep}(R_0, R') \cap \text{sep}(R_0, R))$ . Then, either  $h \in \text{sep}(R_0, R)$ , or  $h \in \text{sep}(R_0, R')$ . Without loss of generality, say  $h \in \text{sep}(R_0, R)$  and  $h \notin \text{sep}(R_0, R')$ . If  $R'$  is on the  $+$  side of  $h$  then so is  $R_0$ ; similarly, if  $R'$  is on the  $-$  side of  $h$  then so is  $R_0$ . Without loss of generality, suppose  $R'$  is on the  $+$  side of  $h$  and therefore so is  $R_0$ . Since  $h \in \text{sep}(R_0, R)$ ,  $R$  must be on the  $-$  side of  $h$ . Hence  $h \in \text{sep}(R, R')$ .  $\square$

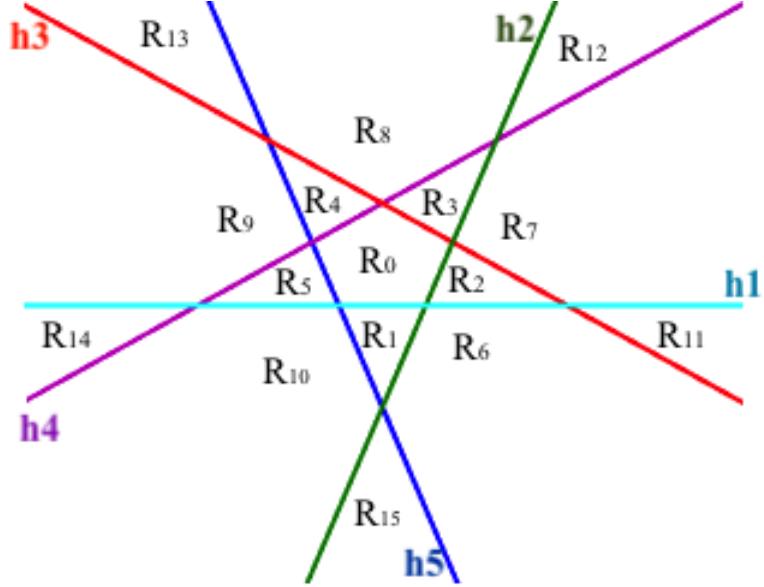
In the cyclic model, the relationship is that for  $1 \leq x \leq n$ ,

$$\begin{aligned} \#\text{sep}(R_{(i-1)n+1}, R_{(j-1)n+x}) &= \#\text{sep}(R_{(i-1)n+2}, R_{(j-1)n+(x+1 \bmod(n))}) \\ &= \#\text{sep}(R_{(i-1)n+3}, R_{(j-1)n+(x+2 \bmod(n))}) \\ &= \dots = \#\text{sep}(R_{(i-1)n+n-1}, R_{(j-1)n+(x+n-2 \bmod(n))}) \\ &= \#\text{sep}(R_{(i-1)n+n}, R_{(j-1)n+(x+n-1 \bmod(n))}). \end{aligned} \quad (3.1.2)$$

The  $(1+np) \times (1+np)$   $q$ -Varchenko matrix for the  $C_n$  arrangement can be written in the following block form where  $E_{ij}$  is an  $n \times n$  circulant matrix, and  $Q_r$  is a  $1 \times n$  row vector  $(q^r, \dots, q^r)$ .  $p$  is as defined in Eq. 3.1.1.

$$V_q = \begin{pmatrix} 1 & Q_1 & Q_2 & Q_3 & \dots & Q_r & \dots & Q_p \\ Q_1^t & E_{11} & E_{12} & E_{13} & \dots & E_{1r} & \dots & E_{1p} \\ Q_2^t & E_{21} & E_{22} & E_{23} & \dots & E_{2r} & \dots & E_{2p} \\ Q_3^t & E_{31} & E_{32} & E_{33} & \dots & E_{3r} & \dots & E_{3p} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ Q_r^t & E_{r1} & E_{r2} & E_{r3} & \dots & E_{rr} & \dots & E_{rp} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ Q_p^t & E_{p1} & E_{p2} & E_{p3} & \dots & E_{pr} & \dots & E_{pp} \end{pmatrix} = \begin{pmatrix} 1 & Q_1 & Q_2 & Q_3 & \dots & Q_r & \dots & Q_p \\ Q_1^t & E_{11} & E_{12} & E_{13} & \dots & E_{1r} & \dots & E_{1p} \\ Q_2^t & E_{12}^t & E_{22} & E_{23} & \dots & E_{2r} & \dots & E_{2p} \\ Q_3^t & E_{13}^t & E_{23}^t & E_{33} & \dots & E_{3r} & \dots & E_{3p} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ Q_r^t & E_{1r}^t & E_{2r}^t & E_{3r}^t & \dots & E_{rr} & \dots & E_{rp} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ Q_p^t & E_{1p}^t & E_{2p}^t & E_{3p}^t & \dots & E_{rp}^t & \dots & E_{pp} \end{pmatrix}. \quad (3.1.3)$$

**Example 3.1.3.** For  $n = 5$ ,



**Figure 3.8** The  $C_5$  hyperplane arrangement.

$$\begin{aligned}
 V_q = & \left( \begin{array}{c|cccccc|cccccc|cccccc}
 1 & q & q & q & q & q & q^2 & q^2 & q^2 & q^2 & q^2 & q^3 & q^3 & q^3 & q^3 & q^3 \\
 q & 1 & q^2 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q^3 & q & q^2 & q^4 & q^4 & q^2 & q^2 \\
 q & q^2 & 1 & q^2 & q^2 & q^2 & q & q & q^3 & q^3 & q^3 & q^2 & q^2 & q^4 & q^4 & q^2 \\
 q & q^2 & q^2 & 1 & q^2 & q^2 & q^3 & q & q & q^3 & q^2 & q^2 & q^2 & q^4 & q^4 & q \\
 q & q^2 & q^2 & q^2 & 1 & q^2 & q^3 & q^3 & q & q & q^3 & q^4 & q^2 & q^2 & q^2 & q^4 \\
 q & q^2 & q^2 & q^2 & q^2 & 1 & q^3 & q^3 & q^3 & q & q & q^4 & q^4 & q^2 & q^2 & q^2 \\
 \hline
 q^2 & q & q & q^3 & q^3 & q^3 & 1 & q^2 & q^4 & q^4 & q^2 & q & q^3 & q^5 & q^3 & q \\
 q^2 & q^3 & q & q & q^3 & q^3 & q^2 & 1 & q^2 & q^4 & q^4 & q & q & q^3 & q^5 & q^3 \\
 q^2 & q^3 & q^3 & q & q & q^3 & q^4 & q^2 & 1 & q^2 & q^4 & q^3 & q & q & q^3 & q^5 \\
 q^2 & q^3 & q^3 & q^3 & q & q & q^4 & q^4 & q^2 & 1 & q^2 & q^5 & q^3 & q & q & q^3 \\
 q^2 & q & q^3 & q^3 & q^3 & q & q^2 & q^4 & q^4 & q^2 & 1 & q^3 & q^5 & q^3 & q & q
 \end{array} \right)_{16 \times 16} \quad (3.1.4) \\
 & = \left( \begin{array}{cccc}
 1 & Q_1 & Q_2 & Q_3 \\
 Q_1^t & E_{11} & E_{12} & E_{13} \\
 Q_2^t & E_{21} & E_{22} & E_{23} \\
 Q_3^t & E_{31} & E_{32} & E_{33}
 \end{array} \right)_{16 \times 16} = \left( \begin{array}{cccc}
 1 & Q_1 & Q_2 & Q_3 \\
 Q_1^t & E_{11} & E_{12} & E_{13} \\
 Q_2^t & E_{12}^t & E_{22} & E_{23} \\
 Q_3^t & E_{13}^t & E_{23}^t & E_{33}
 \end{array} \right)_{16 \times 16}.
 \end{aligned}$$

**Theorem 3.1.4.** For  $1 \leq i \leq j$ ,  $E_{ij}$  is a circulant matrix defined by its first row.

$$E_{ij} = C(q^{j-i}, q^{j-i+2}, q^{j-i+4}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{i+j}, \dots, q^{i+j}}_{n+1-i-j}, q^{j+i-2}, q^{j+i-4}, \dots, q^{j+i-2(i-1)}, \underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i}).$$

*Remark 3.1.5.* Outline of the proof of Theorem 3.1.4: First we will show that the theorem holds for  $i = 1$  and  $j = 1, 2, 3$ . Then we will show that the theorem is true when  $i = 1$  where  $j$  is arbitrary. Next we will continue to let  $j \geq i$  be arbitrary, and show that the theorem is true for  $i = 2$ , and then for  $i = 3$ . Finally we show that the theorem is true for all  $i \leq j \leq p$ .

*Proof.* Let  $n \geq 3$  be arbitrary and  $i \leq j$ .

First we show that the form is true for  $E_{11}$ .

$$\text{sep}(R_0, R_1) = \{h_1\} \text{ and } \text{sep}(R_0, R_x) = \{h_x\}.$$

$$\text{For } 1 < x \leq n, \text{ sep}(R_1, R_0) \cap \text{sep}(R_x, R_0) = \emptyset.$$

By Proposition 3.1.2,  $\text{sep}(R_1, R_x) = \text{sep}(R_1, R_0) \cup \text{sep}(R_0, R_x) = \{h_1, h_x\}$  and  $\#\text{sep}(R_1, R_x) = 2$ .

$$\text{Therefore, } \#\text{sep}(R_1, R_x) = \begin{cases} 0, & \text{for } x = 1, \\ 2, & \text{for } x \neq 1. \end{cases} \text{ Hence } E_{11} = C\left(1, \underbrace{q^2, \dots, q^2}_{n-1}\right).$$

Next we determine  $E_{12}$ .

$$\text{For } 1 \leq x \leq n, \text{ sep}(R_0, R_x) = \{h_x\} \text{ and } \text{sep}(R_0, R_{n+x}) = \{h_x, h_{x+1 \bmod(n)}\}.$$

By Proposition 3.1.2,  $\text{sep}(R_1, R_{n+x}) = \{h_1\} \cup \{h_x, h_{x+1 \bmod(n)}\} - (\{h_1\} \cap \{h_x, h_{x+1 \bmod(n)}\})$ .

$$\text{Therefore, } \#\text{sep}(R_1, R_{n+x}) = \begin{cases} 1, & \text{for } x \in \{1, n\}, \\ 3, & \text{for } x \notin \{1, n\}. \end{cases} \text{ Hence } E_{12} = C(q, \underbrace{q^3, \dots, q^3}_{n-2}, q).$$

Now we determine  $E_{13}$  before moving onto the general case of  $E_{1j}$ .

$$\text{For } 1 \leq x \leq n, \text{ sep}(R_0, R_x) = \{h_x\} \text{ and } \text{sep}(R_0, R_{2n+x}) = \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}\}.$$

By Proposition 3.1.2,

$$\text{sep}(R_1, R_{2n+x}) = \{h_1\} \cup \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}\} - (\{h_1\} \cap \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}\}).$$

$$\text{Therefore, } \#\text{sep}(R_1, R_{2n+x}) = \begin{cases} 2, & \text{for } x \in \{1, n, n-1\} \\ 4, & \text{for } x \notin \{1, n, n-1\}. \end{cases} \text{ Hence } E_{13} = C(q^2, \underbrace{q^4, \dots, q^4}_{n-3}, \underbrace{q^2, q^2}_{3-1=2}).$$

Now consider  $E_{1j}$ .

For  $1 \leq j \leq p$  and  $1 \leq x \leq n$ ,  $\text{sep}(R_0, R_x) = \{h_x\}$  and

$$\text{sep}(R_0, R_{(j-1)n+x}) = \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-1) \bmod(n)}\}.$$

By Proposition 3.1.2,

$$\begin{aligned} \text{sep}(R_1, R_{(j-1)n+x}) &= \{h_1\} \bigcup \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\} \\ &\quad - \left( \{h_1\} \cap \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\} \right). \end{aligned}$$

$$\text{Therefore, } \#\text{sep}(R_1, R_{(j-1)n+x}) = \begin{cases} j-1, & x \in \underbrace{\{1, n, n-1, \dots, n-(j-3), n-(j-2)\}}_{j \text{ terms in the set}}, \\ j+1, & x \notin \{1, n, n-1, \dots, n-(j-3), n-(j-2)\}. \end{cases}$$

$$\text{Hence } E_{1j} = C(q^{j-1}, \underbrace{q^{j+1}, \dots, q^{j+1}}_{n-j}, \underbrace{q^{j-1}, \dots, q^{j-1}}_{j-1}).$$

Now consider  $E_{2j}$ .

For  $2 \leq j \leq p$  and  $1 \leq x \leq n$ ,  $\text{sep}(R_0, R_{n+1}) = \{h_1, h_2\}$  and

$$\text{sep}(R_0, R_{(j-1)n+x}) = \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\}.$$

By Proposition 3.1.2,

$$\begin{aligned} \text{sep}(R_{n+1}, R_{(j-1)n+x}) &= \{h_1, h_2\} \bigcup \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\} \\ &\quad - \left( \{h_1, h_2\} \cap \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\} \right). \end{aligned}$$

There are three possibilities for  $\#\text{sep}(R_{n+1}, R_{(j-1)n+x})$ .

$$\#\text{sep}(R_{n+1}, R_{(j-1)n+x}) = \begin{cases} j-2, & h_1, h_2 \in \text{sep}(R_0, R_{(j-1)n+x}) \\ j, & h_1 \in \text{sep}(R_0, R_{(j-1)n+x}) \text{ but } h_2 \notin \text{sep}(R_0, R_{(j-1)n+x}) \text{ (or vice versa),} \\ j+2, & h_1, h_2 \notin \text{sep}(R_0, R_{(j-1)n+x}). \end{cases}$$

$$\text{Therefore, } \#\text{sep}(R_{n+1}, R_{(j-1)n+x}) = \begin{cases} j-2, & x \in \underbrace{\{1, n, n-1, \dots, n-(j-3)\}}_{j-2 \text{ terms}}, \\ j, & x \in \{2, n-(j-2)\}, \\ j+2, & \text{otherwise.} \end{cases}$$

$$\text{Hence } E_{2j} = C(q^{j-2}, q^j, \underbrace{q^{j+2}, \dots, q^{j+2}}_{n+1-j-2}, q^j, \underbrace{q^{j-2}, \dots, q^{j-2}}_{j-2}).$$

Next consider  $E_{3j}$ .

For  $3 \leq j \leq p$  and  $1 \leq x \leq n$ ,  $\text{sep}(R_0, R_{2n+1}) = \{h_1, h_2, h_3\}$  and

$$\text{sep}(R_0, R_{(j-1)n+x}) = \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\}.$$

By Proposition 3.1.2,

$$\begin{aligned} \text{sep}(R_{2n+1}, R_{(j-1)n+x}) &= \{h_1, h_2, h_3\} \bigcup \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\} \\ &\quad - \left( \{h_1, h_2, h_3\} \cap \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\} \right). \end{aligned}$$

There are four possibilities for  $\#\text{sep}(R_{2n+1}, R_{(j-1)n+x})$ .

$$\#\text{sep}(R_{2n+1}, R_{(j-1)n+x}) = \begin{cases} j-3, & \{h_1, h_2, h_3\} \subseteq \text{sep}(R_0, R_{(j-1)n+x}), \\ j-1, & \text{exactly one of } \{h_1, h_2, h_3\} \notin \text{sep}(R_0, R_{(j-1)n+x}), \\ j+1, & \text{either only } h_1 \in \text{sep}(R_0, R_{(j-1)n+x}) \text{ or only } h_3 \in \text{sep}(R_0, R_{(j-1)n+x}), \\ & \text{Note: only } h_2 \in \text{sep}(R_0, R_{(j-1)n+x}) \text{ is impossible since } j \geq 3. \\ j+3, & h_1, h_2, h_3 \notin \text{sep}(R_0, R_{(j-1)n+x}). \end{cases}$$

$$\text{Therefore, } \#\text{sep}(R_{2n+1}, R_{(j-1)n+x}) = \begin{cases} j-3, & x \in \{1, \underbrace{n, n-1, \dots, n-(j-4)}_{j-3 \text{ terms}}\}, \\ j-1, & x \in \{2, n-(j-3)\}, \\ j+1, & x \in \{3, n-(j-2)\}, \\ j+3, & x \in \{4, \dots, n-(j-1)\}; \text{ equivalently, } 3 < x < n-(j-2). \end{cases}$$

$$\text{Hence } E_{3j} = C(q^{j-3}, q^{j-1}, q^{j+1}, \underbrace{q^{j+3}, \dots, q^{j+3}}_{n+1-j-3}, q^{j+1}, q^{j-1}, \underbrace{q^{j-3}, \dots, q^{j-3}}_{j-3}).$$

Now we consider  $E_{ij}$  where  $i \leq j$ .

For  $i \leq j \leq p$  and  $1 \leq x \leq n$ ,  $\text{sep}(R_0, R_{(i-1)n+1}) = \{h_1, h_2, \dots, h_i\}$  and

$$\text{sep}(R_0, R_{(j-1)n+x}) = \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\}.$$

By Proposition 3.1.2,

$$\begin{aligned} \text{sep}(R_{(i-1)n+1}, R_{(j-1)n+x}) &= \{h_1, h_2, \dots, h_i\} \bigcup \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\} \\ &\quad - \left( \{h_1, h_2, \dots, h_i\} \cap \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(j-2) \bmod(n)}, h_{x+(j-1) \bmod(n)}\} \right). \end{aligned}$$

We will examine three cases (although all can be generalized into one by allowing  $0 \leq k \leq i$ ).

Let  $1 \leq k < i$ .

$$\#sep(R_{(i-1)n+1}, R_{(j-1)n+x}) = \begin{cases} j-i, & \{h_1, h_2, \dots, h_i\} \subseteq sep(R_0, R_{(j-1)n+x}) \\ j-i+2k, & k \text{ elements from } sep(R_0, R_{(i-1)n+1}) \notin sep(R_0, R_{(j-1)n+x}), \\ & \text{ie, exactly one of the following is true:} \\ & h_1, h_2, \dots, h_k \notin sep(R_0, R_{(j-1)n+x}); \\ & h_i, h_{i-1}, \dots, h_{i-(k-1)} \notin sep(R_0, R_{(j-1)n+x}) \\ j+i, & sep(R_0, R_{(i-1)n+1}) \cap sep(R_0, R_{(j-1)n+x}) = \emptyset. \end{cases}$$

Case 1:  $\{h_1, h_2, \dots, h_i\} \subseteq sep(R_0, R_{(j-1)n+x})$  occurs when  $x = 1$  and  $n - (j - i - 1) \leq x \leq n$ .

$$\#sep(R_{(i-1)n+1}, R_{(j-1)n+x}) = j - i \text{ for } x \in \underbrace{\{1, n, n-1, \dots, n-(j-i-1)\}}_{\substack{j-i \text{ terms} \\ j-i+1 \text{ terms in set}}}.$$

Case 2: Exactly  $k$  of  $h_1, h_2, \dots, h_k, \dots, h_i$  are not in  $sep(R_0, R_{(j-1)n+x})$ . This happens when  $x = k + 1$  and when  $x + (j - 1) \bmod(n) = i - k$ .

$$\#sep(R_{(i-1)n+1}, R_{(j-1)n+x}) = j - i + 2k \text{ for } x \in \{k + 1, n - (j - i) - k + 1\}.$$

Case 3:  $sep(R_0, R_{(i-1)n+1}) \cap sep(R_0, R_{(j-1)n+x}) = \emptyset$ . This happens when  $x > i$  and when  $x + (j - 1) \bmod(n) < 1$ ; ie,  $x + (j - 1) < n + 1$ .

$$\#sep(R_{(i-1)n+1}, R_{(j-1)n+x}) = j + i \text{ for } x \in \underbrace{\{i + 1, i + 2, \dots, n - (j - 1)\}}_{n + 1 - i - j \text{ terms in set}}.$$

Therefore, for  $1 \leq k < i$  and  $i \leq j$ ,

$$\#sep(R_{(i-1)n+1}, R_{(j-1)n+x}) = \begin{cases} j - i, & x \in \{1, n, n - 1, \dots, n - (j - i - 1)\}, \\ j - i + 2k, & x \in \{k + 1, n - (j - i - 1) - k\}, \\ j + i, & x \in \{i + 1, i + 2, \dots, n - (j - 1)\}. \end{cases} \quad (3.1.5)$$

This proves the theorem:

$$E_{ij} = C(q^{j-i}, q^{j-i+2}, q^{j-i+4}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{i+j}, \dots, q^{i+j}}_{n+1-i-j}, q^{j+i-2}, q^{j+i-4}, \dots, q^{j+i-2(i-1)}, \underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i}).$$

□

*Remark 3.1.6.*  $E_{ii} = C(c_0, c_1, \dots, c_{n-1})$  is a symmetric matrix.

Observe that for  $k \geq 1$ ,  $c_k = c_{n-k}$  and therefore the first column equals the first row.

We will refer to the row (or column) of the  $q$ -Varchenko matrix given by the vectors  $Q_k$  (or  $Q'_k$ ) with the index  $i = 0$  (or  $j = 0$ ). The row beginning with  $E_{11}$  will denote the index  $i = 1, j = 1$ .

**Proposition 3.1.7.**  $V_q$  for  $C_n$  can be transformed into its Smith Normal Form over  $\mathbb{Z}[q]$  in four steps.

The remainder of this section will prove this proposition.

**Step. 1: Multiply  $V_q$  on the left by matrices  $P_1, \dots, P_k, \dots, P_{p=\lfloor \frac{n+1}{2} \rfloor}$  to obtain  $P_p \dots P_k \dots P_1 V_q$ ,**

where

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \ddots & I_n & 0 \\ 0 & 0 & 0 & \dots & -qI_n & I_n \end{pmatrix}, P_{p-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & -qI_n & I_n & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \ddots & I_n & 0 \\ 0 & 0 & 0 & \dots & 0 & I_n \end{pmatrix}, P_p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -qI_{n \times 1} & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \ddots & I_n & 0 \\ 0 & 0 & 0 & \dots & 0 & I_n \end{pmatrix}.$$

This sequence of left matrix multiplications performs the row operations of beginning with the last block row, sequentially subtracting  $q$  times the previous block row from it, and then performing the same operation on the second to last block row, and so on up to the first block row.

*Remark 3.1.8.* For  $1 \leq k \leq p$

$$Q_k^t - qQ_{k-1}^t = 0. \quad (3.1.6)$$

The first row of the result of multiplying the  $q$ -Varchenko matrix by  $P_p \dots P_1$  on the left is given by  $E_{ij} - q^{j+1} \text{ones}_{n \times n}$ .

*Remark 3.1.9.*

$$E_{11} - qQ_1 = (1 - q^2)I.$$

**Definition 3.1.10.**

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}_{n \times n}, \text{ and } J^t = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 \end{pmatrix}_{n \times n}.$$

$J$  is the matrix associated to the permutation written in cycle notation as  $(n \ n-1 \ \dots \ 2 \ 1)$ .

$J'$  is the matrix associated to the permutation written in cycle notation as  $(1 \ 2 \ \dots \ n-1 \ n)$ .

**Proposition 3.1.11.** At the end of the first step, the original  $E_{ij}$  block matrices have the form

$$E_{ij} - qE_{i-1,j} = \begin{cases} q^{j-1}(1-q^2) \sum_{k=0}^{j-1} (J^t)^k & \text{for } i = 1, \\ q^{j-i}(1-q^2) \left( \left( \sum_{k=0}^{j-i} (J^t)^k \right) + \left( \sum_{m=1}^{i-1} q^{2m} J^m \right) \right), & \text{for } 1 < i \leq j \\ q^{j-i}(1-q^2) J^{j-i} \sum_{k=0}^{i-1} q^{2k} J^k, & \text{for } j > i. \end{cases} \quad (3.1.7)$$

Before we can prove this proposition, we need the following lemmas.

**Lemma 3.1.12.** Let  $qQ_j$  denote the  $n \times n$  matrix with all entries  $q^{j+1}$ .

$$E_{1j} - qQ_j = q^{j-1}(1-q^2) \sum_{k=0}^{j-1} (J^t)^k.$$

*Proof.* Write  $J^t = C(0, \dots, 0, 1)$ . Note that  $J^t = J^{n-1}$ , and that  $(J^t)^n = J^n = I_n$ . For  $k > n$ ,  $J^k = J^{k \bmod(n)}$ , and  $(J^t)^k = J^{(n-1)k \bmod(n)}$ .

For  $1 \leq k \leq n$ ,

$$(J^t)^k = C(\underbrace{0, \dots, 0}_{n-k}, 1, \underbrace{0, \dots, 0}_{k-1}).$$

$$\begin{aligned} E_{1j} &= C(q^{j-1}, \underbrace{q^{j+1}, \dots, q^{j+1}}_{n-j}, \underbrace{q^{j-1}, \dots, q^{j-1}}_{j-1}). \\ E_{1j} - qQ_j &= C(q^{j-1} - q^{j+1}, \underbrace{0, \dots, 0}_{n-j}, \underbrace{q^{j-1} - q^{j+1}, \dots, q^{j-1} - q^{j+1}}_{j-1}) \\ &= q^{j-1}(1-q^2) C(1, \underbrace{0, \dots, 0}_{n-j}, \underbrace{1, \dots, 1}_{j-1}) \\ &= q^{j-1}(1-q^2) \sum_{k=0}^{j-1} (J^t)^k. \end{aligned}$$

□

**Lemma 3.1.13.** For  $i \leq j$ ,

$$E_{ij} - qE_{i-1,j} = q^{j-i}(1-q^2) \left( \left( \sum_{k=0}^{j-i} (J^t)^k \right) + \left( \sum_{m=1}^{i-1} q^{2m} J^m \right) \right).$$

*Proof.*

$$\begin{aligned}
E_{ij} &= \\
C(q^{j-i}, q^{j-i+2}, q^{j-i+4}, \dots, q^{j-i+2(i-2)}, q^{j-i+2(i-1)}, &\underbrace{q^{i+j}, \dots, q^{i+j}}_{n-i-j+1}, q^{j+i-2}, q^{j+i-4}, \dots, q^{j-i+2}, \underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i}). \\
E_{i-1,j} &= \\
C(q^{j-i+1}, q^{j-i+3}, q^{j-i+5}, \dots, q^{j-i+1-2(i-2)}, &\underbrace{q^{i+j-1}, \dots, q^{i+j-1}}_{n-i-j+2}, q^{j+i-3}, q^{j+i-5}, \dots, q^{j-i+3}, \underbrace{q^{j-i+1}, \dots, q^{j-i+1}}_{j-i+1}). \\
E_{ij} - qE_{i-1,j} &= \\
= C(q^{j-i}(1-q^2), q^{j-i+2}(1-q^2), q^{j-i+4}(1-q^2), \dots, q^{j-i+2(i-2)}(1-q^2), q^{j-i+2(i-1)}(1-q^2), &\underbrace{0, \dots, 0}_{n-i-j+1}, \\
0, \dots, 0, &\underbrace{q^{j-i}(1-q^2), \dots, q^{j-i}(1-q^2)}_{j-i}) \\
&= (1-q^2)C(q^{j-i}, q^{j-i+2}, q^{j-i+4}, \dots, q^{j-i+2(i-2)}, q^{j-i+2(i-1)}, &\underbrace{0, \dots, 0}_{n-i-j+1}, 0, \dots, 0, \underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i}) \\
&= q^{j-i}(1-q^2)C(1, q^2, q^4, \dots, q^{2(i-2)}, q^{2(i-1)}, &\underbrace{0, \dots, 0}_{n-j}, \underbrace{1, \dots, 1}_{j-i}) \\
&= q^{j-i}(1-q^2) \left( I + \left( q^2J + q^4J^2 + \dots + q^{2(i-2)}J^{i-2} + q^{2(i-1)}J^{i-1} \right) + ((J^t)^{j-i} + \dots + (J^t)^2 + J^t) \right) \\
&= q^{j-i}(1-q^2) \left( \left( \sum_{k=0}^{j-i} (J^t)^k \right) + \left( \sum_{m=1}^{i-1} q^{2m}J^m \right) \right).
\end{aligned}$$

□

**Lemma 3.1.14.** For  $j > i$ ,

$$E_{ij} - qE_{i-1,j} = (1-q^2) \sum_{k=0}^{i-1} q^{j-i+2k} J^{j-i+k} = q^{j-i}(1-q^2) J^{j-i} \sum_{k=0}^{i-1} q^{2k} J^k.$$

Equivalently,

$$\text{For } i < j, \quad E_{ij}^t - qE_{i,j-1}^t = (1-q^2) \sum_{k=0}^{i-1} q^{j-i+2k} J^{j-i+k} = q^{j-i}(1-q^2) J^{j-i} \sum_{k=0}^{i-1} q^{2k} J^k.$$

*Proof.* We will prove the equivalent statement:

$$\text{For } i < j, \quad E_{ij}^t - qE_{i,j-1}^t = (1-q^2) \sum_{k=0}^{i-1} q^{j-i+2k} J^{j-i+k} = q^{j-i}(1-q^2) J^{j-i} \sum_{k=0}^{i-1} q^{2k} J^k.$$

$$\begin{aligned} \text{For } j > i, \quad E_{ji} &= E_{ij}^t \\ &= C(q^{j-i}, q^{j-i+2}, q^{j-i+4}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{i+j}, \dots, q^{i+j}}_{n+1-i-j}, q^{j+i-2}, q^{j+i-4}, \dots, q^{j+i-2(i-1)}, \underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i})^t. \end{aligned}$$

The proof follows from the definition of the circulant matrix  $E_{ij}$  given in the case  $i \leq j$ , read as the first column vector.

Write  $J = C(0, \dots, 0, 1)^t$ . Note that for  $k \geq 1$ ,  $J^k = C(0, \dots, 0, \underbrace{1, 0, \dots, 0}_{k-1})^t$ .

$$\begin{aligned} E_{ij}^t &= \\ C(q^{j-i}, q^{j-i+2}, q^{j-i+4}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{i+j}, \dots, q^{i+j}}_{n-i-j+1}, q^{j+i-2}, q^{j+i-4}, q^{j+i-6}, \dots, q^{j+i-2(i-1)}, \underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i})^t. \\ E_{i,j-1}^t &= C(q^{j-i-1}, q^{j-i+1}, q^{j-i+3}, \dots, q^{j-i+2(i-1)-1}, \underbrace{q^{i+j-1}, \dots, q^{i+j-1}}_{n-i-j+2}, q^{j+i-3}, q^{j+i-5}, \dots \\ &\quad q^{j+i-2(i-1)+2-1}, q^{j-i+1}, \underbrace{q^{j-i-1}, \dots, q^{j-i-1}}_{j-i-1})^t. \end{aligned}$$

$$\begin{aligned} E_{ij}^t - qE_{i,j-1}^t &= \\ C(0, \dots, 0, \underbrace{0, \dots, 0}_{n-i-j+1}, (1-q^2)q^{j+i-2}, (1-q^2)q^{j+i-4}, (1-q^2)q^{j+i-6}, \dots, (1-q^2)q^{j+i-2}, (1-q^2)q^{j-i}, \underbrace{0, \dots, 0}_{j-i-1})^t \\ &= (1-q^2)C(0, \dots, 0, \underbrace{q^{j-i+2(i-1)}, q^{j-i+2(i-2)}, q^{j-i+2(i-3)}, \dots, q^{j-i+2}, q^{j-i}}_{n-j+1}, \underbrace{0, \dots, 0}_{j-i-1})^t \\ &= (1-q^2) \sum_{k=0}^{i-1} q^{j-i+2k} J^{j-i+k} \\ &= q^{j-i}(1-q^2)J^{j-i} \sum_{k=0}^{i-1} q^{2k} J^k. \end{aligned}$$

□

*Proof.* (Of Proposition 3.1.11):

Case 1 follows from Lemma 3.1.12. Case 2 follows from Lemma 3.1.13. Case 3 follows from Lemma 3.1.14. □

**Example 3.1.15.** In the case  $n = 5$ , after the first step, we have

$$P_3 P_2 P_1 V_q = \begin{pmatrix} 1 & Q_1 & Q_2 & Q_3 \\ 0 & (1-q^2)I_5 & q(1-q^2)(I_5 + J)^t & q^2(1-q^2)(I_5 + J + J^2)^t \\ 0 & q(1-q^2)J & (1-q^2)(I_5 + q^2J) & q(1-q^2)(I_5 + J^t + q^2J) \\ 0 & q^2(1-q^2)J^2 & q(1-q^2)(J + q^2J^2) & (1-q^2)(I_5 + q^2J + q^4J^2) \end{pmatrix}.$$

**Step. 2: Multiply the new matrix on the right by  $P_1^t, \dots, P_p^t$  to get the matrix  $(P_p, \dots, P_1)V_q(P_p, \dots, P_1)^t$ .**

**Proposition 3.1.16.** At the end of the second step, all of the original  $E_{ij}$  blocks have the form

$$(1-q^2)q^{|i-j|} J^{(i-j) \bmod(n)}.$$

$$ie, (E_{ij} - qE_{i-1,j}) - q(E_{i,j-1} - qE_{i-1,j-1}) = (1-q^2)q^{|i-j|}J^{(i-j) \text{ mod}(n)}.$$

Before we can prove the proposition, we need the following corollaries.

**Corollary 3.1.17.** For  $j > 1$ ,

$$E_{1j} - qQ_j - q(E_{1,j-1} - qQ_{j-1}) = q^{j-1}(1-q^2)(J^t)^{j-1}.$$

*Proof.* By Lemma 3.1.12,

$$\begin{aligned} E_{1j} - qQ_j - q(E_{1,j-1} - qQ_{j-1}) &= (1-q^2)q^{j-1} \sum_{k=0}^{j-1} (J^t)^k - q \left( (1-q^2)q^{j-2} \sum_{k=0}^{j-2} (J^t)^k \right) \\ &= q^{j-1}(1-q^2)(J^t)^{j-1}. \end{aligned}$$

□

**Corollary 3.1.18.** For  $1 \leq i = j$ ,

$$E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) = (1-q^2)I.$$

*Proof.*  $E_{11} - qQ_1 = (1-q^2)I$ , and  $Q_1 - qQ_0 = 0$ , so step 2 leaves  $E_{11} = (1-q^2)I$ .

For  $1 < i = j$ , by Lemmas 3.1.13 and 3.1.14,

$$\begin{aligned} E_{ii} - qE_{i-1,i} - q(E_{i,i-1} - qE_{i-1,i-1}) &= (1-q^2) \sum_{k=0}^{i-1} q^{2k} J^k - q \left( (1-q^2) \sum_{k=1}^{i-1} q^{2k-1} J^k \right) \\ &= (1-q^2) \sum_{k=0}^{i-1} q^{2k} J^k - \left( (1-q^2) \sum_{k=1}^{i-1} q^{2k} J^k \right) \\ &= (1-q^2)I. \end{aligned}$$

□

**Corollary 3.1.19.** For  $i < j$ ,

$$E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) = q^{j-i}(1-q^2)(J^t)^{j-i}.$$

*Proof.* By Proposition 3.1.11

$$\begin{aligned} E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) &= (1-q^2)q^{j-i} \left( \sum_{k=0}^{j-i} (J^t)^k + \sum_{m=1}^{i-1} q^{2m} J^m \right) - q \left( (1-q^2)q^{j-i-1} \left( \sum_{k=0}^{j-i-1} (J^t)^k + \sum_{m=1}^{i-1} q^{2m} J^m \right) \right) \\ &= (1-q^2)q^{j-i} \left( \sum_{k=0}^{j-i} (J^t)^k + \sum_{m=1}^{i-1} q^{2m} J^m - \sum_{k=0}^{j-i-1} (J^t)^k + \sum_{m=1}^{i-1} q^{2m} J^m \right) \\ &= q^{j-i}(1-q^2)(J^t)^{j-i}. \end{aligned}$$

□

**Corollary 3.1.20.** For  $i < j$ ,

$$E_{ij}^t - qE_{i,j-1}^t - q(E_{i-1,j}^t - qE_{i-1,j-1}^t) = q^{j-i}(1-q^2)J^{j-i}.$$

And so equivalently, for  $i > j$

$$E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) = q^{i-j}(1-q^2)J^{i-j}.$$

*Proof.* We will prove the first statement,

$$\text{for } i < j, E_{ij}^t - qE_{i,j-1}^t - q(E_{i-1,j}^t - qE_{i-1,j-1}^t) = q^{j-i}(1-q^2)J^{j-i}.$$

By Proposition 3.1.11,

$$\begin{aligned} & E_{ij}^t - qE_{i,j-1}^t - q(E_{i-1,j}^t - qE_{i-1,j-1}^t) \\ &= \left( q^{j-i}(1-q^2)J^{j-i} \sum_{k=0}^{i-1} q^{2k} J^k \right) - q \left( (1-q^2)q^{j-(i-1)} J^{j-(i-1)} \sum_{k=0}^{i-2} q^{2k} J^k \right) \\ &= (1-q^2)q^{j-i} J^{j-i} \left( \left( \sum_{k=0}^{i-1} q^{2k} J^k \right) - q^2 J \sum_{k=0}^{i-2} q^{2k} J^k \right) \\ &= q^{j-i}(1-q^2)J^{j-i} \left( \sum_{k=0}^{i-1} q^{2k} J^k - \sum_{k=0}^{i-2} q^{2(k+1)} J^{k+1} \right) \\ &= q^{j-i}(1-q^2)J^{j-i} \left( \sum_{k=0}^{i-1} q^{2k} J^k - \sum_{k=1}^{i-1} q^{2k} J^k \right) \\ &= q^{j-i}(1-q^2)J^{j-i}. \end{aligned}$$

□

*Proof.* (Of Proposition 3.1.16):

After step 2, by Corollary 3.1.20, for  $i > j$ , the  $E_{ij}$  block matrix has the form  $q^{i-j}(1-q^2)J^{i-j}$ .

By Corollaries 3.1.17, 3.1.19, and 3.1.18, for  $i \leq j$ , the  $E_{ij}$  block has the form  $q^{j-i}(1-q^2)(J^t)^{j-i}$ .

We have the following identities:

$$J^t = J^{n-1}, \text{ and } (J^t)^k = J^{n-k} \text{ since } J^{k(n-1) \bmod(n)} = J^{-k \bmod(n)}. \quad (3.1.8)$$

For  $i \leq j$ ,  $q^{j-i} = q^{|i-j|}$ , and  $(J^t)^{j-i} = J^{i-j} = J^{n+(i-j)} = J^{(i-j) \bmod(n)}$  since  $J^n = I$ .

This proves the proposition that after the second step, each original  $E_{ij}$  block has the form

$$q^{|i-j|}(1-q^2)J^{(i-j) \bmod(n)}.$$

□

Therefore, by Proposition 3.1.16, after step 2 the  $(np + 1) \times (np + 1)$   $q$ -Varchenko matrix has been transformed into

$$(P_p P_{p-1} \dots P_1) V_q (P_p P_{p-1} \dots P_1)^t =$$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & (1-q^2)I_n & (1-q^2)qJ^t & \dots & (1-q^2)q^{r-1}(J^t)^{r-1} & \dots & (1-q^2)q^{p-1}(J^t)^{p-1} \\ 0 & (1-q^2)qJ & (1-q^2)I_n & \dots & (1-q^2)q^{r-2}(J^t)^{r-2} & \dots & (1-q^2)q^{p-2}(J^t)^{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & (1-q^2)q^{r-1}J^{r-1} & (1-q^2)q^{r-2}J^{r-2} & \dots & (1-q^2)I_n & \dots & (1-q^2)q^{p-r}(J^t)^{p-r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & (1-q^2)q^{p-1}J^{p-1} & (1-q^2)q^{p-2}J^{p-2} & \dots & (1-q^2)q^{p-r}J^{p-r} & \dots & (1-q^2)I_n \end{pmatrix}.$$

**Example 3.1.21.** In the case  $n = 5$ , after the second step, the  $q$ -Varchenko matrix is transformed into

$$P_3 P_2 P_1 V_q (P_3 P_2 P_1)^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-q^2)I_5 & q(1-q^2)J^t & q^2(1-q^2)(J^t)^2 \\ 0 & q(1-q^2)J & (1-q^2)I_5 & q(1-q^2)J^t \\ 0 & q^2(1-q^2)J^2 & q(1-q^2)J & (1-q^2)I_5 \end{pmatrix}_{16 \times 16}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-q^2)I_5 & q(1-q^2)J^4 & q^2(1-q^2)J^3 \\ 0 & q(1-q^2)J & (1-q^2)I_5 & q(1-q^2)J^4 \\ 0 & q^2(1-q^2)J^2 & q(1-q^2)J & (1-q^2)I_5 \end{pmatrix}_{16 \times 16}.$$

**Step. 3: Multiply on the left by the matrices  $S_1, S_2, \dots, S_{p-1}$**

**to obtain  $S_{p-1} \dots S_2 S_1 P_p \dots P_1 V_q (P_p \dots P_1)^t$ ,**

where

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & 0 & \dots & 0 \\ 0 & 0 & I_n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & I_n & 0 \\ 0 & \dots & 0 & 0 & -qJ & I_n \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & 0 & \dots & 0 \\ 0 & 0 & I_n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -qJ & I_n & 0 \\ 0 & \dots & 0 & 0 & 0 & I_n \end{pmatrix}, S_{p-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & I_n & 0 & 0 & \dots & 0 \\ 0 & -qJ & I_n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & I_n & 0 \\ 0 & \dots & 0 & 0 & 0 & I_n \end{pmatrix},$$

The resulting matrix is  $S_{p-1} \dots S_1 P_p \dots P_1 V_q (P_p \dots P_1)^t$ . Note that this step does not affect the entries in the first row.

**Corollary 3.1.22.** For  $i > j$ ,

$$E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) - qJ\left(E_{i-1,j} - qE_{i-2,j} - q(E_{i-1,j-1} - qE_{i-2,j-1})\right) = 0.$$

*Proof.* By Proposition 3.1.16,

$$\begin{aligned} E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) - qJ\left(E_{i-1,j} - qE_{i-2,j} - q(E_{i-1,j-1} - qE_{i-2,j-1})\right) \\ = (1-q^2)q^{i-j}J^{i-j} - qJ(1-q^2)q^{i-1-j}J^{i-1-j} = 0. \end{aligned}$$

□

**Corollary 3.1.23.** For  $i = j > 1$ ,

$$E_{jj} - qE_{j-1,j} - q(E_{j,j-1} - qE_{j-1,j-1}) - qJ\left(E_{j-1,j} - qE_{j-2,j} - q(E_{j-1,j-1} - qE_{j-2,j-1})\right) = (1-q^2)^2I.$$

*Proof.* By Proposition 3.1.16,

$$\begin{aligned} E_{jj} - qE_{j-1,j} - q(E_{j,j-1} - qE_{j-1,j-1}) - qJ\left(E_{j-1,j} - qE_{j-2,j} - q(E_{j-1,j-1} - qE_{j-2,j-1})\right) \\ = (1-q^2)I - qJ(1-q^2)q^{j-(j-1)}(J')^{j-(j-1)} \\ = (1-q^2)(I - q^2JJ') = (1-q^2)^2I. \end{aligned}$$

□

**Corollary 3.1.24.** For  $1 < i < j$ ,

$$\begin{aligned} E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) - qJ\left(E_{i-1,j} - qE_{i-2,j} - q(E_{i-1,j-1} - qE_{i-2,j-1})\right) \\ = q^{j-i}(1-q^2)^2(J')^{j-i}. \end{aligned}$$

*Proof.* By Proposition 3.1.16,

$$\begin{aligned} E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) - qJ\left(E_{i-1,j} - qE_{i-2,j} - q(E_{i-1,j-1} - qE_{i-2,j-1})\right) \\ = q^{j-i}(1-q^2)(J')^{j-i} - qJ\left((1-q^2)q^{j-(i-1)}(J')^{j-(i-1)}\right) \\ = (1-q^2)(q^{j-i}(J')^{j-i} - q^{j-i+2}J(J')^{j-i+1}) \\ = q^{j-i}(1-q^2)((J')^{j-i} + q^2JJ'(J')^{j-i}) \\ = q^{j-i}(1-q^2)^2(J')^{j-i} \end{aligned}$$

□

**Example 3.1.25.** In the case  $n = 5$ , after the third step,

$$\begin{aligned} S_2 S_1 P_3 P_2 P_1 V_q (P_3 P_2 P_1)^t &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-q^2)I & q(1-q^2)J^t & q^2(1-q^2)(J^t)^2 \\ 0 & 0 & (1-q^2)^2 I & q(1-q^2)^2 J^t \\ 0 & 0 & 0 & (1-q^2)^2 I \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-q^2)I & q(1-q^2)J^4 & q^2(1-q^2)J^3 \\ 0 & 0 & (1-q^2)^2 I & q(1-q^2)^2 J^4 \\ 0 & 0 & 0 & (1-q^2)^2 I \end{pmatrix}. \end{aligned}$$

Since each entry above the diagonal is a multiple of the diagonal entry on its same row, the matrix can easily be diagonalized by multiplying on the right by  $(S_{p-1} \dots S_1)^t$ . The transition matrix is  $S_{p-1} \dots S_1 P_p \dots P_1$ , and  $(S_{p-1} \dots S_1 P_p \dots P_1) V_q (S_{p-1} \dots S_1 P_p \dots P_1)^t$  is in Smith Normal Form. To verify this, we include one last additional step. Note that this step does not change the block diagonal entries of the matrix.

**Step. 4: Multiply on the right by the transpose of the matrices  $S_1, S_2, \dots, S_{p-1}$  to obtain  $S_{p-1} \dots S_1 P_p \dots P_1 V_q (S_{p-1} \dots S_1 P_p \dots P_1)^t$ ,**

**Corollary 3.1.26.** For  $1 < i < j$ ,

$$\begin{aligned} E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) - qJ(E_{i-1,j} - qE_{i-2,j} - q(E_{i-1,j-1} - qE_{i-2,j-1})) - \\ (E_{i,j-1} - qE_{i-1,j-1} - q(E_{i,j-2} - qE_{i-1,j-2}) - qJ(E_{i-1,j-1} - qE_{i-2,j-1} - q(E_{i-1,j-2} - qE_{i-2,j-2}))) (qJ^t) \\ = 0. \end{aligned}$$

*Proof.* By Corollary 3.1.24,

$$\begin{aligned} E_{ij} - qE_{i-1,j} - q(E_{i,j-1} - qE_{i-1,j-1}) - qJ(E_{i-1,j} - qE_{i-2,j} - q(E_{i-1,j-1} - qE_{i-2,j-1})) - \\ (E_{i,j-1} - qE_{i-1,j-1} - q(E_{i,j-2} - qE_{i-1,j-2}) - qJ(E_{i-1,j-1} - qE_{i-2,j-1} - q(E_{i-1,j-2} - qE_{i-2,j-2}))) (qJ^t) \\ = q^{j-i}(1-q^2)(J^t)^{j-i} - q^{j-1-i}(J^t)^{j-1-i}qJ^t = 0. \end{aligned}$$

□

For any  $n \geq 3$ , the Smith normal form of the  $q$ -Varchenko matrix is

$$\begin{aligned} S_{p-1} \dots S_1 P_p \dots P_1 V_q (S_{p-1} \dots S_1 P_p \dots P_1)^t &= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & (1-q^2)I_n & 0 & \dots & 0 \\ 0 & 0 & (1-q^2)^2 I_n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & (1-q^2)^2 I_n \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-q^2)I_n & 0 \\ 0 & 0 & (1-q^2)^2 I_{n(p-1)} \end{pmatrix}. \end{aligned} \quad (3.1.9)$$

The left transition matrix is  $T = S_{p-1} \dots S_1 P_p \dots P_1$

and the right transition matrix is  $T^t = (S_{p-1} \dots S_1 P_p \dots P_1)^t$ .  $(3.1.10)$

**Example 3.1.27.** In the case  $n = 5$ , the Smith normal form of the  $C_5$   $q$ -Varchenko matrix is

$$\begin{aligned} S_2 S_1 P_3 P_2 P_1 V_q (S_2 S_1 P_3 P_2 P_1)^t &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (1-q^2)I_5 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 I_5 & 0 \\ 0 & 0 & 0 & (1-q^2)^2 I_5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-q^2)I_5 & 0 \\ 0 & 0 & (1-q^2)^2 I_{10} \end{pmatrix}. \end{aligned}$$

## 3.2 The Dihedral Model

The Dihedral Model,  $D_n$ : Now we take the hyperplanes from the cyclic model and move them into  $\mathbb{R}^3$  so that the lines forming edges of the regular  $n$ -gon become planes forming a regular  $n$ -gon shaped cylinder. Then we add one hyperplane  $h$  that is perpendicular to  $h_1, h_2, \dots, h_n$ . This doubles the number of regions. From here on, we will refer to the regions above  $h$  as  $R_x^+$  and the regions below  $h$  as  $R_x^-$ . The  $R_x^+$  regions are labelled according to the cyclic model. Effectively, we label the minus regions by taking a reflection of the cyclic model regions along the axis of symmetry between  $R_1$  and  $R_n$ .

### Algorithm 3.2.1.

1. Label the one hyperplane perpendicular to  $h_1, \dots, h_n$  as  $h$ .
2. Label the  $n$  hyperplanes  $h_1, \dots, h_n$  so that each  $h_x$  forms an edge of the  $n$ -gon that shares vertices with the edges formed by  $h_{x \pm 1 \pmod n}$ , and so that  $h_2$  follows  $h_1$  moving along the edges in a counter-clockwise direction.
3. Label the central region above  $h$  as  $R_0^+$ , and label the central region below  $h$  as  $R_0^-$ .
4. For  $1 \leq x \leq n$ , label the region  $R$  such that  $\text{sep}(R_0^+, R) = h_x$  as  $R_x^+$ .
5. For  $1 \leq x \leq n$ , label the region  $R$  such that  $\text{sep}(R_0^-, R) = h_x$  as  $R_{n+1-x}^-$ .
6. Then, label the remaining regions according the following pseudo-code:

Set  $k = 1$ .

While  $k \leq n\{$

Set  $m = 0$ .

While  $(m+1) \leq p-1\{$

Choose  $R'^+$  such that  $\text{sep}(R_0^+, R_{m+((k+1) \pmod n)}^+) = \text{sep}(R_{mn+k}^+, R'^+)$ .

Label it as  $R_{(m+1)n+k}^+$ .

Choose  $R'^-$  such that  $\text{sep}(R_0^-, R_{m+((k+1) \pmod n)}^-) = \text{sep}(R_{mn+k}^-, R'^-)$ .

Label it as  $R_{(m+1)n+k}^-$ .

$m = m + 1.\}$

$k = k + 1.\}$

The distance enumerator for  $\mathcal{A}$  (with respect to  $R_0^+$ ) is

$$D_{\mathcal{A}, R_0^+}(t) = 1 + t + \sum_{k=1}^p n(t^k + t^{k+1}), \text{ where } p = \lfloor \frac{n+1}{2} \rfloor.$$

To illustrate the algorithm, we go through a couple of steps. Once all of the  $R^+$  regions and  $R_0^-, R_1^-, \dots, R_n^-$  regions  $R'^-$  with  $\#sep(R_0^-, R') = 1$  have been labelled, choose the region  $R'^-$  such that  $sep(R_0^-, R_2^-) = sep(R_1^-, R'^-)$  and label it as  $R_{n+1}^-$ . Next choose the region  $R'^-$  such that  $sep(R_0^-, R_3^-) = sep(R_{n+1}^-, R'^-)$  and label it as  $R_{2n+1}^-$ . Continue until the region  $R_{(p-1)n+1}^-$  has been labelled.

Then, choose the region  $R'^-$  such that  $sep(R_0^-, R_3^-) = sep(R_2^-, R'^-)$  and label it as  $R_{n+2}^-$ . Next choose the region  $R'^-$  such that  $sep(R_0^-, R_4^-) = sep(R_{n+2}^-, R'^-)$  and label it as  $R_{2n+2}^-$ . Continue until the region  $R_{(p-1)n+2}^-$  has been labelled. Repeat the process until regions

$$R_1^-, R_{n+1}^-, \dots, R_{(p-1)n+1}^-, R_2^-, R_{n+2}^-, R_{(p-1)n+2}^-, \dots, R_n^-$$

have been labelled. Then choose the region  $R'^-$  such that  $sep(R_0^-, R_1^-) = sep(R_n^-, R'^-)$  and label it as  $R_{2n}^-$ . Next choose the region  $R'^-$  such that  $sep(R_0^-, R_2^-) = sep(R_{2n}^-, R'^-)$  and label it as  $R_{3n}^-$ . Continue until the region  $R_{(p-1)n+n}^-$  has been labelled.

While the process is identical for the  $R^+$  and  $R^-$  regions, the hyperplane involved in each step is not. In the first part of the algorithm, the hyperplane separating  $R_1^+$  and  $R_{n+1}^+$  is  $h_2$ , and the hyperplane separating  $R_{n+1}^+$  from  $R_{2n+1}^+$  is  $h_3$ . In the second part of the algorithm, the hyperplane separating  $R_1^-$  and  $R_{n+1}^-$  is  $h_{n-1}$ , and the hyperplane separating  $R_{n+1}^-$  and  $R_{2n+1}^-$  is  $h_{n-2}$ .

$$sep(R_0^+, R_1^+) \subseteq sep(R_0^+, R_{n+1}^+) \subseteq sep(R_0^+, R_{2n+1}^+) \subseteq \dots \subseteq sep(R_0^+, R_{(p-1)n+1}^+).$$

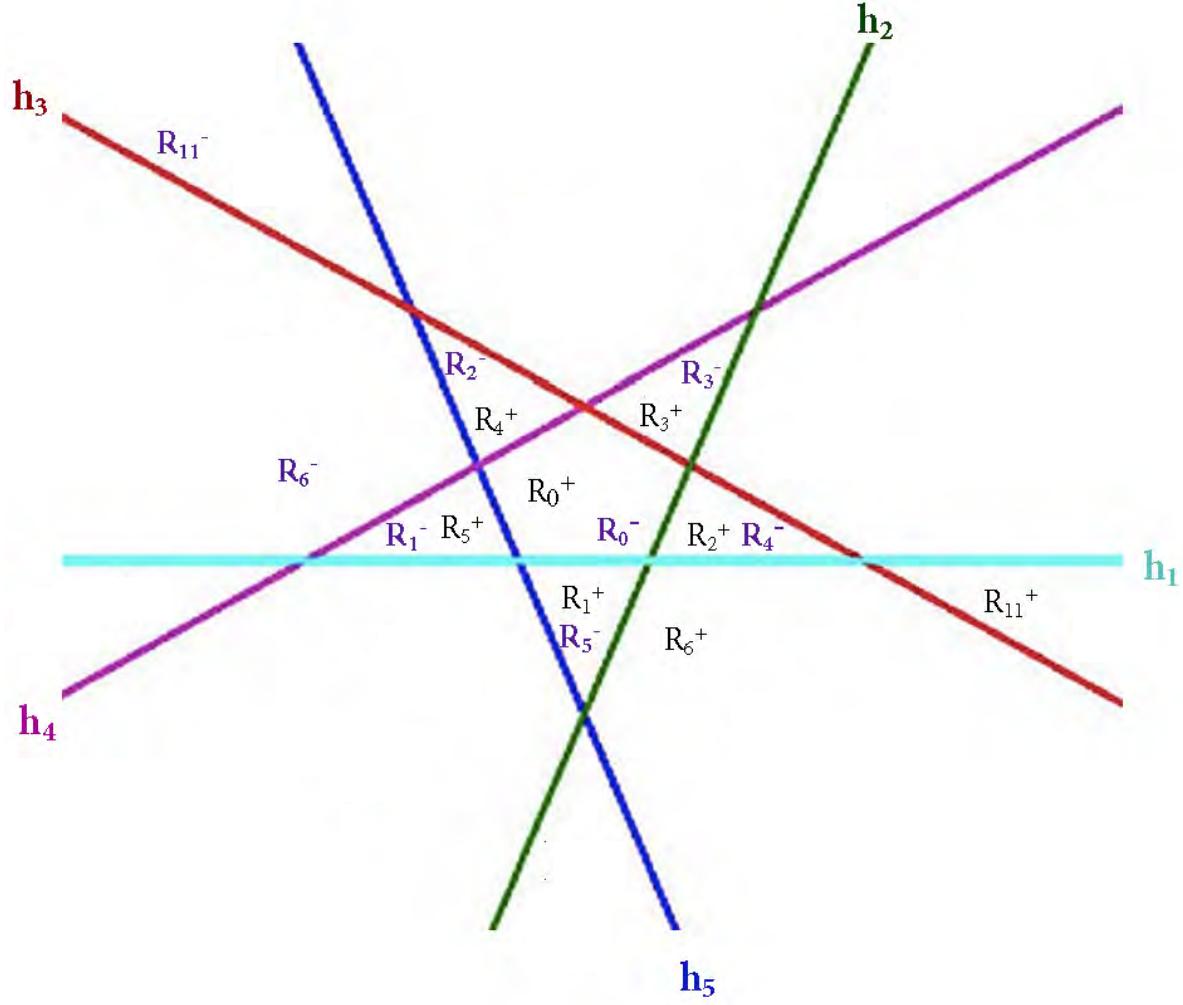
$$sep(R_0^-, R_1^-) \subseteq sep(R_0^-, R_{n+1}^-) \subseteq sep(R_0^-, R_{2n+1}^-) \subseteq \dots \subseteq sep(R_0^-, R_{(p-1)n+1}^-).$$

For  $i, j \leq p$ , we can write:

$$sep(R_0^+, R_{(i-1)n+x}^+) = \{h_x, h_{x+1 \bmod(n)}, h_{x+2 \bmod(n)}, \dots, h_{x+(i-1) \bmod(n)}\}, \quad (3.2.1)$$

$$sep(R_0^-, R_{(j-1)n+x}^-) = \{h_{n+1-x}, h_{n+1-(x+1) \bmod(n)}, h_{n+1-(x+2) \bmod(n)}, \dots, h_{n+1-(x+(j-1)) \bmod(n)}\}. \quad (3.2.2)$$

### Illustration of Algorithm 3.2.1



Note: after the steps:

Choose  $R'^+$  such that  $\text{sep} \left( R_0^+, R_{1+((1+1) \bmod n)}^+ \right) = \text{sep} \left( R_{1(5)+1}^+, R'^+ \right)$ .

Label it as  $R_{(1+1)5+1}^+$ .

Choose  $R'^-$  such that  $\text{sep} \left( R_0^-, R_{1+((1+1) \bmod n)}^- \right) = \text{sep} \left( R_{1(5)+1}^-, R'^- \right)$ .

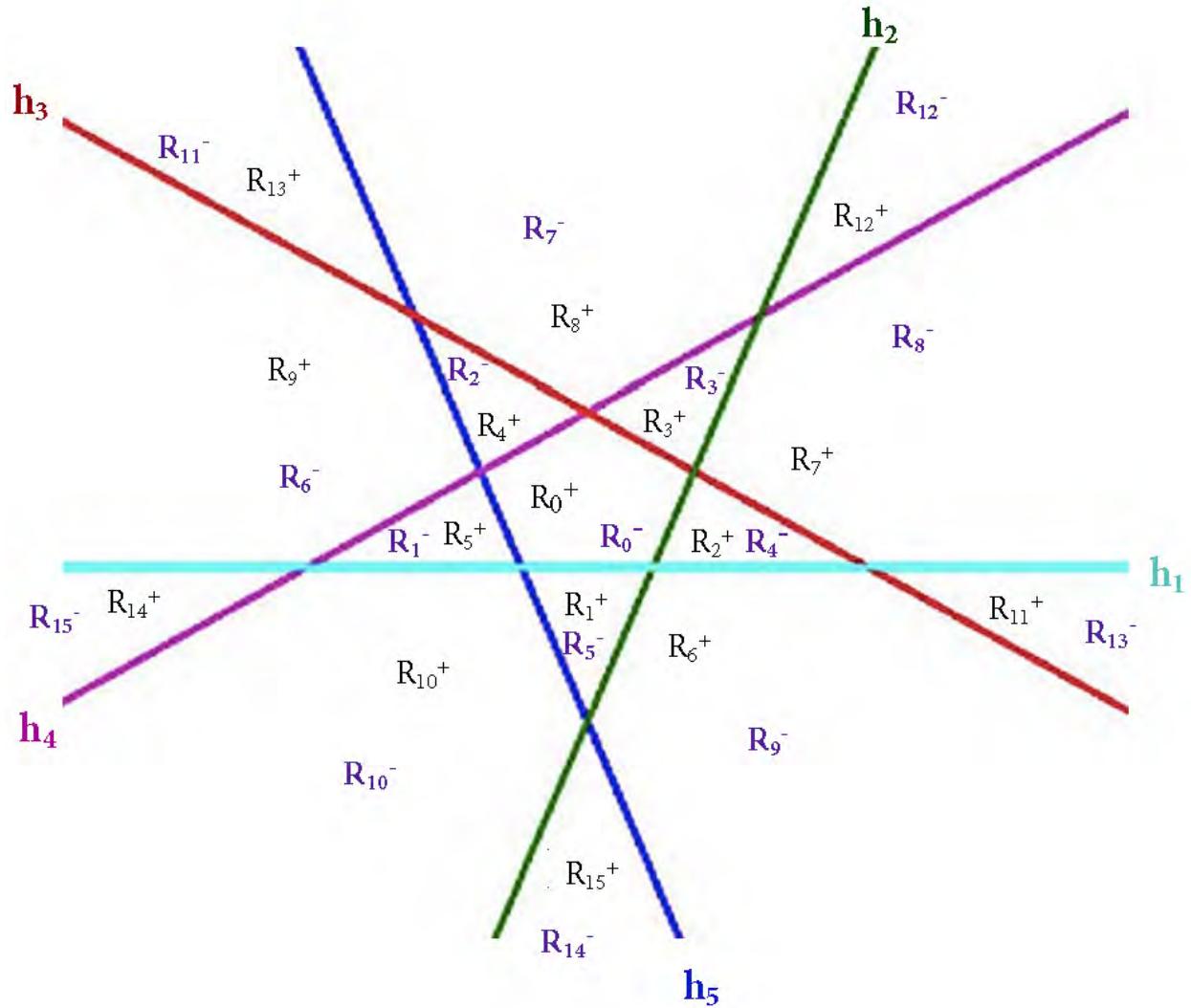
Label it as  $R_{(1+1)5+1}^-$ .

$m = 2$ .

Note: The hyperplane  $h$  is not labelled here - it is the hyperplane consisting of this page, where the + regions are on the front side of the page, and the - regions are on the back side of the page.

**Figure 3.9** The  $D_5$  hyperplane arrangement in step 6 of Algorithm 3.2.1, where  $k = 1$ , at the end of the loop when  $m = 1$  is re-assigned to  $m = 2$ , which begins the  $k = 2$  loop.

### Illustration of results of Algorithm 3.2.1



Note: The hyperplane  $h$  is not labelled here - it is the hyperplane consisting of this page, where the + regions are on the front side of the page, and the - regions are on the back side of the page.

**Figure 3.10** The  $D_5$  hyperplane arrangement after step 6 of Algorithm 3.2.1.

The  $q$ -Varchenko matrix for  $D_n$  is a  $2(np+1) \times 2(np+1)$  matrix with the following form:

$$V_q = \left( \begin{array}{cc|cc|cc|c|cc|cc} 1 & q & Q_1 & Q_2 & Q_2 & Q_3 & \dots & Q_k & Q_{k+1} & \dots & Q_p & Q_{p+1} \\ q & 1 & Q_2 & Q_1 & Q_3 & Q_2 & \dots & Q_{k+1} & Q_k & \dots & Q_{p+1} & Q_p \\ \hline Q_1^t & Q_2^t & E_{1+1+} & E_{1+1-} & E_{1+2+} & E_{1+2-} & \dots & E_{1+k+} & E_{1+k-} & \dots & E_{1+p+} & E_{1+p-} \\ Q_2^t & Q_1^t & E_{1-1+} & E_{1-1-} & E_{1-2+} & E_{1-2-} & \dots & E_{1-k+} & E_{1-k-} & \dots & E_{1-p+} & E_{1-p-} \\ \hline Q_2^t & Q_3^t & E_{2+1-} & E_{2+1-} & E_{2+2+} & E_{2+2-} & \dots & E_{2+k+} & E_{2+k-} & \dots & E_{2+p+} & E_{2+p-} \\ Q_3^t & Q_2^t & E_{2-1+} & E_{2-1-} & E_{2-2+} & E_{2-2-} & \dots & E_{2-k+} & E_{2-k-} & \dots & E_{2-p+} & E_{2-p-} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots & \vdots \\ \hline Q_k^t & Q_{k+1}^t & E_{k+1+} & E_{k+1-} & E_{k+2+} & E_{k+2-} & \dots & E_{k+k+} & E_{k+k-} & \dots & E_{k+p+} & E_{k+p-} \\ Q_{k+1}^t & Q_k^t & E_{k-1+} & E_{k-1-} & E_{k-2+} & E_{k-2-} & \dots & E_{k-k+} & E_{k-k-} & \dots & E_{k-p+} & E_{k-p-} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline Q_p^t & Q_{p+1}^t & E_{p+1+} & E_{p+1-} & E_{p+2+} & E_{p+2-} & \dots & E_{p+k+} & E_{p+k-} & \dots & E_{p+p+} & E_{p+p-} \\ Q_{p+1}^t & Q_p^t & E_{p-1+} & E_{p-1-} & E_{p-2+} & E_{p-2-} & \dots & E_{p-k+} & E_{p-k-} & \dots & E_{p-p+} & E_{p-p-} \end{array} \right), \quad (3.2.3)$$

where  $E_{j^\pm i^\pm} = E_{i^\pm j^\pm}^t$ , and  $E_{j^\mp i^\pm} = E_{i^\pm j^\mp}^t$ .

*Remark 3.2.2.* As immediate consequences of the algorithm for indexing the regions,  $E_{i^+ j^+} = E_{i^- j^-} = E_{ij}$  as it was defined in 3.1.4, since for  $1 \leq x \leq n$  we have relationships like those in the cyclic model:

$$\begin{aligned} \#sep\left(R_{(i-1)n+1}^+, R_{(j-1)n+x}^+\right) &= \#sep\left(R_{(i-1)n+2}^+, R_{(j-1)n+(x+1 \text{ mod}(n))}^+\right) = \dots \\ &= \#sep\left(R_{(i-1)n+n-1}^+, R_{(j-1)n+(x+n-2 \text{ mod}(n))}^+\right) \\ &= \#sep\left(R_{(i-1)n+n}^+, R_{(j-1)n+(x+n-1 \text{ mod}(n))}^+\right), \end{aligned}$$

and similarly,

$$\begin{aligned} \#sep\left(R_{(i-1)n+1}^-, R_{(j-1)n+x}^-\right) &= \#sep\left(R_{(i-1)n+2}^-, R_{(j-1)n+(x+1 \text{ mod}(n))}^-\right) = \dots \\ &= \#sep\left(R_{(i-1)n+n-1}^-, R_{(j-1)n+(x+n-2 \text{ mod}(n))}^-\right) \\ &= \#sep\left(R_{(i-1)n+n}^-, R_{(j-1)n+(x+n-1 \text{ mod}(n))}^-\right). \end{aligned} \quad (3.2.4)$$

Note: here when  $x+k=n$  for some  $1 \leq k < n$ ,  $R_{(j-1)n+(x+k \text{ mod}(n))}^\pm = R_{(j-1)n+n}^\pm$ . Therefore,  $E_{i^+ j^+}$  and  $E_{i^- j^-}$  are circulant matrices.

However, in the dihedral model we also have the relationships

$$\begin{aligned}\#\text{sep}\left(R_{(i-1)n+1}^+, R_{(j-1)n+x}^-\right) &= \#\text{sep}\left(R_{(i-1)n+2}^+, R_{(j-1)n+(x-1 \bmod n)}^-\right) = \dots \\ &= \#\text{sep}\left(R_{(i-1)n+n-1}^+, R_{(j-1)n+(x-(n-2) \bmod n)}^-\right) \\ &= \#\text{sep} = \left(R_{(i-1)n+n}^+, R_{(j-1)n+(x-(n-1) \bmod n)}^-\right),\end{aligned}$$

and similarly,

$$\begin{aligned}\#\text{sep}\left(R_{(i-1)n+1}^-, R_{(j-1)n+x}^+\right) &= \#\text{sep}\left(R_{(i-1)n+2}^-, R_{(j-1)n+(x-1 \bmod n)}^+\right) = \dots \\ &= \#\text{sep}\left(R_{(i-1)n+n-1}^-, R_{(j-1)n+(x-(n-2) \bmod n)}^+\right) \\ &= \#\text{sep}\left(R_{(i-1)n+n}^-, R_{(j-1)n+(x-(n-1) \bmod n)}^+\right).\end{aligned}\tag{3.2.5}$$

Note: here when  $x = (n - k)$  for some  $1 \leq k < n$ ,  $R_{(j-1)n+(x-(n-k) \bmod n)}^\mp = R_{(j-1)n+n}^\mp$ . Therefore  $E_{i^+ j^-}$  and  $E_{i^- j^+}$  are reverse circulant matrices.

### Definition 3.2.3.

$$K = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

$K$  is the matrix associated to the permutation written in cycle notation as  $(1n), (2, n-1), \dots, (k, n+1-k), \dots$  where the last term is  $(\frac{n}{2}, \frac{n}{2}+1)$  if  $n$  is even, and the one-cycle  $(\frac{n+1}{2})$  if  $n$  is odd.

*Remark 3.2.4.* For any  $0 \leq m < n$ ,  $J^m$  commutes with the matrices  $E_{ij}$  because both are circulant matrices.  $K$  does not commute with  $J$ , and  $K$  commutes with  $E_{ij}$  only when  $i = j$ .

$$K^2 = J^n = I, \text{ and } KJ = J^{n-1}K.$$

**Proposition 3.2.5.** *If  $M$  is any  $n \times n$  circulant matrix, then  $KM$  is symmetric, and so is  $MK$ .*

*Proof.* It is sufficient to show that  $KM$  is symmetric, since then  $KM = (KM)^t = M^t K$  and  $M^t$  is circulant if  $M$  is circulant. To show this, write the  $(r, c)$  (row, column) entry of  $K$  as  $K_{rc} = \delta_{r+c, n+1}$ .

$$(KM)_{rc} = \sum_{k=1}^n K_{rk} M_{kc} = \sum_{k=1}^n \delta_{r+k, n+1} M_{kc} = M_{n+1-r, c}. \tag{3.2.6}$$

$$(M^t K)_{rc} = \sum_{k=1}^n M_{rk}^t K_{kc} = \sum_{k=1}^n M_{rk}^t \delta_{k+c, n+1} = M_{r, n+1-c}^t = M_{n+1-c, r}.$$

Since  $M$  is circulant, for any integer  $a$  we have  $M_{ij} = M_{i+a \bmod n, j+a \bmod n}$ .

$$M_{n+1-r, c} = M_{n, c+r-1} = M_{n+1-c, r}.$$

This shows that  $KM = (KM)^t$ .  $\square$

**Proposition 3.2.6.** *Furthermore,  $KM$  and  $MK$  are "reverse circulant" matrices: for an integer  $a$ , the  $(r, c)$ -entry of  $KM$  equals the  $(r + a \bmod(n), c - a \bmod(n))$ -entry.*

*Proof.* Using Equation 3.2.6, we have

$$(KM)_{r+a,c-a} = \sum_{k=1}^n K_{r+a,k} M_{k,c-a} = \sum_{k=1}^n \delta_{r+a+k,n+1} M_{k,c-a} = M_{n+1-r-a,c-a} = M_{n+1-r,c} = (KM)_{rc}.$$

$\square$

*Remark 3.2.7.*  $K$  is symmetric, and  $J^{i-1}KE_{ij}$  is both symmetric and reverse circulant for all  $i, j$ .

The  $E_{i^+ j^-}$  block entries of the  $q$ -Varchenko matrix correspond to the number of separating hyperplanes between the regions in  $R_{(i-1)n+x'}^+$  and the regions in  $R_{(j-1)n+x}^-$ .

**Theorem 3.2.8.**

$$E_{i^+ j^-} = E_{i^- j^+} = qJ^{i-1}KE_{ij}, \text{ where } E_{ij} \text{ is as defined in Theorem 3.1.4.}$$

*Proof.* We proceed as before in the proof of Theorem 3.1.4 where we fixed  $x' = 1$  and determined the  $\#(\text{sep}(R_{(i-1)n+x'}, R_{(j-1)n+x}))$ , but this time we will fix  $x' = n - (i - 1)$  rather than  $x' = 1$ . This will give us the  $n - (i - 1)$  row of  $E_{i^+ j^-}$  instead of the first row. However, as a result of the method used to index the regions, we have the relations given in Eq. 3.2.5 which imply that the  $E_{i^+ j^-}$  blocks are reverse-circulant matrices and therefore they are determined entirely by the first row, or equivalently, any one particular row. For  $1 \leq x \leq n$ , we have the following two equations:

$$\text{sep}\left(R_0^+, R_{(i-1)n+n-(i-1)}^+\right) = \{h_{n-(i-1)}, h_{n-(i-1)+1}, \dots, h_{n-(i-1)+(i-1)}\} = \{h_n, h_{n-1}, h_{n-2}, \dots, h_{n-(i-1)}\}. \quad (3.2.7)$$

$$\text{sep}\left(R_0^+, R_{(j-1)n+x}^-\right) = \{h, h_{n+1-x}, h_{n+1-(x+1) \bmod(n)}, h_{n+1-(x+2) \bmod(n)}, \dots, h_{n+1-(x+(j-1)) \bmod(n)}\}. \quad (3.2.8)$$

Therefore by Proposition 3.1.2,

$$\begin{aligned} \text{sep}\left(R_{(i-1)n+n-(i-1)}^+, R_{(j-1)n+x}^-\right) &= \\ \{h_n, h_{n-1}, h_{n-2}, \dots, h_{n-(i-1)}\} \bigcup \{h, h_{n+1-x}, h_{n+1-(x+1) \bmod(n)}, h_{n+1-(x+2) \bmod(n)}, \dots, h_{n+1-(x+(j-1)) \bmod(n)}\} - \\ (\{h_n, h_{n-1}, h_{n-2}, \dots, h_{n-(i-1)}\} \bigcap \{h, h_{n+1-x}, h_{n+1-(x+1) \bmod(n)}, h_{n+1-(x+2) \bmod(n)}, \dots, h_{n+1-(x+(j-1)) \bmod(n)}\}). \end{aligned} \quad (3.2.9)$$

Let  $1 \leq k < i \leq j$ . We will examine three cases which can be generalized into one by allowing  $0 \leq k \leq i$ .

$$\begin{aligned} \#sep\left(R_{(i-1)n+n-(i-1)}^+, R_{(j-1)n+x}^-\right) \\ = \begin{cases} j+1-i, & sep\left(R_0^+, R_{(i-1)n+n-(i-1)}^+\right) \subseteq sep\left(R_0^+, R_{(j-1)n+x}^-\right) \\ j+1-i+2k, & \#\left(sep\left(R_0^+, R_{(i-1)n+n-(i-1)}^+\right) \cap sep\left(R_0^+, R_{(j-1)n+x}^-\right)\right) = i-k, \\ j+1+i, & sep\left(R_0^+, R_{(i-1)n+n-(i-1)}^+\right) \cap sep\left(R_0^+, R_{(j-1)n+x}^-\right) = \emptyset. \end{cases} \end{aligned}$$

Case 1:

$$sep\left(R_0^+, R_{(i-1)n+n-(i-1)}^+\right) \subseteq sep\left(R_0^+, R_{(j-1)n+x}^-\right).$$

This occurs when  $x = 1$ , and when  $n + 1 - (x + (j-1)) \bmod(n) \leq n - (i-1)$ ; ie,  $n - (j-i-1) \leq x \leq n$ .

$$\text{Hence } \#sep\left(R_{(i-1)n+n-(i-1)}^+, R_{(j-1)n+x}^-\right) = j+1-i \text{ for } x \in \underbrace{\{1, n, n-1, \dots, n-(j-i-1)\}}_{j-i \text{ terms}}.$$

Case 2:

$$\#\left(sep\left(R_{(i-1)n+n-(i-1)}^+, R_0^+\right) \cap sep\left(R_{(j-1)n+x}^-, R_0^+\right)\right) = i-k.$$

This happens when exactly  $i-k$  of  $h_n, h_{n-1}, \dots, h_{n-k}, \dots, h_{n-(i-1)}$  are in  $sep\left(R_0^+, R_{(j-1)n+x}^-\right)$ .

Equivalently, there exists

$$S \subseteq sep\left(R_0^+, R_{(i-1)n+n-(i-1)}^+\right) \text{ such that } \#S = k \text{ and } S \cap sep\left(R_0^+, R_{(j-1)n+x}^-\right) = \emptyset.$$

When  $x = k+1$ :

$$sep\left(R_0^+, R_{(j-1)n+x}^-\right) = \{h, h_{n-k}, h_{n-(k+1)}, h_{n-(k+2)}, \dots, h_{n-(k+(j-1))}\}.$$

$$S = \{h_n, h_{n-1}, \dots, h_{n-(k-1)}\} \subseteq sep\left(R_0^+, R_{(i-1)n+n-(i-1)}^+\right), \#S = k, \text{ and } S \cap sep\left(R_0^+, R_{(j-1)n+x}^-\right) = \emptyset.$$

When  $n + 1 - (x + (j-1)) \bmod(n) = n + 1 - (i-k)$  so that  $x = n - (j-i) - k + 1 = n + 1 - (j-i+k)$ :

$$sep\left(R_0^+, R_{(j-1)n+x}^-\right) = \{h, h_{j-i+k \bmod(n)}, h_{j-i+k-1 \bmod(n)}, \dots, h_{n-(i-1)+(k+1)}, h_{n-(i-1)+k}\}.$$

$$S = \{h_{n-(i-1)+(k-1)}, \dots, h_{n-(i-1)+2}, h_{n-(i-1)+1}, h_{n-(i-1)}\} \subseteq sep\left(R_0^+, R_{(i-1)n+n-(i-1)}^+\right),$$

$$\#S = k, \text{ and } S \cap sep\left(R_0^+, R_{(j-1)n+x}^-\right) = \emptyset.$$

$$\text{Hence } \#sep\left(R_{(i-1)n+n-(i-1)}^+, R_{(j-1)n+x}^-\right) = j+1-i+2k \text{ for } x \in \{k+1, n - (j-i) - k + 1\}.$$

Case 3:

$$\{h_n, h_{n-1}, h_{n-2}, \dots, h_{n-(i-1)}\} \cap \{h, h_{n+1-x}, h_{n+1-(x+1) \text{ mod}(n)}, h_{n+1-(x+2) \text{ mod}(n)}, \dots, h_{n+1-(x+(j-1)) \text{ mod}(n)}\} = \emptyset.$$

This happens when  $x > i$  and when  $x < n - (j - 2)$ ; ie,  $x + (j - 1) < n + 1$ , or  $x + (j - 1) \text{ mod}(n) < 1$ .

$$\text{Hence } \#sep(R_{(i-1)n+n-(i-1)}^+, R_{(j-1)n+x}^-) = j + 1 + i \text{ for } x \in \{i + 1, i + 2, \dots, n - (j - 1)\}.$$

In summary:

$$\#sep(R_{(i-1)n+n-(i-1)}^+, R_{(j-1)n+x}^-) = \begin{cases} j + 1 - i, & x \in \{1, n, n - 1, \dots, n - (j - i - 1)\}. \\ j + 1 - i + 2k, & x \in \{k + 1, n - (j - i - 1) - k\}. \\ j + 1 + i, & x \in \{i + 1, i + 2, \dots, n - (j - 1)\}. \end{cases} \quad (3.2.10)$$

Comparing this to Equation 3.1.5, we see that this is precisely  $1 + \#sep(R_{(i-1)n+1}^+, R_{(j-1)n+x}^+)$ . Therefore, the  $n - (i - 1)$  row of  $E_{i^+ j^-}$  equals  $q$  times the first row of  $E_{ij}$  as it was defined in Theorem 3.1.4.

Left multiplication by  $J^{i-1}K$  acting on the circulant matrix  $E_{ij}$  takes the first row to the  $n$ th row, and then takes the  $n$ th row to the  $n - (i - 1)$  row. Since both  $E_{i^+ j^-}$  and  $qJ^{i-1}KE_{ij}$  are reverse circulant matrices, the equality of one row of each matrix is sufficient to determine the equality of the two matrices. Hence  $qJ^{i-1}KE_{ij} = E_{i^+ j^-}$ .  $\square$

For  $D_n$ , we can write the  $2(np + 1) \times 2(np + 1)$   $q$ -Varchenko matrix in the block form:

$$V_q = \left( \begin{array}{cc|cc|c|cc|cc} 1 & q & Q_1 & qQ_1 & \dots & Q_r & qQ_r & \dots & Q_p & qQ_p \\ q & 1 & qQ_1 & Q_1 & \dots & qQ_r & Q_r & \dots & qQ_p & Q_p \\ \hline Q_1^t & qQ_1^t & E_{11} & qKE_{11} & \dots & E_{1r} & qKE_{1r} & \dots & E_{1p} & qKE_{1p} \\ qQ_1^t & Q_1^t & qKE_{11} & E_{11} & \dots & qKE_{1r} & E_{1r} & \dots & qKE_{1p} & E_{1p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline Q_r^t & qQ_r^t & E_{1r}^t & qKE_{1r} & \dots & E_{rr} & qJ^{r-1}KE_{rr} & \dots & E_{rp} & qJ^{r-1}E_{rp} \\ qQ_r^t & Q_r^t & qKE_{1r} & E_{1r}^t & \dots & qJ^{r-1}KE_{rr} & E_{rr} & \dots & qJ^{r-1}KE_{rp} & E_{rp} \\ \vdots & \ddots & \vdots & \vdots \\ \hline Q_p^t & qQ_p^t & E_{1p}^t & qKE_{1p} & \dots & E_{rp}^t & qJ^{r-1}KE_{rp} & \dots & E_{pp} & qJ^{p-1}KE_{pp} \\ qQ_p^t & Q_p^t & qKE_{1p} & E_{1p}^t & \dots & J^{r-1}KE_{rp} & E_{rp}^t & \dots & qJ^{p-1}KE_{pp} & E_{pp} \end{array} \right). \quad (3.2.11)$$

**Example 3.2.9.** The  $q$ -Varchenko matrix for  $D_5$  is

$$V_q = \left( \begin{array}{cc|cc|cc|cc} 1 & q & Q_1 & qQ_1 & Q_2 & qQ_2 & Q_3 & qQ_3 \\ q & 1 & qQ_1 & Q_1 & qQ_2 & Q_2 & qQ_3 & Q_3 \\ \hline Q_1^t & qQ_1^t & E_{11} & qKE_{11} & E_{12} & qKE_{12} & E_{13} & qKE_{13} \\ qQ_1^t & Q_1^t & qKE_{11} & E_{11} & qKE_{12} & E_{12} & qKE_{13} & E_{13} \\ \hline Q_2^t & qQ_2^t & E_{12}^t & qKE_{12} & E_{22} & qJKE_{22} & E_{23} & qJKE_{23} \\ qQ_2^t & Q_2^t & qKE_{12} & E_{12}^t & qJKE_{22} & E_{22} & qJKE_{23} & E_{23} \\ \hline Q_3^t & qQ_3^t & E_{13}^t & qKE_{13} & E_{23}^t & qJKE_{23} & E_{33} & qJ^2KE_{33} \\ qQ_3^t & Q_3^t & qKE_{13} & E_{13}^t & qJKE_{23} & E_{23}^t & qJ^2KE_{33} & E_{33} \end{array} \right)_{32 \times 32}.$$

The blocks  $E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}$  and  $Q_k$  for  $k = 1, 2, 3$  are as they were defined in Example 3.1.3.

$$qKE_{11} = \begin{pmatrix} q^3 & q^3 & q^3 & q^3 & q \\ q^3 & q^3 & q^3 & q & q^3 \\ q^3 & q^3 & q & q^3 & q^3 \\ q^3 & q & q^3 & q^3 & q^3 \\ q & q^3 & q^3 & q^3 & q^3 \end{pmatrix}, \quad qKE_{12} = \begin{pmatrix} q^4 & q^4 & q^4 & q^2 & q^2 \\ q^4 & q^4 & q^2 & q^2 & q^4 \\ q^4 & q^2 & q^2 & q^4 & q^4 \\ q^2 & q^2 & q^4 & q^4 & q^4 \\ q^2 & q^4 & q^4 & q^4 & q^2 \end{pmatrix}.$$

$$qKE_{13} = \begin{pmatrix} q^5 & q^5 & q^3 & q^3 & q^3 \\ q^5 & q^3 & q^3 & q^3 & q^5 \\ q^3 & q^3 & q^3 & q^5 & q^5 \\ q^3 & q^5 & q^5 & q^3 & q^3 \\ q^3 & q^5 & q^5 & q^3 & q^3 \end{pmatrix}, \quad qJKE_{22} = \begin{pmatrix} q^5 & q^5 & q^3 & q & q^3 \\ q^5 & q^3 & q & q^3 & q^5 \\ q^3 & q & q^3 & q^5 & q^5 \\ q & q^3 & q^5 & q^5 & q^3 \\ q^3 & q^5 & q^5 & q^3 & q \end{pmatrix}.$$

$$qJKE_{23} = \begin{pmatrix} q^6 & q^4 & q^2 & q^2 & q^4 \\ q^4 & q^2 & q^2 & q^4 & q^6 \\ q^2 & q^2 & q^4 & q^6 & q^4 \\ q^2 & q^4 & q^6 & q^4 & q^2 \\ q^4 & q^6 & q^4 & q^2 & q^2 \end{pmatrix}, \quad qJ^2KE_{33} = \begin{pmatrix} q^5 & q^3 & q & q^3 & q^5 \\ q^3 & q & q^3 & q^5 & q^5 \\ q & q^3 & q^5 & q^5 & q^3 \\ q^3 & q^5 & q^5 & q^3 & q \\ q^5 & q^5 & q^3 & q & q^3 \end{pmatrix}.$$

**Proposition 3.2.10.**  $V_q$  for  $D_n$  can be transformed into its Smith normal form over  $\mathbb{Z}[q]$  in seven steps.

The remainder of this section will prove this proposition.

**Step. 1:** Multiply  $V_q$  on the left by matrices  $P_1, P_2, \dots, P_{p-1}, P_{p=\lfloor \frac{n+1}{2} \rfloor}$ , to obtain  $P_p \dots P_1 V_q$ .

where

$$P_1 = \left( \begin{array}{|ccccccccc|} \hline 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -qI_n & 0 & I_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -qI_n & 0 & I_n \\ \hline \end{array} \right),$$
  

$$P_2 = \left( \begin{array}{|ccccccccc|} \hline 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -qI_n & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -qI_n & 0 & I_n \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n \\ \hline \end{array} \right),$$

$$\begin{aligned}
P_{p-1} &= \left( \begin{array}{c|cc|cc|c|cc|cc|cc}
1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & -qI_n & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -qI_n & 0 & I_n & \dots & 0 & 0 & 0 & 0 & 0 \\
\hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & I_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & I_n & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & I_n & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & I_n
\end{array} \right), \\
P_p &= \left( \begin{array}{c|cc|cc|c|cc|cc|cc}
1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\
-q & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\hline -q1 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & -q1 & 0 & I_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 & 0 & 0 \\
\hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & I_n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & I_n & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & I_n & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & I_n
\end{array} \right),
\end{aligned}$$

**Example 3.2.11.** For  $D_5$ , multiply  $V_q$  on the left by

$$P_3 P_2 P_1 = \left( \begin{array}{c|cc|cc|c|cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline -q1 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q1 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & -qI_5 & 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -qI_5 & 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & -qI_5 & 0 & 0 & I_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -qI_5 & 0 & 0 & I_5 & 0 & 0 & 0
\end{array} \right).$$

**Proposition 3.2.12.** At the end of the first step, the original  $qJ^{i-1}KE_{ij}$  block matrices have the form

$$= \begin{cases} q^j(1-q^2) \left( \sum_{k=0}^{j-1} J^k \right) K, & \text{for } i = 1, \\ q^{j-i+1}(1-q^2) \left( \sum_{k=0}^{j-i} J^{j-1-k} + \sum_{m=1}^{i-1} q^{2m} J^{m+j-1} \right) K, & \text{for blocks on or above the block diagonal,} \\ q^{j-i}(1-q^2) \left( \sum_{k=0}^{i-1} q^{2k+1} J^{k+j-1} \right) K, & \text{for blocks below the block diagonal.} \end{cases}$$

Before we can prove this proposition, we need the following lemmas.

**Lemma 3.2.13.** Let  $q^2Q_j$  denote the  $n \times n$  matrix with all entries  $q^{j+2}$ .

$$qKE_{1j} - q^2Q_j = q^j(1-q^2) \left( \sum_{k=0}^{j-1} J^k \right) K.$$

*Proof.* Since all rows of  $qQ_j$  are identical and left multiplication by  $K$  only changes the order of the rows of a matrix,  $qQ_j = KqQ_j$ . From this, the relationships  $J^t = J^{n-1}$ ,  $J^n = I$ ,  $(J^t)^k = J^{-k \bmod(n)}$ ,  $KJ = J^{n-1}K$ , and Lemma 3.1.12,

$$\begin{aligned} qKE_{1j} - q^2Q_j &= qKE_{1j} - qKqQ_j = qK(E_{1j} - qQ_j) \\ &= qK \left( q^{j-1}(1-q^2) \sum_{k=0}^{j-1} (J^t)^k \right) = q^j(1-q^2)K \left( \sum_{k=0}^{j-1} J^k \right)^t = q^j(1-q^2) \left( \sum_{k=0}^{j-1} J^k \right) K. \end{aligned}$$

□

**Proposition 3.2.14.** For  $i \leq j$ ,

$$E_{ij}^t = J^{j-i}E_{ij}.$$

*Proof.* Note that

$$E_{ij}^t = C(\underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i+1}, q^{j-i+2}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{j+i}, \dots, q^{j+i}}_{n+1-i-j}, q^{j-i+2(i-1)}, \dots, q^{j-i+2}).$$

Write  $J_{rc} = \delta_{r+1 \pmod{n}, c}$ , and  $(J^{j-i})_{rc} = \delta_{r+j-i, c}$ .

$$(J^{j-i}E_{ij})_{rc} = \sum_{k=1}^n (J^{j-i})_{rk}(E_{ij})_{kc} = \sum_{k=1}^n \delta_{r+j-i, k}(E_{ij})_{kc} = (E_{ij})_{r+j-i, c}.$$

Since it is a circulant matrix,  $J^{j-i}E_{ij}$  can be written using the first row.

$$(E_{ij})_{1+j-i, c} = C(\underbrace{q^{j-i}, \dots, q^{j-i}}_{j-i+1}, q^{j-i+2}, \dots, q^{j-i+2(i-1)}, \underbrace{q^{j+i}, \dots, q^{j+i}}_{n+1-i-j}, q^{j-i+2(i-1)}, \dots, q^{j-i+2}) = E_{ij}^t.$$

$(E_{ij})_{1+1,c}$  is given by "pushing" the  $(E_{ij})_{1n}$  entry to the  $(E_{ij})_{21}$  entry. Since the  $(E_{ij})_{1,n-(j-i)}$  through  $(E_{ij})_{1n}$  entries are all equal to  $q^{j-i}$ ,  $(E_{ij})_{1+j-i,c}$  is given by taking the  $1 + j - i$  row of  $E_{ij}$  as the first row of an  $n \times n$  circulant matrix, which equivalently produces  $E_{ij}^t$ .  $\square$

**Lemma 3.2.15.** For  $1 < i \leq j$ ,

$$qJ^{i-1}KE_{ij} - qJ^{i-2}KE_{i-1,j} = q^{j-i+1}(1-q^2) \left( \sum_{k=0}^{j-i} J^{j-1-k} + \sum_{m=1}^{i-1} q^{2m} J^{m+j-1} \right) K.$$

*Proof.* By Lemma 3.1.13, Proposition 3.2.14, and commutativity of circulant matrices,

$$qJ^{i-1}KE_{ij} - qJ^{i-2}KE_{i-1,j} = qJ^{i-2}(JKE_{ij} - qKE_{i-1,j}) = qJ^{i-2}K(J^t E_{ij} - qE_{i-1,j})$$

$$\begin{aligned} &= qJ^{i-2}K((J^t E_{ij} - qE_{i-1,j})^t)^t = qJ^{i-2}K(E_{ij}^t J - qE_{i-1,j}^t)^t \\ &= qJ^{i-2}K(J^{j-i}E_{ij}J - qJ^{j-i+1}E_{i-1,j})^t \\ &= qJ^{i-2}K\left(J^{j-i+1}(1-q^2)q^{j-i}\left(\sum_{k=0}^{j-i}(J^t)^k + \sum_{m=1}^{i-1}q^{2m}J^m\right)\right)^t \\ &= qJ^{i-2}\left(J^{j-i+1}(1-q^2)q^{j-i}\left(\sum_{k=0}^{j-i}(J^t)^k + \sum_{m=1}^{i-1}q^{2m}J^m\right)\right)K \\ &= q^{j-i+1}(1-q^2)J^{j-1}\left(\sum_{k=0}^{j-i}(J^t)^k + \sum_{m=1}^{i-1}q^{2m}J^m\right)K \\ &= q^{j-i+1}(1-q^2)\left(\sum_{k=0}^{j-i}J^{j-1-k} + \sum_{m=1}^{i-1}q^{2m}J^{m+j-1}\right)K. \end{aligned}$$

$\square$

**Lemma 3.2.16.** For  $1 \leq i < j$ ,

$$qJ^{i-1}KE_{ij} - q^2J^{i-1}KE_{i,j-1} = q^{j-i}(1-q^2)\left(\sum_{k=0}^{i-1}q^{2k+1}J^{k+j-1}\right)K.$$

*Proof.* By Proposition 3.1.11,

$$\begin{aligned} qJ^{i-1}KE_{ij} - q^2J^{i-1}KE_{i,j-1} &= qJ^{i-1}K(E_{ij} - qE_{i,j-1}) = qJ^{i-1}K(E_{ij}^t - qE_{i,j-1}^t)^t \\ &= qJ^{i-1}K\left((1-q^2)q^{j-i}J^{j-i}\sum_{k=0}^{i-1}q^{2k}J^k\right)^t = q^{j-i+1}(1-q^2)J^{j-i+i-1}\left(\sum_{k=0}^{i-1}q^{2k}J^k\right)K \\ &= q^{j-i}(1-q^2)\left(\sum_{k=0}^{i-1}q^{2k+1}J^{k+j-1}\right)K. \end{aligned}$$

$\square$

*Proof.* (Of Proposition 3.2.12): Case 1 is proved in Lemma 3.2.13. Case 2 is proved in Lemma 3.2.15. Case 3 is proved in Lemma 3.2.16.  $\square$

**Example 3.2.17.** Recalling Proposition 3.1.11, for  $D_5$ , after the first step we have

$$P_3 P_2 P_1 V_q = \left( \begin{array}{c|c|c|c} v_{00} & v_{01} & v_{02} & v_{03} \\ \hline v_{10} & v_{11} & v_{12} & v_{13} \\ \hline v_{20} & v_{21} & v_{22} & v_{23} \\ \hline v_{30} & v_{31} & v_{32} & v_{33} \end{array} \right), \text{ where}$$

$$v_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1-q^2 \end{pmatrix}_{2 \times 2}, \quad v_{10} = v_{20} = v_{30} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_{2 \times 2},$$

$$v_{01} = \begin{pmatrix} Q_1 & qQ_1 \\ qQ_1 & Q_1 \end{pmatrix}_{2 \times 2n}, \quad v_{02} = \begin{pmatrix} Q_2 & qQ_2 \\ qQ_2 & Q_2 \end{pmatrix}_{2 \times 2n}, \quad v_{03} = \begin{pmatrix} Q_3 & qQ_3 \\ qQ_3 & Q_3 \end{pmatrix}_{2 \times 2n},$$

$$v_{11} = \begin{pmatrix} (1-q^2)I & q(1-q^2)K \\ q(1-q^2)K & (1-q^2)I \end{pmatrix}, \quad v_{12} = \begin{pmatrix} q(1-q^2)(I+J)^t & q^2(1-q^2)(I+J)K \\ q^2(1-q^2)(I+J)K & q(1-q^2)(I+J)^t \end{pmatrix},$$

$$v_{13} = \begin{pmatrix} q^2(1-q^2)(I+J+J^2)^t & q^3(1-q^2)(I+J+J^2)K \\ q^3(1-q^2)(I+J+J^2)K & q^2(1-q^2)(I+J+J^2)^t \end{pmatrix},$$

$$v_{21} = \begin{pmatrix} q(1-q^2)J & q^2(1-q^2)JK \\ q^2(1-q^2)JK & q(1-q^2)J \end{pmatrix}, \quad v_{22} = \begin{pmatrix} (1-q^2)(I+q^2J) & q(1-q^2)(J+q^2J^2)K \\ q(1-q^2)(J+q^2J^2)K & (1-q^2)(I+q^2J) \end{pmatrix},$$

$$v_{23} = \begin{pmatrix} q(1-q^2)(I+J^t+q^2J) & q^2(1-q^2)(J+J^2+q^2J^3)K \\ q^2(1-q^2)(J+J^2+q^2J^3)K & q(1-q^2)(I+J^t+q^2J) \end{pmatrix},$$

$$v_{31} = \begin{pmatrix} q^2(1-q^2)J^2 & q^3(1-q^2)J^2K \\ q^3(1-q^2)J^2K & q^2(1-q^2)J^2 \end{pmatrix},$$

$$v_{32} = \begin{pmatrix} q(1-q^2)(J+q^2J^2) & q^2(1-q^2)(J^2+q^2J^3)K \\ q^2(1-q^2)(J^2+q^2J^3)K & q(1-q^2)(J+q^2J^2) \end{pmatrix},$$

$$v_{33} = \begin{pmatrix} (1-q^2)(I+q^2J+q^4J^2) & q(1-q^2)(J^2+q^2J^3+q^4J^4)K \\ q(1-q^2)(J^2+q^2J^3+q^4J^4)K & (1-q^2)(I+q^2J+q^4J^2) \end{pmatrix}.$$

**Step. 2:** Multiply the new matrix on the right by  $P_1^t, \dots, P_p^t$  to get the matrix  $(P_p, \dots, P_1)V_q(P_p, \dots, P_1)^t$ .

**Proposition 3.2.18.** At the end of the second step, we can make a general statement for the block matrix that  $qJ^{i-1}KE_{ij}$  is transformed into:

$$q^{j-i+1}(1-q^2) \left( (1-q^2) \left( \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K.$$

Before we can prove this proposition, we need the following Corollaries.

*Remark 3.2.19.*

$$qKE_{11} - q^2Q_1 = (1-q^2)K, \text{ since } qKE_{11} - q^2Q_1 = qKE_{11} - q^2KQ_1 = qK(E_{11} - qQ_1) = q(1-q^2)K.$$

**Corollary 3.2.20.** For  $j > 1$ ,

$$qKE_{1j} - q^2Q_j - q(qKE_{1,j-1} - q^2Q_{j-1}) = q^j(1-q^2)J^{j-1}K.$$

*Proof.* By Lemma 3.2.13,

$$\begin{aligned} qKE_{1j} - q^2Q_j - q(qKE_{1,j-1} - q^2Q_{j-1}) &= q^j(1-q^2) \left( \sum_{k=0}^{j-1} J^k \right) K - q \left( q^{j-1}(1-q^2) \left( \sum_{k=0}^{j-2} J^k \right) K \right) \\ &= q^j(1-q^2) \left( \sum_{k=0}^{j-1} J^k - \sum_{k=0}^{j-2} J^k \right) K = q^j(1-q^2)J^{j-1}K. \end{aligned}$$

□

**Corollary 3.2.21.** For  $1 < i < j$ ,

$$\begin{aligned} qJ^{i-1}KE_{ij} - q^2J^{i-2}KE_{i-1,j} - q(qJ^{i-1}KE_{i,j-1} - q^2J^{i-2}KE_{i-1,j-1}) \\ = q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K. \end{aligned}$$

*Proof.* By Lemma 3.2.15,

$$\begin{aligned} qJ^{i-1}KE_{ij} - q^2J^{i-2}KE_{i-1,j} - q(qJ^{i-1}KE_{i,j-1} - q^2J^{i-2}KE_{i-1,j-1}) \\ = q^{j-i+1}(1-q^2) \left( \sum_{k=0}^{j-i} J^{j-1-k} + \sum_{m=1}^{i-1} q^{2m} J^{m+j-1} \right) K - q \left( q^{j-i}(1-q^2) \left( \sum_{k=0}^{j-i-1} J^{j-2-k} + \sum_{m=1}^{i-1} q^{2m} J^{m+j-2} \right) K \right) \end{aligned}$$

$$\begin{aligned}
&= q^{j-i+1}(1-q^2) \left( \sum_{k=0}^{j-i} J^{j-k-1} - \sum_{k=0}^{j-i-1} J^{j-k-2} + \sum_{m=1}^{i-1} q^{2m} J^{m+j-1} - \sum_{m=1}^{i-1} q^{2m} J^{m+j-2} \right) K \\
&= q^{j-i+1}(1-q^2) \left( \sum_{k=0}^{j-i} J^{j-k-1} - \sum_{k=1}^{j-i} J^{j-k-1} + \sum_{m=1}^{i-1} q^{2m} J^{m+j-1} - q^2 J^{j-1} - \sum_{m=2}^{i-1} q^{2m} J^{m+j-2} \right) K \\
&= q^{j-i+1}(1-q^2) \left( J^{j-1} - q^2 J^{j-1} + \sum_{m=1}^{i-1} q^{2m} J^{m+j-1} - \sum_{m=1}^{i-2} q^{2(m+1)} J^{m+j-1} \right) K \\
&= q^{j-i+1}(1-q^2) \left( J^{j-1} - q^2 J^{j-1} + q^{2(i-1)} J^{i+j-2} + \sum_{m=1}^{i-2} q^{2m} J^{m+j-1} - \sum_{m=1}^{i-2} q^2 q^{2m} J^{m+j-1} \right) K \\
&= q^{j-i+1}(1-q^2) \left( (1-q^2) J^{j-1} + \sum_{m=1}^{i-2} (1-q^2) q^{2m} J^{m+j-1} + q^{2(i-1)} J^{i+j-2} \right) K \\
&= q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K.
\end{aligned}$$

□

**Corollary 3.2.22.** For  $1 < i < j$ ,

$$\begin{aligned}
qJ^{i-1}KE_{ij} - q^2J^{i-1}KE_{i,j-1} - q(qJ^{i-2}KE_{i-1,j} - q^2J^{i-2}KE_{i-1,j-1}) \\
= q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K.
\end{aligned}$$

*Proof.* By Lemma 3.2.16,

$$\begin{aligned}
qJ^{i-1}KE_{ij} - q^2J^{i-1}KE_{i,j-1} - q(qJ^{i-2}KE_{i-1,j} - q^2J^{i-2}KE_{i-1,j-1}) \\
= q^{j-i}(1-q^2) \left( \sum_{k=0}^{i-1} q^{2k+1} J^{k+j-1} \right) K - q \left( q^{j-(i-1)}(1-q^2) \left( \sum_{k=0}^{i-2} q^{2k+1} J^{k+j-1} \right) K \right) \\
= q^{j-i}(1-q^2) \left( \sum_{k=0}^{i-1} q^{2k+1} J^{k+j-1} - q^2 \sum_{k=0}^{i-2} q^{2k+1} J^{k+j-1} \right) K \\
= q^{j-i}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k+1} J^{k+j-1} \right) + q^{2i-1} J^{i+j-2} \right) K \\
= q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K.
\end{aligned}$$

□

**Corollary 3.2.23.** For  $1 < i$ ,

$$\begin{aligned} qJ^{i-1}KE_{ii} - q^2J^{i-2}KE_{i-1,i} - q(qJ^{i-2}KE_{i-1,i} - q^2J^{i-2}KE_{i-1,i-1}) \\ = q(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+i-1} \right) + q^{2(i-1)} J^{2(i-1)} \right) K. \end{aligned}$$

*Proof.* By Lemma 3.2.16,

$$\begin{aligned} qJ^{i-1}KE_{ii} - q^2J^{i-2}KE_{i-1,i} - q(qJ^{i-2}KE_{i-1,i} - q^2J^{i-2}KE_{i-1,i-1}) \\ = q(1-q^2) \left( J^{i-1} + \sum_{m=1}^{i-1} q^{2m} J^{m+i-1} \right) K - q \left( \left( q(1-q^2) \sum_{k=0}^{i-2} q^{2k+1} J^{k+i-1} \right) K \right) \\ = q(1-q^2) \left( J^{i-1} + \sum_{m=1}^{i-1} q^{2m} J^{m+i-1} - \sum_{k=0}^{i-2} q^{2k+2} J^{k+i-1} \right) K \\ = q(1-q^2) \left( J^{i-1} + \sum_{m=1}^{i-1} q^{2m} J^{m+i-1} - q^2 J^{i-1} - \sum_{k=1}^{i-2} q^{2k+2} J^{k+i-1} \right) K \\ = q(1-q^2) \left( (1-q^2) J^{i-1} + \left( \sum_{k=1}^{i-2} (1-q^2) q^{2k} J^{k+i-1} \right) + q^{2(i-1)} J^{2(i-1)} \right) K \\ = q(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+i-1} \right) + q^{2(i-1)} J^{2(i-1)} \right) K. \end{aligned}$$

□

*Proof.* (Of Proposition 3.2.18): For  $i = j = 1$ , the result is obvious, as noted in Remark 3.2.19. For  $i = 1$  and  $j > 1$ , the proof is given by Corollary 3.2.20.

There are three remaining cases based on the position of the original  $qJ^{i-1}KE_{ij}$  block in the  $q$ -Varchenko matrix. The case when it lies above the block diagonal and  $i > 1$  is proven in Corollary 3.2.21. The case when it lies below the block diagonal is proven in Corollary 3.2.22. The case when it lies on the block diagonal and  $i = j > 1$  is proven in Corollary 3.2.23. □

**Remark 3.2.24.** Therefore, by Proposition 3.1.16, at the end of the second step, the block matrices with the original form  $E_{ij}$  are given by

$$(1-q^2)q^{|i-j|}J^{(i-j) \bmod(n)}, \quad (3.2.12)$$

and the block matrices with the original form  $qJ^{i-1}KE_{ij}$  are given by

$$q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K. \quad (3.2.13)$$

At this point, the  $q$ -Varchenko matrix for  $D_n$  has been transformed into a block matrix form where each block is given by Eq. 3.2.12 and Eq. 3.2.13.

**Example 3.2.25.** For  $D_5$ , after the second step we have

$$\begin{aligned}
P_3 P_2 P_1 V_q (P_3 P_2 P_1)^t &= \left( \begin{array}{c|c|c|c} v_{00} & 0 & 0 & 0 \\ \hline 0 & v_{11} & v_{12} & v_{13} \\ \hline 0 & v_{21} & v_{22} & v_{23} \\ \hline 0 & v_{31} & v_{32} & v_{33} \end{array} \right), \text{ where } v_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1-q^2 \end{pmatrix}, \\
v_{11} &= \begin{pmatrix} (1-q^2)I & q(1-q^2)K \\ q(1-q^2)K & (1-q^2)I \end{pmatrix}, \quad v_{12} = \begin{pmatrix} q(1-q^2)J^4 & q^2(1-q^2)JK \\ q^2(1-q^2)JK & q(1-q^2)J^4 \end{pmatrix}, \\
v_{13} &= \begin{pmatrix} q^2(1-q^2)J^3 & q^3(1-q^2)J^2K \\ q^3(1-q^2)J^2K & q^2(1-q^2)J^3 \end{pmatrix}, \quad v_{21} = \begin{pmatrix} q(1-q^2)J & q^2(1-q^2)JK \\ q^2(1-q^2)JK & q(1-q^2)J \end{pmatrix}, \\
v_{22} &= \begin{pmatrix} (1-q^2)I & q(1-q^2)((1-q^2)J+q^2J^2)K \\ q(1-q^2)((1-q^2)J+q^2J^2)K & (1-q^2)I \end{pmatrix}, \\
v_{23} &= \begin{pmatrix} q(1-q^2)J^4 & q^2(1-q^2)((1-q^2)J^2+q^2J^3)K \\ q^2(1-q^2)((1-q^2)J^2+q^2J^3)K & q(1-q^2)J^4 \end{pmatrix}, \\
v_{31} &= \begin{pmatrix} q^2(1-q^2)J^2 & q^3(1-q^2)J^2K \\ q^3(1-q^2)J^2K & q^2(1-q^2)J^2 \end{pmatrix}, \\
v_{32} &= \begin{pmatrix} q(1-q^2)J & q^2(1-q^2)((1-q^2)J^2+q^2J^3)K \\ q^2(1-q^2)((1-q^2)J^2+q^2J^3)K & q(1-q^2)J \end{pmatrix}, \\
v_{33} &= \begin{pmatrix} (1-q^2)I & q(1-q^2)((1-q^2)J^2+q^2(1-q^2)J^3+q^4J^4)K \\ q(1-q^2)((1-q^2)J^2+q^2(1-q^2)J^3+q^4J^4)K & (1-q^2)I \end{pmatrix}.
\end{aligned}$$

**Step. 3: Multiply  $P_p \dots P_1 V_q (P_p \dots P_1)^t$  on the left by matrices  $S_1, \dots, S_k, \dots, S_{p-1}$  to obtain  $S_{p-1} \dots S_1 P_p \dots P_1 V_q (P_p \dots P_1)^t$ ,**

where

$$S_1 = \left( \begin{array}{|cccc|c|cccc|} \hline 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -qJ & 0 & I_n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -qJ & 0 & I_n \\ \hline \end{array} \right),$$
  

$$S_2 = \left( \begin{array}{|cccc|c|cccc|} \hline 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -qJ & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -qJ & 0 & I_n \\ \hline \end{array} \right),$$

$$S_{p-1} = \left( \begin{array}{c|cc|cc|c|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -qJ & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -qJ & 0 & I_n & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & \dots & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & I_n \end{array} \right),$$

**Example 3.2.26.** For  $D_5$ ,

$$S_2 S_1 = \left( \begin{array}{c|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_5 & 0 & 0 & 0 \\ \hline 0 & 0 & -qJ & 0 & I_5 & 0 & 0 \\ 0 & 0 & 0 & -qJ & 0 & I_5 & 0 \\ \hline 0 & 0 & 0 & 0 & -qJ & 0 & I_5 \\ 0 & 0 & 0 & 0 & 0 & -qJ & 0 \\ \hline \end{array} \right).$$

*Remark 3.2.27.* At this point, we have shown that each block entry originally given in the  $q$ -Varchenko matrix by  $qJ^{i-1}KE_{ij}$  is given by

$$q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K.$$

We will refer to these block matrices as the  **$\mathbf{qJ}^{i-1}\mathbf{KE}_{ij}$ -entry** through the next step. Step 3, left multiplication by the matrices  $S_{p-1} \dots S_1$  has the effect of beginning with the last two rows and subtracting  $qJ$  times the previous two rows; then the same operation is repeated by subtracting  $qJ$  times the third-to-last two rows from the second-to-last two rows, and so on. The  $qKE_{1j}$ -entries remain the same,  $q^j(1-q^2)J^{j-1}K$ , throughout this step.

For all of the  $qJ^{i-1}KE_{ij}$ -entries that are below the diagonal and for which  $i \neq j$ , the third step carries out the operation of subtracting  $qJ$  times the  $qJ^{i-1}KE_{i,j-1}$ -entry from the  $qJ^{i-1}KE_{ij}$ -entry. In this case, the result equals 0.

For all of the  $qJ^{i-1}KE_{ij}$ -entries that are either on or above the diagonal or for which  $i = j$ , the third step

carries out the operation of subtracting  $qJ$  times the  $qJ^{i-2}KE_{i-1,j}$ -entry from the  $qJ^{i-1}KE_{ij}$ -entry.

**Corollary 3.2.28.** *For all of the  $qJ^{i-1}KE_{ij}$ -entries that are below the diagonal and for which  $i \neq j$ , the  $qJ^{i-1}KE_{ij}$ -entry minus  $qJ$  times the  $qJ^{i-1}KE_{i,j-1}$ -entry is zero.*

*Proof.* By Proposition 3.2.18,

$$\begin{aligned} & q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K \\ & \quad - qJ \left( q^{j-i}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-2} \right) + q^{2(i-1)} J^{i+j-3} \right) K \right) \\ \\ & = q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K \\ & \quad - q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K = 0. \end{aligned}$$

□

**Corollary 3.2.29.** *For all of the  $qJ^{i-1}KE_{ij}$ -entries that are either on or above the diagonal or for which  $i = j$ , the  $qJ^{i-1}KE_{ij}$ -entry minus  $qJ$  times the  $qJ^{i-2}KE_{i-1,j}$ -entry equals  $q^{j-i+1}(1-q^2)^2 J^{j-1} K$ .*

*Proof.* By Proposition 3.2.18,

$$\begin{aligned} & q^{j-i+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-2} q^{2k} J^{k+j-1} \right) + q^{2(i-1)} J^{i+j-2} \right) K \\ & \quad - qJ \left( q^{j-(i-1)+1}(1-q^2) \left( \left( (1-q^2) \sum_{k=0}^{i-3} q^{2k} J^{k+j-1} \right) + q^{2(i-2)} J^{i+j-3} \right) K \right) \\ \\ & = \left( \sum_{k=0}^{i-2} q^{2k+j-i+1}(1-q^2)^2 J^{k+j-1} + q^{i+j-1}(1-q^2) J^{i+j-2} \right) K \\ & \quad - qJ \left( \sum_{k=0}^{i-3} q^{2k+j-i+2}(1-q^2)^2 J^{k+j-1} + q^{i+j-2}(1-q^2) J^{i+j-3} \right) K \\ \\ & = \left( q^{j-i+1}(1-q^2)^2 J^{j-1} + \sum_{k=1}^{i-2} q^{2k+j-i+1}(1-q^2)^2 J^{k+j-1} + q^{i+j-1}(1-q^2) J^{i+j-2} \right) K \\ & \quad - \left( \sum_{k=0}^{i-3} q^{2k+j-i+3}(1-q^2)^2 J^{k+j} + q^{i+j-1}(1-q^2) J^{i+j-2} \right) K \end{aligned}$$

$$\begin{aligned}
&= q^{j-i+1}(1-q^2)^2 J^{j-1} K + (1-q^2)^2 \left( \sum_{k=1}^{i-2} q^{2k+j-i+1} J^{k+j-1} - \sum_{k=0}^{i-3} q^{2(k+1)+j-i+1} J^{(k+1)+j-1} \right) K \\
&= q^{j-i+1}(1-q^2)^2 J^{j-1} K + (1-q^2)^2 \left( \sum_{k=1}^{i-2} q^{2k+j-i+1} J^{k+j-1} - \sum_{k=1}^{i-2} q^{2k+j-i+1} J^{k+j-1} \right) K \\
&= q^{j-i+1}(1-q^2)^2 J^{j-1} K.
\end{aligned}$$

□

**Example 3.2.30.** For  $D_5$ , after the third step we have

$$S_2 S_1 P_3 P_2 P_1 V_q (P_3 P_2 P_1)^t =$$

$$\left( \begin{array}{cc|cc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & (1-q^2)I & q(1-q^2)K & q(1-q^2)J^4 & q^2(1-q^2)JK & q^2(1-q^2)J^3 & q^3(1-q^2)J^2K \\
0 & 0 & q(1-q^2)K & (1-q^2)I & q^2(1-q^2)JK & q(1-q^2)J^4 & q^3(1-q^2)J^2K & q^2(1-q^2)J^3 \\
\hline
0 & 0 & 0 & 0 & (1-q^2)^2 I & q(1-q^2)^2 JK & q(1-q^2)^2 J^4 & q^2(1-q^2)^2 J^2 K \\
0 & 0 & 0 & 0 & q(1-q^2)^2 JK & (1-q^2)^2 I & q^2(1-q^2)^2 J^2 K & q(1-q^2)^2 J^4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 I & q(1-q^2)^2 J^2 K \\
0 & 0 & 0 & 0 & 0 & 0 & q(1-q^2)^2 J^2 K & (1-q^2)^2 I
\end{array} \right)$$

**Step. 4: Multiply the new matrix on the right by  $S_1^t, \dots, S_{p-1}^t$  to get the matrix**

$$(S_{p-1} \dots S_1 P_p, \dots, P_1) V_q (S_{p-1} \dots S_1 P_p, \dots, P_1)^t.$$

*Remark 3.2.31.* This step has the effect of beginning with the last two columns and subtracting the previous two columns times  $qJ^t$ ; then the same operation is repeated by subtracting the third-to-last two columns times  $qJ^t$  from the second-to-last two columns, and so on.

The  $qKE_{11}$ -entries remain the same throughout this step, and more generally, the  $qJ^{i-1}KE_{ii}$ -entries remain  $q(1-q^2)^2 J^{i-1} K$ , since the corresponding entries to the left of them are zeros.

**Corollary 3.2.32.** For  $i > j$ , the current  $qJ^{i-1}KE_{ij}$ -entry minus the current  $qJ^{i-1}KE_{i,j-1}$ -entry times  $qJ^t$  equals zero.

*Proof.* Using Corollary 3.2.28 and the identity  $KJ^t = JK$ ,

$$q^{j-i+1}(1-q^2)^2 J^{j-1} K - (q^{(j-1)-i+1}(1-q^2)^2 J^{j-2} K) qJ^t = q^{j-i+1}(1-q^2)^2 (J^{j-1} K - J^{j-2} K J^t) = 0.$$

□

**Example 3.2.33.** For  $D_5$ , after the fourth step we have

$$S_2 S_1 P_3 P_2 P_1 V_q (S_2 S_1 P_3 P_2 P_1)^t =$$

$$\left( \begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (1-q^2)I_5 & q(1-q^2)K & 0 & 0 & 0 & 0 \\ 0 & 0 & q(1-q^2)K & (1-q^2)I_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & (1-q^2)^2 I_5 & q(1-q^2)^2 JK & 0 & 0 \\ 0 & 0 & 0 & 0 & q(1-q^2)^2 JK & (1-q^2)^2 I_5 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 I_5 & q(1-q^2)^2 J^2 K \\ 0 & 0 & 0 & 0 & 0 & 0 & q(1-q^2)^2 J^2 K & (1-q^2)^2 I_5 \end{array} \right).$$

**Step. 5: Multiply the new matrix on the left by the matrices  $T_1, T_2, \dots, T_p$  to obtain**

$$T_p \dots T_1 S_{p-1} \dots S_1 P_p \dots P_1 V_q (S_{p-1} \dots S_1 P_p \dots P_1)^t, \text{ where}$$

$$T_1 = \left( \begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & 0 & 0 & 0 & -qJ^{p-1}K & I_n \end{array} \right),$$

$$T_2 = \left( \begin{array}{|ccccccccc|} \hline & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \hline & 0 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & I_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \hline & 0 & 0 & 0 & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & I_n & \dots & 0 & 0 & 0 & 0 \\ \hline & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ & 0 & 0 & 0 & 0 & 0 & 0 & \dots & I_n & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & \dots & -qJ^{p-2}K & I_n & 0 & 0 \\ \hline & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & I_n & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_n \\ \hline \end{array} \right),$$

$$T_{p-1} = \left( \begin{array}{|cc|cc|cc|cc|cc|} \hline & & & & & & & & \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_n & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -qJK & I_n & \dots & 0 & 0 & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_n \\ \hline \end{array} \right)$$

$$T_p = \left( \begin{array}{cc|cc|cc|c|cc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -qK & I_n & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_n & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & I_n \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{array} \right),$$

**Example 3.2.34.** For  $D_5$ ,

$$T_3 T_2 T_1 = \left( \begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -qK & I_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -qJK & I_5 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & I_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -qJ^2K & I_5 \end{array} \right).$$

*Remark 3.2.35.* This step has the effect of beginning with the last two rows, subtracting  $qJ^{p-1}K$  times the second to last row from the last row; then the same operation is repeated by subtracting  $qJ^{p-2}K$  times the fourth to last row from the third to last row, and so on. The  $qJ^{i-1}KE_{ii}$ -entries above the diagonal remain the same throughout this step, and the ones below the diagonal become zero since each  $qJ^{i-1}KE_{ii}$ -entry equals  $qJ^{i-1}K$  times the corresponding  $E_{ii}$ -entry which is  $(1-q^2)I$  for  $E_{11}$  and  $(1-q^2)^2I$  for  $i \geq 2$ .

Along the diagonal, we get  $(1-q^2)I$  in the first  $E_{11}$  position and  $(1-q^2)^2I$  in the second  $E_{11}$  position since

$$\begin{aligned} (1-q^2)I - qK(q(1-q^2)K) &= (1-q^2)I - q^2(1-q^2)K^2 \\ &= (1-q^2)I - q^2(1-q^2)I \\ &= (1-q^2)^2I. \end{aligned}$$

For  $i \geq 2$ , we get  $(1-q^2)^2I$  in the first  $E_{ii}$  position and  $(1-q^2)^3$  in the second  $E_{ii}$  position since

$$\begin{aligned} (1-q^2)^2I - qJ^{i-1}K(q(1-q^2)^2J^{i-1}K) &= (1-q^2)^2I - q^2(1-q^2)^2J^{i-1}KJ^{i-1}K \\ &= (1-q^2)^2I - q^2(1-q^2)^2J^{i-1}KK(J^{i-1})^t \\ &= (1-q^2)^2I - q^2(1-q^2)^2J^{i-1}(J^{i-1})^t \\ &= (1-q^2)^2I - q^2(1-q^2)^2I \\ &= (1-q^2)^3I. \end{aligned}$$

At this point, the  $q$ -Varchenko matrix for  $D_n$  is transformed into

$$\left( \begin{array}{cc|cc|cc|c|cc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & (1-q^2)I_n & q(1-q^2)K & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^2 I_n & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & (1-q^2)^2 I_n & q(1-q^2)^2 JK & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 I_n & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & (1-q^2)^2 I_n & q(1-q^2)^2 J^{p-1} K \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & (1-q^2)^3 I_n \end{array} \right).$$

**Example 3.2.36.** For  $D_5$ , after the fifth step we have

$$\begin{aligned} T_3 T_2 T_1 S_2 S_1 P_3 P_2 P_1 V_q (S_2 S_1 P_3 P_2 P_1)^t = \\ \left( \begin{array}{cc|cc|cc|c|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (1-q^2)I_5 & q(1-q^2)K & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^2 I_5 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & (1-q^2)^2 I_5 & q(1-q^2)^2 JK & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 I_5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 I_5 & q(1-q^2)^2 J^2 K \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 I_5 \end{array} \right). \end{aligned}$$

**Step. 6: Multiply the new matrix on the right by  $T_1^t, \dots, T_p^t$  to get the matrix**

$$(T_p \dots T_1 S_{p-1} \dots S_1 P_p, \dots, P_1) V_q (T_p \dots T_1 S_{p-1} \dots S_1 P_p, \dots, P_1)^t.$$

*Remark 3.2.37.* This step has the effect of beginning with the last column and subtracting the previous column times  $q(J^{p-1}K)^t = qJ^{p-1}K$ ; then the same operation is repeated by subtracting the fourth-to-last column times  $qJ^{p-1}K$  from the third-to-last column, and so on.

The  $qJ^{i-1}KE_{ii}$ -entries above the diagonal become zero since each  $qJ^{i-1}KE_{ii}$ -entry equals  $qJ^{i-1}K$  times the corresponding  $E_{ii}$ -entry which is  $(1-q^2)I$  for  $E_{11}$  and  $(1-q^2)^2 I$  for  $i \geq 2$ . Now the matrix is in diagonal form.

At this point, the  $q$ -Varchenko matrix for  $D_n$  is transformed into

$$T_p \dots T_1 S_{p-1} \dots S_1 P_p \dots P_1 V_q (T_p \dots T_1 S_{p-1} \dots S_1 P_p \dots P_1)^t =$$

$$\left( \begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & (1-q^2)I_n & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^2 I_n & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & (1-q^2)^2 I_n & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 I_n & \dots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \dots & (1-q^2)^2 I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & (1-q^2)^3 I_n \end{array} \right).$$

**Example 3.2.38.** For  $D_5$ , after the sixth step we have

$$T_3 T_2 T_1 S_2 S_1 P_3 P_2 P_1 V_q (T_3 T_2 T_1 S_2 S_1 P_3 P_2 P_1)^t =$$

$$\left( \begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (1-q^2)I_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^2 I_5 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & (1-q^2)^2 I_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 I_5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 I_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 I_5 & 0 \end{array} \right).$$

**Step. 7: Multiply on the left by the permutation matrix R and on the right by  $R^t$  to put the diagonal matrix into Smith Normal Form.**

For  $D_n$ ,

$$R = \left( \begin{array}{c|ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & I_n & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_n \end{array} \right).$$

$$\text{The permutation matrix is } R = \left( \begin{array}{cccccc} A_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & A_2 & & & \dots & 0 \\ 0 & 0 & \ddots & \dots & \dots & 0 \\ 0 & 0 & \dots & A_j & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & & & A_p \end{array} \right)_{2np+2 \times 2np+2},$$

$$\text{where } A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}_{(2n+2) \times (2n+2)}, \text{ and } A_j = \begin{pmatrix} I_n & 0 & \dots & 0 \\ 0 & \dots & 0 & I_n \end{pmatrix}_{2n \times np} \text{ for } 2 \leq j \leq p.$$

The Smith Normal Form for the  $D_n$  arrangement is

$$\begin{aligned} & RT_p \dots T_1 S_{p-1} \dots S_1 P_p \dots P_1 V_q (RT_p \dots T_1 S_{p-1} \dots S_1 P_p \dots P_1)^t \\ &= \begin{pmatrix} 1 & 0 & & & 0 & & \\ 0 & (1-q^2)I_{n+1} & & & 0 & & \\ 0 & 0 & (1-q^2)^2 I_{np} & & 0 & & \\ 0 & 0 & 0 & (1-q^2)^3 I_{n(p-1)} & & & \end{pmatrix}. \quad (3.2.14) \end{aligned}$$

**Example 3.2.39.** For  $D_5$ ,

$$R = \left( \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_5 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_5 \end{array} \right).$$

The Smith Normal Form is  $RT_3T_2T_1S_2S_1P_3P_2P_1V_q(RT_3T_2T_1S_2S_1P_3P_2P_1)^t$

$$\begin{aligned} &= \left( \begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)I_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^2I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2I_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_5 \end{array} \right) \\ &= \left( \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)I_6 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2I_{15} & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^3I_{10} & 0 \end{array} \right). \end{aligned}$$

## CHAPTER

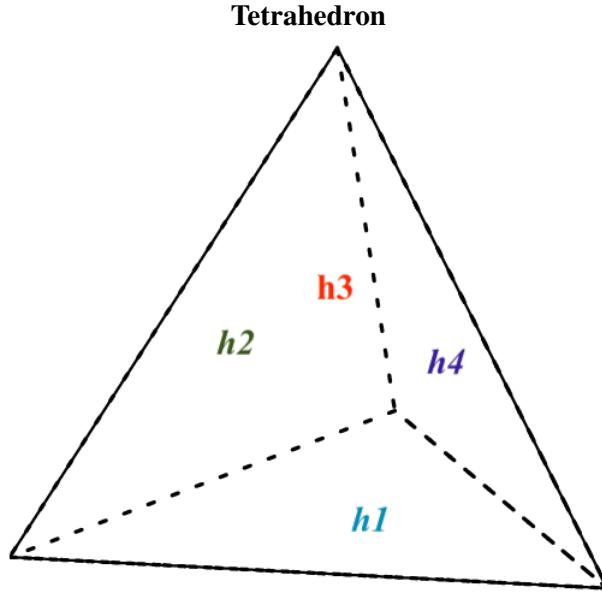
# 4

## TETRAHEDRON

### 4.1 Peelability

Let  $\mathcal{A} = \{h_1, h_2, h_3, h_4\}$  be a hyperplane arrangement in  $\mathbb{R}^3$  that forms a regular tetrahedron. The definition of a peelable hyperplane arrangement was given in [CCM16]. Here, each hyperplane in  $\mathcal{A}$  is peelable; we arbitrarily choose to begin by peeling  $h_1$ . First, assign the regions of the tetrahedron hyperplane arrangement according to their sign vectors here, where a region  $R$  is uniquely identified by the vector  $(\pm \pm \pm \pm)$  that indicates whether it lies on the  $+$  or  $-$  side of  $h_i$  for  $1 \leq i \leq 4$ .

$$\begin{aligned} R_1 &= (---), R_2 = (--+-), R_3 = (--++), R_4 = (-+--), R_5 = (-+-+), \\ R_6 &= (-+++), R_7 = (-+-+-), R_8 = (+---), R_9 = (+-+-), R_{10} = (+-++), \\ R_{11} &= (++-+), R_{12} = (+++-), R_{13} = (++++), R_{14} = (++-+), R_{15} = (+--+). \end{aligned}$$

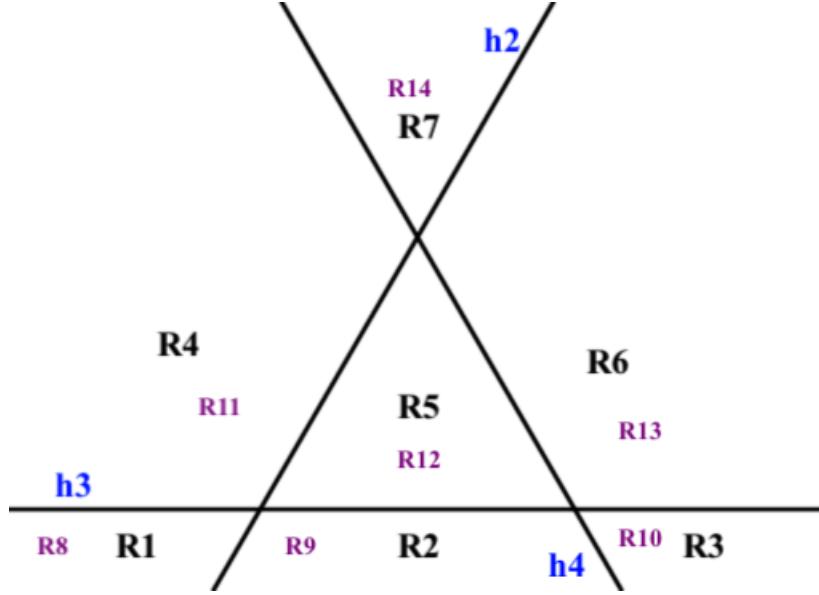


The top side of  $h_1$  is the + side; the right side of  $h_2$  is the + side; the back side of  $h_3$  is the + side; and the right side of  $h_4$  is the + side.

**Figure 4.1** Tetrahedron hyperplane arrangement with labelled hyperplanes.

$$V_q = \left( \begin{array}{ccccccc|ccccccccc|c} 1 & q & q^2 & q & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^4 & q^3 & q^2 & q^2 \\ q & 1 & q & q^2 & q & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^4 & q^3 & q^3 \\ q^2 & q & 1 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q & q^4 & q^3 & q^2 & q^2 & q^3 & q^2 \\ q & q^2 & q^3 & 1 & q & q^2 & q & q^2 & q^3 & q^4 & q & q^2 & q^3 & q^2 & q^3 & q^3 \\ q^2 & q & q^2 & q & 1 & q & q^2 & q^3 & q^2 & q^3 & q^3 & q^2 & q & q^2 & q^3 & q^4 \\ q^3 & q^2 & q & q^2 & q & 1 & q & q^4 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^2 & q^3 \\ q^2 & q^3 & q^2 & q & q^2 & q & 1 & q^3 & q^4 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^2 \\ q & q^2 & q^3 & q^2 & q^3 & q^4 & q^3 & 1 & q & q^2 & q & q^2 & q^3 & q^2 & q & q \\ q^2 & q & q^2 & q^3 & q^2 & q^3 & q^4 & q & 1 & q & q^2 & q & q^2 & q^3 & q^2 & q^2 \\ q^3 & q^2 & q & q^4 & q^3 & q^2 & q^3 & q & q & 1 & q^3 & q^2 & q & q^2 & q & q \\ q^2 & q^3 & q^4 & q & q^2 & q^3 & q^2 & q & q & q^2 & 1 & q & q^2 & q & q^2 & q^2 \\ q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q & q & q & q & 1 & q & q^2 & q^2 & q^3 \\ q^4 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q & q^2 & q & 1 & q & q & q^2 \\ q^3 & q^4 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q^3 & q^2 & q & q^2 & q & 1 & q & q \\ q^2 & q^3 & q^2 & q^3 & q^4 & q^3 & q^2 & q & q & q^2 & q & q^2 & q^3 & q^2 & q & 1 \end{array} \right) \quad (4.1.1)$$

We can write  $V_q$  in the block form  $V_q = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$  where  $[B_1 \ C_1] = q[B_2 \ C_2]$  and  $A_2 = qA_1$ . (4.1.2)

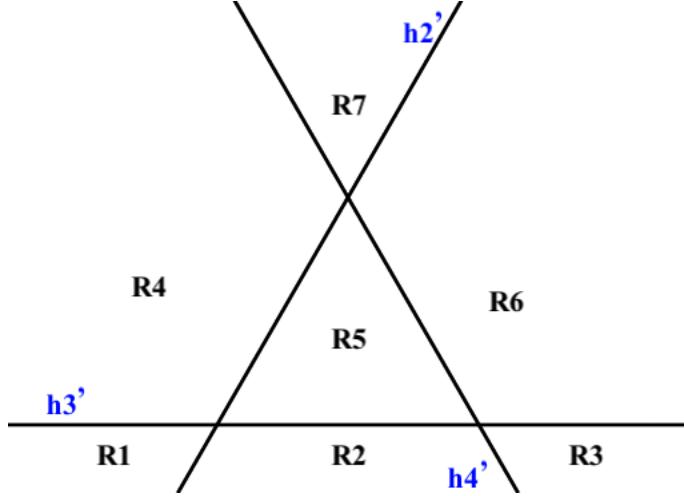


These are the lower and upper regions of the tetrahedron hyperplane arrangement whose intersection with  $h_1$  is nonempty.  $h_1$  is the hyperplane of this page; the regions  $R_1 - R_7$  are on the  $-$  side of  $h_1$  and the regions  $R_8 - R_{14}$  are on the  $+$  side of  $h_1$ .

**Figure 4.2** Two Layers of the Peelable Tetrahedron Hyperplane Arrangement.

### 4.1.1 Triangle

Here  $A_1$  is the  $7 \times 7$   $q$ -Varchenko matrix for the hyperplane arrangement of a regular triangle in  $\mathbb{R}^2$ . This can be "seen" from a viewpoint on the  $-$  side of  $h_1$  by viewing the intersection of the hyperplanes in  $\mathbb{R}^3$  on the  $-$  side of  $h_1$  as hyperplanes in the space  $h_1$ . Define  $\mathcal{A}' = \{h'_2, h'_3, h'_4\}$ , where  $h'_i$  is the intersection of  $h_i$  with  $h_1$ . On the  $-$  side of the hyperplane  $h_1$ ,  $\mathcal{A}'$  is the hyperplane arrangement of a regular triangle. In [CCM16] it was shown that the hyperplane arrangement  $\mathcal{A}'$  is peelable and the first step of block diagonalizing the  $q$ -Varchenko matrix for the  $A_1$  block was given. We will include the full process here. To avoid ambiguity, we will denote the  $q$ -Varchenko matrix for the regular triangle arrangement  $\mathcal{A}'$  as  $V_{q'}$  and break  $A_1$  into a block matrix form using ' notation for the block and transition matrices.



This is  $\mathcal{A}^{h_1}$ ; ie, the view of the hyperplane arrangement from underneath the hyperplane  $h_1$  in the tetrahedron hyperplane arrangement  $\mathcal{A}$  projected onto  $h_1$ .

**Figure 4.3** Triangle hyperplane arrangement with labelled hyperplanes and regions.

$$V_{q'} = \left( \begin{array}{ccc|ccc|c} 1 & q & q^2 & q & q^2 & q^3 & q^2 \\ q & 1 & q & q^2 & q & q^2 & q^3 \\ q^2 & q & 1 & q^3 & q^2 & q & q^2 \\ \hline q & q^2 & q^3 & 1 & q & q^2 & q \\ q^2 & q & q^2 & q & 1 & q & q^2 \\ q^3 & q^2 & q & q^2 & q & 1 & q \\ \hline q^2 & q^3 & q^2 & q & q^2 & q & 1 \end{array} \right) = \begin{pmatrix} A'_1 & B'_1 & C'_1 \\ A'_2 & B'_2 & C'_2 \\ A'_3 & B'_3 & C'_3 \end{pmatrix},$$

$$\text{where } A'_2 = qA'_1, [B'_1 C'_1] = q[B'_2 C'_2], \text{ and } [A'_2 A'_3]^t = q[B'_2 B'_3]^t. \quad (4.1.3)$$

We multiply  $V_{q'}$  on the left by

$$P'_1 = \begin{pmatrix} I_3 & -qI_3 & 0_{3 \times 1} \\ 0_{3 \times 3} & I_3 & 0_{3 \times 1} \\ 0_{1 \times 3} & 0_{1 \times 3} & 1 \end{pmatrix} \quad (4.1.4)$$

to obtain

$$P'_1 V_{q'} = \begin{pmatrix} (1-q^2)A'_1 & 0 & 0 \\ A'_2 & B'_2 & C'_2 \\ A'_3 & B'_3 & C'_3 \end{pmatrix}.$$

Then we multiply  $P'_1 V_{q'} \cdot (P'_1)^t$  on the right by  $(P'_1)^t$  to obtain

$$P'_1 V_{q'} \cdot (P'_1)^t = \begin{pmatrix} (1-q^2)A'_1 & 0 & 0 \\ 0 & B'_2 & C'_2 \\ 0 & B'_3 & C'_3 \end{pmatrix}, \text{ where } \begin{pmatrix} B'_2 & C'_2 \\ B'_3 & C'_3 \end{pmatrix} \text{ is the } q\text{-Varchenko matrix for } \mathcal{A}' - \{h'_3\}. \quad (4.1.5)$$

$A'_1$  is the  $q$ -Varchenko matrix of the hyperplane arrangement consisting of two points on a line (ie, the intersections with  $h'_2$  and with  $h'_4$  on  $h'_3$ ).

Next we re-index the regions using the following permutation matrix

$$R' = \left( \begin{array}{c|cccc} I_3 & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\ \hline 0_{1 \times 3} & 1 & 0 & 0 & 0 \\ 0_{1 \times 3} & 0 & 1 & 0 & 0 \\ 0_{1 \times 3} & 0 & 0 & 0 & 1 \\ 0_{1 \times 3} & 0 & 0 & 1 & 0 \end{array} \right). \quad (4.1.6)$$

$$\begin{aligned} R' P'_1 V_{q'} \cdot (R' P'_1)^t &= \left( \begin{array}{c|cc|cc} (1-q^2)A'_1 & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} & 0_{3 \times 1} \\ \hline 0_{1 \times 3} & 1 & q & q & q^2 \\ 0_{1 \times 3} & q & 1 & q^2 & q \\ 0_{1 \times 3} & q & q^2 & 1 & q \\ 0_{1 \times 3} & q^2 & q & q & 1 \end{array} \right) \\ &= \begin{pmatrix} (1-q^2)A'_1 & 0 & 0 \\ 0 & D'_1 & E'_1 \\ 0 & D'_2 & E'_2 \end{pmatrix} = \begin{pmatrix} (1-q^2)A'_1 & 0 & 0 \\ 0 & D'_1 & qD'_1 \\ 0 & qD'_1 & D'_1 \end{pmatrix}. \end{aligned} \quad (4.1.7)$$

$$\text{Multiply on the left by } P'_2 = \begin{pmatrix} I_3 & 0_{3 \times 2} & 0_{3 \times 2} \\ 0_{2 \times 3} & I_2 & -qI_2 \\ 0_{2 \times 3} & 0_{2 \times 2} & I_2 \end{pmatrix} \quad (4.1.8)$$

$$\text{to obtain } P'_2 R' P'_1 V_{q'} \cdot (R' P'_1)^t = \begin{pmatrix} (1-q^2)A'_1 & 0 & 0 \\ 0 & (1-q^2)D'_1 & 0 \\ 0 & qD'_1 & D'_1 \end{pmatrix}.$$

Then multiply  $P'_2 R' P'_1 V_{q'} \cdot (R' P'_1)^t$  on the right by  $(P'_2)^t$  to obtain

$$P'_2 R' P'_1 V_{q'} \cdot (P'_2 R' P'_1)^t = \begin{pmatrix} (1-q^2)A'_1 & 0 & 0 \\ 0 & (1-q^2)D'_1 & 0 \\ 0 & 0 & D'_1 \end{pmatrix}. \quad (4.1.9)$$

We have:  $A'_1 = \begin{pmatrix} 1 & q & q^2 \\ q & 1 & q \\ q^2 & q & 1 \end{pmatrix}$ , (4.1.10)

$$S'_1 = \begin{pmatrix} 1 & -q & 0 \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{pmatrix}, \text{ and} \quad (4.1.11)$$

$$S'_1 A'_1 (S'_1)^t = \begin{pmatrix} 1-q^2 & 0 & 0 \\ 0 & 1-q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.1.12)$$

For  $S'_2 = \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}$ , we have  $S'_2 D'_1 (S'_2)^t = \begin{pmatrix} 1-q^2 & 0 \\ 0 & 1 \end{pmatrix}$ . (4.1.13)

Therefore, 
$$\begin{aligned} & \left( \begin{array}{ccc} S'_1 & 0 & 0 \\ 0 & S'_2 & 0 \\ 0 & 0 & S'_2 \end{array} \right) \left( \begin{array}{ccc} (1-q^2)A'_1 & 0 & 0 \\ 0 & (1-q^2)D'_1 & 0 \\ 0 & 0 & D'_1 \end{array} \right) \left( \begin{array}{ccc} (S'_1)^t & 0 & 0 \\ 0 & (S'_2)^t & 0 \\ 0 & 0 & (S'_2)^t \end{array} \right) \\ &= \left( \begin{array}{ccc|cc|cc} (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q^2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1-q^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \end{aligned}$$

Since the matrix is in diagonal form it can easily be put into Smith normal form by re-arranging the diagonal entries.

#### 4.1.2 Tetrahedron continued

Now back to the tetrahedron: we will proceed to block-diagonalize  $V_q$  as it was given in Eq. 4.1.1 and Eq. 4.1.2.

$$V_q = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}, \text{ where } A_1 = V_{q'} \text{ as it was defined in Eq. 4.1.3.}$$

First, multiply on the left by  $P_1 = \begin{pmatrix} I_7 & -qI_7 & 0_{7 \times 1} \\ 0_{7 \times 7} & I_7 & 0_{7 \times 1} \\ 0_{1 \times 7} & 0_{1 \times 7} & 1 \end{pmatrix}$  to obtain  $P_1 V_q = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$ .

Since we have  $[A_2 \ A_3]^t = [B_2 \ B_3]^t$ , right multiplication by  $P_1^t$  gives us

$$P_1 V_q P_1^t = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 \\ 0 & B_2 & C_2 \\ 0 & B_3 & C_3 \end{pmatrix}, \text{ where } \begin{pmatrix} B_2 & C_2 \\ B_3 & C_3 \end{pmatrix} \text{ is as defined in Eq. 4.1.1 and Eq. 4.1.2};$$

it is the  $q$ -Varchenko matrix for the hyperplane arrangement in  $\mathbb{R}^3$  given by  $\{h_2, h_3, h_4\}$ .

We can repeat the process of peeling a hyperplane from  $\mathcal{A} - \{h_1\}$  after re-arranging the  $q$ -Varchenko matrix for the regions  $R_8, R_9, \dots, R_{15}$  via left multiplication by a permutation matrix  $R$  and right multiplication by  $R^t$ .  $R$  and  $R^t$  re-arrange the regions according to the following map:

$$R_8 \rightarrow R_8, R_9 \rightarrow R_{10}, R_{10} \rightarrow R_{11}, R_{11} \rightarrow R_{12}, R_{12} \rightarrow R_{14}, R_{14} \rightarrow R_{13}, R_{15} \rightarrow R_9.$$

$$R = \left( \begin{array}{c|cccccccc} I_7 & 0_{7 \times 1} \\ \hline 0_{1 \times 7} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0_{1 \times 7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0_{1 \times 7} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0_{1 \times 7} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0_{1 \times 7} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0_{1 \times 7} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0_{1 \times 7} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0_{1 \times 7} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right). \quad (4.1.14)$$

With the new indexing,

$$\begin{aligned} R_8 &= (+---), R_9 = (+---), R_{10} = (+-++), R_{11} = (+-+-), \\ R_{12} &= (++-+), R_{13} = (++-+), R_{14} = (+++-), R_{15} = (++-+). \end{aligned}$$

The re-indexing step here is unavoidable, since we needed to arrange the  $q$ -Varchenko matrix in such a way that all regions on one side of a chosen hyperplane preceded all regions on the other side of the chosen hyperplane. Now we can peel  $h_2$  from  $\mathcal{A} - \{h_1\}$  since regions  $R_8, R_9, R_{10}, R_{11}$  are on the  $-$  side of  $h_2$  and the rest of the regions are on the  $+$  side of  $h_2$ . Looking one step further ahead, we see that we will be able to peel  $h_3$  next without any further re-indexing of the regions since  $R_{12}, R_{13}$  are on the  $-$  side of  $h_3$  while

$R_{14}, R_{15}$  are on the + side of  $h_3$ .

$$RP_1 V_q(RP_1)^t = \left( \begin{array}{c|ccccc|ccccc} (1-q^2)A_1 & 0_{7 \times 1} \\ \hline 0_{1 \times 7} & 1 & q & q & q^2 & q & q^2 & q^2 & q^3 \\ 0_{1 \times 7} & q & 1 & q^2 & q & q^2 & q & q^3 & q^2 \\ 0_{1 \times 7} & q & q^2 & 1 & q & q^2 & q & q^3 & q^2 \\ 0_{1 \times 7} & q^2 & q & q & 1 & q^3 & q^2 & q^2 & q \\ \hline 0_{1 \times 7} & q & q^2 & q^2 & q^3 & 1 & q & q & q^2 \\ 0_{1 \times 7} & q^2 & q & q^3 & q^2 & q & 1 & q^2 & q \\ 0_{1 \times 7} & q^2 & q & q^3 & q^2 & q & q^2 & 1 & q \\ 0_{1 \times 7} & q^3 & q^2 & q^2 & q & q^2 & q & q & 1 \end{array} \right).$$

$$\text{Write } RP_1 V_q(RP_1)^t = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 \\ 0 & D_1 & E_1 \\ 0 & D_2 & E_2 \end{pmatrix} = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 \\ 0 & D_1 & qD_1 \\ 0 & qD_1 & D_1 \end{pmatrix}.$$

$$\text{Multiply on the left by } P_2 = \begin{pmatrix} I_7 & 0_{7 \times 4} & 0_{7 \times 4} \\ 0_{4 \times 7} & I_4 & -qI_4 \\ 0_{4 \times 7} & 0_{4 \times 4} & I_4 \end{pmatrix}$$

to obtain  $P_2 RP_1 V_q(RP_1)^t = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 \\ 0 & (1-q^2)D_1 & 0 \\ 0 & qD_1 & D_1 \end{pmatrix}.$

Then multiply on the right by  $P_2^t$  to obtain

$$P_2 RP_1 V_q(P_2 RP_1)^t = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 \\ 0 & (1-q^2)D_1 & 0 \\ 0 & 0 & D_1 \end{pmatrix}$$

$$= \left( \begin{array}{c|cc|cc|cc} (1-q^2)A_1 & 0_{7 \times 4} & 0_{7 \times 1} & 0_{7 \times 1} & 0_{7 \times 1} & 0_{7 \times 1} \\ \hline 0_{4 \times 7} & (1-q^2)D_1 & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} & 0_{4 \times 1} \\ 0_{1 \times 7} & 0_{1 \times 4} & 1 & q & q & q^2 \\ 0_{1 \times 7} & 0_{1 \times 4} & q & 1 & q^2 & q \\ \hline 0_{1 \times 7} & 0_{1 \times 4} & q & q^2 & 1 & q \\ 0_{1 \times 7} & 0_{1 \times 4} & q^2 & q & q & 1 \end{array} \right).$$

$$\text{Write } P_2 RP_1 V_q(P_2 RP_1)^t = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 & 0 \\ 0 & (1-q^2)D_1 & 0 & 0 \\ 0 & 0 & F_1 & G_1 \\ 0 & 0 & F_2 & G_2 \end{pmatrix}.$$

The  $q$ -Varchenko matrix for  $\{h_3, h_4\} = \mathcal{A} - \{h_1, h_2\}$  is  $D_1 = \begin{pmatrix} F_1 & G_1 \\ F_2 & G_2 \end{pmatrix} = \begin{pmatrix} F_1 & qF_1 \\ qF_1 & F_1 \end{pmatrix}$ . (4.1.15)

$$\text{Multiply on the left by } P_3 = \begin{pmatrix} I_7 & 0_{7 \times 4} & 0_{7 \times 2} & 0_{7 \times 2} \\ 0_{4 \times 7} & I_4 & 0_{4 \times 2} & 0_{4 \times 2} \\ 0_{2 \times 7} & 0_{2 \times 4} & I_2 & -qI_2 \\ 0_{2 \times 7} & 0_{2 \times 4} & 0_{2 \times 2} & I_2 \end{pmatrix}$$

to obtain  $P_3 P_2 R P_1 V_q (P_2 R P_1)^t = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 & 0 \\ 0 & (1-q^2)D_1 & 0 & 0 \\ 0 & 0 & (1-q^2)F_1 & 0 \\ 0 & 0 & qF_1 & F_1 \end{pmatrix}$ .

Then multiply on the right by  $P_3^t$  to obtain

$$P_3 P_2 R P_1 V_q (P_3 P_2 R P_1)^t = \begin{pmatrix} (1-q^2)A_1 & 0 & 0 & 0 \\ 0 & (1-q^2)D_1 & 0 & 0 \\ 0 & 0 & (1-q^2)F_1 & 0 \\ 0 & 0 & 0 & F_1 \end{pmatrix}.$$

The  $q$ -Varchenko matrix for  $\mathcal{A} - \{h_1, h_2, h_3\} = \{h_4\}$  is  $F_1 = \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}$ .

Now we can diagonalize the  $q$ -Varchenko matrix addressing one block at a time.

To begin with, we can diagonalize the block  $F_1$  by multiplying on the left by  $S'_2$  and on the right by  $S''_2$  where  $S'_2$  is as it was defined in Eq. 4.1.13.

$$S'_2 F_1 (S'_2)^t = \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} = \begin{pmatrix} 1-q^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} S'_2 & 0 \\ 0 & S'_2 \end{pmatrix} \begin{pmatrix} (1-q^2)F_1 & 0 \\ 0 & F_1 \end{pmatrix} \begin{pmatrix} (S'_2)^t & 0 \\ 0 & (S'_2)^t \end{pmatrix} = \begin{pmatrix} (1-q^2)^2 & 0 & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 \\ 0 & 0 & 1-q^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, by Eq. 4.1.15,  $(1-q^2)D_1$  can be first block diagonalized into  $(1-q^2) \begin{pmatrix} (1-q^2)F_1 & 0 \\ 0 & F_1 \end{pmatrix}$ , and

then diagonalized into  $\begin{pmatrix} (1-q^2)^3 & 0 & 0 & 0 \\ 0 & (1-q^2)^2 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 & 0 \\ 0 & 0 & 0 & 1-q^2 \end{pmatrix}$ .

$$\text{Define } P_4 = \left( \begin{array}{c|cc|cc} I_7 & 0_{7 \times 2} & 0_{7 \times 2} & 0_{7 \times 2} & 0_{7 \times 2} \\ \hline 0_{2 \times 7} & I_2 & -qI_2 & 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 7} & 0_{2 \times 2} & I_2 & 0_{2 \times 2} & 0_{2 \times 2} \\ \hline 0_{2 \times 7} & 0_{2 \times 2} & 0_{2 \times 2} & I_2 & 0_{2 \times 2} \\ 0_{2 \times 7} & 0_{2 \times 2} & 0_{2 \times 2} & 0_{2 \times 2} & I_2 \end{array} \right).$$

Multiply on the left by  $P_4$  and on the right by  $P'_4$  to obtain

$$P_4 P_3 P_2 R P_1 V_q (P_4 P_3 P_2 R P_1)^t = \left( \begin{array}{c|cc|cc} (1-q^2)A_1 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)^2 F_1 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)F_1 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)F_1 & 0 \\ 0 & 0 & 0 & 0 & F_1 \end{array} \right).$$

$$\text{Let } S_1 = \begin{pmatrix} P'_2 R' P'_1 & 0_{7 \times 4} & 0_{7 \times 4} \\ 0_{4 \times 7} & I_4 & 0_{4 \times 4} \\ 0_{4 \times 7} & 0_{4 \times 4} & I_4 \end{pmatrix},$$

where  $P'_2, R', P'_1$  are as they were defined in Eq. 4.1.8, Eq. 4.1.6, Eq. 4.1.4 when considering  $V_{q'}$ .

$$S_1 P_4 P_3 P_2 R P_1 V_q (S_1 P_4 P_3 P_2 R P_1)^t$$

$$\begin{aligned}
&= \left( \begin{array}{ccc|cc|cc} (1-q^2)^2 A'_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)^2 D'_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2) E'_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2 F_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2) G_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2) F_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G_2 \end{array} \right) \\
&= \left( \begin{array}{ccc|cc|cc} (1-q^2)^2 A'_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)^2 D'_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2) D'_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2 F_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2) F_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2) F_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F_1 \end{array} \right).
\end{aligned}$$

Let

$$S_2 = \left( \begin{array}{ccc|cc|cc} S'_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S'_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S'_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & S'_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S'_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & S'_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & S'_2 \end{array} \right), \text{ where } S'_1 \text{ and } S'_2 \text{ are as defined in Eq. 4.1.11 and Eq. 4.1.13.}$$

$$S_2 S_1 P_4 P_3 P_2 R P_1 V_q (S_2 S_1 P_4 P_3 P_2 R P_1)^t = \left( \begin{array}{c|c|c} d_{11} & 0 & 0 \\ \hline 0 & d_{22} & 0 \\ \hline 0 & 0 & d_{33} \end{array} \right),$$

$$\text{where } d_{11} = \left( \begin{array}{ccccccc} (1-q^2)^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-q^2 \end{array} \right),$$

$$d_{22} = \left( \begin{array}{cccc} (1-q^2)^3 & 0 & 0 & 0 \\ 0 & (1-q^2)^2 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 & 0 \\ 0 & 0 & 0 & 1-q^2 \end{array} \right), \quad d_{33} = \left( \begin{array}{cccc} (1-q^2)^2 & 0 & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 \\ 0 & 0 & 1-q^2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right).$$

Define  $R_2$  to be the permutation matrix that puts the diagonal form of  $V_q$  into Smith normal form:

$$1 \rightarrow 15, 2 \rightarrow 14, 3 \rightarrow 11, 4 \rightarrow 13, 5 \rightarrow 7, 6 \rightarrow 6, 7 \rightarrow 5, \\ 8 \rightarrow 12, 9 \rightarrow 9, 10 \rightarrow 10, 11 \rightarrow 4, 12 \rightarrow 8, 13 \rightarrow 3, 14 \rightarrow 2, 15 \rightarrow 1.$$

## 4.2 Symmetry

Here we use a different approach to indexing the regions. The region originally indexed as  $R_{12} = (+ + + -)$  using peelability becomes the central region, which we label  $R_0$ .

The hyperplane arrangement for the tetrahedron consists of four hyperplanes in  $R^3$  arranged so that each hyperplane intersects all other three hyperplanes, and they divide  $R^3$  into 15 regions.  $\mathcal{A} = \{h_1, h_2, h_3, h_4\}$ .

The distance enumerator with respect to  $R_0$  is  $D_{\mathcal{A}, R_0}(t) = 1 + 4t + 6t^2 + 4t^3$ .

Labelling the central region of the convex polytope as  $R_0$ , we then label the remaining regions  $R'$  according to the separating sets  $sep(R_0, R')$  as follows:

$$\begin{aligned} sep(R_0, R_1) &= \{h_1\}, & sep(R_0, R_5) &= \{h_1, h_2\}, & sep(R_0, R_{11}) &= \{h_1, h_2, h_3\}, \\ sep(R_0, R_2) &= \{h_2\}, & sep(R_0, R_6) &= \{h_2, h_3\}, & sep(R_0, R_{12}) &= \{h_2, h_3, h_4\}, \\ sep(R_0, R_3) &= \{h_3\}, & sep(R_0, R_7) &= \{h_3, h_4\}, & sep(R_0, R_{13}) &= \{h_1, h_3, h_4\}, \\ sep(R_0, R_4) &= \{h_4\}, & sep(R_0, R_8) &= \{h_1, h_4\}, & sep(R_0, R_{14}) &= \{h_1, h_2, h_4\}. \\ && sep(R_0, R_9) &= \{h_1, h_3\}, \\ && sep(R_0, R_{10}) &= \{h_2, h_4\}, \end{aligned}$$

Then the  $q$ -Varchenko matrix has the following form:

$$V_q = \left( \begin{array}{c|cccc|cccc|cccc} 1 & q & q & q & q & q^2 & q^2 & q^2 & q^2 & q^2 & q^2 & q^3 & q^3 & q^3 & q^3 \\ \hline q & 1 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q & q & q^3 & q^2 & q^4 & q^2 & q^2 \\ q & q^2 & 1 & q^2 & q^2 & q & q & q^3 & q^3 & q^3 & q & q^2 & q^2 & q^4 & q^2 \\ q & q^2 & q^2 & 1 & q^2 & q^3 & q & q & q^3 & q & q^3 & q^2 & q^2 & q^2 & q^4 \\ q & q^2 & q^2 & q^2 & 1 & q^3 & q^3 & q & q & q^3 & q & q^4 & q^2 & q^2 & q^2 \\ \hline q^2 & q & q & q^3 & q^3 & 1 & q^2 & q^4 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q \\ q^2 & q^3 & q & q & q^3 & q^2 & 1 & q^2 & q^4 & q^2 & q^2 & q & q & q^3 & q^3 \\ q^2 & q^3 & q^3 & q & q & q^4 & q^2 & 1 & q^2 & q^2 & q^2 & q^3 & q & q & q^3 \\ q^2 & q & q^3 & q^3 & q & q^2 & q^4 & q^2 & 1 & q^2 & q^2 & q^3 & q^3 & q & q \\ \hline q^2 & q & q^3 & q & q^3 & q^2 & q^2 & q^2 & q^2 & 1 & q^4 & q & q^3 & q & q^3 \\ q^2 & q^3 & q & q^3 & q & q^2 & q^2 & q^2 & q^2 & q^4 & 1 & q^3 & q & q^3 & q \\ \hline q^3 & q^2 & q^2 & q^2 & q^4 & q & q & q^3 & q^3 & q & q^3 & 1 & q^2 & q^2 & q^2 \\ q^3 & q^4 & q^2 & q^2 & q^2 & q^3 & q & q & q^3 & q^3 & q & q^2 & 1 & q^2 & q^2 \\ q^3 & q^2 & q^4 & q^2 & q^2 & q^3 & q^3 & q & q & q & q^3 & q^2 & q^2 & 1 & q^2 \\ q^3 & q^2 & q^2 & q^4 & q^2 & q & q^3 & q^3 & q & q^3 & q & q^2 & q^2 & q^2 & 1 \end{array} \right).$$

$V_q$  has a Smith Normal Form over  $\mathbb{Z}[q]$ . The left and right transformation matrices are (respectively)

$$S_1 R_2 R_1 P_3 P_2 P_1 \text{ and } (S_1 R_2 R_1 P_3 P_2 P_1)^t.$$

$$\begin{aligned}
S_1 R_2 R_1 P_3 P_2 P_1 &= \left( \begin{array}{c|ccccc|ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
q^2 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & 0 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & -q & 0 & 0 & -q & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
q^2 & -q & 0 & -q & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & 0 & -q & 0 & -q & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
-q^3 & q^2 & q^2 & q^2 & 0 & -q & -q & 0 & 0 & -q & 0 & 1 & 0 & 0 & 0 & 0 \\
-q^3 & 0 & q^2 & q^2 & q^2 & 0 & -q & -q & 0 & 0 & -q & 0 & 1 & 0 & 0 & 0 \\
-q^3 & q^2 & 0 & q^2 & q^2 & 0 & 0 & -q & -q & -q & 0 & 0 & 0 & 1 & 0 & 0 \\
-q^3 & q^2 & q^2 & 0 & q^2 & -q & 0 & 0 & -q & 0 & -q & 0 & 0 & 0 & 0 & 1
\end{array} \right) \\
&= \left( \begin{array}{c|c|c|c|c}
1 & 0 & 0 & 0 & 0 \\
\hline
-q1 & I_4 & 0 & 0 & 0 \\
\hline
q^2 1 & -q(I+J) & I_4 & 0 & 0 \\
\hline
q^2 1 & -q(I_2 \mid I_2) & 0 & I_2 & 0 \\
\hline
-q^3 1 & q^2(I+J+J^2) & -q(I+J) & -q(I_2 \mid I_2)^t & I_4
\end{array} \right), \\
\text{where } (I_2 \mid I_2) &= \left( \begin{array}{cc|cc}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array} \right), \text{ and } J = \left( \begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array} \right). \tag{4.2.1}
\end{aligned}$$

$$P_1 = \left( \begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & I_4 & 0 & 0 & 0 \\ \hline 0 & 0 & I_4 & 0 & 0 \\ \hline 0 & 0 & 0 & I_2 & 0 \\ \hline 0 & 0 & -qI_4 & 0 & I_4 \end{array} \right),$$

$$P_2 = \left( \begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & I_4 & 0 & 0 & 0 \\ \hline 0 & -qI_4 & I_4 & 0 & 0 \\ \hline 0 & -qH_1 & 0 & I_2 & 0 \\ \hline 0 & 0 & 0 & 0 & I_4 \end{array} \right), \text{ where } H_1 = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right),$$

$$P_3 = \left( \begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline -q1 & I_4 & 0 & 0 & 0 \\ \hline 0 & 0 & I_4 & 0 & 0 \\ \hline 0 & 0 & 0 & I_2 & 0 \\ \hline 0 & 0 & 0 & 0 & I_4 \end{array} \right).$$

$$P_3 P_2 P_1 V_q = \left( \begin{array}{c|cc|cc|c} 1 & v_{01} & v_{02} & v_{03} & v_{04} \\ \hline 0 & v_{11} & v_{12} & v_{13} & v_{14} \\ 0 & v_{21} & v_{22} & v_{23} & v_{24} \\ \hline 0 & v_{31} & v_{32} & v_{33} & v_{34} \\ 0 & v_{41} & v_{42} & v_{43} & v_{44} \end{array} \right), \text{ where } v_{01} = \begin{pmatrix} q & q & q & q \end{pmatrix}, v_{02} = \begin{pmatrix} q^2 & q^2 & q^2 & q^2 \end{pmatrix}, v_{03} = \begin{pmatrix} q^2 & q^2 \end{pmatrix}, v_{04} = \begin{pmatrix} q^3 & q^3 & q^3 & q^3 \end{pmatrix},$$

$$v_{11} = (1 - q^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad v_{12} = (1 - q^2) \begin{pmatrix} q & 0 & 0 & q \\ q & q & 0 & 0 \\ 0 & q & q & 0 \\ 0 & 0 & q & q \end{pmatrix}, \quad v_{13} = (1 - q^2) \begin{pmatrix} q & 0 \\ 0 & q \\ q & 0 \\ 0 & q \end{pmatrix}, \quad v_{14} = (1 - q^2) \begin{pmatrix} q^2 & 0 & q^2 & q^2 \\ q^2 & q^2 & 0 & q^2 \\ q^2 & q^2 & q^2 & 0 \\ 0 & q^2 & q^2 & q^2 \end{pmatrix},$$

$$v_{21} = (1 - q^2) \begin{pmatrix} 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \\ q & 0 & 0 & 0 \end{pmatrix}, \quad v_{22} = (1 - q^2) \begin{pmatrix} 1 & q^2 & 0 & 0 \\ 0 & 1 & q^2 & 0 \\ 0 & 0 & 1 & q^2 \\ q^2 & 0 & 0 & 1 \end{pmatrix}, \quad v_{23} = (1 - q^2) \begin{pmatrix} 0 & q^2 \\ q^2 & 0 \\ 0 & q^2 \\ q^2 & 0 \end{pmatrix}, \quad v_{24} = (1 - q^2) \begin{pmatrix} q & q^3 & 0 & q \\ q & q & q^3 & 0 \\ 0 & q & q & q^3 \\ q^3 & 0 & q & q \end{pmatrix},$$

$$v_{31} = (1 - q^2) \begin{pmatrix} 0 & 0 & q & 0 \\ 0 & q & 0 & 0 \end{pmatrix}, \quad v_{32} = (1 - q^2) \begin{pmatrix} 0 & q^2 & q^2 & 0 \\ q^2 & q^2 & 0 & 0 \end{pmatrix}, \quad v_{33} = (1 - q^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_{34} = (1 - q^2) \begin{pmatrix} q & q^3 & q & 0 \\ q^3 & q & 0 & q \end{pmatrix},$$

$$v_{41} = (1 - q^2) \begin{pmatrix} 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & q^2 \\ q^2 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \end{pmatrix}, \quad v_{42} = (1 - q^2) \begin{pmatrix} 0 & q & q^3 & 0 \\ 0 & 0 & q & q^3 \\ q^3 & 0 & 0 & q \\ q & q^3 & 0 & 0 \end{pmatrix}, \quad v_{43} = (1 - q^2) \begin{pmatrix} q & 0 \\ 0 & q \\ q & 0 \\ 0 & q \end{pmatrix}, \quad v_{44} = (1 - q^2) \begin{pmatrix} 1 & q^2 & q^2 & 0 \\ 0 & 1 & q^2 & q^2 \\ q^2 & 0 & 1 & q^2 \\ q^2 & q^2 & 0 & 1 \end{pmatrix}.$$

$$P_3 P_2 P_1 V_q (P_3 P_2 P_1)^t = \left( \begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_4 & q(1-q^2)J^3 & v_{13} & q^2(1-q^2)J^2 \\ \hline 0 & q(1-q^2)J & (1-q^2)I_4 & v_{23} & q(1-q^2)J^3 \\ \hline 0 & v_{31} & v_{32} & (1-q^2)I_2 & v_{34} \\ \hline 0 & q^2(1-q^2)J^2 & q(1-q^2)J & v_{43} & (1-q^2)I_4 + q^2(1-q^2)J^2 \end{array} \right),$$

where  $v_{31} = v_{13}^t$ ,  $v_{32} = v_{23}^t$ ,  $v_{34} = v_{43}^t$ ,  $J$  is as defined in Equation 4.2.1, and

$$v_{13} = (1-q^2) \begin{pmatrix} 0 & 0 \\ 0 & q \\ q & 0 \\ 0 & 0 \end{pmatrix}, \quad v_{23} = (1-q^2) \begin{pmatrix} 0 & q^2 \\ q^2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad v_{43} = (1-q^2) \begin{pmatrix} q & 0 \\ 0 & q(1-q^2) \\ q(1-q^2) & 0 \\ 0 & q \end{pmatrix}.$$

$$R_1 = \left( \begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & I_4 & 0 & 0 & 0 \\ \hline 0 & 0 & I_4 & 0 & 0 \\ \hline 0 & 0 & 0 & I_2 & 0 \\ \hline 0 & 0 & -qJ & 0 & I_4 \end{array} \right),$$

$$R_2 = \left( \begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & I_4 & 0 & 0 & 0 \\ \hline 0 & -qJ & I_4 & 0 & 0 \\ \hline 0 & -qH_2 & 0 & I_2 & 0 \\ \hline 0 & 0 & 0 & 0 & I_4 \end{array} \right), \text{ where } H_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$R_2 R_1 P_3 P_2 P_1 V_q (P_3 P_2 P_1)^t = \left( \begin{array}{c|c|c|c|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_4 & q(1-q^2)J^3 & v_{13} & q^2(1-q^2)J^2 \\ \hline 0 & 0 & (1-q^2)I_4 & 0 & q(1-q^2)^2J^3 \\ \hline 0 & 0 & 0 & (1-q^2)^2I_2 & v_{34} \\ \hline 0 & 0 & 0 & v_{43} & (1-q^2)^2I_4 + q^2(1-q^2)^2J^2 \end{array} \right),$$

where

$$v_{13} = q(1-q^2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad v_{43} = q(1-q^2)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_{34} = q(1-q^2)^2 \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

$$R_2 R_1 P_3 P_2 P_1 V_q (R_2 R_1 P_3 P_2 P_1)^t = \left( \begin{array}{c|ccccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_4 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 I_4 & 0 & 0 \\ 0 & 0 & 0 & v_{33} & v_{34} \\ \hline 0 & 0 & 0 & v_{43} & v_{44} \end{array} \right), \text{ where}$$

$$\left( \begin{array}{c|c} v_{33} & v_{34} \\ \hline v_{43} & v_{44} \end{array} \right) = \left( \begin{array}{cc|cccc} (1-q^2)^2 & 0 & q(1-q^2)^2 & 0 & q(1-q^2)^2 & 0 \\ 0 & (1-q^2)^2 & 0 & q(1-q^2)^2 & 0 & q(1-q^2)^2 \\ \hline q(1-q^2)^2 & 0 & (1-q^2)^2 & 0 & q^2(1-q^2)^2 & 0 \\ 0 & q(1-q^2)^2 & 0 & (1-q^2)^2 & 0 & q^2(1-q^2)^2 \\ q(1-q^2)^2 & 0 & q^2(1-q^2)^2 & 0 & (1-q^2)^2 & 0 \\ 0 & q(1-q^2)^2 & 0 & q^2(1-q^2)^2 & 0 & (1-q^2)^2 \end{array} \right).$$

$$S_1 = \left( \begin{array}{c|ccccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & I_4 & 0 & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 \\ \hline 0 & 0 & 0 & I_2 & 0 \\ \hline 0 & 0 & 0 & -q(I_2 | I_2)^t & I_4 \end{array} \right), \text{ where } (I_2 | I_2) \text{ is as defined in 4.2.1.}$$

$$S_1 R_2 R_1 P_3 P_2 P_1 V_q (R_2 R_1 P_3 P_2 P_1)^t = \left( \begin{array}{c|ccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 I_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_{33} & v_{34} & 0 \\ \hline 0 & 0 & 0 & v_{43} & v_{44} & 0 \end{array} \right), \text{ where}$$

$$\left( \begin{array}{c|c} v_{33} & v_{34} \\ \hline v_{43} & v_{44} \end{array} \right) = \left( \begin{array}{cc|ccccc} (1-q^2)^2 & 0 & q(1-q^2)^2 & 0 & q(1-q^2)^2 & 0 & 0 \\ 0 & (1-q^2)^2 & 0 & q(1-q^2)^2 & 0 & q(1-q^2)^2 & 0 \\ \hline 0 & 0 & (1-q^2)^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 & 0 \end{array} \right).$$

$$\begin{aligned}
S_1 R_2 R_1 P_3 P_2 P_1 V_q (S_1 R_2 R_1 P_3 P_2 P_1)^t &= \left( \begin{array}{c|cc|cc|c} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_4 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 I_4 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2 I_2 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^3 I_4 \end{array} \right) \\
&= \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & (1-q^2)I_4 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 I_6 & 0 \\ 0 & 0 & 0 & (1-q^2)^3 I_4 \end{array} \right).
\end{aligned}$$

## CHAPTER

# 5

## CUBE

### 5.1 Peelability

This time we will begin in  $\mathbb{R}^2$  with the hyperplane arrangement for a square before moving onto the cube in  $\mathbb{R}^3$ .

#### 5.1.1 Square

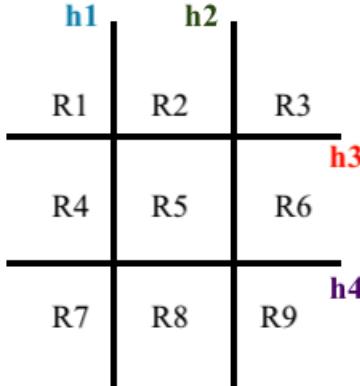
The hyperplane arrangement for a square in  $\mathbb{R}^2$  has nine regions, and is given by

$$\mathcal{A}' = \{h_1, h_2, h_3, h_4\}.$$

The regions are

$$\begin{aligned} R_1 &= (--)++, & R_2 &= (+++)-, & R_3 &= (++++)-, \\ R_4 &= (---+), & R_5 &= (+--+), & R_6 &= (++-+), \\ R_7 &= (----), & R_8 &= (+---), & R_9 &= (++-+). \end{aligned}$$

### Square Hyperplane Arrangement



**Figure 5.1** Square hyperplane arrangement with labelled hyperplanes and regions.

The  $q$ -Varchenko matrix for the square is

$$V_{q'} = \left( \begin{array}{ccc|ccc|ccc} 1 & q & q^2 & q & q^2 & q^3 & q^2 & q^3 & q^4 \\ q & 1 & q & q^2 & q & q^2 & q^3 & q^2 & q^3 \\ q^2 & q & 1 & q^3 & q^2 & q & q^4 & q^3 & q^2 \\ \hline q & q^2 & q^3 & 1 & q & q^2 & q & q^2 & q^3 \\ q^2 & q & q^2 & q & 1 & q & q^2 & q & q^2 \\ q^3 & q^2 & q & q^2 & q & 1 & q^3 & q^2 & q \\ \hline q^2 & q^3 & q^4 & q & q^2 & q^3 & 1 & q & q^2 \\ q^3 & q^2 & q^3 & q^2 & q & q^2 & q & 1 & q \\ q^4 & q^3 & q^2 & q^3 & q^2 & q & q^2 & q & 1 \end{array} \right) = \begin{pmatrix} A'_1 & B'_1 & C'_1 \\ A'_2 & B'_2 & C'_2 \\ A'_3 & B'_3 & C'_3 \end{pmatrix} = \begin{pmatrix} A'_1 & qA'_1 & q^2A'_1 \\ qA'_1 & A'_1 & qA'_1 \\ q^2A'_1 & qA'_1 & A'_1 \end{pmatrix}. \quad (5.1.1)$$

$$\text{Define } P'_1 = \begin{pmatrix} I_3 & -qI_3 & 0 \\ 0 & I_3 & -qI_3 \\ 0 & 0 & I_3 \end{pmatrix}. \quad (5.1.2)$$

$$P'_1 V_{q'} (P'_1)^t = \begin{pmatrix} (1-q^2)A'_1 & 0 & 0 \\ 0 & (1-q^2)A'_1 & 0 \\ 0 & 0 & A'_1 \end{pmatrix}. \quad (5.1.3)$$

$$\text{Define } S' = \begin{pmatrix} 1 & -q & 0 \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } S'_1 = \begin{pmatrix} S' & 0 & 0 \\ 0 & S' & 0 \\ 0 & 0 & S' \end{pmatrix}. \quad (5.1.4)$$

Then, as was shown in the case of the tetrahedron by Equation 4.1.12,

$$S' A'_1 (S')^t = \begin{pmatrix} 1-q^2 & 0 & 0 \\ 0 & 1-q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore  $S'_1 P'_1 V_{q'} (S'_1 P'_1)^t$

$$= \left( \begin{array}{ccc|ccc|ccc} (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-q^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1-q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-q^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) = D'. \quad (5.1.5)$$

$R'$  is the permutation matrix that rearranges the rows and columns so that:

$$1 \rightarrow 9, 2 \rightarrow 8, 3 \rightarrow 5, 4 \rightarrow 7, 5 \rightarrow 6, 6 \rightarrow 4, 7 \rightarrow 3, 8 \rightarrow 2, 9 \rightarrow 1.$$

$$R' = \left( \begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ R' = & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \quad (5.1.6)$$

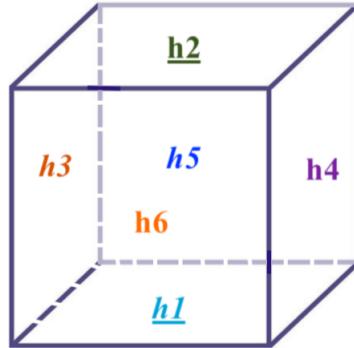
$$\begin{aligned} R'D'(R')^t &= \left( \begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 \end{array} \right) \\ &= \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & (1-q^2)I_4 & 0 \\ 0 & 0 & (1-q^2)^2 I_4 \end{array} \right) = N'. \quad (5.1.7) \end{aligned}$$

### 5.1.2 Cube

Now we consider the cube as an extension of the square. We can take the four hyperplanes of the square arrangement, extend them into planes in  $\mathbb{R}^3$ , then add two additional planes perpendicular to those four to make a cube. Re-labelling the four hyperplanes from the square arrangement as  $\{h_3, h_4, h_5, h_6\}$  and then adding  $\{h_1, h_2\}$  gives us  $\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5, h_6\}$  and the regions defined by

$$\begin{aligned} R_1 &= (- - - + +), & R_2 &= (- - + - + +), & R_3 &= (- - + + + +), \\ R_4 &= (- - - - +), & R_5 &= (- - + - - +), & R_6 &= (- - + + - - +), \\ R_7 &= (- - - - -), & R_8 &= (- - + - - -), & R_9 &= (- - + + - - -), \\ R_{10} &= (+ - - - + +), & R_{11} &= (+ - + - + +), & R_{12} &= (+ - + + + +), \\ R_{13} &= (+ - - - - +) & R_{14} &= (+ - + - - +) & R_{15} &= (+ - + + - - +), \\ R_{16} &= (+ - - - - -), & R_{17} &= (+ - + - - -), & R_{18} &= (+ - + + - - -), \\ R_{19} &= (+ + - - + +), & R_{20} &= (+ + + - + +), & R_{21} &= (+ + + + + +), \\ R_{22} &= (+ + - - - +), & R_{23} &= (+ + + - - - +), & R_{24} &= (+ + + + - - +), \\ R_{25} &= (+ + - - - -), & R_{26} &= (+ + + - - - -), & R_{27} &= (+ + + + - - -). \end{aligned}$$

**Cube Hyperplane Arrangement**



Labels of added hyperplanes  $h_2, h_2$  are underlined. The hyperplanes  $h_2, h_4, h_6$  are visible from this point of view. The hyperplanes are labelled in italics,  $h_1, h_3, h_5$  are not visible from this point of view.

**Figure 5.2** Peelable cube with re-labelled hyperplanes.

Let  $V_{q'}$  be as it was defined in Eq. 5.1.1. The  $q$ -Varchenko matrix for the cube is

$$V_q = \begin{pmatrix} V_{q'} & qV_{q'} & q^2V_{q'} \\ qV_{q'} & V_{q'} & qV_{q'} \\ q^2V_{q'} & qV_{q'} & V_{q'} \end{pmatrix}. \quad (5.1.8)$$

Here  $S'_1, P'_1, D', R'$  and  $N'$  are as they were defined in Eq. 5.1.4, Eq. 5.1.2, Eq. 5.1.5, Eq. 5.1.6 and Eq. 5.1.7.

$$\text{Let } P_1 = \begin{pmatrix} I_9 & -qI_9 & 0 \\ 0 & I_9 & -qI_9 \\ 0 & 0 & I_9 \end{pmatrix}. \text{ Then } P_1 V_q P_1^t = \begin{pmatrix} (1-q^2)V_{q'} & 0 & 0 \\ 0 & (1-q^2)V_{q'} & 0 \\ 0 & 0 & V_{q'} \end{pmatrix}.$$

$$\text{Let } S_1 = \begin{pmatrix} S'_1 P'_1 & 0 & 0 \\ 0 & S'_1 P'_1 & 0 \\ 0 & 0 & S'_1 P'_1 \end{pmatrix}. \text{ Then } S_1 P_1 V_q (S_1 P_1)^t = \begin{pmatrix} (1-q^2)D' & 0 & 0 \\ 0 & (1-q^2)D' & 0 \\ 0 & 0 & D' \end{pmatrix}.$$

$$\text{Let } R_1 = \begin{pmatrix} R' & 0 & 0 \\ 0 & R' & 0 \\ 0 & 0 & R' \end{pmatrix}. \text{ Then } R_1 S_1 P_1 V_q (R_1 S_1 P_1)^t = \begin{pmatrix} (1-q^2)N' & 0 & 0 \\ 0 & (1-q^2)N' & 0 \\ 0 & 0 & N' \end{pmatrix}$$

$$= \left( \begin{array}{ccc|ccc|ccc} 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)^2 I_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^3 I_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2 I_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 I_4 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)I_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 I_4 \end{array} \right).$$

$$\text{Let } P = \left( \begin{array}{cccc|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_4 & 0 \\ 0 & 0 & 0 & 0 & I_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_4 \\ 0 & 0 & 0 & 0 & 0 & I_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

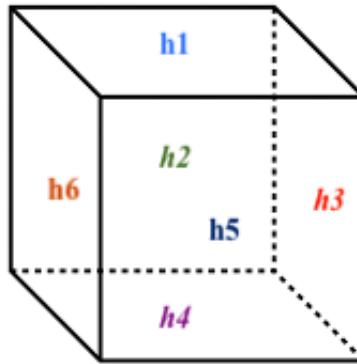
Now left and right multiplication by the permutation matrix  $P$  will arrange  $R_1 S_1 P_1 V_q (R_1 S_1 P_1)^t$  into its Smith

normal form:  $PR_1S_1P_1V_q(PR_1S_1P_1)^t =$

$$\begin{aligned}
& \left( \begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (1-q^2)I_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (1-q^2)^2I_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_4
\end{array} \right) \\
& = \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & (1-q^2)I_6 & 0 & 0 \\
0 & 0 & (1-q^2)^2I_{12} & 0 \\
0 & 0 & 0 & (1-q^2)^3I_8
\end{array} \right).
\end{aligned}$$

## 5.2 Symmetry

Let  $R_0$  denote the central region of the hyperplane arrangement. We number the hyperplanes in  $\mathcal{A}$  by first viewing the cube from a fixed direction so that one hyperplane is between our viewpoint and the central region. This is the front of the cube. Label the hyperplane on the top of the cube  $h_1$ . Then shift the cube forwards 90 degrees so that the back hyperplane becomes the top. Label this one  $h_2$ . Next, shift the cube left 90 degrees and label the hyperplane on top  $h_3$ . This was originally the hyperplane forming the right side of the cube. Next shift the cube forwards 90 degrees and label the hyperplane on top of the cube  $h_4$ . This hyperplane originally formed the bottom of the cube, so  $h_1$  and  $h_4$  are parallel. Shift the cube left 90 degrees and label the top of the cube  $h_5$ . This was the front of the cube, so  $h_2$  and  $h_5$  are parallel. Then shift the cube forwards 90 degrees and label the top of the cube  $h_6$ . This was originally the left side of the cube, so  $h_3$  and  $h_6$  are parallel.



The hyperplanes  $h_1, h_5, h_6$  are visible from this point of view. The hyperplanes labelled in italics,  $h_2, h_3, h_4$  are not visible from this point of view.

**Figure 5.3** Cube with labelled hyperplanes.

$$\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5, h_6\}.$$

The distance enumerator with respect to  $R_0$  is  $D_{\mathcal{A}, R_0}(t) = 1 + 6t + 12t^2 + 8t^3$ .

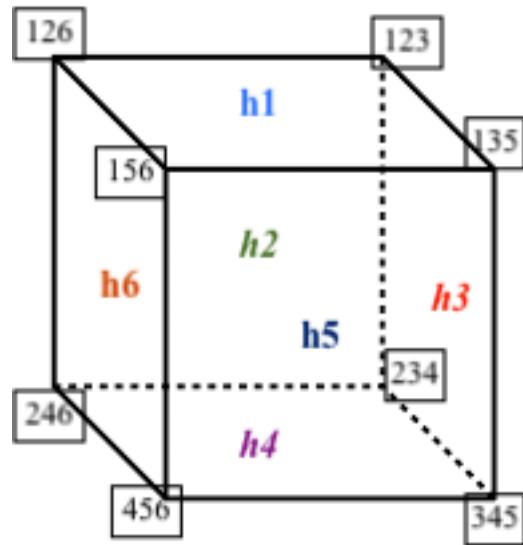
Define each of the 6 regions  $R_i$  such that  $\text{sep}(R_i, R_0) = h_i$  for  $1 \leq i \leq 6$ . Define the remaining regions by their

separating sets as follows:

$$\begin{aligned}
 sep(R_0, R_7) &= \{h_1, h_2\}, & sep(R_0, R_{13}) &= \{h_1, h_3\}, & sep(R_0, R_{19}) &= \{h_1, h_2, h_3\}, \\
 sep(R_0, R_8) &= \{h_2, h_3\}, & sep(R_0, R_{14}) &= \{h_2, h_4\}, & sep(R_0, R_{20}) &= \{h_2, h_3, h_4\}, \\
 sep(R_0, R_9) &= \{h_3, h_4\}, & sep(R_0, R_{15}) &= \{h_3, h_5\}, & sep(R_0, R_{21}) &= \{h_3, h_4, h_5\}, \\
 sep(R_0, R_{10}) &= \{h_4, h_5\}, & sep(R_0, R_{16}) &= \{h_4, h_6\}, & sep(R_0, R_{22}) &= \{h_4, h_5, h_6\}, \\
 sep(R_0, R_{11}) &= \{h_5, h_6\}, & sep(R_0, R_{17}) &= \{h_1, h_5\}, & sep(R_0, R_{23}) &= \{h_1, h_5, h_6\}, \\
 sep(R_0, R_{12}) &= \{h_1, h_6\}, & sep(R_0, R_{18}) &= \{h_2, h_6\}, & sep(R_0, R_{24}) &= \{h_1, h_2, h_6\}, \\
 && && sep(R_0, R_{25}) &= \{h_1, h_3, h_5\}, \\
 && && sep(R_0, R_{26}) &= \{h_2, h_4, h_6\}.
 \end{aligned}$$

The 27 regions are indexed in the following order, and denoted by the hyperplanes in  $sep(R, R_0)$ :

$$0 \mid 1 \ 2 \ 3 \ 4 \ 5 \ 6 \mid 12 \ 23 \ 34 \ 45 \ 56 \ 16 \mid 13 \ 24 \ 35 \ 46 \ 15 \ 26 \mid 123 \ 234 \ 345 \ 456 \ 156 \ 126 \mid 135 \ 246 \quad (5.2.1)$$



**Figure 5.4** Cube with labelled hyperplanes and separating set hyperplane numbers for regions corresponding to vertices.

The  $q$ -Varchenko matrix is  $V_q =$

The left transition matrix is

$$U_6 U_5 U_4 U_3 U_2 U_1$$

and the right transition matrix is

$$(U_6 U_5 U_4 U_3 U_2 U_1)^t.$$

$$U_6 U_5 U_4 U_3 U_2 U_1 =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -q^1 & I_6 & 0 & 0 & 0 & 0 \\ q^2 1 & -q(I_6 + J) & I_6 & 0 & 0 & 0 \\ q^2 1 & -q(I_6 + J^2) & 0 & I_6 & 0 & 0 \\ -q^3 1 & q^2(I_6 + J + J^2) & -q(I_6 + J) & -qI_6 & I_6 & 0 \\ -q^3 1 & q^2(I_2 | I_2 | I_2) & 0 & -q(I_2 | I_2 | I_2) & 0 & I_2 \end{pmatrix},$$

$$\text{where } (I_2 \mid I_2 \mid I_2) = \left( \begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right), \quad (5.2.2)$$

$$\text{and } J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5.2.3)$$

$$U_1 = \left( \begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 \\ \hline 0 & 0 & -qI_6 & 0 & I_6 & 0 \\ \hline 0 & 0 & 0 & -q(I_2 | 0 | 0) & 0 & I_2 \end{array} \right), \text{ where } (I_2 | 0 | 0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$U_2 = \left( \begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & 0 & 0 & 0 \\ \hline 0 & -qI_6 & I_6 & 0 & 0 & 0 \\ \hline 0 & -qI_6 & 0 & I_6 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_6 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_2 \end{array} \right). \quad U_3 = \left( \begin{array}{c|c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -q1 & I_6 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_2 & 0 \end{array} \right).$$

$$U_3 U_2 U_1 V_q = \left( \begin{array}{c|c|c|c|c|c} 1 & Q_1 & Q_2 & Q_2 & Q_3 & Q'_3 \\ \hline 0 & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ \hline 0 & v_{21} & v_{22} & v_{23} & v_{24} & v_{25} \\ \hline 0 & v_{31} & v_{32} & v_{33} & v_{34} & v_{35} \\ \hline 0 & v_{41} & v_{42} & v_{43} & v_{44} & v_{45} \\ \hline 0 & v_{51} & v_{52} & v_{53} & v_{54} & v_{55} \end{array} \right), \text{ where } Q_k = (q^k \ q^k \ q^k \ q^k \ q^k \ q^k) \text{ and } Q'_k = (q^k \ q^k),$$

$$v_{11} = (1 - q^2)I_6, \quad v_{12} = q(1 - q^2)(I_6 + J^5), \quad v_{13} = q(1 - q^2)(I_6 + J^4),$$

$$v_{14} = q^2(1 - q^2)(I_6 + J^4 + J^5), \quad v_{15} = q^2(1 - q^2)(I_2 | I_2 | I_2)^t.$$

$$v_{21} = q(1 - q^2)J, \quad v_{22} = (1 - q^2)(I_6 + q^2J), \quad v_{23} = q(1 - q^2)(J + J^5),$$

$$v_{24} = q(1 - q^2)(I_6 + q^2J + J^5), \quad v_{25} = q^3(1 - q^2) \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

$$v_{31} = q(1 - q^2)J^2, \quad v_{32} = q^2(1 - q^2)(J + J^2), \quad v_{33} = (1 - q^2)(I_6 + q^2J^2),$$

$$v_{34} = q(1 - q^2)(I_6 + q^2J + q^2J^2), \quad v_{35} = q(1 - q^2)(I_2 | I_2 | I_2)^t.$$

$$v_{41} = q^2(1 - q^2)J^2, \quad v_{42} = q(1 - q^2)(J + q^2J^2), \quad v_{43} = q(1 - q^2)(I_6 + q^2J^2),$$

$$v_{44} = (1 - q^2)(I_6 + q^2J + q^4J^2), \quad v_{45} = q^2(1 - q^2)(I_2 | I_2 | I_2)^t.$$

$$v_{51} = q^2(1 - q^2)(0 | 0 | I_2), \quad v_{52} = \begin{pmatrix} 0 & 0 & 0 & q^3(1 - q^2) & q^3(1 - q^2) & 0 \\ 0 & 0 & 0 & 0 & q^3(1 - q^2) & q^3(1 - q^2) \end{pmatrix},$$

$$v_{53} = q(1 - q^2)(0 | I_2 | I_2), \quad v_{54} = \begin{pmatrix} 0 & 0 & q^2(1 - q^2) & q^4(1 - q^2) & q^2(1 - q^2) & 0 \\ 0 & 0 & 0 & q^2(1 - q^2) & q^4(1 - q^2) & q^2(1 - q^2) \end{pmatrix},$$

$v_{55} = (1 - q^2)I_2$ , where  $(0 | 0 | I_2)$ ,  $(0 | I_2 | I_2)$ ,  $(I_2 | I_2 | I_2)$  are defined according to Eq. 5.2.2.

$$U_3 U_2 U_1 V_q (U_3 U_2 U_1)^t = \left( \begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & (1 - q^2)I_6 & q(1 - q^2)J^5 & q(1 - q^2)J^4 & q^2(1 - q^2)J^4 & v_{15} \\ \hline 0 & q(1 - q^2)J & (1 - q^2)I_6 & q^2(1 - q^2)J^5 & q(1 - q^2)J^5 & v_{25} \\ \hline 0 & q(1 - q^2)J^2 & q^2(1 - q^2)J & (1 - q^2)I_6 & q(1 - q^2)I_6 & v_{35} \\ \hline 0 & q^2(1 - q^2)J^2 & q(1 - q^2)J & q(1 - q^2)I_6 & (1 - q^2)I_6 & v_{45} \\ \hline 0 & v_{51} & v_{52} & v_{53} & v_{54} & (1 - q^2)I_2 \end{array} \right),$$

where

$$v_{51} = \begin{pmatrix} 0 & 0 & 0 & 0 & q^2(1 - q^2) & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2(1 - q^2) \end{pmatrix},$$

$$v_{52} = \begin{pmatrix} 0 & 0 & 0 & q^3(1 - q^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & q^3(1 - q^2) & 0 \end{pmatrix},$$

$$v_{53} = \begin{pmatrix} 0 & 0 & q(1-q^2) & 0 & q(1-q^2)^2 & 0 \\ 0 & 0 & 0 & q(1-q^2) & 0 & q(1-q^2)^2 \end{pmatrix},$$

$$v_{54} = \begin{pmatrix} 0 & 0 & q^2(1-q^2) & 0 & q^2(1-q^2)^2 & 0 \\ 0 & 0 & 0 & q^2(1-q^2) & 0 & q^2(1-q^2)^2 \end{pmatrix},$$

and  $v_{15} = v_{51}^t, v_{25} = v_{52}^t, v_{35} = v_{53}^t, v_{45} = v_{54}^t$ .

$$U_4 = \left( \begin{array}{c|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 \\ \hline 0 & 0 & -qJ & 0 & I_6 & 0 \\ \hline 0 & 0 & 0 & -q(0 \mid I_2 \mid 0) & 0 & I_2 \end{array} \right) \text{ where } (0 \mid I_2 \mid 0) = \left( \begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right).$$

$$U_5 = \left( \begin{array}{c|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & -qJ & I_6 & 0 & 0 & 0 \\ \hline 0 & -qJ^2 & 0 & I_6 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_6 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_2 \end{array} \right).$$

$$U_5 U_4 U_3 U_2 U_1 V_q (U_3 U_2 U_1)^t =$$

$$\left( \begin{array}{c|cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_6 & q(1-q^2)J^5 & q(1-q^2)J^4 & q^2(1-q^2)J^4 & q^2(1-q^2)(0 \mid 0 \mid I_2)^t \\ 0 & 0 & (1-q^2)^2I_6 & 0 & q(1-q^2)^2J^5 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2I_6 & q(1-q^2)^2I_6 & q(1-q^2)^2(0 \mid I_2 \mid I_2)^t \\ 0 & 0 & 0 & q(1-q^2)^2I_6 & (1-q^2)^2I_6 & q^2(1-q^2)^2(0 \mid I_2 \mid I_2)^t \\ \hline 0 & 0 & 0 & q(1-q^2)^2(0 \mid 0 \mid I_2) & q^2(1-q^2)^2(0 \mid 0 \mid I_2) & (1-q^2)^2I_2 \end{array} \right).$$

$$U_6 = \left( \begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 \\ \hline 0 & 0 & 0 & -qI_6 & I_6 & 0 \\ \hline 0 & 0 & 0 & -q(0|0|I_2) & 0 & I_2 \end{array} \right), \text{ where } (0|0|I_2) = \left( \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

$$U_6 U_5 U_4 U_3 U_2 U_1 V_q (U_3 U_2 U_1)^t$$

$$= \left( \begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_6 & q(1-q^2)J^5 & q(1-q^2)J^4 & q^2(1-q^2)J^4 & q^2(1-q^2)(0|0|I_2)^t \\ \hline 0 & 0 & (1-q^2)^2I_6 & 0 & q(1-q^2)^2J^5 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2I_6 & q(1-q^2)^2I_6 & q(1-q^2)^2(0|I_2|I_2)^t \\ \hline 0 & 0 & 0 & 0 & (1-q^2)^3I_6 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_2 \end{array} \right).$$

$$U_6 U_5 U_4 U_3 U_2 U_1 V_q (U_6 U_5 U_4 U_3 U_2 U_1)^t$$

$$= \left( \begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & (1-q^2)^3I_6 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_2 \end{array} \right)$$

$$= \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & (1-q^2)I_6 & 0 & 0 \\ 0 & 0 & (1-q^2)^2I_{12} & 0 \\ 0 & 0 & 0 & (1-q^2)^3I_8 \end{array} \right).$$

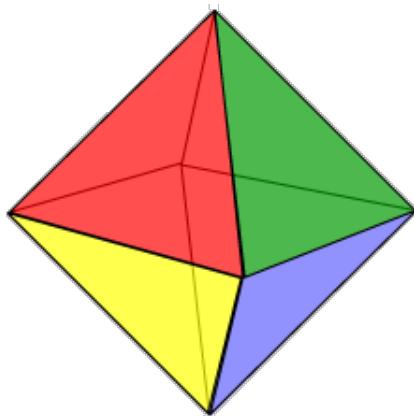
## CHAPTER

# 6

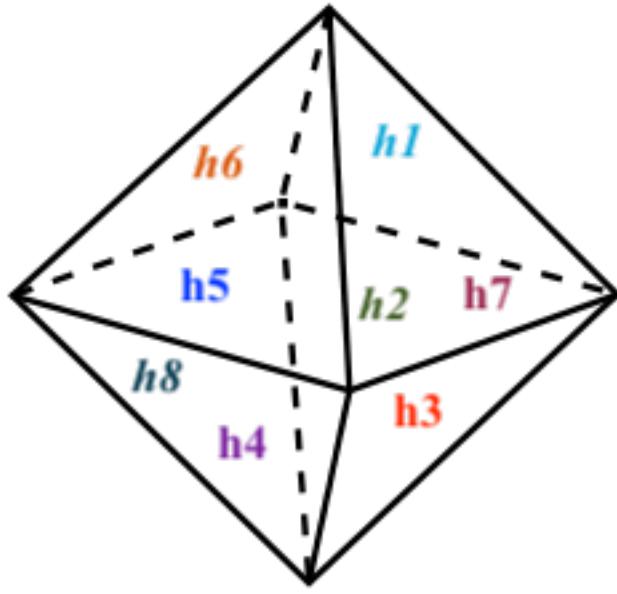
## OCTAHEDRON

$$\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\}.$$

The distance enumerator with respect to  $R_0$  is  $D_{\mathcal{A}, R_0}(t) = 1 + 8t + 12t^2 + 24t^3 + 14t^4$ .



**Figure 6.1** Octahedron.



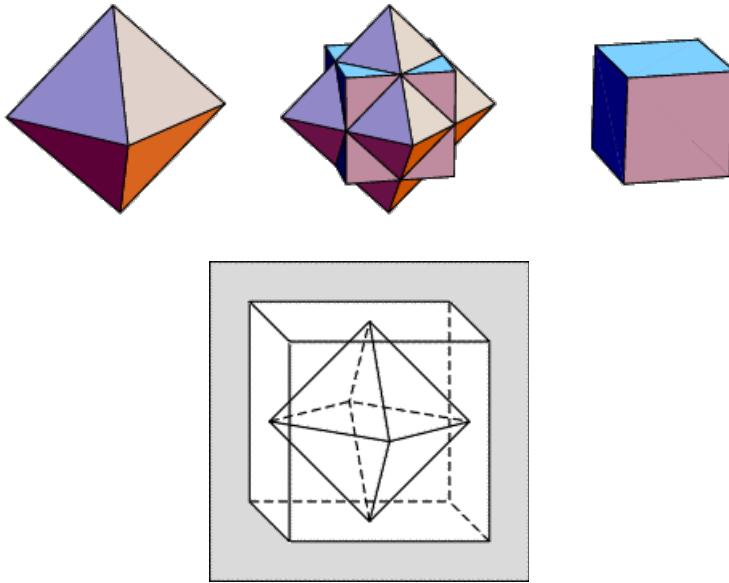
The hyperplanes  $h_3, h_4, h_5, h_7$  are visible from this point of view. Italicized labels on  $h_1, h_2, h_6, h_8$  indicate that the hyperplanes are not visible from this point of view.

**Figure 6.2** Octahedron with labelled hyperplanes.

The hyperplanes  $1 - 8$  are labelled according to the order of the regions corresponding to the vertices (ie, regions  $R$  such that  $\#sep(R_0, R) = 3$ ) in the  $q$ -Varchenko matrix for the cube hyperplane arrangement shown in Fig. 5.4, and the last six regions in the octahedron hyperplane arrangement are ordered according to the labelling of the hyperplanes in the cube arrangement corresponding to the one shown in Fig. 5.3 indexed in the order given in Eq. 5.2.1.

The 59 regions are indexed in the following order, and denoted by the hyperplanes in  $sep(R, R_0)$ :

$$\begin{aligned}
 & 0 \| 1 2 3 4 5 6 | 7 8 \| 16 12 23 34 45 56 | 17 28 37 48 57 68 \| \\
 & 167 128 237 348 457 568 | 157 268 137 248 357 468 | 567 168 127 238 347 458 | 156 126 123 234 345 456 \\
 & \| 1267 1238 2347 3458 4567 1568 | 1357 2468 | 1567 1268 1237 2348 3457 4568
 \end{aligned}$$



**Figure 6.3** Correspondences between indexing of the regions in the cube and octahedron hyperplane arrangements.

The regions  $R$  in the octahedron hyperplane arrangement such that  $\#sep(R_0, R) = 2$  are indexed in the exact order of their counterparts in the cube hyperplane arrangement. For example, the region in the octahedron arrangement whose separating set from  $R_0$  is  $\{h_1, h_6\}$  is listed first because its counterpart in the cube hyperplane arrangement that had its separating set from  $R_0$  as  $\{h_1, h_2\}$  was listed first. All twelve regions  $R$  in the octahedron hyperplane arrangement such that  $\#sep(R_0, R) = 2$  have counterparts in the cube hyperplane arrangement.

The last fourteen regions  $R$  in the octahedron hyperplane arrangement have  $\#sep(R_0, R) = 4$ . The first eight of these lie directly above the hyperplanes  $h_1, \dots, h_8$ , and they are indexed in that order. The remaining six regions correspond to the vertices of the octahedron, and they are indexed in the order corresponding to the hyperplanes in the cube hyperplane arrangement.

The remaining 24 regions  $R$  in the octahedron hyperplane arrangement that have  $\#sep(R_0, R) = 3$  have no corresponding regions in the cube hyperplane arrangement. They are indexed instead into four blocks by choosing one of the four regions adjacent to each of the six regions that correspond to the octahedron's vertices, in the same order that those regions are listed in the last block of the  $q$ -Varchenko matrix.

In the first of these four blocks, the labelled hyperplane omitted goes in the order 561234; in the second, it goes in the order 612345; in the third, it goes in the order 123456; in the fourth, it goes in the order 787878.

$$V_q = \left( \begin{array}{|c|cc|cc|cccc|cc|} \hline & Q_1 & Q'_1 & Q_2 & Q_2 & Q_3 & Q_3 & Q_3 & Q_3 & Q_4 & Q'_4 & Q_4 \\ \hline Q_1^t & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} & v_{17} & v_{18} & v_{19} & v_{1,10} & v_{1,11} \\ Q_1'' & v_{21} & v_{22} & v_{23} & v_{24} & v_{25} & v_{26} & v_{27} & v_{28} & v_{29} & v_{2,10} & v_{2,11} \\ \hline Q_2^t & v_{31} & v_{32} & v_{33} & v_{34} & v_{35} & v_{36} & v_{37} & v_{38} & v_{39} & v_{3,10} & v_{3,11} \\ Q_2' & v_{41} & v_{42} & v_{43} & v_{44} & v_{45} & v_{46} & v_{47} & v_{48} & v_{49} & v_{4,10} & v_{4,11} \\ \hline Q_3^t & v_{51} & v_{52} & v_{53} & v_{54} & v_{55} & v_{56} & v_{57} & v_{58} & v_{59} & v_{5,10} & v_{5,11} \\ Q_3' & v_{61} & v_{62} & v_{63} & v_{64} & v_{65} & v_{66} & v_{67} & v_{68} & v_{69} & v_{6,10} & v_{6,11} \\ Q_3'' & v_{71} & v_{72} & v_{73} & v_{74} & v_{75} & v_{76} & v_{77} & v_{78} & v_{79} & v_{7,10} & v_{7,11} \\ Q_3''' & v_{81} & v_{82} & v_{83} & v_{84} & v_{85} & v_{86} & v_{87} & v_{88} & v_{89} & v_{8,10} & v_{8,11} \\ \hline Q_4^t & v_{91} & v_{92} & v_{93} & v_{94} & v_{95} & v_{96} & v_{97} & v_{98} & v_{99} & v_{9,10} & v_{9,11} \\ Q_4'' & v_{10,1} & v_{10,2} & v_{10,3} & v_{10,4} & v_{10,5} & v_{10,6} & v_{10,7} & v_{10,8} & v_{10,9} & v_{10,10} & v_{10,11} \\ Q_4''' & v_{11,1} & v_{11,2} & v_{11,3} & v_{11,4} & v_{11,5} & v_{11,6} & v_{11,7} & v_{11,8} & v_{11,9} & v_{11,10} & v_{11,11} \\ \hline \end{array} \right).$$

$Q_k$  is a  $1 \times 6$  row vector and  $Q'_k$  is a  $1 \times 2$  row vector.  $Q_k = \begin{pmatrix} q^k & q^k & q^k & q^k & q^k & q^k \end{pmatrix}$ , and  $Q'_k = \begin{pmatrix} q^k & q^k \end{pmatrix}$ .

For all  $v_{ij}, v_{ji} = v_{ij}^t$ . Each  $v_{ij}$  is a circulant matrix.

$$v_{11} = C(1, q^2, q^2, q^2, q^2, q^2)_{6 \times 6}, \quad v_{12} = C(q^2, q^2)_{6 \times 2}, \quad v_{13} = C(q, q, q^3, q^3, q^3, q^3)_{6 \times 6},$$

$$v_{14} = C(q, q^3, q^3, q^3, q^3, q^3)_{6 \times 6}, \quad v_{15} = C(q^2, q^2, q^4, q^4, q^4, q^4)_{6 \times 6}, \quad v_{16} = C(q^2, q^4, q^2, q^4, q^4, q^4)_{6 \times 6},$$

$$v_{17} = C(q^4, q^2, q^2, q^4, q^4, q^4)_{6 \times 6}, \quad v_{18} = C(q^2, q^2, q^2, q^4, q^4, q^4)_{6 \times 6}, \quad v_{19} = C(q^3, q^3, q^5, q^5, q^5, q^3)_{6 \times 6},$$

$$v_{1,10} = C(q^3, q^5)_{6 \times 2}, \quad v_{1,11} = C(q^3, q^3, q^3, q^5, q^5, q^5)_{6 \times 6}, \quad \dots$$

$$v_{22} = C(1, q^2)_{2 \times 2}, \quad v_{23} = C(q^3, q^3, q^3, q^3, q^3, q^3)_{2 \times 6}, \quad v_{24} = C(q, q^3, q, q^3, q, q^3)_{2 \times 6},$$

$$v_{25} = C(q^2, q^4, q^2, q^4, q^2, q^4)_{2 \times 6}, \quad v_{26} = C(q^2, q^4, q^2, q^4, q^2, q^4)_{2 \times 6}, \quad v_{27} = C(q^2, q^4, q^2, q^4, q^2, q^4)_{2 \times 6},$$

$$v_{28} = C(q^4, q^4, q^4, q^4, q^4, q^4)_{2 \times 6}, \quad v_{29} = C(q^3, q^5, q^3, q^5, q^3, q^5)_{2 \times 6}, \quad v_{2,10} = C(q^3, q^5)_{2 \times 2},$$

$$v_{2,11} = C(q^3, q^5, q^3, q^5, q^3, q^5)_{2 \times 6},$$

$$v_{33} = C(1, q^2, q^4, q^4, q^4, q^2)_{6 \times 6}, \quad v_{34} = C(q^2, q^4, q^4, q^4, q^4, q^2)_{6 \times 6}, \quad v_{35} = C(q, q^3, q^5, q^5, q^5, q^3)_{6 \times 6},$$

$$v_{36} = C(q^3, q^3, q^3, q^5, q^5, q^3)_{6 \times 6}, \quad v_{37} = C(q^3, q, q^3, q^5, q^5, q^5)_{6 \times 6}, \quad v_{38} = C(q, q, q^3, q^5, q^5, q^3)_{6 \times 6},$$

$$v_{39} = C(q^2, q^4, q^6, q^6, q^4, q^2)_{6 \times 6}, \quad v_{3,10} = C(q^4, q^4)_{6 \times 2}, \quad v_{3,11} = C(q^2, q^2, q^4, q^6, q^6, q^4)_{6 \times 6},$$

$$\begin{aligned}
v_{44} &= C(1, q^4, q^2, q^4, q^2, q^4)_{6 \times 6}, & v_{45} &= C(q, q^3, q^3, q^5, q^3, q^5)_{6 \times 6}, & v_{46} &= C(q, q^5, q, q^5, q^3, q^5)_{6 \times 6}, \\
v_{47} &= C(q^3, q^3, q, q^5, q^3, q^5)_{6 \times 6}, & v_{48} &= C(q^3, q^3, q^3, q^5, q^5, q^5)_{6 \times 6}, & v_{49} &= C(q^2, q^4, q^4, q^6, q^4, q^4)_{6 \times 6}, \\
v_{4,10} &= C(q^2, q^6)_{6 \times 2}, & v_{4,11} &= C(q^2, q^4, q^2, q^6, q^4, q^6)_{6 \times 6}, \\
v_{55} &= C(1, q^4, q^4, q^6, q^4, q^4)_{6 \times 6}, & v_{56} &= C(q^2, q^4, q^2, q^6, q^4, q^4)_{6 \times 6}, & v_{57} &= C(q^2, q^2, q^2, q^6, q^4, q^6)_{6 \times 6}, \\
v_{58} &= C(q^2, q^2, q^4, q^6, q^6, q^4)_{6 \times 6}, & v_{59} &= C(q, q^5, q^5, q^7, q^3, q^3)_{6 \times 6}, & v_{5,10} &= C(q^3, q^5)_{6 \times 2}, \\
v_{5,11} &= C(q, q^3, q^3, q^7, q^5, q^5)_{6 \times 6}, \\
v_{66} &= C(1, q^6, q^2, q^6, q^2, q^6)_{6 \times 6}, & v_{67} &= C(q^2, q^4, q^2, q^6, q^4, q^4)_{6 \times 6}, & v_{68} &= C(q^2, q^4, q^4, q^6, q^4, q^4)_{6 \times 6}, \\
v_{69} &= C(q^3, q^5, q^5, q^5, q^3, q^3)_{6 \times 6}, & v_{6,10} &= C(q, q^7)_{6 \times 2}, & v_{6,11} &= C(q, q^5, q^3, q^7, q^3, q^5)_{6 \times 6}, \\
v_{77} &= C(1, q^4, q^4, q^6, q^4, q^4)_{6 \times 6}, & v_{78} &= C(q^2, q^4, q^6, q^6, q^4, q^2)_{6 \times 6}, & v_{79} &= C(q^3, q^7, q^5, q^5, q, q^3)_{6 \times 6}, \\
v_{7,10} &= C(q^3, q^5)_{6 \times 2}, & v_{7,11} &= C(q, q^5, q^5, q^7, q^3, q^3)_{6 \times 6}, \\
v_{88} &= C(1, q^2, q^4, q^6, q^4, q^2)_{6 \times 6}, & v_{89} &= C(q^3, q^5, q^7, q^5, q^3, q)_{6 \times 6}, & v_{8,10} &= C(q^3, q^5)_{6 \times 2}, \\
v_{8,11} &= C(q, q^3, q^5, q^7, q^5, q^3)_{6 \times 6}, \\
v_{99} &= C(1, q^4, q^4, q^8, q^4, q^4)_{6 \times 6}, & v_{9,10} &= C(q^4, q^4)_{6 \times 2}, & v_{9,11} &= C(q^2, q^2, q^2, q^6, q^6, q^6)_{6 \times 6}, \\
v_{10,10} &= C(1, q^2)_{2 \times 2}, & v_{10,11} &= C(q^2, q^6, q^2, q^6, q^2, q^6)_{2 \times 6}, \\
v_{11,11} &= C(1, q^4, q^4, q^8, q^4, q^4)_{6 \times 6}.
\end{aligned}$$

Throughout the rest of this chapter,

$$J = \left( \begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right)_{6 \times 6}, \quad J^t = J^5, \quad \text{and} \quad \left( \begin{array}{c|c|c} I_2 & I_2 & I_2 \end{array} \right) = \left( \begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right)_{2 \times 6}.$$

The first part of the left multiplication transition matrix  $U$  which has  $U^t$  as a corresponding right transition matrix is

$$U = U_{10}U_9U_8U_7U_6U_5U_4U_3U_2U_1. \quad (6.0.1)$$

Let the numbers of the labelled hyperplanes in  $\text{sep}(R_0, R)$  denote the rows of the block matrix in  $V_q$  which begins with that region indexed there. For example, 167 indicates the 5<sup>th</sup> block row of  $V_q$ ; similarly, 1567 indicates the 11<sup>th</sup> block row of  $V_q$ . Then the steps taken by multiplying  $V_q$  on the left by the matrices  $U_{10}U_9U_8U_7U_6U_5U_4U_3U_2U_1$  can also be described by the set of operations listed below each of these matrices:

$$U_1 = \left( \begin{array}{c|ccc|cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -qJ^2 & 0 & 0 & I_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q(I_2|0|0) & 0 & 0 & 0 & I_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -qI & 0 & 0 & 0 & 0 \\ \end{array} \right) .$$

$$\left\{ \begin{array}{l} 1567 - qI(157), \\ 1267 - qJ^2(567), \\ 1357 - q(I_2 | 0 | 0)(157). \end{array} \right\} \quad (6.0.2)$$

$$U_2 = \left( \begin{array}{c|ccc|ccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -qI & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -qJ^4 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q(J^5 - J^4) & -qJ^4 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -qI & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 \\ \end{array} \right) .$$

$$\left\{ \begin{array}{l} 167 - qI(17), \\ 157 - qJ^4(17), \\ 567 - qJ^4(17) \text{ and } 567 - q(J^t - J^4)(16), \\ 156 - qI(16). \end{array} \right\} \quad (6.0.3)$$

$$U_3 = \left( \begin{array}{c|ccc|ccccc|ccc} 1 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -qI & 0 & & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q(I_2 | I_2 | I_2)^t & & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ \hline 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 \\ 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 \\ 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 \end{array} \right) .$$

$$\left\{ \begin{array}{l} 16 - qI(1), \\ 17 - q(I_2 | I_2 | I_2)^t(7). \end{array} \right\} \quad (6.0.4)$$

$$U_4 = \left( \begin{array}{c|ccc|ccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -qI & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q(0|I_2|0) & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -qI & 0 & 0 & 0 & 0 & 0 & I_6 \end{array} \right) .$$

$$\left\{ \begin{array}{l} 1567 - qI(167), \\ 1267 - qI(167), \\ 1357 - q(0 | I_2 | 0)(157). \end{array} \right\} \quad (6.0.5)$$

$$U_5 = \left( \begin{array}{c|ccc|ccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -qI & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -qI & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -qJ^4 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -qJ^5 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 \end{array} \right) .$$

(6.0.6)

$$U_6 = \left( \begin{array}{c|ccc|ccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -q1 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q1 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 \end{array} \right) .$$

(6.0.7)

$$U_7 = \left( \begin{array}{c|ccc|ccccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & -qJ^5 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -qI & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6
\end{array} \right) .$$

$\left\{ \begin{array}{l} 16 - qJ^t(1), \\ 17 - qI(1). \end{array} \right\}$  (6.0.8)

$$U_8 = \left( \begin{array}{c|ccc|ccccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -qJ & 0 & 0 & I_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -q(0|0|I_2) & 0 & 0 & 0 & I_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6
\end{array} \right) .$$

$\left\{ \begin{array}{l} 1267 - qJ(156), \\ 1357 - q(0|0|I_2)(157). \end{array} \right\}$  (6.0.9)

$$U_9 = \left( \begin{array}{c|ccc|ccccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\
0 & 0 & 0 & -q^2 J^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6
\end{array} \right) \cdot \{1567 - q^2 J^t(16).\} \quad (6.0.10)$$

$$\begin{array}{c}
U_9 U_8 U_7 U_6 U_5 U_4 U_3 U_2 U_1 \\
\\
= \left( \begin{array}{|ccc|ccc|ccc|} \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -q1 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q1 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline q^2 1 & -q(I+J^5) & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ q^2 1 & -qI & -q(I_2|I_2|I_2)^t & 0 & I_6 & 0 & 0 & 0 & 0 \\ \hline 0 & q^2 I & 0 & -qI & -qI & I_6 & 0 & 0 & 0 \\ 0 & 0 & q^2(I_2|I_2|I_2)^t & 0 & -q(I+J^4) & 0 & I_6 & 0 & 0 \\ 0 & q^2 J^4 & 0 & -qJ^5 & -qJ^4 & 0 & 0 & I_6 & 0 \\ 0 & q^2 J^5 & 0 & -q(I+J^5) & 0 & 0 & 0 & I_6 & 0 \\ \hline 0 & -q^3 I & 0 & q^2(I+J) & q^2 I & -qI & 0 & -qJ^2 & -qJ \\ 0 & 0 & -q^3 I_2 & 0 & q^2(I_2|I_2|I_2) & 0 & -q(I_2|I_2|I_2) & 0 & 0 \\ -q^4 1 & q^3(J^4+J^5) & 0 & -q^2 J^5 & q^2 I & -qI & -qI & 0 & 0 \\ \hline \end{array} \right).
\end{array}$$

After multiplying  $V_q$  on the left by  $U_9U_8U_7U_6U_5U_4U_3U_2U_1$  and on the right by the transpose, the  $q$ -Varchenko matrix is transformed into

$$U_9U_8U_7U_6U_5U_4U_3U_2U_1V_q(U_9U_8U_7U_6U_5U_4U_3U_2U_1)^t =$$

$$\left( \begin{array}{ccc|ccc|cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & q^2(1-q^2)^2I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & q^2(1-q^2)^2I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2(1-q^2)^2I_6 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & q(1-q^2)^3I_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^2(1-q^2)^2I_6 & 0 & (1-q^2)^2I_6 & 0 & 0 & q(1-q^2)^3I_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q(1-q^2)^3I_6 & q(1-q^2)^3I_6 & 0 & 0 & 0 & (1+q^2)(1-q^2)^3I_6 \end{array} \right). \quad (6.0.11)$$

$$U_{10} = \left( \begin{array}{c|ccc|cccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6
\end{array} \right) .$$

$$\left\{ \begin{array}{l} 567 - q^2 I(167), \\ 156 - q^2 I(157). \end{array} \right\} \quad (6.0.12)$$

$$U_{10} U_9 U_8 U_7 U_6 U_5 U_4 U_3 U_2 U_1 =$$

$$\left( \begin{array}{c|cc|cc|ccccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q1 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q1 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
q^21 & -q(I+J^5) & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^21 & -qI & -q(I_2|I_2|I_2)^t & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^2I & 0 & -qI & -qI & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^2(I_2|I_2|I_2)^t & 0 & -q(I+J^4) & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q^4I + q^2J^4 & 0 & q^3I - qJ^5 & q^3I - qJ^4 & -q^2I & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\
0 & q^2J^5 & -q^4(I_2|I_2|I_2)^t & -q(I+J^5) & q^3(I+J^4) & 0 & -q^2I & 0 & I_6 & 0 & 0 & 0 & 0 \\
\hline
0 & -q^3I & 0 & q^2(I+J) & q^2I & -qI & 0 & -qJ^2 & -qJ & I_6 & 0 & 0 \\
0 & 0 & -q^3I_2 & 0 & q^2(I_2|I_2|I_2) & 0 & -q(I_2|I_2|I_2) & 0 & 0 & 0 & I_2 & 0 \\
-q^41 & q^3(J^4 + J^5) & 0 & -q^2J^5 & q^2I & -qI & -qI & 0 & 0 & 0 & 0 & I_6
\end{array} \right) .$$

After multiplying  $V_q$  on the left by  $U_{10}U_9U_8U_7U_6U_5U_4U_3U_2U_1$  and on the right by the transpose, the  $q$ -Varchenko matrix is transformed into

$$U_{10}U_9U_8U_7U_6U_5U_4U_3U_2U_1V_q(U_{10}U_9U_8U_7U_6U_5U_4U_3U_2U_1)^t =$$

$$\left( \begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1+q^2)(1-q^2)^3I_6 & 0 & 0 & 0 & q(1-q^2)^3I_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1+q^2)(1-q^2)^3I_6 & 0 & 0 & q(1-q^2)^3I_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q(1-q^2)^3I_6 & q(1-q^2)^3I_6 & 0 & 0 & 0 & (1+q^2)(1-q^2)^3I_6 \end{array} \right) \quad (6.0.13)$$

Now the problem of transforming  $V_q$  into Smith Normal form is reduced to transforming the following  $3 \times 3$  matrix  $M$  into Smith Normal form.

$$M = \begin{pmatrix} (1+q^2)(1-q^2)^3 & 0 & q(1-q^2)^3 \\ 0 & (1+q^2)(1-q^2)^3 & q(1-q^2)^3 \\ q(1-q^2)^3 & q(1-q^2)^3 & (1+q^2)(1-q^2)^3 \end{pmatrix}. \quad (6.0.14)$$

Left multiplication by

$$L' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & q^5 & 1 \end{pmatrix} \begin{pmatrix} 1 & q^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -q & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & q^2 & -q \\ 0 & 1 & 0 \\ -q & q^5 & (1+q^2) \end{pmatrix} \quad (6.0.15)$$

and right multiplication by

$$R' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -q & 1 \end{pmatrix} \begin{pmatrix} 1 & -q^4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -q \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -q^4 & q^5 \\ 0 & 1 & -q \\ 0 & -q & (1+q^2) \end{pmatrix} \quad (6.0.16)$$

transform  $M$  into its Smith Normal form:

$$\begin{pmatrix} (1-q^2)^3 & 0 & 0 \\ 0 & (1-q^2)^3 & 0 \\ 0 & 0 & (1-q^2)^3(1+q^2)(1+q^4) \end{pmatrix}. \quad (6.0.17)$$

Therefore, the last part of the left transformation matrix for  $V_q$  is

$$L = \left( \begin{array}{c|ccc|ccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & q^2I_6 & 0 & 0 & -qI_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -qI_6 & q^5I_6 & 0 & 0 & 0 & (1+q^2)I_6 \end{array} \right)$$

and the last part of the right transformation matrix for  $V_q$  is

$$R = \left( \begin{array}{c|ccc|cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^4I_6 & 0 & 0 & q^5I_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 & -qI_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -qI_6 & 0 & 0 & (1+q^2)I_6 \end{array} \right).$$

Then, using Equation 6.0.1 we have that the Smith normal form of  $V_q$  is

$$(LU)V_q(U^tR) =$$

$$\left( \begin{array}{c|ccc|cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)I_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)I_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2(1-q^8)I_6 \end{array} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)I_8 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2I_{24} & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^3I_{20} & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2(1-q^8)I_6 \end{pmatrix}.$$

## CHAPTER

# 7

## PYRAMIDS

### 7.1 Square Base (n=4)

Let  $\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5\}$  be a hyperplane arrangement in  $\mathbb{R}^3$  of a pyramid with a square base, where  $h_5$  is the hyperplane that forms the base. Then the  $q$ -Varchenko matrix for  $\mathcal{A}$  has a Smith normal form over  $\mathbb{Z}[q]$ .

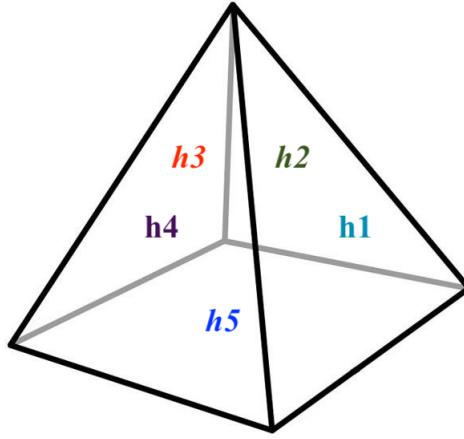
Here the regions are indexed in the following order by the hyperplanes in their separating sets from  $R_0$ :

5	15	25	35	45	125	235	345	145	1235	2345	1345	1245	12345
0	1	2	3	4	12	23	34	41	123	234	341	412	1234

Since  $h_5$  is peelable from  $\mathcal{A}$ , by [CCM16] there exists a matrix  $P$  with entries in  $\mathbb{Z}[q]$  such that  $\det(P) = 1$  and

$$PV_q(\mathcal{A})P^t = \begin{pmatrix} (1-q^2)V_q(\mathcal{A}^{h_5}) & 0 \\ 0 & V_q(\mathcal{A} - \{h_5\}) \end{pmatrix}. \quad (7.1.1)$$

$$\text{In this case, } P = \begin{pmatrix} I_9 & -qI_9 & 0 \\ 0 & I_9 & 0 \\ 0 & 0 & I_5 \end{pmatrix}.$$



The hyperplanes  $h_1, h_4$  are visible from this point of view. The hyperplanes labelled in italics,  $h_2, h_3, h_5$  are not visible from this point of view.

**Figure 7.1** Square base pyramid with labelled hyperplanes.

By Eq. 2.0.1,  $V_q(\mathcal{A}^{h_5}) = V_q$  for the  $C_4$  hyperplane arrangement.

$$V_q(\mathcal{A} - \{h_5\}) = \left( \begin{array}{c|cccc|cccc|cccc|c} 1 & q & q & q & q & q^2 & q^2 & q^2 & q^2 & q^3 & q^3 & q^3 & q^3 & q^4 \\ \hline q & 1 & q^2 & q^2 & q^2 & q & q^3 & q^3 & q & q^2 & q^4 & q^2 & q^2 & q^3 \\ q & q^2 & 1 & q^2 & q^2 & q & q & q^3 & q^3 & q^2 & q^2 & q^4 & q^2 & q^3 \\ q & q^2 & q^2 & 1 & q^2 & q^3 & q & q & q^3 & q^2 & q^2 & q^2 & q^4 & q^3 \\ q & q^2 & q^2 & q^2 & 1 & q^3 & q^3 & q & q & q^4 & q^2 & q^2 & q^2 & q^3 \\ \hline q^2 & q & q & q^3 & q^3 & 1 & q^2 & q^4 & q^2 & q & q^3 & q^3 & q & q^2 \\ q^2 & q^3 & q & q & q^3 & q^2 & 1 & q^2 & q^4 & q & q & q^3 & q^3 & q^2 \\ q^2 & q^3 & q^3 & q & q & q^4 & q^2 & 1 & q^2 & q^3 & q & q & q^3 & q^2 \\ q^2 & q & q^3 & q^3 & q & q^2 & q^4 & q^2 & 1 & q^3 & q^3 & q & q & q^2 \\ \hline q^3 & q^2 & q^2 & q^2 & q^4 & q & q & q^3 & q^3 & 1 & q^2 & q^2 & q^2 & q \\ q^3 & q^4 & q^2 & q^2 & q^2 & q^3 & q & q & q^3 & q^2 & 1 & q^2 & q^2 & q \\ q^3 & q^2 & q^4 & q^2 & q^2 & q^3 & q^3 & q & q & q^2 & q^2 & 1 & q^2 & q \\ q^3 & q^2 & q^2 & q^4 & q^2 & q & q^3 & q^3 & q & q^2 & q^2 & q^2 & 1 & q \\ \hline q^4 & q^3 & q^3 & q^3 & q^3 & q^2 & q^2 & q^2 & q^2 & q & q & q & q & 1 \end{array} \right).$$

Here the regions are indexed in the following order by the hyperplanes in their separating sets from  $R_0$ :

$$0 \mid 1 \ 2 \ 3 \ 4 \mid 12 \ 23 \ 34 \ 41 \mid 123 \ 234 \ 341 \ 412 \mid 1234$$

The first part of the left transition matrix for  $V_q(\mathcal{A} - \{h_5\})$  is  $U_2U_1$  and the first part of the right transition matrix for  $V_q(\mathcal{A} - \{h_5\})$  is  $(U_2U_1)^t$ , where

$$U_2U_1 = \left( \begin{array}{c|cccc|cccc|cccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -q & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline q^2 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^2 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^2 & -q & 0 & 0 & -q & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & q^2 & 0 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^2 & 0 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -q^4 & 0 & q^2 & q^3 & q^3 & -q & -q & -q^2 & 0 & 1 & 0 & 0 \\ 0 & q^2 & 0 & -q^4 & 0 & -q & q^3 & q^3 & -q & 0 & -q^2 & 0 & 1 & 0 \\ \hline -q^4 & q^3 & 0 & 0 & q^3 & 0 & q^2 & 0 & -q^2 & -q & -q & 0 & 0 & 1 \end{array} \right) .$$

$$U_1 = \left( \begin{array}{c|cccc|cccc|cccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -q & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline q^2 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^2 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^2 & -q & 0 & 0 & -q & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & q^2 & 0 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q^2 & 0 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q^2 & 0 & 0 & -q & -q & 0 & 0 & 1 & 0 & 0 \\ 0 & q^2 & 0 & 0 & 0 & -q & 0 & 0 & -q & 0 & 0 & 0 & 1 & 0 \\ \hline -q^4 & q^3 & 0 & 0 & q^3 & 0 & q^2 & 0 & -q^2 & -q & -q & 0 & 0 & 1 \end{array} \right) , U_2 = \left( \begin{array}{c|ccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q^2 I_2 & I_2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) .$$

Let the numbers of the labelled hyperplanes in  $\text{sep}(R_0, R)$  denote the rows of the block matrix in  $V_q$  which begins with that region indexed there. Then the steps taken by multiplying  $V_q$  on the left by the matrices  $U_2U_1$  can also be described as follows:

1.

$$1234 - q(123). \quad (7.1.2)$$

Note: here (123) means only the 123 row, and not the entire block matrix.

2.

$$123 - qI(12). \quad (7.1.3)$$

3.

$$12 - qI(1). \quad (7.1.4)$$

4.

$$1 - q(0). \quad (7.1.5)$$

Note: here (0) means the  $R_0$  row.

5.

$$1234 - q(234). \quad (7.1.6)$$

Note: here (234) means only the 234 row, and not the entire block matrix.

6.

$$123 - qJ(12). \quad (7.1.7)$$

7.

$$12 - qJ(1). \quad (7.1.8)$$

8.

$$1234 - q^2(14). \quad (7.1.9)$$

Note: here (14) means only the 14 row, and not an entire block matrix; ie, this is the last row of the 12 region block.

9.

$$(341 \ 412) - q^2I(123 \ 234). \quad (7.1.10)$$

Note: here (341 412) denotes the  $2 \times 2$  lower right corner block of 123; (123 234) denotes the  $2 \times 2$  upper left corner block of 123.

Here, Eq. 7.1.2 corresponds to the first part of Eq. 6.0.2 and Eq. 7.1.6 corresponds to the first part of Eq. 6.0.5; the correspondences can be seen by viewing the 1234 region of the pyramid arrangement as the 1567 region of the  $q$ -Varchenko matrix for the octahedron arrangement. Then the 123 and 234 regions of the pyramid arrangement correspond to the regions 157 and 167, respectively, in the octahedron arrangement. Eq. 7.1.9 corresponds to Eq. 6.0.10.  $U_2$  is the last step given in Eq. 7.1.10, and it corresponds to Eq. 6.0.12, which gave the step  $U_{10}$  in Eq. 6.0.1 for the octahedron hyperplane arrangement.

$$U_1 V_q (\mathcal{A} - \{h_5\}) U_1^t =$$

$$\left( \begin{array}{|cccccc|cccccc|c|} \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & q^2(1-q^2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & q^2(1-q^2)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2(1-q^2)^2 & 0 & (1-q^2)^2 & 0 & q(1-q^2)^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2(1-q^2)^2 & 0 & (1-q^2)^2 & q(1-q^2)^3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q(1-q^2)^3 & q(1-q^2)^3 & (1-q^2)^3(1+q^2) \\ \end{array} \right).$$

Note the similarity of this matrix to Equation 6.0.11

$$U_2 U_1 V_q (\mathcal{A} - \{h_5\}) (U_1 U_2)^t =$$

$$\left( \begin{array}{|cccccc|cccccc|c|} \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3(1+q^2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q(1-q^2)^3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q(1-q^2)^3 & q(1-q^2)^3 & (1-q^2)^3(1+q^2) \\ \end{array} \right).$$

Note the similarity of this matrix to Equation 6.0.13

Recalling Equations 6.0.14 and 6.0.17, and the definitions of  $L'$  given in Eq. 6.0.15 and  $R'$  given in Eq. 6.0.16, we see that the last part of the left and right transformation matrices for  $V_q(\mathcal{A} - \{h_5\})$  are

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & I_4 & 0 & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & L' \end{pmatrix}, \text{ and } R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & I_4 & 0 & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & R' \end{pmatrix}$$

$$\begin{aligned} & L(U_2U_1)V_q(\mathcal{A} - \{h_5\})(U_2U_1)^tR \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)I_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2I_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^2I_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^3I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^2(1-q^8) \end{pmatrix}. \end{aligned} \quad (7.1.11)$$

To tie everything together, let  $T'_{9 \times 9}$  be the left transformation matrix for  $V_q(\mathcal{A}^{h_5}) = V_q$  for  $C_4$  as it was defined in Equation 3.1.10.

$$\text{Let } T = \begin{pmatrix} T' & 0 \\ 0 & I_{14} \end{pmatrix}, \text{ let } U = \begin{pmatrix} I_9 & 0 \\ 0 & U_2U_1 \end{pmatrix}, L_1 = \begin{pmatrix} I_9 & 0 \\ 0 & L \end{pmatrix}, \text{ and let } R_1 = \begin{pmatrix} I_9 & 0 \\ 0 & R \end{pmatrix}.$$

Then, by Equations 3.1.9 and 7.1.11,  $L_1(UTP)V_q(\mathcal{A})(UTP)^tR_1$

$$= \left( \begin{array}{ccc|ccccc} (1-q^2) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)^2I_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^3I_4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)I_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^2I_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3I_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2(1-q^8) \end{array} \right).$$

Then left and right multiplication by the permutation matrix

$$S = \left( \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_4 & 0 & 0 & 0 \\ \hline 0 & I_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_6 & 0 & 0 \\ \hline 0 & 0 & I_4 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & I_2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array} \right).$$

arranges the diagonal matrix  $L_1(UTP)V_q(\mathcal{A})(UTP)^tR_1$  into its Smith normal form:

$$SL_1(UTP)V_q(\mathcal{A})(UTP)^tR_1S^t = \left( \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)I_5 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2I_{10} & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^3I_6 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2(1-q^8) \end{array} \right).$$

## 7.2 Pentagonal Base (n=5)

Let  $\mathcal{A} = \{h_1, h_2, h_3, h_4, h_5, h_6\}$  be a hyperplane arrangement in  $\mathbb{R}^3$  of a pyramid with a regular pentagonal base, where  $h_6$  is the hyperplane that forms the base. Then the  $q$ -Varchenko matrix for  $\mathcal{A}$  has a Smith normal form over  $\mathbb{Z}[q]$ .

Since  $h_6$  is peelable from  $\mathcal{A}$ , by [CCM16] there exists a matrix  $P$  with entries in  $\mathbb{Z}[q]$  such that  $\det(P) = 1$  and

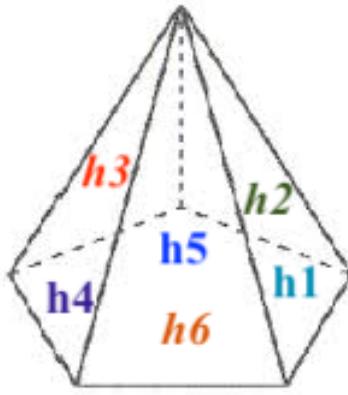
$$PV_q(\mathcal{A})P^t = \begin{pmatrix} (1-q^2)V_q(\mathcal{A}^{h_6}) & 0 \\ 0 & V_q(\mathcal{A} - \{h_6\}) \end{pmatrix}. \quad (7.2.1)$$

Here the regions are indexed in the following order by the hyperplanes in their separating sets from  $R_0$ :

6	16	26	36	46	56	126	236	346	456	156
0	1	2	3	4	5	12	23	34	45	15
<hr/>										
1236	2346	3456	1456	1256	12346	23456	13456	12456	12356	123456
123	234	345	145	125	1234	2345	1345	1245	1235	12345

In this case,

$$P = \begin{pmatrix} I_{16} & -qI_{16} & 0 \\ 0 & I_{16} & 0 \\ 0 & 0 & I_6 \end{pmatrix}.$$



The hyperplanes  $h_1, h_4, h_5$  are visible from this point of view. The hyperplanes labelled in italics,  $h_2, h_3, h_6$  are not visible from this point of view.

**Figure 7.2** Pentagonal base pyramid with labelled hyperplanes.

$V_q(\mathcal{A}^{h_6}) = V_q$  for the  $C_5$  hyperplane arrangement.

$$V_q(\mathcal{A} - \{h_6\}) =$$

1	$q$	$q$	$q$	$q$	$q$	$q^2$	$q^2$	$q^2$	$q^2$	$q^2$	$q^3$	$q^3$	$q^3$	$q^3$	$q^3$	$q^4$	$q^4$	$q^4$	$q^4$	$q^4$	$q^5$	
$q$	1	$q^2$	$q^2$	$q^2$	$q^2$	$q$	$q$	$q^3$	$q^3$	$q^3$	$q$	$q^2$	$q^4$	$q^4$	$q^2$	$q^2$	$q^3$	$q^5$	$q^3$	$q^3$	$q^3$	$q^4$
$q$	$q^2$	1	$q^2$	$q^2$	$q^2$	$q$	$q$	$q$	$q^3$	$q^3$	$q^3$	$q^2$	$q^2$	$q^4$	$q^4$	$q^2$	$q^3$	$q^3$	$q^5$	$q^3$	$q^3$	$q^4$
$q$	$q^2$	$q^2$	1	$q^2$	$q^2$	$q^3$	$q$	$q$	$q$	$q^3$	$q^3$	$q^2$	$q^2$	$q^2$	$q^4$	$q^4$	$q^3$	$q^3$	$q^5$	$q^3$	$q^5$	$q^4$
$q$	$q^2$	$q^2$	$q^2$	1	$q^2$	$q^3$	$q^3$	$q$	$q$	$q^3$	$q^3$	$q^4$	$q^2$	$q^2$	$q^2$	$q^4$	$q^3$	$q^3$	$q^3$	$q^5$	$q^3$	$q^4$
$q$	$q^2$	$q^2$	$q^2$	$q^2$	1	$q^3$	$q^3$	$q^3$	$q$	$q$	$q$	$q^4$	$q^4$	$q^2$	$q^2$	$q^2$	$q^5$	$q^3$	$q^3$	$q^3$	$q^3$	$q^4$
$q^2$	$q$	$q$	$q^3$	$q^3$	$q^3$	1	$q^2$	$q^4$	$q^4$	$q^2$	$q$	$q^3$	$q^5$	$q^3$	$q$	$q^2$	$q^4$	$q^4$	$q^2$	$q^2$	$q^2$	$q^3$
$q^2$	$q^3$	$q$	$q$	$q^3$	$q^3$	$q^2$	1	$q^4$	$q^4$	$q^2$	$q$	$q$	$q^3$	$q^5$	$q^3$	$q^2$	$q^2$	$q^4$	$q^4$	$q^2$	$q^2$	$q^3$
$q^2$	$q^3$	$q^3$	$q$	$q$	$q^3$	$q^2$	$q^2$	1	$q^4$	$q^4$	$q^3$	$q^3$	$q$	$q$	$q^3$	$q^5$	$q^2$	$q^2$	$q^4$	$q^4$	$q^3$	
$q^2$	$q^3$	$q^3$	$q^3$	$q$	$q$	$q^4$	$q^2$	$q^2$	1	$q^4$	$q^5$	$q^3$	$q$	$q$	$q^3$	$q^3$	$q^4$	$q^2$	$q^2$	$q^4$	$q^3$	
$q^2$	$q$	$q^3$	$q^3$	$q^3$	$q$	$q^4$	$q^4$	$q^2$	$q^2$	1	$q^3$	$q^5$	$q^3$	$q$	$q$	$q^4$	$q^4$	$q^2$	$q^2$	$q^2$	$q^3$	
$q^3$	$q^2$	$q^2$	$q^2$	$q^4$	$q^4$	$q$	$q$	$q$	$q^3$	$q^5$	$q^3$	1	$q^2$	$q^4$	$q^4$	$q^2$	$q$	$q^3$	$q^3$	$q^3$	$q$	$q^2$
$q^3$	$q^4$	$q^2$	$q^2$	$q^2$	$q^4$	$q^3$	$q$	$q$	$q$	$q^3$	$q^5$	$q^2$	1	$q^2$	$q^4$	$q^4$	$q$	$q$	$q^3$	$q^3$	$q^3$	$q^2$
$q^3$	$q^4$	$q^4$	$q^2$	$q^2$	$q^2$	$q^5$	$q^3$	$q$	$q$	$q^3$	$q^3$	$q^4$	$q^2$	1	$q^2$	$q^4$	$q^3$	$q$	$q$	$q^3$	$q^2$	
$q^3$	$q^2$	$q^4$	$q^4$	$q^2$	$q^2$	$q^3$	$q^5$	$q^3$	$q$	$q$	$q$	$q^4$	$q^4$	$q^2$	1	$q^2$	$q^3$	$q$	$q$	$q^3$	$q^2$	
$q^3$	$q^2$	$q^2$	$q^4$	$q^4$	$q^2$	$q$	$q^3$	$q^5$	$q^3$	$q$	$q$	$q^2$	$q^4$	$q^4$	$q^2$	1	$q^3$	$q^3$	$q^3$	$q$	$q^2$	
$q^4$	$q^3$	$q^3$	$q^3$	$q^3$	$q^5$	$q^2$	$q^2$	$q^2$	$q^4$	$q^4$	$q$	$q$	$q$	$q^3$	$q^3$	$q^3$	1	$q^2$	$q^2$	$q^2$	$q^2$	
$q^4$	$q^5$	$q^3$	$q^3$	$q^3$	$q^3$	$q^4$	$q^2$	$q^2$	$q^2$	$q^4$	$q^3$	$q$	$q$	$q$	$q^3$	$q^3$	$q^2$	1	$q^2$	$q^2$	$q^2$	
$q^4$	$q^3$	$q^5$	$q^3$	$q^3$	$q^3$	$q^4$	$q^4$	$q^2$	$q^2$	$q^2$	$q^2$	$q^3$	$q^3$	$q$	$q$	$q^3$	$q^2$	$q^2$	1	$q^2$	$q^2$	
$q^4$	$q^3$	$q^3$	$q^5$	$q^3$	$q^3$	$q^2$	$q^4$	$q^4$	$q^2$	$q^2$	$q^3$	$q^3$	$q$	$q$	$q$	$q^2$	$q^2$	$q^2$	1	$q^2$	$q$	
$q^4$	$q^3$	$q^3$	$q^3$	$q^5$	$q^3$	$q^2$	$q^2$	$q^4$	$q^4$	$q^2$	$q$	$q^3$	$q^3$	$q^3$	$q$	$q$	$q^2$	$q^2$	$q^2$	1	$q$	
$q^5$	$q^4$	$q^4$	$q^4$	$q^4$	$q^4$	$q^3$	$q^3$	$q^3$	$q^3$	$q^3$	$q^3$	$q^2$	$q^2$	$q^2$	$q^2$	$q^2$	$q$	$q$	$q$	$q$	$q$	1

Here the regions are indexed in the following order by the hyperplanes in their separating sets from  $R_0$ :

$$0 \mid 1\ 2\ 3\ 4\ 5 \mid 12\ 23\ 34\ 45\ 15 \mid 123\ 234\ 345\ 145\ 125 \mid 1234\ 2345\ 1345\ 1245\ 1235 \mid 12345$$

The first part of the left and right transition matrices for  $V_q(\mathcal{A} - \{h_6\})$  are  $U_2U_1$  and  $(U_2U_1)^t$ , where

$$U_1 =$$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$-q$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$-q$	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$-q$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
$-q$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$-q$	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
$q^2$	$-q$	$-q$	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$q^2$	0	$-q$	$-q$	0	0	0	1	0	0	0	0	0	0	0	0	0
$q^2$	0	0	$-q$	$-q$	0	0	0	1	0	0	0	0	0	0	0	0
$q^2$	0	0	0	$-q$	$-q$	0	0	0	1	0	0	0	0	0	0	0
$q^2$	$-q$	0	0	0	$-q$	0	0	0	0	1	0	0	0	0	0	0
0	0	$q^2$	0	0	0	$-q$	$-q$	0	0	0	1	0	0	0	0	0
0	0	0	$q^2$	0	0	0	$-q$	$-q$	0	0	0	1	0	0	0	0
0	0	0	0	$q^2$	0	0	0	$-q$	$-q$	0	0	0	1	0	0	0
0	0	0	0	0	$q^2$	0	0	0	$-q$	$-q$	0	0	0	1	0	0
0	$q^2$	0	0	0	0	$-q$	0	0	0	$-q$	0	0	0	1	0	0
0	0	0	0	0	$-q^5$	0	$q^2$	0	$q^4$	$q^4$	$-q$	$-q$	0	$-q^3$	0	1
0	$-q^5$	0	0	0	0	$q^4$	0	$q^2$	0	$q^4$	0	$-q$	$-q$	0	$-q^3$	0
0	0	$-q^5$	0	0	0	$q^4$	$q^4$	0	$q^2$	0	$-q^3$	0	$-q$	$-q$	0	0
0	0	0	$-q^5$	0	0	0	$q^4$	$q^4$	0	$q^2$	0	$-q^3$	0	$-q$	$-q$	0
0	0	0	0	$-q^5$	0	$q^2$	0	$q^4$	$q^4$	0	$-q$	0	$-q^3$	0	$-q$	0
$-q^5$	$q^4$	0	0	0	$q^4$	0	0	0	0	$-q^3$	0	$q^2$	0	0	0	$-q$

$$\text{and } U_2 = \left( \begin{array}{c|ccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & I_5 & 0 & 0 & 0 & 0 \\
0 & 0 & I_5 & 0 & 0 & 0 \\
0 & 0 & 0 & I_5 & 0 & 0 \\
0 & 0 & 0 & 0 & I_5 & 0 \\
0 & 0 & 0 & u & 0 & 1
\end{array} \right), \text{ where } u = \left( \begin{array}{ccccc} 0 & 0 & 0 & -q^2(1-q^2) & -q^2(1-q^2) \end{array} \right).$$

Let the numbers of the labelled hyperplanes in  $\text{sep}(R_0, R)$  denote the rows of the block matrix in  $V_q$  which begins with that region indexed there. Then the steps taken by multiplying  $V_q$  on the left by the matrices  $U_2U_1$  can also be described as follows:

1.

$$12345 - q(1234). \quad (7.2.2)$$

Note: here (1234) means only the 1234 row, and not the entire block matrix.

2.

$$1234 - qI(123). \quad (7.2.3)$$

3.

$$123 - qI(12). \quad (7.2.4)$$

4.

$$12 - q(1). \quad (7.2.5)$$

5.

$$1 - q(0). \quad (7.2.6)$$

Note: here (0) means the  $R_0$  row.

6.

$$12345 - q(2345). \quad (7.2.7)$$

Note: here (2345) means only the 2345 row.

7.

$$1234 - qJ(123). \quad (7.2.8)$$

8.

$$123 - qJ(12). \quad (7.2.9)$$

9.

$$12 - qJ(1). \quad (7.2.10)$$

10.

$$1234 - q^3J^3(123). \quad (7.2.11)$$

11.

$$12345 - q^3(15). \quad (7.2.12)$$

Note: here (15) means only the last row of the (12) block.

12.

$$12345 - q^2(1 - q^2)(145) \text{ and } 12345 - q^2(1 - q^2)(125). \quad (7.2.13)$$

Note: here (145) and (125) mean only the last two rows of the (123) block.

$$U_1 V_q (\mathcal{A} - \{h_6\})(U_1)^t =$$

$$\left( \begin{array}{c|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 I_5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2 I_5 & 0 & v_{35} \\ 0 & 0 & 0 & 0 & v_{44} & v_{45} \\ \hline 0 & 0 & 0 & v_{53} & v_{54} & v_{55} \end{array} \right),$$

where  $v_{53} = \begin{pmatrix} 0 & 0 & 0 & q^2(1-q^2)^3 & q^2(1-q^2)^3 \end{pmatrix}$ ,  $v_{35} = v_{53}^t$ ,

$$\text{and } \left( \begin{array}{c|c} v_{44} & v_{45} \\ \hline v_{54} & v_{55} \end{array} \right) =$$

$$\left( \begin{array}{ccccc|c} (1-q^2)^2(1-q^6) & 0 & q^2(1-q^2)^3 & q^2(1-q^2)^3 & 0 & -q^5(1-q^2)^3 \\ 0 & (1-q^2)^2(1-q^6) & 0 & q^2(1-q^2)^3 & q^2(1-q^2)^3 & -q^5(1-q^2)^3 \\ q^2(1-q^2)^3 & 0 & (1-q^2)^2(1-q^6) & 0 & q^2(1-q^2)^3 & q(1-q^2)^3 \\ q^2(1-q^2)^3 & q^2(1-q^2)^3 & 0 & (1-q^2)^2(1-q^6) & 0 & q(1-q^2)^4 \\ 0 & q^2(1-q^2)^3 & q^2(1-q^2)^3 & 0 & (1-q^2)^2(1-q^6) & q(1-q^2)^3 \\ \hline -q^5(1-q^2)^3 & -q^5(1-q^2)^3 & q(1-q^2)^3 & q(1-q^2)^4 & q(1-q^2)^3 & (1-q^2)^2(1-q^6) \end{array} \right).$$

$$U_2 U_1 V_q(\mathcal{A} - \{h_6\})(U_2 U_1)^t =$$

$$\left( \begin{array}{c|cc|ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & (1-q^2)I_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 I_{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & (1-q^2)^2(1-q^6) & 0 & q^2(1-q^2)^3 & q^2(1-q^2)^3 & 0 & -q^5(1-q^2)^3 \\ 0 & 0 & 0 & 0 & (1-q^2)^2(1-q^6) & 0 & q^2(1-q^2)^3 & q^2(1-q^2)^3 & -q^5(1-q^2)^3 \\ 0 & 0 & 0 & q^2(1-q^2)^3 & 0 & (1-q^2)^2(1-q^6) & 0 & q^2(1-q^2)^3 & q(1-q^2)^3 \\ 0 & 0 & 0 & q^2(1-q^2)^3 & q^2(1-q^2)^3 & 0 & (1-q^2)^2(1-q^6) & 0 & q(1-q^2)^4 \\ 0 & 0 & 0 & 0 & q^2(1-q^2)^3 & q^2(1-q^2)^3 & 0 & (1-q^2)^2(1-q^6) & q(1-q^2)^3 \\ \hline 0 & 0 & 0 & -q^5(1-q^2)^3 & -q^5(1-q^2)^3 & q(1-q^2)^3 & q(1-q^2)^4 & q(1-q^2)^3 & (1-q^2)^3(1+q^2-q^4+2q^6) \end{array} \right).$$

Now the problem of putting  $V_q(\mathcal{A} - \{h_6\})$  into Smith normal form has been reduced to putting  $M$ , which is defined to be the lower right corner  $6 \times 6$  matrix, into Smith normal form. This can be done over  $\mathbb{Z}[q]$ .

$$Left = \begin{pmatrix} 1 & 0 & -1 & -q^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -(1+q^2) & 0 \\ 0 & q^2 & 1 & q^8 - (1+q^4) & -q^2(1+q^2) & q^5 - q \\ 0 & -q^3 & -q & -q^9 + q^5 + q^3 + q & q^5 + q^3 & -q^6 + q^2 + 1 \\ 0 & -q^2(1+q^2) & -q^2 & q^2(1-q^6 - q^8) & q^6 + 2q^2(1+q^2) + 1 & -q^5(1+q^2) \\ -q^2 & -2q^2 & 0 & 1 + q^2 - q^8 & q^2(1+q^2) + 1 & -q(1+q^4) \end{pmatrix}.$$

$$Right = \begin{pmatrix} 1 & q^4 & q^6(1+q^2) + 1 + q^4 & q(q^{14} + 2q^{12} + 2q^{10} + q^8 + 3q^6 + 2q^4 + 2) & q^{10} + 2q^8 + 2q^2 - 1 & 1 - q^2 + q^6 \\ 0 & 1 & q^2(1+q^2) & q^3(q^8 + 2q^6 + q^2(1+q^2) + 2) & q^6 + q^2(1+q^2) + 1 & -q^2(1+q^2) \\ 0 & 0 & 1 & q^7 + q & q^4 - 1 & q^2 + 1 \\ 0 & 0 & 0 & 0 & -1 & q^2 + 1 \\ 0 & 0 & 0 & -q(1-q^2) & 1 - q^2 & q^2 \\ 0 & 0 & 0 & 1 & q & -q(2+q^2) \end{pmatrix}.$$

We present the matrices *Left* and *Right* as one possible set of left and right transition matrices with determinants both equal to  $-1$  and entries in  $\mathbb{Z}[q]$ . Simpler, more elegant transition matrices may exist. Nonetheless,

$$\text{Left } M \text{ Right} = \begin{pmatrix} (1-q^2)^3 I_3 & 0 \\ 0 & (1-q^2)^2(1-q^{10})I_3 \end{pmatrix}. \quad (7.2.14)$$

To tie everything together, let  $T'_{16 \times 16}$  be the left transformation matrix for  $V_q(\mathcal{A}^{h_6}) = V_q$  for  $C_5$  as it was defined in Equation 3.1.10.

$$\text{Let } T = \begin{pmatrix} T' & 0 \\ 0 & I_{22} \end{pmatrix}, U = \begin{pmatrix} I_{16} & 0 \\ 0 & U_2 U_1 \end{pmatrix}, L_1 = \begin{pmatrix} I_{32} & 0 \\ 0 & \text{Left} \end{pmatrix}, \text{ and } R_1 = \begin{pmatrix} I_{32} & 0 \\ 0 & \text{Right} \end{pmatrix}.$$

Then, by Equations 3.1.9 and 7.2.14,  $L_1(UTP)V_q(\mathcal{A})(UTP)^tR_1 =$

$$\left( \begin{array}{ccc|ccc|c} (1-q^2) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)^2 I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^3 I_5 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2) I_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-q^2)^2 I_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^3 I_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-q^2)^2(1-q^{10})I_3 \end{array} \right).$$

Then left and right multiplication by the permutation matrix  $S =$

$$\left( \begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_5 & 0 & 0 \\ 0 & I_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{10} & 0 \\ 0 & 0 & I_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \end{array} \right)$$

arranges the diagonal matrix  $L_1(UTP)V_q(\mathcal{A})(UTP)^tR_1$  into its Smith normal form:

$$SL_1(UTP)V_q(\mathcal{A})(UTP)^tR_1S^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (1-q^2)I_6 & 0 & 0 & 0 \\ 0 & 0 & (1-q^2)^2 I_{15} & 0 & 0 \\ 0 & 0 & 0 & (1-q^2)^3 I_8 & 0 \\ 0 & 0 & 0 & 0 & (1-q^2)^2(1-q^{10})I_3 \end{pmatrix}.$$

## BIBLIOGRAPHY

- [CCM16] Cai, T. W., Chen, Y. & Mu, L. “On the Smith Normal Form of the q-Varchenko Matrix of a Real Hyperplane Arrangement”. (*preprint*) (2016).
- [DH97] Denham, G. & Hanlon, P. “On the Smith normal form of the Varchenko bilinear form of a hyperplane arrangement”. *Pacific Journal of Mathematics* **181**, No. 3 (1997), pp. 123–146.
- [GZ18] Gao, Y. & Zhang, Y. “Diagonal Form of the the Varchenko Matrices”. *Journal of Algebraic Combinatorics* <https://doi.org/10.1007/s10801-018-0813-7> (2018), pp. 1–18.
- [Grü71] Grünbaum, B. “Arrangements of hyperplanes”. *Proc. Second Lousiana Conf. on Combinatorics and Graph Theory, Baton Rouge* (1971), pp. 41–106.
- [OT92] Orlik, P. & Terao, H. *Arrangements of Hyperplanes*. Springer-Verlag, 1992.
- [PS09] Pach, J. & Sharir, M. *Combinatorial Geometry and Its Algorithmic Applications: The Alcalà Lectures, Mathematical Surveys and Monographs, Volume 152*. American Mathematical Society, 2009.
- [Shi04] Shiu, W. C. “Invariant Factors of Graphs associated with Hyperplane Arrangements”. *Discrete Mathematics* **288**.1-3 (2004), pp. 135–148.
- [Sta04] Stanley, R. P. “An Introduction to Hyperplane Arrangements”. *Geometric Combinatorics* **13** (2004), pp. 389–496.
- [Sta16] Stanley, R. P. “Smith Normal Form In Combinatorics”. *Journal of Combinatorial Theory, Series A* **144** (2016), pp. 476–495.
- [Var93] Varchenko, A. “Bilinear form of real configuration of hyperplanes”. *Advances in Mathematics* **97** (1993), pp. 110–144.