ABSTRACT

BOYLE, KRISTEN JOHNSON. On Derivations of Leibniz Algebras. (Under the direction of Dr. Kailash Misra and Dr. Ernest Stitzinger.)

Leibniz algebras are non-commutative generalizations of Lie algebras. Since the introduction of Leibniz algebras in 1993, researchers have been working to determine which properties of Lie algebras have analogs for Leibniz algebras. In this work we study the properties of derivations of Leibniz algebras. We determine a basis for the derivation algebra of an n-dimensional cyclic Leibniz algebra. Based on George Seligman's work on Lie algebras we investigate characteristic ideals of Leibniz algebras. By modifying the proof techniques for Lie algebras we are able to prove the invariance of the radical and nilradical under derivations of a Leibniz algebra over fields of characteristic 0 and over some finite fields. We introduce the notion of completeness for Leibniz algebras, and prove that all semisimple Leibniz algebras are complete. It is known that nilpotent Lie algebras are not complete. However, there exist nilpotent Leibniz algebras which are complete. We define the holomorph for a Leibniz algebra, and study properties of the holomorph for complete Leibniz algebras. We also introduce the notion of semicompleteness for Leibniz algebras. We show that all complete Leibniz algebras are semicomplete, but not conversely.

On Derivations of Leibniz Algebras

by
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DEDICATION

To my husband, to my mom and brother, to my father's memory, and to my beloved friends and family who have molded me into the person I am today.

BIOGRAPHY

Kristen Leigh Johnson Boyle was born in Morganton, North Carolina to Keith and Jane Johnson. She grew up in Lexington, North Carolina with brother Matt Johnson, and attended North Davidson High School. After graduating in 2006, she was awarded the North Carolina Teaching Fellowship and attended Appalachian State University, where she earned a B.S. in Mathematics, Secondary Education in 2010, and an M.A. in Mathematics in 2011. Kristen married her husband Dean in the summer of 2011, and both taught at Lexington Senior High School until she began further graduate studies at North Carolina State University in 2015. Kristen earned an M.S. in Mathematics from N.C. State in December of 2016 and then continued into the doctoral program under the advisement of Dr. Misra and Dr. Stitzinger. Kristen will begin her career in academia in the fall of 2018 as an Assistant Professor of Mathematics at Longwood University.

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Chapter 1

Introduction

Lie algebras have been studied since the mid-nineteenth century, appearing in connection with the study of Lie groups. Lie algebras are a widely researched area of mathematics, and have applications in applied mathematics and physics. Jean-Louis Loday, known for his work in Lie algebra homology, proved in 1989 that the Chevalley-Eilenberg boundary map on the exterior algebra of a Lie algebra can be lifted to the tensor algebra into a new boundary map. In [11], Loday shows that this lift gives rise to a new complex whose homology is called *non-commutative homology of the Lie algebra*. Loday noticed that the Leibniz identity was the only property necessary for the chain complex to be well-defined. This discovery motivated him to define Leibniz algebras as non-commutative generalizations of Lie algebras.

Since the introduction of Leibniz algebras in 1993, researchers have been working to determine which properties of Lie algebras can be extended to Leibniz algebras. Among the results which have been proven for Leibniz algebras are analogs of Lie's Theorem, Engel's Theorem, Levi decomposition, and Cartan's criterion [8]. Classification of finite dimensional Leibniz algebras is a difficult open problem. Many researchers are working to classify finite dimensional Leibniz algebras which are not Lie [1], [3], [7], [9], [17]. An important class of non-Lie Leibniz algebras are the cyclic Leibniz algebras generated by a single element [5]. We use cyclic Leibniz algebras and other low-dimensional examples to illustrate important concepts for Leibniz algebras throughout this work.

In this work we study the properties of derivations of Leibniz algebras. In Chapter 2 we recall important notions for Leibniz algebras in the same manner as Demir, Misra, and Stitzinger in [8]. In Chapter 3 we discuss cyclic Leibniz algebras. We compute the derivation algebras of cyclic Leibniz algebras up to dimension 5, and find their bases. We determine a basis for the derivation algebra of an n-dimensional cyclic Leibniz algebra. Since the cyclic Leibniz algebras are significant examples of algebras which are not Lie, having a basis for the derivation algebra is useful to investigate properties of derivations of Leibniz algebras.

George Seligman [19] studied the properties of characteristic ideals of Lie algebras. In Chapter 4, we prove corresponding results for Leibniz algebras. We use techniques similar to those shown by Petravchuk [16] and Maksimenko [13] to prove the invariance of the radical and nilradical under derivations of a Leibniz algebra over fields of characteristic 0 and over some finite fields.

In Chapter 5, we introduce the notion of completeness for Leibniz algebras generalizing the notion of completeness for Lie algebras. In particular, any complete Lie algebra is also complete as a Leibniz algebra. We prove that all semisimple Leibniz algebras are complete. Using our results from Chapter 2, we show that a cyclic Leibniz algebra is not complete. Demir, Misra, and Stitzinger in [8] and [9] provide lists of all 3-dimensional non-Lie Leibniz algebras, and all 4-dimensional non-split non-Lie nilpotent Leibniz algebras, up to isomorphism. We determine which of these Leibniz algebras are complete and which are not complete. Jacobson proves in [10] that all nilpotent Lie algebras are not complete. We give an example of a nilpotent Leibniz algebra which is complete. We also define the holomorph for a Leibniz algebra. We show that the holomorph of a Leibniz algebra is a Leibniz algebra. In [14], Meng proves that a Lie algebra L is complete if and only if the holomorph of L is a direct sum between L and the centralizer of L in the holomorph. When A is a complete Leibniz algebra, we provide a decomposition of the holomorph, and show by example that this decomposition does not imply that A is a complete Leibniz algebra.

In Chapter 6 we introduce the notion of semicompleteness for Leibniz algebras. We prove that all cyclic Leibniz algebras are semicomplete. We show that all complete Leibniz algebras are semicomplete, but not all semicomplete Leibniz algebras are complete.

Chapter 2

Preliminaries

Leibniz algebras are non-commutative generalizations of Lie algebras. In particular, a *(left) Leibniz algebra* is a vector space over a field $\mathbb{F}$ equipped with a bilinear product $(,) : A \times A \to A$ that satisfies the Leibniz identity, $a(bc) = (ab)c + b(ac)$ for all $a, b, c \in A$. Recall, it is the alternativity of the Lie bracket that yields the antisymmetry of the product. So, when a Leibniz algebra A also satisfies the condition $a^2 = 0$ for all $a \in A$, the Leibniz algebra is a Lie algebra and the Leibniz identity becomes the Jacobi identity.

Example 2.0.1. Let $A = \text{span}\{a, b, c\}$ with nonzero multiplications defined by $a^2 = b$ and $ab = c$. A is not a Lie algebra since $a^2 \neq 0$. Let $x, y, z \in A$ where
$x = \alpha_1 a + \alpha_2 b + \alpha_3 c$, $y = \beta_1 a + \beta_2 b + \beta_3 c$, and $z = \gamma_1 a + \gamma_2 b + \gamma_3 c$ for $\alpha_i, \beta_i, \gamma_i \in \mathbb{F}$, $1 \leq i \leq 3$. Then

$$x(yz) = (\alpha_1 a + \alpha_2 b + \alpha_3 c)(\beta_1\gamma_1 b + \beta_1\gamma_2 c) = \alpha_1\beta_1\gamma_1 c.$$

And

$$\begin{aligned}(xy)z + y(xz) &= (\alpha_1\beta_1 b + \alpha_1\beta_2 c)(\gamma_1 a + \gamma_2 b + \gamma_3 c) + (\beta_1 a + \beta_2 b + \beta_3 c)(\alpha_1\gamma_1 b + \alpha_1\gamma_2 c) \\ &= 0 + \beta_1\alpha_1\gamma_1 c \\ &= \alpha_1\beta_1\gamma_1 c.\end{aligned}$$

Thus, the Leibniz identity holds; and A is a Leibniz algebra.

A linear map $\delta : A \to A$ is a *derivation* of a Leibniz algebra A if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. For any derivation δ of a Leibniz algebra A, define $\delta^0(A) = A$ and

$\delta^k(A) = \delta(\delta^{k-1}(A))$. We denote Der$(A)$ to be the set of all derivations of A.

Example 2.0.2. Let A be a Leibniz algebra. Der(A) is closed under linear combinations, so it is a subspace of $gl(A)$, the algebra of all linear operators on A with the product given by composition. Let $\delta_1, \delta_2 \in$ Der(A), $x, y \in A$, and consider the commutator of δ_1 and δ_2 applied to the product xy.

$$\begin{aligned}
[\delta_1, \delta_2](xy) &= \delta_1\delta_2(xy) - \delta_2\delta_1(xy) = \delta_1(\delta_2(x)y + x\delta_2(y)) - \delta_2(\delta_1(x)y + x\delta_1(y)) \\
&= \delta_1(\delta_2(x)y) + \delta_1(x\delta_2(y)) - \delta_2(\delta_1(x)y) - \delta_2(x\delta_1(y)) \\
&= \delta_1(\delta_2(x))y + \delta_2(x)\delta_1(y) + \delta_1(x)\delta_2(y) + x\delta_1(\delta_2(y)) \\
&- \delta_2(\delta_1(x))y - \delta_1(x)\delta_2(y) - \delta_2(x)\delta_1(y) - x\delta_2(\delta_1(y)) \\
&= \delta_1(\delta_2(x))y + x\delta_1(\delta_2(y)) - \delta_2(\delta_1(x))y - x\delta_2(\delta_1(y)) \\
&= \delta_1\delta_2(x)y + x\delta_1\delta_2(y) - \delta_2\delta_1(x)y - x\delta_2\delta_1(y) \\
&= [\delta_1, \delta_2](x)y + x[\delta_1, \delta_2](y).
\end{aligned}$$

Therefore, $[\delta_1, \delta_2] \in$Der(A). Hence, Der(A) of any Leibniz algebra A is a Lie algebra.

For a Leibniz algebra A, and for any $a \in A$, we define the left multiplication operator $L_a : A \to A$ by $L_a(b) = ab$ for all $b \in A$.

Remark 1. Left multiplication L_a is a derivation.
Let $a, b, c \in A$. $L_a(bc) = a(bc) = \underbrace{(ab)c + b(ac)}_{\text{by Leibniz identity}} = L_a(b)c + bL_a(c)$.

We denote L(A) =span$\{L_a|a \in A\}$. Since all left multiplications are derivations, for every Leibniz algebra we have the containment L$(A) \subseteq$Der(A). For a Leibniz algebra A, and for any $a \in A$, we define the right multiplication operator $R_a : A \to A$ by $R_a(b) = ba$ for all $b \in A$.

Remark 2. Right multiplication R_a is not a derivation.
Let $a, b, c \in A$. $R_a(bc) = (bc)a = \underbrace{b(ca) - c(ba)}_{\text{by Leibniz identity}} = bR_a(c) - cR_a(b) \neq R_a(b)c + bR_a(c)$.

We defined the Leibniz identity in such a way that left multiplication is a derivation. Notice that the derivation property is exactly Leibniz's rule for the product of derivatives. This is one of the reasons Leibniz algebras were named for Gottfried Leibniz. A right Leibniz algebra is a vector space over a field $\mathbb{F}$ equipped with a bilinear product such that right multiplication is a derivation. In this paper, a Leibniz algebra always refers to a left Leibniz algebra.

For any subspace I of a Leibniz algebra A, if $I^2 \subseteq I$, then I is a *subalgebra* of A. I is a *left ideal* of A if $A \cdot I \subseteq I$, and I is a *right ideal* of A if $I \cdot A \subseteq I$. I is an *ideal* of A if it is both a left ideal and a right ideal. A particularly important subalgebra of A is Leib(A) =span$\{a^2 | a \in A\}$.

Remark 3. Leib(A) is an ideal of A.
Let $a \in A$ and let $b' \in$Leib(A). Then $b' = b^2$ for some $b \in A$.
Consider $b'a = b^2a = (bb)a = b(ba) - b(ba) = 0 \in$Leib$(A)$; thus, Leib$(A)$ is a right ideal.
Now consider the element $(a + b^2)(a + b^2) - a^2 \in$Leib$(A)$.

$$\begin{aligned}(a + b^2)(a + b^2) - a^2 &= a(a + b^2) + b^2(a + b^2) - a^2 = a^2 + ab^2 + b^2(a + b^2) - a^2 \\ &= ab^2 + (bb)(a + b^2) = ab^2 + b(b(a + b^2)) - b(b(a + b^2)) \\ &= ab^2 = ab'.\end{aligned}$$

Thus, $ab' \in$Leib(A), which implies Leib(A) is a left ideal. Therefore, Leib(A) is an ideal of A.

For any ideal I of A, we define the quotient Leibniz algebra A/I by $a \mapsto a+I$. Leib(A) is the minimal ideal such that $A/$Leib(A) is a Lie algebra, since the quotient by Leib(A) implies $a^2 = 0$ for every $a \in A$.

Example 2.0.3. Let A =span$\{a, b, c, d\}$ with nonzero multiplications defined by $a^2 = d$, $ab = c$, $ac = d$, $ba = -c$, and $ca = -d$. Let $x \in A$ where $x = \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d$. Then

$$\begin{aligned}x^2 &= (\alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d)(\alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d) \\ &= \alpha_1^2 d + \alpha_1\alpha_2 c + \alpha_1\alpha_3 d - \alpha_2\alpha_1 c - \alpha_3\alpha_1 d \\ &= \alpha_1^2 d.\end{aligned}$$

Thus, Leib(A) =span$\{d\}$. Hence, $A/$Leib(A) =span$\{a+$Leib$(A), b+$Leib$(A), c+$Leib$(A)\}$ =span$\{\overline{a}, \overline{b}, \overline{c}\}$ with $[\overline{a}, \overline{b}] = \overline{c}$, the 3-dimensional Heisenberg Lie algebra.

Let A be a Leibniz algebra. The *left center* of A is $Z^l(A) = \{x \in A | xa = 0$ for all $a \in A\}$. The *right center* of A is $Z^r(A) = \{x \in A | ax = 0$ for all $a \in A\}$. The *center* of

A is $Z(A) = Z^l(A) \cap Z^r(A)$. By definition, the center of a Leibniz algebra is an abelian ideal.

Remark 4. Let A be a Leibniz algebra, and let $a, b \in A$. By the Leibniz identity, $a(ab) = (a^2)b + a(ab)$. Therefore, $a^2b = a(ab) - a(ab) = 0$.

Since left multiplication by any square element of A is zero, we have the containment $\mathrm{Leib}(A) \subseteq Z^l(A)$ for every Leibniz algebra. The *derived series* of a Leibniz algebra A is the sequence of ideals $A \supseteq A^{(2)} \supseteq A^{(3)} \supseteq \ldots$ where $A^{(1)} = A^2$ and $A^{(n)} = A^{(n-1)} \cdot A^{(n-1)}$ for all $n \geq 1$. A Leibniz algebra A is *solvable* if $A^{(k+1)} = 0$ for some $k+1 \geq 0$ such that $A^{(k)} \neq 0$. In this case, we say the derived length of A is k.

The *lower central series* of a Leibniz algebra A is the sequence of ideals $A \supseteq A^{\mathbf{1}} \supseteq A^{\mathbf{2}} \supseteq \ldots$ where $A^{\mathbf{1}} = A \cdot A = A^2, A^{\mathbf{k+1}} = A \cdot A^{\mathbf{k}}$. A Leibniz algebra A is nilpotent if $A^{\mathbf{k+1}} = 0$ for some $k \geq 0$ such that $A^{\mathbf{k}} \neq 0$. In this case, we say the nilpotency class of A is k. We have the containment $A^{(n)} \subseteq A^{\mathbf{n}}$ for all n. Therefore, every nilpotent Leibniz algebra is a solvable Leibniz algebra. But every solvable Leibniz algebra is not nilpotent, as seen in Example 2.0.5

Example 2.0.4. Let A be the Leibniz algebra with basis $B = \{a, b, c\}$ and nonzero multiplications defined by $a^2 = b$ and $ab = c$. Then $Z^l(A)$ =span$\{b, c\}$ =Leib(A), $Z^r(A)$ =span$\{c\}$, and $Z(A)$ =span$\{c\}$. Let us also consider the ideals in the derived series of A: $A^{(1)}$ =span$\{b, c\}$, $A^{(2)} = 0$. Thus, A is solvable with derived length 1. Now consider the ideals in the lower central series of A: $A^{\mathbf{1}}$ =span$\{b, c\}$, $A^{\mathbf{2}}$ =span$\{c\}$, $A^{\mathbf{3}} = 0$. Thus, A is nilpotent with nilpotency class 2.

Example 2.0.5. Let A be the Leibniz algebra $B = \{a, b, c\}$ with nonzero multiplications defined by $a^2 = b$, $ab = c$, and $ac = c$. The terms in the derived series of A are: $A^{(1)}$ =span$\{b, c\}$, $A^{(2)} = 0$. Thus, A is solvable of derived length 1. The terms in the lower central series of A are: $A^{\mathbf{1}}$ =span$\{b, c\}$, $A^{\mathbf{2}}$ =span$\{c\} = A^{\mathbf{3}}$. Therefore, $A^{\mathbf{k}}$ =span$\{c\}$ for all $k \geq 2$. Hence, A is not nilpotent.

Chapter 3

Derivations of Cyclic Leibniz Algebras

3.1 Cyclic Leibniz Algebras

Consider $A = \text{span}\{a, a^2, \ldots, a^n\}$ with multiplications $a \cdot a^i = a^{i+1}$, $1 \leq i \leq n-1$, and $a \cdot a^n = \alpha_1 a + \alpha_2 a^2 + \ldots + \alpha_n a^n$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. Then A is generated by the single element a, and A is called a *cyclic* Leibniz algebra. Cyclic Leibniz algebras are not Lie algebras since $a^2 \neq 0$.

Example 3.1.1. Let A be an n-dimensional cyclic Leibniz algebra. Consider $a^2 \cdot a^k$ for $1 \leq k \leq n$. By the Leibniz identity, $a^2 \cdot a^k = (a \cdot a) \cdot a^k = a \cdot (a \cdot a^k) - a \cdot (a \cdot a^k) = 0$. Therefore, multiplication on the left by a^2 is zero. Now assume $a^i \cdot a^k = 0$ for $2 \leq i \leq n-1$ and $1 \leq k \leq n$. Then $a^{i+1} \cdot a^k = (a \cdot a^i) \cdot a^k = a \cdot (a^i \cdot a^k) - a^i \cdot (a \cdot a^k) = a \cdot 0 - 0 = 0$. Therefore, multiplication on the left by any basis element other than a is zero.

Example 3.1.2. Let A be an n-dimensional cyclic Leibniz algebra. Since $a^n \cdot a = 0$, $a \cdot (a^n \cdot a) = a \cdot 0 = 0$. And, by the Leibniz identity, $a \cdot (a^n \cdot a) = (a \cdot a^n) \cdot a + a^n \cdot (a \cdot a) = (\alpha_1 a + \alpha_2 a^2 + \ldots + \alpha_n a^n) \cdot a + 0 = \alpha_1 a \cdot a + \alpha_2 a^2 \cdot a + \ldots + \alpha_n a^n \cdot a = \alpha_1 a^2$. Thus, $\alpha_1 a^2 = 0$, which implies $\alpha_1 = 0$.

It follows from Example 3.1.2 that for any n-dimensional cyclic Leibniz algebra A, $\text{Leib}(A) = A^2 = \text{span}\{a^2, a^3, \ldots, a^n\}$. And in fact, for any Leibniz algebra A, the dimension of A^2 is $n-1$ if and only if A is cyclic.

Note: the multiplications $a \cdot a^i = a^{i+1}$, $1 \leq i \leq n-1$ are defined in the same way for every cyclic Leibniz algebra. Thus, we only need to describe the multiplication for $a \cdot a^n$ when defining a cyclic Leibniz algebra.

3.2 Nilpotent Cyclic Leibniz Algebras

Consider the n-dimensional cyclic Leibniz algebra A with basis $\{a, a^2, \ldots, a^n\}$ and multiplication $a \cdot a^n = 0$. The terms in the lower central series of A are: $A^{\mathbf{1}}$ =span$\{a^2, \ldots, a^n\}$, $A^{\mathbf{2}}$ =span$\{a^3, \ldots, a^n\}$, $\ldots$, $A^{\mathbf{n}}$ =span$\{a^n\}$, $A^{\mathbf{n+1}} = 0$. Thus, A is nilpotent with nilpotency class $n+1$.

Example 3.2.1. Let A be the cyclic Leibniz algebra with basis $B = \{a, a^2, a^3\}$ and multiplication $a \cdot a^3 = 0$. Let $\delta \in$Der(A) and define the action of δ on the basis elements as follows: $\delta(a) = \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3$, $\delta(a^2) = \beta_1 a + \beta_2 a^2 + \beta_3 a^3$, $\delta(a^3) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3$.
Therefore, $[\delta]_B = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}$.
By the derivation property, $\delta(a \cdot a) = \delta(a)a + a\delta(a)$
$= (\alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3)a + a(\alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3) = 2\alpha_1 a^2 + \alpha_2 a^3$.
And $\delta(a \cdot a) = \delta(a^2) = \beta_1 a + \beta_2 a^2 + \beta_3 a^3$.
Hence, by the linear independence of the basis vectors, $\beta_1 = 0$, $\beta_2 = 2\alpha_1$, and $\beta_3 = \alpha_2$.
Continuing this process for every combination of basis vectors yields:

$$[\delta]_B = \begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 2\alpha_1 & 0 \\ \alpha_3 & \alpha_2 & 3\alpha_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, Der(A) =span$\{\delta_1, \delta_2, \delta_3\}$, where

- $\delta_1(a) = a, \delta_1(a^2) = 2a^2, \delta_1(a^3) = 3a^3$
- $\delta_2(a) = a^2, \delta_2(a^2) = a^3, \delta_2(a^3) = 0$
- $\delta_3(a) = a^3, \delta_3(a^2) = 0, \delta_3(a^3) = 0$.

Example 3.2.2. Let A be the cyclic Leibniz algebra with basis $B = \{a, a^2, a^3, a^4\}$ and multiplication $a \cdot a^4 = 0$. Let $\delta \in$Der(A) and define the action of δ on the basis elements as follows: $\delta(a) = \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 + \alpha_4 a^4$, $\delta(a^2) = \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4$, $\delta(a^3) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_4 a^4$, $\delta(a^4) = \omega_1 a + \omega_2 a^2 + \omega_3 a^3 + \omega_4 a^4$.

Therefore, $[\delta]_B = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \omega_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \omega_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \omega_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \omega_4 \end{pmatrix}$.

By applying δ to the product of every combination of basis vectors, we can show:

$$[\delta]_B = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ \alpha_2 & 2\alpha_1 & 0 & 0 \\ \alpha_3 & \alpha_2 & 3\alpha_1 & 0 \\ \alpha_4 & \alpha_3 & \alpha_2 & 4\alpha_1 \end{pmatrix}$$
$$= \alpha_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$, where

- $\delta_1(a) = a, \delta_1(a^2) = 2a^2, \delta_1(a^3) = 3a^3, \delta_1(a^4) = 4a^4$
- $\delta_2(a) = a^2, \delta_2(a^2) = a^3, \delta_2(a^3) = a^4, \delta_2(a^4) = 0$
- $\delta_3(a) = a^3, \delta_3(a^2) = a^4, \delta_3(a^3) = 0, \delta_3(a^4) = 0$
- $\delta_4(a) = a^4, \delta_4(a^2) = 0, \delta_4(a^3) = 0, \delta_4(a^4) = 0$

Example 3.2.3. Let A be the cyclic Leibniz algebra with basis $B = \{a, a^2, a^3, a^4, a^5\}$ and multiplication $a \cdot a^5 = 0$. Let $\delta \in$Der(A) and define the action of δ on the basis elements as follows: $\delta(a) = \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 + \alpha_4 a^4 + \alpha_5 a^5$, $\delta(a^2) = \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4 + \beta_5 a^5$, $\delta(a^3) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_4 a^4 + \gamma_5 a^5$, $\delta(a^4) = \omega_1 a + \omega_2 a^2 + \omega_3 a^3 + \omega_4 a^4 + \omega_5 a^5$, $\delta(a^5) = \lambda_1 a + \lambda_2 a^2 + \lambda_3 a^3 + \lambda_4 a^4 + \lambda_5 a^5$.

Therefore, $[\delta]_B = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \omega_1 & \lambda_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \omega_2 & \lambda_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \omega_3 & \lambda_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \omega_4 & \lambda_4 \\ \alpha_5 & \beta_5 & \gamma_5 & \omega_5 & \lambda_5 \end{pmatrix}$.

By applying δ to the product of every combination of basis vectors, we can show:

$$[\delta]_B = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_2 & 2\alpha_1 & 0 & 0 & 0 \\ \alpha_3 & \alpha_2 & 3\alpha_1 & 0 & 0 \\ \alpha_4 & \alpha_3 & \alpha_2 & 4\alpha_1 & 0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & 5\alpha_1 \end{pmatrix}$$

$$= \alpha_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$+ \alpha_4 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \alpha_5 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$, where

- $\delta_1(a) = a, \delta_1(a^2) = 2a^2, \delta_1(a^3) = 3a^3, \delta_1(a^4) = 4a^4, \delta_1(a^5) = 5a^5$
- $\delta_2(a) = a^2, \delta_2(a^2) = a^3, \delta_2(a^3) = a^4, \delta_2(a^4) = a^5, \delta_2(a^5) = 0$
- $\delta_3(a) = a^3, \delta_3(a^2) = a^4, \delta_3(a^3) = a^5, \delta_3(a^4) = 0, \delta_3(a^5) = 0$
- $\delta_4(a) = a^4, \delta_4(a^2) = a^5, \delta_4(a^3) = 0, \delta_4(a^4) = 0, \delta_4(a^5) = 0$
- $\delta_5(a) = a^5, \delta_5(a^2) = 0, \delta_5(a^3) = 0, \delta_5(a^4) = 0, \delta_5(a^5) = 0$

By completing enough examples of derivations of low-dimensional nilpotent cyclic Leibniz algbebras, we establish a pattern among the basis vectors for the derivation algebras. We generalize the result to define the derivation algebra of any n-dimensional nilpotent cyclic Leibniz algebra.

Theorem 1. Let A be a nilpotent cyclic Leibniz algebra with basis $B = \{a, a^2, \ldots, a^n\}$. Let $\delta \in$Der(A), and let $(d_{ij})_{n\times n}$ be the representation matrix of δ with respect to the basis B. Then $(d_{ij})_{n\times n} = \sum_{k=0}^{n-1} \alpha_k E^{(k)}$ where

- $E^{(0)} = (e_{ij}^{(0)}) = \begin{cases} i & i = j \\ 0 & i \neq j \end{cases}$
- $E^{(k)} = (e_{ij}^{(k)}) = \begin{cases} 1 & i = j + k \\ 0 & i \neq j + k \end{cases}$, $1 \leq k \leq n-1$.

Proof. Let $B = \{a, a^2, \ldots, a^n\}$ be a basis for cyclic Leibniz algebra A, and let $a \cdot a^n = 0$. Let $\delta \in$Der(A), and consider $[\delta]_B = (d_{ij})_{n\times n}$.
To check for linear independence, consider $\sum_{k=0}^{n-1} \alpha_k E^{(k)} = 0$.
Observe column one in $E^{(k)}$. Column one of $E^{(k)}$ is a column of zeros with a one in the $a_{k+1,1}$ entry for every $0 \leq k \leq n-1$.
Thus, column one in the matrix $\sum_{k=0}^{n-1} \alpha_k E^{(k)}$ is $\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$, and $\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$
implies $\alpha_0 = \alpha_1 = \ldots = \alpha_{n-1} = 0$. Hence, the vectors $E^{(k)}$ are linearly independent.
Now we need to show spanning. Let $\overline{\delta}$ be any derivation in Der(A).
We need to show $[\overline{\delta}]_B = (\overline{d}_{ij})_{n\times n}$ can be written as a linear combination of the $E^{(k)}$. $\overline{\delta}(a)$ is a linear combination of the basis elements of A, so let $\overline{\delta}(a) = \sum_{i=0}^{n-1} \beta_i a^{i+1}$.
Note $[\overline{\delta}]_B = \Big([\overline{\delta}(a)]_B| \quad [\overline{\delta}(a^2)]_B| \quad \ldots| \quad [\overline{\delta}(a^n)]_B\Big)_{n\times n}$.
Thus, column one in the matrix $[\overline{\delta}]_B$ is $\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix}$.

This implies:

$$[\overline{\delta}(a)]_B = \beta_0 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \beta_{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \beta_0 \begin{pmatrix} e_{11}^{(0)} \\ e_{21}^{(0)} \\ \vdots \\ e_{n1}^{(0)} \end{pmatrix} + \beta_1 \begin{pmatrix} e_{11}^{(1)} \\ e_{21}^{(1)} \\ \vdots \\ e_{n1}^{(1)} \end{pmatrix} + \ldots + \beta_{n-1} \begin{pmatrix} e_{11}^{(n-1)} \\ e_{21}^{(n-1)} \\ \vdots \\ e_{n1}^{(n-1)} \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{n-1} \beta_k e_{11}^{(k)} \\ \sum_{k=0}^{n-1} \beta_k e_{21}^{(k)} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e_{n1}^{(k)} \end{pmatrix}.$$

Assume $[\overline{\delta}(a^i)]_B = \begin{pmatrix} \sum_{k=0}^{n-1} \beta_k e_{1i}^{(k)} \\ \sum_{k=0}^{n-1} \beta_k e_{2i}^{(k)} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e_{ni}^{(k)} \end{pmatrix}$ for $i \geq 1$.

By definition, $\begin{pmatrix} \sum_{k=0}^{n-1} \beta_k e_{1i}^{(k)} \\ \sum_{k=0}^{n-1} \beta_k e_{2i}^{(k)} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e_{ni}^{(k)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{k=0}^{n-1} \beta_k e_{ii}^{(k)} \\ \sum_{k=0}^{n-1} \beta_k e_{i+1,i}^{(k)} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e_{n,i}^{(k)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ i \cdot \beta_0 \\ \sum_{k=0}^{n-1} \beta_k e_{i+1,i}^{(k)} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e_{n,i}^{(k)} \end{pmatrix}.$

Since $\overline{\delta}$ is a derivation $\overline{\delta}(a^{i+1}) = \overline{\delta}(a \cdot a^i)$
$= \overline{\delta}(a) \cdot a^i + a \cdot \overline{\delta}(a^i)$
$= (\sum_{i=0}^{n-1} \beta_i a^{i+1}) \cdot a^i + a \cdot (i \cdot \beta_0 a^i + \sum_{k=0}^{n-1} \beta_k e_{i+1,i}^{(k)} a^{i+1} + \ldots + \sum_{k=0}^{n-1} \beta_k e_{ni}^{(k)} a^n)$
$= \beta_0 a^{i+1} + i \cdot \beta_0 a^{i+1} + \sum_{k=0}^{n-1} \beta_k e_{i+1,i}^{(k)} a^{i+2} + \ldots + \sum_{k=0}^{n-1} \beta_k e_{n-1,i}^{(k)} a^n$
$= (i+1) \cdot \beta_0 a^{i+1} + \sum_{k=0}^{n-1} \beta_k e_{i+1,i}^{(k)} a^{i+2} + \ldots + \sum_{k=0}^{n-1} \beta_k e_{n-1,i}^{(k)} a^n.$

This implies $[\overline{\delta}(a^{i+1})]_B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ (i+1)\beta_0 \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{i+1,i} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{n-1,i} \end{pmatrix}$.

Since $n-1 = i+k$ if and only if $n = (i+1)+k$ we can shift the indices to show:

$$[\overline{\delta}(a^{i+1})]_B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ (i+1)\beta_0 \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{i+1,i} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{n-1,i} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \sum_{k=0}^{n-1} \beta_k e_{i+1,i+1} \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{i+2,i+1} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{n,i+1} \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{n-1} \beta_k e^{(k)}_{1,i+1} \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{2,i+1} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{n,i+1} \end{pmatrix}.$$

Hence, $[\overline{\delta}(a^j)]_B = \begin{pmatrix} \sum_{k=0}^{n-1} \beta_k e^{(k)}_{1j} \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{2j} \\ \vdots \\ \sum_{k=0}^{n-1} \beta_k e^{(k)}_{nj} \end{pmatrix}$ for $1 \le j \le n$, which implies $[\overline{\delta}]_B = \sum_{k=0}^{n-1} \beta_k E^{(k)}$.

Therefore, the vectors $E^{(k)}$ form a basis for Der(A). $\square$

Each of the basis vectors $E^{(k)}$ corresponds to a representation matrix for a derivation in Der(A) with respect to the given basis B.

Corollary 1. Let A be a nilpotent cyclic Leibniz algebra with basis $B = \{a, a^2, \ldots, a^n\}$. Then Der$(A)$ =span$\{\delta_1, \delta_2, \ldots, \delta_n\}$, where

- $\delta_1(a^i) = ia^i$ for $1 \le i \le n$
- $\delta_k(a^i) = a^{i+k-1}$ if $i+k-1 \le n$ and $\delta_k(a^i) = 0$ if $i+k-1 > n$ for $1 < k \le n$.

3.3 Nonnilpotent Cyclic Leibniz Algebras

Consider the n-dimensional cyclic Leibniz algebra A with basis $\{a, a^2, \ldots, a^n\}$ and multiplication $a \cdot a^n = \alpha_1 a^2 + \ldots + \alpha_{n-1} a^n$, where at least one of the $\alpha_i \neq 0$ for $1 \leq i \leq n-1$. The terms in the derived series of A are: $A^{(1)} = \text{span}\{a^2, \ldots, a^n\}$, $A^{(2)} = 0$. Hence, A is solvable of derived length 2. However, A is nonnilpotent since $a \cdot a^n \neq 0$ implies there is not an ideal in the lower central series equal to 0.

Example 3.3.1. Let A be the cyclic Leibniz algebra with basis $B = \{a, a^2, a^3\}$ and multiplication $a \cdot a^3 = b_1 a^2 + b_2 a^3$ for $b_1, b_2 \in \mathbb{F}$. Let $\delta \in$Der(A) and define the action of δ on the basis elements as follows: $\delta(a) = \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3$, $\delta(a^2) = \beta_1 a + \beta_2 a^2 + \beta_3 a^3$, $\delta(a^3) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3$.

Therefore, $[\delta]_B = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix}$.

By the derivation property, $\delta(a \cdot a) = \delta(a)a + a\delta(a)$
$= (\alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3)a + a(\alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3)$
$= 2\alpha_1 a^2 + \alpha_2 a^3 + \alpha_3(b_1 a^2 + b_2 a^3) = (2\alpha_1 + \alpha_3 b_1)a^2 + (\alpha_2 + \alpha_3 b_2)a^3$.
And $\delta(a \cdot a) = \delta(a^2) = \beta_1 a + \beta_2 a^2 + \beta_3 a^3$.
Hence, by the linear independence of the basis vectors,
$\beta_1 = 0$, $\beta_2 = 2\alpha_1 + \alpha_3 b_1$, and $\beta_3 = \alpha_2 + \alpha_3 b_2$.
It is important to note that $\delta(a \cdot a^3) = \delta(a) \cdot a^3 + a \cdot \delta(a^3)$ and
$\delta(a \cdot a^3) = \delta(b_1 a^2 + b_2 a^3) = b_1 \delta(a^2) + b_2 \delta(a^3)$.
This equality yields the relationships $2\alpha_1 b_1 = 0$ and $\alpha_1 b_2 = 0$.
If $\alpha_1 \neq 0$, then $b_1 = b_2 = 0$, which implies A is nilpotent.
Thus, for the nonnilpotent case, $\alpha_1 = 0$.
Continuing this process for every combination of basis vectors yields:

$$[\delta]_B = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 & \alpha_3 b_1 & \alpha_2 b_1 + \alpha_3 b_2 b_1 \\ \alpha_3 & \alpha_2 + \alpha_3 b_2 & \alpha_2 b_2 + \alpha_3(b_1 + b_2^2) \end{pmatrix} = \alpha_2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & b_1 \\ 0 & 1 & b_2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_1 & b_2 b_1 \\ 1 & b_2 & b_1 + b_2^2 \end{pmatrix}.$$

Therefore, Der(A) =span$\{\delta_1, \delta_2\}$, where

- $\delta_1(a) = a^2, \delta_1(a^2) = a^3, \delta_1(a^3) = b_1 a^2 + b_2 a^3$
- $\delta_2(a) = a^3, \delta_2(a^2) = b_1 a^2 + b_2 a^3, \delta_2(a^3) = b_2 b_1 a^2 + (b_1 + b_2^2)a^3$.

Example 3.3.2. Let A be the cyclic Leibniz algebra with basis $B = \{a, a^2, a^3, a^4\}$ and multiplication $a \cdot a^4 = b_1a^2 + b_2a^3 + b_3a^4$ for $b_1, b_2, b_3 \in \mathbb{F}$. Let $\delta \in$Der(A) and define the action of δ on the basis elements as follows: $\delta(a) = \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 + \alpha_4 a^4$,
$\delta(a^2) = \beta_1 a + \beta_2 a^2 + \beta_3 a^3 + \beta_4 a^4$, $\delta(a^3) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + \gamma_4 a^4$,
$\delta(a^4) = \omega_1 a + \omega_2 a^2 + \omega_3 a^3 + \omega_4 a^4$.

Therefore, $[\delta]_B = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \omega_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \omega_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \omega_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \omega_4 \end{pmatrix}$.

By applying δ to the product of every combination of basis vectors, we can show:

$$\begin{aligned}
[\delta]_B &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \alpha_2 & \alpha_4 b_1 & \alpha_3 b_1 + \alpha_4 b_1 b_3 & \alpha_2 b_1 + \alpha_3 b_1 b_3 + \alpha_4(b_1 b_2 + b_1 b_3^2) \\ \alpha_3 & \alpha_2 + \alpha_4 b_2 & \alpha_3 b_2 + \alpha_4(b_1 + b_2 b_3) & \alpha_2 b_2 + \alpha_3(b_1 + b_2 b_3) + \alpha_4(b_2^2 + b_1 b_3 + b_2 b_3^2) \\ \alpha_4 & \alpha_3 + \alpha_4 b_3 & \alpha_2 + \alpha_3 b_3 + \alpha_4(b_2 + b_3^2) & \alpha_2 b_3 + \alpha_3(b_2 + b_3^2) + \alpha_4(b_1 + 2b_2 b_3 + b_3^3) \end{pmatrix} \\
&= \alpha_2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_1 b_3 \\ 1 & 0 & b_2 & b_1 + b_2 b_3 \\ 0 & 1 & b_3 & b_2 + b_3^2 \end{pmatrix} \\
&+ \alpha_4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & b_1 b_3 & b_1 b_2 + b_1 b_3^2 \\ 0 & b_2 & b_1 + b_2 b_3 & b_1 b_3 + b_2^2 + b_2 b_3^2 \\ 1 & b_3 & b_2 + b_3^2 & b_1 + 2b_2 b_3 + b_3^3 \end{pmatrix}.
\end{aligned}$$

Therefore, Der(A) =span$\{\delta_1, \delta_2, \delta_3\}$, where

- $\delta_1(a) = a^2, \delta_1(a^2) = a^3, \delta_1(a^3) = a^4, \delta_1(a^4) = b_1a^2 + b_2a^3 + b_3a^4$
- $\delta_2(a) = a^3, \delta_2(a^2) = a^4, \delta_2(a^3) = b_1a^2 + b_2a^3 + b_3a^4$,
  $\delta_2(a^4) = b_1b_3a^2 + (b_1 + b_2b_3)a^3 + (b_2 + b_3^2)a^4$
- $\delta_3(a) = a^4, \delta_3(a^2) = b_1a^2 + b_2a^3 + b_3a^4$,
  $\delta_3(a^3) = b_1b_3a^2 + (b_1 + b_2b_3)a^3 + (b_2 + b_3^2)a^4$,
  $\delta_3(a^4) = (b_1b_2 + b_1b_3^2)a^2 + (b_1b_3 + b_2^2 + b_2b_3^2)a^3 + (b_1 + 2b_2b_3 + b_3^3)a^4$

By completing enough examples of derivations of low-dimensional nonnilpotent cyclic Leibniz algbebras, we establish a pattern among the basis vectors for the derivation algebras. We generalize the result to define the derivation algebra of any n-dimensional nonnilpotent cyclic Leibniz algebra. In particular, it is important to note from the previous example that:

$$[\delta_1]_B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_1b_3 \\ 1 & 0 & b_2 & b_1 + b_2b_3 \\ 0 & 1 & b_3 & b_2 + b_3^2 \end{pmatrix} = [\delta_2]_B$$

and

$$[\delta_1]_B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & b_1b_3 & b_1b_2 + b_1b_3^2 \\ 0 & b_2 & b_1 + b_2b_3 & b_1b_3 + b_2^2 + b_2b_3^2 \\ 1 & b_3 & b_2 + b_3^2 & b_1 + 2b_2b_3 + b_3^3 \end{pmatrix} = [\delta_3]_B.$$

Theorem 2. Let A be a cyclic Leibniz algebra with basis $B = \{a, a^2, \dots, a^n\}$ and let $a \cdot a^n = \sum_{i=1}^{n-1} b_i a^{i+1}$, where at least one of the $b_i \neq 0$. Let $\delta \in$Der(A), and let $(d_{ij})_{n\times n}$ be the representation matrix of δ with respect to the basis B. Then $(d_{ij})_{n\times n} = \sum_{k=0}^{n-1} \alpha_k E^{(k)}$ where

- $E^{(1)} = (e_{ij}^{(1)}) = \begin{cases} 1 & i = j+1 \\ b_{i-1} & j = n \\ 0 & \text{otherwise} \end{cases}$

- $E^{(k)} = (e_{ij}^{(k)}) = \begin{cases} (e_{i,j+1}^{(k-1)}) & 1 \leq j \leq n-1 \\ \sum_{l=1}^{n-1} b_l e_{i,l}^{(k)} & j = n \\ 0 & \text{otherwise} \end{cases} = (E^{(1)})^k,\ 2 \leq k \leq n-1.$

Proof. Let $B = \{a, a^2, \dots, a^n\}$ be a basis for cyclic Leibniz algebra A, and let $a \cdot a^n = \sum_{i=1}^{n-1} b_i a^{i+1}$, where at least one of the $b_i \neq 0$. Let $\delta \in$Der(A), and $[\delta]_B = (d_{ij})_{n\times n}$.
To check for linear independence consider $\sum_{k=1}^{n-1} \alpha_k E^{(k)} = 0$.
Column one in $E^{(k)}$ is a column of zeros with a 1 in the $a_{k+1,1}$ entry for $1 \leq k \leq n-1$.

Thus, column one in the matrix $\sum_{k=1}^{n-1} \alpha_k E^{(k)}$ is $\begin{pmatrix} 0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$.

Therefore, $\begin{pmatrix} 0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ implies $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0$.

Hence, the vectors $E^{(k)}$ are linearly independent. Now we need to show spanning.
Let $\overline{\delta}$ be any derivation in $\mathrm{Der}(A)$. We need to show $[\overline{\delta}]_B = (\overline{d}_{ij})_{n\times n}$ can be written as a linear combination of the vectors $E^{(k)}$, $1 \le k \le n-1$.
$\overline{\delta}(a)$ is a linear combination of the basis elements of A, so let $\overline{\delta}(a) = \sum_{i=1}^{n} \beta_i a^i$.

Thus, column one in the matrix $[\overline{\delta}]_B$ is $\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix}$. Which implies

$$[\overline{\delta}(a)]_B = \beta_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \beta_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$= \beta_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} e_{11}^{(1)} \\ e_{21}^{(1)} \\ \vdots \\ e_{n1}^{(1)} \end{pmatrix} + \ldots + \beta_n \begin{pmatrix} e_{11}^{(n-1)} \\ e_{21}^{(n-1)} \\ \vdots \\ e_{n1}^{(n-1)} \end{pmatrix} = \begin{pmatrix} \beta_1 + \sum_{j=1}^{n-1} \beta_{j+1} e_{11}^{(j)} \\ \sum_{j=1}^{n-1} \beta_{j+1} e_{21}^{(j)} \\ \vdots \\ \sum_{j=1}^{n-1} \beta_{j+1} e_{n1}^{(j)} \end{pmatrix}.$$

Since $\overline{\delta}$ is a derivation $\overline{\delta}(a^2) = \overline{\delta}(a \cdot a) = \overline{\delta}(a) \cdot a + a \cdot \overline{\delta}(a)$
$= (\sum_{i=1}^{n} \beta_i a^i) \cdot a + a \cdot (\sum_{i=1}^{n} \beta_i a^i)$
$= \beta_1 a^2 + \sum_{i=1}^{n-1} \beta_i a^{i+1} + \beta_n \sum_{i=1}^{n-1} b_i a^{i+1}$
$= (2\beta_1 + \beta_n b_1)a^2 + (\beta_2 + \beta_n b_2)a^3 + \ldots + (\beta_{n-1} + \beta_n b_{n-1})a^n$

This implies

$$[\overline{\delta}(a^2)]_B = \begin{pmatrix} 0 \\ 2\beta_1 + \beta_n b_1 \\ \beta_2 + \beta_n b_2 \\ \vdots \\ \beta_{n-1} + \beta_n b_{n-1} \end{pmatrix}$$

$$= \beta_1 \begin{pmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \beta_{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} + \beta_n \begin{pmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix}$$

$$= \beta_1 \begin{pmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} e_{12}^{(1)} \\ e_{22}^{(1)} \\ e_{32}^{(1)} \\ \vdots \\ e_{n2}^{(1)} \end{pmatrix} + \ldots + \beta_n \begin{pmatrix} e_{12}^{(n-1)} \\ e_{22}^{(n-1)} \\ \vdots \\ e_{n2}^{(n-1)} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n-1} \beta_{j+1} e_{12}^{(j)} \\ 2\beta_1 + \sum_{j=1}^{n-1} \beta_{j+1} e_{22}^{(j)} \\ \vdots \\ \sum_{j=1}^{n-1} \beta_{j+1} e_{n2}^{(j)} \end{pmatrix}.$$

Assume $[\overline{\delta}(a^i)]_B = \begin{pmatrix} \sum_{j=1}^{n-1} \beta_{j+1} e_{1i}^{(j)} \\ \sum_{j=1}^{n-1} \beta_{j+1} e_{2i}^{(j)} \\ \vdots \\ i\beta_1 + \sum_{j=1}^{n-1} \beta_{j+1} e_{ii}^{(j)} \\ \vdots \\ \sum_{j=1}^{n-1} \beta_{j+1} e_{ni}^{(j)} \end{pmatrix}$ for $2 \leq i \leq n$.

Then, by the derivation property, $\overline{\delta}(a^{i+1}) = \overline{\delta}(a \cdot a^i) = \overline{\delta}(a) \cdot a^i + a \cdot \overline{\delta}(a^i)$

$= (\sum_{i=1}^{n} \beta_i a^i) \cdot a^i + a(\sum_{j=1}^{n-1} \beta_{j+1} e_{1i}^{(j)} a + \sum_{j=1}^{n-1} \beta_{j+1} e_{2i}^{(j)} a^2 + \ldots + \sum_{j=1}^{n-1} \beta_{j+1} e_{ni}^{(j)} a^n)$

$= \beta_1 a^{i+1} + \sum_{j=1}^{n-1} \beta_{j+1} e_{1i}^{(j)} a^2 + \sum_{j=1}^{n-1} \beta_{j+1} e_{2i}^{(j)} a^3 + \ldots$

$+ \sum_{j=1}^{n-1} \beta_{j+1} e_{n-1,i}^{(j)} a^n + (\sum_{j=1}^{n-1} \beta_{j+1} e_{ni}^{(j)})(\sum_{i=1}^{n} b_i a^{i+1})$

$=(\sum_{j=1}^{n-1}\beta_{j+1}e_{1i}^{(j)}+b_1\sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)})a^2+(\sum_{j=1}^{n-1}\beta_{j+1}e_{2i}^{(j)}+b_2\sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)})a^3+\ldots+(\beta_1+i\beta_1+\sum_{j=1}^{n-1}\beta_{j+1}e_{ii}^{(j)}+b_i\sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)})a^{i+1}+\ldots+(\sum_{j=1}^{n-1}\beta_{j+1}e_{n-1,i}^{(j)}+b_{n-1}\sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)})a^n.$

Since $e_{1,i+1}^{(j)}=0$ for $1\le j\le n-1$, and since $e_{k,i+1}^{(j)}=e_{k-1,i}+b_{k-1}e_{n,i}$ for $2\le k\le n$ and $1\le j\le n-1$, we have:

$$[\overline{\delta}(a^{i+1})]_B=\begin{pmatrix} 0 \\ \sum_{j=1}^{n-1}\beta_{j+1}e_{1i}^{(j)}+b_1\sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)} \\ \sum_{j=1}^{n-1}\beta_{j+1}e_{2i}^{(j)}+b_2\sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)} \\ \vdots \\ (i+1)\beta_1+\sum_{j=1}^{n-1}\beta_{j+1}e_{ii}^{(j)}+b_i\sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)} \\ \vdots \\ \sum_{j=1}^{n-1}\beta_{j+1}e_{n-1,i}^{(j)}+b_{n-1}\sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)} \end{pmatrix}$$
$$=\begin{pmatrix} \sum_{j=1}^{n-1}\beta_{j+1}e_{1,i+1}^{(j)} \\ \sum_{j=1}^{n-1}\beta_{j+1}e_{2,i+1}^{(j)} \\ \sum_{j=1}^{n-1}\beta_{j+1}e_{3,i+1}^{(j)} \\ \vdots \\ (i+1)\beta_1+\sum_{j=1}^{n-1}\beta_{j+1}e_{i,i+1}^{(j)} \\ \vdots \\ \sum_{j=1}^{n-1}\beta_{j+1}e_{n,i+1}^{(j)} \end{pmatrix}$$

Therefore, by induction, $[\overline{\delta}(a^i)]_B=\begin{pmatrix} \sum_{j=1}^{n-1}\beta_{j+1}e_{1i}^{(j)} \\ \sum_{j=1}^{n-1}\beta_{j+1}e_{2i}^{(j)} \\ \vdots \\ i\beta_1+\sum_{j=1}^{n-1}\beta_{j+1}e_{ii}^{(j)} \\ \vdots \\ \sum_{j=1}^{n-1}\beta_{j+1}e_{ni}^{(j)} \end{pmatrix}$ for all $2\le i\le n$.

Now consider $\overline{\delta}(a\cdot a^n)$.
$\overline{\delta}(a\cdot a^n)=\overline{\delta}(a)\cdot a^n+a\cdot\overline{\delta}(a^n)$
$=(\sum_{i=1}^{n}\beta_i a^i)\cdot a^n+a\cdot(\sum_{j=1}^{n-1}\beta_{j+1}e_{1n}^{(j)}a+\sum_{j=1}^{n-1}a\beta_{j+1}e_{2n}^{(j)}a^2+\ldots+(n\beta_1+\sum_{j=1}^{n-1}\beta_{j+1}e_{nn}^{(j)})a^n)$

$= \beta_1(\sum_{i=1}^{n-1} b_i a^{i+1}) + \sum_{j=1}^{n-1} \beta_{j+1} e_{1n}^{(j)} a^2 + \sum_{j=1}^{n-1} \beta_{j+1} e_{2n}^{(j)} a^3 + \ldots + \sum_{j=1}^{n-1} \beta_{j+1} e_{n-1,n}^{(j)} a^n + (n\beta_1 + \sum_{j=1}^{n-1} \beta_{j+1} e_{nn}^{(j)})(\sum_{i=1}^{n-1} b_i a^{i+1})$

$= [(n+1)\beta_1 b_1 + \sum_{j=1}^{n-1} \beta_{j+1}(e_{1n}^{(j)} + b_1 e_{nn}^{(j)})]a^2 + [(n+1)\beta_1 b_2 + \sum_{j=1}^{n-1} \beta_{j+1}(e_{2n}^{(j)} + b_2 e_{nn}^{(j)})]a^3 + \ldots + [(n+1)\beta_1 b_{n-1} + \sum_{j=1}^{n-1} \beta_{j+1}(e_{n-1,n}^{(j)} + b_{n-1} e_{nn}^{(j)})]a^n$

Therefore, the coefficient of a^{i+1} can be written:
$(n+1)\beta_1 b_i + \sum_{j=1}^{n-1} \beta_{j+1}(e_{in}^{(j)} + b_i e_{nn}^{(j)})$
$= (n+1)\beta_1 b_i + \sum_{j=1}^{n-1} \beta_{j+1}(\sum_{l=1}^{n-1} b_l e_{i,l}^{(j)}) + \sum_{j=1}^{n-1} \beta_{j+1}(\sum_{l=1}^{n-1} b_i b_l e_{n,l}^{(j)})$
$= (n+1)\beta_1 b_i + b_1[\sum_{j=1}^{n-1} \beta_{j+1} e_{i1}^{(j)} + b_i \sum_{j=1}^{n-1} \beta_{j+1} e_{n1}^{(j)}] + b_2[\sum_{j=1}^{n-1} \beta_{j+1} e_{i2}^{(j)} + b_i \sum_{j=1}^{n-1} \beta_{j+1} e_{n2}^{(j)}] + \ldots + b_{n-1}[\sum_{j=1}^{n-1} \beta_{j+1} e_{i,n-1}^{(j)} + b_i \sum_{j=1}^{n-1} \beta_{j+1} e_{n,n-1}^{(j)}]$
$= (n+1)\beta_1 b_i + b_1 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,2}^{(j)} + b_2 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,3}^{(j)} + \ldots + b_{n-1} \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,n}^{(j)}$

Also, $\overline{\delta}(a \cdot a^n) = \overline{\delta}(\sum_{i=1}^{n-1} b_i a^{i+1}) = \sum_{i=1}^{n-1} b_i \overline{\delta}(a^{i+1})$.
Collecting the coefficients of a^{i+1} in $\sum_{i=1}^{n-1} b_i \overline{\delta}(a^{i+1})$ we have:
$b_1 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,2}^{(j)} + b_2 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,3}^{(j)} + \ldots + b_i[(i+1)\beta_1 + \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,i+1}^{(j)}] + \ldots + b_{n-1} \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,n}^{(j)}$
$= (i+1)\beta_1 b_i + b_1 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,2}^{(j)} + b_2 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,3}^{(j)} + \ldots + b_{n-1} \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,n}^{(j)}$

Since the a^i are linearly independent for $1 \leq i \leq n$, setting the two equations for $\overline{\delta}(a \cdot a^n)$ equal and collecting the coefficients of a^{i+1} on both sides of the equation yields:
$(n+1)\beta_1 b_i + b_1 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,2}^{(j)} + b_2 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,3}^{(j)} + \ldots + b_{n-1} \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,n}^{(j)}$
$= (i+1)\beta_1 b_i + b_1 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,2}^{(j)} + b_2 \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,3}^{(j)} + \ldots + b_{n-1} \sum_{j=1}^{n-1} \beta_{j+1} e_{i+1,n}^{(j)}$.
Hence, $(n-i)\beta_1 b_i = 0$.
Since $1 \leq i \leq n-1$, $(n-i) \neq 0$. Therefore, $\beta_1 = 0$ or $b_i = 0$ for all i. But we assumed that at least one of the $b_i \neq 0$. Hence, $\beta_1 = 0$.

Thus, $[\overline{\delta}(a^k)]_B = \begin{pmatrix} \sum_{j=1}^{n-1} \beta_{j+1} e_{1k}^{(j)} \\ \sum_{j=1}^{n-1} \beta_{j+1} e_{2k}^{(j)} \\ \vdots \\ \sum_{j=1}^{n-1} \beta_{j+1} e_{nk}^{(j)} \end{pmatrix}$ for $1 \leq k \leq n$.
Which implies $[\overline{\delta}]_B = \left(\; [\overline{\delta}(a)]_B | [\overline{\delta}(a^2)]_B | \ldots | [\overline{\delta}(a^n)]_B \; \right)_{n \times n} = \sum_{k=1}^{n-1} \beta_{k+1} E^{(k)}$.

$\square$

As was true for the nilpotent case, each of the basis vectors $E^{(k)}$ corresponds to a representation matrix for a derivation in $\mathrm{Der}(A)$ with respect to the given basis B.

Corollary 2. Let A be a cyclic Leibniz algebra with basis $B = \{a, a^2, \ldots, a^n\}$, and let $a \cdot a^n = \sum_{i=1}^{n-1} b_i a^{i+1}$, where at least one of the $b_i \neq 0$. Then $\mathrm{Der}(A)$ =span$\{\delta_1, \delta_2, \ldots, \delta_{n-1}\}$, where

- $\delta_1(a^i) = a^{i+1}$ for $1 \leq i \leq n-1$, $\delta_1(a^n) = \sum_{j=1}^{n-1} b_j a^{j+1}$
- $\delta_k(a^i) = \delta_1^k(a^i)$ for $1 \leq i \leq n$ and $1 < k \leq n-1$.

Chapter 4

Characteristic Ideals of Leibniz Algebras

Definition 1. Let A be a Leibniz algebra. An ideal B of A is called *characteristic* if it is invariant under all derivations of A.

Example 4.0.1. The derived algebra of A is characteristic.
Let $a, b \in A$ and let $\delta \in$Der(A). $\delta(ab) = \delta(a)b + a\delta(b) \in A^2$. This implies $\delta(A^2) \subseteq A^2$. Thus, A^2 is characteristic.

Using induction, we can show that every ideal in the lower central series and in the derived series of A is characteristic.

Example 4.0.2. $Z^l(A)$ is a characteristic ideal of A.
Let $\delta \in$Der(A), and let $x \in Z^l(A)$, $y \in A$. Then $xy = 0$ for every $y \in A$, which implies $\delta(xy) = \delta(x)y + x\delta(y) = \delta(x)y = 0$. Therefore, $\delta(x) \in Z^l(A)$; and hence, $\delta(Z^l(A)) \subseteq Z^l(A)$. Thus, $Z^l(A)$ is characteristic.

Lemma 1. The sum of characteristic ideals is characteristic.

Proof. Let $B_1, B_2, \ldots, B_n$ be characteristic ideals of a Leibniz algebra A. Let $\delta \in$Der(A). Then $\delta(B_1 + B_2 + \ldots + B_n) = \delta(B_1) + \delta(B_2) + \ldots + \delta(B_n) \subseteq B_1 + B_2 + \ldots + B_n$, since δ is a linear operator. □

4.1 Properties of Derivations

In this section we study properties of derivations and ideals. These results will be useful in determining under which conditions an ideal is characteristic.

Lemma 2. Let A be a Leibniz algebra over a field $\mathbb{F}$, let I be any ideal of A, and let $\delta \in$Der(A). Then $A \cdot \delta^n(I) \subseteq I + \delta(I) + \ldots + \delta^n(I)$ and $\delta^n(I) \cdot A \subseteq I + \delta(I) + \ldots + \delta^n(I)$ for all $n \geq 1$.

Proof. Let $x \in A$ and $y \in \delta(I)$. This implies $y = \delta(z)$ for some $z \in I$. Then

$$xy = x\delta(z) = \delta(\underbrace{xz}_{\in I}) - \underbrace{\delta(x)z}_{\in I} \in I + \delta(I) \text{ and}$$
$$yx = \delta(z)x = \delta(\underbrace{zx}_{\in I}) - \underbrace{z\delta(x)}_{\in I} \in I + \delta(I).$$

Hence, $A \cdot \delta(I) \subseteq I + \delta(I)$ and $\delta(I) \cdot A \subseteq I + \delta(I)$.
Now assume $A \cdot \delta^{n-1}(I) \subseteq I + \delta(I) + \ldots + \delta^{n-1}(I)$ and $\delta^{n-1}(I) \cdot A \subseteq I + \delta(I) + \ldots + \delta^{n-1}(I)$ for $n - 1 \geq 1$. Let $x \in A$ and let $y \in \delta^n(I) = \delta(\delta^{n-1}(I))$. This implies $y = \delta(z)$ for some $z \in \delta^{n-1}(I)$. Then

$$\begin{aligned} xy &= x\delta(z) = \delta(xz) - \delta(x)z \\ &\in \delta(I + \delta(I) + \ldots + \delta^{n-1}(I)) + I + \delta(I) + \ldots + \delta^{n-1}(I) \\ &\subseteq I + \delta(I) + \ldots + \delta^{n-1}(I) + \delta^n(I). \end{aligned}$$

And

$$\begin{aligned} yx &= \delta(z)x = \delta(zx) - z\delta(x) \\ &\in \delta(I + \delta(I) + \ldots + \delta^{n-1}(I)) + I + \delta(I) + \ldots + \delta^{n-1}(I) \\ &\subseteq I + \delta(I) + \ldots + \delta^{n-1}(I) + \delta^n(I). \end{aligned}$$

Therefore, by induction, $A \cdot \delta^n(I) \subseteq I + \delta(I) + \ldots + \delta^n(I)$ and $\delta^n(I) \cdot A \subseteq I + \delta(I) + \ldots + \delta^n(I)$ for all $n \geq 1$. $\square$

Lemma 3. Let A be a Leibniz algebra over a field $\mathbb{F}$, and let $\delta \in$Der(A). Then for any ideal I of A, the $\mathbb{F}$-subspace $I + \delta(I) + \ldots + \delta^n(I)$ is an ideal of A for all $n \geq 1$.

Proof. Let $x \in A$ and let $y \in I + \delta(A)$. This implies $y = z + \delta(w)$ for some $z, w \in I$. Then

$$\begin{aligned} xy &= x(z + \delta(w)) = xz + x\delta(w) \\ &= \underbrace{xz}_{\in I} + \delta(\underbrace{xw}_{\in I}) - \underbrace{\delta(x)w}_{\in I} \in I + \delta(I) \text{ and} \\ yx &= (z + \delta(w))x = zx + \delta(w)x \\ &= \underbrace{zx}_{\in I} + \delta(\underbrace{wx}_{\in I}) - \underbrace{w\delta(x)}_{\in I} \in I + \delta(I). \end{aligned}$$

Thus, $I + \delta(I)$ is an ideal of A. Assume $I + \delta(I) + \ldots + \delta^{n-1}(I)$ is an ideal for $n - 1 \geq 1$. Note $A \cdot (I + \delta(I) + \ldots + \delta^n(I)) = A \cdot (I + \delta(I) + \ldots + \delta^{n-1}(I)) + A \cdot \delta^n(I)$.
Also, by the induction hypothesis, $A \cdot (I + \delta(I) + \ldots + \delta^{n-1}(I)) \subseteq I + \delta(I) + \ldots + \delta^{n-1}(I)$, and by Lemma 2, $A \cdot \delta^n(I) \subseteq I + \delta(I) + \ldots + \delta^n(I)$.
Therefore, $A \cdot (I + \delta(I) + \ldots + \delta^n(I)) \subseteq I + \delta(I) + \ldots + \delta^n(I)$.
Similarly,

$$\begin{aligned} (I + \delta(I) + \ldots + \delta^n(I)) \cdot A &= (I + \delta(I) + \ldots + \delta^{n-1}(I)) \cdot A + \delta^n(I) \cdot A \\ &\subseteq (I + \delta(I) + \ldots + \delta^{n-1}(I)) + (I + \delta(I) + \ldots + \delta^n(I)) \\ &\subseteq I + \delta(I) + \ldots + \delta^n(I). \end{aligned}$$

Thus, $I + \delta(I) + \ldots + \delta^n(I)$ is an ideal of A for all $n \geq 1$. $\square$

Lemma 4. Let A be a Leibniz algebra over a field $\mathbb{F}$, and let $\delta \in$Der(A). Then for every $x, y \in A$ the general Leibniz product rule holds:

$$\delta^k(xy) = \textstyle\sum_{s=0}^{k} \binom{k}{s} \delta^s(x)\delta^{k-s}(y).$$

Proof. Let $x, y \in A$. Then

$$\begin{aligned} \delta(xy) &= x\delta(y) + \delta(x)y \\ &= \binom{1}{0}\delta^0(x)\delta^{1-0}(y) + \binom{1}{1}\delta^1(x)\delta^{1-1}(y) \\ &= \sum_{s=0}^{1} \binom{1}{s}\delta^s(x)\delta^{1-s}(y). \end{aligned}$$

Thus, the product rule holds when $k = 1$. Now assume:

$$\delta^{k-1}(xy) = \sum_{s=0}^{k-1} \binom{k-1}{s} \delta^s(x)\delta^{k-1-s}(y).$$

Then

$$\begin{aligned}
\delta^k(xy) = \delta(\delta^{k-1}(xy)) = \delta(\sum_{s=0}^{k-1} \binom{k-1}{s} \delta^s(x)\delta^{k-1-s}(y))& \\
= \sum_{s=0}^{k-1} \binom{k-1}{s} \delta(\delta^s(x)\delta^{k-1-s}(y))& \\
= \sum_{s=0}^{k-1} \binom{k-1}{s} (\delta^{s+1}(x)\delta^{k-1-s}(y) + \delta^s(x)\delta^{k-s}(y))& \\
= \sum_{s=0}^{k-1} \binom{k-1}{s} \delta^{s+1}(x)\delta^{k-1-s}(y) + \sum_{s=0}^{k-1} \binom{k-1}{s} \delta^s(x)\delta^{k-s}(y)&
\end{aligned}$$

Taking the $s = k-1$ term from the first summand and the $s = 0$ term from the second summand yields:

$$= \delta^k(x)y + x\delta^k(y) + \sum_{s=0}^{k-2} \binom{k-1}{s} \delta^{s+1}(x)\delta^{k-1-s} + \sum_{s=1}^{k-1} \binom{k-1}{s} \delta^s(x)\delta^{k-s}(y)$$

Re-indexing the first summand yields:

$$= \delta^k(x)y + x\delta^k(y) + \sum_{s=1}^{k-1} \binom{k-1}{s-1} \delta^s(x)\delta^{k-s} + \sum_{s=1}^{k-1} \binom{k-1}{s} \delta^s(x)\delta^{k-s}(y)$$

Recall $\binom{k-1}{s-1} + \binom{k-1}{s} = \binom{k}{s}$, which yields:

$$= \delta^k(x)y + x\delta^k(y) + \sum_{s=1}^{k-1} \binom{k}{s} \delta^s(x)\delta^{k-s}(y) = \sum_{s=0}^{k} \binom{k}{s} \delta^s(x)\delta^{k-s}(y).$$

Therefore, Leibniz's product rule holds for all $n \geq 1$.

$\square$

The previous result can be generalized to Leibniz's rule for differentiation of several multipliers:

$$\delta(x_1 \cdot x_2 \cdots x_m) = \sum_{k_1+k_2+\ldots+k_m=n} \frac{n!}{k_1!k_2!\ldots k_m!}\delta^{k_1}(x_1)\cdot\delta^{k_2}(x_2)\cdots\delta^{k_m}(x_m).$$

Now we consider the action of derivations on ideals from the derived series of a Leibniz algebra A.

Lemma 5. Let A be a Leibniz algebra over a field $\mathbb{F}$, let I be any ideal of A, and let $\delta \in$Der(A). Then $\delta^m(I^{(s)}) \subseteq I$ for all $m \leq 2^s - 1$.

Proof. First consider $s = 1$. Then $m \leq 2^1 - 1 = 1$, implies $m = 1$. And

$$\delta(I^{(1)}) = \delta(I^2) \subseteq \delta(I)\cdot I + I \cdot \delta(I) \subseteq I,$$

since I is an ideal of A. Thus, the base case holds. Assume $\delta^m(I^{(s-1)}) \subseteq I$ for all $m \leq 2^{s-1} - 1$. Take an arbitrary $m \leq 2^s - 1$, and consider $\delta^m(I^{(s)}) = \delta^m(I^{(s-1)} \cdot I^{(s-1)})$. For any elements $x, y \in I^{(s-1)}$, by Lemma 4,

$$\delta^m(xy) = \sum_{i=0}^{m} \binom{m}{i} \delta^i(x)\delta^{m-i}(y).$$

Since $i+(m-i) = m \leq 2^s - 1$, and since $\frac{1}{2}(2^s-1) = 2^{s-1} - \frac{1}{2}$ at least one of the numbers i or $m - i$ does not exceed $2^{s-1} - 1$, since $i, m \in \mathbb{Z}$. Thus, by the induction hypothesis, either $\delta^i(x)$ or $\delta^{m-i}(y)$ is an element of I. Hence, $\delta^i(x)\delta^{m-i}(y) \in I$ for each $1 \leq i \leq m$. Therefore, $\delta^m(I^{(s)}) \subseteq I$ for all $m \leq 2^s - 1$. □

Lemma 6. Let A be a Leibniz algebra over a field $\mathbb{F}$, let I be any ideal of A, and let $\delta \in$Der(A). If char$(\mathbb{F})$ does not divide the binomial coefficient $\binom{2^k}{2^{k-1}}$, then $\delta^{2^{k-1}}(I^{(k-1)}) \cdot \delta^{2^{k-1}}(I^{(k-1)}) \subseteq \delta^{2^k}(I^{(k)}) + I$.

Proof. Let $x, y \in I^{(k-1)}$. By Lemma 4, $\delta^{2^k}(xy) = \sum_{s=0}^{2^k} \binom{2^k}{s} \delta^s(x)\delta^{2^k-s}(y)$.
Taking the $s = 2^{k-1}$ term from the summand yields:

$$\delta^{2^k}(xy) = \binom{2^k}{2^{k-1}} \delta^{2^{k-1}}(x)\delta^{2^k-2^{k-1}}(y) + \sum_{s=0;s\neq 2^{k-1}}^{2^k} \binom{2^k}{s} \delta^s(x)\delta^{2^k-s}(y)$$

Since $2^k - 2^{k-1} = 2^k(1-2^{-1}) = 2^k(\frac{1}{2}) = 2^{k-1}$, we can simplify the exponent in the first summand to get:

$$\delta^{2^k}(xy) = \binom{2^k}{2^{k-1}}\delta^{2^{k-1}}(x)\delta^{2^{k-1}}(y) + \sum_{s=0;s\neq 2^{k-1}}^{2^k}\binom{2^k}{s}\delta^s(x)\delta^{2^k-s}(y).$$

Now, in the second summand, $s + (2^k - s) = 2^k$ and $\frac{1}{2}(2^k) = 2^{k-1}$, but $s \neq 2^{k-1}$; hence, either the number s or the number $2^k - s$ does not exceed $2^{k-1} - 1$.
Therefore, by Lemma 5, the second summand is in I.
Also, since $\mathrm{char}(\mathbb{F}) \nmid \binom{2^k}{2^{k-1}}$, the first summand is not 0 and $\binom{2^k}{2^{k-1}}$ has a multiplicative inverse in $\mathbb{F}$. Thus,

$$\binom{2^k}{2^{k-1}}\delta^{2^{k-1}}(x)\delta^{2^{k-1}}(y) = \delta^{2^k}(xy) - \sum_{s=0;s\neq 2^{k-1}}^{2^k}\binom{2^k}{s}\delta^s(x)\delta^{2^k-s}(y),$$

which implies

$$\delta^{2^{k-1}}(I^{(k-1)})\delta^{2^{k-1}}(I^{(k-1)}) \subseteq \delta^{2^k}(I^{(k)}) + I.$$

□

Lemma 7. Let A be a Leibniz algebra over a field of characteristic 0 or characteristic $p \neq 2$, let I be an abelian ideal of A, and let $\delta \in$Der(A). Then $\delta(I)\cdot\delta(I) \subseteq I$.

Proof. Let $x, y \in I$. I abelian implies $xy = 0$. Thus, $\delta^2(xy) = 0$. This implies

$$0 = \delta^2(xy) = \delta^2(x)y + 2\delta(x)\delta(y) + x\delta^2(y).$$

Thus,

$$2\delta(x)\delta(y) = -\delta^2(x)y - x\delta^2(y) \in I.$$

Since the characteristic of the field is not 2, and since x and y are arbitrary elements from I, this implies $\delta(I)\cdot\delta(I) \subseteq I$. □

Now we consider the action of derivations on ideals from the lower central series of a Leibniz algebra A.

Lemma 8. Let A be a Leibniz algebra over a field $\mathbb{F}$, let I be any ideal of A, and let $\delta \in$Der(A). Then for any $x_1, \dots, x_s \in I$ and for any nonnegative number $m < s$:

$$\delta^m(x_1 \cdots x_s) \in I^{\mathbf{s\text{-}m}}.$$

Proof. Denote $l = s - m > 0$. By Leibniz's rule for differentiation of several multipliers,

$$\delta^m(x_1 \cdots x_s) = \sum_{k_1+\ldots+k_s=m} \frac{m!}{k_1! \ldots k_s!} \delta^{k_1}(x_1) \cdots \delta^{k_s}(x_s).$$

Since $m < s$, and since each k_i is nonnegative, there must be at least l of the $k_i = 0, 1 \leq i \leq s$. Recall $\delta^0(x_i) = x_i$; thus, each term of the sum $\delta^{k_1}(x_1)\cdots\delta^{k_s}(x_s)$ contains at least l elements of I and m elements of A. Therefore, $\delta^m(x_1 \cdots x_s) \in I^{\mathbf{l}} = I^{\mathbf{s\text{-}m}}$. $\square$

Lemma 9. Let I be a nilpotent ideal with nilpotency class n of a Leibniz algebra A over a field of characteristic 0 or characteristic $p > n + 1$, and let $\delta \in$ Der(A).
Then $(I + \delta(I))^{\mathbf{n+1}} \subseteq I$.

Proof. Any term of $(I+\delta(I))^{\mathbf{n+1}}$ with elements from I will be contained in I by definition. So it is enough to show $\underbrace{\delta(I) \cdots \delta(I)}_{n+1} \subseteq I$. Since the nilpotency class of I is n,
$x_1 \cdots x_n \cdot x_{n+1} = 0$ for all elements $x_1, \dots, x_n, x_{n+1} \in I$.
By Leibniz's rule for differentiation of several multipliers,

$$\delta^{n+1}(x_1 \cdots x_{n+1}) = \sum_{k_1+\ldots+k_{n+1}=n+1} \frac{(n+1)!}{k_1! \ldots k_{n+1}!} \delta^{k_1}(x_1) \cdots \delta^{k_{n+1}}(x_{n+1}) = 0.$$

The sum of $n+1$ nonnegative integers to $n+1$ can be split into two cases: the case where each $k_i = 1$, and the case where at least one of the $k_i > 1$. In the second case, this leaves at least 1 of the $k_i = 0$. Thus,

$$\delta^{n+1}(x_1 \cdots x_{n+1}) =$$
$$\frac{(n+1)!}{1! \ldots 1!} \delta(x_1) \cdots \delta(x_{n+1}) + \sum_{k_1+\ldots+k_{n+1}=n+1} \frac{(n+1)!}{k_1! \ldots k_{n+1}!} \delta^{k_1}(x_1) \cdots \delta^{k_{n+1}}(x_{n+1}) = 0.$$

and in each summand there is at least one $\delta^{k_i}(x_i) = \delta^0(x_i) = x_i$. Hence, each summand belongs to I. Therefore,

$$\frac{(n+1)!}{1!\ldots 1!}\delta(x_1)\cdots\delta(x_{n+1}) = -\sum_{k_1+\ldots+k_{n+1}=n+1}\frac{(n+1)!}{k_1!\ldots k_{n+1}!}\delta^{k_1}(x_1)\cdots\delta^{k_{n+1}}(x_{n+1})\in I.$$

Since the characteristic of the field is zero or greater than $n+1$, $\delta(x_1)\cdots\delta(x_{n+1})\in I$. $\square$

Lemma 10. Let I be a nilpotent ideal of nilpotency class n from a Leibniz algebra A over a field of characteristic 0 or characteristic $p > n+1$, and let $\delta \in$Der(A).
Then $I\cdot\underbrace{\delta(I)\cdots\delta(I)}_{n+1}\subseteq I^2$.

Proof. Let $x_1,\ldots,x_{n+2}$ be arbitrary elements from I. For convenience, denote

$$t_i = \delta(x_1)\cdots x_i\cdots\delta(x_{n+2}) \text{ and}$$
$$u_s = x_1\cdot x_2\cdots\delta(x_s)\cdots x_{n+2}$$

for $1\le i,s\le n+2$. Note $u_s\in I^{n+1}=0$ implies $u_s=0$. Therefore,

$$\begin{aligned}0 &= \delta^n(u_s) = \delta^n(x_1\cdot x_2\cdots\delta(x_s)\cdots x_{n+2})\\ &= \sum_{k_1+\ldots+k_{n+2}=n}\frac{n!}{k_1!\ldots k_{n+2}!}\delta^{k_1}(x_1)\cdots\delta^{k_s+1}(x_s)\cdots\delta^{k_{n+2}}(x_{n+2}).\end{aligned}$$

All of the k_j are nonnegative, so $k_1+\ldots+k_{n+2}=n$ implies that at least 2 of the $k_j=0$.

- If more than 2 of the $k_j=0$, then at least 2 of the degrees of the derivations are 0 and the summand lies in I^2.
- If exactly 2 $k_i,k_j=0$ such that $i,j\neq s$, then, as before, the summand lies in I^2.

So, we have left to consider the case when one of the indices $i,j=s$. Without loss of generality, let $k_s=0, k_j=0, j\neq s$. This implies all other $k_m=1$. Therefore, the summand is equal to

$$\frac{n!}{1!\ldots 1!}\delta(x_1)\cdots\delta(x_{j-1})\cdot\delta^0(x_j)\cdot\delta(x_{j+1})\cdots\delta(x_{n+2}) = n!t_j.$$

Since $i=s$ is fixed, j is arbitrarily chosen, and since all other cases can be reduced to an element in I^2, we have

$$0 = \delta^n(u_s) = n!(t_1 + \ldots + t_{s-1} + t_{s+1} + \ldots + t_{n+2}) + z_s$$

for some $z_s \in I^2$. Define

$$v_s = t_1 + \ldots + t_{s-1} + t_{s+1} + \ldots + t_{n+2}.$$

Since $0 = n!v_s + z_s$, this implies $n!v_s = -z_s$. And since the characteristic of the field is greater than $n + 1$, this implies $v_s \in I^2$. Consider

$$v = \sum_{s=1}^{n+2} v_s = (n + 1) \sum_{k=1}^{n+2} t_k.$$

Then $v \in I^2$, since each $v_s \in I^2$. Thus, $t = t_1 + t_2 + \ldots + t_{n+2} \in I^2$, again because $p > n + 1$. But then, since $t = t_1 + v_1$ and $t, v_1 \in I^2$, we have $t_1 \in I^2$. Also, $x_1, \ldots, x_{n+2}$ were chosen arbitrarily and $t_1 = x_1 \cdot \delta(x_2) \cdots \delta(x_{n+2})$. Therefore, $I \cdot \underbrace{\delta(I) \cdots \delta(I)}_{n+1} \subseteq I^2$. $\square$

Lemma 11. Let I be a nilpotent ideal of nilpotency class n from a Leibniz algebra A over a field of characteristic 0 or characteristic $p > n+1$, and let $\delta \in Der(A)$. Then there exists a function $f_n(m)$ such that $f_n(m) = f_n(m-1) + n - m + 1$ and $I^m \cdot \underbrace{\delta(I) \cdots \delta(I)}_{f_n(m)} \subseteq I^{m+1}$, for $1 \leq m \leq n$.

Proof. Let n be a fixed natural number. When $m = 1$, $f_n(1) = n + 1$.
Thus, by Lemma 10 we have the relation $I \cdot \underbrace{\delta(I) \cdots \delta(I)}_{n+1} \subseteq I^2$.

Assume the function $f_n(m - 1)$ satisfies the condition $I^{m-1} \cdot \underbrace{\delta(I) \cdots \delta(I)}_{f_n(m-1)} \subseteq I^m$ for some $m > 1$.
We need to show $I^m \cdot \underbrace{\delta(I) \cdots \delta(I)}_{m(n+1) - \frac{m(m+1)}{2} + 1} \subseteq I^{m+1}$.

For convenience, denote $N = m(n+1) - \frac{m(m+1)}{2} + 2$ and take arbitrary elements $x_1 \in I^m$ and $x_2, \ldots, x_N \in I$.
Let $s = f_n(m - 1) + 1$ and $t = n - m + 1$. Note: $N = t + s$.

Also, $I^m \subseteq I^{m-1}$, which implies $x_1 \in I^{m-1}$.
Therefore, by induction, $x_1 \cdot \delta(x_2) \cdots \delta(x_s) \in I^m$.
Thus, $\underbrace{x_1 \cdot \delta(x_2) \cdots \delta(x_s)}_{\in I^m} \cdot \underbrace{x_{s+1} \cdots x_N}_{n-m+1} \in I^{m+(n-m+1)} = I^{n+1} = 0$.

Consider $\delta^{n-m+1}(x_1 \cdot \delta(x_2) \cdots \delta(x_s) \cdot x_{s+1} \cdots x_N) = \delta^{n-m+1}(0) = 0$.
Also, using Leibniz's Rule, $\delta^{n-m+1}(x_1 \cdot \delta(x_2) \cdots \delta(x_s) \cdot x_{s+1} \cdots x_N) =$
$\sum \frac{t!}{k_1! \ldots k_N!} \delta^{k_1}(x_1) \cdot \delta^{k_2+1}(x_2) \cdots \delta^{k_s+1}(x_s) \cdot \delta^{k_{s+1}}(x_{s+1}) \cdots \delta^{k_N}(x_N) = 0$, where $k_1, \ldots, k_N$ are nonnegative and $k_1 + \ldots + k_N = t = n - m + 1$.
Recall $N = s + t$, so there are at least s of the $k_i = 0$.
We will show that all summands of this sum either lie in I^{m+1} or have the form $t!\delta^0(x_1) \cdot \delta(x_2) \cdots \delta(x_s) \cdot \delta(x_{s+1}) \cdots \delta(x_N)$. Consider the following possible cases:

1. There are exactly s of the $k_i = 0$.
   If these numbers are $k_1, \ldots, k_s$, then $k_{s+1} = \ldots = k_N = 1$, which reduces the summand to $t!\delta^0(x_1) \cdot \delta(x_2) \cdots \delta(x_s) \cdot \delta(x_{s+1}) \cdots \delta(x_N)$.
   If at least one of the numbers $k_1, \ldots, k_s$ is nonzero, then at least one of the numbers $k_{s+1}, \ldots, k_N$ is 0. First assume that $k_1 = 0$.
   Then $\delta^{k_1}(x_1) = \delta^0(x_1) = x_1 \in I^m$ and if at least one of the numbers $k_{s+1}, \ldots, k_N$ is 0, then the summand $t!\delta^0(x_1) \cdot \delta(x_2) \cdots \delta(x_s) \cdot \delta(x_{s+1}) \cdots \delta(x_N) \in I^{m+1}$.
   Consider now the case where $k_1 = 1$. If $k_2 = \ldots = k_s = 0$, then $\delta(x_1) \in I^{m-1}$, by Lemma 8. Thus, $\delta(x_1) \cdot \underbrace{\delta(x_2) \cdots \delta(x_s)}_{f_n(m-1)} \in I^m$.

   And, since at least one of the numbers $k_{s+1}, \ldots, k_N$ is 0, then the element in this case lies in I^{m+1}.

2. There are exactly $s + j$ of the $k_i = 0$, where $j \geq 1$.
   Since $N = s + t$, we have at least j of the $k_{s+1}, \ldots, k_N$ equal to 0.
   First assume there are exactly j such numbers.
   This implies $k_1 = \ldots = k_s = 0$. Thus, $x_1 \cdot \delta(x_2) \cdots \delta(x_s) \in I^m$.
   Since $j \geq 1$ at least one of the $k_{s+1}, \ldots, k_N$ is equal to 0, which means $t!x_1 \cdot \delta(x_2) \cdots \delta(x_s) \cdot \delta(x_{s+1}) \cdots \delta(x_N) \in I^{m+1}$.
   Now suppose at least $j + 1$ of the $k_{s+1}, \ldots, k_N$ are equal to 0.
   Let the quantity of the $k_i = 0$, $s + 1 \leq i \leq N$, be called r.
   Then according to our assumption $r \geq j + 1$.

Hence, the quantity of nonzero numbers among $k_{s+1}, \ldots, k_N$ is $t - r$.

Thus $\sum_{s+1}^{N} k_i \geq t - r$, which implies $\sum_{1}^{s} k_i \leq t - (t - r) = r$. In particular, this implies $k_1 \leq r$. First, assume $\sum_{1}^{s} k_i < r$.

Then $k_1 \leq r - 1$, and therefore, $\delta^{k_1}(x_1) \in I^{m-(r-1)} = I^{m-r+1}$, by Lemma 8.

Since there are at least r of the $k_{s+1}, \ldots, k_N$ equal to 0, this implies

$\underbrace{\delta^{k_1}(x_1)}_{\in I^{m-r+1}} \cdot \delta^{k_2+1}(x_2) \cdots \delta^{k_s+1}(x_s) \cdot \underbrace{\delta^{k_{s+1}}(x_{s+1}) \cdots \delta^{k_N}(x_N)}_{\in I^r} \in I^{m-r+1+r} = I^{m+1}$.

Now assume $\sum_{1}^{s} k_i = r$.

If $k_1 \leq r - 1$, then, as above, one can show that the element lies in I^{m+1}.

So let $k_1 = r$.

Then $k_2 = \ldots = k_s = 0$ and by induction $\underbrace{\delta^r(x_1)}_{\in I^{m-r}} \cdot \underbrace{\delta(x_2) \cdots \delta(x_s)}_{f_n(m-1)} \in I^{m-r+1}$.

Since at least r of the elements among $\delta^{k_{s+1}}(x_{s+1}), \ldots, \delta^{k_N}(x_N)$ are in I, we have

$\underbrace{\delta^r(x_1) \cdot \delta(x_2) \cdots \delta(x_s)}_{\in I^{m-r+1}} \cdot \underbrace{\delta^{k_{s+1}}(x_{s+1}) \cdots \delta^{k_N}(x_N)}_{\in I^r} \in I^{m+1}$.

Therefore, $\sum \frac{t!}{k_1! \ldots k_N!} \delta^{k_1}(x_1) \cdot \delta^{k_2+1}(x_2) \cdots \delta^{k_s+1}(x_s) \cdot \delta^{k_{s+1}}(x_{s+1}) \cdots \delta^{k_N}(x_N) = x_k + t!\delta^0(x_1) \cdot \delta(x_2) \cdots \delta(x_s) \cdot \delta(x_{s+1}) \cdots \delta(x_N) = 0$, for $x_k \in I^{m+1}$.

Since $t = n - m + 1$ and the characteristic $p > n + 1$, this implies

$x_1 \cdot \underbrace{\delta(x_2) \cdots \delta(x_s)}_{f_n(m-1)} \cdot \underbrace{\delta(x_{s+1}) \cdots \delta(x_N)}_{n-m+1} \in I^{m+1}$.

And since the elements $x_1 \in I^m, x_2, \ldots, x_N \in I$ can be chosen arbitrarily, this implies

$I^m \cdot \underbrace{\delta(I) \cdots \delta(I)}_{f_n(m-1)+n-m+1} \subseteq I^{m+1}$, for $1 \leq m \leq n$.

Thus, $f_n(m) = f_n(m-1) + n - m + 1$.

$\square$

Remark An explicit formula can be written $f_n(m) = m(n+1) - \frac{m(m+1)}{2} + 1$.

4.2 Invariance of Solvable Ideals

Recall a Leibniz algebra A is solvable if $A^{(k)} = 0$ for some $k \geq 0$ such that $A^{(k-1)} \neq 0$. The sum of solvable ideals of a Leibniz algebra is solvable. Hence, the sum of solvable characteristic ideals is a solvable characteristic ideal of A.

Definition 2. The *characteristic radical*, or *c-radical*, of A is the maximal solvable characteristic ideal of A.

Definition 3. A Leibniz algebra A is *characteristic semisimple* if its c-radical is 0.

Equivalently, A is characteristic semisimple if its only solvable characteristic ideal is 0. Let J be a solvable ideal of A. Then $J^{(i)} = J^{(i-1)} \cdot J^{(i-1)} = 0$ for some i such that $J^{(i-1)} \neq 0$. Thus, $J^{(i-1)}$ is a nonzero characteristic ideal. And if we consider the first term in the derived series of $J^{(i-1)}$, $(J^{(i-1)})^{(1)} = J^{(i-1)} \cdot J^{(i-1)} = 0$, we can see that $J^{(i-1)}$ is a nonzero solvable characteristic ideal. This implies $J^{(i-1)}$ is in the c-radical of A. Therefore, A is characteristic semisimple if and only if the only abelian characteristic ideal of A is 0.

Theorem 3. If R is the c-radical of A, then A/R is characteristic semisimple.

Proof. Let S be a solvable characteristic ideal of A/R,
and let $\Phi : A \to A/R$ be defined by $\Phi(a) = a + R$, for all $a \in A$.
By the correspondence theorem, we know that the ideals of A/R are in $1-1$ correspondence with the ideals of A containing R.
Thus, $S = S^*/R$, for an ideal S^* of A.
Furthermore, since S and R are both solvable, we know that S^* is a solvable ideal of A.
Let δ be any derivation of A. We want to show S^* is characteristic in A, so consider $\overline{\delta} : A/R \to A/R$, the induced derivation in the quotient space defined by:
$\overline{\delta}(a + R) = \delta(a) + R$.
Now consider $\overline{\delta} : S \to S$, and let $u^* \in S^*$.
This implies $u^* + R \in S$, and since S is a characteristic ideal of A/R,
$\overline{\delta}(u^* + R) = \delta(u^*) + R \in S = S^*/R$.
Thus, $\delta(u^*) \in S^*$, for all $u^* \in S^*$. Hence, S^* is a solvable characteristic ideal of A.
This implies $S^* \subseteq R$. But we chose S^* such that $R \subseteq S^*$; thus, $S^* = R$.
Hence, $S = S^*/R = S^*/S^* = \{0\}$, which implies A/R is characteristic semisimple. $\square$

Lemma 12. If A is characteristic semisimple, then every characteristic ideal in A is characteristic semisimple.

Proof. Let A be characteristic semisimple, and let B be a characteristic ideal in A. Consider S, any solvable characteristic ideal of B.
Let δ be any derivation of A. Since B is a characteristic ideal of A, $\delta(B) \subseteq B$; so, $\delta : B \to B$ is a derivation of B. Thus, $\delta(S) \subseteq S$.
Hence, S is a characteristic ideal in A. Since A is characteristic semisimple, $S = \{0\}$.
Therefore, B is characteristic semisimple. $\square$

Definition 4. A Leibniz algebra A is called *characteristic simple*, or *c-simple*, if its only characteristic ideals are A and 0, and if $A^2 = A$.

$A^2 = A$ implies that A is not a solvable Leibniz algebra. Therefore, the only solvable characteristic ideal of a c-simple Leibniz algebra is 0. Hence, every c-simple Leibniz algebra is characteristic semisimple.

Definition 5. A Leibniz algebra A is called *completely semisimple* if A can be written as $A = A_1 \oplus A_2 \oplus \ldots \oplus A_k$ where each A_i is a characteristic simple algebra.

It is not clear from the definition, but we can show that each A_i is a characteristic ideal of A. Consider

$$U_{A_2+\ldots+A_k} = \{a \in A | a(A_2 + \ldots + A_k) = (A_2 + \ldots + A_k)a = 0\},$$

the annihilator of $A_2 + \ldots + A_k$ in A. Let $a_1 \in A_1$, and consider $a_1(A_2 + \ldots + A_k)$. Since A is a direct sum of the A_i's, $A_iA_j = 0$ when $i \neq j$. Thus,

$$a_1(A_2 + \ldots + A_k) = (A_2 + \ldots + A_k)a_1 = 0,$$

which implies $a_1 \in U_{A_2+\ldots+A_k}$. Hence, $A_1 \subseteq U_{A_2+\ldots+A_k}$. Now let

$$a = a_1 + a_2 + \ldots + a_k \in U_{A_2+\ldots+A_k},$$

such that $a_i \in A_i$. Then $a_j \in Z_j(A_j)$ for $2 \leq j \leq k$. But, the center of a Leibniz algebra is a characteristic ideal, and A_j characteristic simple implies $Z_j(A_j) = 0$ for $2 \leq j \leq k$. Therefore, $a_j = 0$ for $2 \leq j \leq k$, which implies $a = a_1 \in A_1$. Thus, $U_{A_2+\ldots+A_k} \subseteq A_1$, which implies $U_{A_2+\ldots+A_k} = A_1$. Now let $a_1 \in A_1$, $\delta \in$Der(A), and $y \in A_2 + \ldots + A_k$. Then

$$\delta(a_1)y = \delta(a_1y) - a_1\delta(y) = -a_1\delta(y),$$

which implies $\delta(a_1)y \in (A_2 + \ldots + A_k) \cap A_1 = 0$. Therefore, $\delta(a_1) \in U_{A_2+\ldots+A_k} = A_1$; and hence, $\delta(A_1) \subseteq A_1$. Thus, A_1 is a characteristic ideal of A, and the same holds for each A_i.

Theorem 4. Every completely semisimple Leibniz algebra is characteristic semisimple.

Proof. Let A be a completely semisimple Leibniz algebra, and let B be a characteristic ideal of A. Thus, $A = A_1 \oplus \ldots \oplus A_k$, where each A_i is a characteristic simple ideal. Let δ be any derivation of A. Then $\delta(BA_i) \subseteq \delta(B)A_i + B\delta(A_i) \subseteq BA_i$; hence, BA_i is a characteristic ideal in A for each i. Let δ_i be a derivation of A_i. By defining $\delta_i(A_j) = 0$ when $i \neq j$, we can extend δ_i to be a derivation of A. Thus, BA_i is a characteristic ideal of A_i, and A_i characteristic simple implies $BA_i = A_i$ or $BA_i = 0$. If $BA_i = 0$, then $B \subseteq A_1 + \ldots + A_{i-1} + A_{i+1} + \ldots + A_k$, the annihilator of A_i. If $BA_i = A_i$, then $A_i \subseteq B$. Therefore, B is the direct sum of the A_i such that $BA_i = A_i$, and in particular, $B^2 = B$. Thus, the only solvable characteristic ideal of A is 0. Hence, A is characteristic semisimple. □

For any ideal I of a Leibniz algebra A, recall by Lemma 3, $I + \delta(I)$ is an ideal of A.

Theorem 5. Let A be a Leibniz algebra over a field $\mathbb{F}$, and let I be a solvable ideal of A of derived length n. Then the ideal $I + \delta(I)$ is solvable and its derived length is $\leq 2n$ in the following cases:

1. $\text{char}(\mathbb{F}) = 0$
2. $\text{char}(\mathbb{F}) > 2^n$

Proof. If $\text{char}(\mathbb{F}) = 2$, then $2 > 2^n$ implies $n = 0$, and hence, $I = 0$.
So assume $\text{char}(\mathbb{F}) = 0$ or $\text{char}(\mathbb{F}) = p > 2$.
Let $J = I + \delta(I)$.

$$\begin{aligned} J^{(1)} = J^2 = (I + \delta(I))(I + \delta(I)) &= I^2 + I\delta(I) + \delta(I)I + \delta(I)\delta(I) \\ &\subseteq I + \delta(I)\delta(I). \end{aligned}$$

Let $x, y \in I$. Then $xy \in I^2 = I^{(1)}$.
Note: $\frac{1}{2}(\delta^2(xy) - \delta^2(x)y - x\delta^2(y)) \in \delta^2(I) + I$. And by Lemma 4,

$$\begin{aligned}\delta^2(xy) &= \binom{2}{0}\delta^0(x)\delta^2(y) + \binom{2}{1}\delta(x)\delta(y) + \binom{2}{2}\delta^2(x)\delta^0(y)\\ &= x\delta^2(y) + 2\delta(x)\delta(y) + \delta^2(x)y.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}(\delta^2(xy) - \delta^2(x)y - x\delta^2(y)) &= \frac{1}{2}(x\delta^2(y) + 2\delta(x)\delta(y) + \delta^2(x)y - \delta^2(x)y - x\delta^2(y))\\ &= \frac{1}{2}(2\delta(x)\delta(y)) = \delta(x)\delta(y).\end{aligned}$$

Hence, $\delta(x)\delta(y) \in \delta^2(I^{(1)}) + I$ for all $x, y \in I$.
Therefore, $\delta(I)\delta(I) \subseteq \delta^2(I^{(1)}) + I$, which implies $J^{(1)} \subseteq \delta^{2^1}(I^{(1)}) + I$.
Assume $J^{(k-1)} \subseteq \delta^{2^{k-1}}(I^{(k-1)}) + I$, for $k \geq 2$.
This implies

$$\begin{aligned}J^{(k)} &= J^{(k-1)}J^{(k-1)} \subseteq (\delta^{2^{k-1}}(I^{(k-1)}) + I)(\delta^{2^{k-1}}(I^{(k-1)}) + I)\\ &= \delta^{2^{k-1}}(I^{(k-1)})\delta^{2^{k-1}}(I^{(k-1)}) + \delta^{2^{k-1}}(I^{(k-1)})I + I\delta^{2^{k-1}}(I^{(k-1)}) + I^2\\ &\subseteq \delta^{2^{k-1}}(I^{(k-1)})\delta^{2^{k-1}}(I^{(k-1)}) + I.\end{aligned}$$

By a result from combinatorics, an odd prime number p does not divide the binomial coefficients $\binom{2}{1}, \binom{2^2}{2}, \ldots, \binom{2^n}{2^{n-1}}$ if and only if $p > 2^n$.
Thus, p does not divide the binomial coefficients $\binom{2^k}{2^{k-1}}$. So, by Lemma 6,

$$\delta^{2^{k-1}}(I^{(k-1)})\delta^{2^{k-1}}(I^{(k-1)}) \subseteq \delta^{2^k}(I^{(k)}) + I.$$

This implies $J^{(k)} \subseteq \delta^{2^k}(I^{(k)}) + I$ for all $k \geq 1$. Thus, when $k = n =$derived length of I, we have

$$J^{(n)} \subseteq \delta^{2^n}(I^{(n)}) + I = \delta^{2^n}(0) + I = I.$$

But, $J^{(n)} \subseteq I$ implies $J^{(2n)} \subseteq I^{(n)} = 0$. Hence, $J = I + \delta(I)$ is a solvable ideal of derived length $\leq 2n$. $\square$

Theorem 6. The radical of a Leibniz algebra A, denoted Rad(A), is a characteristic ideal of A in the following cases:

1. $\text{char}(\mathbb{F}) = 0$
2. $\text{char}(\mathbb{F}) > 2^n$

Proof. Let $\delta \in$Der(A), and assume the characteristic of the field meets the above criteria. Rad(A) an ideal implies by Lemma 3 that Rad(A) + δ(Rad(A)) is an ideal. Rad(A) is a solvable ideal so, by Theorem 5, the ideal Rad(A) + δ(Rad(A)) is a solvable ideal. Thus, Rad(A) + δ(Rad(A)) $\subseteq$Rad(A); and hence, δ(Rad(A)) $\subseteq$Rad(A) for every $\delta \in$Der(A). Therefore, Rad(A) is characteristic. $\square$

Thus, if the field has characteristic 0 or if $\text{char}(\mathbb{F}) > 2^n$ where n is the derived length of Rad(A), then the c-radical is equal to the radical of a Leibniz algebra A. It is unknown whether the radical is a characteristic ideal for other characteristics of the ground field. George Seligman [19] provides an example of a Lie algebra that has a radical which is not characteristic when the field has prime characteristic > 2. Since all Lie algebras are Leibniz algebras we can use this example to show that the radical of a Leibniz algebra is not always a characteristic ideal.

4.3 Invariance of Nilpotent Ideals

Recall a Leibniz algebra A is nilpotent if $A^{\mathbf{k}} = 0$ for some $k \geq 0$ such that $A^{\mathbf{k\text{-}1}} \neq 0$. The sum of nilpotent ideals of a Leibniz algebra is nilpotent. Hence, the sum of nilpotent characteristic ideals is a nilpotent characteristic ideal of A.

Definition 6. The *nilradical* of A is the maximal nilpotent characteristic ideal of A.

Theorem 7. Every proper ideal in a characteristic simple Leibniz algebra A is nilpotent.

Proof. Let I be a nonzero, proper ideal of A. Let $\delta : A \to A$ be any derivation of A. Since I is an ideal of A, it is a left ideal and a right ideal. Hence,

$$\delta(I^{\mathbf{1}}) = \delta(I^2) \subseteq \delta(I)I + I\delta(I) \subseteq AI + IA \subseteq I.$$

Assume $\delta(I^{\mathbf{k}}) \subseteq I^{\mathbf{k\text{-}1}}$. Then

$$\begin{aligned}\delta(I^{\mathbf{k+1}}) &= \delta(I \cdot I^{\mathbf{k}}) \subseteq \delta(I)I^{\mathbf{k}} + I\delta(I^{\mathbf{k}})\\ &\subseteq A \cdot I^{\mathbf{k}} + I \cdot I^{\mathbf{k\text{-}1}} \subseteq I^{\mathbf{k}}.\end{aligned}$$

Therefore, by induction, $\delta(I^{\mathbf{k+1}}) \subseteq I^{\mathbf{k}}$ for all $k \in \mathbb{Z}^+$. Since the lower central series is a descending chain of ideals, $I^{\mathbf{m+1}} = I^{\mathbf{m}}$ for some $m \in \mathbb{Z}^+$. This implies

$$\delta(I^{\mathbf{m}}) = \delta(I^{\mathbf{m+1}}) \subseteq I^{\mathbf{m}}.$$

Hence, $I^{\mathbf{m}}$ is a characteristic ideal. But, A is characteristic semisimple and I is a proper ideal, so $I^{\mathbf{m}} = 0$. Therefore, I is nilpotent. □

Theorem 8. Let I be a nilpotent ideal of nilpotency class n of a Leibniz algebra A over a field of characteristic 0 or characteristic $p > n+1$, and let $\delta \in Der(A)$. Then $I + \delta(I)$ is a nilpotent ideal of the Leibniz algebra A of nilpotency class at most $\frac{n(n+1)(2n+1)}{6} + 2n$.

Proof. Let $k = \sum_{m=1}^{n} f_n(m)$. Using Lemma 11, we can show that

$$\begin{aligned}&I \cdot \underbrace{\delta(I) \cdots \delta(I)}_{k}\\ &= I \cdot \underbrace{\delta(I) \cdots \delta(I)}_{f_n(1)} \cdot \underbrace{\delta(I) \cdots \delta(I)}_{f_n(2)} \cdots \underbrace{\delta(I) \cdots \delta(I)}_{f_n(n)}\\ &= \underbrace{\underbrace{\underbrace{I \cdot \delta(I) \cdots \delta(I)}_{\subseteq I^2} \cdot \delta(I) \cdots \delta(I)}_{\subseteq I^3} \cdots \delta(I) \cdots \delta(I)}_{\subseteq I^{n+1}} \subseteq I^{n+1} = 0.\end{aligned}$$

By Lemma 9, $(I + \delta(I))^{k+n+1} = ((I + \delta(I))^{n+1})^k \subseteq (I)^k = 0$, since $k \geq n+1$. Recall: $f_n(1) = n+1$. Therefore, the ideal $I + \delta(I)$ is nilpotent of nilpotency class at most $k+n$.

Direct calculation yields:

$$\begin{aligned}
k+n &= n+\sum_{m=1}^{n} m(n+1) - \sum_{m=1}^{n} \frac{(m-1)(m+2)}{2} \\
&= n+(n+1)\sum_{m=1}^{n} m - \frac{1}{2}[\sum_{m=1}^{n} m^2 + \sum_{m=1}^{n} m - 2\sum_{m=1}^{n} 1] \\
&= n+\frac{n(n+1)^2}{2} - \frac{1}{2}[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - 2n] \\
&= 2n+\frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

□

Theorem 9. Let A be a Leibniz algebra over a field $\mathbb{F}$, let $\delta \in$Der(A), and let $N(L)$ be the nilradical of A with nilpotency class n. $N(L)$ is characteristic in the following cases:

- char($\mathbb{F}$)=0
- char$(\mathbb{F}) = p > n+1$.

Proof. $N(L)$ is a nilpotent ideal, so when the conditions on the characteristic of the field are met, by Theorem 8, $N(L)+\delta(N(L))$ is a nilpotent ideal for any derivation $\delta \in$Der(A). Therefore, $N(L)+\delta(N(L)) \subseteq N(L)$; and thus, $\delta(N(L)) \subseteq N(L)$.
Hence, $N(L)$ is a characteristic ideal of A. □

Chapter 5

Complete Leibniz Algebras

A Lie algebra L is called *complete* if it has a trivial center and if all derivations are left multiplications. Here we present a definition of complete Leibniz algebras that agrees with the theory of complete Lie algebras when we consider $A/\mathrm{Leib}(A)$.

Definition 7. Let A be a Leibniz algebra. A is *complete* if the following conditions hold:

- $Z^l(A) = \mathrm{Leib}(A)$
- For every $\delta \in \mathrm{Der}(A)$ there is an element $a_\delta \in A$ such that $\mathrm{im}(\delta - L_{a_\delta}) \subseteq \mathrm{Leib}(A)$
- There is some $\delta \in \mathrm{Der}(A)$ such that $\mathrm{im}(\delta) \nsubseteq \mathrm{Leib}(A)$.

Recall that when $\mathrm{Leib}(A) = 0$, A is a Lie algebra. So the first condition guarantees $Z(A) = 0$ when A is a Lie algebra, the second condition is equivalent to $\mathrm{im}(\delta) = \mathrm{im}(L_{a_\delta})$ when A is a Lie algebra, and the final condition guarantees the existence of a nontrivial derivation of A when A is a Lie algebra.

Example 5.0.1. Let A be a complete Leibniz algebra. Then for every $\delta \in \mathrm{Der}(A)$ there is an $a \in A$ such that $\delta(b) - \mathrm{L}_a(b) \in \mathrm{Leib}(A)$ for every $b \in A$. Let $b \in \mathrm{Leib}(A) \subseteq A$. Then $\delta(b) - \mathrm{L}_a(b) \in \mathrm{Leib}(A)$, which implies $\delta(b) - ab \in \mathrm{Leib}(A)$. But, $ab \in \mathrm{Leib}(A)$ since $\mathrm{Leib}(A)$ is an ideal. Hence, $\delta(b) \in \mathrm{Leib}(A)$ for every $\delta \in \mathrm{Der}(A)$ and for every $b \in \mathrm{Leib}(A)$. Therefore, when A is a complete Leibniz algebra, $\mathrm{Leib}(A)$ is a characteristic ideal.

Theorem 10. Let A be a Leibniz algebra such that $\mathrm{Leib}(A)$ is a characteristic ideal. If $A/\mathrm{Leib}(A)$ is a complete Lie algebra, then A is a complete Leibniz algebra.

Proof. Let A be a Leibniz algebra such that Leib(A) is a characteristic ideal, and assume $A/\text{Leib}(A)$ is a complete Lie algebra. Let $\delta \in$Der(A). Since Leib(A) is invariant under derivations, δ induces a derivation $\overline{\delta} \in$Der($A/$Leib(A)) defined by $\overline{\delta}(a+\text{Leib}(A)) = \delta(a)+\text{Leib}(A)$ for every $a \in A$. Since $A/\text{Leib}(A)$ is a complete Lie algebra, $\overline{\delta} = L_{\overline{x}}$ for some $\overline{x} \in A/\text{Leib}(A)$ such that $\overline{x} = x+\text{Leib}(A)$ for some $x \in A$. This implies

$$\begin{aligned}\delta(b) + \text{Leib}(A) &= \overline{\delta}(b + \text{Leib}(A)) \\ &= L_{\overline{x}}(b + \text{Leib}(A)) = L_{\overline{x}}(b) + \text{Leib}(A) = L_x(b) + \text{Leib}(A).\end{aligned}$$

Therefore, $\delta(b) - L_x(b) \in$Leib(A) for every $b \in A$. Thus, A is complete.

□

Recall, all semisimple Lie algebras are complete [14].

Corollary 3. All semisimple Leibniz algebras are complete.

Proof. Let A be a semisimple Leibniz algebra. Then

$$A = (S_1 \oplus S_2 \oplus \ldots \oplus S_k) \dot{+} \text{Leib}(A),$$

where S_i is a simple Lie algebra for all $1 \leq i \leq k$. This implies $\text{Leib}(A) = Z^l(A)$, which is a characteristic ideal. Since

$$A/\text{Leib}(A) = S_1 \oplus S_2 \oplus \ldots \oplus S_k$$

is a semisimple Lie algebra, $A/\text{Leib}(A)$ is a complete Lie algebra. Thus, by Theorem 10, A is a complete Leibniz algebra. □

Example 5.0.2. Let A be an n-dimensional nilpotent Leibniz algebra. By Corollary 1, Der(A) =span$\{\delta_1, \delta_2, \ldots, \delta_n\}$, where

- $\delta_1(a^i) = ia^i$ for $1 \leq i \leq n$
- $\delta_k(a^i) = a^{i+k-1}$ if $i + k - 1 \leq n$ and $\delta_k(a^i) = 0$ if $i + k - 1 > n$ for $1 < k \leq n$.

Thus, im(δ_k) $\subseteq$Leib(A) for $1 < k \leq n$. But there is not an element $b \in A$ such that $\delta_1(a) - L_b(a) \in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.0.3. Let A be an n-dimensional nonnilpotent Leibniz algebra. By Corollary 2, Der(A) =span$\{\delta_1, \delta_2, \ldots, \delta_{n-1}\}$, where

- $\delta_1(a^i) = a^{i+1}$ for $1 \leq i \leq n-1$, $\delta_1(a^n) = \sum_{j=1}^{n-1} b_j a^{j+1}$
- $\delta_k(a^i) = \delta_1^k(a^i)$ for $1 \leq i \leq n$ and $1 < k \leq n-1$.

Thus, im(δ_k) $\subseteq$Leib(A) for $1 \leq k \leq n-1$. Hence, A is not complete.

By these examples, all cyclic Leibniz algebras are not complete.

5.1 3-Dimensional Nilpotent Leibniz Algebras

Let A be a non-split non-Lie nilpotent Leibniz algebra, A =span$\{x, y, z\}$. Then, by Demir, Misra, and Stitzinger [8], A is isomorphic to one of the following algebras defined by the given nonzero multiplications. We will define a basis for $Z^l(A)$, Leib(A), $L(A)$, and Der(A) of each algebra to show all 3-dimensional nilpotent Leibniz algebras are not complete.

Example 5.1.1. $x^2 = y, xy = z$.
These nonzero multiplications define a nilpotent cyclic Leibniz algebra. Thus, Leib(A) = $Z^l(A)$ =span$\{y, z\}$, and by Corollary 1, Der(A) =span$\{\delta_1, \delta_2, \delta_3\}$ where

- $\delta_1(x) = x, \delta_1(y) = 2y, \delta_1(z) = 3z$
- $\delta_2(x) = y, \delta_2(y) = z, \delta_2(z) = 0$
- $\delta_3(x) = z, \delta_3(y) = 0, \delta_3(z) = 0$

We can write L(A)=span$\{L_x\}$ =span$\{\delta_2\}$. Thus, im$(\delta_2 - L_x) = 0$ $\subseteq$Leib(A), and im$(\delta_3 - L_0)$ $\subseteq$Leib(A); but there is not an element $b \in A$ such that $\delta_1(a) - L_b(a)$ $\in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.1.2. $x^2 = z$
By definition of the nonzero multiplications $L(A)$ =span$\{L_x\}$ and $Z^l(A)$=span$\{y, z\}$. Let $a = \alpha_1 x + \alpha_2 y + \alpha_3 z$. Then $a^2 = (\alpha_1 x + \alpha_2 y + \alpha_3 z)(\alpha_1 x + \alpha_2 y + \alpha_3 z) = \alpha_1^2 z$, which implies Leib$(A)$ =span$\{z\} \neq Z^l(A)$. Hence, A is not complete.

Example 5.1.3. $x^2 = z; y^2 = z$

By definition of the nonzero multiplications $L(A)$ =span$\{L_x, L_y\}$ and $Z^l(A)$ =span$\{z\}$. Let $a = \alpha_1 x + \alpha_2 y + \alpha_3 z$. Then $a^2 = (\alpha_1 x + \alpha_2 y + \alpha_3 z)(\alpha_1 x + \alpha_2 y + \alpha_3 z) = \alpha_1^2 z + \alpha_2^2 z$, which implies Leib$(A)$ =span$\{z\}$. Let $\delta \in$Der(A), and define $\delta(x) = a_1 x + a_2 y + a_3 z$, $\delta(y) = b_1 x + b_2 y + b_3 z$, and $\delta(z) = c_1 x + c_2 y + c_3 z$. We will apply δ to the product of every combination of basis vectors to find a basis for Der(A).

$\delta(x^2) = (a_1 x + a_2 y + a_3 z)x + x(a_1 x + a_2 y + a_3 z) = 2a_1 z = c_1 x + c_2 y + c_3 z = \delta(z)$.

$\implies c_1 = 0, c_2 = 0, c_3 = 2a_1$.

$\delta(xy) = (a_1 x + a_2 y + a_3 z)y + x(b_1 x + b_2 y + b_3 z) = a_2 z + b_1 z = 0$.

$\implies b_1 = -a_2$.

$\delta(xz) = (a_1 x + a_2 y + a_3 z)z + x(2a_1 z) = 0$

$\delta(yx) = (-a_2 x + b_2 y + b_3 z)x + y(a_1 x + a_2 y + a_3 z) = -a_2 z + a_2 z0$

$\delta(y^2) = (-a_2 x + b_2 y + b_3 z)y + y(-a_2 x + b_2 y + b_3 z) = 2b_2 z = 2a_1 z = \delta(z)$

$\implies b_2 = a_1$

$\delta(yz) = (-a_2 x + a_1 y + b_3 z)z + y(2a_1 z) = 0$

$\delta(zx) = (2a_1 z)x + z(a_1 x + a_2 y + a_3 z) = 0$

$\delta(zy) = (2a_1 z)y + z(-a_2 x + a_1 y + b_3 z) = 0$

$\delta(z^2) = (2a_1 z)z + z(2a_1 z) = 0$

Thus,

$$
\begin{aligned}
[\delta]_B &= \begin{pmatrix} a_1 & -a_2 & 0 \\ a_2 & a_1 & 0 \\ a_3 & b_3 & 2a_1 \end{pmatrix} \\
&= a_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + a_2 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned}
$$

Therefore, Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$, where

- $\delta_1(x) = x, \delta_1(y) = y, \delta_1(z) = 2z$
- $\delta_2(x) = y, \delta_2(y) = -x, \delta_2(z) = 0$
- $\delta_3(x) = z, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(x) = 0, \delta_4(y) = z, \delta_4(z) = 0$

We can write L(A)=span$\{L_x, L_y\}$ =span$\{\delta_3, \delta_4\}$. Thus, im$(\delta_3 - L_x) = 0$ $\subseteq$Leib(A), and im$(\delta_4 - L_y) = 0$ $\subseteq$Leib(A); but there is not an element $b \in A$ such that $\delta_1(a) - L_b(a)$ $\in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.1.4. $xy = z; yx = -z; y^2 = z$
Using techniques similar to those used in Example 5.1.3 we can show
$Z^l(A)$ =span$\{z\}$ =Leib(A), and we can find a basis for the derivation algebra of A. Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(x) = x, \delta_1(y) = y, \delta_1(z) = 2z$
- $\delta_2(x) = z, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(x) = 0, \delta_3(y) = x, \delta_3(z) = 0$
- $\delta_4(x) = 0, \delta_4(y) = z, \delta_4(z) = 0$

We can write L(A)=span$\{L_x, L_y\}$ =span$\{\delta_4, \delta_4 - \delta_2\}$ =span$\{\delta_2, \delta_4\}$. Hence, there is not an element $b \in A$ such that $\delta_1(a) - L_b(a)$ $\in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.1.5. $xy = z; yx = \alpha z, \alpha \in \mathbb{F} \setminus \{-1, 1\}$
For all $\alpha \in \mathbb{F} \setminus \{-1, 1\}$, Leib($A$) =span$\{z\}$. But, when $\alpha = 0$, $Z^l(A)$ =span$\{y, z\}$, which implies A is not complete. So assume $\alpha \neq \{-1, 0, 1\}$. Then we can show
Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(x) = x, \delta_1(y) = 0, \delta_1(z) = z$
- $\delta_2(x) = z, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(x) = 0, \delta_3(y) = y, \delta_3(z) = z$
- $\delta_4(x) = 0, \delta_4(y) = z, \delta_4(z) = 0$

We can write L(A)=span$\{L_x, L_y\}$ =span$\{\delta_4, \delta_2\}$. Hence, there is not an element $b \in A$ such that $\delta_1(a) - L_b(a)$ $\in$Leib(A) for all $a \in A$. Therefore, A is not complete.

5.2 4-Dimensional Nilpotent Leibniz Algebras

Let A be a non-split non-Lie nilpotent Leibniz algebra, $A=$span$\{w,x,y,z\}$. Then, by Demir, Misra, and Stitzinger [9], A is isomorphic to one of the following algebras defined by the given nonzero multiplications. We will define a basis for $Z^l(A)$, Leib(A), $L(A)$, and Der(A) of each algebra to determine the completeness of all 4-dimensional nilpotent Leibniz algebras.

Example 5.2.1. $wy=z$, $yx=z$
We can show Leib(A)=span$\{z\}$ and $Z^l(A)=$span$\{x,z\}$. Hence, Leib$(A)\neq Z^l(A)$, which implies A is not complete.

Example 5.2.2. $wy=z$, $x^2=z$, $xy=z$, $yw=z$, $yx=-z$
We can show Leib$(A)=$span$\{z\}=Z^l(A)$, and Der$(A)=$span$\{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5\}$ where

- $\delta_1(w)=w,\delta_1(x)=x,\delta_1(y)=y,\delta_1(z)=2z$
- $\delta_2(w)=z,\delta_2(x)=0,\delta_2(y)=0,\delta_2(z)=0$
- $\delta_3(w)=0,\delta_3(x)=w-2x,\delta_3(y)=x-4y,\delta_3(z)=-4z$
- $\delta_4(w)=0,\delta_4(x)=z,\delta_4(y)=0,\delta_4(z)=0$
- $\delta_5(w)=0,\delta_5(x)=0,\delta_5(y)=z,\delta_5(z)=0$

We can write L(A)=span$\{L_w,L_x,L_y\}$ =span$\{\delta_5,\delta_4+\delta_5,\delta_2-\delta_4\}$. Thus, there is not an element $b\in A$ such that im$(\delta_1-L_b)\subseteq$Leib(A). Hence, A is not complete.

Example 5.2.3. $wx=z$, $xw=-z$, $y^2=z$
We can show $Z^l(A)=$span$\{z\}=$Leib(A), and Der$(A)=$span$\{\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6,\delta_7\}$ where

- $\delta_1(w)=w-\frac{1}{2}y,\delta_1(x)=0,\delta_1(y)=-\frac{1}{2}x+\frac{1}{2}y,\delta_1(z)=z$
- $\delta_2(w)=x,\delta_2(x)=0,\delta_2(y)=0,\delta_2(z)=0$
- $\delta_3(w)=z,\delta_3(x)=0,\delta_3(y)=0,\delta_3(z)=0$
- $\delta_4(w)=0,\delta_4(x)=w,\delta_4(y)=0,\delta_4(z)=0$
- $\delta_5(w)=-\frac{1}{2}y,\delta_5(x)=x,\delta_5(y)=-\frac{1}{2}x+\frac{1}{2}y,\delta_5(z)=z$

- $\delta_6(w) = 0, \delta_6(x) = z, \delta_6(y) = 0, \delta_6(z) = 0$
- $\delta_7(w) = 0, \delta_7(x) = 0, \delta_7(y) = z, \delta_7(z) = 0$

We can write L(A)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_3, \delta_6, \delta_7\}$. Thus, there is not an element $b \in A$ such that $\delta_1(a) - L_b(a) \in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.2.4. $wx = z$, $xw = -z$, $x^2 = z$, $y^2 = z$
We can show Leib(A) =span$\{z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

- $\delta_1(w) = w, \delta_1(x) = x, \delta_1(y) = y, \delta_1(z) = 2z$
- $\delta_2(w) = z, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = 0, \delta_3(x) = w, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = z, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = 0, \delta_5(y) = z, \delta_5(z) = 0$

We can write L(A)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_4\delta_4 - \delta_2, \delta_5\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b) \subseteq$Leib($A$). Hence, A is not complete.

Example 5.2.5. $wx = z$, $xw = \alpha z$, $y^2 = z$, $\alpha \in \mathbb{C}/\{-1, 1\}$
For all $\alpha \in \mathbb{C}/\{-1, 1\}$, Leib($A$) =span$\{z\}$, but when $\alpha = 0$, $Z^l(A)$ =span$\{x, z\}$, which means A is not complete. So assume $\alpha \in \mathbb{C}/\{-1, 0, 1\}$. Then we can show Leib($A$) =span$\{z\} = Z^l(A)$, and Der($A$) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

- $\delta_1(w) = w, \delta_1(x) = 0, \delta_1(y) = \frac{1}{2}y, \delta_1(z) = z$
- $\delta_2(w) = z, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = 0, \delta_3(x) = x, \delta_3(y) = \frac{1}{2}y, \delta_3(z) = z$
- $\delta_4(w) = 0, \delta_4(x) = z, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = 0, \delta_5(y) = z, \delta_5(z) = 0$

We can write L(A)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_4, \alpha\delta_2, \delta_5\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b) \subseteq$Leib($A$). Hence, A is not complete.

Example 5.2.6. $w^2 = z$, $x^2 = z$, $y^2 = z$.
We can show Leib(A) =span$\{z\}$ = $Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$ where

- $\delta_1(w) = w, \delta_1(x) = x, \delta_1(y) = y, \delta_1(z) = 2z$
- $\delta_2(w) = x, \delta_2(x) = -w, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = -w, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = y, \delta_5(y) = -x, \delta_5(z) = 0$
- $\delta_6(w) = 0, \delta_6(x) = z, \delta_6(y) = 0, \delta_6(z) = 0$
- $\delta_7(w) = 0, \delta_7(w) = 0, \delta_7(y) = z, \delta_7(z) = 0$

We can write L(A)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_4, \delta_6, \delta_7\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b)$ $\subseteq$Leib(A). Hence, A is not complete.

Example 5.2.7. $w^2 = x$, $wx = y$, $wy = z$.
These nonzero multiplications define a nilpotent cyclic Leibniz algebra. Thus, Leib(A) = $Z^l(A)$ =span$\{x, y, z\}$, and by Corollary 1, Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(w) = w, \delta_1(x) = 2x, \delta_1(y) = 3y, \delta_1(z) = 4z$
- $\delta_2(w) = x, \delta_2(x) = y, \delta_2(y) = z, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = z, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$

We can write L(A)=span$\{L_w\}$ =span$\{\delta_2\}$. Thus, im$(\delta_2 - L_w) = 0$ $\subseteq$Leib(A), but there is not an element $b \in A$ such that $\delta_1(a) - L_b(a)$ $\in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.2.8. $w^2 = z$, $wx = y$, $xw = -y$.
We can show Leib(A) =span$\{z\}$, but $Z^l(A)$ =span$\{y, z\}$. Hence, A is not complete.

Example 5.2.9. $w^2 = z$, $wx = y$, $xw = -y$, $x^2 = z$.
We can show Leib(A) =span$\{z\}$, but $Z^l(A)$ =span$\{y, z\}$. Hence, A is not complete.

Example 5.2.10. $w^2 = z$, $wx = y$, $xw = -y$, $wy = z$, $yw = -z$.
We can show Leib(A) =span$\{z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$ where

- $\delta_1(w) = w, \delta_1(x) = 0, \delta_1(y) = y, \delta_1(z) = 2z$
- $\delta_2(w) = x, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = y, \delta_5(y) = z, \delta_5(z) = 0$
- $\delta_6(w) = 0, \delta_6(x) = z, \delta_6(y) = 0, \delta_6(z) = 0$

We can write L(A)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_4 + \delta_5, -\delta_3, -\delta_4\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b) \subseteq$Leib$(A)$. Hence, A is not complete.

Example 5.2.11. $wx = y$, $xw = -y$, $x^2 = z$, $wy = z$, $yw = -z$.
We can show Leib(A) =span$\{z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

- $\delta_1(w) = w, \delta_1(x) = 2x, \delta_1(y) = 3y, \delta_1(z) = 4z$
- $\delta_2(w) = y, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = z, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = y, \delta_4(y) = z, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = z, \delta_5(y) = 0, \delta_5(z) = 0$

We can write L(A)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_4, \delta_5 - \delta_2, -\delta_3\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b) \subseteq$Leib$(A)$. Hence, A is not complete.

Example 5.2.12. $w^2 = z$ $wx = y$, $xw = -y + z$, $wy = z$, $yw = -z$.
We can show Leib(A) =span$\{z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

- $\delta_1(w) = x, \delta_1(x) = x, \delta_1(y) = y, \delta_1(z) = z$
- $\delta_2(w) = y, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = z, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$

- $\delta_4(w) = 0, \delta_4(x) = y, \delta_4(y) = z, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = z, \delta_5(y) = 0, \delta_5(z) = 0$

We can write L(A)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_3 + \delta_4, \delta_3 - \delta_2, -\delta_3\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b) \subseteq$Leib$(A)$. Hence, A is not complete.

Example 5.2.13. $wx = y$, $xw = -y + z$, $x^2 = z$, $wy = z$, $yw = -z$.
We can show Leib(A) =span$\{z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(w) = y, \delta_1(x) = 0, \delta_1(y) = 0, \delta_1(z) = 0$
- $\delta_2(w) = z, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = 0, \delta_3(x) = y, \delta_3(y) = z, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = z, \delta_4(y) = 0, \delta_4(z) = 0$

We can write L(A)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_3, -\delta_1 + \delta_2 + \delta_4, -\delta_2\}$. The im$(\delta_2) \subseteq$Leib$(A)$ and im$(\delta_4) \subseteq$Leib(A). Also, $\delta_3 = L_w$ implies im$(\delta_3 - L_w) = 0 \subseteq$Leib$(A)$. Finally, im$(\delta_1 + L_x)$ =im$(\delta_1 - \delta_1 + \delta_2 + \delta_4)$ =im$(\delta_2 + \delta_4) \subseteq$Leib$(A)$. Hence, by linearity, for any $\delta \in$Der(A) there exists a $b \in A$ such that im$(\delta - L_b) \subseteq$Leib(A). Therefore, A is complete.

Example 5.2.14. $w^2 = y$, $wx = z$.
We can show Leib(A) =span$\{y, z\}$, but $Z^l(A)$ =span$\{x, y, z\}$. Hence, A is not complete.

Example 5.2.15. $w^2 = y$, $xw = z$.
We can show Leib(A) =span$\{y, z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$ where

- $\delta_1(w) = w, \delta_1(x) = 0, \delta_1(y) = 2y, \delta_1(z) = z$
- $\delta_2(w) = x, \delta_2(x) = 0, \delta_2(y) = z, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = x, \delta_5(y) = 0, \delta_5(z) = z$
- $\delta_6(w) = 0, \delta_6(x) = y, \delta_6(y) = 0, \delta_6(z) = 0$

- $\delta_7(w) = 0, \delta_7(x) = z, \delta_7(y) = 0, \delta_7(z) = 0$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_3, \delta_4\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b) \subseteq$Leib$(A)$. Hence, A is not complete.

Example 5.2.16. $xw = y$, $wx = z$, $x^2 = -y$
We can show $Z^l(A)$ =span$\{y, z\}$=Leib(A), and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(w) = y, \delta_1(x) = 0, \delta_1(y) = 0, \delta_1(z) = 0$
- $\delta_2(w) = z, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = 0, \delta_3(x) = y, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = z, \delta_4(y) = 0, \delta_4(z) = 0$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_4, \delta_1 - \delta_3\}$. Thus, there is not an element $b \in A$ such that $\delta_2(a) - L_b(a) \in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.2.17. $w^2 = y$, $xw = z$.
We can show Leib(A) =span$\{y, z\}$ = $Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$ where

- $\delta_1(w) = w, \delta_1(x) = 0, \delta_1(y) = 2y, \delta_1(z) = z$
- $\delta_2(w) = x, \delta_2(x) = 0, \delta_2(y) = (\alpha + 1)z, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = x, \delta_5(y) = 0, \delta_5(z) = z$
- $\delta_6(w) = 0, \delta_6(x) = y, \delta_6(y) = 0, \delta_6(z) = 0$
- $\delta_7(w) = 0, \delta_7(x) = z, \delta_7(y) = 0, \delta_7(z) = 0$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_3 + \delta_7, \alpha\delta_4\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b) \subseteq$Leib$(A)$. Hence, A is not complete.

Example 5.2.18. $w^2 = y$, $xw = z$, $wx = -\alpha y$, $x^2 = -z$, $\alpha \in \mathbb{F}$
We can show $Z^l(A)$ =span$\{y, z\}$ =Leib(A), and Der(A)=span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

- $\delta_1(w) = w, \delta_1(x) = x, \delta_1(y) = 2y, \delta_1(z) = 2z$
- $\delta_2(w) = y, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = z, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = y, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = z, \delta_5(y) = 0, \delta_5(z) = 0$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_2 - \alpha\delta_4, \delta_3 - \delta_5\}$. Thus, there is not an element $b \in A$ such that $\delta_1(a) - L_b(a) \in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.2.19. $w^2 = y$, $wx = y$, $xw = y + z$, $x^2 = z$.
We can show Leib(A) =span$\{y, z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$ where

- $\delta_1(w) = w, \delta_1(x) = x, \delta_1(y) = 2y, \delta_1(z) = 2z$
- $\delta_2(w) = x, \delta_2(x) = 2x, \delta_2(y) = 2y + z, \delta_2(z) = 2z$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = y, \delta_5(y) = 0, \delta_5(z) = 0$
- $\delta_6(w) = 0, \delta_6(x) = z, \delta_6(y) = 0, \delta_6(z) = 0$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_3 + \delta_5, \delta_3 + \delta_4 + \delta_6\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b) \subseteq$Leib($A$). Hence, A is not complete.

Example 5.2.20. $wx = y$, $wy = z$.
We can show Leib(A) =span$\{y, z\}$, but $Z^l(A)$ =span$\{x, y, z\}$. Hence, A is not complete.

Example 5.2.21. $wx = y$, $x^2 = z$, $wy = z$.
We can show Leib(A) =span$\{y, z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(w) = w, \delta_1(x) = 2x, \delta_1(y) = 3y, \delta_1(z) = 4z$
- $\delta_2(w) = z, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = 0, \delta_3(x) = y, \delta_3(y) = z, \delta_3(z) = 0$

- $\delta_4(w) = 0, \delta_4(x) = z, \delta_4(y) = 0, \delta_4(z) = 0$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_3, \delta_4\}$. Thus, there is not an element $b \in A$ such that im$(\delta_1 - L_b)$ $\subseteq$Leib(A). Hence, A is not complete.

Example 5.2.22. $wx = y$, $xw = z$, $wy = z$.
We can show Leib(A) =span$\{y, z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(w) = z, \delta_1(x) = 0, \delta_1(y) = 0, \delta_1(z) = 0$
- $\delta_2(w) = 0, \delta_2(x) = x, \delta_2(y) = y, \delta_2(z) = z$
- $\delta_3(w) = 0, \delta_3(x) = y, \delta_3(y) = z, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = z, \delta_4(y) = 0, \delta_4(z) = 0$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_3, \delta_1\}$. Thus, there is not an element $b \in A$ such that im$(\delta_2 - L_b)$ $\subseteq$Leib(A). Hence, A is not complete.

Example 5.2.23. $wx = y$, $xw = z$, $x^2 = z$, $wy = z$.
We can show Leib(A) =span$\{y, z\} = Z^l(A)$, and Der(A) =span$\{\delta_1, \delta_2, \delta_3\}$ where

- $\delta_1(w) = z, \delta_1(x) = 0, \delta_1(y) = 0, \delta_1(z) = 0$
- $\delta_2(w) = 0, \delta_2(x) = y, \delta_2(y) = z, \delta_2(z) = 0$
- $\delta_3(w) = 0, \delta_3(x) = z, \delta_3(y) = 0, \delta_3(z) = 0$

Therefore, every derivation of A has an image in Leib(A). Hence, A is not complete.

Example 5.2.24. $w^2 = y$, $xw = z$, $wy = z$
We can show $Z^l(A)$ =span$\{y, z\}$ =Leib(A), and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

- $\delta_1(w) = w, \delta_1(x) = 2x, \delta_1(y) = 2y, \delta_1(z) = 3z$
- $\delta_2(w) = x, \delta_2(x) = 0, \delta_2(y) = z, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = z, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = z, \delta_5(y) = 0, \delta_5(z) = 0$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_3, \delta_4\}$. Thus, there is not an element $b \in A$ such that $\delta_1(a) - L_b(a)$ $\in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Example 5.2.25. $w^2 = y$, $x^2 = z$, $wy = z$
We can show $Z^l(A)$ =span$\{y, z\}$=Leib(A), and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(w) = w, \delta_1(x) = \frac{3}{2}x, \delta_1(y) = 2y, \delta_1(z) = 3z$
- $\delta_2(w) = y, \delta_2(x) = 0, \delta_2(y) = z, \delta_2(z) = 0$
- $\delta_3(w) = z, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = z, \delta_4(y) = 0, \delta_4(z) = 0.$

We can write L(A)=span$\{L_w, L_x\}$ =span$\{\delta_2, \delta_4\}$. Thus, there is not an element $b \in A$ such that $\delta_1(a) - L_b(a)$ $\in$Leib(A) for all $a \in A$. Therefore, A is not complete.

Jacobson provides a proof in [10] that all nilpotent Lie algebras have an outer derivation; and hence, all nilpotent Lie algebras are not complete. By Example 5.2.13, nilpotent Leibniz algebras can be complete. It is also important to note that when we consider A/Leib(A), for A as defined in Example 5.2.13, we get the 3-dimensional Heisenberg Lie algebra which is not complete. Therefore, when A is a complete Leibniz algebra, the Lie algebra A/Leib(A) is not necessarily complete.

5.3 3-Dimensional Solvable Leibniz Algebras

Let A be a non-split non-Lie non-nilpotent Leibniz algebra, A =span$\{x, y, z\}$. Then A is solvable, and by Demir, Misra, and Stitzinger [8], A is isomorphic to one of the following algebras defined by the given nonzero multiplications. We will define a basis for $Z^l(A)$, Leib(A), $L(A)$, and Der(A) of each algebra to show 3-dimensional non-nilpotent Leibniz algebras can be either complete or not complete.

Example 5.3.1. $xz = z$
We can show $Z^l(A)$=span$\{y, z\}$ and Leib(A) =span$\{z\}$. Hence, A is not complete.

Example 5.3.2. $xz = \alpha z, \alpha \in \mathbb{F} \setminus \{0\}, xy = y, yx = -y$
We can show $Z^l(A)$=span$\{z\}$=Leib(A), and Der(A) =span$\{\delta_1, \delta_2, \delta_3\}$ where

- $\delta_1(x) = y, \delta_1(y) = 0, \delta_1(z) = 0$
- $\delta_2(x) = 0, \delta_2(y) = y, \delta_2(z) = 0$
- $\delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = z$

We can write L(A)=span$\{L_x, L_y\}$ =span$\{\delta_2 + \alpha\delta_3, -\delta_1\}$. Thus, im$(\delta_1 - L_{-y}) = 0$ $\subseteq$Leib(A), im$(\delta_2 - L_x)$ $\subseteq$Leib(A), and im$(\delta_3 - L_0)$ $\subseteq$Leib(A). Therefore, A is complete.

Example 5.3.3. $xy = y; yx = -y; x^2 = z$
We can show $Z^l(A)$=span$\{z\}$=Leib(A), and Der(A) =span$\{\delta_1, \delta_2, \delta_3\}$ where

- $\delta_1(x) = y, \delta_1(y) = 0, \delta_1(z) = 0$
- $\delta_2(x) = z, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(x) = 0, \delta_3(y) = y, \delta_3(z) = 0$

We can write L(A)=span$\{L_x, L_y\}$ =span$\{\delta_2 + \delta_3, -\delta_1\}$. Thus,
im$(\delta_1 - L_{-y}) = 0$ $\subseteq$Leib(A), im$(\delta_2 - L_0)$ $\subseteq$Leib(A), and im$(\delta_3 - L_x)$ $\subseteq$Leib(A).
Therefore, A is complete.

Example 5.3.4. $xz = 2z; y^2 = z; xy = y; yx = -y; x^2 = z$
We can show $Z^l(A)$=span$\{z\}$=Leib(A), and Der(A) =span$\{\delta_1, \delta_2\}$ where

- $\delta_1(x) = y, \delta_1(y) = -z, \delta_1(z) = 0$
- $\delta_2(x) = z, \delta_2(y) = y, \delta_2(z) = 2z$

We can write L(A)=span$\{L_x, L_y\}$ =span$\{\delta_2, -\delta_1\}$. Thus, im$(\delta_1 - L_{-y}) = 0$ $\subseteq$Leib(A), and im$(\delta_2 - L_x) = 0$ $\subseteq$Leib(A). Therefore, A is complete.

Example 5.3.5. $xz = \alpha z, \alpha \in \mathbb{F} \setminus \{0\}, xy = y$
We can show $Z^l(A)$=span$\{y, z\}$=Leib(A), and Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(x) = 0, \delta_1(y) = y, \delta_1(z) = 0$
- $\delta_2(x) = 0, \delta_2(y) = z, \delta_2(z) = 0$
- $\delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = y$

- $\delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = z$

There is not a $\delta \in$Der(A) whose image is not in Leib(A). Therefore, A is not complete.

Example 5.3.6. $xz = z + y; xy = y$
We can show $Z^l(A)$=span$\{y, z\}$=Leib(A), and Der(A) =span$\{\delta_1, \delta_2\}$ where

- $\delta_1(x) = 0, \delta_1(y) = y, \delta_1(z) = z$
- $\delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = y$

There is not a $\delta \in$Der(A) whose image is not in Leib(A). Therefore, A is not complete.

Example 5.3.7. $xz = y; xy = y; x^2 = z$
We can show $Z^l(A)$=span$\{y, z\}$=Leib(A), and Der(A) =span$\{\delta_1, \delta_2, \delta_3\}$ where

- $\delta_1(x) = y, \delta_1(y) = 0, \delta_1(z) = y$
- $\delta_2(x) = z, \delta_2(y) = 0, \delta_2(z) = y$
- $\delta_3(x) = 0, \delta_3(y) = y, \delta_3(z) = z$

There is not a $\delta \in$Der(A) whose image is not in Leib(A). Therefore, A is not complete.

5.4 The Holomorph of a Leibniz Algebra

Meng [14] provides the following definition for the holomorph $h(L)$ of a Lie algebra L. Let L be a Lie algebra, then $h(L) = L \dot{+}$Der(L), where the bracket in $h(L)$ is defined by $[x + \delta_1, y + \delta_2] = [x, y] + \delta_1(y) - \delta_2(x) + [\delta_1, \delta_2]$, where $x, y \in L$ and $\delta_1, \delta_2 \in$Der(L). But,

$$\begin{aligned}[x, \delta_2](y) &= (x\delta_2 - \delta_2 x)(y) = x\delta_2(y) - \delta_2(xy) \\ &= x\delta_2(y) - \delta_2(x)y - x\delta_2(y) = -\delta_2(x)y\end{aligned}$$

by properties of the commutator bracket. Therefore, we need to show how elements from a Leibniz algebra will interact with elements of the derivation algebra before we can define the holomorph of a Leibniz algebra.

Example 5.4.1. Let A be a Leibniz algebra. Let $x \in A$ and let $\delta \in$Der(A). Der(A) is a Lie algebra with the commutator bracket, so for any $z \in A$,

$$\begin{aligned}[L_x, \delta](z) &= (L_x\delta - \delta L_x)(z) = L_x(\delta(z)) - \delta(L_x(z)) \\ &= x\delta(z) - \delta(xz) = x\delta(z) - \delta(x)z - x\delta(z) = -\delta(x)z.\end{aligned}$$

Thus, $[L_x, \delta] = -\delta(x)$.

Example 5.4.2. Let A be a Leibniz algebra. Let $x, y, z \in A$.

$$\begin{aligned}L_{xy}(z) &= (xy)z = x(yz) - y(xz) = L_x(L_y(z)) - L_y(L_x(z)) \\ &= (L_xL_y - L_yL_x)(z) = [L_x, L_y](z).\end{aligned}$$

Therefore, $L_{xy} = [L_x, L_y]$.

Example 5.4.3. Let A be a Leibniz algebra. Let $x \in A$ and let $\delta \in$Der(A). Then $\delta(x) \in A$, and for any $z \in A$ we have:

$$\begin{aligned}L_{\delta(x)}(z) &= \delta(x)z = \delta(xz) - x\delta(z) = \delta(L_x(z)) - L_x(\delta(z)) \\ &= (\delta L_x - L_x\delta)(z) = [\delta, L_x](z).\end{aligned}$$

Thus, $L_{\delta(x)} = [\delta, L_x]$.

Now we are ready to provide a definition for the holomorph of a Leibniz algebra.

Definition 8. Let A be a Leibniz algebra. The *holomorph* of A is $h(A) = A \dot{+} \mathrm{Der}(A)$ with the product defined by $(x+\delta_1)(y+\delta_2) = xy+\delta_1(y)+[L_x, \delta_2]+[\delta_1, \delta_2]$ for all $x, y \in A$ and $\delta_1, \delta_2 \in$Der(A).

By Example 5.4.1, this definition for the product in $h(A)$ is equivalent to
$(x + \delta_1)(y + \delta_2) = xy + \delta_1(y) - \delta_2(x) + [\delta_1, \delta_2]$.

Proposition 1. The holomorph of a Leibniz algebra A is a Leibniz algebra.

Proof. Let $x, y, z \in A$ and let $\delta_1, \delta_2, \delta_3 \in$Der$(A)$. We need to show multiplication in $h(A)$ satisfies the Leibniz identity:

$$(x + \delta_1)((y + \delta_2)(z + \delta_3)) = ((x + \delta_1)(y + \delta_2))(z + \delta_3) + (y + \delta_2)((x + \delta_1)(z + \delta_3)).$$

The left hand side of the Leibniz identity expands as follows:

$$
\begin{aligned}
&(x+\delta_1)((y+\delta_2)(z+\delta_3))\\
&=(x+\delta_1)(yz+\delta_2(z)+[L_y,\delta_3]+[\delta_2,\delta_3])\\
&=x(yz+\delta_2(z))+\delta_1(yz+\delta_2(z))+[L_x,[L_y,\delta_3]+[\delta_2,\delta_3]]+[\delta_1,[L_y,\delta_3]+[\delta_2,\delta_3]]\\
&=\underbrace{x(yz)}_{1}+\underbrace{x\delta_2(z)}_{2}+\underbrace{\delta_1(yz)}_{3}+\underbrace{\delta_1(\delta_2(z))}_{4}+\underbrace{[L_x,[L_y,\delta_3]]}_{5}+\underbrace{[L_x,[\delta_2,\delta_3]]}_{6}+\underbrace{[\delta_1,[L_y,\delta_3]]}_{7}\\
&+\underbrace{[\delta_1,[\delta_2,\delta_3]}_{8}.
\end{aligned}
$$

The right hand side of the Leibniz identity expands as follows:

$$
\begin{aligned}
&((x+\delta_1)(y+\delta_2))(z+\delta_3)+(y+\delta_2)((x+\delta_1)(z+\delta_3))\\
&=(xy+\delta_1(y)+[L_x,\delta_2]+[\delta_1,\delta_2])(z+\delta_3)+(y+\delta_2)(xz+\delta_1(z)+[L_x,\delta_3]+[\delta_1,\delta_3])\\
&=(xy+\delta_1(y))z+([L_x,\delta_2]+[\delta_1,\delta_2])(z)+[L_{xy+\delta_1(y)},\delta_3]+[[L_x,\delta_2]+[\delta_1,\delta_2],\delta_3]\\
&+y(xz+\delta_1(z))+\delta_2(xz+\delta_1(z))+[L_y,[L_x,\delta_3]+[\delta_1,\delta_3]]+[\delta_2,[L_x,\delta_3]+[\delta_1,\delta_3]]\\
&=(xy)z+\delta_1(y)z+[L_x,\delta_2](z)+[\delta_1,\delta_2](z)+[L_{xy},\delta_3](z)+[L_{\delta_1(y)},\delta_3]+[[L_x,\delta_2],\delta_3]\\
&+[[\delta_1,\delta_2],\delta_3]+y(xz)+y\delta_1(z)+\delta_2(xz)+\delta_2(\delta_1(z))+[L_y,[L_x,\delta_3]]+[L_y,[\delta_1,\delta_3]]\\
&+[\delta_2,[L_x,\delta_3]]+[\delta_2,[\delta_1,\delta_3]]\\
&=\underbrace{(xy)z+y(xz)}_{1}+\underbrace{[L_x,\delta_2](z)+\delta_2(xz)}_{2}+\underbrace{\delta_1(y)z+y\delta_1(z)}_{3}+\underbrace{[\delta_1,\delta_2](z)+\delta_2(\delta_1(z))}_{4}\\
&+\underbrace{[L_{xy},\delta_3]+[L_y,[L_x,\delta_3]]}_{5}+\underbrace{[[L_x,\delta_2],\delta_3]+[\delta_2,[L_x,\delta_3]]}_{6}+\underbrace{[L_{\delta_1(y)},\delta_3]+[L_y,[\delta_1,\delta_3]]}_{7}\\
&+\underbrace{[[\delta_1,\delta_2],\delta_3]+[\delta_2,[\delta_1,\delta_3]]}_{8}
\end{aligned}
$$

1. $x(yz)=(xy)z+y(xz)$ by the Leibniz identity.

2. $[L_x,\delta_2](z)+\delta_2(xz)=-\delta_2(x)z+\delta_2(xz)=-\delta_2(x)+\delta_2(x)z+x\delta_2(z)=x\delta_2(z)$ by Example 5.4.1 and the derivation property.

3. $\delta_1(yz)=\delta_1(y)z+y\delta_1(z)$ by the derivation property.

4. $[\delta_1,\delta_2](z)+\delta_2(\delta_1(z))=(\delta_1\delta_2-\delta_2\delta_1)(z)+\delta_2(\delta_1(z))=\delta_1(\delta_2(z))-\delta_2(\delta_1(z))+\delta_2(\delta_1(z))=\delta_1(\delta_2(z))$ by properties of the Lie bracket.

5. $[L_{xy}, \delta_3]+[L_y, [L_x, \delta_3]] = [[L_x, L_y], \delta_3]+[L_y, [L_x, \delta_3]] = [L_x, [L_y, \delta_3]]$ by Example 5.4.2 and the Jacobi identity.

6. $[L_x, [\delta_2, \delta_3]] = [[L_x, \delta_2], \delta_3] + [\delta_2, [L_x, \delta_3]]$ by the Jacobi identity.

7. $[L_{\delta_1(y)}, \delta_3]+[L_y, [\delta_1, \delta_3]] = [[\delta_1, L_y], \delta_3]+[L_y, [\delta_1, \delta_3]] = [\delta_1, [L_y, \delta_3]]$ by Example 5.4.3 and the Jacobi identity.

8. $[\delta_1, [\delta_2, \delta_3] = [[\delta_1, \delta_2], \delta_3] + [\delta_2, [\delta_1, \delta_3]]$ by the Jacobi identity.

Thus, the Leibniz identity holds, and $h(A)$ is a Leibniz algebra. □

Example 5.4.4. Let $x + \delta \in h(A)$ and let $y \in A$. Then $(x + \delta)(y) = xy + \delta(y) \in A$. Thus, A is a left ideal of $h(A)$.

Example 5.4.5. Let $x + \delta \in h(A)$ and let $y \in A$. Then $y(x + \delta) = yx - L_{\delta(y)} \in h(A)$. Thus, A is not a right ideal of $h(A)$; and hence, A is not an ideal of $h(A)$.

Definition 9. Let A be a Leibniz algebra and let B be a subalgebra of A. The *left centralizer* of B in A is the set $C^l_A(B) = \{x \in A | L_x(y) = 0 \text{ for all } y \in B\}$.

Example 5.4.6. Let A be a Leibniz algebra, and let $x \in C^l_{h(A)}(A)$. Then $x \in h(A)$ implies $x = a + \delta$ for some $a \in A$ and $\delta \in$Der(A). And $x \in C^l_{h(A)}(A)$ implies $L_x(y) = 0$ for all $y \in A$. Thus,

$$L_x(y) = L_{a+\delta}(y) = (a + \delta)(y) = ay + \delta(y) = 0.$$

Therefore, $\delta(y) = -ay$, which implies $\delta(y) = -L_a(y)$. Therefore, the left centralizer of A in $h(A)$ is $C^l_{h(A)}(A) = \{a - L_a | a \in A\}$.

Example 5.4.7. Let A be a Leibniz algebra. Let $x \in h(A)$, and let $y \in C^l_{h(A)}(A)$. Then $x = a + \delta_1$ and $y = b - L_b$ for some $a, b \in A$ and $\delta_1 \in$Der(A). Consider

$$\begin{aligned} xy &= (a + \delta_1)(b - L_b) = ab + \delta_1(b) - L_{-L_b(a)} + [\delta_1, -L_b] \\ &= ab + \delta_1(b) + L_{ba} - [\delta_1, L_b] = ab + \delta_1(b) + [L_b, L_a] - L_{\delta_1(b)} \\ &= ab + \delta_1(b) - [L_a, L_b] - L_{\delta_1(b)} = ab + \delta_1(b) - L_{ab} - L_{\delta_1(b)} \\ &= ab + \delta_1(b) - L_{ab+\delta_1(b)} \in C^l_{h(A)}. \end{aligned}$$

Thus, $C^l_{h(A)}(A)$ is a left ideal of $h(A)$.

Example 5.4.8. Let A be a Leibniz algebra, and let $x \in h(A)$. Then $x = a + \delta$ for some $a \in A$ and $\delta \in$Der(A). If $y \in A$, then $L_x(y) = L_{a+\delta}(y) = (a+\delta)(y) = ay + \delta(y) \in A$. Therefore, $L_x|_A \in$Der(A) for all $x \in h(A)$.

Example 5.4.9. Let A be a complete Leibniz algebra, and let $x \in h(A)$. Then $x = a + \delta$ for some $a \in A$ and $\delta \in$Der(A). If $y \in A/\mathrm{Leib}(A)$, then $y = b+$Leib(A) for some $b \in A$. Then

$$\begin{aligned} L_x(y) &= L_{a+\delta}(b + \mathrm{Leib}(A)) = ab + a\mathrm{Leib}(A) + \delta(b + \mathrm{Leib}(A)) \\ &= ab + \delta(b) + \mathrm{Leib}(A) \in A/\mathrm{Leib}(A), \end{aligned}$$

since A complete implies Leib(A) is a characteristic ideal. Therefore, $L_x|_{A/\mathrm{Leib}(A)} \in$Der$(A/\mathrm{Leib}(A))$ for all $x \in h(A)$.

Lemma 13. For any Leibniz algebra A, $A \cap C^l_{h(A)}(A) = Z^l(A)$.

Proof. Let A be a Leibniz algebra, and let $a \in A \cap C^l_{h(A)}(A)$. $a \in C^l_{h(A)}(A)$ implies $a = x - L_x$ for some $x \in A$. But $x - L_x \in A$ implies $L_x = 0$. Thus, $L_x(b) = 0$ for every $b \in A$, which implies $x \in Z^l(A)$. Therefore, $a = x - L_x = x \in Z^l(A)$ for every $a \in A \cap C^l_{h(A)}(A)$. Hence, $A \cap C^l_{h(A)}(A) = Z^l(A)$. □

Meng proved in **??** that L is a complete Lie algebra if and only if $h(L)$ has the decomposition $h(L) = L \oplus C_{h(L)}(L)$. We have just shown that for any Leibniz algebra A, $A \cap C^l_{h(A)}(A) = Z^l(A)$. When A is a complete Leibniz algebra $Z^l(A) =$Leib(A), so we know that the decomposition of the holomorph into $A \oplus C_{h(A)}(A)$ is not possible. We modify Meng's result for Lie algebras to prove the following theorem.

Theorem 11. Let A be a complete Leibniz algebra. Then $h(A) = A + (C^l_{h(A)}(A) \oplus I)$, where $I = \{\delta \in \mathrm{Der}(A) | \mathrm{im}(\delta) \subseteq \mathrm{Leib}(A)\}$.

Proof. By definition $A + (C^l_{h(A)}(A) \oplus I) \subseteq h(A)$. Let $a + \delta \in h(A)$. Since A is complete, im$(\delta - L_b) \subseteq$Leib(A) for some $b \in A$. This implies $\delta - L_b \in I$. Therefore,

$$a + \delta = a + b - (b - L_b) + (\delta - L_b) \in A + C^l_{h(A)}(A)) + I,$$

which implies $h(A) \subseteq A + C^l_{h(A)}(A) + I$. Assume $\overline{\delta} \in C^l_{h(A)}(A) \cap I$. $\overline{\delta} \in C^l_{h(A)}(A)$ implies $\overline{\delta}(a) = 0$ for all $a \in A$. But this implies, $\delta = 0$. Thus, $C^l_{h(A)}(A) \cap I = \{0\}$.
Hence, $h(A) = A + (C^l_{h(A)}(A) \oplus I)$. □

By Theorem 11,

$$
\begin{aligned}
\dim(h(A)) &= \dim(A \cup (C^l_{h(A)}(A) \cup I)) \\
&= \dim(A) + \dim(C^l_{h(A)}(A) \cup I) - \dim(A \cap (C^l_{h(A)}(A) \cup I)) \\
&= \dim(A) + \dim(C^l_{h(A)}(A)) + \dim(I) - \dim[(A \cap C^l_{h(A)}(A)) \cup (A \cap I)] \\
&= \dim(A) + \dim(C^l_{h(A)}(A)) + \dim(I) \\
&- [\dim(A \cap C^l_{h(A)}(A)) + \dim(A \cap I) - \dim((A \cap C^l_{h(A)}(A)) \cap (A \cap I))].
\end{aligned}
$$

By Lemma 13, $A \cap C^l_{h(A)}(A) = Z^l(A)$, and when A is complete $Z^l(A) =$Leib(A). Also, $A \cap I = \{0\}$, since $A\cap$Der$(A) = \{0\}$. Thus,

$$
\begin{aligned}
\dim(h(A)) &= \dim(A) + \dim(C^l_{h(A)}(A)) + \dim(I) - \dim(\text{Leib}(A)) \\
&= \dim(A/\text{Leib}(A)) + \dim(C^l_{h(A)}(A)) + \dim(I).
\end{aligned}
$$

And, if A is a Lie Algebra, this implies $\dim(h(A)) =\dim(A)+\dim(C_{h(A)}(A))$. And since Leib$(A) = 0$ when A is a Lie algebra, $A \cap C_{h(A)}(A) = 0$. Thus, $h(A) = A \oplus C_{h(A)}(A)$, agreeing with Meng's result. Meng also proved in his paper that a Lie algebra L is complete if and only if $h(L)$ has the decomposition $h(L) = L \oplus C_{h(L)}(L)$. However, as shown in the following example, the decomposition of the holomorph to $h(A) = A + (C^l_{h(A)}(A) \oplus I)$ does not imply that A is a complete Leibniz algebra.

Example 5.4.10. Let A be the 3-dimensional cyclic Leibniz algebra with basis $\{a, a^2, a^3\}$ and with the multiplication $a \cdot a^3 = a^3$. Recall Leib$(A) = A^2 =$span$\{a^2, a^3\}$. And by Corollary 2, Der$(A) =$span$\{\delta_1, \delta_2\}$ with $\delta_1(a) = a^2, \delta_1(a^2) = a^3, \delta_1(a^3) = a^3$ and $\delta_2(a) = a^3, \delta_2(a^2) = a^3, \delta_2(a^3) = a^3$. Therefore, the image of all derivations of A are contained in Leib(A), which means Der$(A) = I$. We can also show that $C^l_{h(A)}(A) =$span$\{a^2, a^3, a - L_a\}$; hence $C^l_{h(A)}(A) \cap I = \{0\}$. Thus, $h(A) = A+$Der$(A) = A+(C^l_{h(A)}(A) \oplus I)$, but by Example A is not complete.

Chapter 6

Semicomplete Leibniz Algebras

In this section we introduce the notion of semicompleteness for Leibniz algebras.

Definition 10. A derivation δ of a Leibniz algebra A is called an *ID-derivation* if its image is contained in the derived algebra of A. $\mathrm{ID}(A)$ is the set of all ID-derivations of A.

A Lie algebra L is called *semicomplete* if all ID-derivations are left multiplications. Given a Leibniz algebra A, to guarantee $A/\mathrm{Leib}(A)$ is a semicomplete Lie algebra we need the image of all ID-derivations A to differ from a left multiplication of A by $\mathrm{Leib}(A)$.

Definition 11. A Leibniz algebra A is called *semicomplete* if for every $\delta \in\mathrm{ID}(A)$ there is an $a \in A$ such that $\delta(b) - \mathrm{L}_a(b) \in\mathrm{Leib}(A)$ for every $b \in A$.

We have the containment $\mathrm{L}(A) \subseteq \mathrm{ID}(A) \subseteq \mathrm{Der}(A)$ for all Leibniz algebras. Therefore, all complete Leibniz algebras are semicomplete.

Proposition 2. If $A^2 =\mathrm{Leib}(A)$, then A is semicomplete.

Proof. Let A be a Leibniz algebra such that $A^2 =\mathrm{Leib}(A)$. Let $\delta \in\mathrm{ID}(A)$.
Then $\delta(a) \in\mathrm{Leib}(A)$ for all $a \in A$.
Thus, $\delta(a) - L_0(a) \in\mathrm{Leib}(A)$, and A is semicomplete. □

Recall from Chapter 3 that when A is a cyclic Leibniz algebra, $A^2 =\mathrm{Leib}(A)$. Thus, all cyclic Leibniz algebras are semicomplete.

Proposition 3. Let $A = I \oplus J$ be the direct sum of two ideals I and J.
If A is semicomplete, then both I and J are semicomplete.

Proof. Let $A = I \oplus J$, and let A be semicomplete.
Let $\delta \in$ID(I), and consider the linear map $\overline{\delta} : A \to A$ defined by
$\overline{\delta}(i + j) = \delta(i)$, for all $i \in I$ and $j \in J$.
Then $\overline{\delta}((i_1 + j_1)(i_2 + j_2)) = \overline{\delta}(i_1 i_2 + j_1 j_2) = \delta(i_1 i_2) = \delta(i_1)i_2 + i_1\delta(i_2)$
$= \overline{\delta}(i_1 + j_1)(i_2 + j_2) + (i_1 + j_1)\overline{\delta}(i_2 + j_2)$.
Thus, $\overline{\delta}$ is a derivation of A.
Also, since $\delta \in$ ID(I) and $A^2 = I^2 \oplus J^2$, this implies
$\overline{\delta}(i + j) = \delta(i) \in I^2 \subseteq A^2$ for all $i \in I$ and $j \in J$.
Thus, $\overline{\delta} \in$ ID(A), which implies $\overline{\delta} \in$L(A), since A is semicomplete.
Therefore, there exist $i \in I$ and $j \in J$ such that $\overline{\delta} = L_{i+j}$.
Hence, for all $x \in I$ and $y \in J$, $\overline{\delta}(x + y) = \delta(x) = L_{i+j}(x) = L_i(x) + L_j(x) \in I^2$.
Thus, $\delta(x) = L_i(x)$, which implies $\delta = L_i \in$L(I).
Therefore, since $\delta \in$Der(A) was arbitrary, I is semicomplete.
Similarly, if we define a linear map $\overline{\alpha} : A \to A$ by $\overline{\alpha}(i + j) = \delta(j)$ for all $i \in I$, $j \in J$, and $\delta \in$ ID(J), then we can show J is semicomplete. □

Example 6.0.1. $A = sl_2 \oplus \mathbb{C}^2 =$span$\{e, f, h, e_1, e_2\}$
Recall the multiplications: $ef = h, fe = -h, he = 2e, eh = -2e, hf = -2f,$
$fh = 2f, ee_1 = 0, e_1e = 0, e_2e = 0, ee_2 = e_1, fe_1 = e_2, e_1f = 0, fe_2 = 0, e_2f = 0, he_1 = e_1,$
$he_2 = -e_2, e_1h = 0, e_2h = 0$
Let $x = a_1e + a_2f + a_3h + a_4e_1 + a_5e_2$ and $y = b_1e + b_2f + b_3h + b_4e_1 + b_5e_2$.
This implies $xy = -2a_1b_3e + 2a_2b_3f + (a_1b_2 - a_2b_1)h + a_1b_5e_1 + a_2b_4e_2$.
Thus, $A^2 =$span$\{e, f, h, e_1, e_2\} = A$; and hence, all derivations are ID-derivations.
Also, Leib$(A) = \mathbb{C}^2$.
Since multiplication on the left by Leib(A) is zero, L$(A) =$span$\{L_e, L_f, L_h\}$.
Let $\delta \in$Der(A), and define:
$\delta(e) = a_1e + a_2f + a_3h + a_4e_1 + a_5e_2,$
$\delta(f) = b_1e + b_2f + b_3h + b_4e_1 + b_5e_2,$
$\delta(h) = c_1e + c_2f + c_3h + c_4e_1 + c_5e_2,$
$\delta(e_1) = d_1e + d_2f + d_3h + d_4e_1 + d_5e_2,$
$\delta(e_2) = n_1e + n_2f + n_3h + n_4e_1 + n_5e_2.$

$\delta(ef) = -2b_3e - 2a_3f + (a_1 + b_2)h + b_5e_1 = c_1e + c_2f + c_3h + c_4e_1 + c_5e_2 = \delta(h).$
$\implies -2b_3 = c_1, -2a_3 = c_2, a_1 + b_2 = c_3, b_5 = c_4, c_5 = 0.$

$\delta(fe) = 2b_3e + 2a_3f + (-a_1 - b_2)h + a_4e_2 = -c_1e - c_2f - c_3h - c_4e_1 = -\delta(h)$
$\implies a_4 = 0, c_4 = 0 = b_5.$
$\delta(eh) = (-2a_1 - 2c_3)e + 2a_2f + c_2h = -2a_1e - 2a_2f - 2a_3h - 2a_5e_2 = -2\delta(e)$
$\implies c_3 = 0(a_1 = -b_2), a_2 = 0, a_5 = 0.$
$\delta(fh) = -2b_1e + 2b_2f - c_1h = 2b_1e + 2b_2f + 2b_3h + 2b_4e_1 = 2\delta(f)$
$\implies b_1 = 0, b_4 = 0.$
$\delta(ee_1) = -2d_3e + d_2h + (a_3 + d_5)e_1 = 0$
$\implies d_3 = 0, d_2 = 0, a_3 = -d_5.$
$\delta(ee_2) = -2n_3e + n_2h + (a_1 + n_5)e_1 - a_3e_2 = d_1e + d_4e_1 + d_5e_2 = \delta(e_1)$
$\implies -2n_3 = d_1, n_2 = 0, a_1 + n_5 = d_4.$
$\delta(e_2e) = 2n_3e = 0$
$\implies n_3 = 0(d_1 = 0).$
$\delta(fe_1) = -d_1h + b_3e_1 + (b_2 + d_4)e_2 = n_1e + n_4e_1 + n_5e_2 = \delta(e_1)$
$\implies n_1 = 0, b_3 = n_4.$
Thus,

$$[\delta]_B = \begin{pmatrix} a_1 & 0 & -2b_3 & 0 & 0 \\ 0 & -a_1 & -2a_3 & 0 & 0 \\ a_3 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4 & b_3 \\ 0 & 0 & 0 & -a_3 & d_4 - a_1 \end{pmatrix}$$

$$= a_1 \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} + a_3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ d_4 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, Der(A)=span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

- $\delta_1(e) = e, \delta_1(f) = -f, \delta_1(h) = 0, \delta_1(e_1) = 0, \delta_1(e_2) = -e_2$

- $\delta_2(e) = h, \delta_2(f) = 0, \delta_2(h) = -2f, \delta_2(e_1) = -e_2, \delta_2(e_2) = 0$
- $\delta_3(e) = 0, \delta_3(f) = h, \delta_3(h) = -2e, \delta_3(e_1) = 0, \delta_3(e_2) = e_1$
- $\delta_4(e) = 0, \delta_4(f) = 0, \delta_4(h) = 0, \delta_4(e_1) = e_1, \delta_4(e_2) = e_2$

Now consider each derivation in the basis of Der(A) applied to
$x = a_1e + a_2f + a_3h + a_4e_1 + a_5e_2$.

- $\delta_1(x) - L_{\frac{h}{2}}(x) = (a_1e - a_2f - a_5e_2) - (a_1e - a_2f + \frac{1}{2}a_4e_1 - \frac{1}{2}a_5e_2)$
  $= -\frac{1}{2}a_4e_1 - \frac{1}{2}a_5e_2 \in$Leib$(A)$.
- $\delta_2(x) - L_{-f} = (a_1h - 2a_3f - a_4e_2) - (a_1h - 2a_3f - a_4e_2) = 0 \in$Leib$(A)$.
- $\delta_3(x) - L_e = (a_2h - 2a_3e + a_5e_1) - (a_2h - 2a_3e + a_5e_1) = 0 \in$Leib$(A)$.
- $\delta_4(x) - L_0(x) = a_4e_1 + a_5e_2 \in$Leib$(A)$.

Since there is an element $x_i \in A$ for each derivation δ_i in the basis for Der(A), such that $\delta_i(x) - L_{x_i}(x) \in$Leib$(A)$, we know that for every $\delta \in$Der(A) there is an element $a \in A$ such that $\delta(x) - \mathrm{L}_a(x) \in$Leib$(A)$ for every $x \in A$. Therefore, A is complete; and thus, A is semicomplete.

6.1 3-Dimensional Nilpotent Leibniz Algebras

Let A be a non-split non-Lie nilpotent Leibniz algebra, $A =$span$\{x, y, z\}$, with the given nonzero multiplications. We have shown in section 5.1 that the 3-dimensional nilpotent Leibniz algebras are not complete. We will find a basis for A^2 for each algebra to show that all 3-dimensional nilpotent Leibniz algebras are semicomplete.

Example 6.1.1. $x^2 = y, xy = z$.
Let $a = \alpha_1x + \alpha_2y + \alpha_3z$ and let $b = \beta_1x + \beta_2y + \beta_3z$. Consider $ab = (\alpha_1x + \alpha_2y + \alpha_3z)(\beta_1x + \beta_2y + \beta_3z) = \alpha_1\beta_1y + \alpha_1\beta_2z$. Thus, $A^2 =$span$\{y, z\}$. Recall from Example 5.1.1, Leib(A)=span$\{y, z\}$. Since $A^2 =$Leib(A), A is semicomplete.

Example 6.1.2. $x^2 = z$
By Example 5.1.2 Leib(A)=span$\{z\}$, and using techniques similar to those in Example 6.1.1, we can show $A^2 =$span$\{z\}$=Leib(A). Therefore, A is semicomplete.

Example 6.1.3. $x^2 = z; y^2 = z$
We can show A^2 =span$\{z\}$=Leib(A). Therefore, A is semicomplete.

Example 6.1.4. $xy = z; yx = -z; y^2 = z$
We can show A^2 =span$\{z\}$=Leib(A). Therefore, A is semicomplete.

Example 6.1.5. $xy = z; yx = \alpha z, \alpha \in \mathbb{F} \setminus \{-1, 1\}$
We can show A^2 =span$\{z\}$=Leib(A). Therefore, A is semicomplete.

6.2 4-Dimensional Nilpotent Leibniz Algebras

Let A be a non-split non-Lie nilpotent Leibniz algebra, A =span$\{w, x, y, z\}$, with the given nonzero multiplications. We have shown in section 5.2 that the 4-dimensional nilpotent Leibniz algebras can be either complete or not complete. We will find a basis for A^2 for each algebra to show that all 4-dimensional nilpotent Leibniz algebras are semicomplete.

Example 6.2.1. $wy = z$, $yx = z$
We can show Leib(A)=span$\{z\} = A^2$. Hence, A is semicomplete.

Example 6.2.2. $wy = z$, $x^2 = z$, $xy = z$, $yw = z$, $yx = -z$
We can show Leib(A) =span$\{z\} = A^2$. Hence, A is semicomplete.

Example 6.2.3. $wx = z$, $xw = -z$, $y^2 = z$
We can show A^2 =span$\{z\}$ =Leib(A). Hence, A is semicomplete.

Example 6.2.4. $wx = z$, $xw = -z$, $x^2 = z$, $y^2 = z$
We can show Leib(A) =span$\{z\} = A^2$. Hence, A is semicomplete.

Example 6.2.5. $wx = z$, $xw = \alpha z$, $y^2 = z$, $\alpha \in \mathbb{C}/\{-1, 1\}$
For all $\alpha \in \mathbb{C}/\{-1, 1\}$, Leib($A$) =span$\{z\} = A^2$. Hence, A is semicomplete.

Example 6.2.6. $w^2 = z$, $x^2 = z$, $y^2 = z$.
We can show Leib(A) =span$\{z\} = A^2$. Hence, A is semicomplete.

Example 6.2.7. $w^2 = x$, $wx = y$, $wy = z$.
These nonzero multiplications define a nilpotent cyclic Leibniz algebra. Thus, Leib(A) = A^2; and hence, A is semicomplete.

Example 6.2.8. $w^2 = z$, $wx = y$, $xw = -y$.
We can show Leib(A) =span$\{z\}$, but A^2 =span$\{y, z\}$. Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$ where

- $\delta_1(w) = w, \delta_1(x) = 0, \delta_1(y) = y, \delta_1(z) = 2z$
- $\delta_2(w) = x, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = y, \delta_5(y) = 0, \delta_5(z) = 0$
- $\delta_6(w) = 0, \delta_6(x) = x, \delta_6(y) = y, \delta_6(z) = 0$
- $\delta_7(w) = 0, \delta_7(x) = z, \delta_7(y) = 0, \delta_7(z) = 0$

Therefore, ID(A)=span$\{\delta_3, \delta_4, \delta_5, \delta_7\}$, and we can write L($A$) =span$\{L_w, L_x\}$ =span$\{\delta_4 + \delta_5, -\delta_3\}$. So, im($\delta_4$) $\subseteq$Leib(A) and im(δ_7) $\subseteq$Leib(A); and im($\delta_5 - L_w$) $\subseteq$Leib(A) and im($\delta_3 + L_x$) $\subseteq$Leib(A). Hence, A is semicomplete.

Example 6.2.9. $w^2 = z$, $wx = y$, $xw = -y$, $x^2 = z$.
We can show Leib(A) =span$\{z\}$, but A^2 =span$\{y, z\}$. Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$ where

- $\delta_1(w) = w, \delta_1(x) = x, \delta_1(y) = 2y, \delta_1(z) = 2z$
- $\delta_2(w) = x, \delta_2(x) = -w, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = y, \delta_5(y) = 0, \delta_5(z) = 0$
- $\delta_6(w) = 0, \delta_6(x) = z, \delta_6(y) = 0, \delta_6(z) = 0$

Therefore, ID(A)=span$\{\delta_3, \delta_4, \delta_5, \delta_6\}$, and we can write L($A$) =span$\{L_w, L_x\}$ =span$\{\delta_4 + \delta_5, \delta_6 - \delta_3\}$. So, im($\delta_4$) $\subseteq$Leib(A) and im(δ_6) $\subseteq$Leib(A); and im($\delta_5 - L_w$) $\subseteq$Leib(A) and im($\delta_3 + L_x$) $\subseteq$Leib(A). Hence, A is semicomplete.

Example 6.2.10. $w^2 = z$, $wx = y$, $xw = -y$, $wy = z$, $yw = -z$.
We can show Leib(A) =span$\{z\}$, but A^2 =span$\{y, z\}$. Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6\}$ where

- $\delta_1(w) = w, \delta_1(x) = 0, \delta_1(y) = y, \delta_1(z) = 2z$
- $\delta_2(w) = x, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = y, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = z, \delta_4(x) = 0, \delta_4(y) = 0, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = y, \delta_5(y) = z, \delta_5(z) = 0$
- $\delta_6(w) = 0, \delta_6(x) = z, \delta_6(y) = 0, \delta_6(z) = 0$

Therefore, ID(A)=span$\{\delta_3, \delta_4, \delta_5, \delta_6\}$. Since L($A$)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_4 + \delta_5, -\delta_3, -\delta_4\}$,$A$ is semicomplete.

Example 6.2.11. $wx = y$, $xw = -y$, $x^2 = z$, $wy = z$, $yw = -z$.
We can show Leib(A) =span$\{z\}$, but A^2 =span$\{y, z\}$. Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

- $\delta_1(w) = w, \delta_1(x) = 2x, \delta_1(y) = 3y, \delta_1(z) = 4z$
- $\delta_2(w) = y, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = z, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = y, \delta_4(y) = z, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = z, \delta_5(y) = 0, \delta_5(z) = 0$

Therefore, ID(A) =span$\{\delta_2, \delta_3, \delta_4, \delta_5\}$. Since L($A$)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_4, \delta_5 - \delta_2, -\delta_3\}$, A is semicomplete.

Example 6.2.12. $w^2 = z$ $wx = y$, $xw = -y + z$, $wy = z$, $yw = -z$.
We can show Leib(A) =span$\{z\}$, but A^2=span$\{y, z\}$. Der(A) =span$\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

- $\delta_1(w) = x, \delta_1(x) = x, \delta_1(y) = y, \delta_1(z) = z$

- $\delta_2(w) = y, \delta_2(x) = 0, \delta_2(y) = 0, \delta_2(z) = 0$
- $\delta_3(w) = z, \delta_3(x) = 0, \delta_3(y) = 0, \delta_3(z) = 0$
- $\delta_4(w) = 0, \delta_4(x) = y, \delta_4(y) = z, \delta_4(z) = 0$
- $\delta_5(w) = 0, \delta_5(x) = z, \delta_5(y) = 0, \delta_5(z) = 0$

Therefore, ID(A) =span$\{\delta_2, \delta_3, \delta_4, \delta_5\}$. Since L($A$)=span$\{L_w, L_x, L_y\}$ =span$\{\delta_3 + \delta_4, \delta_3 - \delta_2, -\delta_3\}$, A is semicomplete.

Example 6.2.13. $wx = y$, $xw = -y + z$, $x^2 = z$, $wy = z$, $yw = -z$.
We can show A is complete. Therefore, A is semicomplete.

Example 6.2.14. $w^2 = y$, $wx = z$.
We can show Leib(A) =span$\{y, z\} = A^2$. Hence, A is semicomplete.

Example 6.2.15. $w^2 = y$, $xw = z$.
We can show Leib(A) =span$\{y, z\} = A^2$. Hence, A is semicomplete.

Example 6.2.16. $xw = y$, $wx = z$, $x^2 = -y$
We can show A^2 =span$\{y, z\}$=Leib(A). Hence, A is semicomplete.

Example 6.2.17. $w^2 = y$, $xw = z$.
We can show Leib(A) =span$\{y, z\} = A^2$. Hence, A is semicomplete.

Example 6.2.18. $w^2 = y$, $xw = z$, $wx = -\alpha y$, $x^2 = -z$, $\alpha \in \mathbb{F}$
We can show A^2 =span$\{y, z\}$ =Leib(A). Hence, A is semicomplete.

Example 6.2.19. $w^2 = y$, $wx = y$, $xw = y + z$, $x^2 = z$.
We can show Leib(A) =span$\{y, z\} = A^2$. Hence, A is semicomplete.

Example 6.2.20. $wx = y$, $wy = z$.
We can show Leib(A) =span$\{y, z\} = A^2$. Hence, A is semicomplete.

Example 6.2.21. $wx = y$, $x^2 = z$, $wy = z$.
We can show Leib(A) =span$\{y, z\} = A^2$. Hence, A is semicomplete.

Example 6.2.22. $wx = y$, $xw = z$, $wy = z$.
We can show Leib(A) =span$\{y, z\} = A^2$. Hence, A is semicomplete.

Example 6.2.23. $wx = y$, $xw = z$, $x^2 = z$, $wy = z$.
We can show Leib(A) =span$\{y, z\} = A^2$. Hence, A is semicomplete.

Example 6.2.24. $w^2 = y$, $xw = z$, $wy = z$
We can show A^2 =span$\{y, z\}$ =Leib(A). Hence, A is semicomplete.

Example 6.2.25. $w^2 = y$, $x^2 = z$, $wy = z$
We can show A^2 =span$\{y, z\}$=Leib(A). Hence, A is semicomplete.

6.3 3-Dimensional Solvable Leibniz Algebras

Let A be a non-split non-Lie non-nilpotent Leibniz algbera, such that A =span$\{x, y, z\}$ with the following nonzero multiplications. We will find a basis for A^2 for each algebra to show that all 3-dimensional solvable Leibniz algebras are semicomplete.

Example 6.3.1. $xz = z$
We can show A^2 =span$\{z\}$=Leib(A). Therefore, A is semicomplete.

Example 6.3.2. $xz = \alpha z, \alpha \in \mathbb{F} \setminus \{0\}, xy = y, yx = -y$
We can show A is complete. Therefore, A is semicomplete.

Example 6.3.3. $xy = y; yx = -y; x^2 = z$
We can show A is complete. Therefore, A is semicomplete.

Example 6.3.4. $xz = 2z; y^2 = z; xy = y; yx = -y; x^2 = z$
We can show A is complete. Therefore, A is semicomplete.

Example 6.3.5. $xz = \alpha z, \alpha \in \mathbb{F} \setminus \{0\}, xy = y$
We can show A^2 =span$\{y, z\}$=Leib(A). Therefore, A is semicomplete.

Example 6.3.6. $xz = z + y; xy = y$
We can show A^2 =span$\{y, z\}$=Leib(A). Therefore, A is semicomplete.

Example 6.3.7. $xz = y; xy = y; x^2 = z$
We can show A^2 =span$\{y, z\}$=Leib(A). Therefore, A is semicomplete.

Therefore, all 3-dimensional non-split non-Lie Leibniz algebras and all 4-dimensional non-split non-Lie nilpotent Leibniz algebras are semicomplete. Recall from Chapter 5 that many of these examples were not complete. Thus, while completeness implies semicompleteness for Leibniz algebras, not all semicomplete Leibniz algebras are complete.

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