ABSTRACT

PAUL, ANDRAS GEORGES. Student Learning Experiences in a Community College Precalculus Algebra Course: A Mixed Methods Study of the Influence of Structure Sense Instruction on Students' Structure Sense and Algebraic Proficiency. (Under the direction of Dr. Lee V. Stiff).

The purpose of the study was to investigate "what students' work and explanation of their work would show about their development of structure sense and their algebraic proficiency after receiving structure-sense teaching for eight weeks in a community college precalculus algebra course." Structure sense is a group of abilities formally described by Hoch and Dreyfus (2004; 2006; 2007) to refer to recognition and use of structure when reasoning through algebraic statements. A review of the literature suggested that algebraic proficiency can be described on two levels: (a) a level pertaining to mathematics, in general and (b) a level pertaining to algebra in particular. The first level characterizes the broad processes and practices through which mathematical proficiency is shown according to CCSSM (2011), NCTM (2000), and NRC (2001). The second level of algebraic proficiency involves aspects that are more specific to the nature of algebra as described by Carlson et al.(2010), Driscoll (1999), Kieran (2007) McCallum (2007), and NRC (2001). In the quantitative strand, students' test scores showed improvement in structure sense and in algebraic proficiency. However, improvements in both areas needed to be unpacked. In the qualitative strand of the study, students' work and explanation of their work showed development of structure sense in various ways and more in some aspects than others. Their work also suggested other patterns. The results and analyses of the study have potentially important information for teaching and redesigning precalculus algebra courses.

Student Learning Experiences in a Community College Precalculus Algebra Course: A Mixed Methods Study of the Influence of Structure Sense Instruction on Students' Structure Sense and Algebraic Proficiency

by
Andras Georges Paul

A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Mathematics Education

Raleigh, North Carolina

2018

APPROVED BY:

Dr. Lee V. Stiff Committee Chair	Dr. Karen A. Keene
Dr. Allison McCulloch	Dr. Ernest Stitzinger

BIOGRAPHY

Andras Georges Paul was born near Cap Haitien, Haiti in 1975, the oldest son of George and Marie-Suzie Paul. In 1994, he traveled to the United States where he finished his secondary studies. He then attended University of South Florida, where he graduated with a bachelor's degree in chemical engineering in 2002. After graduation, he spent two years teaching mathematics at the middle school level in the Norlina school district in North Carolina, where he developed a love for teaching. Around this time he also began teaching mathematics as an adjunct professor at Durham Technical Community College, which introduced him to teaching mathematics at the post-secondary level.

In 2009, Andras began graduate work in Mathematics Education at North Carolina State University, earning a Master's degree in 2011. He subsequently enrolled in the Ph.D. program at North Carolina State University and has been working on the degree part-time since 2012, as off the final review of this document in fall 2018.

Andras Paul has been a faculty member at Wake Technical Community College in Raleigh, North Carolina since 2012, where he is currently an Assistant Professor in the mathematics and physics department. There, he teaches a range of classes, including precalculus algebra, precalculus trigonometry, quantitative literacy, and linear algebra.

Andras is a talented soccer player and an avid soccer fan; he belongs to a soccer team that plays weekly. He is married to Kathrynne Paul and has a son, Zachary.

ACKNOWLEDGMENTS

I dedicate this work to my spouse, Kathrynne Homicile Paul, for her prayers, her advice, and for all the support she has given to me.

I give thanks and honor to Christ, Jesus, God in Whom I live, move, and have my being. I thank God for blessing me with an intelligent, competent, but modest spouse, and a dedicated mother to our son. I thank God for blessing me with such a lovely son, Zachary. I thank Dr. Lee Stiff, my committee chair, for faithfully supporting me through the process, even starting with my masters' degree. I enjoyed advising time (so did Zachary); I am so grateful for the opportunity to learn from you. I thank you, Dr. Karen Keene; I had positive learning experiences in your courses, and I am so thankful for your commitment to see me through the entire process. I thank you, Dr. Allison McCulloch, for your kindness and at the same time for getting me to work harder for a product that is better in quality. I am so thankful for you, for your patience, and your encouragement. I thank you Dr. Ernest Stitzinger for calmly and quietly encouraging and supporting me through the process. I thank Dr. Karen Norwood for supporting me as my advisor through the beginning and early stage of this process. Thank you for being the first person to make me think that I could consider a doctoral degree. I am so grateful for the opportunity to learn from you. I am thankful for your patience. I thank my supervisors and colleagues at Wake Technical Community College for their support. I know that it would have been very difficult to complete this degree without the kindness of my supervisors and of some of my colleagues at Wake Technical Community College. I thank my students for their support and collaboration; I am so grateful for them. I thank my friends for their loyal support. I thank several of my fellow graduate students for their kindness and generous support. I do not

want to miss anyone; so I will not list any names. I thank my dad for his sacrifice for me (as well as my siblings) and for his motivation so that we can go further than he has in every aspect of life, especially in christianity, education, and finance. I thank my mother for always caring for me; her prayers have moved mountains and caused miracles in my life. I thank my older sister Claire (Nathania Paul) for always being a strong supporter of me; I thank all my other siblings for their support in every way. I also thank my father-in-law for teaching me a lot of things about leadership. I thank my mother-in-law for her support with my son during the hard years when I was taking doctoral courses and working full time. I thank Dr. Esther Ogirri-Burgess and my colleague Mrs. Karen Klein for their support. I thank Mr. G. for being the first person who told me that I was capable of performing at a higher level academically than I was. I was 16 years old then, and Mr. G.'s words never left me. I thank Mr. Fadi Semaan for giving my first big break in employing me in United States when I still had a lot of English to learn.

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Chapter 1: Introduction

Students' low algebraic proficiency is a worldwide concern (Van Stiphout, Drijvers, & Gravemeijer, 2011). Specifically, students have traditionally been low performing in algebra at all levels from the middle grades to college (Kieran, 1992, 2007). In the United States, at the undergraduate level, students' passing rates in algebra courses are often unsatisfactory. Since students taking mathematics courses at community colleges account for about half of all undergraduate students taking mathematics (Rodi, 2007), examining the algebraic proficiency of students in community colleges is an important undertaking. In community colleges, students experience great difficulty in completing precalculus algebra successfully (Barnes, Cerrito, & Levi, 2004; Reyes, 2010). In fact, passing rates in precalculus algebra in community colleges are low (Nguyen, 2015)--they can be less than 50% (Berry, 2003). Indeed, a large percentage of community college students repeat precalculus algebra up to three times and even more (Reyes, 2010) if allowed. Low achievement in precalculus algebra is a large problem for most community colleges.

Given the issue of low achievement in precalculus algebra and the emphasis that the National Council of Teachers of Mathematics (NCTM, 1989; 2000; 2014), the National Mathematics Advisory Panel (NMAP, 2008), and the Common Core State Standards (CCSS, 2010) placed on algebra, the teaching and learning of algebra continues to be an area that requires more investigation. The teaching and learning of algebra requires more attention and efforts from mathematics education researchers in order to address the on-going concern about students' low performance in algebra at all levels from the middle grades to college. In part, this work involves understanding the causes of students' difficulties and addressing these causes through appropriate curricula and instruction. However, because of

philosophical disagreements about what algebra is and how to effectively incorporate technology in algebra instruction (Kieran, 2007), characterizing students' difficulty and remedying them have become part of the problem.

Students' low algebraic proficiency and passing rates in community colleges not only affect their prospects for studies and careers in the STEM fields (NMAP, 2008; Zorn, 2002), but also underscore the need to further address algebra instruction and research in post-secondary mathematics. Although more research on mathematics instruction is necessary to inform policy makers, curriculum developers, teachers, and researchers (Hiebert & Grouws, 2007), mathematics education research has not adequately considered mathematics instruction in the community college setting (Mesa et al., 2014) as a means to better understand teaching and learning in algebra. Kieran (1992; 2007) has shown that students' lack of structural conception of algebra is one reason responsible for their low algebraic proficiency. Hoch and Dreyfus (2006) and Van Stiphout et al. (2011) have attributed the issue of students' algebraic proficiency to a lack of "structure sense" and have considered how structure sense instruction can help develop algebraic proficiency in students.

The term 'structural' refers to operations carried out on algebraic expressions which result in other algebraic expressions (Kieran, 1992). These operations go beyond the four basic operations of adding, subtracting, multiplying, and dividing. They include simplifying, factoring, rationalizing, solving, differentiating, and so on. This idea of structural conception only applies to algebraic representation, not to numerical and graphical representations. Similarly, structure sense has been defined for algebraic representation of mathematical ideas (See Hoch and Dreyfus (2004, 2006)). Structure sense refers to the gestalt view or global

perception of algebraic expressions and equations (Arcavi, 1994; Hoch & Dreyfus, 2004; 2005; 2006; 2007; 2010; Kieran, 1992; 2007; Stiphout et al., 2011).

Structure sense represents a group of learned abilities for dealing with concepts germane to algebraic expressions and equations, but such abilities are separate from manipulative abilities (Hoch & Dreyfus, 2004). Hoch and Dreyfus (2004) defined the term structure as a "broad view analysis of the way in which an entity is made up of its parts… Any algebraic expression or sentence represents an algebraic structure (p. 50)." More specifically, Hoch and Dreyfus (2006) **defined structure sense** as being able to:

> recognize a familiar structure in its simplest form **(SS1)**; deal with a compound term as a single entity and through an appropriate substitution recognize a familiar structure in a more complex form **(SS2)**; choose appropriate manipulations to make best use of a structure **(SS3)** (p. 306).

The goal of this study was to use the "structure sense" theoretical framework described in Arcavi (1994; 2005), Hoch and Dreyfus (2004; 2006; 2007; 2010), and Van Stiphout et al. (2011) for the sake of improving algebra instruction. More specifically, the study will use a "structure sense" instructional intervention in a precalculus algebra course taught at a community college to investigate the influence on students' structure sense and algebraic proficiency.

The precalculus algebra course is the first of a two-course series that many students take in community colleges. Students either have to successfully complete developmental mathematics courses (denoted by DMA) up to DMA 080 or place out of them in order to get enrolled in this precalculus algebra course. Starting with DMA 010 and up to DMA 030, only arithmetic topics are covered. DMA 040 through DMA 080 cover algebra topics such

as expressions, linear equations and inequalities, graphs and equations of lines, polynomials and quadratic applications, rational expressions and equations, and radical expressions and equations. The precalculus algebra course covers these topics in more depth and covers additional topics like exponential and logarithmic equations and applications, rational inequalities and applications, absolute value equations and inequalities, systems of linear and nonlinear equations and inequalities. The instructional intervention was implemented on those topics covered in DMA courses beside other topics.

Instruction can be described as a function of four elements: classroom environment, type of tasks, type of discourse, and analysis of teaching (NCTM, 1991). In a similar way, the Standards-Based Learning Environment (SBLE) emphasized classroom environment, type of tasks, and discourse for describing instruction (Tarr, Reys, R. E., Reys, B. J., Chavez, Shih, Osterlind, 2008). The background of the instructional intervention that was implemented in the study consists of (a) a definition of structure sense instruction (SI) and (b) a description of tasks that illustrate structure sense instruction and that contrast it with traditional instruction (TI).

Structure Sense Instruction (SI). Drawing on NCTM's (1991) and Tarr et al's (2008) description of instruction, I define structure sense instruction as a teaching practice that seeks to develop structure sense among students through deliberate use of tasks that lend themselves to the definition of structure sense articulated in Hoch and Dreyfus (2004, 2006, 2007). And in keeping with NCTM's (1991) and Tarr et al's (2008) description of instruction, structure sense instruction also involves classroom discourse around those special tasks in order to emphasize structure, including how to recognize them, how to use them, and how to talk about them. The goal of structure sense instruction is to help students develop

structure sense, which is the ability to recognize structure in algebraic statements, to use them, and to communicate about them.

The goal of structure sense instruction is mediated by special tasks, several of which have been recommended in the literature (e.g., Arcavi, 1994, 2005; Hoch & Dreyfus, 2004; 2005; 2006; 2007; 2010; Stiphout et al., 2011). These tasks give students opportunities to recognize structure in algebraic equations and to communicate their reasoning about how they recognize and use structure (e.g., Arcavi (2005), Hoch and Dreyfus (2004, 2005, 2010)). Tables 1-8 show excerpts of those tasks that I used as part of the actual intervention in the study in order to elaborate on structure. Several examples are given to illustrate how the structure sense intervention was implemented across a variety of topics. The complete set of lessons that comprises the intervention can be found in Appendix J.

The first illustration, seen in Table 1, focuses on function notation. The goal is to help students recognize and use function notations as single entities and function expressions as single entities. This task speaks to the second component of the definition of structure sense (SS2 part a): “deal with a compound term as a single entity ”. It is also conducive to classroom discourse.

Table 1: Task on function notation adapted from Arcavi (2005)

The function $p(x) = 10\sqrt{x}$ is used to adjust original test scores, x, to new test scores *p(x)*, in percent. 1. In light of this situation, what does the 90 in the equation $90 = 10\sqrt{x}$ represent? Why? 2. How about the expression $10\sqrt{x}$? What does it represent? Why? 3. What is wrong with the equation $90(x) = 10\sqrt{x}$? 4. Based on your responses to questions 1 and 2, what is the equation $90 = 10\sqrt{x}$ stating or saying about the relationship between p(x) and $10\sqrt{x}$?

In this task, students were given the opportunity to learn that the function expression "$10\sqrt{x}$" represents a single entity (i.e. adjusted grade) even though that expression involves several operations. In a similar way the function notation p(x) represents a single entity because of the equivalence property of the equal sign even though the notation p(x) is composed of four characters. Through this task, the instructor led students to espouse a gestalt (global) view of function notation and function expression regardless of the number of terms/symbols in the expression and regardless of the structure of the expression. This view is different from the operational (or procedural) view of function expression. And in the same way, students can be lead to see function notation as a single entity because the equal sign between function notation and function expression. The use of this task intended to help increase students' structural view.

The second illustration, seen in Table 2, focuses on the "average of change" and the "difference quotient." The goal was to help students recognize that new and more complex expressions sometimes have the same structure as some familiar expressions. It is important for students to know that those recognitions can inform their work and understanding in

positive ways. Seeing the familiar in new and complex situations is a key ability for problem solving (NRC, 2001; Schoenfeld, 2007). This task speaks to the second component of the definition of structure sense (SS2 part b): "through an appropriate substitution recognize a familiar structure in a more complex form.

Table 2: Task from lesson on average rate of change

1. The average rate of change (ARC) of the function y = f(x) between x_1 and x_2 is $\frac{change\ in\ y}{change\ in\ x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Use appropriate substitution(s) to show that the ARC formula comes from a familiar formula.	2. Use appropriate substitution(s) to show that the difference quotient, $\frac{f(a+h)-f(a)}{h}$ Comes from a familiar structure.

In the task shown in Table 2, students went through the process of writing out the substitution $f(x_1) = y_1$ and $f(x_2) = y_2$ to show that the average rate of change is a more complex form of the familiar slope formula or structure,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

In a similar way, students went through the process of writing out the substitution $f(a) = y_1$, $f(a+h) = y_2$, $h = (a+h) - a$ to show that the difference quotient is a more complex form of the familiar slope formula or structure,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Hoch and Dreyfus (2007) highlighted a supporting element for the three-component definition of structure sense called the "substitution principle." The substitution principle states "if a parameter is replaced by a compound term, or if a compound term is replaced by a parameter, the structure remains the same" (Hoch & Dreyfus, 2007, p. 436).

Table 3: Task from lesson on difference quotient

1. Write the appropriate substitution to show that $f(a+h) = 3(a+h)^2 - 5(a+h) + 8$ and $f(x) = 3x^2 - 5x + 8$ have the same structure or the same type.	2. Apply suitable substitutions to simplify $(x+h)^2 + 4(x+h) - (x^2 + 4x)$ using function notation.

The two items in Table 3 were used to help students see how structure is preserved when one applies the substitution principle. These items were also used to help students practice interchanging function expressions and function notations. These interchanges are useful when students to learn to combine and decompose functions.

The third illustration focuses on transforming, combining, and decomposing functions. The goal was to help students recognize structures within other structures as shown in the items of Table 4.

Table 4: Transformation and combination of functions

In explicitly defined functions **y =f(x)** or y = name of the function (Indep. Var.) 1. If f(x) = x^2 , then _____ = -x^2. 2. If g(x) = x+3, then _____ = -(x+3). 3. If $j(x) = \sqrt{x-7}$, then $-2\sqrt{x-7}+8 =$ _____. 4. If $q(x) = x^2$, then _____ = $(2x-3)^2$
Given $f(x) = 4x+1,\ g(x) = x^2-5$, and $h(x) = \sqrt{x-8}$ 5. Write the expression that g(x) represents _____ 6. Write the expression $\frac{x^2-5}{4x+1}$ in terms of the function notations above _____ 7. The function notation $(f \circ g)(x)$ is the same as $f(g(x))$. Explain briefly why $f(g(x))$ is the same as $f(x^2-5)$. 8. Since $f(x) = 4x+1$, then $f(x^2-5)$ =_____ . Use the substitution principle to justify your answer.

In Item 3, students were supposed to learn to see the $\sqrt{x-7}$ structure within $-2\sqrt{x-7}+8$ as a single entity. In a similar way, students could learn to recognize the quadratic structure in Item 4 and use that recognition to see that the answer to Item 4 is $q(2x-3)$. In Item 8, students could learn to deal with x^2-5 as a single entity which replaces x in $f(x) = 4x+1$ to yield

$f(x^2-5) = 4(x^2-5)+1$. Therefore the structure of "4 times a quantity +1" is preserved, but only that the quantity is a compound term "x^2-5" instead of x.

Table 5: Decomposition of functions

1. (a) State all the operations involved in $Q(x) = \frac{5}{3x+4}$. (b) Do the same for $R(x) = \sqrt{4x-3}$.

2. Which of the operations you listed for question 1 describes $\frac{5}{3x+4}$ entirely or a as whole? How about $\sqrt{4x-3}$?

Use this format and what you learn about function notation (i.e. notation $(f \circ g)(x)$ is the same as $f(g(x))$) to decompose functions:

- 1st, write the structure (operation) that captures the expression as a **whole**.
- 2nd, write smaller or **inner** operation(s) of the expression.
  For example, the 1st function can be $\frac{5}{x}$ and the 2nd function can be 3x+4.

The task shown in Table 5 were used to help draw students' attention to taking a gestalt or broad view analysis of expressions. "A broad view analysis in which an entity is made up of its parts…" is an earlier definition of structure given in Hoch and Dreyfus (2004) prior to the three-component definition given in Hoch and Dreyfus (2006).

McCallum (2007) provided a real-world model of the weekly profit made by a vendor of t-shirts (See Table 6). This task was part of a lesson on quadratic functions. Students were asked to reason about the weekly profit through quadratic expressions given in factored, vertex, and standard forms.

Table 6: Structure sense through forms of quadradic functions adapted from McCallum (2007)

A street vendor of t-shirts finds that if the price of a t-shirt is set at x dollars, the profit from a week's sales can be modeled by any of the quadratic expressions below.

A. $(x - 6)(900 - 15x)$

B. $-15(x - 33)^2 + 10935$

C. $-15(x - 6)(x - 60)$

D. $-15x^2 + 990x - 5400$

1. Which form of these quadratic expressions shows most clearly the maximum profit and the price that gives that maximum? Explain (any explanation i.e. graphic or numeric).

2. Which form of these quadratic expressions shows most immediately the break-even price? Explain.

3. Why is \$10,935 the maximum profit and \$33 is the price which gives that maximum profit? Provide an explanation based on the **terms** of the expression $-15(x - 33)^2 + 10935$, (beyond using the vertex, or graph, or guess & check).

These items in Table 6 were used to further reinforce the importance of taking a gestalt or broad view analysis of algebraic expressions so that ultimately students would form this kind of habit of mind.

The next illustration deals with framing conceptual understanding of solving equations algebraically around the idea of structure. The first part of Table 7 shows this framing in three steps. The second part of Table 7 shows some of the types of equations that were used as examples, but the actual task also included other types of equations such as rational, radical, and exponential equations; see Appendix J.

Table 7: Solving equations algebraically

Part 1:
Solving an equation means to find the values for the unknown(s) for which the equation is true. It is true that some equations can be solved by inspection, by guess-and-check, graphically, or numerically. But solving equations algebraically (analytically) involves the following three key ideas.

1. Identify the defining structure of the equation.
2. Collect, isolate, or single out that defining structure.
3. Break that defining structure by its inverse, or by factoring, or by applying its definition.

Part 2

Equations	Defining Structure	Breaking the Structure
1. $4x - (x + 1) = 7 + x$	Linear	Multiply by a multiplicative inverse or add an additive inverse. For example • $2x = 8$ can be solved by multiplying by the multiplicative inverse of 2 which is $\frac{1}{2}$. • $x - 5 = 11$ can be solved by adding the additive inverse of -5 which is +5 on each side.
2. $x^2 + 3x - 10 = 0$	Quadratic	Either by • Completing the square and taking the square root • Factoring into linear factors • Quadratic formula
3. $3\|4x - 7\| + 2 = 41$	Absolute value	Apply the definition of abs. value which eliminates the absolute value bars and splits the equation into two different equations: • One for the distance to the left of zero • The other for the distance to the right of zero.
4. $\log_3(x + 15) = 2 + \log_3(x - 1)$	Logarithmic	Convert to an exponential statement, which is the inverse of log.

<u>Note</u>: Before breaking a structure the second idea must be applied, meaning (isolate, collect, or single out the defining structure).

The idea of inverse structure of an equation for breaking a corresponding structure was emphasized to help students further assimilate how structure plays a key role in the rationale behind the steps taken in solving equations algebraically. The discussion of those three ideas was followed by solving Equation #3 as an example to illustrate the process. This process for solving equations underlies the third component of the structure sense definition given in Hoch and Dreyfus (2006): "choose appropriate manipulations to make best use of a structure," (SS3).

The last illustration focuses on solving quadratic inequalities. It speaks to the idea of structure as a "broad view analysis of the way in which an entity is made up of its part" (Hoch & Dreyfus, 2004, p. 50). This lesson on solving quadratic inequalities started with the following four examples shown in Table 8.

Table 8: Solving quadratic inequalities

In examples 1-4, provide responses based on the terms and meaning of the inequalities (without resorting to their graphs).

1. Explain why x on $(-\infty, \infty)$ is the solution of the inequality $x^2 \geq 0$?

2. Explain why the inequality $x^2 < 0$ has no solutions?

3. Explain why x on $(-\infty, \infty)$ is the solution of the inequality $-x^2 - 6 < 0$? How about $-(x+5)^2 - 6 < 0$?

4. Explain why the inequality $x^2 + 1 < 0$ has no solutions?

Solving polynomial inequalities involves collecting all nonzero terms on one side of the inequality symbol making the other side zero. For example,

polynomial < 0

This allows us to think about the resulting polynomial as a single entity, say y. For example,

y < 0

This rephrasing of an inequality statement allows us to think about when is y negative. In other cases, when is y positive, and when is y = 0 (x-intercept).

5. Solve $x^2 - 6x \geq 7$.

These examples shown in Tables 1-8 do not describe any lesson in their entirety; they merely represent the start of most of lessons.

Thus far, a definition of structure sense instruction (SI) and a descripton of tasks that illustrate structure sense instruction has been provided. The third point about the SI intervention is that the deliberate and persistent use of such tasks (e.g., Tables 1-8) differentiate it significantly from traditional instruction (TI). The consistent focus on structure or types of expressions/equations and the three components of Hoch and Dreyfus' (2006) definition is not done in traditional instruction. Those tasks differ from tasks used in traditional instruction because the reasoning approaches used in them center around Hoch and Dreyfus' (2004, 2006, 2007) definitions of structure sense. Structure sense instruction emphasizes structure in more elaborate ways than in traditional instruction. Traditional instruction attends to structure sense to a lesser extent without a primary goal or any stated priority of developing structure sense among students. Whereas structure sense instruction has as a target to help students develop the abilities to recognize and use structure of algebraic expressions, equations, and inequalities, traditional instruction does not. The following statement from Hiebert and Grouws (2007) lends credence to this argument about emphasis as a differentiating factor between types of instruction:

> But teaching, as we have defined it, plays a major role in shaping students' learning opportunities. The emphasis teachers place on different learning goals and different topics, the expectations for learning that they set, the time they allocate for particular topics, the kinds of tasks they pose, the kinds of questions they ask and responses they accept, the nature of the discussions they lead—all are part of teaching and all influence the opportunities students have to learn (p. 379).

The goal of this study is to describe structure sense instruction by assessing the influence of a "structure sense" intervention on students' structure sense and algebraic proficiency in the context of a community college precalcalus algebra course.

The research questions are:

1. Is there significant difference in structure sense between students' Pretest-A and Posttest-A performance in a community college precalculus course?
2. Is there significant difference in algebraic proficiency between students' Pretest-B and Posttest-B performance in a community college precalculus course?
3. In what ways do students' work show development of structure sense in a community college precalculus algebra course?

Theoretical Framework

This section describes how my perspective shapes the research study. It points out my philosophical position or inquiry worldview, my personal view on the subject under study (students' algebraic proficiency), and the structure sense theory I used to interpret the result of the study. As the theoretical framework of the study, these components form the basis for examining how structure sense instruction influences students' algebraic proficiency in precalculus algebra courses taught at a community college.

Inquiry worldview. I consider my approach to this study to be seated in a pragmatism worldview as described in Creswell (2013). According to Creswell, this paradigm does not hold to any specific system of philosophy and reality; the world is not an absolute unity. There are various ways of collecting and analyzing data; the focus is not on methods but on the problem being studied and the question asked about this problem (Creswell, 2013). Pragmatism acknowledges "an external world independent of the mind as

well as those lodged in the mind;" it admits "that research always occurs in social, historical, political, and other contexts" (Creswell, 2013, p. 28).

Subjectivity statement. I want to acknowledge an aspect of my educational background that may have influenced my outlook on algebra learning and teaching. I completed an undergraduate degree in chemical engineering. This training involved considerable mathematical modeling of processes with systems of linear and non-linear equations. It has impressed me with great respect for the power of mathematical symbols (algebraic representation) for describing highly complex dynamic systems. In a similar way, completing a master's degree in mathematics education with a post-secondary emphasis has impressed me with even more appreciation for the power of mathematical symbols for encoding and expressing abstract concepts. I experienced that power of mathematical symbols which allows someone to reason on an abstract level and to glean conceptual information mostly through working with proofs. Upon completing the master's degree in mathematics education with a post-secondary emphasis, I began teaching precalculus algebra and trigonometry at a community college. After about two years teaching mostly precalculus algebra and trigonometry, it occurred to me that many students struggle with the course not because of reasoning ability, but primarily because of the challenge of mathematical symbols, especially the letter-symbolic representation. This observation along with my appreciation for the power of mathematical symbolism has greatly motivated my interest in studying the issue of students' algebraic proficiency as it relates to the structure sense framework.

Substantive content theory. Besides pragmatism as an inquiry worldview and my personal background, research on the teaching and learning of algebra, as well as the theory

of structure sense, have also shaped this study. More specifically, research on the teaching and learning of algebra and especially that research about students' experience with learning algebra point primarily to the lack of structural conception among students in middle and high school, and even in college (Kieran, 1992, 2007; McNeil, 2006; Knuth et al 2006; Knuth et al., 2011; Stephens et al. 2013; Welder, 2012; Wang, 2015). The term 'structural' refers to operations performed on algebraic expressions which themselves result in other algebraic expressions (Kieran, 1992). The operations on algebraic expressions include simplifying, factoring, rationalizing, solving, differentiating, and so on, beyond the four basic operations of adding, subtracting, multiplying and dividing (Kieran 1992). It is important to point out that Kieran's (1992) description of "structural" only applies to symbolic representations, while Sfard's operational-structural theory (Sfard, 1991) covered all representational modes.

Sfard's (1991) operational-structural theory departed from other theories of mathematical thinking (e.g., Hiebert & Lefevre, 1986) by arguing that there is a duality instead of a dichotomy between procedures and concepts (Kieran, 2013). The term operational refers to actions, processes, and results. The term structural refers to processes as unified entities and integrated wholes. The concept development she proposed has three stages: interiorization, condensation, and reification. These stages will discussed later in the literature review. Suffices it to illustrate at this point the comment in the last paragraph that Sfard's description of "structural also applies to graphical, geometric, numerical, and verbal representations, not only in algebraic representation. For instance, symmetry can be conceived structurally as the property of a geometrical shape. Natural numbers can be

conceived structurally as “the property of a set or the class of all sets of the same finite cardinality” (Sfard, 1991, p. 5), beside the process of counting step by step.

Structural conception of algebra requires structure sense. Structure sense refers to the gestalt view or global perception of algebraic concepts and objects (Arcavi, 1994; Hoch & Dreyfus, 2004; 2005; 2006; 2007; 2010; Kieran, 1992; 2007; Van Stiphout et al., 2011). Therefore, structure sense is essential to algebraic proficiency. According to the research cited earlier, the structural conception of algebra is crucial to algebraic proficiency because it is a dominant deficiency found among students (Kieran, 1992, 2007). To improve algebraic proficiency among students, it is important to consider the development of structural conception through structure sense. According to Arcavi (1994), Hoch and Dreyfus (2004; 2005; 2006; 2007; 2010), and Van Stiphout et al. (2011), developing students’ structure sense is an effective means for improving structural conception of algebra, and in turn, algebraic proficiency. In particular, this study is based on two main ideas: (i) structure sense can be taught (Hoch & Dreyfus, 2004) and (ii) structure sense instruction can help develop algebraic proficiency (Arcavi, 1994; Hoch & Dreyfus, 2004; 2005; 2006; 2007; 2010; Kieran, 1992; 2007; Van Stiphout et al., 2011).

Purpose Statement

This section reiterates the important topic of students’ algebraic proficiency and the associated research problem (how to improve algebraic proficiency among students). It outlines the goals of the research in investigating the influence of structure sense instruction. It also provides a brief rationale for considering the structure sense instruction for addressing the issue of students’ poor performance in algebra and for studying algebraic proficiency further.

Statement of the problem. Students struggle with learning algebra in middle and high school (Kieran, 1992) as well as at the college level (Kieran, 2007; Barnes, Cerrito, & Levi, 2004; Reyes, 2010). Large and increasing numbers of students enroll in college algebra (Nguyen, 2015; Reyes, 2010) and a large percentage of them repeat the course up to three times and even more (Reyes, 2010) if allowed. Since college algebra is required for most majors (Reyes, 2010), low proficiency in algebra impedes the progress of students toward STEM fields (NCTM, 2014; NMAP, 2008; Packard, Gagnon, & Senas, 2012; Zorn, 2002).

A significant portion of students' difficulty stems from the letter-symbolic representation according to literature syntheses in Kieran (1992; 2007). All age groups (including community college students) have shown difficulty translating from graphical to algebraic representation and from word-problems to algebraic representation (Kieran, 2007), or from generalization of patterns to algebraic representation (Warren, Trigueros, & Ursini, 2016). According to Kieran (2007, 2013), the NCR (2001), Rittle-Johnson et al. (2015), and Sfard (1991), fluency in the letter-symbolic representation is as important as conceptual understanding. This statement regarding procedural fluency and conceptual understanding is consistent with the NRC's (2001) description of mathematical proficiency. NRC (2001) maintained that the five strands of mathematical proficiency are mutually supportive and bidirectional. However, while the NRC's (2001) description of mathematical proficiency is useful and comprehensive (Hiebert & Grouws, 2007), there is a significant lack of information on how mathematical thinking theories and teaching approaches connect to the development of the strands of mathematical proficiency.

There are several approaches for teaching algebra, but it has not been explained how they develop the five strands of mathematical proficiency described by the NRC (2001). Examples of such teaching approaches include those emphasizing multiple representations of functions with technology, functions and covariation, and structure-sense. Although structure-sense has not been explicitly linked to the five strands, it is the instructional approach that most directly addresses algebraic reasoning in symbolic representations. As mentioned above, a large measure of students' difficulty with algebra stems from the letter-symbolic representation according to Kieran (1992, 2007) and NMAP (2008). However, the meaning of structure sense is still debatable (Warren et al., 2016); that is, the description of structure sense varies to some extent among researchers (e.g., Novotnà and Hoch, 2008; Van Stiphout et al., 2011).

Goal of the research. The purpose of this study is to contribute more data toward evaluating the influence of structure sense instruction on students' algebraic proficiency. Thus far, structure sense instruction has been delivered to high school students as seen in Hoch and Dreyfus (2004; 2005; 2006; 2007; 2010) and Van Stiphout et al. (2011). Structure sense has been considered theoretically in abstract algebra at the university level by Novotnà and Hoch (2008). Structure sense has also been considered in middle grades level by Linchevsky and Lineh (1999). This study may be the first investigation of structure sense instruction at the community college level. It is in this context and through these aspects that this study will add data toward assessing the influence of structure sense instruction on students' algebraic proficiency.

Rationale for conducting a mixed methods study. A mixed methods research approach involves collecting and analyzing both quantitative and qualitative data, mixing the

data, and using a design to frame the procedures (Creswell & Clark, 2011 p. 112). The study integrated quantitative and qualitative data in a mixed methods design – "research in which the investigator collects and analyzes data, integrates the findings, and draws inferences using both qualitative and quantitative approaches or methods in a single study or program of inquiry" (Tashakkori & Creswell, 2007, p. 4). This research study employed an embedded mixed methods design QUAL (+ quan) = enhance study (Creswell & Clark, 2011), where quantitative data helped answer the first and second research questions. The qualitative strand complemented the qualitative strand by addresses the third research question. The qualitative strand was a case study. The results on how community college students experience the structure sense intervention helped describe structure sense instruction.

Obtaining quantitative data allowed for assessing the effect of the teaching approach in ways that support the interpretation of qualitative data. Because the study took place over an eight-week period and because it takes time for students to adapt to new instruction, quantitative data may not adequately capture the impact of the intervention.

Specifically, quantitative data could not provide detailed description about how students experience the teaching intervention. At the same time, quantitative data did help assess the improvement of students whose algebraic proficiency was low at the beginning of the course. That way, the quantitative data helped evaluate the efficacy of the intervention based on the progress shown by students in the eight-week period. Thus, both quantitative and qualitative data were necessary to more adequately assess the influence of the structure sense intervention on community college students' structure sense and algebraic proficiency. Therefore an embedded design was appropriate since the qualitative and quantitative strand targeted separate research questions (Creswell & Clark, 2011 p. 91). The quantitative strand

was necessary for measuring algebra proficiency (research questions 1 and 2). The qualitative strand focused on structure sense (research question 3).

Significance of the study. This study carries practical significance for structure sense instruction as it relates to community college students' algebraic proficiency. By using Hoch and Dreyfus' structure sense theory, it addresses the lack of research on community college students' algebraic proficiency based on mathematical thinking theories. It provides information that can be used for future studies. In particular, it may inform how researchers consider the implementation of structure sense instruction as the independent variable or part of an independent variable. It may inform researchers on how to consider the dependent variable, algebraic proficiency. It may inform the type (qualitative or mixed methods) of future studies conducted about structure sense instruction.

Other relevant implications that are not directly connected to the research questions are addressed in a discussion section after the study was implemented. These implications include the application of the structure sense definition by Hoch and Dreyfus (2004, 2006) to additional concepts. Thus far, the definition of structure sense has been applied mostly to concepts of coefficients, factors, and types of algebraic expressions and equations (e.g., $ax + b = 0$; and $ax^2 + bx + c = 0$), and to special algebraic identities (e.g., $a^2 - b^2$; $a^2 + 2ab + b^2$; $ab + ac + ad$) as seen in Hoch & Dreyfus (2004; 2005; 2006; 2007; 2010) and Van Stiphout et al. (2011). The structure sense definition applies to solving equations and inequalities (See Ursini & Trigueros, 2009) and to letter-symbolic descriptions of functions. Some class handouts are provided in the appendices after implementation. This information may help others in their implementation of structure sense instruction, which will be useful for further

developing this type of instruction. Finally, these implications include contributions of the literature review toward reconciling different conceptions of algebraic proficiency.

Chapter 2: Literature Review

This review of the literature examines information regarding mathematical proficiency as it relates to precalculus algebra. It also synthesizes findings about community college students' algebraic proficiency and how such findings relate to algebraic proficiency. In addition, this review shows how some framework and teaching approaches apply to algebraic proficiency and community college students' algebraic proficiency. Students' low algebraic proficiency is a critical issue (Van Stiphout et al., 2011) in the teaching and learning of mathematics. Students' algebraic proficiency must be evaluated in light of effective conception of successful mathematical learning. However, the existence of multiple conceptions of mathematical and algebraic proficiency adds complexity to this task of evaluating students' algebraic proficiency. Thus, understanding the issue of students' proficiency in algebra also involves understanding how these multiple conceptions mesh together. Figure 1 is a diagram of findings and syntheses that warrant the use of structure sense as a theoretical framework.

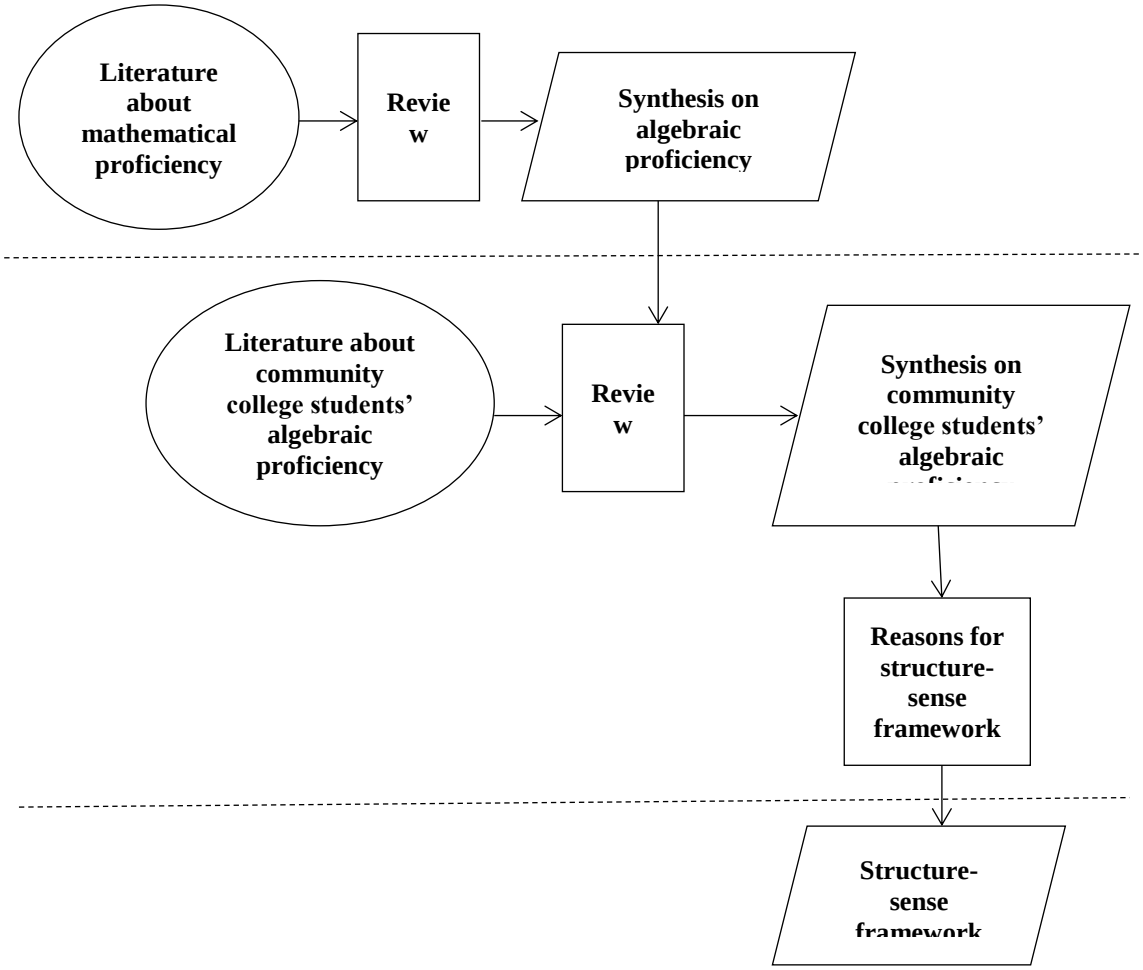


Figure 1. Findings and syntheses that warrant the use of structure sense as a theoretical framework.

What is algebra? There is much debate about this question; not everyone agrees on one definition. There is the term "algebra" in the context of the various algebra courses that mathematics majors take, and more generally, the abstract algebra that professional mathematicians use (Drijvers et al., 2010). Then, there is another use of "algebra" in the context of school algebra and precalculus college algebra. School algebra is algebra concepts and skills taught from the early grades through high school. Precalculus algebra in college involves the same topics and in some cases a few more topics than those covered in high school. However, coverage of these topics tends to focus a lot more on algebraic

representations. In both cases, there is no consensus on what should constitute the main content and teaching approach for this algebra (Drijvers et al., 2010; Kieran, 2007). There are several perspectives of algebra and algebraic thinking such as algebra as generalized arithmetic and abstraction from computation, algbra as the study of functions, relations, and co-variation; algebra as the study of structure and processes; and algebra as the study and generalization of patterns. Consequently, there are various views of algebraic proficiency is (Driscoll, 1999). Thus, algebraic proficiency depends on one's definition of algebra.

Algebraic Proficiency

There are at least three major interpretations of mathematical proficiency which represent a first pass at understanding algebraic proficiency. The three major interpretations of mathematical proficiency are provided by NCTM (2000), NRC (2001), and the *Common Core State Standards for Mathematics*, (CCSSM, 2011). These interpretations of mathematical proficiency have contributed considerably to what algebraic proficiency is considered to be (NCTM, 2000; NRC, 2001; CCSSM, 2011). In the context of this literature review, it is important to point out how each of these conceptions has contributed to the evolution of mathematical and algebraic proficiency.

In *Principles and Standards for School Mathematics*, NCTM (2000) has provided an interpretation of mathematical proficiency (Schoenfeld, 2007) in two parts. The first part maintained that algebra is one of the content standards, meaning that students must learn it. In this view of algebra, the following ideas are areas of focus: "understand patterns, relations, and functions; represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to represent and understand quantitative relationships; analyze change in various contexts" (NCTM, 2000, p. 37). The second part

maintained that proficiency is demonstrated through five process standards: Problem solving, reasoning and proof, communication, connection, and representation. Hence, algebraic proficiency is demonstrated in these focus areas of content through the five process standards.

According to the NCTM (2000), "problem solving means engaging in a task which the solution method is not known in advance (p. 52)." It stated that proofs are formal ways of expressing particular kinds of reasoning and justification. These proofs are arguments involving "logically rigorous deductions of conclusions from hypotheses (p. 56)." Communication in this context is a means of sharing ideas for the purpose of showing understanding; it also includes organization, coherence, analysis, evaluation, and precision (NCTM, 2000). Connections of mathematical ideas reveal deeper and more lasting understanding (NCTM, 2000). Representations reveal understanding of abstract concepts, communication of ideas, and application of mathematics in realistic problem situations (NCTM, 2000).

The NRC (2001) provided an interpretation on mathematical proficiency consisting of five strands that are similar to the five process standards of the NCTM (2000). The strands are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. They apply to all branches of mathematics (NRC, 2001), including algebra. Therefore, these strands are ways of demonstrating successful algebra learning. Conceptual understanding refers to comprehension of algebraic concepts, operations, and relations. Procedural fluency refers to skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. Strategic competence refers to the ability to formulate, represent, and solve mathematical problems. Adaptive reasoning refers to the

capacity for logical thought, reflection, explanation, and justification. And productive disposition refers to a habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NRC, 2001).

The Common Core State Standards for Mathematics, CCSSM (2011) has also provided its interpretation on algebraic proficiency based on the NCTM's (2000) five process standards and the NRC's (2001) five interwoven strands of mathematical proficiency. It stipulated the types of expertise that students should acquire and would demonstrate. The Common Core State Standards for Mathematics captured this expertise through eight standards of mathematical practice. The eight standards of mathematical practices (MPs) emphasized making sense of problems and persevering in solving them (MP1), reasoning abstractly and quantitatively (MP2), communicating critically (MP3), modeling with mathematics (MP4), using appropriate tools strategically (MP5, attending to precision (MP6), using structure (MP7), identifying and expressing regularity in repeated reasoning (MP8) (CCSSM, 2011). Hence, successful learners of algebra show proficiency through these eight practices.

In all, these three documents offer a more comprehensive conception of mathematical proficiency at large. While the NRC (2001) lacked explicit reference to the role of communication and to the role of tools (and technology), both the NCTM (2000) and CCSSM (2011) highlighted them. While the NRC (2001) lacked explicit reference to the use of structure, NCTM (2000) and CCSSM (2011) highlighted it as a practice of proficient students. CCSSM (2011) also emphasized attending to precision, an important mathematical practice (Milgram, 2007) while the NRC (2001) only referred to precision in the context of carrying out procedures. Of the three conceptions of what mathematical proficicency should

be, the NRC (2001) more explicitly emphasized the mutual interweaving or interrelatedness of strands and process standards in proficiency. Likewise, NCTM (2000) more explicitly described the process standard of problem solving. Together, they share a large area of overlap; all of them emphasized problem solving, reasoning and proof, and representation.

Evolution of mathematical proficiency. Before the NCTM (2000), NRC (2001), and CCSSM (2011) documents, mathematical proficiency was mainly a debate about procedural and conceptual knowledge (or understanding). Hiebert & Carpenter (1992) and Kieran (2013) acknowledged the initial phase of this debate about teaching and learning mathematics by referring to the works of Brownell (1935), Bruner (1960), Gagne (1977), McLellan & Dewey (1895), and Thorndike (1922). Procedural knowledge tended to come in second place behind conceptual knowledge in terms of relative importance for students. For example, Skemp (1976) contrasted relational understanding ('knowing both what to do and why') to instrumental understanding ('knowing both what to do'). Skemp's instrumental understanding lends itself toward the aspects of procedural fluency such as 'carrying out procedures efficiently and appropriately' to the extent that it emphasizes the use of rules and techniques for the sake of usage itself. In addition, it is important to note that Skemp (1976) did not consider instrumental understanding as part of a framework for successful mathematical learning as did the NRC (2001). Skemp (1976) did not acknowledge a relationship between the two types of understandings or knowledge, as if relational understanding would essentially equate proficiency. Although Skemp would later revise his stance on the role of instrumental understanding (Sfard, 1991), underestimating the importance of procedural skill does reflect a phase of the debate over which conceptual knowledge took precedence (Hiebert & Carpenter, 1992).

Instead of extending this contest between procedures and concepts, the NCTM (2000), NRC (2001), and CCSSM (2011) documents have improved the conception of mathematical (hence algebraic) proficiency tremendously. The NRC (2001) improved the description of these two types of knowledge and effectively set them as mutually supportive strands of mathematical proficiency among three other strands. This conception of proficiency began with Hiebert and Lefevre (1986) and Hiebert and Carpenter (1992). Hiebert and Lefevre (1986) defined conceptual knowledge as meaningfully connected information. Their type of conceptual knowledge corroborated with the idea of relations from the NRC's (2001) description of conceptual understanding, especially when such connections capture broad and common features. Moreover, Hiebert and Lefevre (1986) offered a perspective consonant with that provided by the NRC (2001) in which the two aspects relating to mathematical (in this case algebraic) proficiency intertwine mutually. On one hand, Hiebert & Lefevre (1986) elaborated on how linking conceptual knowledge to procedures help with (i) retention and effective uses of procedures and (ii) strategic decision-making. On the other hand, they argued that procedural knowledge also supports conceptual knowledge because symbols provide powerful means for dealing with complex ideas and because procedures facilitate application of conceptual knowledge.

Hiebert and Carpenter (1992), Zaski and Liljedahl (2002), and Bokhove and Drijvers (2010) stressed that both conceptual knowledge and procedural skill are crucial, go hand in hand, and therefore are required for proficiency. Hiebert and Carpenter (1992) encouraged substantiating the forms of the relationships between the two kinds of knowledge and pointed out that research on these relationships are more promising than earlier research that tried to show the importance of one over the other. The NRC (2001), by describing proficiency as

interwoven strands, has certainly contributed toward substantiating the relationships between procedural fluency and conceptual understanding.

However, Hiebert and Carpenter (1992) also commented that "both theory and data available at the time recommend emphasis on understanding prior to skill" (p. 79). The issue with such results is that they generalized and promoted a foundational role for conceptual understanding. In fact, generalizing that conceptual understanding comes first regardless of topics or contexts is in keeping with a line of research promoted by Grouws and Cebulla (2000) and NCTM (2014). According to Rittle-Johnson et al. (2015) this assertion of 'conceptual understanding prior to procedural fluency' undermines the bidirectional relations between conceptual and procedural knowledge. Rittle-Johnson et al. (2015) maintained that there is a stronger relationship in which the two kinds of knowledge support each other more equitably rather than simply acknowledge that they go hand in hand. Their argument is consistent with the NRC's (2001) report about the interdependence of conceptual understanding and procedural fluency. Rittle-Johnson et al. (2015) added also that the NMAP's (2008) wording 'mutually reinforcing' implied a bidirectional relationship between conceptual understanding and procedural fluency.

Grouws and Cebulla (2000) and NCTM (2014) advocated that instruction should follow a conceptual-then-procedural approach. But considering the reverse approach meaning procedural-then-conceptual, as well as an iterative sequencing approach, and a simultaneous approach, as Rittle-Johnson et al. (2015) suggested, may advance what is known about the relationship between conceptual and procedural knowledge. Rittle-Johnson et al. (2015) extended a view of procedural and conceptual knowledge consistent with Kieran (2013), NRC (2001), and Sfard (1991). Sfard (1991) called the interdependence between

procedural and conceptual knowledge a "vicious circle" highlighting the necessity of skills with algorithms for grasping mathematical objects and conversely the necessity of perceiving mathematical objects for mastering skills.

Kieran (2013) provided convincing support for the perspective of NRC (2001) and Sfard (1991) by establishing the interdependence and bidirectional relationship between conceptual and procedural knowledge. Kieran (2013) illustrated a false dichotomy between conceptual understanding and procedural fluency in mathematics education using algebra as an example. Kieran's analysis showed that researchers have been rethinking the relationships between the two abilities and laid out two major arguments against the dichotomy. The first argument featured Lagrange's (2000) observation that Heid's (1988) conceptual activity involved considerable technical work. It also featured Ruthven's (2002) comment that techniques accomplish an epistemic purpose in conceptual development in mathematics. The point is "even procedures that have become automatized are regularly being updated by the constitution of new techniques that have been elaborated conceptually (Kieran, 2013, p.161)." Hence, instead of dichotomy or general ordering of concept-then-procedure, the interaction between procedural and conceptual knowledge is "on-going recursive."

In summary, the NRC (2001) further advanced the concept of proficiency not only by maintaining that procedural knowledge is an equally important component, but also by extending its description. Procedural knowledge became the awareness of when and how to use procedures, which go beyond the four basic operations, but include algorithms as important concepts in their own right. Mathematical proficiency is not one-dimensional; it requires mutually supportive interactions between five strands. For instance, conceptual

understanding and procedural fluency can interrelate in ways that make it impossible and unnecessary to distinguish one from the other (Hiebert & Carpenter, 1992; NRC, 2001). Using these mutually supportive interactions between the strands is what constitutes mathematical proficiency.

Amplification of mathematical and algebraic proficiency. While the NCTM (2000), NRC (2001), and CCSSM (2011) documents have improved the conception of mathematical and algebraic proficiency, other works have provided further elaborations. Schoenfeld (2007) built on the five strands proposed in the NRC's (2001) description to provide a four-part conceptualization of algebraic proficiency. The four parts in this version are knowledge base, strategies, metacognition, and beliefs and dispositions. Schoenfeld's (2007) description of mathematical proficiency also subsumes algebraic proficiency. Therefore, it can be appropriated to algebraic proficiency. Schoenfeld (2007) drew on the NRC's (2001) description for each of the four parts but focused on problem solving, especially on new problem situations. According to him, mathematical proficiency is the effective and efficient use of knowledge base in algebra for solving new problems. Hence, Schoenfeld (2007) amplified the NRC's (2001) strand of strategic competence--the ability to formulate, represent, and solve problems. It also amplified problem solving, which is one of the NCTM (2001) process standards.

Moreover, Schoenfeld noted an important point about issues and tensions in the assessment of mathematical proficiency. He pointed out that drilling students on tasks similar to those found on their assessment may give the illusion of competence. In other words, students who are only strong on procedural skills will demonstrate competence on assessments that focus on such skills. But, they will not show proficiency on holistic

assessments—those that evaluate all of the NRC's strands (NRC, 2001). The understanding of these students is deficient. Conversely, a similar issue may play out for students with low procedural fluency. As Zazkis and Liljedahl (2002) stated, there are usually gaps between students' understanding of mathematical concepts and their ability to express that understanding symbolically. Zazkis' and Liljedahl's (2002) point highlights the importance of communication and representation, which are process standards. The implication is that assessments must be balanced; they must be holistic because of their role in determining proficiency.

Like Schoenfeld (2007), Milgram (2007) emphasized problem solving as a major aspect of mathematical and algebraic proficiency. Applying knowledge in novel ways is the key element for proficiency in the context of problem solving. Applying knowledge in novel ways is consistent with the NCTM's (2001) definition of problem solving (a task which the solution path is not known before hand). But Milgram's main contribution to the description of mathematical (algebraic) proficiency is that it incorporated precision. According to Milgram, this precision stems from mathematics being the study of precisely defined objects. In sum, Milgram couched mathematical (algebraic) proficiency within a three-part characterization of mathematics involving definitions, stages (context), and problem solving.

Given the backdrop of mathematical proficiency, the next step is to look at what other researchers have said in particular about algebraic proficiency which represents a second pass at understanding algebraic proficiency.

Carlson, Oehrtman, and Engelke (2010) identified reasoning abilities and understandings central to precalculus and foundational for beginning calculus. The reasoning abilities consist of the following: a process view of function, covariational reasoning, and

computational abilities. A process view of function refers to the function taking inputs and converting them into outputs. This process view (relational view) is desired over the action view or procedural view. Covariational reasoning refers to the analysis of various ways those input and output values change together. Computational abilities refer to facility with manipulations and procedures. The understandings consist of specific function concepts. These concepts are function evaluation, rate of change, function composition, function inverse, growth patterns of various function types, and meanings of various representations and the connections between those representations (graphical, algebraic, numerical, and contextual). It is important to note that this conception of algebraic proficiency centers on functions. This conception of algebraic proficiency also emphasized connections between representations.

Driscoll (1999) framed his own perspective about algebraic proficiency in terms of facility with algebraic thinking. The focus of algebraic thinking is on functions and on how a system's structure affects calculations. The 'algebraic-thinking' conception of proficiency consists of three habits of mind Driscoll (1999) described as doing-undoing, building rules to represent functions, and abstracting from computation. In particular, Driscoll (1999) emphasized that proficient algebraic thinkers are those who reason flexibly among different algebraic representations. Driscoll (1999) also emphasized that symbolic representation and manipulation are the lifeblood of algebra. Both of these assertions translate automatically into components of algebraic proficiency.

McCallum (2007) acknowledged algebraic proficiency as represented by the five strands of the NRC's (2001) description. But McCallum considered algebraic proficiency to be for the most part synonymous with proficiency with symbolic representations. Algebra is

about proficiency with symbolic representations in the context of studying algebraic expressions and equations (McCallum, 2007). According to him, this study of algebra would include algebraic form, properties of values and solutions, as well as equivalence and transformations. McCallum (2007) viewed algebra in terms of such study, but not as the study of function and covariational reasoning.

The NRC's (2001) viewed school algebra through two underlying aspects. The first aspect is algebra as a systematic way of expressing generality and abstraction, including algebra as generalized arithmetic. The second aspect is algebra as syntactically guided transformations of symbols. According to the NRC's (2001), these two aspects of algebra can be seen through three kinds of activities in school and precalculus algebra: (a) representational activities (translating to symbolic forms), (b) transformational (rule-based) activities, and (b) generalizing and justifying activities (noting structure, proving, and justifying). The NRC (2001) maintained that representational activities involve conceptual understanding and strategic competence. It maintained that aspects of conceptual understanding, strategic competence, and procedural fluency interact together in transformational (rule-based) activities. It also maintained that all the strands of algebraic proficiency come together in generalizing and justifying activities, especially adaptive reasoning.

In a similar way, algebraic proficiency is described in terms of models of algebra as various kinds of activities. As such, these activities represent how proficiency is demonstrated. Kieran (2007) described a model of algebra as activities similar to those provided by the NRC (2001). Kieran's (2007) model was built on major ideas from Bell (1996), Usiskin (1998), Kaput (1995), Pimm (1995), and Kieran (1996). These ideas

summarized algebra as generalization, study of structures, study of functions and of relationships among quantities, syntactically guided manipulations, relations, joint variation, and modeling. Kieran (2007) described algebra in terms of three types of activities: generational, transformational, and global/meta-level. The generational type of activities comprises of synthesizing algebraic expressions and equations. The transformational type of activities consists of manipulating algebraic expressions and equations. The global/meta-level type of activities covers modeling, problem-solving, proving, and other tasks as well as the motivation and purpose behind these tasks.

Within the context of algebra as an activity, the use of technology may be considered as another dimension of algebraic proficiency because of the role it plays in those activities mentioned above. However, for the most part it is treated in the context of teaching and learning (e.g., NCTM, 2000) as opposed to a means of showing evidence of learning. Yet, using appropriate tools strategically is one of the mathematical practices of proficient students (CCSSM, 2011). Thus, using technology strategically may be integrated as a process, one of the ways algebraic proficiency is demonstrated. For example, algebraic expectation is needed for effective use of Computer Algebra Systems (Pierce & Stacey, 2002, 2004). Students' understanding of concepts allows them to correctly use technology to enter information and to anticipate results and outcome.

In summary, algebraic proficiency can be described on two levels which represent a first and a second pass at understanding it: (a) a level pertaining to mathematics, in general and (b) a level pertaining to algebra in particular. The first level characterizes the broad processes through which mathematical proficiency is shown. According to NCTM (2000), these processes are problem solving, proof and reasoning, communication, representation,

and connections. According to NRC (2001), these processes are interwoven strands such as conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. CCSSM (2011) provided the eight mathematical practices (MPs). These broad processes and practices are not specific to algebra, but they provide comprehensiveness to the idea of proficiency. They also provide context for the aspects of algebraic proficiency at the second level.

The second level of the description of algebraic proficiency involves those aspects that are more specific to the nature of algebra. These aspects include the conceptions of algebra such as those described earlier (e.g., Carlson et al., 2010; Driscoll, 1999; Kieran, 2007; McCallum, 2007; NRC, 2001). Functions (including covariation, patterns, and representation) and forms (i.e., structure, equivalence, and transformation) emerged as two dominant but competing themes of these conceptions (Kieran, 2007). Like the debates of yesteryears that contrasted procedural knowledge and conceptual knowledge, pitting functions against forms (structure) is a forced and unnatural separation that contradicts the broad description of mathematical proficiency at the first level. The conceptions from Carlson et al. (2010) and Driscoll (1999) have brought functions and forms together, and to some extent, have indicated their constituent roles.

Therefore, it is all of these five conceptions that more accurately represent algebraic proficiency. At the same time, because proficiency is not all or nothing (NRC, 2001, p. 135), any one of the five descriptions also represents algebraic proficiency. These two notions of were used in the study for describing algebraic proficiency.

Community College Students' Algebraic Proficiency

The issue of students' proficiency in community college precalculus algebra is given little attention in research (Barnes et al., 2004; Mesa et al., 2014). Hence, it is useful to include the work on undergraduate university students in order to provide some context. Moreover, a brief summary of research on younger students' work should provide some background as well (Warren et al., 2016, p. 73).

Kieran's (1992) review of the literature on the teaching and learning of algebra provided two significant characterizations of younger students' work in algebra. Both characterizations deal mainly with topics such as expressions and literal terms, simplifying expressions, equations, solving equations, and word problems. The first characterization was that procedural interpretations of algebra were more accessible for students. The term 'procedural' refers to "arithmetic operations carried out on numbers to yield numbers (p. 392)" even with the context of evaluating algebraic expressions. The second characterization was that acquiring structural conceptions of algebra was very difficult for students.

The research synthesized in Kieran (1992) showed significant evidence that younger students (middle and high school age) predominantly think procedurally (operationally). Findings of this operational view were largely about middle and high school students (See Knuth et al., 2011; Rittle-Johnson et al., 2011). But, Welder (2012) commented that some research has shown that the equal sign is misunderstood by students at all levels of education because they missed its equivalence role.

Kieran's (2007) review reported that middle, high school, and college students alike showed difficulties with the analytic (algebraic) representation. Unlike middle and high school students, college students then, relied on using the symbolic (algebraic) representation

(Kieran, 2007). However, like the younger students, they had difficulties using the algebraic representation. All age groups had difficulty translating from graphical to algebraic representation and from word-problems to algebraic representation (Kieran, 2007), or from generalization of patterns to algebraic representation (Warren et al., 2016). The discussion about how the equal sign, variables, and expressions can be treated procedurally or structurally are based on definitions provided earlier from Kieran (1992), which were adapted from Sfard (1991). Research related to this area include Kieran (2007), Warren et al. (2016), Mielicki & Wiley (2016), Godfrey and Thomas (2008), Weinberg et al. (2016).

College students' proficiency in terms of algebra as an activity. Kieran (2007) investigated research on algebra conducted with students at the college level through the framework of generational, transformational, and global/meta-level activities mentioned earlier. Within the generational type with focus on the letter-symbolic form, Kieran identified three areas. The first area is "form and structure" where students showed lack of ability in recognizing laws and forms, and in getting abstract understanding. Even those who demonstrated structure sense did it inconsistently. The second area is about "parameters", where few students could differentiate between parameters, unknowns, and variables. The third area is about "multiple representations" where students have been found to prefer their own intuition and process-oriented interpretations of functions (different from the process view Carlson et al., 2010 described regarding functions). Other misconceptions include neglecting domain and range. Some studies discovered significant use of letter-symbolic representations even when graphical representations seemed more suitable. But researchers believe that this tendency is a consequence of conditioning and that it is superficial.

Kieran (2007) considered the experience of college students within the transformational type of activities as they relate to several algebraic concepts and objects. For the concept of equivalence, researchers found significant lack of ability in recognizing and using equivalence as well as other structural aspects of algebraic expressions. With regards to equations and inequalities, researchers found difficulty and misconceptions surrounding the forms and solutions of quadratic equations. And like younger students at middle school and lower secondary levels, college students applied arithmetic reasoning to word problems involving linear inequalities.

Kieran (2007) considered college students' work within the global/meta-level type of activities. Their work on proving and problem solving, which involved technology, showed that most of the students in those studies preferred numerical and graphical approach. Evidence from their work revealed that they could not express aspects of problems in algebraic language as they do in natural language. Yet, they demonstrated ability to produce valid proofs and problem solutions.

Finally, Kieran (2007) considers the impact of technology on algebraic activity with students at the upper secondary and college levels. Some studies involving graphing calculators reported significantly better understanding of functions and better student performance on modeling, interpreting, and translating. At the same time, some of these studies pointed out that graphing calculators did not improve traditional algebra skills for students. But they helped with understanding of graphing concepts. Computer Algebra Systems (CAS) contributed to students' performance in technology-based courses as well as in subsequent courses.

Overall, a number of the studies that looked at students' notions and use of symbolic and graphical representations found that the students in that group (college) preferred to use the symbolic form. They differed from the younger students who avoided symbolic approaches. According to Kieran's (2007) review, college students exhibited more productive disposition than younger students with respect to reasoning in the algebraic representational mode. While useful, the research synthesized in Kieran (2007) concerned college students in general, and not community college students only.

Community college students' learning of precalculus algebra. Research on community college mathematics education has increased recently between 2005 and 2014 (Mesa, 2016). This research has targeted several different areas such as curriculum reform, faculty, content, students, and low success rates in developmental courses (Mesa, 2016). In particular, a significant portion of the research has been allotted to developmental mathematics courses. Even within this area, some of that research has come through doctoral dissertation and unpublished manuscripts. There is still a tremendous lack of research on community college students' algebraic proficiency. However, it is possible to glean some worthwhile information about community college students' learning of algebra from the paucity of research available. The content will be much broader because of the paucity of information specifically on proficiency in that student population.

Mesa, Suh, Blake, & Whittemore (2013) analyzed 10 typical textbooks used for college algebra in community colleges and for four-year higher institutions. Their analysis of what community college students could learn from typical textbooks is based on the examples provided therein as opposed to the expositions and exercises. The analysis was performed along four dimensions: cognitive demand, type of response required, types of

representations, and strategies for controlling the correctness of solutions. In terms of cognitive demand, their analysis revealed that 90% of the examples were about procedures without connections. For type of response, 'answer only' was the most frequently expected response. Justification and explanation responses were the least expected; in only half of the books were such responses expected. With respect to types of representations, 76% of the statements were symbolic; verbal and graphical statements were few. In the answers provided to these statements, single numbers were the most frequent followed by symbolic representation. Finally, in terms of strategies for check solutions, three-quarters of the books did not illustrate any strategies for verifying solutions.

Considering the fact that textbooks do play a role in students' learning, the findings reported in Mesa et al. (2013) show that their learning from such textbooks would fall short of the standards of proficiency articulated in the NCTM (2000), NRC (2001), and CCSSM (2011) documents. For example, a disproportionate amount of emphasis is given to procedural fluency while the NCR (2001) and CCSSM (2011) have promoted five interwoven and mutually dependent strands. A disproportionate amount of emphasis is given to symbolic representation while NCTM (2000), CCSSM (2011), Driscoll (1999), and Carlson et al. (2010) have all promoted flexible connections among representations and concepts. Other components that are integral to proficiency such as cognitive demand, reasoning and proof, and communication were also neglected. However, these findings do not tell the whole story about community college students' of algebra because the content of textbooks is only partially implemented in the classroom. What takes place in the classroom, the interactions between instructors and students provide additional information about community college students' learning of algebra.

Mesa, Celis, & Lande (2014) reported on the teaching of mathematics in community colleges based on interactions between instructors (14 of them) and students in the classroom. They investigated two research questions. The first question asked how teaching approaches (beliefs and values) influence mathematics instruction at a community college. The second question asked whether "faculty who use mostly student-centered approaches create opportunities for students to also engage with more complex mathematical questions" (p. 12). They identified three teaching main approaches (1) Traditional, (2) Meaning Making, and (3) Student Support. Each teaching approach has two aspects. The Traditional approach has Traditional Distancing (content-oriented) and Traditional Adapting (student oriented). The Meaning-Making approach consists of Meaning Making Relating (driven by deeper meaning and real world contexts, not adapting), and Meaning Making Clarifying (adapting to students). The Student-Support approach consists of Student-Support Relating (focus on improving students' self-confidence through personal relationships and of Student-Support Clarifying (same goal but through rules and structure). Only three of the 14 instructors were not using mostly a traditional approach; most instructors were using more than one approach with the traditional approach being the most used.

In response to the first question, it was found that instructors' approaches mentioned earlier (Traditional, Meaning Making, and Student Support) did not match the way they delivered mathematical content during instruction. With regards to the second question, no pattern was found indicating that instructors with mostly student-centered approaches gave their students more novel or complex questions than instructors with content-centered approaches. According to Mesa et al. (2014), although instructors possessed the right beliefs to train students, they are limited by the constraints of time, large amount of material, and

lack of knowledge about how to pose novel questions. Thus, community college students in this study may represent an example of a larger group of similar students who are not getting sufficient opportunities to conjecture, reason, justify, connect and communicate mathematical ideas as recommended in the NCTM (2000), NCR (2001), and CCSSM (2011) documents.

Although the findings of Mesa et al. (2014) were not only about college algebra, they may reveal a lot about the opportunity that community college students have to develop algebraic proficiency. They showed that teachers do espouse approaches (hold beliefs) that are consistent with the processes of mathematical proficiency delineated in NCTM (2000), NCR (2001), and CCSSM (2011) documents. However these approaches do not transpire, they are not actualized in the classroom. Hence, there may be a lack of opportunity for these students to develop algebraic proficiency, especially considering also the findings from the previous study discussed earlier in the context of textbooks.

While the two previous studies focused on textbooks and on how course content is discussed by teachers and students, Mesa (2011, April) reported findings about community college students' goal and orientations about learning as well as findings about the instructors' perceptions of those goals. "In achievement goal theory, achievement goal orientations refer to students' reasons or purposes for engaging in academic behaviors together with the standards used to assess performance" (Friedel, et al., 2010; Meece, Blumenfeld, & Hoyle, 1988; Midgley, et al., 2000 as cited in Mesa, 2011, April). Mesa surveyed 777 community college students, some taking remedial courses and others taking curriculum courses. She also interviewed the instructors of those students. Results of the analysis showed that students' goals were oriented towards understanding. In other words, the results indicated that these students saw themselves as having positive and productive

dispositions about mastering the material, as exhibiting constructive behaviors, and as being open to challenging work. However, interviews with instructors indicated that they perceived that their students are more concerned about rote performance than real mastery; that they have a poor sense of their ability; they engage in unproductive behaviors; have low mathematics self-concept; usually ask that cognitive demand of tasks be lowered.

The findings from Mesa (2011, April) are evidently not generalizable because the study only surveyed students at one community college and was based on students' self-reported data. Yet, it is suggestive of what may be the case in some mathematics courses at other community colleges as well. It suggested that instructors' perceptions of students' orientation goals may be one reason instructors settled for content-oriented approach or for using a traditional-distancing approach (See Mesa et al., 2014). It suggested that there may be more students than expected who have productive disposition toward mathematics. This idea is important because productive disposition is one of the five strands of mathematical proficiency stipulated by the NRC (2001). If it is true that there are students who are more likely to bring productive and constructive orientation goals to mathematics courses, then instructors miss a real opportunity to capitalize on that potential by resorting to a traditional-distancing approach due to their perceptions of students' goals. Hence, it is possible that productive disposition towards algebra for many community college students may not be cultivated.

Like Mesa (2011, April), Wheeler & Montgomery (2009) reported findings from community college students' perspectives that directly relate to the strands of productive disposition, a component of mathematical proficiency according to NRC 2001). The study was conducted at a small Midwestern community college. Students' responses to open-

ended items were collected. And analysis of the data revealed that students' responses described them as active learners, skeptical learners, and confident learners according to their views toward mathematics. Confident learners think they have always been good at mathematics; have positive attitude toward mathematics and are not affected by anxiety. Active learners do not necessarily have positive attitude toward mathematics, but they value and believe that hard work and good study skills will bring success. Skeptical learners do not have positive attitude toward mathematics; they see their success in mathematics as strongly dependent upon what teachers do. Results of the study suggested that learners' views toward mathematics significantly influenced what they actually experienced in their mathematics courses. Much like Mesa (2011, April), these findings from Wheeler & Montgomery (2009) indicated that teachers' own disposition and attitude have significant influence on their students' disposition and attitude towards mathematics. This idea reinforces the need for teachers to overcome constraints and limitations (Mesa et al., 2014) in order to deliver instruction according to their approaches (Mesa et al., 2014).

With respect to the crucial role of teachers, Mesa (2010) showed there may be a trend that even successful teachers pose lower-cognitive questions by studying this behavior in seven successful mathematics instructors at a community college. These instructors used activities that required factual and procedural knowledge because it is important to them that they ask questions their students can answer (Mesa, 2010). Mesa (2010) has suggested that this behavior on the part of instructors stemmed from their perceptions of students' ability and goals. But more specifically, Mesa suggested that this behavior was the result of instructors' adaptation to a specific group of students (e.g., adults returning students) in lower level mathematics courses, especially remedial courses, and to some extent in college

algebra. In that sense, Mesa put this trend in perspective indicating that it may not be present in calculus courses and more advanced courses, at least not to the same extent.

The trend from the findings of these studies is that many teachers may be lowering the level of instruction to accommodate under-prepared students. Consequently, many community college students (even those who are ready for higher-cognitive tasks and questions) may not be receiving the opportunity to develop algebraic proficiency. This situation may be helped by differentiated instruction. Differentiated instruction is a more effective way to deal with situations in which students' preparedness in the same class varies significantly. On one hand, going at a slower pace and reviewing too much may be unprofitable to advanced students. On the other hand going too fast can be frustrating to many students leading to issues of motivation (Nguyen, 2015).

Nguyen (2015) is a mixed methods study that analyzed data from a larger, previously conducted study. Nguyen (2015) analyzed this data in order to answer two research questions. The first question asked what were students' experiences in a college algebra course. Some 613 students were enrolled in college at the time of the study. Seventy-two percent of them completed demographics survey. Some of them were interviewed. Three of the 11 teachers agreed to be interviewed. The second question asked how did students' experiences influence their motivation with respect to an ARCS (Attention, Relevance, Confidence, and Satisfaction) framework.

In regards to the first question, preliminary results from the sample of students showed that a large portion of students repeated the course. Still, students' actual grades received were significantly lower than the grades they expected. Students' responses revealed they felt frustrated and overwhelmed because of too much material, covered too

rapidly sometimes within six-week summer semesters. These experiences are the results of students' lack of prerequisite knowledge as well (Nguyen, 2015). And with respect to offering both shorter eight-week sections alongside sixteen-week sections of algebra (Reyes, 2010), students who lack prerequisite knowledge would likely have similar negative experiences in eight-week sections of algebra. Therefore, shorter semesters as advocated by Reyes (2010) may be advantageous to advanced students only.

In regards to the second question, Nguyen (2015) found that instead of frustration, students can experience satisfaction when teachers' actions and behaviors promote relevance, confidence, and attention for them (students). Relevance occurs when students perceive that course content has real connections to their goals. Confidence occurs when students realize they can succeed in mastering a task. Attention occurs when students recognize a gap in their knowledge. Nguyen found that these three factors have statistically significant impact on students' satisfaction, or in promoting positive experiences in college algebra. This finding is consistent with other studies reviewed earlier in that it strongly emphasizes the critical role of students' dispositions and behavior such as active, confident, skeptical learners (Wheeler & Montgomery, 2009). More importantly, Nguyen (2015) echoed a similar message as Mesa (2011, April) and (Wheeler & Montgomery, 2009) that teachers' own disposition and attitude have significant influence on their students' disposition and attitude towards mathematics. This trend in the findings seems to highlight a direct connection between teachers' disposition and students' development of productive disposition (NCTM, 2000; NRC, 2001) and perseverance (CCSSM, 2011).

Insofar as it concerns community college, these studies have barely addressed students' work in explicit ways (e.g., problem solving, representation, functions, or

variables), with the exception of Mesa et al. (2013), Mesa et al. (2014), and Mesa (2010). Consequently, information about students' work in precalculus algebra is implicitly deduced based on the idea that students do not learn what is not taught to them, as Schoenfeld (2007) noted. Findings from Mesa et al. (2013), Mesa et al. (2014), and especially Mesa (2010) have showed indirectly through textbook and teachers that students' conception of algebra may be more factual and procedural than anything else. This information would be consistent with findings about middle and high school and college students reported by Kieran (1992, 2007) mentioned earlier in the second section of this review. This information that community college students' conception would be action-based (procedural), not based on process-view of function and covariational reasoning is consistent with findings reported by Carlson et al. (2010). Mesa et al. (2013), Mesa et al. (2014), and Mesa (2010) reported findings on potential aspects of students' experiences which mainly involve the procedural fluency strand of algebraic proficiency. With the exception of productive disposition, the other strands of mathematical proficiency have not been targeted within these studies. The reason is that the studies tended to focus on psychological affects as opposed to discipline-oriented topics.

From reviewing the literature on algebra learning in the community college setting, the trend that emerges highlights psychological affects and broader aspects of students' experiences such as motivation, orientation goals, disposition, and opportunity to learn. However, with respect to studies that investigate students' work with aspects that are intrinsic to algebra, there is a notable lack because practically all such studies have been conducted within school algebra and in the university setting. This situation may have to do with the fact that community colleges are primarily teaching instead of research institutions, coupled

with the reality that mathematics education research has traditionally focused on the K-12 school system. Yet, given the large number of students enrolling in precalculus algebra in community colleges and given the emphasis on improving passing rates, it seems that this setting would be also an appropriate environment to conduct research studies. More importantly, it seems sensible that such studies would employ theoretical frameworks that focus on systematic thinking such as operational-structural, procept, or structure sense.

Theoretical Framework for Understanding Students' Algebraic Proficiency

Although they have provided significant information about students' experience vis-à-vis algebraic proficiency (especially productive disposition), findings on students' affect do not paint a holistic picture. It is necessary to also have more studies that utilize theoretical frameworks which focus on systematic thinking such as Sfard's (1991) operational-structural,

Gray's and Tall's (1994) procept, and Hoch and Dreyfus' (2004, 2006) structure sense.

Sfard's (1991) theory stated that any mathematical concept can conceived both structurally and operationally. Sfard also mentioned that these two conceptions are not mutually exclusive. She argued that procedures and concepts are complementary. The procedural conception is sequential and detailed. The structural conception is integrative; it is about gaining insight as a whole without attending to the details. According to Sfard's theory, the operational and structural views are linked through a concept development of three stages: interiorization, condensation, and reification. Interiorization refers to learners' acquisition of procedural skills. Condensation describes learners' management of several procedures as a single unit. Reification describes learners' discovery of the static structure of procedures, their spontaneous discovery of a series of procedures captured as an object.

These three stages represent the process of development from operational to structural thinking. They can also be considered as the path or trajectory of concept development in the context of student learning.

Gray's and Tall's (1994) procept theory stated that there is a duality, ambiguity, and flexibility between process and object. The same symbolism used to indicate a process also captures the product of that process. That product is call the concept. The mental structure of process and concept is called procept. The authors distinguished between process and procedure. Process is the actions or steps of carrying an operation, or mathematical activity (e.g., solving an equation). Procedure refers to specific algorithm (e.g., count-on, count-all, synthetic division) used within a process. Thus, the notion of procept refers to the combination of process and concept represented by the same symbol. A basic or elementary procept has a process, which becomes an object, and a symbol which represent either a process or an object. Then, a procept consists of elementary procepts that have the same objects.

Suffices it to say that there are several other theories that can be used to address issues pertaining to algebra as an activity, as a subject, or as a discipline in community colleges. These theories are only two of them. I selected Hoch and Dreyfus' (2004, 2006) structure sense theory for the study because it applies to both the first and second levels of algebraic proficiency as described earlier. I begin by illustrating structure sense and then explain my rationale for choosing Hoch and Dreyfus' (2004, 2006) structure sense theory in the next section.

The following tasks are examples that illustrate situations where structure sense can significantly inform someone's responses. There is an example task for each of the three

parts of the definition of structure sense. The first set of examples is taken from the literature (Table 9).

Table 9: Examples illustrating definitions taken from Hoch and Dreyfus (2006, p. 307)

Part of Definition	Ability	Example	Comments
SS1	Recognize structure in simplest form	Factor $81 - x^2$	No formula given
SS2	Deal with compound term (sum) as single entity, recognize structure in complex form	Factor $(x-3)^4 - (x+3)^4$	Deal with $(x-3)^2$ and $(x+3)^2$ as single entities
SS3	Choose appropriate manipulations to make best use of structure	Prove that $(x+y)^4 = (x-y)^4 + 8xy(x^2+y^2)$	Subtract $(x-y)^4$ from both sides of equation; deal with $(x+y)^2$; $(x-y)^2$ as single entities

The second set of examples is an extension of structure sense used when teaching algebra that was implemented in this study. These extensions incorporate function notation, "f(x)", as terms and apply structure sense to function compositions. Further applications of structure sense are possible, especially in the context of solving equations and inequalities (See Warren et al., 2016). Those applications can be instrumental in helping students understand the idea of solving equations and inequalities algebraically.

Table 10: Other examples of the definition structure sense from author's teaching

Part of Definition	Ability	Example	Comments
SS1	Recognize structure in simplest form	Describe $x^2 + y^2 = 3^2$ and the set of points it generates.	$x^2 + y^2 = 3^2$ instead of $(x+1)^2 + (y-2)^2 = 3^2$ The Pythagorean theorem
SS2	Deal with compound term (sum) as single entity, recognize structure in complex form	Use appropriate substitution(s) to show that the difference quotient, $\frac{f(a+h)-f(a)}{h}$ comes from a familiar structure	Deal with $f(a+h)$ and $f(a)$ as single entities, y_2 y_1. Replace h by $x_2 - x_1$ to show the slope formula $\frac{y_2-y_1}{x_2-x_1}$
SS2		Given $f(x)=5x^2-4x+1$ and $g(x)=2x-3$. Show two ways to simplify $5(2x-3)^2 -4(2x-3)+1$.	Using function notation, $5[g(x)]^2-4g(x)+1$, which is $f(g(x))$.
SS3	Choose appropriate manipulations to make best use of structure	Solve the inequality $(x-3)^2 + 4 \leq 68$	Subtract 68 from both sides of inequality; deal with the structure $(x-3)^2 -64$ as a single entity that is negative, positive, or 0.

Before the work of Hoch and Dreyfus on structure sense, Arcavi (1994) was inspired by Fey's (1990) idea of number and symbol sense to propose the notion of symbol sense (Arcavi, 2002). It is important to also consider symbol sense because Arcavi's (1994) description of symbol sense can inform structure sense instruction. Arcavi's (1994) description of symbol sense consists of a list of eight behaviors. These behaviors are consistent with the definition of structure sense given by Hoch and Dreyfus (2004, 2006, 2007), especially Behaviors 2, 4, 6, and 8. The eight behaviors are: 1. A feel for when to use

symbols and when to abandon them; 2. Reading through symbols; 3. The ability to successfully construct algebraic expressions to match specific goals; 4. The confidence in symbols that inform and encourage the search for new aspects of the original meanings of expressions; 5. A premonitory feeling for an optimal choice of symbols; 6. The realization of (a) a potential circularity of symbolic manipulation, (b) the gestalt view of expressions, and (c) manipulations directed towards formal targets; 7. The ability to recognize meaning in intermediate steps of algebraic manipulation; and 8. The resourcefulness to sort out nuances in the multiplicity of roles that symbols can play based on contexts (Arcavi, 1994).

The term structure which is described as a broad view analysis in Hoch and Dreyfus (2004) reflects Behavior 6b, the gestalt (or global) view of expressions. The realizations of Behavior 6, which are (a) a potential circularity of symbolic manipulation, (b) the gestalt view of expressions, and (c) manipulations directed towards formal targets are the results of Behavior 2 (reading through symbols). None of the realizations of Behavior 6 is possible without reading through symbols, which is Behavior 2. Similarly, none of the three recognitions of structure (SS1, SS3, SS3) stated in Hoch & Dreyfus (2006) is possible without reading through symbols (Behavior 2) and without taking a gestalt (or global) view of expressions (Behavior 6b). The three recognitions of structure (SS1, SS3, SS3) were mentioned earlier in the first part section of the introduction.

> Hoch and Dreyfus' (2006) definition of structure sense is reiterated as the ability to: recognize a familiar structure in its simplest form **(SS1)**; deal with a compound term as a single entity and through an appropriate substitution recognize a familiar structure in a more complex form **(SS2)**; choose appropriate manipulations to make best use of a structure **(SS3)** (p. 306).

Looking at those three recognitions of structure, it makes sense logically to see them as specific extensions of reading through symbols (Behavior 2) and taking a gestalt (or global) view of expressions (Behavior 6b). The position for this study as in Van Stiphout et al. (2011) is that structure sense is part of symbol sense.

Rationale for Selecting the Structure Sense Theoretical Framework

Hoch and Dreyfus' (2004, 2006) structure sense theory is an effective theoretical framework for understanding community college students' algebraic proficiency for three important reasons. First, structure sense aligns well with the two major theories discussed earlier (i.e., operational-structural and procept). In some ways, structure sense is a direct application of various aspects of these theories, especially the object aspect. In the operational-structural theory, the "interiorization-condensation-reification" stages describe a progressive sequence from individual steps of a process to object entity. The connection between the three stages and structure sense is the condensation stage by virtue of their commonality. According to the definition provided earlier, structure sense is about recognizing structure in algebraic expressions and treating them as a single entity. Similarly, condensation describes learners' management of several procedures combined as a single unit. In that sense, structure sense instruction is an application of the condensation stage in the algebraic representational mode.

The second reason Hoch and Dreyfus' (2004, 2006) structure sense theory is an effective framework for algebraic proficiency is that it influences both the first and second levels of algebraic proficiency as described earlier. It can be shown with appropriate tasks how structure sense relates to the models of mathematical proficiency discussed in the NCTM (2000), NRC (2001), and CCSSM (2011) documents. Nevertheless, looking at the

definition of structure sense (Hoch & Dreyfus, 2004, 2006), it is not hard to see that it applies directly to MP7 – about structure found in the CCSSM (2011), which is also an aspect of the 'generalizing and justifying' activities (NRC, 2001, p. 258). More importantly, it is not hard to see how the definition of structure sense applies to 'representational' and especially 'transformational' activities of algebra as described in Kieran (2007) and NRC (2001). And, the NRC maintained that representational activities involve both conceptual understanding and strategic competence (p. 257). Likewise, it also maintained that transformational activities involve aspects of conceptual understanding, strategic competence, and procedural fluency working together (p. 257). Furthermore, it is evident how the definition of structure sense is instrumental to the second-level description of algebraic proficiency in (e.g., Driscoll, 1999; Kieran, 2007; McCallum, 2007; NCTM, 2000; NRC, 2001), which highlights the importance of algebraic representation.

The third reason Hoch & Dreyfus' (2004, 2006) structure sense theory is an effective framework for algebraic proficiency is that it relates directly to the largest area of difficulty students have faced with learning algebra, namely symbolic representation as shown in Kieran (1992, 2007) and NMAP (2008). The structure sense definition applies specifically to central concepts like variables, equivalence, factors, terms, and coefficients that are germane to the symbolic/algebraic representational mode (Schoenfeld & Arcavi, 1988; Ursini & Trigueros, 2011; Usiskin, 1988). Trigueros and Ursini (2008) and Ursini and Trigueros (2009) have incorporated understanding of variables into the definition of structure sense.

After discussing two possible frameworks that would address the problem of algebraic proficiency in community college algebra, Hoch and Dreyfus' structure sense

theory provides the best framework. It helps implement the other frameworks in specific ways that are readily useful for algebra instruction. The definition of structure sense provides specific instructional means for developing proficiency in terms of the perspectives described by NCTM (2000), NRC (2001), CCSSM (2011), Carlson et al. (2010), Driscoll (1999), Kieran (2007), McCallum (2007), NRC (2001).

Chapter 3: Methodology

This chapter describes the procedures for data collection and analysis that were used to answer the research questions with respect to the structure sense theory. The research questions were identified at the beginning of the introduction chapter. Again, they are:

1. Is there significant difference in structure sense between students' Pretest-A and Posttest-A performance in a community college precalculus course?
2. Is there significant difference in algebraic proficiency between students' Pretest-B and Posttest-B performance in a community college precalculus course?
3. In what ways do students' work show development of structure sense in a community college precalculus algebra course?

This study aimed to describe students' work regarding structure sense in response to the qualitative question (3). It addressed hypotheses relative to the quantitative research questions (1 and 2). The research hypotheses are that:

- the structure sense intervention would have significant effect on students' structure sense in a college precalculus course.
- the structure sense intervention would have significant effect on students' algebraic proficiency in a college precalculus course.
- out-of-class factors also would be responsible for variation in students' achievement (Hiebert & Grouws, 2007), in this case, students' structure sense and algebraic proficiency in a college precalculus course.

The primary purpose of this study was to describe structure sense instruction in the context of a community college precalcalus algebra course by assessing the influence of a

“structure sense” intervention on students’ structure sense and algebraic proficiency. Because describing structure sense instruction was the primary purpose of this research, it was determined that a case study (qualitative) would be the appropriate approach.

Research Design

The study was a mixed methods investigation with a quantitative and qualitative strand. The qualitative strand was a case study, an instrumental case study (Stake, 1995 as cited in Creswell, 2013). The case study provided data on how community college students experience the structure sense intervention. It was an instrumental case study in the sense that the data collected about students’ experiences would inform how structure sense instruction can be improved. The qualitative strand would provide detailed information about the structure sense intervention through students’ work. The qualitative strand is the overarching framework in terms of the goal of the study to describe structure sense instruction (in the context of a community college precalcalus algebra course). Consequently, the quantitative strand is said to be embedded; it is a supportive strand. In addition, since neither strand depended on the results of the other, the strands are said to be concurrent. Hence, this type of design is called a concurrent embedded design. The quantitative strand helped assess the intervention along with other factors that affect students’ learning.

As the researcher, I taught two sections of a college precalculus algebra course, which is the first course in a two-semester precalculus sequence. The first course involved the study of polynomials; absolute value; radicals; and rational, step, exponential, and logarithmic functions. One section started with 30 students and the other with 26 students. Some students withdrew or were withdrawn and others opted not to participate in the study. As a result, the study was conducted on a sample size of 33 participants, 20 from one section

and 13 from the other. Instruction was markedly different in that I paid special attention to the development of structure sense. I used the tasks that I refered to in the background section (See also appendix J) for instruction. The use of those special tasks and the amount of time given to them in class are what characterize this teaching as structure sense instruction (SI). The content of the precalculus course was divided into four units, but the study was implemented during the first two units. Both unit 1 and unit 2 taught to the sections implemented structure sense instruction.

Table 11 includes a list of topics covered in the first two units of the course. Unit 1 emphasized algebraic reasoning in graphic form. Unit 2 emphasized algebraic reasoning in symbolic form.

Table 11: Topics covered in the two units

Unit 1 contains the topics:	Interpreting graphs, piecewise functions, obtaining information from a functions graph, increasing and decreasing parts of graphs, end behavior and asymptotes from graphs, transformations of functions, inverses of functions, converting between exponential form and logarithmic form, graphical approach of average rate of change, combining functions graphically, and solving systems of equations graphically.
Unit 2 contains the topics:	Domain of functions, combining functions symbolically, determining inverse functions symbolically, symbolic approach of average rate of change, symbolic approach of difference quotient, equations of polynomial functions, equations of exponential functions, analyzing rational functions, analyzing quadratic functions, solving quadratic inequalities, and solving absolute value equations.

Setting and Participants

The setting was a large community college in an urban area in the southeastern United States. Participants in this study were students enrolled in MAT 171, a Precalculus

Algebra course offered over a sixteen-week semester at this community college. MAT 171 is the first of a two-course series for students who are STEM majors (mostly). The second course in this two-course series is Precalculus Trigonometry. The participants were students who had all of the prerequisites for the course or had a 2.5 GPA from high school. The study had 33 participants--20 males and 13 females. The majority of the students were caucasian, with most of them between the ages of 18 and 21 years old. There were no special enrollment criteria for these sections. All participants registered as they normally would. Approvals to conduct the research were obtained from the mathematics department at the community college and from the North Carolina State University Institutional Review Board (IRB).

In choosing the cases for the qualitative strand of the study, I could not implement purposeful sampling for selecting cases because there were too few students who volunteered to be interviewed at the end of the semester. Even though the study ended at midterm, the Institutional Review Board (IRB) did not allow me to recruit interviewees until after final grades were submitted. I could not select participants who represented a cross-section of ability level because I would not have adequate data for such determination. However, due to the constraints of the IRB, I had to asked students to participate in the interviews after I had calculated their final grade for the course. The cases were students who agreed to volunteer when I asked them at the end of the semester. I asked all students in both sections, and six volunteered. The first two cases were students who had little interest in mathematics. The other four cases were students who had moderate to strong interest in mathematics. All cases were motivated, studious, and very responsible students.

Pseudonyms were used to refer to students in the interview sample. That sample consisted of 6 interviewees or cases for the case study, the qualitative strand, whose purpose was to answer the first research question. The quantitative strand was incorporated in order to answer the 2nd, 3rd, and 4th research questions and to provide further descriptive information about students' work relative to the structure sense intervention. Table 12 shows all four research questions along with their respective data collection instrument and type of analysis that was performed.

Table 12: Research questions, data collection, and analyses

Research questions	**Data collection**	**Analyses**
1. Is there significant difference in structure sense between students' pretest A and posttest A performance in a community college precalculus course?	Pretest A and Posttest A	Quantitative: Paired t-test
2. Is there significant difference in algebraic proficiency between students' pretest B and posttest B performance in a community college precalculus course?	Pretests A and B, Posttests A and B, background survey, midterm reflection questionnaire, mathematics interest and beliefs questionnaire	Quantitative: Paired t-test Correlation ANOVA Multiple linear regression
3. In what ways do students' work show development of structure sense in a community college precalculus algebra course?	Task-based interviews from Posttest B	Qualitative: Within-case and cross-case analyses

Pretest A and Posttest A are the set of items that lend themselves to structure sense on two alternate forms of the Unit 2 test. Pretest B and Posttest B represent Unit 2 test in its entirety, but in two alternate forms. Pre and post represent alternate forms of the Unit 2 test.

A and B represent content; A is structure sense content; B is the entire content of the Unit 2 test.

Implementing the Structure Sense Instruction

This section include a description of how the structure sense instruction was actually implememted. Structure sense instruction (SI) was the instructional intervention that was implemented in those two sections of precalculus algebra. Some background information about this type of instruction was provided earlier at the end of the background section of Chapter 1, the Introduction. This section contains also some general information about each of the two classes that were involved in the research study.

Students in those two classes self-reported that their study skills were average. In general, it was their first time taking precalculus algebra in college. In general, it had been about two to three years since they had taken an algebra course. For some, it had been as little as half a year, for others six years, and still for others more than 20 years. They self-reported moderate level of interest, motivation, and confidence in algebra. Class attendance in both sections was very good; so most students were exposed consistently to the intervention. However, the homework system was not conducive to the structure sense intervention. That homework system was supposed to be same for every precalculus algebra course across the college; it could not be altered.

At the time, as the instructor I was completing my sixth year as a mathematics instructor at that community college. I had taught precalculus algebra there almost every semester since 2013. A subjectivity statement is included in the 'Theoretical Framework" section of the Introduction detailing additional information about me as the teacher and resesearcher. The precalculus course had been redesigned in 2016. The redesign organized

topics in a way such that they would be revisited several times, each time with increasing level of sophistication, building on previous learning. This approach was termed the spiral approach. The first unit of the semester focused on analyzing relations and solving equations in their graphical representations. The second unit introduced reasoning about relations/functions in symbolic representation through translating between graphs of functions and their corresponding equations. That unit also involved solving equations and inequalities symbolically. The third unit continued with objectives/tasks similar to those in the second unit, but more in depth and with more advanced forms of the functions from the second unit. Whereas the third unit focused more on analysis of polynomials, the fourth unit dealt with different functions and equations like logarithmic, exponential, rational, radical, and absolute value; it also involved systems of linear and nonlinear equations; partial fraction decomposition, absolute value and rational inequalities, mostly in symbolic representation. All units included application problems.

Another component of this redesign was scaffolding, which meant in this case that at the beginning there is a lot of responsibility on teachers to model algebra and less responsibility on students for instant learning. Therefore, there is significant scaffolding. However, as teachers promote learning through skill building, active learning, and formative assessment, students are required to be more responsible toward summative assessment. Therefore, scaffolding ends with students having full responsibility to demonstrate learning on summative assessment. Throughout this process, teachers are supposed to flip the classroom by assigning preparatory activities to students, especially in Desmos, a mathematical software. Students were not allowed to use a graphing calculator to graph on tests. They could only use Desmos for graphing while they learned the concepts. The

semester I implemented the intervention was the second semester I had taught the course since it had been completely redesigned.

First, implementating the intervention involved teaching the topics in the second unit in a different sequence than outlined in the course calendar. The purpose of reorganizing the other of covering the topics was to better emphasize the structure and forms of different functions and equations. The left side of Table 13 lists the topics in the original sequence taken from the course calendar. The right side of Table 13 lists the topics in the sequence I taught them according to the intervention.

Table 13: Topic sequence according to structure

	Original sequence	Intervention sequence
Week 1	• Combining and composing functions graphically and symbolically • Difference quotient • Inverses symbolically	• Equations of polynomials from graphs • Quadratic functions applications with models given, and graphing • Rational functions exploration
Week 2	• Equations of exponential functions from graphs / equations of polynomials from graphs • Absolute value equations • Quadratic functions applications with models given, and graphing	• Graphing simple rational without slant asymptotes • Equations of exponential functions from graphs • Quadratic inequalities
Week 3	• Quadratic inequalities • Rational functions exploration • Graphing simple rational without slant asymptotes	• Piecewise functions and modeling (linear equations) • Combining and composing functions graphically and symbolically • Difference quotient
Week 4	• Piecewise functions and modeling • Review • Test 2	• Inverses symbolically • Absolute value equations / Review • Test 2

I resequenced these topics so that the second unit would be divided into two strategic parts. The first part would involve tasks that require students to identify parameters of different functions (structures). Once these parameters are identified, the final step in the task is to substitute the parameters into the general form of a function/structure to obtain a specific instance of that type of structure/function. Each kind of function is a structure (Hoch and

Dreyfus, 2004). A structure or a type of function may have several different forms. For example, the linear function has at least four different forms (i.e. slope-intercept, point-slope, standard, intercept); the quadratic function has at least three (i.e. expanded, vertex, factored). When they do not have specific numbers for the parameters, I presented the types of functions in Table 13 as structures. I called them structure (type) when discussing them. An example of a task for the first part of the unit would be: Given the graph of some parabola, write the equation of the function. Students are required to know which structure to invoke and which form of that structure to use. For example, if the vertex and a random point are readily available from that graph, then the vertex form of the quadratic structure should be used. On the other hand if the *x*-intercepts and a random point are readily available, then the factored form should be used.

The first part of the second unit focused on analyzing parameters of the structures in some of their different forms, which I called general forms or templates; as well as emphasizing that *x* and *y* represent ordered pairs, independent and dependent variables that remain as such after the equations of those functions have been determined by fixing specific values for parameters. This part also involved converting between different forms of the same structure. A example is converting ax^2+bx+c into $a(x-h)^2+k$. I put a lot of emphasis on getting familiar with and recognizing structures and their forms as well meaning of parameters based on the terms and factors of those structures. I built on Driscoll's (1999) idea of doing-undoing to emphasize the concept of opposite/inverse structures for manipulating symbols in the context of solving equations and in the context of writing the equation of an inverse function. This first part of the unit took about 7 class meetings. The second part took the remaining 4 class meetings.

The second strategic part of the second unit focused on tasks that dealt significantly with function notation such as combining and composing functions, difference quotient symbolically, and inverse functions symbolically. I thoroughly explained function notation in relation to the expressions they label, stressing the concept of equivalence and interchangeability of function notation and function expression.

Handouts were given in class for each lesson to model structure sense (See Tables 1 through 8 for excerpts and Appendix J for complete handouts). The tasks in the first part of the unit dealt primarily with the general definition of structure sense (Gen defn), taking a gestalt view of algebraic statements. Examples include comparing structures with each other highlighting similarities and differences as well as extracting details through mental calculations (without writing anything down). For instance, picking out x-intercepts, y-intercepts, vertex, asymptotes, orientation of graphs, degree, solutions, or veracity of statements simply by looking at statements.

They dealt with the first part of Hoch and Dreyfus' (2006) definition of structure sense (SS1), recognizing structures in their simplest form. An example is recognizing types and forms of functions/equations. For instance, in a task that is based on an expression like $3(2)^x - 4$, students were taught to recognize the structure as exponential. Students were taught to be precise about what part of an expression defines as being a specific structure. For instance, simply having an exponent in an equation does not make that equation an exponential equation. Or simply have fraction in an equation does not make it a rational equation for that matter. All of those nuances went in teaching students how to recognize structures.

The tasks in the second part of the unit dealt primarily with the first clause of the second part of Hoch and Dreyfus' (2006) definition of structure sense (SS2-A), which is to deal with a compound term as a single entity. An example would be applying the difference quotient formulat to the equation of a function. For instance, the compound term ($x+h$) is substituted in for x as if it were a single number/character. Similar instances occur when composing functions symbolically.

Those tasks also dealt with the third part of Hoch and Dreyfus' (2006) definition of structure sense (SS3), which is choose appropriate manipulations to make best use of a structure. Students were taught that moving symbols around not only depends on structure but it is done for the purpose of exploiting structure. For instance, when all nonzero terms are moved on the same same side, it is done based on maintaining equivalence between the two sides of the equation. At the same time, it is also for the purpose of breaking/undoing that structure so that new information may be revealed (e.g., solutions an the equation, a simpler equation, and so on). Isolating the absolute value part of an equation is done so that the definition of absolute value may be applied to yield two different equations. Isolating the absolute value part is choosing appropriate manipulations so that the absolute value structure may be better addressed. Equivalence across the equal sign or equivalence between two sides of a statement is also an aspect of structure sense. In the context of SS3, students were taught equivalence of full statements. For example, restating an original equation into subsequent equivalent statements illustrate the process of choosing appropriate manipulations to make best use of structure.

Second, this same approach was implemented in the course project with the same purpose of fostering structure sense abilities in students. The project was assigned at the

beginning of the second unit, February 2nd . The due date was February 23rd. Students were given three weeks to complete the project. I graded students' work on the project and returned them with feedback before Test 2. Table 14 list some of the tasks on that project along with the structure sense ability needed.

Table 14: Example tasks from course project

Task number	Task	Structure sense ability
1a	Without solving for x and without graphing, explain why there exist no x-value for which $\frac{2x+3}{4x+6} = 2$?	• General definition (gestalt view analysis)
3	Explain how you can tell (without graphing it) that the function $$r(x) = \frac{x^6 + 10}{x^4 + 8x^2 + 15}$$ has no horizontal intercept and no horizontal, vertical, or slant asymptote. What is its end behavior?	• General definition (gestalt view analysis)
4	Why is \$10,935 the maximum profit and \$33 is the price which gives that maximum profit? Provide an explanation based on the terms of the expression $-15(x-33)^2 + 10935$, (beyond using the vertex, or graph, or guess & check).	• General definition (gestalt view analysis) • Deal with a compound term as a single entity (SS2-A)
5i	The following expressions are equivalent to $-15(x-33)^2 + 10935$, which is called the transformation or vertex form for quadratic polynomials. a. $-15(x-6)(x-60)$ b. $-15x^2 + 990x - 5400$ (i) Identify the form of both expressions (a and b) above (i.e., expanded, factored/root, standard).	• Recognize structure in its simplest form (SS1)

Appendix K contains a copy of the entire project as it was assigned to students. I created the structure sense project using tasks from the literature. It was assigned as an individual, not a group project. The project recommended for the course was a group project on applications of exponential functions in Unit 4, toward the end of the semester. I chose to assign the structure sense project much earlier in the second unit in to order to reinforce the teachings of the intervention.

Variables

With regards to the quantitative strand of the study, there are two dependent variables. The first one is students' structure sense, the extent students demonstrate structure sense relative to specific items of the Pretest A and Posttest A. The definitions of structure sense provided in Hoch and Dreyfus (2004, 2006, 2007) were used to operationalize the construct of structure sense into a quantitative measure. These definitions were presented in the 'background' section of Chapter 1 and in the 'rationale' section of Chapter 2. The second construct is students' algebraic proficiency which was partly operationalized at the end of the first section of the literature review.

Operationalizations. Structure sense was operationalized using the three definitions of structure sense provided in Hoch and Dreyfus (2004, 2006, 2007). Structure sense was measured on a subset of the pretest/posttest items that lend themselves to the use of structure sense. More specifically, structure sense was assessed only on items that require the use of at least one of the components of the definition for their execution. The number of points awarded for structure sense on an item is proportional to the total number of points for the item. The number of points awarded for structure sense on an item is also proportional to other actions required for execution of the item. If an item is worth 3 points and only

requires the use of the definition, then all 3 is awarded for structure sense on that item. By contrast, if an item requires other actions besides the use of the definition of structure sense, then a portion of the total number points is awarded for structure sense. The following item from the pretest/posttest is an example,

"3. Using the function $f(x) = x^2 - 3x + 4$. (a) Construct and simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

for the function $f(x)$ (4 points)." In this case, the item is worth 4 points. Two points are to be awarded for structure sense when the compound term (x+h) is substituted in f(x) twice, one for each instance of x and when the expression x^2-3x+4 is substituted in for the function notation, f(x). In this context, both (x+h) and x^2-3x+4 are considered as compound term used as single entities. The pretest/posttest items that lend themselves to the use of structure sense are: 1a, 1d, 2a, 3a, 4a, 6, 7a, and 8d. These items total 25 points for structure sense. So 25 points are possible for structure sense. Therefore a student's score for structure sense is the number of points they earned on these items divided by 25, expressed in percent. Table 15 lists the pretest items that lend themselves to the use of structure sense and the number of points for structure sense associated with each of them. The analysis section in this chapter contains a list of these items with the definition of structure that corresponds with their completion (Table 15).

Table 15: Point distribution for structure sense

Item number	**Item**	**Total number of points for item**	**Points for structure sense**
1a	Use $f(x)=x^2-3x$, $g(x)=7-2x^2$ and $h(x)=\sqrt{9-x}$ to complete the following. • $(f-g)(x)=$	3	1
1d	Use $f(x)=x^2-3x$, $g(x)=7-2x^2$ and $h(x)=\sqrt{9-x}$ to complete the following. • $(f\circ g)(x)=$	4	4
2a	If $h(x)=\sqrt{1+(x+2)^3}$, determine two non-trivial functions *f(x)* and *g(x)* such that $h(x)=(f\circ g)(x)$	4	4
3a	Using the function $f(x)=x^2-3x+4$ Construct and simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the function $f(x)$.	4	4
4a	Given $f(x)=\sqrt{x-4}$. Determine the inverse, $f^{-1}(x)$	5	1
6	Solve the quadratic inequality and write your solution using interval notation. $x^2-2x\geq 15$	5	5
7a	Suppose the quadratic function, $g(x)=-2x^2+24x+7$ represents the height of a projectile in feet after x seconds. Determine the transformation form of $g(x)$.	3	3
8d	Use the rational function $R(x)=\frac{4(x-1)(x+3)}{(x-1)(x+2)}$ to complete the following. • Write the **equation(s)** of the horizontal asymptote(s).	3	3

This set of items make up Pretest A. These items allowed me to calculate values for students' potential structure sense as the first dependent variable, thus operationalizing it.

With regards to algebraic proficiency as the second dependent variable and operationalizing it, the review of the literature indicates that it is best represented by all five descriptions provided by Carlson et al. (2010), Driscoll (1999), Kieran (2007), McCallum (2007), and NRC (2001). However, any one of the five descriptions also represents algebraic proficiency. The skills and concepts assessed on the pretest and posttest fall within the scope of those five descriptions of algebraic proficiency. In that sense, the score on each of those tests represents the operationalization (or a proxy) of students' algebraic proficiency relative to the topics covered and concepts assessed.

Students' structure sense is the measure of students' performance on Pretest A and Posttest A; it is represented by the variable names Pretest A and Posttest A. Similarly, students' algebraic proficiency is the measure of students' performance on pretest B and posttest B; it is represented by the variable names Pretest B and Posttest B. Validity and reliability for both measures are discussed in the analysis section of this chapter.

The influence of the "structure sense" intervention on students' structure sense and algebraic proficiency was measured quantitatively relative to other factors that affect learning. Structure sense as a dependent variable is students' score on selected items from a pretest and a posttest. Algebraic proficiency as the other dependent variable is students' scores on a pretest and posttest. Tables 16-19 list explanatory variables with their descriptions.

Table 16: Explanatory variables from background survey

Variable name	Instrument	Item (Item number)	Calculation of variable
Timesince	Background survey on the first day of semester	• How long has it been since students took an algebra course? (3)	Students entries were converted in weeks.
Firsttime	Background survey on the first day of semester	• Is this your first time taking precalculus in college? (4)	Yes = 1 and no = 0.
Load_start	Background survey on the first day of semester	• How many credit hours are you taking? (6) • How many hours a week on average students work? (8)	Load_start is the number of credit hours plus the number of hours working per week.

Table 17 lists explanatory variables which were generated from affective factors. The first variable in that group is INTMAT; values of 4 through 8 would indicate that students are interested in and enjoy mathematics while values of 9 through 16 would indicate no interest or enjoyment in mathematics in varying degree. The second variable in that group is INSTMOT; values of 4 through 8 would indicate that students are motivated in mathematics for some external rewards while values of 9 through 16 would indicate no motivation in mathematics, in varying degree. Because of the way the items were designed in the Students' Mathematics Interest and Beliefs instrument, low values for these affectives variables mean high interest and enjoyment in mathematics, high motivation in mathematics, and high self-efficacy in mathematics.

Table 17: Explanatory variable based on affective factors

Variable name	Instrument	Item (Item number)	Calculation of variable
INTMAT (interest in and enjoyment of mathematics)	Students' mathematics interest and beliefs questionnaire on the first day of the semester	• I enjoy reading about mathematics. (a) • Making an effort in mathematics is worth it because it will help me in the work that I want to do later. (b) • I look forward to my mathematics lessons. (c) • I do mathematics because I enjoy it. (d)	INTMAT is the sum of the values of items a, b, c, and d where 1= strongly agree, 2=agree, 3=disagree, and 4=strongly disagree. The minimum and maximum values for INTMAT are 4 and 16 respectively.
INSTMOT (Instrumental motivation)	Students'mathematics interest and beliefs questionnaire on the first day of the semester	• Learning mathematics is worthwhile for me because it will improve my career prospects. (e) • I am interested in the things I learn in mathematics. (f) • Mathematics is an important subject for me because I need it for what I want study later on. (g) • I will learn many things in mathematics that will help me get a job. (h)	INSTMOT is the sum of the value of items e, f, g, and h where 1= strongly agree, 2=agree, 3=disagree, and 4=strongly disagree. The minimum and maximum values for INSTMOT are 4 and 16 respectively.

The third variable about affective factors is MATHEFF; values of 8 through 16 indicated that students felt confident about completing the task in the items while values of

17 through 32 indicated that students did not feel confident about completing the task in the items in varying degree, Table 18.

Table 18: Additional explanatory variable based on affective factors

Variable name	Instrument	Item (Item number)	Calculation of variable
MATHEFF (Students' self-efficacy in mathematics)	Students'mathematics interest and beliefs questionnaire on the first day of the semester	• Using a train schedule to figure out how long it would take to get from one place to another. (a) • Calculating how much cheaper a TV would be after a 30% discoutn. (b) • Calculating how many square feet of tile you need to cover a floor. (c) • Understanding graphs presented in newspapers. (d) • Solving an equation like 3x+5=17. (e) • Finding the actual distance between two places o a map with a 1:10,000 scale. (f) • Solving an equation like 2(x+3)=(x+3)(x-3). (g) • Calculating the gas mileage of a car. (h)	MATHEFF is the sum of the values of items a through h regarding the question "how do you feel about having to the following mathematics tasks" where 1= very confident, 2=confident, 3=not very confident, and 4=not at all confident. The minimum and maximum values for MATHEFF are 8 and 32 respectively.

By midterm, students were asked to complete a midterm reflection questionnaire (Appendix F). This questionnaire helped uncover other explanatory variables that may have contributed to students learning. The questionnaire focused on factors related to activities that may take place outside the classroom (See Table 19 and Appendix F). For example, if

they hired a personal tutor, how many hours per week on average did they spend getting tutored. If they participated in study group sessions, how many hours per week on average have they spent. If they received assistance from any on-campus resource, how many hours per week on average did they use that resource. These out-of-class activities also involve watching videos and visiting instructors' office hours.

Table 19: Explanatory variables from a midterm reflection questionnaire

Variable name	**Instrument**	**Item (Item number)**	**Calculation of variable**
Effort	Midterm reflection questionnaire	• How many hours per week on average did you spend getting tutored?_____ (2) • Did you participate in study groups? (yes - no). How many hours per week on average have you spent?_____ (3) • If any, what campus resources did you use thus far (e.g., office hours, STEM Center, ILC)? _____ How many hours per week on average did you use that resource? _____ (4) • How many hours a week on average have you spent on MAT 171 doing homework, reading class notes, watching instructional videos to learn the concepts in this course? (9)	Effort is the sum of hours per week from each of these four items.
Load_end	Midterm reflection questionnaire	• How many credit hours are you taking?_____ (6) • On average how many hours per week have you been working in the last 7 weeks? (8)	Load_end is the sum of class and work hours per week from each of these two items.
Load_average	Background survey and the Midterm reflection questionnaire	Items 6 and 8 on both instruments.	Load_average is the average hours from items 6 and 8 for the start and the midterm term point of the semester.

Data Collection

The data collection supported several analyses, both quantitative and qualitative (Table 12). For the quantitative analyses, I collected data through five primary instruments: a background survey, a mathematics interest and beliefs questionnaire, a pretest, a posttest (Test 2), and a midterm reflection questionnaire.

Background survey. The Background Survey (Appendix A) was used to collect information pertaining to individual students like gender, age, and ethnicity, program of study; but most importantly, to gather information on whether they were enrolled full-time or part-time, whether they were unemployed or were working full time or part time, and the level of access they had to computer and internet. The information collected from this survey and the mathematics interest/self-efficacy questionnaire would provide values for the explanatory variables mentioned above.

Students' mathematics interest and beliefs questionnaire. A portion of the 2012 PISA questionnaire, Q31), found in Appendix B, was used to collect data about students' mathematics interest and beliefs about the utility of mathematics. This questionnaire has 16 items. The first four items make up the "interest in and enjoyment of mathematics (INTMAT): The intensity and continuity of engagement and enjoyment in mathematics" (*OECD, 2004a, 2004b* as cited in Lee & Stankov, 2013). The second set of four items comes from the "Instrumental motivation (INSTMOT): The extent to which students make an effort to learn for external rewards" (*Deci & Ryan, 1992* as cited in Lee & Stankov, 2013). These two sets of items together make up the mathematics-interest part (Part 1) of the questionnaire. The second half of the questionnaire is the "Students' self-efficacy in mathematics (MATHEFF): The extent to which students believe that they can do the task

being asked" (*Bandura, 1986* as cited in Lee & Stankov, 2013). It contains the other eight items.

Pretest A and B (Test 2). A pretest (Appendix C), which is an alternate form of the posttest (Test 2 was used to assess students' proficiency on the topics listed in Table 11. This instrument was administered only as part of the study, but not as a regular part of the course.

Posttest A and B (Test 2, alternate form). This test as shown in Appendix D is an equivalent form of the pretest. The posttest or Test 2 is part of the course and will be administered as such. Like the pretest, it covered the topics listed in Table 11. Unlike the pretest, Test 2 is just one the four class tests administered to students prior to the final exam. The pretest and the posttest being equivalent, would help control for confounding variables. Students were not allowed to use a graphing calculator to graph on tests. They could only use Desmos for graphing while they learned the concepts.

Midterm reflection questionnaire. One purpose of this questionnaire (Appendix F) was to confirm the values students provided for some of the explanatory variables generated from the background survey at the beginning of the course. Another purpose of this questionnaire was to discover other possible variables that may influence students' learning. These two instruments, the background survey and the midterm questionnaire were to help triangulate information with the final interviews after the end of the semester.

Procedures. I described how these instruments were used to collect data for analysis; they are summarized in Table 20. I administered the background survey, the students' mathematics interest and beliefs questionnaire, and the pretest on February 1st, 2018 for one section and on February 2nd for the other section. The Midterm reflection questionnaire was given to students to fill out in class (March 2nd, for both section).

Table 20: Summary of the data collection process for the quantitative strand

Time	Instrument	Collection method	Reason
February 1st, 2018/ February 2nd	• Informed consent form • Background survey • Students' mathematics interest and beliefs questionnaire • Pretest A and B	Hard copies handed out to students to be completed in class	• Complying with human subjects requirements • Collecting values for explanatory variables • Collecting values for explanatory variables on students' mathematics interest and beliefs • Comparing students within groups
February 28th/March 1st	• Posttest A and B (Test 2)	Hard copies of test 2 handed out to students to be completed in class	• Comparing students within groups
March 2nd	• Midterm reflection questionnaire	Hard copies handed out to students to be completed in class	• For discovering other explanatory variables

The informed consent form is mentioned in Table 20 for the timeline and to indicate that students will fill it out first before any other questionnaires.

For the qualitative analyses, I collected data mainly from three (3) sources: Posttest B (Test 2), a reflection on Test 2 assignment (not for grade), and task-based interviews with six (6) students who were willing to be interviewed. Posttest B (Test 2) was administered on

Wednesday February 28th, 2018 for one section and on Thursday March 1st, 2018 for the other section.

The reflection on Test 2 assignment was administered on Friday March 2nd to all students in for both sections. Test 2 was graded and returned to students that Friday so they could use in class to reflect on selected items. This reflection activity took place during the first half hour at the beginning of the class period. The primary purpose of this reflection activity was to collected some initial information on their work from Test 2 before to much time elapsed.

The IRB protocol did not allow me teacher-researcher to know the students who were willing to be interviewed until after final grades were determined. Once final grades were determined, task-based interviews were conducted with six students and audiovisual recordings of each interview session were obtained. These interviews were based on selected items from the second unit test (Posttest B or Test 2).

Table 21 summarizes the data collection process for the qualitative strand. In particular, students were asked questions about their written work to specific items on Posttest A.

Table 21: Summary of the data collection process for the qualitative strand

Time	Instrument	Collection method	Reason
February 28th and March 1st	• Posttest B (test 2)	Paper copies administered in class	To capture students' work and how their reasoning correlated to the structure sense definitions
March 2nd	• Reflection assignment on posttest B	Paper copies administered in class	To capture specific details about students' reasoning as they related/or not to the structure sense definitions
May 4th, May 8th, and May 14th	• Task-based interview protocol	Audiovisual recordings	To capture specific details about students' reasoning as they related/or not to the structure sense definitions

Appendix D has the full content of Posttest B (Test 2). Table 22 the tasks from the reflection on Test 2 assignment (not for grade). Appendix E has the full content of the reflection on Test 2.

Table 22: Tasks from reflection on Test 2 assignment

Items
1. In item #2a on test 2, how do you know which function has an inverse and which one doesn't? 2. In item #2b, did you expect the kind of function or structure you would get for the inverse of g(x)? Yes or no. If yes, how? 3. In item #4, how did you think about decomposing h(x)? 4. In item #5, how did you think about handling the absolute value bars or solving the equation? Why? 5. In items #8 and #9, what difficulty, if any, did you have? 6. In item #11c, how did you come up with your response? What helped you? 7. In item #12a, how did you come up with your response? What helped you? 8. In item #12d, how did you come up with your response? What helped you? 9. In item #12e, how did you come up with your response? What helped you? 10. In item #13, how did you come up with your response? What helped you?

Table 23 shows some of items from the interview protocol. Appendix G has the full interview protocol including instructions given to students before questioning began.

Table 23: Interview protocol

Items	Main questions
1a/b:	• Why did you put parentheses in that statement? (ss2a)
1c	• How did you know that f(x) and g(x) are inverses of each other? • How else could you have noticed that they are inverses?
2a	• What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered?
2b	• Why did you square both sides of the equation? (gen def, 2004) • Looking at the inverse function that you came up with, how do you know it is the correct inverse for the original function?
4	• How did you think about decomposing that function? Which part of the original expression did you think could be a component? Why?
5	• Why did you / why didn't you distribute that factor through the absolute value bars to get rid of them (i.e. absolute value bars?
6	• How did you think about or what helped you get this task right?
11b	• Why didn't you/did you choose the last term in the statement as the y-intercept?
11c	• What was/is your thinking process for determining the vertex from the statement?
11d	• How did/do you understand what the idea of symmetry was/is referring to in that question? What was/is your thinking process? Why?
12a	• How did you determine the x-intercept(s) from the given equation? What was/is your thinking process? Why?
12d	• How did you determine the horizontal asymptote from the given equation? What was/is your thinking process? Why?
13a	• Using the given equation, how did you determine what score the student would earn without studying? What was/is your thinking process? Why?
13b	• What was/is you thinking process? Why?

During the interview sessions, I (the interviewer) followed the directions written on the interview protocol. The video camera only captured students' work on paper and students' hand gestures, what they wrote, and pointed to on their test, but their faces was

never videotaped. Students were given time to read each item and their work for that item before questions were asked on that item. Questions were clarified and additional questions were asked by both interviewer and interviewees as needed. This portion of each interview session was videotaped and lasted 30 minutes on average. The other portion of the interview sessions that was not videotaped included some background questions and some instructions about the interview process itself (Appendix G).

Data Analysis

In this section, I describe how I validated, organized, and analyzed data in order to answer the research questions for the study. The quantitative analyses address the first and second research questions. Results and analyses are presented in the next two chapters: qualitative results and analyses in Chapter 4, then quantitative results and analyses in Chapter 5. The qualitative analyses address the third research question: In what ways do students' work show development of structure sense in a community college precalculus algebra course?

Procedures for quantitative analyses. Using STATA, a statistical software package, I entered and coded the data collected from the instruments listed in the data collection section. Coding served to organize the data and facilitated analyses. For example, students who took a remedial course in college were coded 0 and those who did not were coded 1. Full time work status was code 1; and part time or unemployed coded 0. "Yes" was coded 1. Full time course enrollment status was code 1; and part time coded 0. I used STATA to run all the quantitative analyses.

Besides calculating basic descriptive statistics, I performed a related-samples t-test for Pretest A and Posttest A and another related-samples t-test for Pretest B and Posttest B. The purpose of these t-tests was to address research questions 1 and 2. To answer the second research question, I also ran an analysis of variance (ANOVA) for both pre-post tests across groups of students with different circumstances (i.e. work, class schedule) as described earlier in Table 14. Then, I ran a correlation on the affective variables defined earlier in Tables 15, 16, and 17. Finally, I ran a full regression model.

The related-samples t-test for Pretest A and Posttest A addressed whether students' structure sense improved from when they took the pretest to the time they took the posttest at midterm. The related-samples t-test for Pretest B and Posttest B addressed whether students' algebraic proficiency improved from when they took the pretest to the time they took the posttest at midterm. The analysis of variance (ANOVA) addressed whether students with more favorable circumstances improved more in structure sense and algebraic proficiency. Then, the correlation on the affective variables addressed how students' posttest performance were distributed based on affective factors such as interest, beliefs, and motivation. Students' structure sense is the measure of students' performance on Pretest A and Posttest A; it is represented by the variable names Pretest A and Posttest A. Similarly, students' algebraic proficiency is the measure of students' performance on Pretest B and Posttest B; it is represented by the variable names Pretest B and Posttest B.

Table 24: Summary of quantitative analysis

Phase	Phase description
1	• Ran and interpreted related samples t-tests
2	• Ran and interpreted independent samples t-tests based on students' circumstances • Ran and interpreted ANOVA • Ran and interpreted correlation on affective variables
3	• Performed multiple linear regression (MLR

Validity. Shadish et al. (2002) underlined the importance of validity. Internal validity refers "to inferences about whether "the experimental treatments make a difference in this specific instance" (Campbell and Stanley, 1963) as cited in (Shadish et al. 2002)." External validity addresses the issue of generalization (Campbell and Stanley, 1963) as cited in (Shadish et al. 2002). Shadish and colleagues described four components of validity from Campbell and Stanley (1979): Statistical conclusion validity, internal validity, construct validity, and external validity.

Statistical conclusion validity has to do with inferences about the correlation or covariation between the treatment (independent variable) and the outcome or dependent variables (i.e. students' structure sense and algebraic proficiency). In this case, I chose a p-value of 0.05 or less in order to reject the null hypothesis and minimize the likelihood of type I error by paying attention to the assumptions of the statistical tests. Type I error occurs when the research wrongly reject the null hypothesis, concluding that there is covariation when there is not. The other side is the type II error, which is when the researcher fails to detect a

real effect thereby incorrectly accepting the null hypothesis. The second part of this type of validity deals with the overestimating or underestimating the size of the covariation. Low statistical power is the major threat to this validity.

Internal validity addresses the causal aspect of the covariation between the treatment (independent variable) and the outcome or dependent variables (i.e. students' structure sense and algebraic proficiency). It requires that the treatment precedes the posttest and that no other explanations are plausible for the relationship between the treatment and students' algebraic proficiency at the end of the study. It is not possible to determine this, but the analysis did quantify the contribution of the intervention through the estimation of effect size. Selection threat is present when one group differs initially from the other on the outcome variable.

Another threat to internal validity is history. That is, other events that may occur alongside the treatment which can affect the outcome. I asked students in the midterm reflection questionnaire to document such possible events. For instance, did they have a personal tutor; did they participate in study groups, etc. I asked them to document anything they believe may have affected their learning experience outside of what took place in class. These questions also helped to address maturation threat, whereby change in students' structure sense and algebraic proficiency may increase because they naturally gain new insight. The pretest and posttest will be different forms, and it is unlikely that students will remember test items over a 6-week period. Therefore, testing threat to internal validity was managed.

Construct validity and external validity affect generalization. One threat to construct validity is the failure to adequately explicate the construct. In the study the constructs are

students' structure sense and algebraic proficiency measured from the pretest and posttest. I described structure sense with various illustrations in Chapter 1, the Introduction. I thoroughly described algebraic proficiency in Chapter 2, the Literature Review. Specifically, construct validity in the study depend on students' score. This precalculus course transfers to any four-year college or university. Therefore, the scores from every test is a valid measure of algebraic proficiency relative to concepts covered. Another mathematics instructor beside me graded the tests based on a rubric; I ran the analyses on the average the two scores.

External validity deals with whether causal relationship holds for elements within the experiment and for elements outside the experiment. As already mentioned, the study did not concern generalization beyond the participants.

Reliability. The students' mathematics interest and beliefs about the utility of mathematics is a reliable instrument. The first four items make up the "interest in and enjoyment of mathematics (INTMAT) has a realibility of 0.90. The second set of four items comes from the Instrumental motivation (INSTMOT) has a realibility of 0.87. The Students' self-efficacy in mathematics (MATHEFF) has a realibility of 0.82. However, these measures were not used in their respective entirety. So the reliability of these variables do not necessarily apply to the study. I compared students' responses from the reflection assignment with their responses from the students' mathematics interest and beliefs and their performances on tests.

Procedures for qualitative analyses. The qualitative data analyses were performed in two main parts: a description of each case followed by a cross-case analysis. The first part describes the background and work of each case (within-case analysis). Then the second part describes a comparison of the cases (cross-case analysis). The purpose of these analyses were

to provide an in-depth understanding of how the structure sense instruction may have influenced the cases, the six student participants.

Within-case analysis. This analysis involved four steps. First I presented additional background information on each case. Second, I transcribed the data from video recordings for the interviews into text files in MS Word. Third, I read every transcript several times to get a general sense of each as a whole (Creswell, 2013, p. 183) and making notes in the margins. Fourth, I used a mixed approach for my coding scheme. I used "prefigured" or a priori codes based on the definitions of structure sense provided by Hoch and Dreyfus (2004; 2006; 2007) in order to categorize the items of the posttest. Then I also used additional codes that emerged during the analysis and the literature review as recommended by Creswell (2013, p. 185). I used the additional codes for noting the transcripts in order to track how student responses correspond to both the structure sense (see Table 24). Both prefigured codes and additional codes are described in this section. After coding the data from the reflection on Posttest B and the video recordings into meaningful segments with labels, I combined the number of codes into a fewer number of broader themes. I compressed the coded data into broader themes in the next part of the analysis, the cross-case analysis.

Students were interviewed individually. Every part of their work was coded for evidence of structure sense using the additional codes. The analyses were performed on (a) data from the reflection on Posttest B and on (b) data transcribed from the interview recordings.

"Prefigured" coding was determined by breaking Hoch and Dreyfus' (2006) definition of structure sense into elememtary parts. Each elementary part were given an abbreviated label. For example, the second component of Hoch and Dreyfus' (2006)

definition, which is “deal with a compound term as a single entity and through an appropriate substitution recognize a familiar structure in a more complex form (SS2)” was broken into two elememtary parts: (1) “deal with a compound term as a single entity (SS2-A)” and (2) “through an appropriate substitution recognize a familiar structure in a more complex form (SS2-B)”. The “prefigured” coding also involved applying the codes to individual students’ work. The general definition and the third component of the Hoch and Dreyfus’ (2006) definition were not broken into elememtary parts.

Along with their descriptions, Table 24 lists the additional codes that I used to make notes about student responses regarding the way they used structure sense or attend to aspects of algebraic proficiency.

Table 25: Codes used in making notes about student responses

Code	Description	Purpose
IDT	Identification: Pertinent actions/words that separate or point to various referents or structural concepts (e.g., pointing, circling, underlining, other gesturing).	To note student responses in the transcripts regarding how they used structure sense.
LAN	Language: Pertinent words/terminology used that point to all the definitions of structure sense.	
REA	Rationale: Reasoning provided through explanation involving concepts in ways that pertain to all the definitions or some aspects of structure sense.	
PROC	Procedure: Approaches or ways of thinking that focus and rely on steps.	To note student responses in the transcripts regarding algebraic proficiency.
GRP	Graph: Reasoning through graphs or that is based on graphs.	
COM	Communication: Explanation of ideas that show understanding in an organized, coherence, precise, analytic, and evaluative way (NCTM, 2000).	
CON	Connections: Connections of mathematical ideas reveal deeper and more lasting understanding (NCTM, 2000).	
MAN	Manipulation: Correct manipulations of symbols.	

Tables 26, 28, 30, and 32 organize the items of the posttest according to the primary code that applies to them. The general definition applies to the items in Table 25 as a primary code, which means that the abilities described by that code is more readily applicable than other structure sense abilities that may be used on the items. The abbreviated label for the general definition code is Gen Defn. Hoch and Dreyfus' (2004) general definition states that structure is a "broad view analysis of the way in which an entity is made up of its parts… Any algebraic expression or sentence represents an algebraic structure (p. 50)." Following each of those tables is an accompanied table containing examples of how students' work and explanation of their work were coded using those additional codes

(Tables 27, 29, 31, and 33). Hence, four pairs of tables are presented in regards to the coding process.

Table 26: Posttest items based on the general definition of structure sense

Item	Item description	Interview question
1c	Use the functions $f(x) = 2x + 8$ and $g(x) = \frac{x}{2} - 4$ to answer the parts that follow. Using the results of (a) and (b), what conclusion can you make about $f(x)$ and $g(x)$? Please provide your answer in a complete sentence.	How did you know that f(x) and g(x) are inverses of each other?
2a	Use the functions $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ to answer the parts that follow. One of the functions provided above is one-to-one, and the other is not. Which of the functions **<u>is</u>** a one-to-one function?	What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered?
2b	Determine the inverse of the one-to-one function identified in (a).	Why did you square both sides of the equation? Looking at the inverse function that you came up with, how do you know it is the correct inverse for the original function
12a	Use the function below to answer the parts that follow. If the requested quantity does not exist, please write DNE. $f(x) = \frac{3(x+1)(x-2)}{(x-2)(x+4)}$ Determine the x-intercept(s) of the function.	How did you determine with the x-intercept(s) from the given equation? What was/is your thinking process? Why?
12d	Determine the **<u>equation</u>** of any horizontal asymptote of the function.	How did you determine the horizontal asymptote from the given equation? What was/is your thinking process? Why?

Table 27 illustrates how some of the items for the general definition were coded for one of the cases. The texts that are underlined and labeled IDT refer to specific part of their

work students were addressing as explain their reasoning. The texts that are underlined and labeled LAN refer specific language used to describe their thinking process. The texts that are underlined and labeled REA refer students' actual reasoning process or explanation.

Table 27: How some of the items for the general definition were coded

Item	Coding	Interview question
1c	01:58 of f would be if this was divided by one 02:06 so I mean okay so if F divided by G no I 02:15 mean let me ask you a question how else PROC IDT 02:34 because one is because when you multiply 02:37 them together they equal X okay so if I 02:42 multiply this with this I'm gonna get X 02:47 IDT IDT so if I multiply that by that I'm gonna 02:53 get one as an example that's an example 02:55 PROC COM/LAN not precise	How did you know that f(x) and g(x) are inverses of each other?
2b	Gen Defn item 08:17 another minute or so 08:47 2b because a an output value for this one 08:53 mm-hmm 08:55 would produce this the output value for 09:02 this one would be the same as the infant 09:04 PROC	Looking at the inverse function that you came up with, how do you know it is the correct inverse for the original function?
12a	Gen Defn item 20:32 what was what is your thinking process 20:34 why so whatever whatever makes the 20:41 REA 12a function of X whatever makes the f of X 20:46 REA zero is what's ~~Canadian~~ my x-intercept 20:50 REA so I chose negative 1 & 2 because 20:53 negative 1 is going to make the entire 20:56 LAN equation 0 & 2 is going to make the 20:59 entire equation 0 21:00 LAN REA	How did you determine the x-intercept(s) from the given equation? What was/is your thinking process? Why?

The first part of Hoch and Dreyfus'(2006) definition (SS1) applies to the items in Table 28 as a primary code, which means that the abilities described by that code are more readily applicable than other structure sense abilities that may be used on the items. The abbreviated label for this code is SS1. The first part of Hoch and Dreyfus'(2006) definition is the ability to recognize structure in simplest form.

Table 28: Posttest items based on the Hoch and Dreyfus' (2006) definition of structure sense (SS1)

Item	Item description	Interview question
11b	Use the function $f(x) = -4(x-3)^2 + 16$ to answer the parts that follow. If the requested quantity does not exist, please write DNE. Determine the y-intercept of the function.	Why didn't you/did you choose the last term in the statement as the y-intercept?
11c	Determine the vertex of the function.	What was/is your thinking process for determining the vertex from the statement?
11d	Determine the **equation** of the axis of symmetry of the function.	How did/do you understand what the idea of symmetry was/is referring to in that question? What was/is your thinking process? Why?
13a	The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 12h + 64$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours.What score will Alex earn without studying?	Using the given equation, how did you determine what score the student would earn without studying? What was/is your thinking process? Why?

Table 29 illustrates how some of the items for the SS1 component of the definition of structure sense was coded for one of the cases. The texts that are underlined and labeled IDT refer to specific part of their work students were addressing as explain their reasoning. The

texts that are underlined and labeled LAN refer specific language used to describe their thinking process. The texts that are underlined and labeled REA refer students' actual reasoning process or explanation.

Table 29: Examples of coding for some of the SS1 items

Item	Coding	Interview question
11c	SS1 item 17:56 that statement okay 18:02 11c so the x value that gives me 0 that 0 is 18:09 this out is my x value so that's 3 and 18:16 IDT then my Y value is what this function 18:23 IDT equals when I'm zeroed out this 18:25 REA IDT REA expression here so can I write here okay 18:31 IDT so I got a 4 3 minus 3 so that becomes 0 18:38 11c then what I have leftover is 16 so 18:41 getting circling	What was/is your thinking process for determining the vertex from the statement?

The first clause of the second part of Hoch and Dreyfus'(2006) definition (SS2-A) applies to the items in Table 30 as a primary code, which means that the abilities described by that code are more readily applicable than other structure sense abilities that may be used on the items. The abbreviated label for this code is SS2-A. The second part of Hoch and Dreyfus'(2006) definition is the ability to deal with a compound term (sum) as a single entity. The second part of Hoch and Dreyfus'(2006) definition also involves another clause, which I labeled SS2-B. It is the ability to recognize structure in its simplest form. However, SS2-B is not sufficiently represented on the posttest; so it was excluded from the analysis. So

was any coding for the substitution principle, Hoch and Dreyfus'(2007) addition to the description of structure sense.

Table 30: Posttest items based on Hoch and Dreyfus' (2006) definition of structure sense (SS2-A)

Item	Item description	Interview question
1a/b	Use the functions $f(x) = 2x + 8$ and $g(x) = \frac{x}{2} - 4$ to answer the parts that follow. Determine $(f \circ g)(x)$. Determine $(g \circ f)(x)$.	Why did you put parentheses in that statement? (in their answer)
4	Suppose that $h(x) = \sqrt{x^2 + 6} - 12$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f\big(g(x)\big)$. $f(x) =$ _____ $g(x) =$ _____	How did you think about decomposing that function? Which part of the original expression did you think could be a component? Why?
6	Construct and simplify the difference quotient for the function $f(x) = x^2 - 3x + 2$. For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.	How did you think about or what helped you get this task right?

Table 31 illustrates how some of the items for the SS2-A component of the definition of structure sense was coded for one of the cases. The texts that are underlined and labeled IDT refer to specific part of their work students were addressing as explain their reasoning. The texts that are underlined and labeled LAN refer specific language used to describe their thinking process. The texts that are underlined and labeled REA refer students' actual reasoning process or explanation.

Table 31: Examples of coding for some of the SS2-A items

Item	Coding	Interview question
1a/b	1a/b 00:21 these buttons I put a parenthesis here 00:25 in here because the function of G 00:32 substitutes the X for the function of F 00:37 so the parenthesis everything within 00:42 REA LAN LAN/IDT REA that is a substitution for X yeah that's 00:49 REA why I did that 00:49 okay item 1 C and because yeah because I 01:03 PROC I want to be able to distribute to to 01:06 this entire expression here and then the 01:11 LAN	Why did you put parentheses in that statement? (in your answer)

The third part of Hoch and Dreyfus'(2006) definition (SS3) applies to the items in Table 32 as a primary code, which means that the abilities described by that code are more readily applicable than other structure sense abilities that may be used on the items. The abbreviated label for this code is SS3. The third part of Hoch and Dreyfus'(2006) definition is the ability to choose appropriate manipulations to make best use of structure.

Table 32: Posttest items based on Hoch and Dreyfus' (2006) definition of structure sense (SS3)

Item	Coding	Interview question
5	Solve the absolute value equation below. $$13 - 2\|x - 5\| = 9$$	Why did you / why didn't you distribute that factor through the absolute value bars to get rid of them (i.e. absolute value bars?
13b	The grade that Alex receives on a particular test can be modeled by the function below. $$G(h) = -h^2 + 12h + 64$$ Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)	What was/is your thinking process? Why?

Table 33 illustrates how some of the items for the SS3 component of the definition of structure sense was coded for one of the cases.

Table 33: Examples of coding for some of the SS3 items

Item	Coding	Interview question
5	5 12:14 because whatever is inside the absolute 12:18 IDT value bars needs to be by itself on one 12:22 side the equation before it can be dealt 12:24 with okay and that is because whatever's 12:29 inside the absolute value bars is 12:33 the distance from zero it so if I were 12:38 REA to distribute that to it would it would 12:41 skew like the meaning of the absolute 12:44 LAN REA REA MAN MAN 5	Why did you/why didn't you distribute that factor through the absolute value bars to get rid of them?
13b	that this equation is a degree of two so 24:54 LAN it's a quadratic and so I know that I'm 24:57 LAN dealing with a parabola and so I get 25:04 GRP everything on the one side so I subtract 25:08 by 91 and I get this expression here 25:13 IDT mm-hmm and I see that I can factor yeah 25:20 REA I can the factors for this expression 25:24 REA are X minus 9 and X minus 3 which means GRP REA	What was/is your thinking process? Why?
	hours because it's okay so the parabola 26:03 GRP is inverted and so there's nine and 26:08 there's three and I'm not gonna pretend 26:11 it was like ninety one so so that's 26:18 right here these two roots here 26:22 IDT represent where ninety one percent would 26:24 GRP be and so this portion right in here is 26:28 GRP/IDT between nine and three and that's gonna 26:31 be where he's gonna have ninety-one 26:33 percent or greater on his score 26:40 GRP pointing to graph (parabola)	

Item 5 can be included in the SS2-A category, but I placed it in the SS3 category so that the SS3 category would have two items instead of just one. At least two items per code provided a minimum amount of information for evaluating students' thinking.

These a priori or "prefigured codes along with the additional emergent codes organized the data into meaning segments. In the next part, which is a 'cross-case' analysis, I collapsed these meaning segments into broader themes. Thus, by comparing the cases, the 'cross-case' analysis offered a more concise description of the impact the intervention may have had on those students who were interviewed. Qualitative studies are not generalizable; however, the results pertaining students' work is applicable to the structure sense instruction in precalculus algebra.

'Cross-case' analysis. This part focused on comparing and contrasting individual participants. The analysis in this part involved identifying similarities and differences between the six cases to help provide thematic results in Chapter 4. Assertions or interpretations made from both the 'within-case' and 'cross-case' analyses are presented in Chapter 4.

Validity. Creswell (2013) highlighted eight validation strategies that are frequently used by qualitative researchers. These strategies are prolonged engagement and persistent observation in the field, triangulation, peer review or debriefing, negative case analysis, clarifying researcher bias, member checking, rich-thick description, and external audits. I used three of these strategies: prolonged engagement and persistent observation in the field, clarifying researcher bias, and rich-thick description.

Prolonged engagement and persistent observation in the field applies by virtue of my role as instructor and researcher. In that capacity, I was able to closely watch participants'

work every class meeting. In regards to clarifying researcher bias, I provided a subjectivity statement in Chapter 1, theoretical framework section. Rich-thick descriptions focused on providing interconnected details about the cases and the themes that emerge by using strong action verbes (Creswell, 2013).

Reliability. I attended to reliability by taking detailed field notes recording and transcribing every detail as recommended by Silverman (2005) as cited in Creswell. "In qualitative research, reliability often refers to the stability of responses to multiple coders of data sets (Creswell, 2013, p. 253)." There was only one coder, but using Hoch and Dreyfus' definition as "prefigured" codes supported the reliability of the coding process. Moreover, collecting data on the posttest twice, once from the reflection on Test 2 at midterm and then again in the interview at the end of the semester, also supported the stability of responses.

Chapter 4: Quantitative Results and Analysis

I answer the quantitative research questions using results generated from a number of statistical tests, including independent and paired samples t-tests, analysis of variance (ANOVA), and correlation. The first and second research questions were:

RQ1. Is there significant difference in structure sense between students' Pretest-A and Posttest-A performance in a community college precalculus course?

RQ2. Is there significant difference in algebraic proficiency between students' Pretest-B and Posttest-B performance in a community college precalculus course?

Phase 1: Related Samples T-test for Research Questions 1 and 2

To address the first research question, I conducted a paired samples *t*-test to assess whether students' structure sense changed from the beginning of the semester to the middle of the semester. My prediction was that students' structure sense would improve. Results support that prediction. Students' structure sense at the midterm mark ($M = 66.97$, $SD = 21.01$) was higher than at the beginning ($M = 4.48$, $SD = 7.26$), $t = 17.81$, $p < .001$, two-tailed, mean difference = 62.49. A list of means and standard deviations is provided in Table 34.

Table 34: Means and standard deviations for structure sense

	Beginning of semester	Midterm mark
Structure sense	$M = 4.48$ $SD = 7.26$	$M = 66.97$ $SD = 21.01$

Error bars in Figure 2 are 95% confidence intervals.

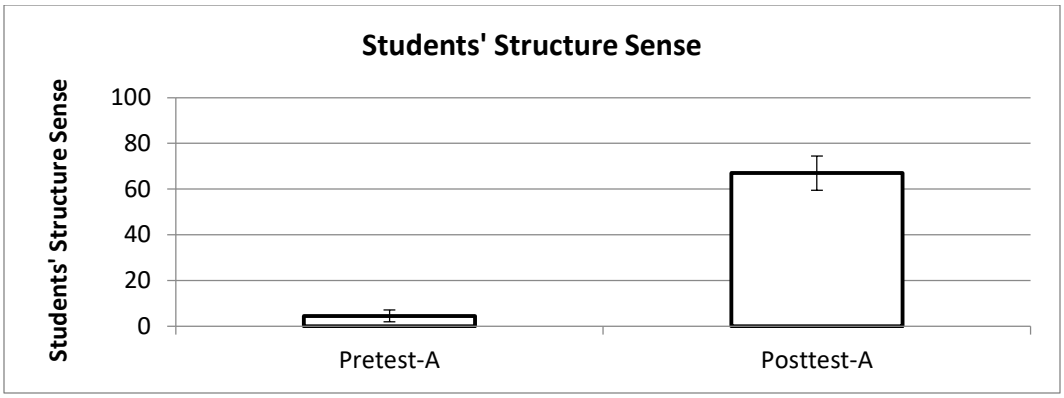


Figure 2: Students' structure sense at the start of the semester versus at the midterm mark.

Hence, there is significant difference in structure sense between students' Pretest-A and Posttest-A performance in the context of these community college precalculus courses. The low structure sense on the pretest may be due to a lack of emphasis on structure sense in previous algebra courses taken by the students or to a lack of retention or genuine understanding of concepts. The larger percentage in structure sense at the midterm mark indicates that students' structure sense increased.

To address the second research question, I conducted a paired samples *t*-test to determine whether students' algebraic proficiency change from the beginning of the semester to the middle of the semester. My prediction was that students' algebraic proficiency would improve. Results support that prediction. Students' algebraic proficiency at the midterm mark ($M = 61.87$, $SD = 22.78$) was higher than at the beginning ($M = 3.71$, $SD = 4.26$), $t = 15.11$, $p < .001$, two-tailed, mean difference = 58.16. A list of means and standard deviations is provided Table 35.

Table 35: Means and standard deviations for algebraic proficiency

	Beginning of semester	Midterm mark
Algebraic proficiency	*M = 3.71*	*M = 61.87*
	SD = 4.26	*SD = 22.78*

Error bars in Figure 3 are 95% confidence intervals.

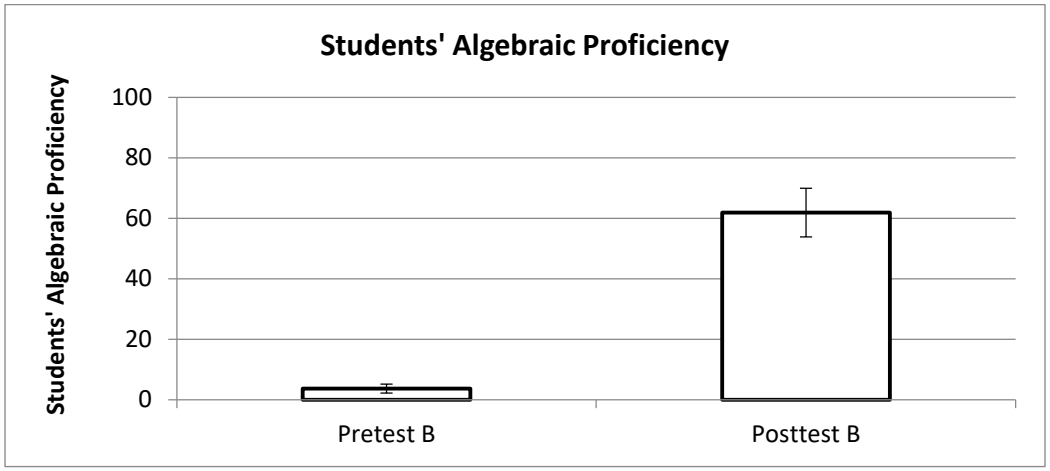


Figure 3: Students' algebraic proficiency at the beginning of the semester versus at the midterm mark.

Hence, there is significant difference in algebraic proficiency between students' pretest-B and posttest-B performance in the context of these community college precalculus courses. The low algebraic proficiency may be due to lack of emphasis on structure sense also since structure sense contributes to algebraic proficiency. The low algebraic proficiency on the pretest may be due in part to lack of retention or genuine understanding. Students have come in contact with some on these topics in development mathematics courses (DMA). The large difference could probably be attenuated by retention if students had gained deeper understanding from previous algebra courses.

For both of the dependent variables, structure sense and algebraic proficiency, standard deviations are high in Pretests A and B in relation to the means. This relationship is

the effect of very low performance at the beginning of the semester, which is an indication of students' entry knowledge. According to Hiebert and Grouws (2007), student entry knowledge affects their opportunity to learn, which is considered the single most important predictor of student learning outcome, according to NRC (2001). On the other hand the standard deviations in Posttest A and B were not as close to their respective means because the means were significantly higher than at the beginning of the semester. Other factors notwithstanding, teaching certainly contributes to student learning outcome. As NMAP (2008) found, teaching and especially effective teachers matter for student learning outcome. The lower student entry knowledge is, the more teaching is necessary and the more effective teachers need to be. Moreover, the large values of standard deviation signal that even at the midterm mark, there was still sizable gaps among students in their structure sense and algebraic proficiency. This result supports the fact that beside student entry knowledge and the teaching students received, there are other factors that influence learning outcomes (e.g., background, life circumstance). I considered some of those factors in the following section, which is the purpose of the fourth research question.

Phase 2: Various Analyses on the Results for Research Questions 1 and 2

I conducted other analyses involving additional variables in regards to the results for the first and second research questions for instance: "To what extent is variation in students' structure sense and algebraic proficiency explained by other factors beyond classroom teaching." I started considering how students' background may have affected their learning. Then I examined how students' life circumstances may have affected their learning.

Starting with students' background, I performed a couple of independent samples t-test to find out if there was a difference in performance between students who took remedial

mathematics courses in college and those who did not. For structure sense, analysis of the data showed that students who took remedial courses in college made essentially the same gain over that period of instruction with $M=67.60$, $SD=22.60$ for those who did not take remedial courses and $M=65.73$, $SD=18.36$ for those who took remedial courses, $t(31)= 0.2373$; $p > 0.8140$. Error bars in Figure 4 are 95% confidence intervals.

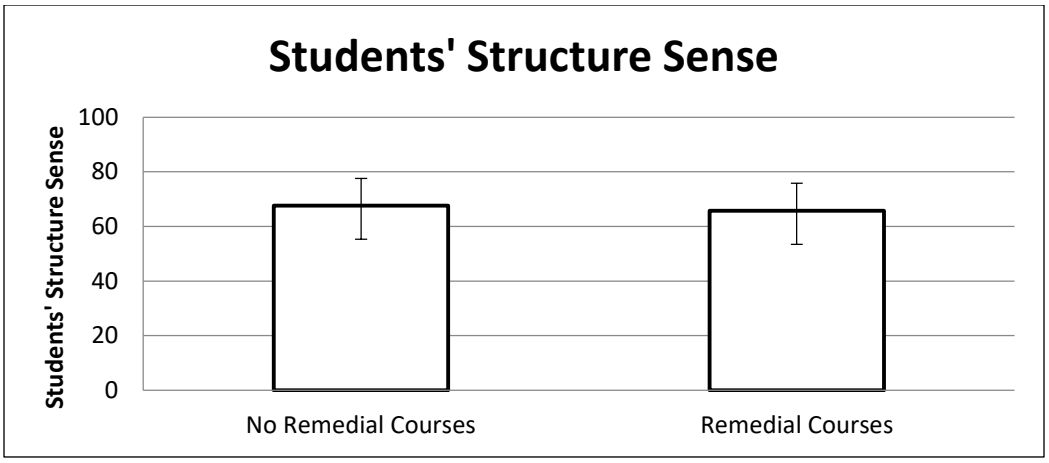


Figure 4: Students' structure sense based on whether they took remedial algebra

Similarly, for algebraic proficiency, analysis of the data showed that students who took remedial algebra courses in college made essentially the same gain over that period of instruction with $M=64.74$, $SD=22.40$ for those who did not take remedial algebra courses and $M=56.14$, $SD=23.52$ for those who took remedial courses, $t(31)=1.0232$, $p>0.31$. Error bars in Figure 5 are 95% confidence intervals.

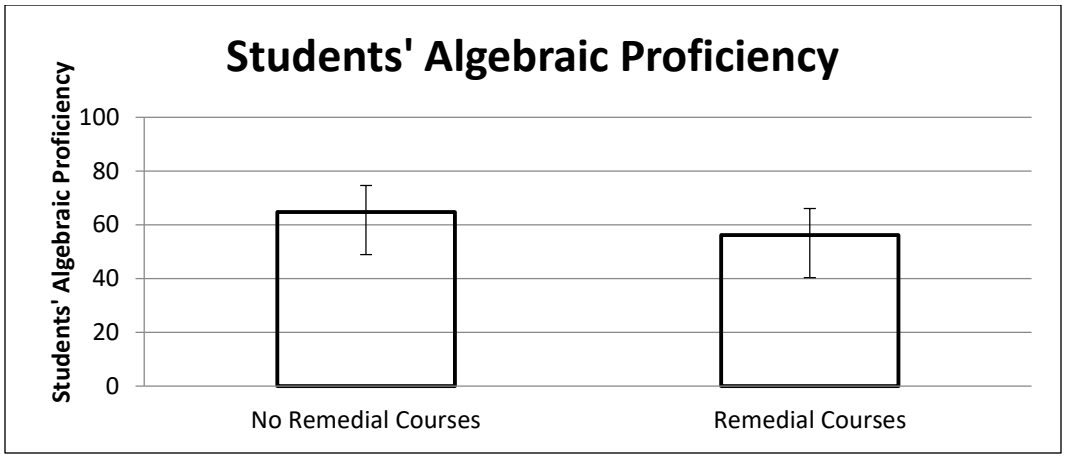


Figure 5: Students' algebraic proficiency based on whether they took remedial algebra

One observation is that the group who took remedial algebra courses scored slightly lower that the group who did not take remedial algebra courses especially in algebraic proficiency, albeit not significantly lower. This difference may hint that those who did not take remedial algebra courses are stronger students. This difference is consonant with the knowledge that students with higher prior achievement tend to outperform peers with lower prior achievement.

Still considering students' background, I performed independent samples t-tests to determine whether students who were repeating precalculus algebra differed from those who were taking the course for the first time. Results showed that their performances were essentially the same for structure sense with $M=72.23$, $SD=24.87$ for those who were repeating the course and $M=65.01$, $SD=19.60$ for those who took the course for the first time, $t(31)=0.8766$, $p>0.38$. Error bars in Figure 6 are 95% confidence intervals.

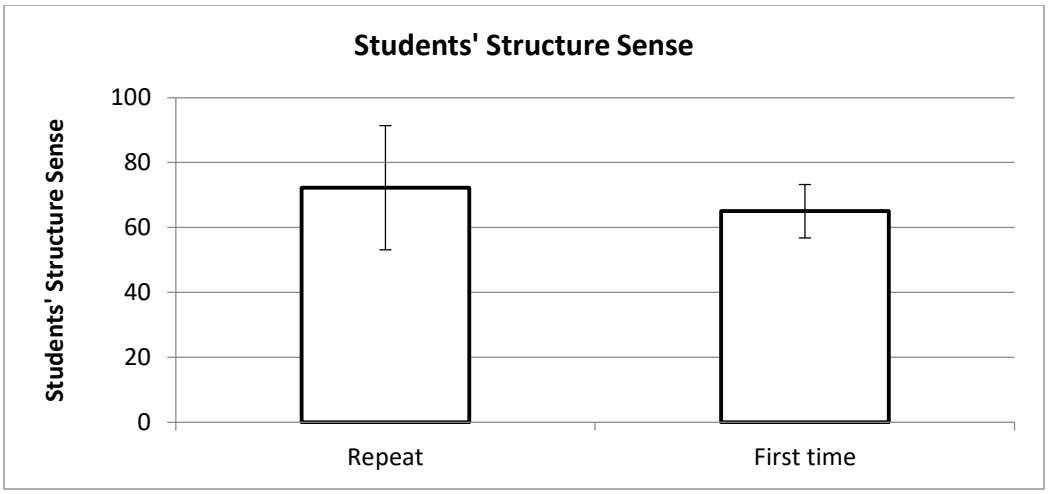


Figure 6: Students' structure sense based on whether they were repeating the course

Similarly, results showed that their performances were essentially the same for algebraic proficiency with *M=67.36, SD=19.85* for those who were repeating the course and *M=59.81, SD=23.85* for those who took the course for the first time, *t(31)=0.8766, p>0.8438*. Error bars in Figure 7 are 95% confidence intervals.

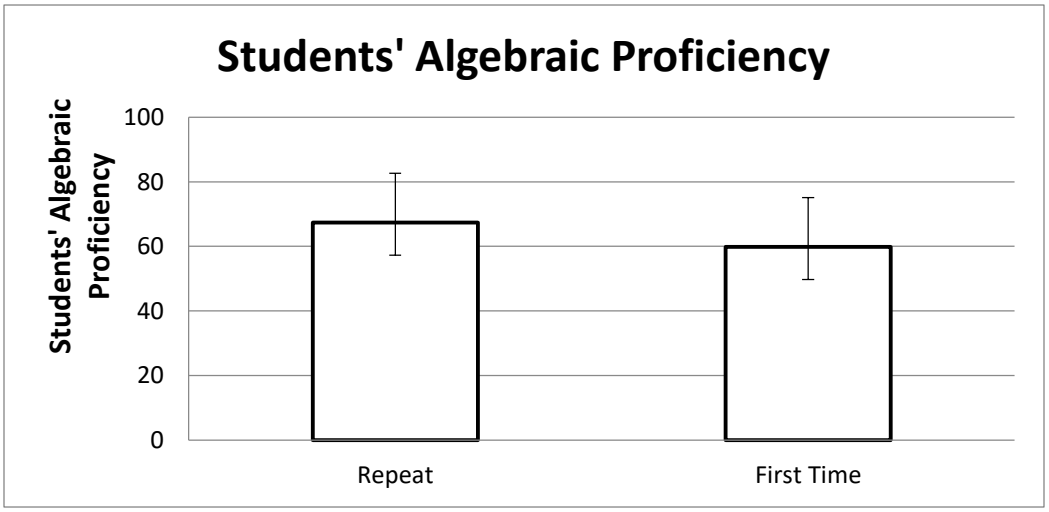


Figure 7: Students' algebraic proficiency if they were repeating the course

I conducted an ANOVA analysis to test whether students' structure sense differed as a function of students' study skills (poor, average, good). In general, one would expect study skills to matter. However, results showed their structure sense did not differ based on study skills, $F = 0.11$, $p > .90$. Table 36 shows means and 95% confidence intervals.

Table 36: Students' structure sense based on their study skills

Study Skills	Mean	Standard Error	95% Confidence Interval	
Poor	63.93	12.38	38.71	89.16
Average	67.09	4.05	58.84	75.34
Good	71.13	13.66	43.30	98.96

The upper limit of the confidence interval is high for the category of poor study skills, indicating that students' opinion of the study skills may not be accurate. Consequently, it also put in question the practical significance of the mean value. The reliability of these results depends on the sample size and on the instrument items to which students responded. Error bars in Figure 8 are 95% confidence intervals.

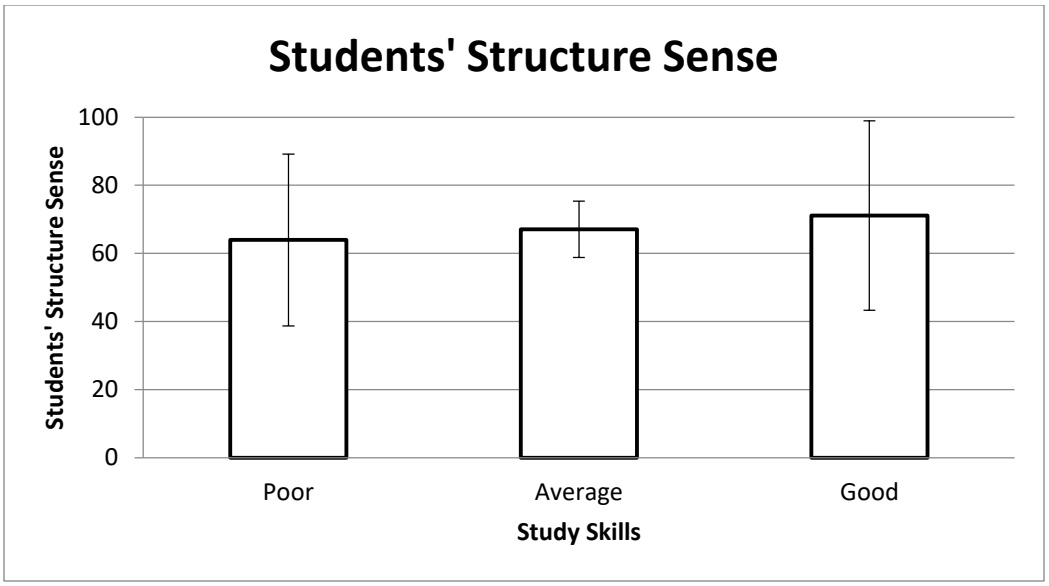


Figure 8: Students' structure sense based on their study skills

I conducted an ANOVA analysis to test whether students' algebraic proficiency differ as a function of students' study skills (poor, average, good). In general, one would expect study skills to matter. However, results show that their algebraic proficiency does not differ based on study skills, $F = .45$, $p > .64$. Table 37 shows means and 95% confidence intervals.

Table 37: Students' algebraic proficiency based on their study skills

Study Skills	Mean	Standard Error	95% Confidence Interval	
Poor	53.1	14.05	24.49	81.71
Average	63.09	4.30	54.32	71.86
Good	66.33	13.37	39.09	93.57

The reliability of these results depends on the sample size and on the instrument items to which students responded. Error bars in Figure 9 are 95% confidence intervals.

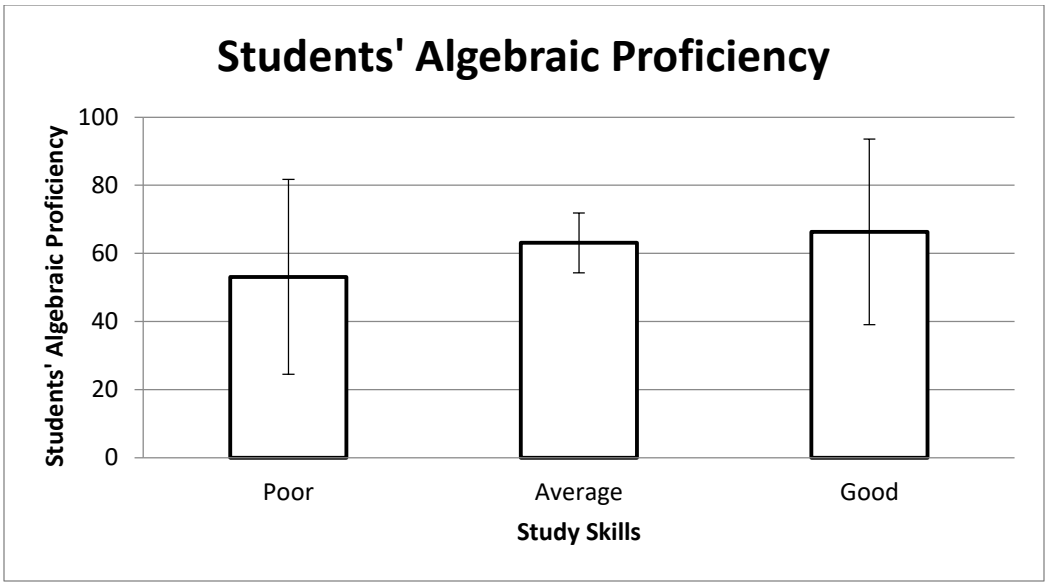


Figure 9: Students' algebraic based on their study skills

There are other variables that should matter also, like the affective variables for example, but the following results did not indicate that. The sample size being so small has a significantly adverse impact on the results.

Next, I ran and interpreted correlation based on the affective variables such as students' interest in and enjoyment of mathematics (INTMAT), their instrumental motivation (INSTMOT), and self-efficacy in mathematics (MATHEFF). Table 38 shows weak, negative correlation coefficients indicating minimal association between students' structure sense and affective factors. These correlation coefficients go in the opposite direction than expected because the instrument items that describe the affective variables were written such that the lower the value the more interest, motivation, or self-efficacy. In other words, the lower values for INTMAT, INSTMOT, and MATHEFF, the higher a student's interest, motivation, and self-efficacy are. In this case, one would expect that as INTMAT, INSTMOT, and MATHEFF increase, students' structure sense and algebraic proficiency decrease. This

inverse relationship stemmed from the way the instrument was designed. Otherwise, lower INTMAT, INSTMOT, and MATHEFF would normally associate with lower students' structure sense and algebraic proficiency. Therefore, the correlation is not truly negative.

Table 38: Correlation coefficients between students' structure sense and affective factors

	Structure Sense
Students interest and enjoyment in mathematics (INTMAT)	-0.155
Students' motivation in mathematics for some external rewards (INSTMOT)	-0.1045
Students confidence about completing task in instrument (MATHEFF)	-0.1254

Likewise, Table 39 shows weak, "negative" correlation coefficients indicating minimal association between students' algebraic proficiency and affective factors. These correlation coefficients go in the opposite direction than expected but the association between these affectives variables and algebraic proficiency should be direct, as discussed earlier.

Table 39: Correlation coefficients between students' algebraic proficiency and affective factors

	Algebraic Proficiency
Students interest and enjoyment in mathematics (INTMAT)	-0.1283
Students' motivation in mathematics for some external rewards (INSTMOT)	-0.076
Students confidence about completing task in instrument (MATHEFF)	-0.118

Phase 3: Multiple Linear Regression (MLR), Assumptions, and Post-hoc

Obtaining STATA outputs of Multiple Linear Regression (MLR) tests regarding hypotheses and coefficient estimates including post-hoc test are not necessary because the results in phase 2 already indicated that other factors beyond classroom teaching did not affect students' structure sense and algebraic proficiency significantly. Thus, these analyses in phase 2 answer to the fourth research question. I still ran the multiple linear regressions anyway for structure sense and for algebraic proficiency. The results confirmed earlier results in phases 2. Every confidence interval contains zero for additional variables, except for the constant term. So with the exception of the constant term, none of the coefficient estimates were significant for students' structure sense as seen in Table 40. Therefore, the results have no impact for inference to a larger population of students and the sizes of the estimates do not matter on a larger scale. In other words, there is no meaningful statistical or practical significance from these results.

The coefficient estimate for the variable "Load_end" is negative, which makes sense because learning performance can decrease for students with heavier load in terms of the number of credit hours they enroll in and hours per week they work. This result may not speak for certain high achieving students, who do well regardless of the course load and workload. However, it does speak for those in the study; that means for every one additional credit or work hour, students' structure sense performance decreased by 0.45 of a point on Posttest-A (ceteris paribus). The variable "Effort" is a composite of several other variables. These variables are the number of hours a week students participated in study groups (Group study), the number of hours per week students used campus resources to help them study the course material (CampusRes), the number of hours students study on their own (Selfstudy),

and other times students spent toward learning the course material. The coefficient estimate associated with the variable "Effort" should be positive, but instead it is negative. Perhaps struggling students with low prior achievement in algebra put a lot of time studying. One hour "Effort" spent for the course corresponds to a decrease of 0.04 in students' structure sense (ceteris paribus). Then finally, the coefficient estimate associated with the constant term is high at 84.23 and significant indicating students' average score when all variables are zero or otherwise; a large portion of students' structure sense is not explained by these variables.

Table 40: Results from ordinary least squares estimator for Posttest A

Posttest-A	Coefficient	Standard Error	t	P>|t|	95% Confidence Interval	
Load_end	-0.45	0.31	-1.45	0.151	-1.09	0.19
Effort	-0.04	0.51	-0.07	0.944	-1.08	1.00
Constant	84.23	11.22	7.51	0.00	61.12	107.35

Post-hoc would be necessary if the ANOVA and MLR yielded statistically significant estimates. Lots of variables and a small sample size are reasons for some of the estimates to have counter-intuitive orientation and larger standard error. However, these variables were needed to account for other aspects of students' learning. These variables would be needed not only in a study about curriculum and teaching such as this one, but they can also help in evaluating programs at educational institutions.

Chapter 5: Qualitative Results and Analysis

The qualitative results and analysis include within-case and cross-case analyses. The cross-case analysis includes a condensed response to the qualitative research question: "In what ways do students' work show development of structure sense in a community college precalculus algebra course?"

To study how the intervention may have impacted students' learning, I considered an in-depth analysis of students' work on an individual basis, against the backdrop of what I just described earlier about the two classes I taught, the precalculus course (redesign), and the way I taught the material. The in-depth analysis of students' work on an individual basis is termed 'within-case' analysis in the literature, e.g., Creswell (2013).

Within-Case Analysis

Case 1: Jim

Jim, a Caucasian male, was a fourth year part-time student at the time of data collection. He was pursuing an associates degree in science with the goal of getting a bachelor's degree in psychology. Ultimately, he wanted to become a doctor or physician's assistant. He had not taken any remedial math courses prior to taking MAT 171, and it had been 8 years since he had taken an algebra course in high school. At the time, he was taking 8 credit hours and working 30 hours a week. He rated his study skills as very good. He indicated that it had taken outstanding effort to complete all of his coursework with satisfactory grades, and that being used to the intuitive nature of the social sciences, the transition into STEM work required a change in mindset for him. He stated that he had always enjoyed math classes and that he had very good access to the internet and computers in general. Jim stated that he did not enjoy reading about mathematics, nor did he do

mathematics because he enjoyed it. However, he looked forward to his mathematics lessons and was interested in the things he learned in mathematics. He also believed that he would learn things in mathematics that would help him get a job. He strongly agreed with the idea that making an effort in mathematics was worth it because he needed it for what he wanted to study later on, that it would help him in the work he wanted to do later on, and that it would improve his career prospects. In terms of completing certain mathematics tasks, Jim felt confident that he could solve an equation like $2(x+3)=(x+3)(x-3)$. However, he felt very confident that he could use a train schedule to figure out how long it would take to get from one place to another, calculate how much cheaper a TV would be after a 30% discount, calculate how many square feet of tile would be needed to cover a floor, understand graphs presented in newspapers, solve an equation like $3x+5=17$, find the actual distance between two places on a map with a 1:10,000 scale, and calculate the gas mileage of a car.

In the Midterm Reflection Questionnaire, Jim indicated that he had not had a personal tutor help him with the course material, nor had he participated in any study groups. In addition, he had not used any campus resource. He was currently enrolled full-time and was taking 14 credit hours. He was working part-time—30 hours a week on average for the past 7 weeks. He spent about 12 hours a week on MAT 171 doing homework, reading class notes, and watching instructional videos to learn the concepts of the course. Jim believed that doing WebAssign and watching YouTube videos on how to solve problems covered in class were things he did outside of class that may have affected his learning experience.

Jim expressed in the Final Interview at the end of the semester that he enjoyed MAT 171 very much. Jim stated, “The flow from graphical interpretation and understanding the material conceptually helped me remember the detailed processes and synthesize reasonable

answers as opposed to remembering and regurgitating only the details that were presented." He also stated that "This math class was different from previous ones because the focus seemed to be on making sense of the problems instead of blindly performing calculations. Many of the problems required us to come to conclusions in the context of real world application of math, which is different than previous algebra courses."

Results regarding their work are organized in two parts – those related to the 'prefigured' codes and those related to emergent codes. Both results based on the 'prefigured' codes and the emergent codes are responses taken from the posttest reflection and the interview. The posttest reflection was administered to all students 24 hours after the posttest for one section and after 48 hours for the other section. The interview was administered 9 weeks after the posttest only to students who volunteered for the interview. The results are presented according to instruments and codes (i.e. results from posttest reflection first, followed by results from the interview). Within that order, responses are considered by codes in the same sequence outlined in the data analysis section.

Therefore, the results start with the posttest reflection and responses to items that correspond to general definition of structure sense. The label for that 'prefigured' code is Gen Defn. The posttest items for that code are 1c, 2a, 2b, 12a, and 12d as listed in Table 19a in the data analysis section, Chapter 3.

Jim's responses to general definition items from the posttest reflection. Item 1c was not addressed on the posttest reflection; no questions were posed to students regarding that item. When asked the following question regarding item 2a "how do you know which function has an inverse and which one doesn't?" The two functions are $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$. As shown in Figure 10, Jim replied that "I knew only g(x) has

inverse because it was a 1-to-1; f(x), being a quadratic fn was not 1-to-1, so there was no inverse." Jim's structure sense, in this case, based on my previous interactions with him, is likely informed by the graph of the quadratic function. However, it is plausible that he thought about one-to-one functions and how they appear in their tabular form where repeating y-values have different x-values. Or, it is also possible that he recognized that in any quadratic equation, there may be two distinct x-values for the same y-value.

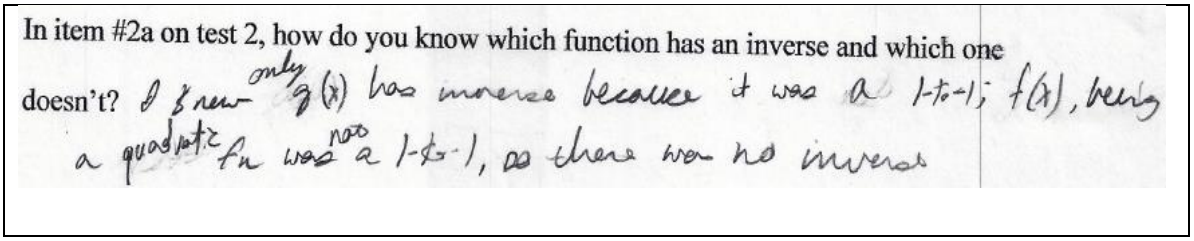

In item #2a on test 2, how do you know which function has an inverse and which one doesn't? I knew only g(x) has inverse because it was a 1-to-1; f(x), being a quadratic fn was not a 1-to-1, so there was no inverse

Figure 10: Jim's posttest reflection response about Item 2a

When asked the following question regarding Item 2b "did you expect the kind of function or structure you get for the inverse of g(x), $g(x) = \sqrt{2x-7}$? Yes or no. If yes, how?" As shown in Figure 11, Jim replied no. His answer shows that his structure sense does not associate the radical function with its opposite structure, quadratic function (half of the graph), perhaps because a full quadratic graph does not have an inverse (or does not pass the horizontal line test).

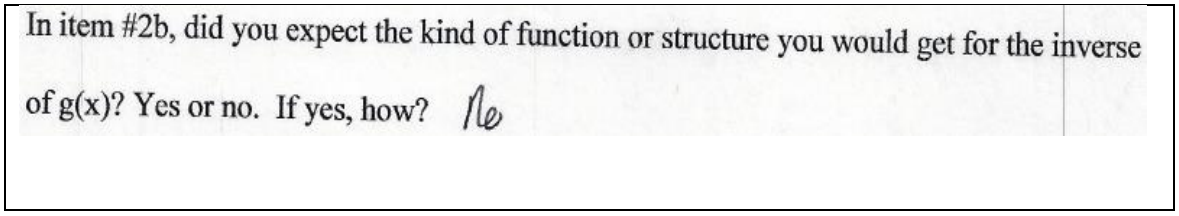
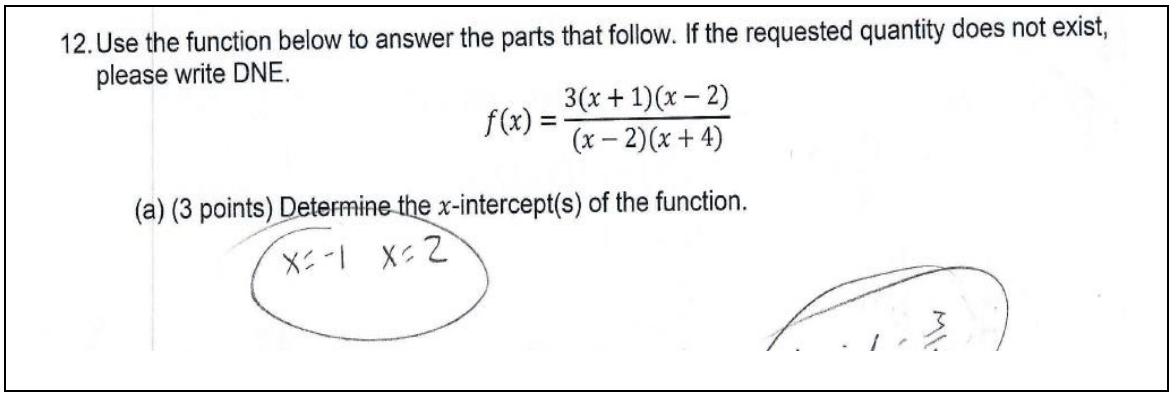

In item #2b, did you expect the kind of function or structure you would get for the inverse of g(x)? Yes or no. If yes, how? No

Figure 11: Jim's posttest reflection response about Item 2b

When asked about item 12a, “how did you come up with your response? What helped you?” Figure 12 shows Jim’s response to Item 12a on the posttest.


12. Use the function below to answer the parts that follow. If the requested quantity does not exist, please write DNE.

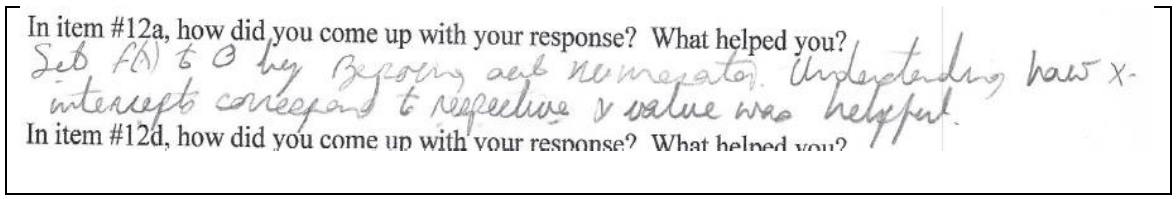
$$f(x) = \frac{3(x+1)(x-2)}{(x-2)(x+4)}$$

(a) (3 points) Determine the x-intercept(s) of the function.

x=-1 x=2

Figure 12: Jim's response on the posttest to Item 12a

Jim wrote “Set f(x) to 0 by zeroing out the numerator. Understanding how x-intercepts correspond to respective y value was helpful.” This answer seems to indicate his structure sense in terms of function notation because he mentioned setting f(x) = 0. This answer seems to indicate his structure sense in terms of knowing what part of equation to focus upon.

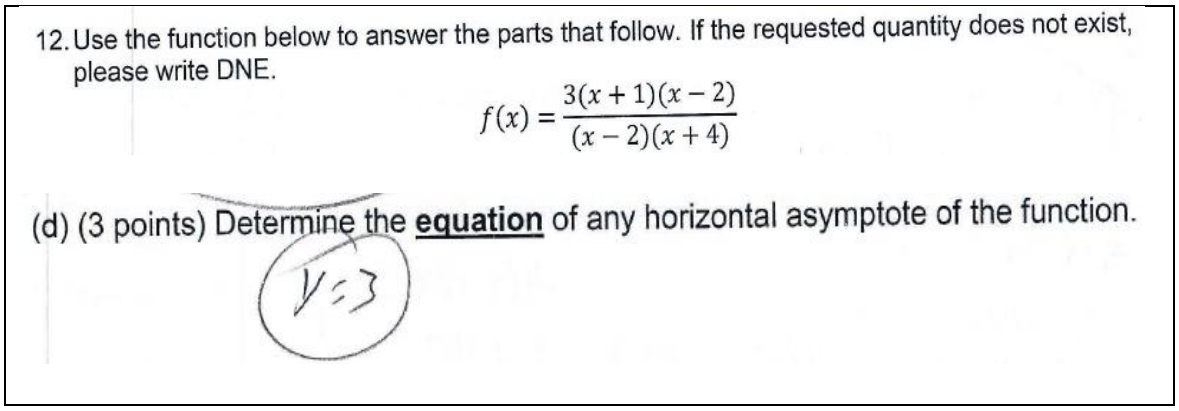

In item #12a, how did you come up with your response? What helped you?

Set f(x) to 0 by zeroing out the numerator. Understanding how x-intercepts correspond to respective y value was helpful.

In item #12d, how did you come up with your response? What helped you?

Figure 13: Jim's posttest reflection response about Item 12a

When asked about Item 12d, “How did you com up with your response? What helped you?’


12. Use the function below to answer the parts that follow. If the requested quantity does not exist, please write DNE.

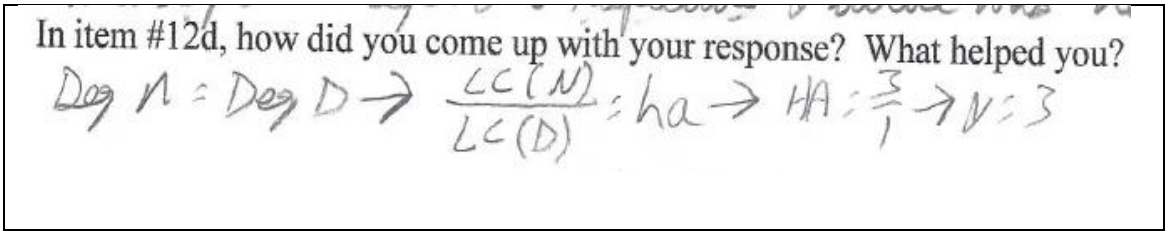
$$f(x) = \frac{3(x+1)(x-2)}{(x-2)(x+4)}$$

(d) (3 points) Determine the **equation** of any horizontal asymptote of the function.

$y=3$

Figure 14: Jim's response on the posttest to Item 12d

Jim replied using one of the conditions for horizontal asymptote, the degree of the numerator must be equal to the degree of the denominator. When that condition is met then the horizontal asymptote is y=(leading coefficient of the numerator divided by the leading coefficient of the denominator).


In item #12d, how did you come up with your response? What helped you?

Deg N = Deg D → $\frac{LC(N)}{LC(D)}$ = ha → HA = $\frac{3}{1}$ → y = 3

Figure 15: Jim's response on the posttest to Item 12d

This answer shows Jim's structure sense in terms of reading through the expressions and extracting the right details (e.g., degrees, leading coefficients).

Jim's interview responses to general definition items. This part of the results describes Jim's explanation in the interview for his work on the posttest. The first interview question for this category of items was "How did you know that f(x) and g(x) are inverses of each other? ($\boldsymbol{f(x) = 2x + 8}$ and $\boldsymbol{g(x) = \frac{x}{2} - 4}$). In Table 41 lines 02:26 through 02:55, Jim

said “when you multiply together they equal X okay so if I multiply this with this I’m gonna get X so if I multiply that by that I’m gonna get one…” When he said “this and that” he referring to the f(x) and g(x). Then, Jim was referring to the identity function or element that results when inverses are combined under the appropriate relationship, function composition. However, his explanation while effective, does not point to the exact opposite operations in the two functions f(x) and g(x) below. His language lacks precision; he should have said function composition instead of multiplication.

Table 41: Justification of the general definition in Jim’s work – Item 1c

Item	Interview question	Jim’s response
1c	How did you know that f(x) and g(x) are inverses of each other? $f(x) = 2x + 8$ $g(x) = \frac{x}{2} - 4$	because the inverse would be the inverse 01:58 of f would be if this was divided by one 02:06 so I mean okay so if F divided by G no I 02:15 mean let me ask you a question how else 02:21 could you have noticed that by looking 02:26 at the two functions because one is because when you multiply 02:37 them together they equal X okay so if I 02:42 multiply this with this I'm gonna get X 02:47 so if I multiply that by that I'm gonna 02:53 get one as an example that's an example 02:55 of why they're inverses

The interview question for Item 2a asked “What if the task had asked you to explain if these functions are one-to-one or not, instead of which function is one-to-one? How would you have answered?” The two functions are $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$. His

reasoning here for Item 2a in Table 42 is based on the structure of f(x) and g(x) for applying the 1-to-1 rule in a numerical way. His response does not connect the type of equations with the shape of their graph. His response also show a rationale that is rule/definition/structure driven as indicated in his comments in lines 04:02 through 04:35: "I would say G of X is one-to-one because for every x value there is one y value I would say that f of X is not one-to-one because for every x value except for zero there are two Y values." This reasoning is informed by the structure or type of the functions, especially the x squared term which transformed two different x-values into the same y-value.

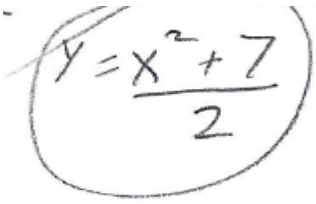
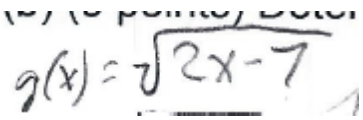
Table 42: Justification of the general definition in Jim's work – Item 2a

Item	Interview question	Jim's response
2a	What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered? Use the functions $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ to answer the parts that follow. One of the functions provided above is one-to-one, and the other is not. Which of the functions **<u>is</u>** a one-to-one function?	okay if it asked me if either these 04:02 functions was one-to-one I would say G 04:05 of X is one-to-one because for every x 04:12 value there is one y value I would say 04:16 that f of X is not one-to-one because 04:20 for every x value except for zero there 04:26 are two Y values for every Y value there 04:35 are two X values that's what I mean just 04:37 looking okay yeah okay yeah so for every 04:40 Y value there are two X values for f of 04:42 X but G of X there's only one y value 04:45 for each x value

The interview question for Item 2b asked "Looking at the inverse function that you came up with $y = \frac{x^2+7}{2}$ how do you know it is the correct inverse for the original function, $y = \sqrt{2x - 7}$?" His explanation in Table 43 shows that structure sense was not

used, neither the general type of the function as opposite structure nor the opposite operations in them. Instead, in lines 06:17 through 07:28 Jim's explanation is procedural as his comment centers around multiplying.

Table 43: Justification of the general definition in Jim's work – Item 2b

Item	Interview question	Jim's response
2b	Looking at the inverse function that you came up with,  $y = \frac{x^2+7}{2}$ how do you know it is the correct inverse for the original function?  $g(x) = \sqrt{2x-7}$	the because if I 06:17 multiply it by the original function I'm 06:23 going to get a second one or an X 06:33 because I multiply it by the original 06:35 one can I do that if you yes you I can't 06:42 do half time okay I've been six minutes 06:44 so far okay so when you're writing you 06:47 can something you can all right so yeah 06:58 equals 2x minus 7 07:11 I guess 07:14 how's this hard how do we multiply 07:18 yeah it's multiply these two together 07:20 but I know that they are inverses 07:23 because if I didn't multiply them 07:24 together I would get like a common very 07:28 simple answer

The interview question for Item 12a was "How did you determine the x-intercept? What was/is your thinking process? Why?" Jim's response in the interview regarding his work on Item 12a is the same as in the posttest reflection. His explanation directly applies to the general definition of structure sense. He chose the negative 1 and the 2 from looking at the variable factors in the numerator instead of solving x+1=0 and x-2=0 mechanically. His comment in Table 44 lines 20:50 through 20:59 shows that he read through the function expression and took a gestalt, global view. He said "I chose negative 1 & 2 because negative

1 is going to make the entire equation 0 and 2 is going to make the entire equation 0." The word "entire" is indicative of that gestalt analysis.

Table 44: Justification of the general definition in Jim's work – Item 12a

Item	Interview question	Jim's response
12a	How did you determine with the x-intercept(s) from the given equation? What was/is your thinking process? Why? $$f(x) = \frac{3(x+1)(x-2)}{(x-2)(x+4)}$$	so whatever whatever makes the 20:41 function of X whatever makes the f of X 20:46 zero is what's going to be my x-intercept 20:50 so I chose negative 1 & 2 because 20:53 negative 1 is going to make the entire 20:56 equation 0 & 2 is going to make the 20:59 entire equation 0

The interview question for Item 12d was "How did you determine the horizontal asymptote? What was/is your thinking process? Why?" Jim's response regarding his work on Item 12d in the interview is the same as in the posttest reflection. As shown in Table 45 lines 22:09 through 22:25, his explanation directly applies to the general definition of structure sense in terms of taking a gestalt (global) view of the equation (a broad view analysis), first through the numerator and then through the denominator. During the intervention, students were taught that the degree of an expression is a global attribute.

Table 45: Justification of the general definition in Jim's work – Item 12d

Item	Interview question	Jim's response
12d	How did you determine the horizontal asymptote from the given equation? What was/is your thinking process? Why? $$f(x) = \frac{3(x+1)(x-2)}{(x-2)(x+4)}$$	from the given equation the 22:09 degree of the numerator and the 22:10 denominator are equal and so I divide or 22:16 I divided the leading coefficient of the 22:22 denominator by the numerator 22:24 leading coefficient of the numerator 22:25 which is 3 over 1 which is 3

Jim's responses to SS1 coded items from the posttest reflection. This part of the results describes Jim's explanation in the posttest reflection for his work on the posttest. Students were asked only about Items 11c and 13a on the posttest reflection. The other item in that rubric is 11b. Regarding Item 11c, Jim was asked, "how did you come up with your response? What helped you?" Item 11c was "determine the vertex of the function $f(x) = -4(x-3)^2 + 16$." Jim replied "knowing said equation helped." His answer indicates that he recognize that structure of the quadratic function in vertex form.

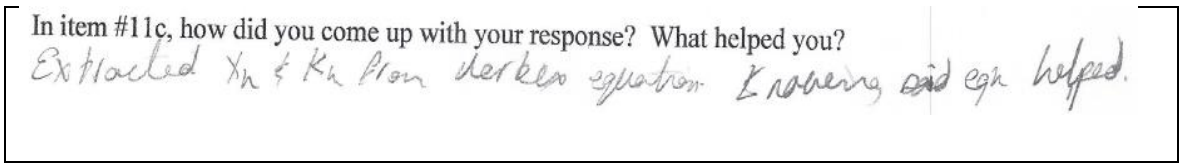
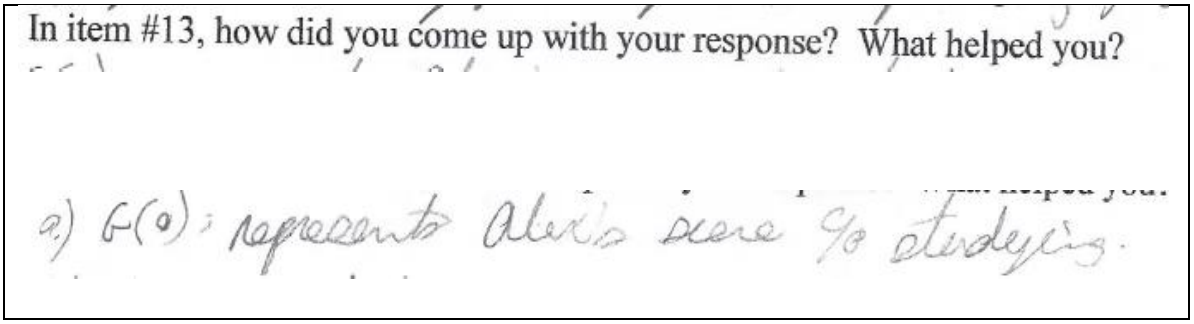

In item #11c, how did you come up with your response? What helped you?
Extracted x_h & K_h from vertex equation. Knowing said eqn helped.

Figure 16: Jim's posttest reflection response about Item 11c

Regarding Item 13a, Jim was asked, "how did you come up with your response? What helped you?" Item 13a states "The grade that Alex receives on a particular test can be modeled by the function below $G(h) = -h^2 + 12h + 64$ where h represents the number of

hours spent studying, and $G(h)$ represents the score earned by studying h hours. What score will Alex earn without studying?" Jim replied "G(0) represents Alexis score without studying". This response indicates that Jim did not use structure sense to answer the question, though he must have recognized the quadratic structure of the expression.


In item #13, how did you come up with your response? What helped you?

a) G(0): represents Alexis score w/o studying.

Figure 17: Jim's posttest reflection response about Item 13a

Jim's interview responses to SS1 coded items. This part of the results describes Jim's explanation in the interview for his work on the posttest. Jim's response to Item 11b clearly demonstrates his structure sense in recognizing the vertex form of the quadratic function. In lines 16:59 and 17:07, he said "that because that's the y-value for the vertex… that's the y part y-coordinate for the vertex". His response here in Table 46 is consistent with his answer on the posttest reflection regarding Item 11c.

Table 46: Justification of SS1 coded items in Jim's work – Item 11b

Item	Interview question	Jim's response
11b	Why didn't you/did you choose the last term in the statement as the y-intercept? Use the function $f(x) = -4(x-3)^2 + 16$ to answer the parts that follow. Determine the y-intercept of the function.	why did I not well I 16:59 didn't choose that because that's the 17:00 the y-value for the vertex you mean this 17:05 one right yes yeah that's yeah because 17:07 that's the y part y-coordinate for the 17:12 vertex

The interview question for Item 11c was "What was/is your thinking process for determining the vertex from the statement $f(x) = -4(x-3)^2 + 16$?" As shown in Table 47 in line 18:23 through 18:38, Jim's response to Item 11c clearly demonstrates his structure sense in recognizing the vertex for the graph of the equation. "When I zeroed out this expression $-4(x-3)^2$, gesturing, circling around the term $-4(x-3)^2$ shows his ability to make sense of the vertex information that he recognized in the equation. Circling around expressions, part of expressions, and terms was a habit that I modeled throughout the intervention.

Table 47: Justification of SS1 coded items in Jim's work – Item 11c

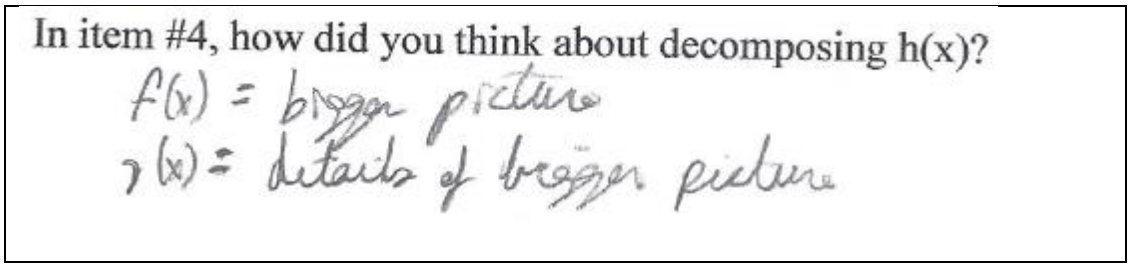
Item	Interview question	Jim's response
11c	What was/is your thinking process for determining the vertex from the statement? Determine the vertex of the function $f(x) = -4(x-3)^2 + 16$	so the x value that gives me 0 that 0 is 18:09 this out is my x value so that's 3 and 18:16 then my Y value is what this function 18:23 equals when I'm zeroed out this 18:25 expression here so can I write here okay 18:31 so I got a 4 3 minus 3 so that becomes 0 18:38 then what I have leftover is 16

As shown in Table 46 lines 22:53 through 23:33, Jim's response to Item 13a is procedural because there is no reference to the form of the quadratic equation. He explains his answer by talking about plugging in zero; there is no indication of any kind of recognition regarding the equation, its type, or form. His response here is consistent with the response he gave on the posttest reflection.

Table 48: Justification of SS1 coded items in Jim's work – Item 13a

Item	Interview question	Jim's response
13a	Using the given equation, how did you determine what score the student would earn without studying? What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 12h + 64$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours.What score will Alex earn without studying?	so 8 represents the number of 22:53 hours spent studying and the score is G 22:58 of H so if Alex doesn't study H is going 23:02 to be 0 so plug 0 into where each was 23:09 and the answer looks like it's 64 so the 23:20 score earned by studying t hours well it 23:23 doesn't say that the score is a 23:24 percentage but I made it one by uh yeah 23:33 okay

Jim's responses to SS2-A coded items from the posttest reflection. Students were asked only about Items 4 on the posttest reflection. The other items in that rubric are 1a/b and 6. Regarding Item 4, Jim was asked, "how did you think about decomposing h(x)?" Item 4 states "Suppose that $h(x) = \sqrt{x^2 + 6} - 12$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f\big(g(x)\big)$." Jim replied "by seeing f(x) as a bigger picture of h(x) and g(x) as details of the bigger picture." It is inconclusive whether Jim is dealing with a compound term as an entity. However what he mentions is in line with that type of thinking.


In item #4, how did you think about decomposing h(x)?

f(x) = bigger picture

g(x) = details of bigger picture

Figure 18: Jim's posttest reflection response about Item 4

Jim's interview responses to SS2-A coded items. This part of the results describes Jim's explanation in the interview for his work on the posttest. As shown in Table 49 lines 00:25 through 00:49, Jim's response on Item 1a/b clearly shows his structure sense in dealing with a compound term as a single entity. This structure sense ability is evidenced by his use of the phrase "everything within that is a substitution for x" and by his circling around the expression of the substituted function.

Table 49: Justification of SS2-A coded items in Jim's work – Item 1a/b

Item	Interview question	Jim's response
1a/b	Why did you put parentheses in that statement? (Statement written by the student) Use the functions $f(x) = 2x + 8$ and $g(x) = \frac{x}{2} - 4$ to answer the parts that follow. Determine $(f \circ g)(x)$. Determine $(g \circ f)(x)$.	I put a parenthesis here 00:25 in here because the function of G 00:32 substitutes the X for the function of F 00:37 so the parenthesis everything within 00:42 that is a substitution for X yeah that's 00:49 why I did that. $2\left(\frac{x}{2} - 4\right) + 8$

As shown in Table 50 lines 10:43 through 11:15, Jim's response to the interview question regarding Item 4 clearly demonstrates his structure sense ability to deal with a

compound term as a single entity. This ability is evidenced by his use of the word "umbrella", which combines several terms as one, x, as x=sqrt(x^2+6).

Table 50: Justification of SS2-A coded items in Jim's work – Item 4

Item	Item description	Jim's response
4	How did you think about decomposing that function? Which part of the original expression did you think could be a component? Why? Suppose that $h(x) = \sqrt{x^2+6} - 12$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$. $f(x) =$ _____ $g(x) =$ _____	so I chose so X minus 12 is a component 10:43 of f of X because it it's like the 10:50 umbrella that the G of X is under so you 10:54 have I thought of it this way yeah X 10:58 minus 4x minus 12 and G of X is gonna be 11:06 going in here so G of X is x squared 11:11 plus 6 then I could just that's my X 11:15 minus 12 $f(x) = x - 12$ $g(x) = \sqrt{x^2+6}$

The interview question for Item 6 was "How did you think about or what helped you get this task right?" Item 6 required students to construct and simplify the difference quotient for $f(x) = x^2 - 3x + 2$. As shown in Table 51 lines 15:10 through 15:36, Jim's response to the interview question regarding Item 6 also demonstrates his structure sense ability to deal with a compound term as a single entity. He talks about the expression of f(x), x^2-3x+4 by circling around it. He talks about (x+h) by mentioning the parentheses around it and by circling around it. He talks about showing x^2-3x+4 as a single entity by putting parentheses around it and acting upon it by multiplying it with a negative sign, mentioning the word "whole".

Table 51: Justification of SS2-A coded items in Jim's work – Item 6

Item	Interview question	Jim's response
6	How did you think about or what helped you get this task right? Construct and simplify the difference quotient for the function $f(x) = x^2 - 3x + 2$. For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.	right now looking at this it seems to 15:10 help me understand it I'm just rewriting this function here 15:23 adding a plus h everywhere that there 15:27 was the variable X and that helps make 15:34 sense with the the parentheses there 15:36 because I can isolate where the X plus h 15:38 goes and then I just have to distribute 15:42 the two exponent I have to distribute 15:46 the negative three and then I guess I 15:49 should have put or not should've but I 15:51 could have put a parentheses around this 15:52 so that I know that I'm in distributing 15:54 that negative to the whole f of X okay 15:57 but yeah that is how I think I did that 16:12

Jim's responses to SS3 coded items from the posttest reflection. Regarding Item 5, Jim was asked "how did you think about handling the absolute value bars or solving the equation? Why?" Item 5 asked students to solve $13 - 2|x - 5| = 9$. When asked about this item, Jim replied that "Absolute value bars must be isolated because it only refers to distance from 0. There are two possible results, - integer and + integer." His answer does indicate that the manipulations involved in isolating the absolute value bars are for the purpose of rightly applying the two possible results he mentioned.

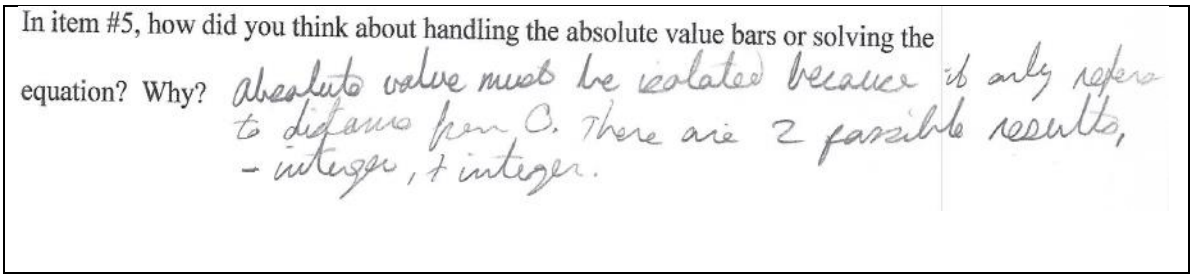
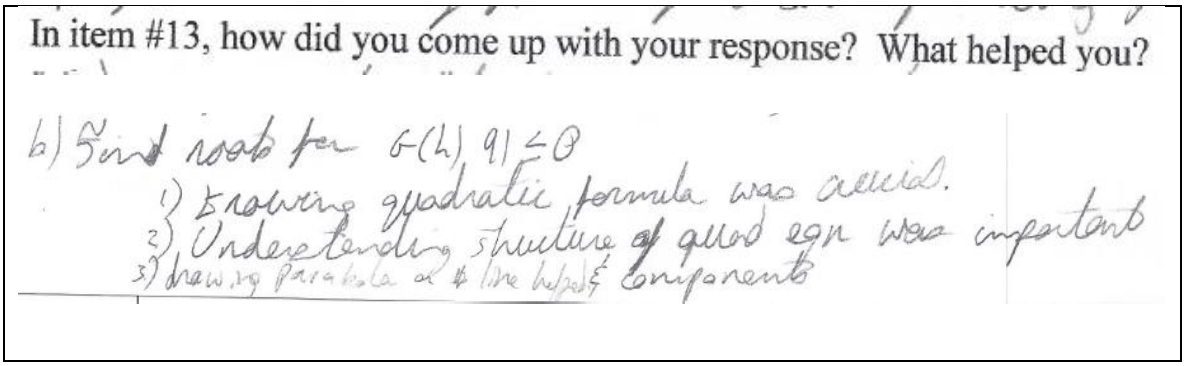

In item #5, how did you think about handling the absolute value bars or solving the equation? Why? absolute value must be isolated because it only refers to distance from 0. There are 2 possible results, - integer, + integer.

Figure 19: Jim's posttest reflection response about Item 5

Regarding Item 13b, when asked "how did you come up with your response? What helped you?" Jim replied that understanding structure of quadratic equation was important and drawing the parabola graph also. His answer here does not show less awareness about why he performed the symbol manipulations. His answer to number 5 shows more awareness about why he performed the symbol manipulations.


In item #13, how did you come up with your response? What helped you?

b) Find roots for $6(h)\,91 \leq 0$

1) Knowing quadratic formula was crucial.
2) Understanding structure of quad eqn was important
3) drawing parabola or # line helped components

Figure 20: Jim's posttest reflection question about Item 13

Jim's interview responses to SS3 coded items. The interview question for Item 5 was "Why did you / why didn't you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars?)" As shown in Table 52 lines 12:18 through 12:44, Jim's response to the interview question regarding Item 5 demonstrates his structure sense awareness that some symbol manipulations were necessary in order to properly deal

with the absolute value bars. He used words and phrases like “Whatever is inside the absolute value bars”, “by itself”, “one side”, and “distance from zero” to show his awareness of the meaning of that structure and how it ought to be manipulated.

Table 52: Justification of SS3 coded items in Jim’s work – Item 5

Item	Interview question	Jim’s response
5	Why did you / why didn’t you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars? Solve the absolute value equation below. $13 - 2\|x - 5\| = 9$	because whatever is inside the absolute 12:18 value bars needs to be by itself on one 12:22 side the equation before it can be dealt 12:24 with okay and that is because whatever's 12:29 inside the absolute value bars is 12:33 the distance from zero it so if I were 12:38 to distribute that to it would it would 12:41 skew like the meaning of the absolute 12:44 value so that's my answer

The interview question for Item 13b was “What was/is your thinking process? Why?” Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 12h + 64$. Jim’s response in the interview, which was a couple of months after the posttest reflection show more awareness of structure sense than at first when asked about Item 13b on the posttest reflection. As shown in Table 53 lines 24:46 through 25:20, he is attending to structure in a graphical way and by labeling the statement as quadratic. He mentions that the symbol manipulation of subtracting 91 on both sides allows him to get a new expression on one side of the inequality. Then he mentions that he sees that he can factor this new expression. Although it is not very explicit, Jim’s response does show a level of awareness of the symbol manipulations he performed for 13b.

Table 53: Justification of SS3 coded items in Jim's work – Item 13b

Item	Interview question	Jim's response
13b	What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 12h + 64$ Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)	so I want the Y 24:22 value to be 91 or greater and so I plug 24:26 91 in there which would be the test 24:30 score and 24:39 now I want to find the H or the amount 24:43 of time he needs to study to be greater 24:46 than or equal to that 91 mm-hmm so I see 24:50 that this equation is a degree of two so 24:54 it's a quadratic and so I know that I'm 24:57 dealing with a parabola and so I get 25:04 everything on the one side so I subtract 25:08 by 91 and I get this expression here 25:13 mm-hmm and I see that I can factor yeah 25:20 I can the factors for this expression 25:24 are X minus 9 and X minus 3 which means that at nine hours and at 25:41 three hours of studying he is going to 25:45 receive 91 percent or greater on his 25:50 test yeah and I know it's between those 25:58 hours because it's okay so the parabola 26:03 is inverted and so there's nine and 26:08 there's three

Summary. I summarize the results and analysis on Jim's work both in regards to the four prefigured codes in terms of development of structure sense and in regards to emergent codes in terms of algebraic proficiency. The prefigured codes correspond to the four definitions of structure sense used for the results and analyses. I used those prefigured codes to sort and organize posttest items. Then to make notes about student responses on the transcripts regarding structure sense and algebraic proficiency, I used eight other codes that are listed in Table 24. For structure sense, those codes are identification (IDT), language (LAN), and rationale (REA). For algebraic proficiency, they are procedure (PROC), graph (GRP), communication (COM), connections (CON), and manipulation (MAN).

First, for the general definition of structure sense, Jim showed development of structure sense by thinking about an algebraic expression in terms of its graph. He showed development of structure sense also by extracting information from algebraic expressions such as zeros, degrees, leading coefficients. He showed development of structure sense in ways that are associate with prescribed steps for completing a task (i.e., the structure sense is an integral part of a set of steps, e.g., performing the horizontal line test, finding horizontal asymptotes). However, Jim's development of structure sense did not involve recognizing or connecting inverse structures such as a quadratic function versus a square root function. Second, for the first part of Hoch and Dreyfus' (2006) definition, Jim showed development of structure sense in recognizing structures in their simplest form, though he made use of that structure in instances that required the use of structure sense or in instances where the use of structure sense is significantly more efficient. Third, for the SS2-A part of Hoch and Dreyfus' (2006) definition, Jim showed development of structure sense by using words/phrases like "umbrella", whole, "everything within the parentheses" and by circling around parts of algebraic expressions. Fourth, for the SS3 part of Hoch and Dreyfus' (2006) definition, Jim showed development of structure sense by mentioning that he recognized the structure of the equations and by mentioning how to handle those structures effectively.

In regards to emergent codes, several of Jim's explanations were procedural for items which were readily interpretable by structure sense (e.g., knowing to expect a quadratic function as the inverse of a square root function). His responses to the interview questions and his work on the posttest showed some flexibility in the way he performed tasks. For example, his reasoning involved procedural, graphical, and structural approaches. His reasoning involved several habits that I modeled during the intervention such as the use of

words/phrases “everything inside”, “whole”, “entire”, “umbrella”, and “big picture” and the use of gestures like circling around algebraic expressions to highlight them and to reason about them. These phrases were instrumental language for identifying referents and structures in order to convey reasoning.

Case 2: Tom

Tom, an African-American male, was a full-time student at the time of data collection. He already had three degrees in the computer field and was planning to transfer to a local university to pursue a degree in Industrial Technology. His career goal was to continue working in the computer field and ultimately move into management. Tom indicated that he had a very curious mind. At the beginning of the course, he was taking 12 credit hours and was not working. He had taken remedial math courses in the past, and it had been one semester since he had taken an algebra course. This was not his first time taking pre-calculus in college. He indicated that his overall experience was very good in the two math classes he completed. He rated his study skills as average and his access to computers and the internet as very good. Tom stated that he looked forward to his mathematics lessons, but that he did not enjoy reading about mathematics. He disagreed with the statement that making an effort in mathematics was worth it because it would help him in the work that he wanted to do later on. He stated that he does not do mathematics because he enjoys it, and he did not believe that learning math would improve his career prospects, help him get a job, or help him in future studies. He stated that he was not interested in the things he learned in mathematics. In terms of his confidence level in completing certain mathematical tasks, Tom felt confident that he could calculate how much cheaper a TV would be after a 30% discount, understand graphs presented in newspapers, solve an equation like $3x+5=17$, find the actual

distance between two places on a map with a 1:10,000 scale, solve an equation like $2(x+3)=(x+3)(x-3)$, and calculate the gas mileage of a car. However, he did not feel very confident in using a train schedule to figure out how long it would take to get from one place to another, or calculating how many square feet of tile would be needed to cover a floor.

Tom indicated in the Midterm Reflection Questionnaire that he had not had a personal tutor, nor had he participated in a study group. He did use the Individualized Learning Center (ILC), a campus resource, 2 hours a week on average. Currently, he was enrolled part time, taking 4 credit hours. He was also working part time, averaging 20 hours a week over the past 7 weeks. He indicated that he spent on average 24-30 hours a week doing MAT 171 homework, reading class notes and watching instructional videos to learn the concepts in the course. He also indicated that watching YouTube math videos was affecting his learning experience outside of class.

In the Final Interview at the end of the semester, Tom reported that "the learning experience in class was great." However, he indicated that the online work from WebAssign was overly burdensome, citing as an example one section that had 72 different assignments. In comparing MAT 171 to his previous math courses, Tom stated, "This class MAT 171 moved much faster with more material to cover than my previous class. With the speed of the class along with the online portion it was a big challenge just to keep up and perform well. It seems as if this class is structured for students that are moving on to higher levels of math and not those of us who simply have to complete this class because it is required of us….the class should be restructured in a way that it is slowed down and add more class time….I think this would increase the success rate for the class overall." Tom also indicated

that if the homework were only what the instructor posted online, the “methods, techniques and training will be a constant and less confusing for the students.”

Tom’s responses to general definition items from the posttest reflection. Item 1c was not addressed on the posttest reflection; no questions were posed to students regarding that item. When asked “how do you know which function has an inverse and which one doesn’t?” The two functions are $f(x) = x^2 - 5x + 3$ and $g(x) = \sqrt{3x - 8}$**.** As shown in Table 13, Tom replied that looking at the radical function I knew it was the inverse and I stop and only solve that equation.

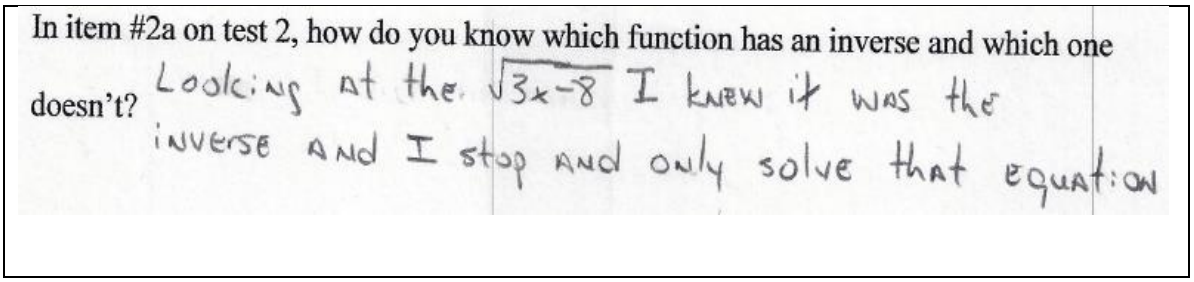

In item #2a on test 2, how do you know which function has an inverse and which one doesn’t? Looking at the $\sqrt{3x-8}$ I knew it was the inverse and I stop and only solve that equation

Figure 21: Tom's posttest reflection response about Item 2a

Tom’s response here needs more details in order to be commented upon beside the important point he made about “looking” and being able to a decision about the inverse.

Tom was asked the following question regarding Item 2b “did you expect the kind of function or structure you get for the inverse of g(x), $g(x) = \sqrt{3x - 8}$? Yes or no. If yes, how?” As shown in Table 14, Tom replied no.

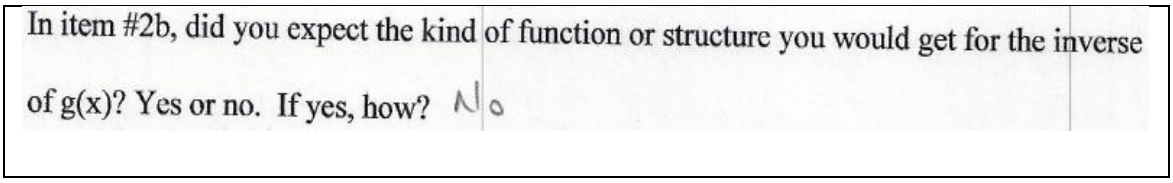
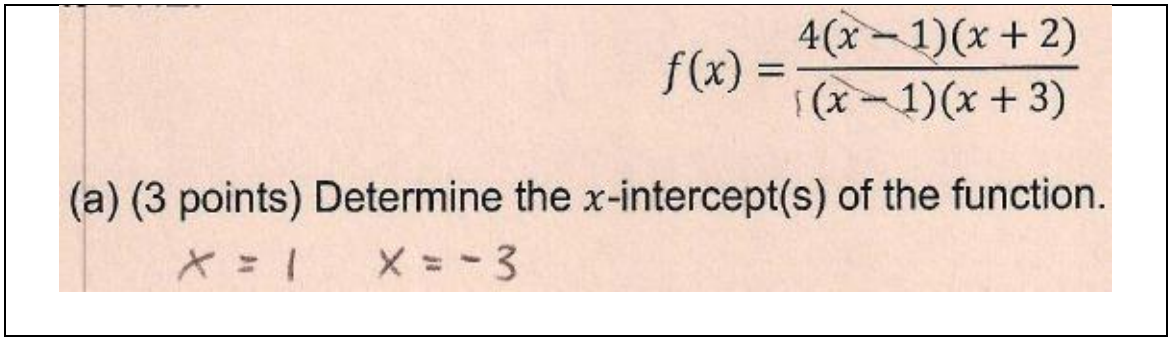

In item #2b, did you expect the kind of function or structure you would get for the inverse of g(x)? Yes or no. If yes, how? No

Figure 22: Tom's posttest reflection response about Item 2b

His answer shows that his structure sense does not associate the radical function with its opposite structure, the quadratic function (half of the graph), perhaps because a full quadratic graph does not have an inverse.

Tom was asked about Item 12a, "how did you come up with your response? What helped you?" His response to Item 12a on the posttest is shown in Figure 12a.

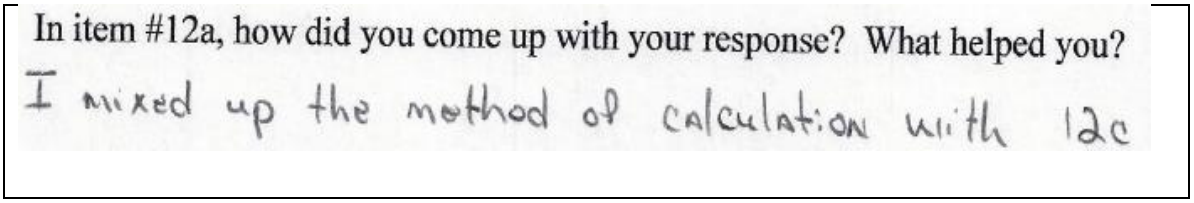

$$f(x) = \frac{4\cancel{(x-1)}(x+2)}{1\cancel{(x-1)}(x+3)}$$

(a) (3 points) Determine the x-intercept(s) of the function.

$x = 1$ $x = -3$

Figure 23: Tom's response on the posttest to Item 12a

As shown in Figure 24, Tom replied "I mixed up the method of calculation with 12c." Item 12c to which he refers asked about vertical asymptotes.


In item #12a, how did you come up with your response? What helped you?

I mixed up the method of calculation with 12c

Figure 24: Tom's posttest reflection response about Item 12a

This answer indicates lack of structure sense in terms of knowing what part of equation to focus upon in the structure of the rational equation.

When asked about Item 12d, “How did you come up with your response? What helped you?’ Figure 25 shows Item 12d from the posttest.

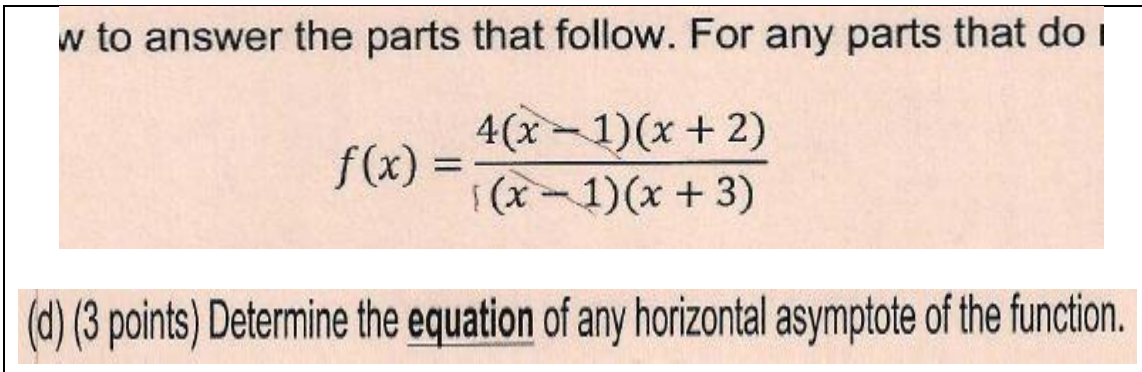


Figure 25: Item 12d on the posttest

As shown in Figure 26, Tom’s response provides the horizontal intercept instead of the horizontal asymptote.

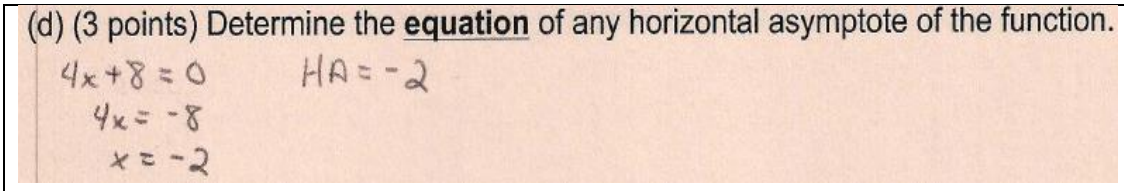


Figure 26: Tom's posttest reflection response about Item 12d

In this response, it is evident that his approach is procedural. Its appropriateness notwithstanding, his answer could be determined by inspection, by seeing that a -2 would

zero out the numerator. At best, his structure sense or the concepts at play were entangled, not clear for him.

Tom's interview responses to general definition items. This part of the results describes Tom's explanation in the interview for his work on the posttest. The first interview question for this category of items was "How did you know that f(x) and g(x) are inverses of each other? ($\mathbf{f}(\boldsymbol{x}) = \mathbf{3}\boldsymbol{x} + \mathbf{12}$ and $\boldsymbol{g}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\mathbf{3}} - \mathbf{4}$)." Tom does show some structure sense in terms of reading through the expressions of f(x) and g(x) and identifying opposite operations. He is not sure of the approach of reading through the expressions because he relies more on knowing the proper steps that are specific to inverses as seen in Table 54, lines 02:14 and 02:30.

Table 54: Justification of the general definition in Tom's work - Item 1c

Item	Interview question	Tom's response
1c	How did you know that f(x) and g(x) are inverses of each other? $f(x) = 3x + 12$ $g(x) = \frac{x}{3} - 4$	02:00 actually I started I got a little 02:02 confused between what I was taught in 02:07 class what was in WebAssign and YouTube 02:11 because I was trying to remember exactly 02:14 which steps that was supposed to take 02:16 and it was just something that I 02:19 remember from class but I couldn't 02:20 actually put it together and when I just 02:23 kept looking at I said these two have to 02:26 be inverses but I couldn't really 02:27 understand how they were in I mean I 02:30 couldn't go through the steps to 02:32 actually be able to solve it to show 02:34 that they were inverses but by looking 02:35 at I could tell that they were inverses 02:37 I just didn't know mmm 02:39 yeah and that's why I sent it these two 02:44 must be inverses of each other 02:47 because you got 3x here and you got x 02:50 over 3 and I said whenever is it is it 02:54 because of the plus sign or a minus sign 02:56 or is it because I'm looking at that 03:00 this is that the G of G of X has been 03:04 reduced 03:05 in comparison to f of X

The interview question for Item 2a asked "What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered?" The two functions are $f(x) = x^2 - 5x + 3$ and $g(x) = \sqrt{3x - 8}$. In Table 55 lines 09:02 through 09:27, his reasoning for item 2a is based on the structure of f(x) and g(x) for applying the 1-to-1 rule in a graphical way. However, he is not sure about which test, vertical line test or horizontal line test. As shown in Table 53 lines 05:27 through 05:48, he reiterates that his thinking has been unclear due to following several approaches which

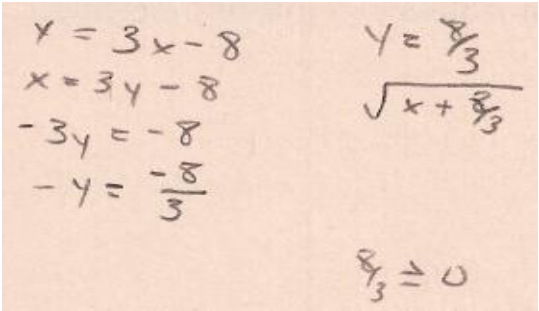
were different from classroom instruction. Nevertheless, his attention to the alternate representation—graph—in order to reason indicates a positive aspect of his learning.

Table 55: Justification of the general definition in Tom's work - Item 2a

Item	Interview question	Tom's response
2a	What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered? Use the functions $f(x) = x^2 - 5x + 3$ and $g(x) = \sqrt{3x - 8}$ to answer the parts that follow. One of the functions provided above is one-to-one, and the other is not. Which of the functions **is** a one-to-one function?	I would look at the first one in the 04:54 first one it looks like it's a parabola 04:57 and I think it was it's the vertical 05:02 line test that you used to find out if 05:04 it's a one-to-one I think I can't 05:07 remember okay and I said when when it 05:18 came to this particular test that was 05:21 because this was the one where 05:24 everything it was getting really 05:27 confusing for me okay and it wasn't it 05:32 was what we learned in class and like I 05:36 said when I was doing them - well WebAssign 05:38 they would throw a lot of different 05:41 stuff at you mm-hm 05:42 completely different from what was it 05:44 class mm-hmm and the methods that they 05:48 used to solve some of the … just trying to get to that 09:00 point to be able to figure out exactly 09:02 how to get to where I can actually 09:05 visualize the test where I can either do 09:08 the horizontal or vertical line test to 09:10 actually do it but I had to put it take 09:14 the G of X and put it in a form where I 09:17 can figure out what shape is gonna be on 09:20 a graph okay to be able to do it 09:22 (inaudible) you know if I was gonna sit and 09:24 draw it or if I was just gonna do you 09:27 know try to picture it mentally

The interview question for Item 2b asked "Looking at the inverse function that you came up with $y = \frac{x^2+8}{3}$ how do you know it is the correct inverse for the original function, $y = \sqrt{3x-8}$?" As shown in Table 56 lines 07:53 through 07:59, Tom's response echoes the same difficulty stemming from following several other approaches which were different from classroom instruction. His thinking about this concept was still in a developmental stage, not mature enough to reconcile different approaches or at best keep them separate. His response here is consistent his response in the posttest reflection.

Table 56: Justification of the general definition in Tom's work - Item 2b

Item	Interview question	Tom's response
2b	Looking at the inverse function that you came up with,  how do you know it is the correct inverse for the original function? $g(x) = \sqrt{3x-8}$	I was looking for the inverse to the two 07:39 G of X and the method that I used that I 07:44 thought was the correct one to be able 07:47 to be able to get the inverse 07:53 and again it goes back to all of 07:56 different things that I was just trying 07:59 to figure out to get to this step okay

The interview question for Item 12a was "How did you determine the x-intercept? What was/is your thinking process? Why?" Tom's response regarding his work on Item 12a in the interview provides more details, but it is similar to the response given in the posttest reflection. He has mistaken horizontal or x- intercepts with vertical asymptotes. Table 57 shows his work for that item.

Table 57: Justification of the general definition in Tom's work - Item 12a

Item	Interview question	Tom's response
12a	How did you determine with the x-intercept(s) from the given equation? What was/is your thinking process? Why? $f(x) = \frac{4\cancel{(x-1)}(x+2)}{1\cancel{(x-1)}(x+3)}$	the given illusion I was looking at 27:16 the denominator and then when I was I 27:20 was looking at while I was looking at 27:22 I saw that X minus 1 X plus 3 so if I 27:27 take the X minus 1 and I set the one 27:30 equal to X it's gonna be a positive 1 27:33 and if I set that 3 to the X it's gonna 27:35 be a negative 3 (a) (3 points) Determine the x-intercept(s) of the function. $x = 1$ $x = -3$

The interview question for Item 12d was "How did you determine the horizontal asymptote? What was/is your thinking process? Why?" Tom's response regarding his work on Item 12d in the interview reveals another misperception similar to the one in Item 12a, this time between horizontal asymptote and x-intercept. This mix-up notwithstanding, the approach evident in his work is procedural as shown in Table 58 in lines 28:25 through 28:39. Perhaps the lack of structure sense on those two items (12a and 12d) caused the mix-up and the lack of structure sense may be due to an overemphasis on steps (algorithm).

Table 58: Justification of the general definition in Tom's work - Item 12d

Item	Interview question	Tom's response
12d	How did you determine the horizontal asymptote from the given equation? What was/is your thinking process? Why? $f(x) = \frac{4(x-1)(x+2)}{(x-1)(x+3)}$	looks 28:14 like what happened what I did is I still 28:17 had the four from the original equation 28:21 where I had already cancelled out the X 28:23 minus 1 from the numerator and 28:24 denominator 28:25 mm-hmm so I took the 4 and I said that 28:28 with the X plus 2 and I multiply to get 28:31 4x Plus 8 equals and then I set it to 0 28:34 and move the 8 the positive 8 over to 28:39 make it a negative 8 and then solved it 28:40 to get X equal negative 2 ok which 28:47 equals negative 2 okay that was 12 D 28:52 (d) (3 points) Determine the **equation** of any horizontal asymptote $4x+8=0$ $HA=-2$ $4x=-8$ $x=-2$

Tom's responses to SS1 coded items from the posttest reflection. This part of the results describes Tom's explanation in the posttest reflection for his work on the posttest. Students were asked only about items 11c and 13a on the posttest reflection. The other item in that rubric is 11b. Regarding item 11c, Tom was asked, "how did you come up with your response? What helped you?" Item 11c was "determine the vertex of the function $f(x) = -4(x-3)^2 + 16$." As shown in Figure 27 Tom replied "I couldn't remember how to do the vertex."

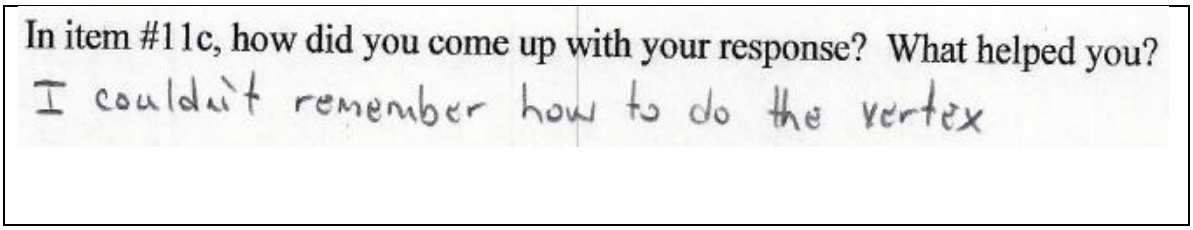

In item #11c, how did you come up with your response? What helped you?
I couldn't remember how to do the vertex

Figure 27: Tom's posttest reflection about Item 11c

Regarding Item 13a, Tom was asked, "how did you come up with your response? What helped you?" Tom replied "I can't remember". Even though he had his work on the posttest with him while completing out the reflection.

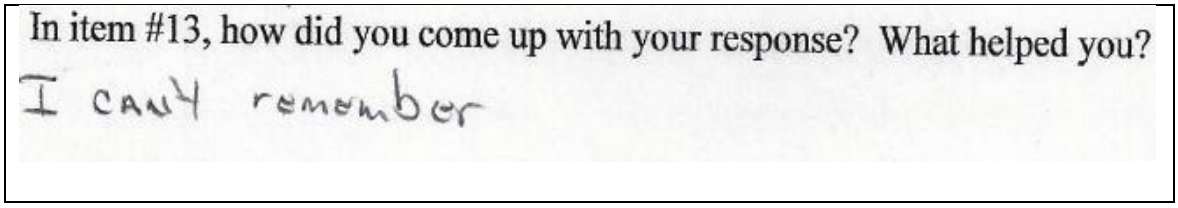

In item #13, how did you come up with your response? What helped you?
I can't remember

Figure 28: Tom's posttest reflection about Item 13

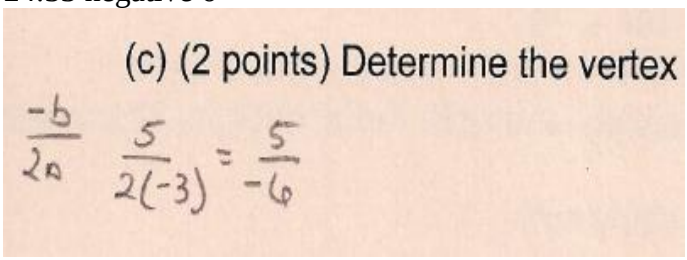
Tom's interview responses to SS1 coded items. This part of the results describes Tom's explanation in the interview for his work on the posttest. The interview question for Item 11b was "Why didn't you/did you choose the last term in the statement $f(x) = -3(x-5)^2 + 12$ as they y-intercept?" Tom's response to item 11b shows that he recognizes the vertex form of the quadratic function in Table 59, middle column. In Table 57 line 22: 47 he mentions the words "format, x sub h, and y sub k", which are the labels for the coordinates of the vertex. Yet, his response shows a mix-up between the coordinates of the vertex and the coordinates of the y-intercept.

Table 59: Justification of SS1 coded items in Tom's work – Item 11b

Item	Interview question	Tom's response
11b	Why didn't you/did you choose the last term in the statement as the y-intercept? Use the function $f(x) = -3(x-5)^2 + 12$ to answer the parts that follow. Determine the y-intercept of the function.	because in that particular finding well 22:43 in this format if you said where I put 22:47 the X with the base of H that's gonna be 22:51 my five and then the y with the base 22:53 of K is gonna be my 12 so that that 22:55 tells me that 12 is gonna 22:57 be my y-intercept just like their x 23:00 the negative five is my x-intercept 23:03 okay okay so you did choose 12 as 23:06 your y-intercept yeah okay.

The interview question for Item 11c was "What was/is your thinking process for determining the vertex from the statement $f(x) = -3(x-5)^2 + 12$?" Tom's response to item 11c reveals the same difficulty of not being sure of the form of the quadratic equation and the same difficulty of misidentifying information from the specific forms of the quadratic equation. As shown in Table 60 lines 23:49 through 24:20, he talks about the equation as the vertex form (See response in 11c) and as the expanded form (ax^2+bx+c) at the same time.

Table 60: Justification of SS1 coded items in Tom's work – Item 11c

Item	Interview question	Tom's response
11c	What was/is your thinking process for determining the vertex from the statement? Determine the vertex of the function $f(x) = -3(x - 5)^2 + 12$	I was looking at the way the way the 23:45 function was actually set up and from 23:49 that I was trying to determine how to go 23:51 out to get the vertex which pieces was 23:53 actually that would actually go into 23:56 that formula of negative B over 2a okay 24:00 and when I was looking at it I was I 24:05 thought it was gonna be the X with base of 24:12 H she was gonna be my B where my 3 24:18 negative 3 would be a negative a be B 24:20 plus 12 be C mm-hmm and then I just pulled 24:24 out five to be my negative P time not to 24:27 a which is negative three okay so you 24:31 come up with that good 5 divided by 24:33 negative 6 

Item 13a states "The grade that Alex receives on a particular test can be modeled by the function below $G(h) = -h^2 + 10h + 75$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours. What score will Alex earn without studying?" Tom's response regarding Item 13a is interesting because of his perspective and how he expresses his thinking. His response does not exhibit structure sense in the manner of SS1, but it shows him thinking through an expression term by term, which is something that was taught in class within the design of the intervention. Pointing to the first two terms, -h^2 and 10h, in the expression $-h^2 + 10h + 75$, Tom said "these two parts at the beginning of the equation is actually asking if he did something". His thinking here as shown in Table 61 is a good example of reading through symbols (Gen Defn), though he did

not recognize the expanded form of the quadratic equation (SS1). This inconsistency is a positive sign of development.

Table 61: Justification of SS1 coded items in Tom's work – Item 13a

Item	Interview question	Tom's response
13a	Using the given equation, how did you determine what score the student would earn without studying? What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 10h + 75$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours.What score will Alex earn without studying?	these the two parts at the 30:11 beginning of the equation is actually 30:13 asking if he did something else but if 30:16 you get nothing at all he's gonna get 75 30:19 75 is where it's like he would that 30:23 would be the score he would get if he 30:25 did nothing

Tom's responses to SS2-A coded items from the posttest reflection. Students were asked only about Item 4 on the posttest reflection. The other items in that rubric are 1a/b and 6. Regarding Item 4, Tom was asked, "how did you think about decomposing h(x), $h(x) = \sqrt{x^2 + 13} - 11$?" As shown in Figure 29, Tom replied "I just wanted to make it as simple as possible." It is inconclusive whether Tom is dealing with a compound term as an entity without getting more information from him.

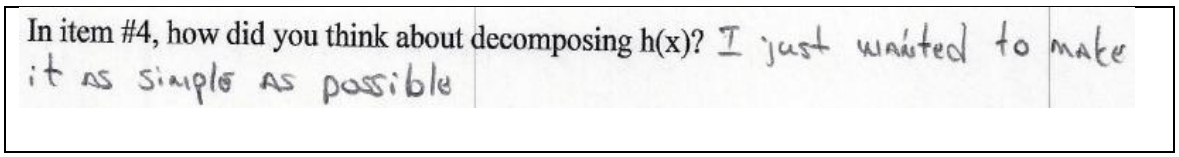
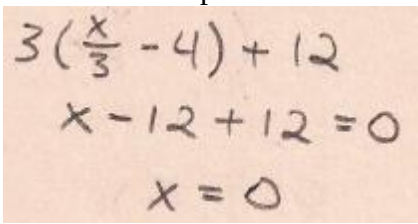

In item #4, how did you think about decomposing h(x)? I just wanted to make it as simple as possible

Figure 29: Tom's posttest reflection response about Item 4

Tom's interview responses to SS2-A coded items. The interview question for Item 1a/b was "Why did you put parentheses in that statement?" (Statement written by the student). Tom's response on Item 1a/b clearly shows his structure sense in dealing with a compound term as a single entity. This structure sense ability is evidenced by his use of the phrase (language) in Table 62, starting in line 00:38 "to help separate or help me keep track of the x over 3 minus 4…"

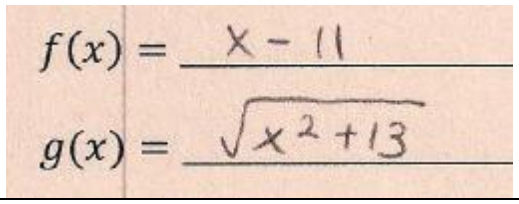
Table 62: Justification of SS2 coded items in Tom's work – Item 1a/b

Item	Interview question	Tom's response
1a/b	Why did you put parentheses in that statement? (Statement written by the student) Use the functions $f(x) = 3x + 12$ and $g(x) = \frac{x}{3} - 4$ to answer the parts that follow. Determine $(f \circ g)(x)$. Determine $(g \circ f)(x)$.	well 00:22 I chose to use the parentheses because 00:25 it's when you have 3x plus 12 which was 00:31 my f of X and then with the G of X is x 00:35 over 3 minus 4 I chose to use the 00:38 parentheses to help separate or help me 00:41 keep track of the x over 3 minus 4 00:45 that's what to replace the X that's why 00:47 I use the parentheses 

The interview question for Item 4 was "How did you think about decomposing that function, $h(x) = \sqrt{x^2 + 13} - 11$? Which part of the original expression did you think could be a component? Why?" Tom's response to the interview question regarding item 4 clearly demonstrates his structure sense ability to deal with a compound term as a single entity. This

ability is evidenced by his use of the words “first piece, second piece”, as he focused on sqrt(x^2+13) and calls it the first piece. Then he pointed out the negative 11 as the second piece (line 11:12 in Table 63), so that sqrt(x^2+13) becomes a single entity, which he calls x.

Table 63: Justification of SS2 coded items in Tom’s work – Item 4

Item	Item description	Tom’s response
4	How did you think about decomposing that function? Which part of the original expression did you think could be a component? Why? Suppose that $h(x) = \sqrt{x^2 + 13} - 11$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f\big(g(x)\big)$. $f(x) =$ _____ $g(x) =$ _____	well just by 10:51 looking at the way it was set up that 10:52 what was under the square root sign even 10:56 can you can be open to that just what 11:00 was I just gonna say mm-hmm that this 11:02 was gonna be a component this was the 11:05 first piece of it okay and then I when I 11:08 was asked I could see that I had on the 11:10 negative eleven that was gonna be 11:12 second piece to it so this was that's 11:15 why I separated it that way and I pulled 11:19 out and I think yeah and here X minus 11:23 eleven as f of X which was the minus 11:28 eleven that was that wasn't under the 11:30 square root 11:31 mm-hmm and then x squared plus thirteen 11:33 that was gonna be my second component 

The interview question for Item 6 was "How did you think about or what helped you get this task right?" Item 6 required students to construct and simplify the difference quotient for $f(x) = x^2 - 2x + 3$. Tom's response to the interview question regarding Item 6 also demonstrates his structure sense ability to deal with a compound term as a single entity. As shown in line 20:23, Table 64, he talks about substituting (x+h) in for x in the x^2 term and in the -2x term. His response shows that he clearly understands the essential part of the task is substitution (line 20:42). Then he demonstrates his ability to think term by term as he shifts his focus from f(x+h) toward f(x) in the numerator of the difference quotient expression (line 20:52). And in line 20:58, he refers to the f(x) part of the difference quotient as the last piece, clearly reasoning term by term, sizing up the whole formula, and seeing whole expressions of two and three terms as single entities. One evidence of this reasoning can be seen in line 20:52 through his use of the word "representative".

Table 64: Justification of SS2 coded items in Tom's work – Item 6

Item	Interview question	Tom's response
6	How did you think about or what helped you get this task right? Construct and simplify the difference quotient for the function $f(x) = x^2 - 2x + 3.$ For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.	we'll go to the difference quotient 20:15 where you have X plus h so I know that 20:19 was in there parenthesis I can 20:21 substitute that for their X and then 20:23 it's got X plus h squared the same thing 20:29 with 2x where like what I did here was a 20:32 X plus h I know I can substitute that X 20:36 plus h for their X and then I can 20:40 start to solve once I finish those 20:42 substitutions and then I get where I'm 20:45 at plus 3 and we're right here we have 20:49 the negative and the f of X that this X 20:52 is going to be representative of the x 20:55 squared minus 2x plus 3 so that will be 20:58 the last piece to actually setting this 21:01 up to be able to solve it

Tom's responses to SS3 coded items from the posttest reflection. Regarding Item 5, Tom was asked "how did you think about handling the absolute value bars or solving the equation? Why?" Item 5 asked students to solve $14 - 3|x - 5| = 2$. As shown in Figure 30 Tom replied that "I multiplied the number to the left of the first bar and complete the math." His answer indicates he is not aware that the absolute bars are not merely for separation or for grouping purposes as is often the case for parentheses. He is not aware that the absolute value expression must be treated in a special way based on its meaning as a structure.

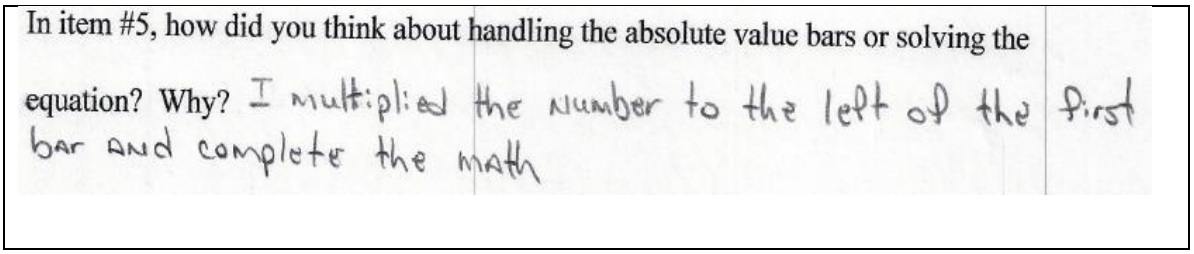

In item #5, how did you think about handling the absolute value bars or solving the equation? Why? I multiplied the number to the left of the first bar and complete the math

Figure 30: Tom's posttest reflection response about Item 5

Regarding Item 13b, Tom was asked "how did you come up with your response? What helped you?" Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 10h + 75$.

As seen in Figure 31, Tom replied "I can't remember."

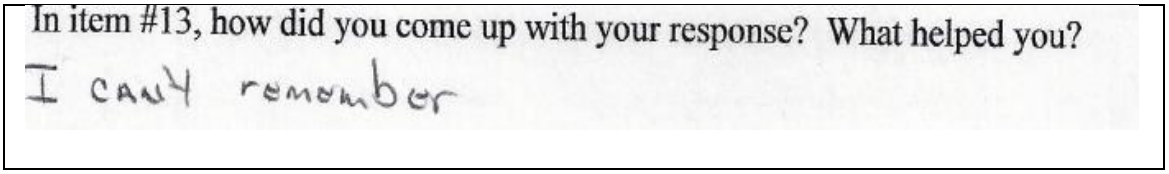

In item #13, how did you come up with your response? What helped you?
I can't remember

Figure 31: Tom's posttest reflection response about Item 13b

Tom's interview responses to SS3 coded items. The interview question for Item 5 was "Why did you / why didn't you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars?)" Tom's response to the interview question regarding Item 5 shows again a lack of awareness about the absolute value structure. Not knowing what steps, the number of steps necessary, and a mix-up with other approaches from YouTube and WebAssign seem to resonate through his explanation about the items with which he struggles. As shown in Table 63, he talks about a different approach in line 13:20. Then he talks about steps, right steps, in lines 14:46, 14:53, 14:56, 15:05, 15:27, 15:31, and 15:55. It is not quoted here because it would be lengthy to include.

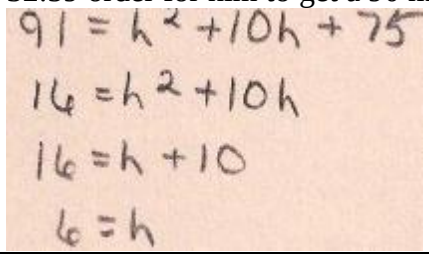
Table 65: Justification of SS3 coded items in Tom's work – Item 5

Item	Interview question	Tom's response
5	Why did you / why didn't you distribute that factor (-3) through the absolute value bars to get rid of them (i.e. absolute value bars? Solve the absolute value equation below. $14 - 3\|x - 5\| = 2$	well this was one 13:06 of those ones where I looked at it more 13:09 than one way but I the way that I did it 13:13 here was that I thought what was gonna 13:17 be the correct way I could have done it 13:20 a different way where I had a I start I 13:26 could have started with the addition 13:28 first but I didn't and I decided that to 13:32 do what was inside the absolute value 13:36 bars was to do the multiplication first 13:38 to try to make it a little easy on 13:40 myself to be able to solve or get to the 13:43 solution 13:49 and again as I guess out this this is 13:53 another one of those where I watched 13:54 somebody do it completely different on 13:57 YouTube where they did a

The interview question for Item 13b was "What was/is your thinking process? Why?" Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 10h + 75$. Tom's response in the interview shows a correct substitution from the function notation, G(h) and the function expression, $-h^2 + 10h + 75$, though the inequality sign is mistakenly switched to an equal sign. As shown in Table 66 lines 32:36 through 32:56, the steps that follow do not show the awareness described in SS3, making appropriate symbol manipulations in order

to take advantage of structure. His reasoning is evident here in this response or in the response regarding Item 5.

Table 66: Justification of SS3 coded items in Tom's work – Item 13b

Item	Interview question	Tom's response
13b	What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 10h + 75,$ Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)	so 91 is the goal and to get the goal 32:16 you have to figure out exactly how many 32:18 hours he would need to put in to to get 32:23 a 91 so that's when I started 32:28 so from here that's when I took he if he 32:32 did nothing he would get a 75 mm-hmm so 32:36 then I subtracted 75 from 91 hmm and 32:39 then once I got here I factored I 32:43 factored out an H I didn't put it in I 32:47 didn't put their stuff in but I factored 32:48 out age mm-hmm and by doing that then I 32:51 can move the 10 to subtract and that's 32:56 how I got to 6 hours if he to be in 32:59 order for him to get a 90 more 

Summary. I summarize the results and analysis on Tom's work both in regards to the four prefigured codes in terms of development of structure sense and in regards to emergent codes in terms of algebraic proficiency. The prefigured codes correspond to the four definitions of structure sense used for the results and analyses. I used those prefigured codes to sort and organize posttest items. Then to make notes about student responses on the transcripts regarding structure sense and algebraic proficiency, I used eight other codes that are listed in Table 24. For structure sense, those codes are identification (IDT), language

(LAN), and rationale (REA). For algebraic proficiency, they are procedure (PROC), graph (GRP), communication (COM), connections (CON), and manipulation (MAN).

First, for the general definition of structure, Tom showed development of structure sense by thinking about an algebraic expression in terms of its graph. He showed development of structure sense also by reading through algebraic expressions term by term in some instances. However, Tom's development of structure sense did not involve recognizing or connecting inverse structures such as quadratic function versus a square root function. Second, for the first part of Hoch and Dreyfus' (2006) definition, Tom showed development of structure sense in recognizing structures in their simplest form, though not in a consistent manner, and not at the level of consistently distinguishing between structures. Third, for the SS2-A part of Hoch and Dreyfus' (2006) definition, Tom showed development of structure sense in a consistent manner by using words/phrases like "first piece", "second piece", "to separate", "last piece", "representative", "substitution", and by pointing to parts of algebraic expressions. Fourth, for the SS3 part of Hoch and Dreyfus' (2006) definition, Tom showed little development of structure sense by substituting the expression of the function for the function notation.

In regards to emergent codes, several of Tom's explanations were procedural for items which were readily interpretable by structure sense (e.g., knowing to expect a quadratic function as the inverse of a square root function). His responses to the interview questions and his work on the posttest do not show much flexibility in the way he performed tasks. For example, his reasoning is mostly procedural, sometimes graphical, and structural in terms of his approaches in instances of dealing with a compound term as a single entity. As a computer programming major, the procedural aspect of his reasoning is strong. His strength

as a process thinker and different approaches he encountered on YouTube or WebAssign may have prevented him from espousing more of the reasoning habits that I modeled during the intervention such as the use of words/phrases “everything inside”, “whole”, “entire”, “umbrella”, and “big picture” and the use of gestures like circling around algebraic expressions to highlight them and reason about them. Given more time, Tom would have made key connections and reconcile the different approaches he encountered.

Case 3: Stacia

At the time of data collection, Stacia, a self-identified Hispanic woman, was a full-time second year student majoring in Biology. She wanted to become an occupational therapist. She was taking 12 credit hours and had a part time job where she worked 15 hours a week. She stated that up to that point, her time in college had been pleasant although she struggled sometimes to balance the demands of her schoolwork and her part-time job. She had not taken any remedial mathematics courses and it had been two semesters since she had taken an algebra course. Her learning experience in prior math courses had been good. She stated that the teachers were always ready to assist their students if they needed help with the material. That semester was her first time taking pre-calculus in college. She rated her study skills as average and indicated that she had very good access to computers and the internet. She stated that she did not enjoy reading about mathematics and that she did not enjoy doing mathematics, but that she believed making an effort in mathematics was worth it because it would help her in the work she wanted to do later in life and improve her career prospects. She also indicated that mathematics was an important subject for her because she would need it for what she wanted to study later on and that it would help her get a job. She looked forward to her mathematics lessons and she was interested in the things she learned in

mathematics. In terms of her confidence level in doing mathematics tasks, Stacia indicated that she did not feel very confident in calculating how many square feet of tile would be needed to cover a floor, finding the actual distance between two places on a map with a 1:10,000 scale, or calculating the gas mileage of a car. However, she felt confident that she could use a train schedule to figure out how long it would take to get from one place to another, calculate how much cheaper a TV would be after a 30% discount, understand graphs presented in newspapers, solve an equation like $3x+5=17$, and solve an equation like $2(x+3)=(x+3)(x-3)$.

In the Midterm Reflection Questionnaire, Stacia indicated that she had not had a personal tutor help her with the course material, nor had she participated in a study group. She was enrolled part-time and was currently taking 10 credit hours. She also was working part-time and had been working on average 19 hours a week for the previous 7 weeks. She indicated that she spent on average 1 ½ hours a week doing MAT 171 homework, reading class notes and watching instructional videos to learn the concepts in the course. She also indicated that other classes were affecting her learning experience outside of class.

Stacia reported on the Final Interview at the end of the semester that her learning experience was "great." She declared, "I realized that I have the potential to learn material that I thought was impossible for me to learn, if I really set my mind to it and study." She also stated that "Math 171 this semester went better than my math courses in the previous semesters. What really helped was the way my proffesor [sic] introduced different materials. He would provide different practice problems for us to work out and even analogies used to remember some of the terms more easily."

Stacia's responses to general definition items from the posttest reflection. Stacia was asked the following question regarding Item 2a, "how do you know which function has an inverse and which one doesn't?" The two functions are $f(x) = x^2 - 5x + 3$ and $g(x) = \sqrt{3x - 8}$**.** As shown in Figure Stacia replied that "Because after it simplifies the equations both equations equal out to zero." Though correct, the response does not show structure sense that is based on a broad view analysis of the way an equation is put together.

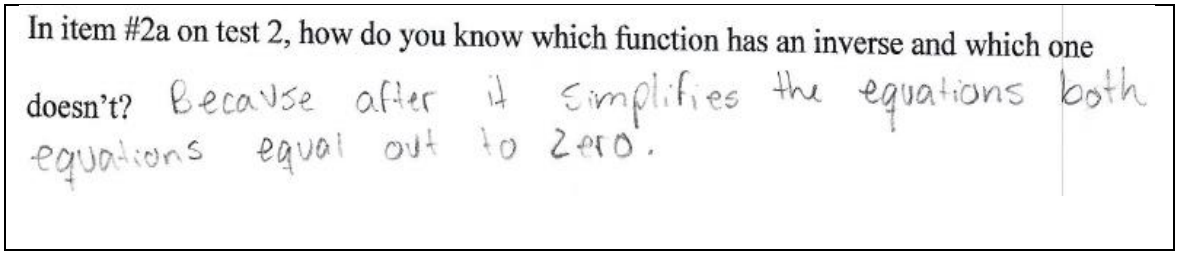

In item #2a on test 2, how do you know which function has an inverse and which one doesn't? Because after it simplifies the equations both equations equal out to zero.

Figure 32: Stacia's posttest reflection response about Item 2a

Stacia was asked the following question regarding Item 2b, "did you expect the kind of function or structure you get for the inverse of g(x)? Yes or no. If yes, how?" As shown in Figure 33, Stacia replied "Yes, because the domain and range are inverses of each other." Her answer does not indicate that she reads through the expressions and pays attention to how those expressions are put together. Her answer does show some insight about the fact that the range of the inverse function is the same as the domain of the original function, or the other inverse.

In item #2b, did you expect the kind of function or structure you would get for the inverse of g(x)? Yes or no. If yes, how?

Yes, because the domain and range are inverses of eachother

Figure 33: Stacia's posttest reflection response about Item 2a

Stacia was asked about Item 12a, "how did you come up with your response? What helped you?" Figure 34 shows Item 12a from the posttest and Stacia's response. Stacia did not provide any answer.

12. Use the function below to answer the parts that follow. For any parts that do not exist, please write DNE.

$$f(x) = \frac{4\cancel{(x-1)}(x+2)}{\cancel{(x-1)}(x+3)}$$

$$\frac{4(x+2)}{x+3}$$

(a) (3 points) Determine the x-intercept(s) of the function.

The x intercepts are at x=1 and x=-3 or (1,0) and (-3,0)

Figure 34: Stacia's response to Item 12a on the posttest

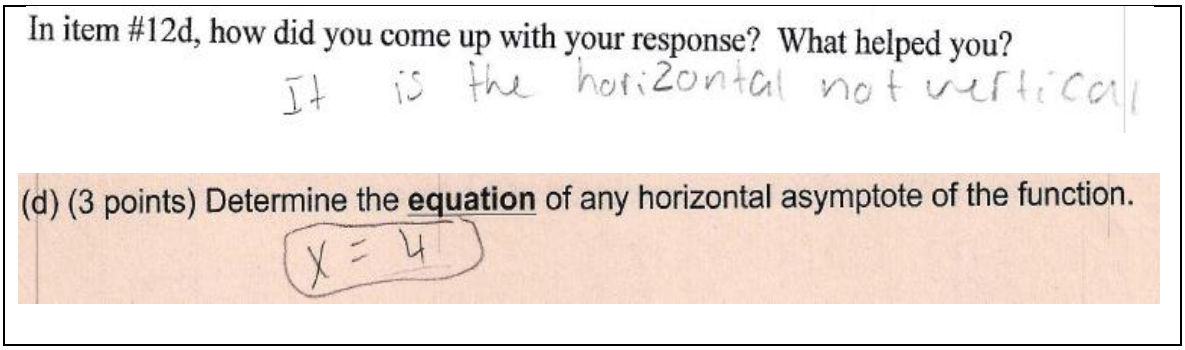
Stacia was asked about Item 12d, "How did you come up with your response? What helped you?' Figure 35 shows Item 12d from the posttest.

$$f(x) = \frac{4(x-1)(x+2)}{(x-1)(x+3)}$$

(d) (3 points) Determine the **equation** of any horizontal asymptote of the function.

Figure 35: Item 12d from the posttest

Stacia replied “It is the horizontal not vertical.” Her work on the posttest on that item is shown below. Her response reveals that she read through the symbols, taking a gestalt view analysis to obtain the degrees of the two expressions. However, instead of writing y = 4, she wrote x = 4. She realized and pointed out the error saying “horizontal not vertical”.


In item #12d, how did you come up with your response? What helped you?

It is the horizontal not vertical

(d) (3 points) Determine the **equation** of any horizontal asymptote of the function.

x = 4

Figure 36: Stacia's response to Item 12d on the posttest

Stacia's interview responses to general definition items. The first interview question for this category of items was "How did you know that f(x) and g(x) are inverses of each other?

$\boldsymbol{f(x) = 3x + 12}$ and $\boldsymbol{g(x) = \frac{x}{3} - 4}$." In Table 67 lines 02:24 through 02:30, Stacia indicates how she knew f(x) and g(x) are inverses by pointing out that the function compositions f(g(x)) and g(f(x)) cancel each other out. She mentioned also in lines 03:07 through 03:17 that the x-values in one of the equation will be y-values in the other. Though correct, neither of these two arguments point to the exact opposite operations in the two functions f(x) and g(x), which would have been a clear instance of structure sense by way of reading through symbolic expressions. Rather, both explanations provided are only procedural.

Table 67: Justification of the general definition in Stacia's work - Item 1c

Item	Interview question	Stacia's response
1c	How did you know that f(x) and g(x) are inverses of each other? $f(x) = 3x + 12$ $g(x) = \frac{x}{3} - 4$	I don't really 02:24 know exactly what it was, but I just felt like they were inverses because 02:30 they canceled out in both problems. Hmm hmm... ok... in both 1A and 1B If you replace, i think, the...or if you plug 03:07 in the values for x and y in one of the equations, then x from one of the equations, the x terms 03:17 will be the y terms for the second equation.

The interview question for Item 2a asked "What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would

you have answered?" The two functions are $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$**.** Her reasoning here about item 2a in Table 68, lines 05:00 through 06:13, is based on the structure of f(x) and g(x) using the 1-to-1 rule in a graphical way. Based on my previous interactions with Stacia, there was no reason to think that she was not thinking about graphs when she mentioned the word parabola and said "imagine them in my head".

Table 68: Justification of the general definition in Stacia's work - Item 2a

Item	Interview question	Stacia's response
2a	What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered? Use the functions $f(x) = x^2 - 5x + 3$ and $g(x) = \sqrt{3x - 8}$ to answer the parts that follow. One of the functions provided above is one-to-one, and the other is not. Which of the functions **is** a one-to-one function?	Well, I just looked at the equations themselves, 05:00 the x to the second means it's gonna be like a parabola, and this one is a radical, so 05:11 for the...yeah, I would just look at them and imagine them in my head umm and see here 05:20 I wrote VTL but I meant HLT for the horizontal line it says so if it touches more than one 05:32 point when you're looking at it from the bottom to the top then it fails the 1 to 1 rule. 05:44 And so the parabola is out of the question and the radical...the radical is usually pass 05:58 the 1 to 1 because there's no U curve or there's no other graph that would make it not pass, 06:13 unless there's like two of them in the same grid.

The interview question for Item 2b ask, "Looking at the inverse function that you came up with, $y = \frac{x^2+8}{3}$, how do you know it is the correct inverse for the original function, $y = \sqrt{3x - 8}$?" Her explanation in Table 69 shows that structure sense was not

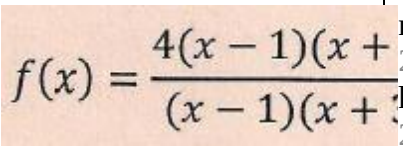
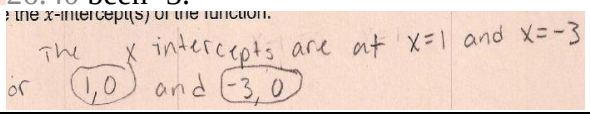
used, neither the general type of the function as opposite structure nor the opposite operations in them. Instead, Stacia talks about substituting x-values in one of the functions and getting the same values for y in the other function. This explanation is an ordered-pair argument; it is based on the fact the x-coordinate and y-coordinate swap position between two inverse functions.

Table 69: Justification of the general definition in Stacia's work - Item 2b

Item	Interview question	Stacia's response
2b	Looking at the inverse function that you came up with $g(x)^{-1} = \frac{x^2+8}{3}$, how do you know it is the correct inverse for the original function? $g(x) = \sqrt{3x-8}$	Umm...I think you can just check by plugging in the values for x and then ...umm...and 09:53 then if you get maybe that same value for the y term and go original, then that would 10:13 mean that's the correct inverse function for it.

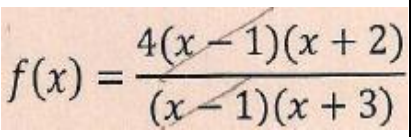
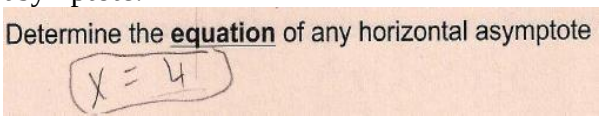
The interview question for Item 12a was "How did you determine the x-intercept? What was/is your thinking process? Why?" As shown in Table 70 line 26:30, Stacia realized that she misread the equation as she missed the common variable factor. She did not read through the symbols in the numerator to see that a -2 would zero out the numerator. So structure sense was not used, or properly used.

Table 70: Justification of the general definition in Stacia's work - Item 12a

Item	Interview question	Stacia's response
12a	How did you determine with the x-intercept(s) from the given equation? What was/is your thinking process? Why?  $f(x) = \frac{4(x-1)(x+}{(x-1)(x+}$	would have to look at since it is a rational equation, you would have to look at the denominator 26:09 for your x intercepts and...I see... what I did wrong...I didn't...I saw that 1 was 26:20 1 x intercept and then -3 but I didn't realize there was a hole here or I did realize, I 26:30 still put that intercept there (inaudible) a hole, I shouldn't have. It would have just 26:40 been -3.  the x-intercept(s) of the function. The x intercepts are at x=1 and x=-3 or (1,0) and (-3,0)

The interview question for Item 12d was "How did you determine the horizontal asymptote? What was/is your thinking process? Why?" Stacia's explanation in Table 71 directly applies to the general definition of structure sense in terms of taking a gestalt (global) view of the equation (a broad view analysis), of the numerator and denominator. The idea of degree is a global concept for algebraic expressions (polynomials). But instead of using $y = 4$, she mistakenly used $x = 4$.

Table 71: Justification of the general definition in Stacia's work - Item 12d

Item	Interview question	Stacia's response
12d	How did you determine the horizontal asymptote from the given equation? What was/is your thinking process? Why?  $f(x) = \frac{4(x-1)(x+2)}{(x-1)(x+3)}$	The horizontal asymptote I put x=4 because If you look at the first Xs here, since they 27:35 are... and you look at the degree also. If you distribute these, you would end up with 27:42 an x squared quadratic formula and then same for this one, it would have the same degree. 27:49 And so when it has the same degree, you just look at the coefficient and whatever the coefficient 27:54 is, you just...that's your horizontal asymptote.  Determine the **equation** of any horizontal asymptote $x = 4$

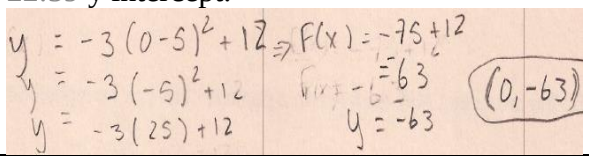
Stacia's responses to SS1 coded items from the posttest reflection. Regarding Item 11c, Stacia was asked, "how did you come up with your response? What helped you?" Item 11c was "determine the vertex of the function $f(x) = -4(x-3)^2 + 16$." Stacia did not reply to the question.

Regarding Item 13a, Stacia was asked "how did you come up with your response? What helped you?" Item 13a states "The grade that Alex receives on a particular test can be modeled by the function below $G(h) = -h^2 + 10h + 75$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours. Stacia did not reply to the question.

Stacia's interview responses to SS1 coded items. The interview question for Item 11b was "Why didn't you/did you choose the last term in the statement $f(x) = -3(x-5)^2 + 12$ as they y-intercept?" Stacia's response in Table 72 lines 21:33 through 22:59

shows that she did not mix up the vertex form and the expanded form of the quadratic equation. She points out, “this is a vertex formula, and the vertex formula provides the vertex point.” Then in lines 22:50 through 22:59, her response clearly indicates her structure sense ability to recognize the vertex format of the quadratic structure; her response also indicates basic understanding of the concept of y-intercept. She said if only x were equal to zero, but x=5, then it is not possible.

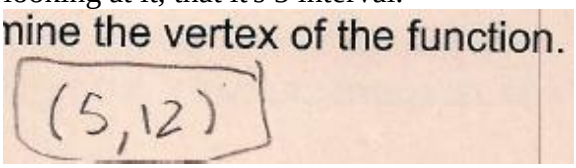
Table 72: Justification of SS1 coded item in Stacia’s work - Item 11b

Item	Interview question	Stacia’s response
11b	Why didn’t you/did you choose the last term in the statement as the y-intercept? Use the function $f(x) = -3(x-5)^2 + 12$ to answer the parts that follow. Determine the y-intercept of the function.	This is a vertex formula, and the vertex formula 21:33 provides the vertex point. So this would--this is the x term and this is the y term but since 21:46 it's in the parentheses it's opposites, so it's actually 5 and 12 for the vertex point. Because the vertex point doesn't necessarily mean the y values, the y intercept because 22:32 it has to touch um...If the maybe if the x were 0, and then they had that value, then 22:50 it would be a y intercept. But since the x was 5, then there's no way that can be the 22:59 y intercept. 

The interview question for Item 11c was “What was/is your thinking process for determining the vertex from the statement $f(x) = -3(x-5)^2 + 12$?” Stacia’s response to the interview question demonstrates her structure sense in recognizing the vertex for the equation. In Table 73 lines 23:27 through 23:39, Stacia says “I just looked” and “you can just tell by looking.” These phrases are key indicators of structure sense or structure sense

approach. Her response clearly shows that she recognized the vertex format of the quadratic function and that this recognition allowed to answer the question properly.

Table 73: Justification of SS1 coded item in Stacia's work - Item 11c

Item	Interview question	Stacia's response
11c	What was/is your thinking process for determining the vertex from the statement? Determine the vertex of the function $f(x) = -3(x-5)^2 + 12$	I just looked at the equation, the vertex 23:31 equation that provides the vertex for you with out having to do any input or do the 23:39 vertex formula; you can just tell by looking at it, that it's 5 interval. 

The interview question for Item 13a was "Using the given equation, $G(h) = -h^2 + 10h + 75$, how did you determine what score the student would earn without studying? What was/is your thinking process? Why?" In Table 74 lines 28:34 through 28:59 she explains her answer by talking about replacing h with zero and obtaining 75 upon calculation; there is no indication of any kind of recognition regarding the equation, its type, or form.

Table 74: Justification of SS1 coded item in Stacia's work - Item 13a

Item	Interview question	Stacia's response
13a	Using the given equation, how did you determine what score the student would earn without studying? What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 10h + 75$, where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours.What score will Alex earn without studying?	For here, so basically, if they didn't study, that's the same thing as saying 0 hours. 28:41 So you just plug in 0 for the h values for the hours spent, and if you do that, what 28:52 you have left is just the seventy-five, so without studying, that is the score he would 28:59 earn. 75

Stacia's responses to SS2-A coded items from the posttest reflection. Regarding Item 4, when asked, "how did you think about decomposing h(x)?", Stacia replied "decompose $\sqrt{x}$ from the whole equation" Her response does not contain sufficient information for analyzing it.

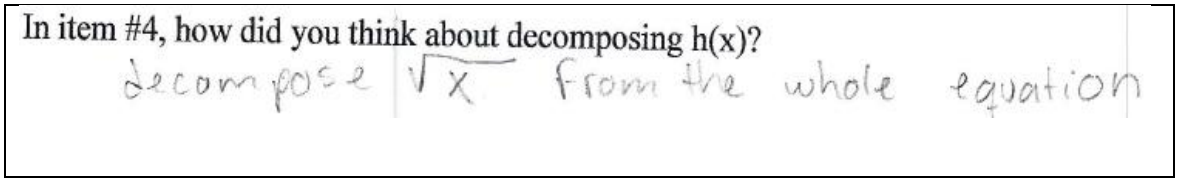
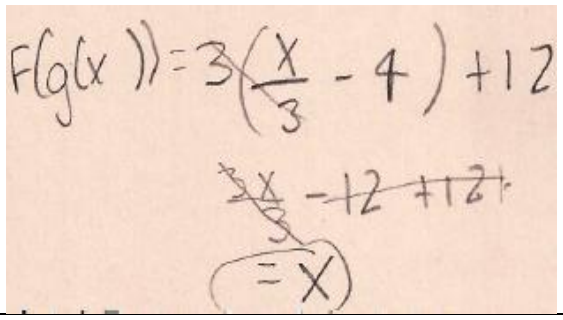

In item #4, how did you think about decomposing h(x)?
decompose $\sqrt{x}$ from the whole equation

Figure 37: Stacia's posttest reflection response about Item 4

Stacia's interview responses to SS2-A coded items. The interview question for Item 1a/b was "Why did you put parentheses in that statement?" (Statement written by the student). Stacia's response in Table 75 lines 01:02 through 01:14 regarding her work for Item 1a/b clearly that shows her structure sense in dealing with a compound term as a single entity. Her structure sense is evidenced starting in line 01:02 by her saying "parentheses just

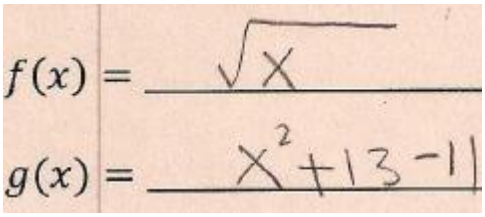
separate that equation into the new one…," as she pointed to the expression of g(x), which was substituted into f(x).

Table 75: Justification of SS2 coded item in Stacia's work - Item 1a/b

Item	Interview question	Stacia's response
1a/b	Why did you put parentheses in that statement? (Statement written by the student) Use the functions $f(x) = 3x + 12$ and $g(x) = \frac{x}{3} - 4$ to answer the parts that follow. Determine $(f \circ g)(x)$. Determine $(g \circ f)(x)$.	Well here it's saying, it's asking us to determine f of g so g has to go g of x has to go into 01:02 f of x, like that, and since x so g of x goes into x of the first equation, so parentheses 01:14 just separates that equation um into the new one in order for us to work it out. 

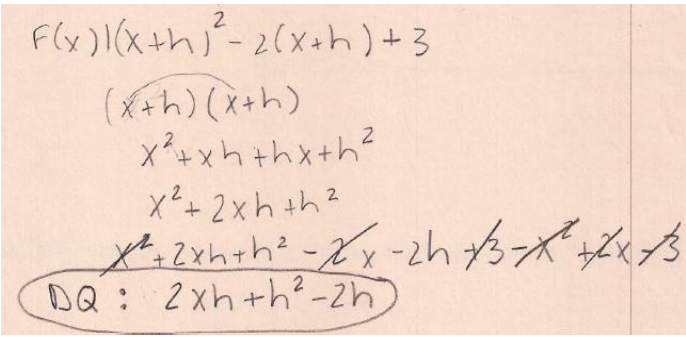
The interview question for Item 4 was "How did you think about decomposing that function, $h(x) = \sqrt{x^2 + 13} - 11$? Which part of the original expression did you think could be a component? Why?" Stacia's response to the interview question regarding Item 4 does not show evidence of dealing with a compound term as a single entity. Her work and comments in Table 76 show that she partially saw the radical term as single entity as it is missing the negative 11 term.

Table 76: Justification of SS2 coded item in Stacia's work - Item 4

Item	Item description	Stacia's response
4	How did you think about decomposing that function? Which part of the original expression did you think could be a component? Why? Suppose that $h(x) = \sqrt{x^2 + 13} - 11$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f\big(g(x)\big)$. $f(x) =$ _____ $g(x) =$ _____	Uh...I thought long trivial, um yeah, I just 11:56 thought about separating the radical from the rest of the equation, so I can have two. 12:07 And so I did the radical and then I did an x, and then I wrote down the quadratic equation 12:20 underneath. That's how I thought about it. Um...but I don't remember if I... 

The interview question for Item 6 was "How did you think about or what helped you get this task right?" Stacia's response to the interview question regarding this item also does demonstrate the ability to deal with a compound term as a single entity. In Table 77, starting in line 17:42, Stacia talks about substituting the (x+h) for x into the expression for f(x) pointing and circling around the compound term (x+h).

Table 77: Justification of SS2 coded item in Stacia's work - Item 6

Item	Interview question	Stacia's response
6	How did you think about or what helped you get this task right? Construct and simplify the difference quotient for the function $f(x) = x^2 - 2x + 3$ For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.	17:42 example, here I just plugged in the (inaudible) ...this part and then the ...ok...I just squared 18:12 this part for this part and then I came up with this equation and then I um subtracted 18:23 here from this equation um...and then I came up with this, but I missed the second part 18:38 for the division by h. So, yeah, I just didn't do it. 

Stacia's responses to SS3 coded items from the posttest reflection. Item 5 asked students to solve $14 - 3|x - 5| = 2$. When asked about this item "how did you think about handling the absolute value bars or solving equation? Why?" Stacia replied that "I subtracted instead of dividing the answers should have been x=1 x = -9." This response acknowledges that she knew the correct symbol manipulation even though she mistakenly did otherwise. However, her response is not sufficient to ascertain her awareness about the rationale for symbol manipulation. Her response in the interview may shed more light on her reasoning.

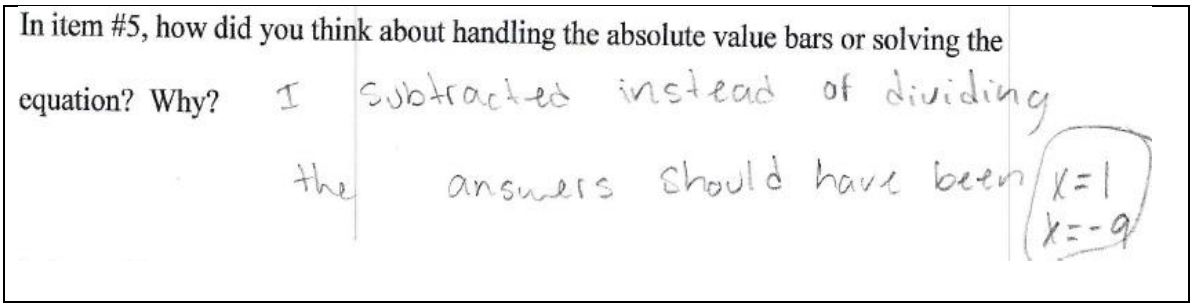

In item #5, how did you think about handling the absolute value bars or solving the equation? Why? I subtracted instead of dividing the answers should have been x=1 x=-9

Figure 38: Stacia's posttest reflection response about Item 5

Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 10h + 75$. When asked this item "how did you come up with your response? What helped you?" Stacia did not reply to this question.

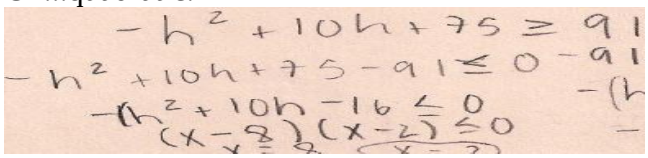
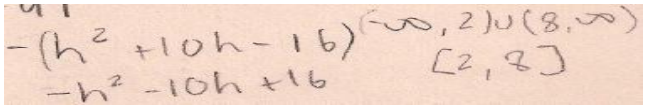
Stacia's interview responses to SS3 coded items. The interview question for Item 5 was "Why did you / why didn't you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars?)." Stacia's response to the interview question regarding this item demonstrates her structure sense in terms of her awareness that some symbol manipulations were necessary in order to properly deal with the absolute value bars. In Table 78 starting line 14:00, Stacia points out that she wanted to get rid of everything else that was outside the absolute value bars. She adds "and then since absolute value just means distance from zero, I just do like plus that or minus that so I can (inaudible)…negative or positive or like both just for distance; I ended up with these two answers." Taking together her response shows awareness that symbol manipulation was necessary in order to make best use of structure.

Table 78: Justification of SS3 coded item in Stacia's work - Item 5

Item	Interview question	Stacia's response
5	Why did you / why didn't you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars?) Solve the absolute value equation below. $14 - 3\|x - 5\| = 2$	I...careless mistake, I think. I thought it was...at the moment I thought it was this 13:37 was just minus 3 and then here, I didn't think it was multiplying, so I just added the three 13:49 to the other side. But I should have divided by the negative 3 and so I would have ended 14:00 up with a different x answer, x value. I got rid of everything else outside 15:04 of the bars, and then since absolute value just means distance from zero, I just did 15:25 like plus that or minus that so I can (inaudible)...negative or positive or like both just for distance; 15:42 I ended up with these two answers.

The interview question for Item 13b was "What was/is your thinking process? Why?" Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 10h + 75$. Stacia's response in the interview as shown in Table 79 lines 29:54 through 33:22 shows awareness that symbol manipulation was necessary in order to make best use of structure. In lines 30:11 through 30:21, she mentions her rationale for subtracting 91 on both sides of the inequality; "I just substracted the 91 to the other side os it could be equal to zero." Her statement here indicates some awareness that the other side of quadratic structures needed to be zero. Her next statement provides further evidence of structure sense in terms of manipulating symbols for the purpose of taking advantage of structure. The evidence is in lines 30:39 where she said "I set apart the negative so I could have just a regular quadratic equation and I factored it and I got these two answers, x=8, x=2." In this case, factoring is the best use of structure for which she performed the symbol manipulations.

Table 79: Justification of SS3 coded item in Stacia's work - Item 13b

Item	Interview question	Stacia's response
13b	What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 10h + 75,$ Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)	So, when I first looked at what it was asking, 29:54 so it was asking us to solve the inequality where g of x is greater than 91. And so, I 30:05 just set it up like that, like I set up the original equation and Iset it up greater than 30:11 or equal to 91, and then I just subtracted the 91 to the other side so it could be equal 30:21 to zero. And so I ended up with this equation here. And then ... 30:33 Let me move this a little bit so the camera can catch what you're pointing to...yeah, 30:39 that's perfect. And then I...let's see...I set apart the 30:51 negative so I could have just a regular quadratic equation and I factored it and I got these 31:05 two answers, x=8, x=2. And then here...I think I just wrote it twice...and then I...and so 31:32 it gave me two answers and after that I think I just checked with my calculator if the numbers, 31:39 these numbers, (inaudible) picked a number in this negative infinity to 2, to see if 31:49 it would make the statement true, greater than or equal to 91, and so I chose that and 31:58 then I did the same for the rest of the equations. Ithink I just did it by instinct; it looked 33:04 right. But yeah...I'm sure there's a reason for it, I just don't remember or maybe it 33:16 was so I can get an equation like this What do you mean like this? 33:22 Like just like this quadratic equation. Ok...quadratic.  

Summary. I summarize the results and analysis on Stacia's work both in regards to the four prefigured codes in terms of development of structure sense and in regards to emergent codes in terms of algebraic proficiency. The prefigured codes correspond to the four definitions of structure sense used for the results and analyses. I used those prefigured codes to sort and organize posttest items. Then to make notes about student responses on the transcripts regarding structure sense and algebraic proficiency, I used eight other codes that are listed in Table 24. For structure sense, those codes are identification (IDT), language (LAN), and rationale (REA). For algebraic proficiency, they are procedure (PROC), graph (GRP), communication (COM), connections (CON), and manipulation (MAN).

First, for the general definition of structure, Stacia showed development of structure sense by thinking about algebraic expressions in terms of their graphs. However, Stacia's development of structure sense did not involve recognizing or connecting inverse structures such as quadratic function versus a square root function. So for this part of the definition, she did not show much development in structure sense. Second, for the first part of Hoch and Dreyfus' (2006) definition, Stacia showed development of structure sense in recognizing structures in their simplest form, though she only made use of that structure in instances that required the use of structure sense or in instances where the use of structure sense is significantly more efficient. Her development in structure sense in regards to that definition is strong. Third, for the SS2-A part of Hoch and Dreyfus' (2006) definition, Stacia showed development of structure sense by using words/phrases like "separate", "substitute" and by circling around parts of algebraic expressions (e.g., x+h). Her structure sense was consistent for the most part, except on the decomposition task. Fourth, for the SS3 part of Hoch and Dreyfus' (2006) definition, Stacia showed development of structure sense by mentioning that

she recognized the structure of the equations and by mentioning how to handle those structures effectively. This development is consistent through both items for the SS3 part of the definition.

In regards to emergent codes, a few of her explanations were procedural for items which were readily interpretable by structure sense or by reading though symbols (E.g., knowing to expect a quadratic function as the inverse of a square root function). Her responses to the interview questions and her work on the posttest showed some flexibility in the way she performed tasks. For example, her reasoning involved procedural, graphical, and structural approaches, but not necessarily on the same tasks. Her reasoning involved several habits that I modeled during the intervention like circling around algebraic expressions to highlight them and reason about them.

Case 4: Ellen

Ellen, a Hispanic woman, was a first semester college student pursuing an associates degree in science. Her plan was to get a bachelor's degree in nutrition science and become a clinical dietitian. She had a previous bachelor's degree in marketing. In the past, she had taken remedial mathematics courses and it had been 10 years since she had taken an algebra course. This was her first time taking precalculus in college. She was a full-time student, taking 16 credit hours. She was not working. She rated her access to computers and the internet as very good and rated her study skills as average. She felt she was more prepared for her classes because this was her second degree. Although she felt challenged because she was now taking classes in a different country and in a different language, she felt that her work ethic ensured that she learned all the material and helped her maintain a 4.0 GPA. Ellen stated that she had always enjoyed math and in high school, her highest grades

were always in math. She felt that working in the business field also improved her analytical skills. Ellen did not think that learning mathematics would improve her career prospects or that she would need it for what she would study later on. She also did not think that learning many things in mathematics would help her get a job. However, she enjoyed reading about mathematics, she looked forward to her mathematics lessons, she did mathematics because she enjoyed it, and she was interested in the things she learned in mathematics. She strongly felt that making an effort in mathematics was worth it because it would help her in the work she wanted to do later on. In terms of her ability to do certain mathematics tasks, Ellen felt that she was not at all confident that she could find the actual distance between two places on a map with a 1:10,000 scale. She was not very confident that she could calculate how many square feet of tile would be needed to cover a floor, nor was she very confident that she could calculate the gas mileage of a car. However, she was confident that she could use a train schedule to figure out how long it would take to get from one place to another and that she could understand graphs presented in newspapers. She was very confident that she could calculate how much cheaper a TV would be after a 30% discount, that she could solve an equation like $3x+5=17$, and that she could solve an equation like $2(x+3)=(x+3)(x-3)$.

In her Midterm Reflection Questionnaire, Ellen indicated that she had not used a personal tutor for the course, not had she participated in study groups. She had not used any campus resource, either. She was still enrolled full-time, taking 16 credit hours, and she was not working. She spent about 4 hours a week on average doing MAT 171 homework, reading class notes, and watching instructional videos to learn the concepts in the course. She did not indicate anything that may have affected her learning experience outside of what took place in the class.

Ellen indicated in her Final Interview that MAT 171 was enjoyable yet challenging. She said, “there were some moments that I had to really [force] myself to think more critically and not only learn the mecanic [sic] of solving a problem but also the foundation. I have realized that when you really know the concept, you can apply it to any problem and solve it without any fixed mecanism [sic].” She felt that being older and more mature contributed to her discipline of studying. Although the content in this class was more difficult than that of her previous math courses, the challenge kept her focused and disciplined.

Ellen’s responses to general definition items from the posttest reflection. Ellen was asked the following question regarding Item 2a “how do you know which function has an inverse and which one doesn’t?” The two functions are $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$. As shown in Figure 38, Ellen replied that “because the function f(x) is a quadratic function, and this kind of function cannot have an inverse because it is not a one-to-one function.” Ellen’s structure sense is informed by the graph of the quadratic function. Sometimes students think about one-to-one functions within the context of numerical values with repeating x-values or within equations containing x-squared terms. However, there was not enough reason to think that she was not thinking graphically in this situation.

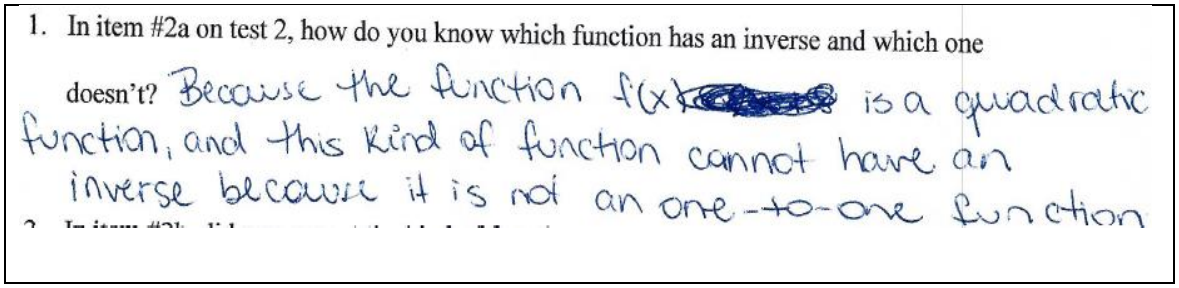
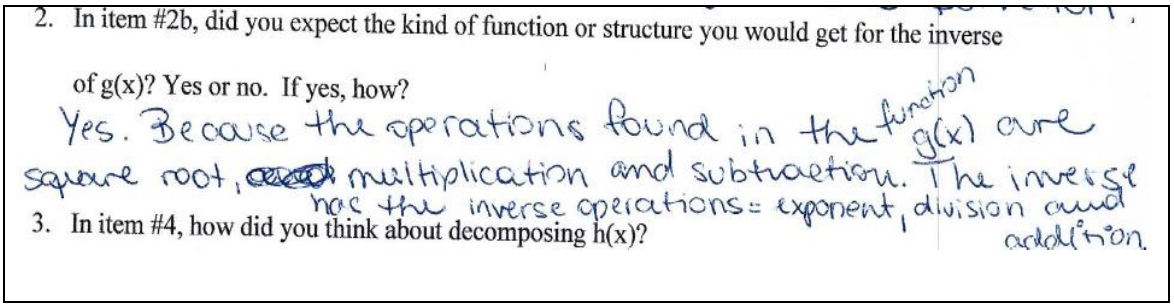

1. In item #2a on test 2, how do you know which function has an inverse and which one doesn’t? Because the function f(x) is a quadratic function, and this kind of function cannot have an inverse because it is not an one-to-one function

Figure 39: Ellen's posttest reflection response about Item 2a

Ellen was asked the following question regarding Item 2b “did you expect the kind of function or structure you get for the inverse of g(x), $g(x) = \sqrt{2x - 7}$? Yes or no. If yes, how?” As shown in Figure 40, Ellen replied “Yes. Because the operations found in the function g(x) are square root, multiplication and subtraction. The inverse has the inverse operations = exponent, division and addition.” Her answer shows that she reads through the expressions and pays attention to how those expressions are put together. Her answer is consistent with the teaching intervention.

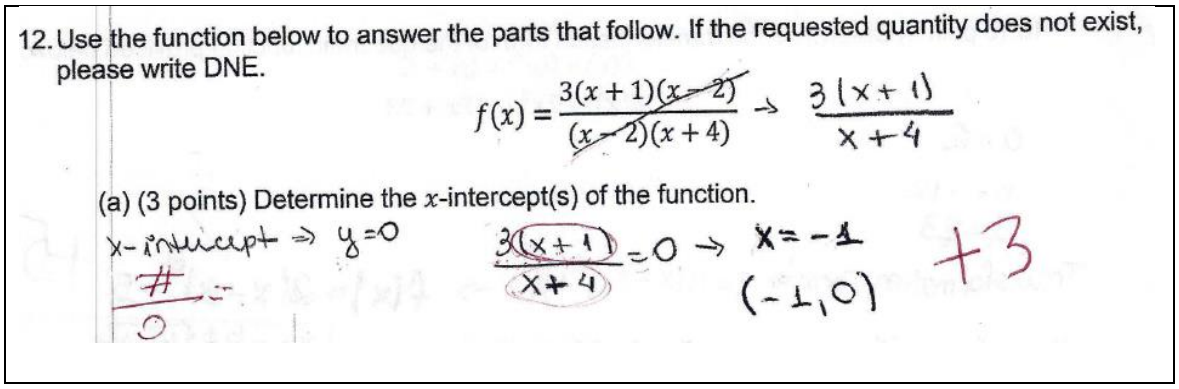

2. In item #2b, did you expect the kind of function or structure you would get for the inverse of g(x)? Yes or no. If yes, how?

Yes. Because the operations found in the function g(x) are square root, multiplication and subtraction. The inverse has the inverse operations = exponent, division and addition.

3. In item #4, how did you think about decomposing h(x)?

Figure 40: Ellen's posttest reflection response about Item 2b

Ellen was asked about Item 12a, “how did you come up with your response? What helped you?” Figure 41 shows her response to that item on the posttest.


12. Use the function below to answer the parts that follow. If the requested quantity does not exist, please write DNE.

$$f(x) = \frac{3(x+1)\cancel{(x-2)}}{\cancel{(x-2)}(x+4)} \rightarrow \frac{3(x+1)}{x+4}$$

(a) (3 points) Determine the x-intercept(s) of the function.

x-intercept ⇒ y=0

$\frac{\#}{0} =$

$\frac{3(x+1)}{x+4} = 0 \rightarrow x = -1$

$(-1, 0)$

+3

Figure 41: Ellen's response to Item 12a on the posttest

As shown in Figure 42, Ellen wrote “I knew that to find the x-intercept the ordered pair is always the x-value and y = 0. So I substituted y by 0 and solved for x.” This answer seems to indicate her structure sense in terms of knowing what part of equation to focus upon, in this case the numerator.

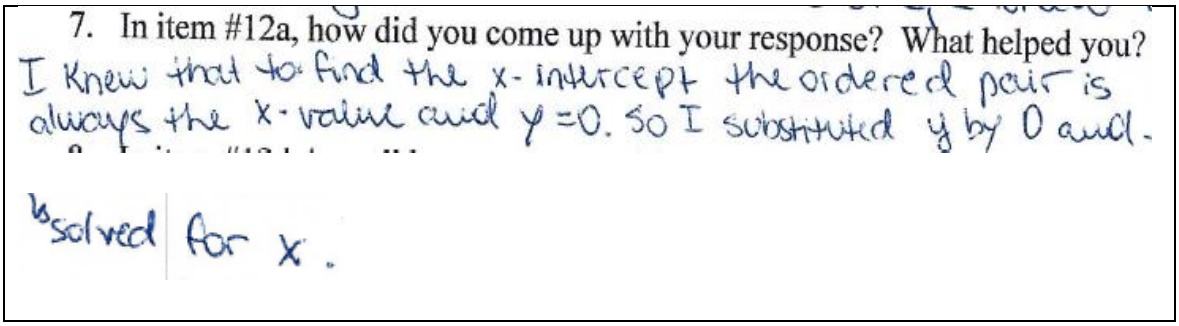

7. In item #12a, how did you come up with your response? What helped you?
I Knew that to find the x-intercept the ordered pair is always the x-value and y=0. So I substituted y by 0 and solved for x.

Figure 42: Ellen's posttest reflection response about Item 12a

When asked about Item 12d, “How did you com up with your response? What helped you?’ Figure 43 shows the equation of the items of question 12.

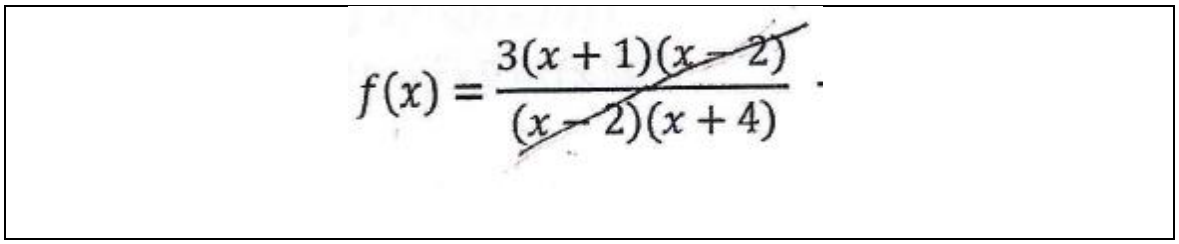

$$f(x) = \frac{3(x+1)\cancel{(x-2)}}{\cancel{(x-2)}(x+4)}.$$

Figure 43: Equation for Item 12 on the posttest

As shown in Figure 44, Ellen replied “I first analyzed the degree of the numerator and denominator, once I found it was the same, I divided the LC of the numerator by the LC of the deominator.” LC stands for leading coefficient. Her work on the posttest on that item is

shown below. Her response reveals that she read through the symbols, taking a gestalt view analysis to obtain the degrees of the two expressions.

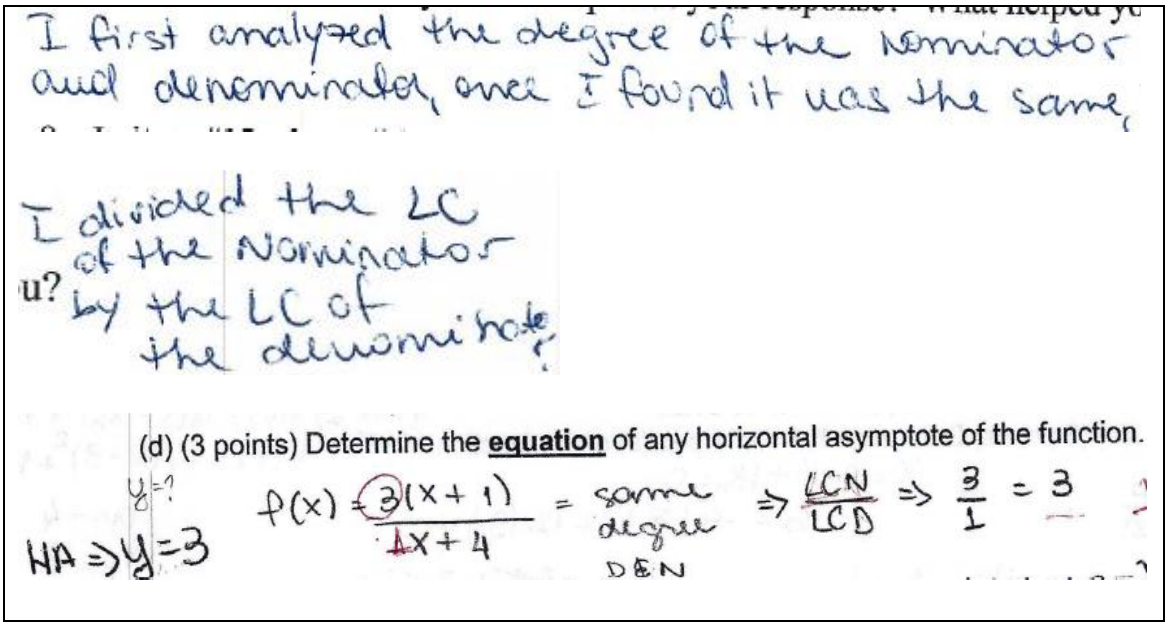


Figure 44: Ellen's posttest reflection response about Item 12d

This answer shows Ellen's structure sense in terms of reading through the expressions and extracting the right details (e.g., degrees, leading coefficients).

Ellen's interview responses to general definition items. The first interview question for this category of items was "How did you know that f(x) and g(x) are inverses of each other? ($\boldsymbol{f(x) = 2x + 8}$ and $\boldsymbol{g(x) = \frac{x}{2} - 4}$). In Table 80 lines 02:33 to 02:54, Ellen refers to the function compositions f(g(x)) and g(f(x)) both yielding x as result and proof that f and g are inverses of each other. Then starting about line 03:00 down to line 03:20, she mentions that she could also tell that they are inverses of each other because of the operations and numbers involved in them. She continues by saying you multiply x in one function and in the other you divide x by 2 and you have plus 8 in one function and minus 4 in the other.

Her explanation points to the exact opposite operations in the two functions f(x) and g(x) below, a clear instance of structure sense by way of reading through symbolic expressions. Her second response here in the interview is consistent with her response on the posttest reflection and with the teaching of the intervention.

Table 80: Justification of the general definition in Ellen's work - Item 1c

Item	Interview question	Ellen's response
1c	How did you know that f(x) and g(x) are inverses of each other? $f(x) = 2x + 8$ $g(x) = \frac{x}{2} - 4$	because for the letter A I 02:33 need F of G of X and then I got X as 02:37 an answer and the same for letter B G of 02:41 f of X I got the same result so I can't 02:46 remember if I have these 02:49 problem here and the answer is always 02:52 the same then that means that they are 02:54 inverse of each other I think also I could tell the inverse 03:03 of each other because of the operations 03:05 you have here so you have to multiply X 03:08 and here you have X dividing dividing by 03:12 2 and you have plus 8 and you have minus 03:16 4 so those are inverse of inverses 03:20 operations yeah I think that's what and also about the 03:45 both answers being X there was also 03:49 something that you explained before the 03:51 test on classes inverse if you get the 03:55 same if you have this situation when you 03:59 have F of G of X and then G of f of X 04:02 you have if you have the same if you 04:05 have both X or whatever is the variable 04:08 that means that they are inverse of each 04:10 other I think

The interview question for Item 2a asked "What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered?" The two functions are $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$. Her

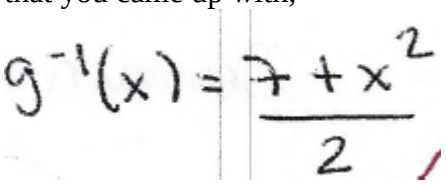
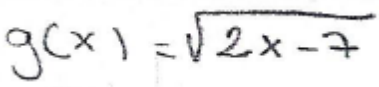
reasoning here about Item 2a in Table 81, lines 4:51 through 5:51, is based on the structure of f(x) and g(x) for applying the 1-to-1 rule in a graphical way. She stated in lines 4:59 that the curve is a parabola.

Table 81: Justification of the general definition in Ellen's work - Item 2a

Item	Interview question	Ellen's response
2a	What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered? Use the functions $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ to answer the parts that follow. One of the functions provided above is one-to-one, and the other is not. Which of the functions **is** a one-to-one function?	so I knew that only one is 04:51 one-to-one because of the type of 04:53 function so I know that a quadratic 04:56 function cannot be one-to-one because 04:59 the curve it's always a parabola so you 05:03 will have the line crossing the same it 05:14 fails the horizontal line test so you 05:18 have for example here a parabola here so 05:21 you have two points crossing the same 05:25 the same line so then it cannot be a 05:29 one-to-one which one it's only if you 05:31 you have only one point only one 05:33 crossing just once at this point for 05:36 example like this so the second one it's 05:39 um it's a radical function which is 05:43 always something like this so this can 05:46 be in one-to-one so that's why I knew 05:48 that the quadratic function cannot be 05:51 one-to-one

The interview question for Item 2b asked "Looking at the inverse function that you came up with $y = \frac{x^2+7}{2}$ how do you know it is the correct inverse for the original function, $y = \sqrt{2x-7}$?" Her explanation in Table 82 shows that structure sense was used. Pointing to opposite operations on single component and on compound component in lines 10:07 through 10:20 is evidence of her structure sense in terms of reasoning about the way expressions are put together.

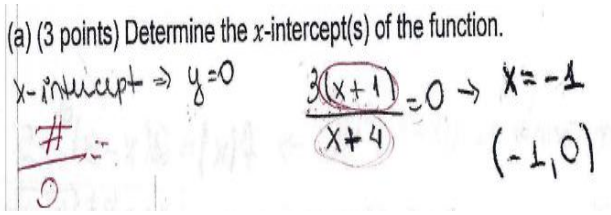
Table 82: Justification of the general definition in Ellen's work - Item 2b

Item	Interview question	Ellen's response
2b	Looking at the inverse function that you came up with,  how do you know it is the correct inverse for the original function? 	I think is the same as here because the 10:06 operations are the inverse of each 10:07 other so here I have two times X and 10:12 here I have everything divided by two 10:14 and then instead of subtracting a have 10:17 addition and then instead of square root 10:20 I have just x squared so it's all 10:25 inverse operations of this

The interview question for Item 12a was "How did you determine the x-intercept? What was/is your thinking process? Why?" Table 81 shows the equation for Item 12a. Her explanation directly applies to the general definition of structure sense in terms of the way the whole expression or part it corresponds to specific concepts. For instance, in Table 83 line 28:16 she refers to the whole expression equaling to zero; in lines 28:19 to 28:25, she mentions that setting the denominator equal to zero corresponds to an undefined expression.

In line 28:28, she says she know then to set the numerator equal to zero. The writing in red ink are notes that Ellen made as she replied to interview questions, not when she actually took the test. And although, she misspoke about 3(x+1) being 3x+1, she still obtained the correct x-intercept because she used structure to complete the task. Her final comment starting about lines 29:53 through 29:57 contains further evidence of her structure sense and of adherence to the teaching of the intervention. More specifically, she mentions "so just looking at it" in line 29:53. Extracting information from symbolic expressions by merely looking at them was a hallmark of the teaching intervention.

Table 83: Justification of the general definition in Ellen's work - Item 12a

Item	Interview question	Ellen's response
12a	How did you determine with the x-intercept(s) from the given equation? What was/is your thinking process? Why? $f(x) = \frac{3(x+1)(x-2)}{(x-2)(x+4)}$	I always know that 27:51 the x intercept y is always equal to 27:55 zero so then I had a function here and 27:57 after finding the holes so I canceled 28:02 the repeated factors in the denominator 28:05 and no me later and then I got the the 28:09 final expression here so then I just I 28:16 did these expression equals to zero and 28:19 then I knew that anything that is 28:22 divided by zero is undefined so then I 28:25 could not just use the denominator so 28:28 that's why I use numerator and then so 28:30 I just need three times X plus one 28:33 equals to zero and which is x equals to 28:36 negative one I know that it's X minus 1 29:53 it's always the opposite so just looking 29:55 at it I knew that the x intercept would 29:57 be the opposite of it 

The interview question for Item 12d was "How did you determine the horizontal asymptote? What was/is your thinking process? Why?" Table 82 shows the equation for that item. Ellen's response regarding her work on Item 12d in the interview is the same as in the posttest reflection. Her explanation directly applies to the general definition of structure sense in terms of taking a gestalt (global) view of the equation (a broad view analysis), first

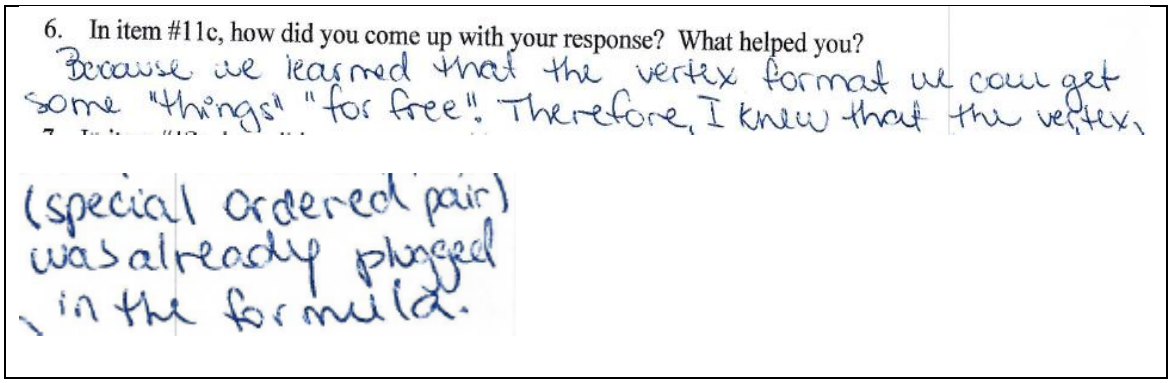
through the numerator and then through the denominator, as shown in Table 72 lines 31:09 through 31:13. During the intervention, students were taught that the degree of an expression is a global attribute.

Table 84: Justification of the general definition in Ellen's work - Item 12d

Item	Interview question	Ellen's response
12d	How did you determine the horizontal asymptote from the given equation? What was/is your thinking process? Why? $f(x) = \frac{3(x+1)(x-2}{(x-2)(x+4}$	I thought 30:33 about the rules of finding and I knew 30:43 that I the first reason is because I 30:48 knew that the degree was the same so 30:50 here you have these two equations and 30:53 then the degree here is the same so when 30:56 you have this situation then you you get 30:58 the leading coefficient from the 31:00 denominator and then from the 31:02 denominator which here was 1 and then 31:05 you just do these leading coefficient 31:09 coefficient after the 31:10 numerator divided by the leading coefficient 31:13 of the denominator so that's what I take 31:15 here 3 divided by 1 and it's 3

Ellen's responses to SS1 coded items from the posttest reflection. This part of the results describes Ellen's explanation in the posttest reflection for her work on the posttest. Students were asked only about Items 11c and 13a on the posttest reflection. The other item in that rubric is 11b. Regarding Item 11c, Ellen was asked, "how did you come up with your response? What helped you?" As shown in Figure 45, Ellen replied "Because we learned that the vertex format we can get some 'things' 'for free'. Therefore, I know that the vertex,

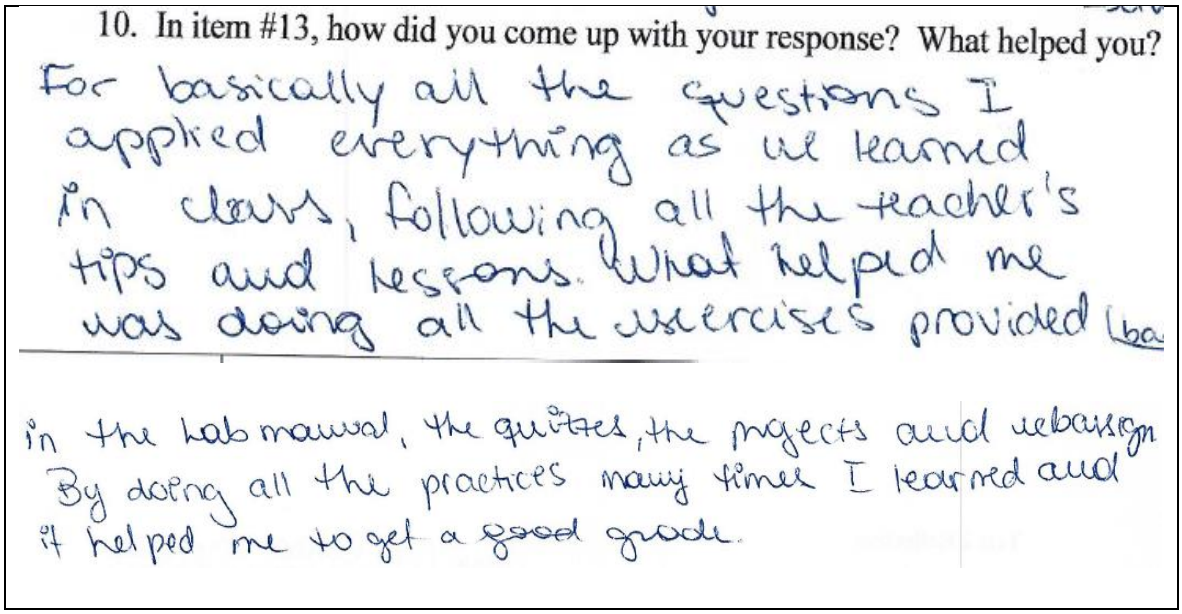
(special ordered pair) was already plugged in the formula." Her answer indicates that she recognize that structure of the quadratic function in vertex form, which is a clear instance of the structure sense labeled as SS1.


6. In item #11c, how did you come up with your response? What helped you?

Because we learned that the vertex format we can get some "things" "for free". Therefore, I knew that the vertex, (special ordered pair) was already plugged in the formula.

Figure 45: Ellen's posttest reflection response about Item 11c

Regarding Item 13a, Ellen was asked, "how did you come up with your response? What helped you?" As shown in Figure 46, Ellen replied "For basically all the questions I applied everything as we learned in class, following all the teacher's tips and lessons. What helped me was doing all the exercises provided in the lab manual, the quizzes, the projects and the webassign. By doing all the practices many times I learned and it helped me to get a good grade". Her response is general, but it speaks to her attentiveness and adherence to the intervention.


10. In item #13, how did you come up with your response? What helped you?

For basically all the questions I applied everything as we learned in class, following all the teacher's tips and lessons. What helped me was doing all the exercises provided in the lab manual, the quizzes, the projects and webassign. By doing all the practices many times I learned and it helped me to get a good grade.

Figure 46: Ellen's posttest reflection response about Iitem 13

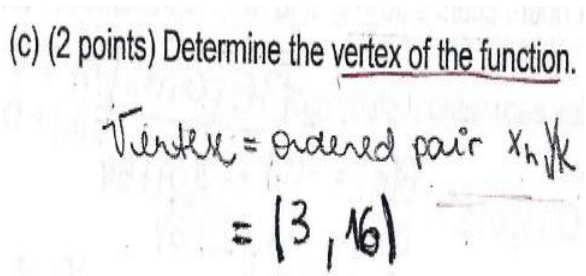
Ellen's interview responses to SS1 coded items. This part of the results describes Ellen's explanation in the interview for her work on the posttest. The interview question for Item 11b was "Why didn't you/did you choose the last term in the statement $f(x) = -4(x-3)^2 + 16$ as they y-intercept?" As shown in Table 85 lines 23:08 through 23:15, Ellen's response to item 11b on the posttest clearly shows a mix up between the vertex form and the expanded form of the quadratic equation. However, during the interview she realized her error on the posttest in regards to recognizing and using structure appropriately.

Table 85: Justification of SS1 coded items in Ellen's work - Item 11b

Item	Interview question	Ellen's response
11b	Why didn't you/did you choose the last term in the statement as the y-intercept? Use the function $f(x) = -4(x-3)^2 + 16$ to answer the parts that follow. Determine the y-intercept of the function.	I used instead of 22:45 converting to the the other format and 22:49 then get the y intercept which is x 22:55 equals zero so I would just put zero 22:58 work instead of here instead of X I was 23:03 confused and then I used Y I used a y 23:08 the the vertex I mean I shouldn't 23:11 because it's the max max my point for a 23:15 minimum or again so I should I should 23:18 have converted to these format and then 23:21 solve for y when X is equal to zero I also understood about the 25:11 vertex formula so here I think I didn't 25:13 really I didn't know the they come the 25:16 concept 100% so maybe that's why I got 25:19 wrong (b) (3 points) Determine the y-intercept of the fu y - intercept = C therefore = (0, 16)

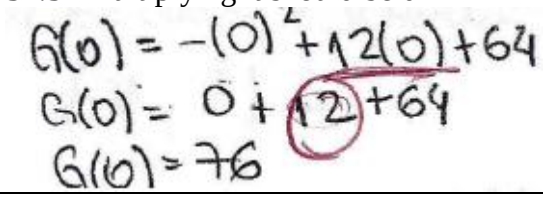
The interview question for Item 11c was "What was/is your thinking process for determining the vertex from the statement $\boldsymbol{f(x) = -4(x-3)^2 + 16}$?" Ellen's response to Item 11c demonstrates her structure sense in recognizing the vertex for the graph of the equation. In Table 86 lines 26:07 through 26:20 she said "I knew that when we have these format we have something that is given so which is the vertex pair so I jus needed to look at here…" Those two things she mentioned were habits I modeled throughout the intervention—recognizing format and knowing with information can be gleaned by merely looking at an expression.

Table 86: Justification of SS1 coded items in Ellen's work - Item 11c

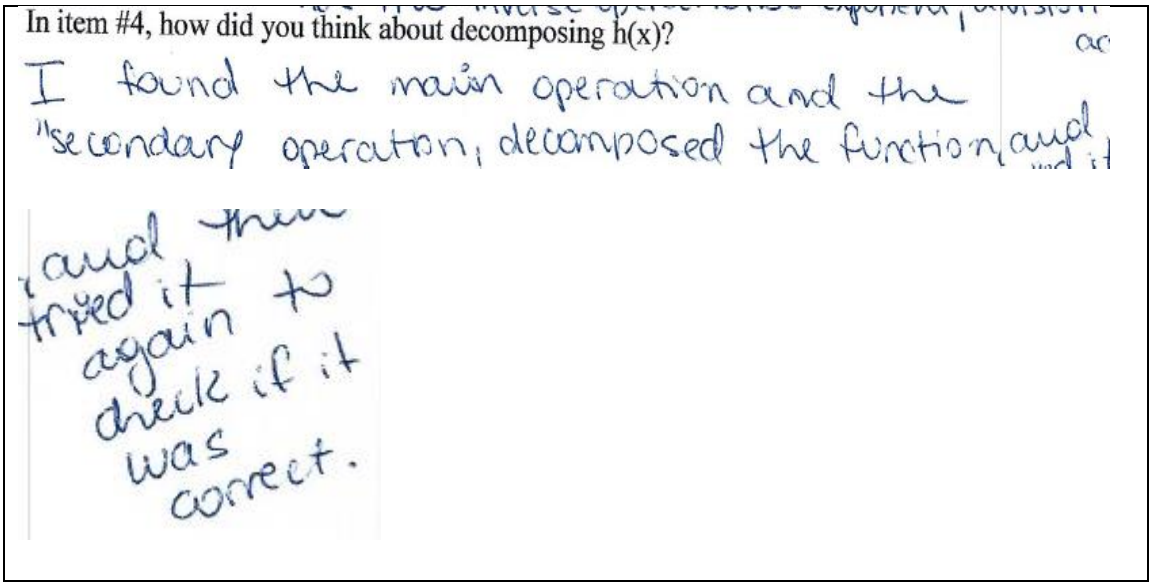
Item	Interview question	Ellen's response
11c	What was/is your thinking process for determining the vertex from the statement? Determine the vertex of the function $f(x) = -4(x-3)^2 + 16$	so I knew this was the the 25:57 vertex format mm-hm and then when asking 26:01 about the vertex of the function I know 26:03 it's the special ordered pair X H and Y 26:07 and then I knew that when we have these 26:12 format we have something that is given 26:15 so which is the vertex (inaudible) so I just 26:18 needed to look at here and 26:20 and get the information from the date 26:22 from the function so I already knew that 26:25 it was 16 and 3 

Item 13a states "The grade that Alex receives on a particular test can be modeled by the function below $G(h) = -h^2 + 12h + 64$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours. What score will Alex earn without studying?" Ellen's response to item 13a on the posttest and regarding Item 13a in the interview are procedural because there is no reference to the form of the quadratic equation. She explains her answer by talking about plugging in zero; in Table 87 lines 31:53 through 32:31 there is no indication of any kind of recognition regarding the equation, its type, or form.

Table 87: Justification of SS1 coded items in Ellen's work - Item 13a

Item	Interview question	Ellen's response
13a	Using the given equation, how did you determine what score the student would earn without studying? What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 12h + 64$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours.What score will Alex earn without studying?	that means he 31:53 didn't spend any our study so then here 31:56 H is equals number of hours spent 32:00 studying so then I knew that H was 32:03 equal to zero because without studying 32:05 math I mean he didn't study at all so 32:08 it's no any hours so that's what I there 32:15 are so that's what I wrote here is 32:16 without studying means that he will 32:19 study for zero hours so then it just 32:22 solved the equation saying G of 0 is 32:27 equals to 10 and then here I got room I 32:29 just didn't pay attention that was 32:31 multiplying it should be 0 

Ellen's responses to SS2-A coded items from the posttest reflection. Students were asked only about item 4 on the posttest reflection. The other items in that rubric are 1a/b and 6. Regarding Item 4, Ellen was asked, "how did you think about decomposing h(x), $h(x) = \sqrt{x^2 + 6} - 12$?" As shown in Figure 38, Ellen replied "I found the main operation and the secondary operation, decomposed the function, and then tried it again to check if it was correct." It is inconclusive whether Ellen is dealing with a compound term as an entity. However what he mentions is in line with that type of thinking and the teaching of the intervention.


In item #4, how did you think about decomposing h(x)?

I found the main operation and the "secondary operation, decomposed the function, and tried it again to check if it was correct.

Figure 47: Ellen's posttest reflection response about Item 4

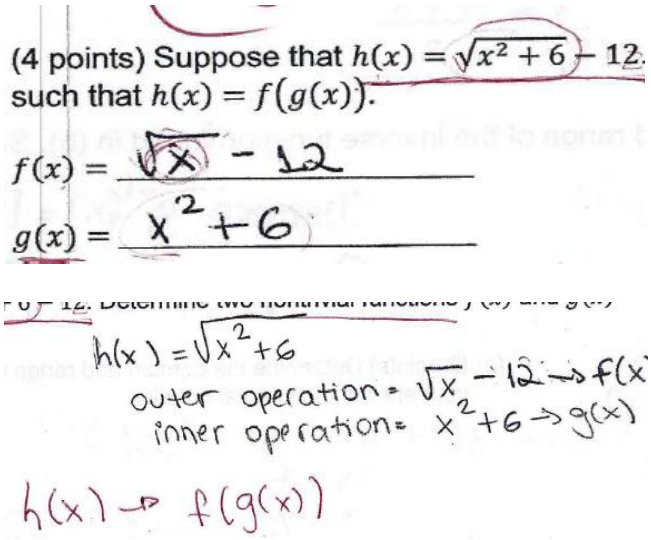
Ellen's interview responses to SS2-A coded items. This part of the results describes Ellen's explanation in the interview for his work on the posttest. The interview question for Item 1a/b was "Why did you put parentheses in that statement?" (Statement written by the student). Ellen's response in Table 88 lines 01:09 through 01:20 regarding her work for Item 1a/b clearly that shows her structure sense in dealing with a compound term as a single entity. This structure sense ability is evidenced starting in line 01:12 by his use of the phrase "so if I if I hadn't used parentheses then I you'd only multiply two by x over two and not the whole expression." Albeit this explanation is procedural in nature.

Table 88: Justification of SS2-A coded items in Ellen's work - Item 1a/b

Item	Interview question	Ellen's response
1a/b	Why did you put parentheses in that statement? (Statement written by the student) Use the functions $f(x) = 2x + 8$ and $g(x) = \frac{x}{2} - 4$ to answer the parts that follow. Determine $(f \circ g)(x)$. Determine $(g \circ f)(x)$.	okay so because you have the first 00:51 function f of X and then you need to 00:55 determine F of G of X so I substituted 01:03 these I mean the X mm-hmm by 01:09 everything that I have here so if I if I 01:12 hadn't used parentheses then I you'd 01:15 only multiply two by x over two and not 01:20 the whole expression so that's why I 01:22 used $f(g(x)) = \frac{2}{1}\left(\frac{x}{2} - \frac{4}{1}\right) + 8$ $= \frac{\cancel{2}x}{\cancel{2}} \cancel{-8} \cancel{+8}$ $= \underline{x}$

The interview question for Item 4 was "How did you think about decomposing that function, $h(x) = \sqrt{x^2 + 6} - 12$? Which part of the original expression did you think could be a component? Why?" Ellen's response to the interview question regarding Item 4 clearly demonstrates her structure sense ability to deal with a compound term as a single entity. This ability is evidenced in Table 89 starting at line 12:36 by her explanation "everything that was inside this square root I defined as X", which combines several terms as one, x, as x=sqrt(x^2+6).

Table 89: Justification of SS2-A coded items in Ellen's work - Item 4

Item	Item description	Ellen's response
4	How did you think about decomposing that function? Which part of the original expression did you think could be a component? Why? Suppose that $h(x) = \sqrt{x^2+6} - 12$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$. $f(x) =$ _____ $g(x) =$ _____	so I think what I 12:15 thought here was the main operation here 12:19 was where I have the variable I think 12:23 mm-hmm so then everything that was 12:36 yes everything that was inside this 12:39 square root I defined as X and then 12:43 minus 2 I opened then as it's like a 12:52 secondary operation I think so then I 12:55 used x squared plus 6 so I think what I 13:01 did was what is the main operation here 

The interview question for Item 6 was "How did you think about or what helped you get this task right?" Item 6 required students to construct and simplify the difference quotient for $f(x) = x^2 - 3x + 2$. Ellen's response to the interview question regarding Item 6 also demonstrates her structure sense ability to deal with a compound term as a single entity. In Table 90, starting at line 18:57 she talks about the terms of difference quotient equation in terms of pieces pointing to the parentheses and square brackets she used to separate the pieces to which she refers.

Table 90: Justification of SS2-A coded items in Ellen's work - Item 6

Item	Interview question	Ellen's response
6	How did you think about or what helped you get this task right? Construct and simplify the difference quotient for the function $f(x) = x^2 - 3x + 2$. For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.	so I 18:57 did the first piece I had the function 18:59 here so I did the first piece that is f 19:02 of X plus h and then I got this result 19:05 and then I already had f of X so then 19:09 the second thing I did was subtract each 19:12 other so then I used the result that I 19:15 got here minus f of X and then I got 19:20 this year and then finally I did the 19:23 different quotient is just dividing by H big it was basically 20:22 easier to me to do each each piece and 20:27 then put them together can I can break 21:13 it down pieces and then put them 21:14 together it's easier

Ellen's responses to SS3 coded items from the posttest reflection. Regarding Item 5, Ellen was asked "how did you think about handling the absolute value bars or solving equation? Why?" Item 5 asked students to solve $14 - 3|x - 5| = 2$. As shown in Figure 47, Ellen replied that "I thought it as the equation I should first isolate and then solve for it." Her answer indicates that symbol manipulation must be performed first in order to isolate the absolute value. However, this information is not sufficient to ascertain her awareness of about the rationale for isolating the absolute value bars first. Her response in the interview may shed more light on her reasoning.

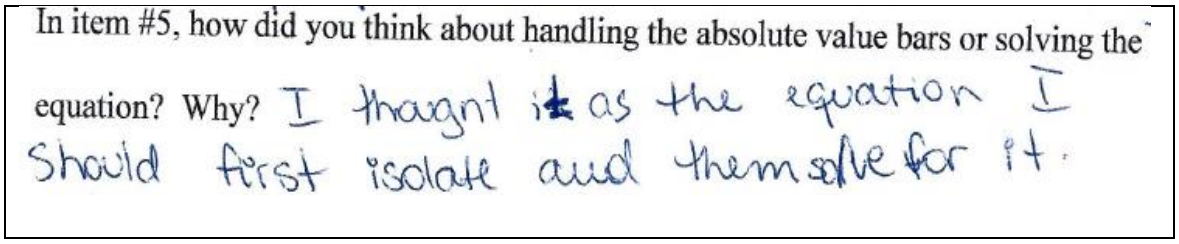
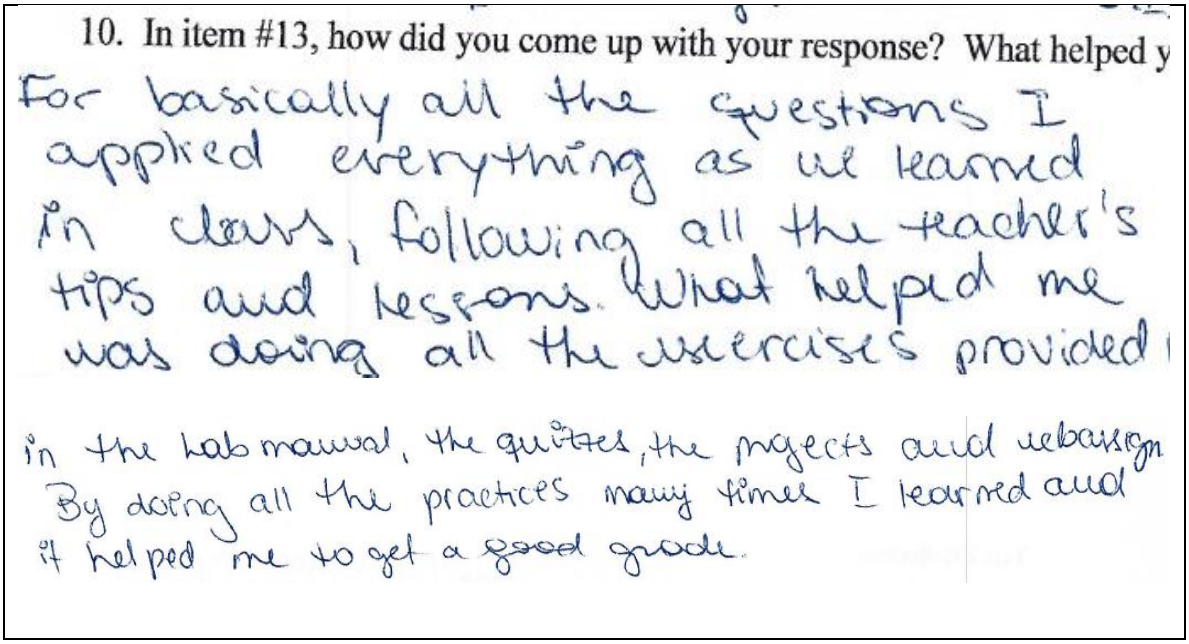

In item #5, how did you think about handling the absolute value bars or solving the equation? Why? I thought it as the equation I Should first isolate and them solve for it.

Figure 48: Ellen's posttest reflection response about Item 5

Regarding Item 13b, Ellen was asked "how did you come up with your response? What helped you?" Ellen gave a general response that is not readily applicable to her reasoning regarding Item 13, as shown in Figure 48.


10. In item #13, how did you come up with your response? What helped y

For basically all the questions I applied everything as we learned in class, following all the teacher's tips and lessons. What helped me was doing all the usercises provided in the lab manual, the quizzes, the projects and webassign By doing all the practices many times I learned and it helped me to get a good grade.

Figure 49: Ellen's posttest reflection response about Item 13

Ellen's interview responses to SS3 coded items. The interview question for Item 5 was "Why did you / why didn't you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars?)" Item 5 asked students to solve $13 - 2|x - 5| = 9$. In Table 91 line 15:52, Ellen acknowledges the type of equation, meaning the structure of

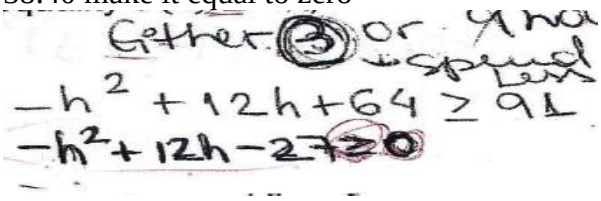
the equation. As shown in Table 79 lines 16:26 through 16:36, Ellen's response to the interview question regarding Item 5 demonstrates her structure sense in terms of her awareness that some symbol manipulations were necessary in order to properly deal with the absolute value bars. She refers to her knowing that the absolute bars yield two results in Table 91 lines 16:40 through 16:50.

Table 91: Justification of SS3 coded items in Ellen's work - item 5

Item	Interview question	Ellen's response
5	Why did you / why didn't you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars?) Solve the absolute value equation below. $13 - 2\|x - 5\| = 9$	it's an 15:21 absolute value equation in one side and 15:23 everything else in the other side and 15:25 then I could just eliminate the absolute 15:27 value sign up and then solve for X so I 15:31 couldn't just just (inaudible) this because 15:33 it's not parentheses if I had to 15:45 like this X minus 5 I cannot just 15:49 distribute you here because this is one 15:52 type of equation equation it's not that 15:54 I have the variable just a variable here 15:56 like between parentheses so that I had the absolute value 16:26 equation isolated then I took off the 16:31 how did you do that and then and then I 16:36 solve so then I know that I have I have two 16:40 options but I need to yeah I have 16:45 two results so I need to do a positive 16:48 and a negative because it's absolute 16:50 value so I I should have two answers

The interview question for Item 13b was “What was/is your thinking process? Why?” Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 12h + 64$. Ellen’s response in the interview shows her understanding about arranging the terms of the expression in a specific structure (manner) in order to facilitate a certain reasoning. In Table 91, starting just before line 37:30, she speaks about putting everything in one side and then make it (all nonzero terms) equal to zero so that she could use the quadratic formula. What she talks about here is exactly what the structure sense labeled SS3 describes.

Table 92: Justification of SS3 coded items in Ellen’s work - Item 13b

Item	Interview question	Ellen’s response
13b	What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 12h + 64$ Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)	so I knew that the whole 35:10 expression the whole function is so I 35:15 know that G of X is the score that you 35:18 earn and he needed at least 91 so that's 35:22 why we needed to solve the inequality 35:23 because he need something greater than 35:27 or equal to 91 to put everything on one side and then 37:30 it goes to zero so then I could use the 37:33 quadratic formula to find the variable I think I just know that I need 38:38 to put everything in one side and then 38:40 make it equal to zero 

Summary. I summarize the results and analysis on Ellen’s work both in regards to the four prefigured codes in terms of development of structure sense and in regards to

emergent codes in terms of algebraic proficiency. The prefigured codes correspond to the four definitions of structure sense used for the results and analyses.

First, for the general definition of structure, Ellen showed development of structure sense by thinking about algebraic expressions in terms of their graphs. Ellen's development of structure sense consistently involved reading through symbols and recognizing, then connecting inverse structures such as quadratic function versus a square root function, those of addition versus substraction, and those of multiplication versus division. So for this part of the definition, she showed development in structure sense is strong and consistent. Second, for the first part of Hoch and Dreyfus' (2006) definition, Ellen showed development of structure sense in recognizing structures in their simplest form, but there was a bit of mix-up with some concepts. Her development in structure sense in regards to that definition is not consistent and not strong. Third, for the SS2-A part of Hoch and Dreyfus' (2006) definition, Ellen showed development of structure sense by using words/phrases like "whole", "type" and by circling around parts of algebraic expressions (e.g., x+h). Her structure sense in regards to the SS2-A definition was consistent and strong. Fourth, for the SS3 part of Hoch and Dreyfus' (2006) definition, Ellen showed development of structure sense by mentioning that she recognized the structure of equations and by mentioning how to handle those structures effectively. This development is consistent through both items for the SS3 part of the definition.

In regards to emergent codes, her explanations were not procedural for items which were readily interpretable by structure sense or by reading though symbols. For example, she knew to expect a quadratic function as the inverse of a square root function). Her responses to the interview questions and her work on the posttest showed some flexibility in the way

she performed tasks. For example, her reasoning involved procedural, graphical, and structural approaches, but not necessarily on the same tasks. Her reasoning consistently involved several habits that I modeled during the intervention such as the use of gestures like circling around algebraic expressions to highlight them and reason about them.

Case 5: Pete

Pete was a Caucasian male who was a third year student in exercise science. His career goal was to open a gym and do some personal training. He had attended other colleges prior to this one but had had some brain injuries that forced him to change his goals. Most of his classes were health related. It had been 8 years since he last took an algebra course and this was his first time taking precalculus in college. He had taken remedial mathematics courses prior to precalculus. He described his experience in previous math classes as positive and found that math was relatively easy for him to understand because he loved having a designated right answer that he could work through. Pete described himself as being enrolled full-time, taking 9 credit hours. He worked part-time for an average of 15 hours a week. He rated his study skills as average and his level of access to a computer and the internet as very good. He disagreed with the ideas that math was important because he needed it for what he would study later on and that learning many things in mathematics would help him get a job. However, he stated that he enjoyed reading about mathematics, that making an effort in math was worth it because it would help him in the work he wanted to do later on, that he looked forward to his mathematics lessons, that he did mathematics because he enjoyed it, that learning mathematics was worthwhile because it would improve his career prospects, and that he was interested in the things he learned in mathematics. Regarding his ability to complete certain mathematics tasks, Pete felt confident

that he could find the actual distance between two places on a map with a 1:10,000 scale and that he could solve an equation like 2(x+3)=(x+3)(x-3). He felt very confident that he could use a train schedule to figure out how long it would take to get from one place to another, calculate how much cheaper a TV would be after a 30% discount, calculate how many square feet of tile would be needed to cover a floor, understand graphs presented in newspapers, solve an equation like 3x+5=17, and calculate the gas mileage of a car.

In his Midterm Reflection Questionnaire, Pete indicated that he had not had a personal tutor, he had not participated in study groups, and he had not used other campus resources. He was still taking 9 credit hours and was working part time; however, he had not worked at all in the past 7 weeks. He was spending on average 2 hours a week doing MAT 171 homework, reading class notes, and watching instructional videos to learn the concepts in the course. He did not indicate anything that may have affected his learning experience outside of what took place in the class.

At the end of the course in his Final Interview, Pete indicated that although he slacked off toward the end of the course, he still learned a lot and was able to understand 90% of the material. He said, "This semester was very good. [I] learned a lot of different ways to understand problems & functions & I learned new ways to think through issues. I hope to remember some of this in the future."

Pete's responses to general definition items from the posttest reflection. Pete was asked the following question regarding Item 2a "how do you know which function has an inverse and which one doesn't?" The two functions are $f(x) = x^2 - 5x + 3$ and $g(x) = \sqrt{3x - 8}$**.** As shown in Figure 50, Pete replied that "You can put the fxn into each other + whichever will equal ϕ is the inverse fxn." It seems that Pete was referring to function

composition of the original and inverse function, which is correct, but this response does not show structure sense that is based on a broad view analysis of the way an equation is put together.

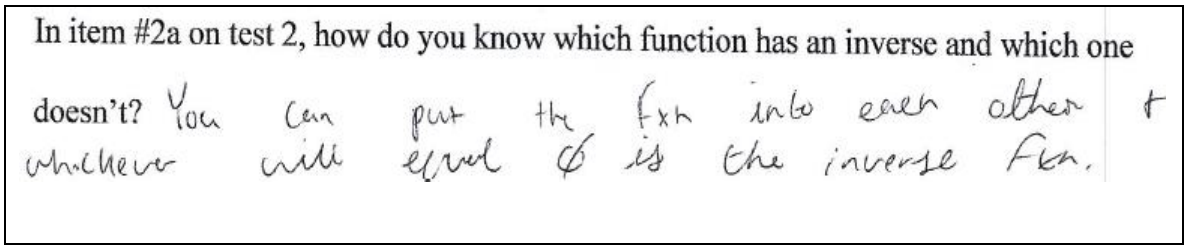

In item #2a on test 2, how do you know which function has an inverse and which one doesn't? You can put the fxn into each other + whichever will equal ∅ is the inverse fxn.

Figure 50: Pete's posttest reflection response about Item 2a

Pete was asked the following question regarding Item 2b "did you expect the kind of function or structure you get for the inverse of g(x), $(x) = \sqrt{3x - 8}$? Yes or no. If yes, how?" Pete replied "No, I would assume it would be a radical going another direction. I thought this because simple inverse is a flip in my mind." His answer does not indicate that he analyzes the expressions carefully and pays attention to how those expressions are put together. His answer does strongly indicate that he is thinking graphically about the inverse. His statement that, "A radical going in another direction" refers to graphical thinking.

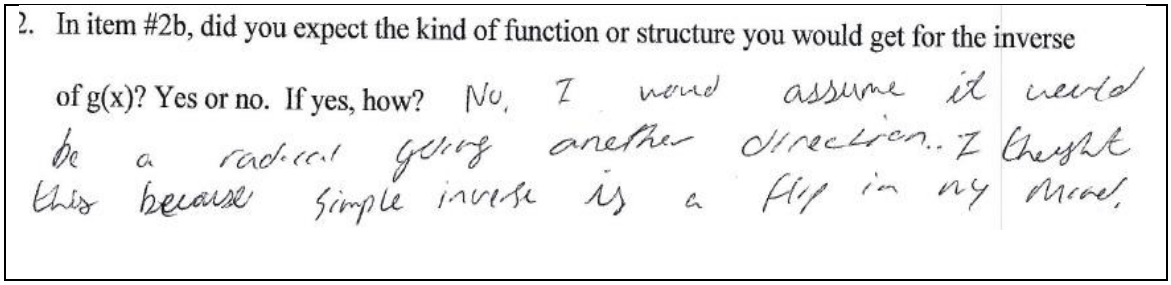
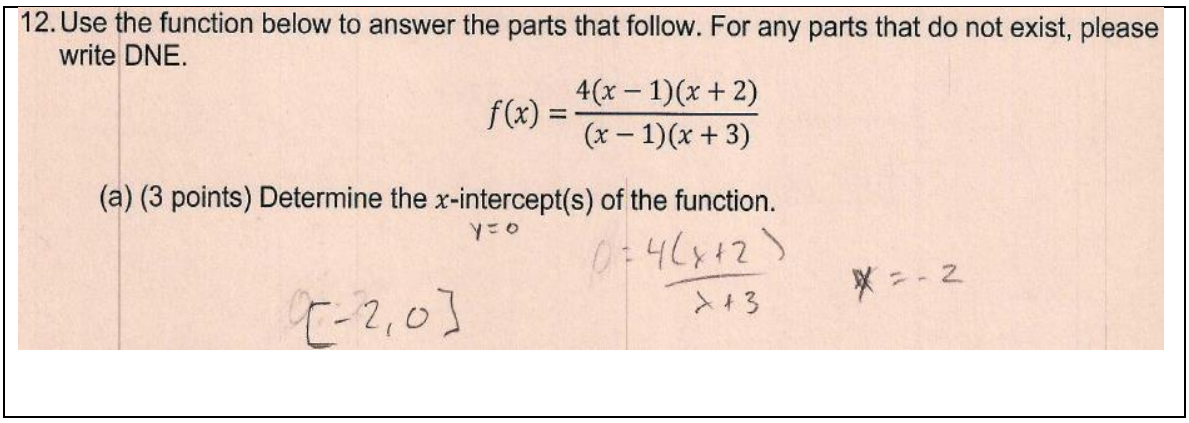

2. In item #2b, did you expect the kind of function or structure you would get for the inverse of g(x)? Yes or no. If yes, how? No, I would assume it would be a radical going another direction.. I thought this because simple inverse is a flip in my mind.

Figure 51: Pete's posttest reflection response about Item 2b

Pete was asked about Item 12a, "how did you come up with your response? What helped you?" Figure 52 shows Item 12a from the posttest and Pete's response.



12. Use the function below to answer the parts that follow. For any parts that do not exist, please write DNE.

$$f(x) = \frac{4(x-1)(x+2)}{(x-1)(x+3)}$$

(a) (3 points) Determine the x-intercept(s) of the function.

y=0

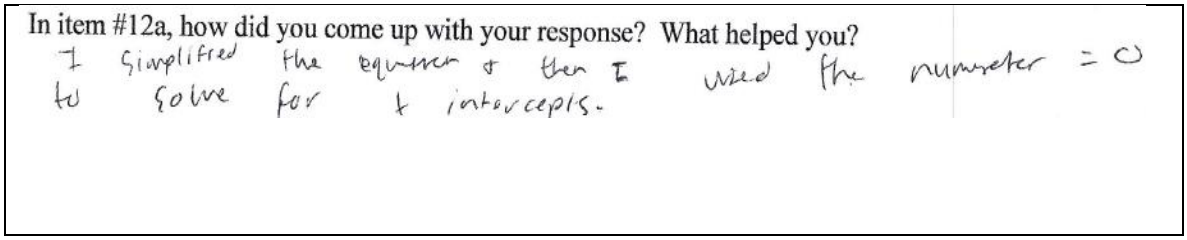
$0 = \frac{4(x+2)}{x+3}$

$x = -2$

[-2, 0]

Figure 52: Pete's response to Item 12a on the posttest

As shown in Figure 53, Pete wrote "I simplified the equation + then I used the numerator = 0 to solve for x intercepts." This answer seems to indicate his structure sense in terms of knowing what part of equation to focus upon, in this case the numerator.



In item #12a, how did you come up with your response? What helped you?

I simplified the equation + then I used the numerator = 0 to solve for x intercepts.

Figure 53: Pete's posttest reflection response about Item 12a

Pete was asked about Item 12d, "How did you com up with your response? What helped you?' Figure 54 shows the equation for Item 12.

$$f(x) = \frac{4(x-1)(x+2)}{(x-1)(x+3)}$$

Figure 54: Equation for Item 12 on the posttest

As shown in Figure 55, Pete replied “I used rules because the degree num = degree den, LC*A = 4 .” LC stands for leading coefficient. His work on the posttest on that item is shown below. His response reveals that he read through the symbols, taking a gestalt view analysis to obtain the degrees of the two expressions.

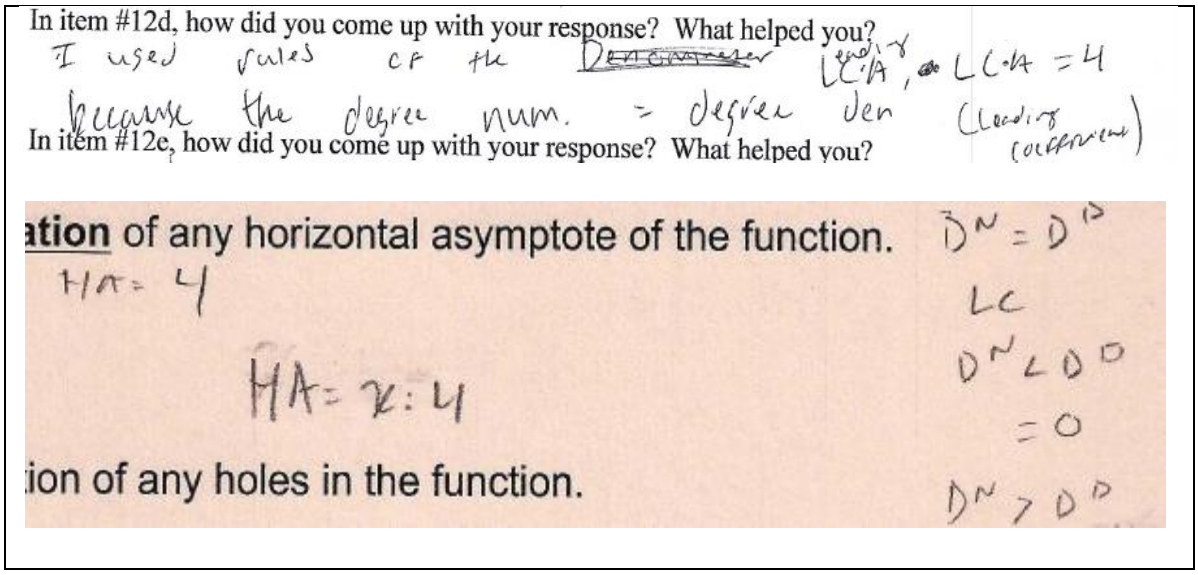


Figure 55: Pete's posttest reflection response about Item 12d

This answer shows Pete’s structure sense in terms of reading through the expressions and extracting the right details (e.g., degrees, leading coefficients). Though he mistakenly used x=4 instead of y=4.

Pete’s interview responses to general definition items. The first interview question for this category of items was “How did you know that f(x) and g(x) are inverses of each other? The two functions are $\boldsymbol{f(x) = 3x + 12}$ and $\boldsymbol{g(x) = \frac{x}{3} - 4}$.” So in Table 93 lines 01:43 through 01:52, Pete indicates how he knew f(x) and g(x) are inverses by pointing out that the function compositions f(g(x)) and g(f(x)) are both equal to x, the identity function. He mentioned “I can understand that they’re equal to each other and so they are an identity

function when they're equal", as he pointed to x. Then Pete continued "which means they are on top of each other perfectly" referring to the graphs of f(x) and g(x) being on each side a diagonal line that drew. His response in lines 01:43 through 01:52 reveals that his understanding of inverses in terms of the symbolic composition of those functions and in terms of graphs. Besides the composition and the graphical arguments, Pete did not have another argument or approach that would clearly show that f(x) and g(x) are inverses as seen in lines 3:18 through 3:27 in Table 93. His explanation does not point to the exact opposite operations in the two functions f(x) and g(x), which would have been a clear instance of structure sense by way of reading through symbolic expressions.

Table 93: Justification of the general definition in Pete's work - Item 1c

Item	Interview question	Pete's response
1c	How did you know that f(x) and g(x) are inverses of each other? $f(x) = 3x + 12$ $g(x) = \frac{x}{3} - 4$	I can understand that they're equal 01:43 to each other and so there are an 01:46 identity function when they're equal 01:49 which means they're on top of each other 01:52 perfectly but if I got X alone possibly it could 03:18 be 30 X + 12 + 3 X 3 X - 12 well maybe I 03:25 mean if you distribute that to be a gtp 03:28 that we think so we should not have 03:37 anything else to mention about that we

The interview question for Item 2a asked "What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered?" The two functions are $f(x) = x^2 - 5x + 3$ and $g(x) = \sqrt{3x - 8}$. His reasoning here about Item 2a in Table 94, lines 4:21 through 5:03, is based on the structure of

f(x) and g(x) for applying the 1-to-1 rule in a graphical way. He said "I would have graphed it"

Table 94: Justification of the general definition in Pete's work - Item 2a

Item	Interview question	Pete's response
2a	What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered? Use the functions $f(x) = x^2 - 5x + 3$ and $g(x) = \sqrt{3x - 8}$ to answer the parts that follow. One of the functions provided above is one-to-one, and the other is not. Which of the functions **<u>is</u>** a one-to-one function?	I would have 04:21 graphed it most likely because that 04:23 helps me but with wonder ones but a 04:27 one-to-one it can't cross 04:31 I might done that wrong yeah it's what 04:39 it's one of these okay yeah yeah so it 04:42 helps me to graph it that's why I always 04:44 do the graphs but I have an x squared 04:46 here so it's gonna be up like this and 04:49 then when I cross it it's gonna hit 04:52 twice in the same the same function and 04:57 that's not usually a one-to-one or it's 04:59 not a one-to-one so that's how yeah I 05:01 would I would use the graph method for 05:03 both of them

The interview question for Item 2b ask "Looking at the inverse function that you came up with $y = \frac{x^2+8}{3}$ how do you know it is the correct inverse for the original function, $y = \sqrt{3x - 8}$?" His explanation in Table 95 shows that structure sense was not used, neither the general type of the function as opposite structure nor the opposite operations in them. Instead, in lines 07:04 through 07:36 Pete talks about replacing y in $y = \sqrt{3x - 8}$

by $\frac{x^2+8}{3}$, which would essentially equate the original function with its inverse. After making this statement, Pete was not sure how to continue and complete that line of reasoning.

Table 95: Justification of the general definition in Pete's work - Item 2b

Item	Interview question	Pete's response
2b	Looking at the inverse function that you came up with $g^{-1}(x) = y = \frac{x^2+8}{3}$, how do you know it is the correct inverse for the original function? $y = \sqrt{3x-8}$	so 07:04 trying to remember but 07:19 I'm assuming a it has to it has to equal 07:23 itself so it has to equal one equals one 07:27 or something so I'm assuming I plug it 07:31 back in 07:31 this whole Y back into here and if this 07:36 equals this I think it's inverse

The interview question for Item 12a was "How did you determine the x-intercept? What was/is your thinking process? Why?" Table 96 shows the equation for Item 12a. Pete's response regarding his work on item 12a in the interview is similar to his response in the posttest reflection in showing some initial structure sense. In the posttest reflection, Pete knew and wrote about setting the numerator equal to zero, showing some structure sense about the rational structure. In the interview Pete talks about remembering to set y equal to zero; then from setting the equation equal to zero he takes a procedural approach to completing his line of reasoning. In Table 96, lines 20:05 through 20:19, Pete explains how he cleared the denominator, divided by 4 and obtained $x + 2 = 0$. His final answer of $x = -2$ was obtained in a procedural manner. He did not read through the symbols in the numerator to see that a -2 would zero out the numerator and in effect the entire equation.

Table 96: Justification of the general definition in Pete's work - Item 12a

Item	Interview question	Pete's response
12a	How did you determine with the x-intercept(s) from the given equation? What was/is your thinking process? Why? $f(x) = \frac{4(x-1)(x+2}{(x-1)(x+3)}$	I can see here from my notes 18:55 I have y equals 0 18:56 mm-hmm so whenever I get it an 18:59 x-intercept or a y-intercept question I 19:02 think okay if it's the x intercept the y 19:04 has to be 0 it can't be up or down it 19:08 has to be on on the x axis so why 19:11 wouldn't have to equal 0 and same 19:13 difference for the y-intercept X would 19:15 have to be 0 so that it can cross the 19:19 intercept okay so that if I have y 19:22 equals 0 I used it in the formula right 19:28 here and from this formula I took out 19:31 the X x minus ones because those are 19:33 holes we learned and then I just set 19:36 this equal to 0 20:05 oh yeah ok so do you have 0 there tries 20:10 to multiply this out it's going to 0 20:13 divided by 4 is still gonna be 0 X plus 20:16 2 equals 0 and then I solve for X and I 20:19 get the negative 2 intercept $y=0$ $0=\frac{4(x+2)}{x+3}$ $x=-2$ $[-2,0]$

The interview question for Item 12d was "How did you determine the horizontal asymptote? What was/is your thinking process? Why?" Table 97 shows the equation for Item 12d. Pete's response regarding his work on Item 12d in the interview is the same as in the posttest reflection. His explanation directly applies to the general definition of structure sense in terms of taking a gestalt (global) view of the equation (a broad view analysis), first

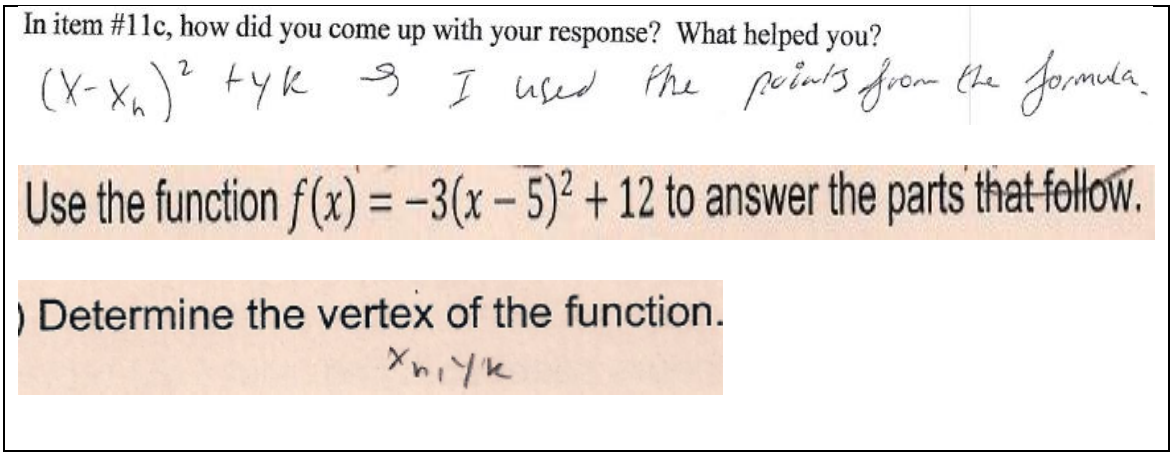
through the numerator and then through the denominator, in Table 97 lines 20:58 through 21:39. But instead of using $y = 4$, he mistakenly used $x = 4$.

Table 97: Justification of the general definition in Pete's work - Item 12d

Item	Interview question	Pete's response
12d	How did you determine the horizontal asymptote from the given equation? What was/is your thinking process? Why? $f(x) = \frac{4(x-1)(x+2)}{(x-1)(x+3)}$	20:46 certain there's certain rules for them 20:49 to find asymptotes and that is so that 20:53 they are the numerator over the 20:56 denominator and you're gonna have an X 20:58 here and an X here and you base which 21:04 one you're looking for based on the the 21:06 powers of the exponents in total so to 21:14 do to do that I think each of these if 21:20 you look up here this one's gone so I 21:23 have one power and one power so that 21:26 equal and I think I had yeah they're 21:33 equal right here and so therefore I said 21:35 I made myself a note and I said leading 21:37 coefficient and so my leading 21:39 coefficient was 4 and that's I just made 21:45 that my horizontal asymptote $D^N = D^D$ LC $D^N < D^D$ $= 0$ $D^N > D^D$

Pete's responses to SS1 coded items from the posttest reflection. This part of the results describes Pete's explanation in the posttest reflection for his work on the posttest. Students were asked only about Items 11c and 13a on the posttest reflection. The other item

in that rubric is 11b. Regarding item 11c, Pete was asked, “how did you come up with your response? What helped you?” Item 11c was “determine the vertex of the function $f(x) = -3(x-5)^2 + 12$.” As shown in Figure 47, Pete replied “$(x - x_h)^2 + y_k$ and I used the points from the formula.” His answer indicates that he recognized that structure of the quadratic function in vertex form, which is a clear instance of the structure sense labeled as SS1.


In item #11c, how did you come up with your response? What helped you?

$(X-X_h)^2 + yk$ → I used the points from the formula.

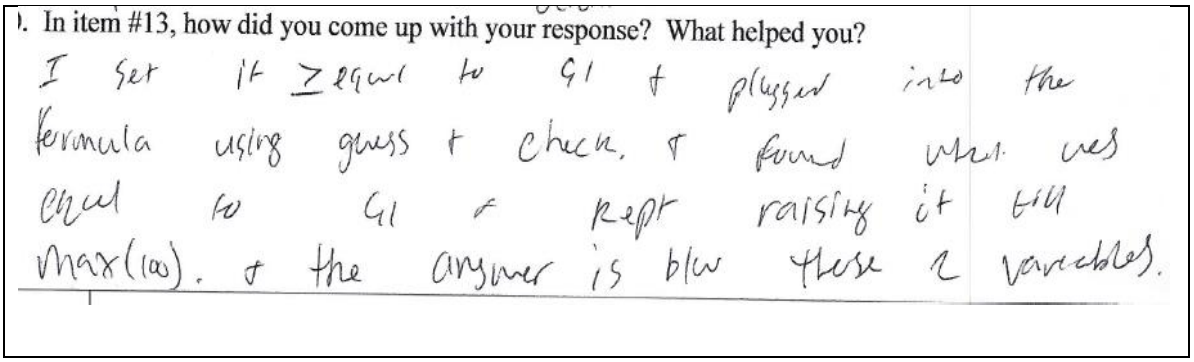
Use the function $f(x) = -3(x-5)^2 + 12$ to answer the parts that follow.

) Determine the vertex of the function.

x_h, y_k

Figure 56: Pete's posttest reflection response about Item 11c

Regarding Item 13a, when asked, “how did you come up with your response? What helped you?”, As shown in Figure 57, Pete replied “I set it ≥ equal to 91 and plugged in using the formula and used guess and check and found what was equal to 91 and kept on raising it till it max out (∞). And the answer is b/w these 2 variables.” His response addressed Item 13b, not 13a, which was categorized as an SS1-coded item.

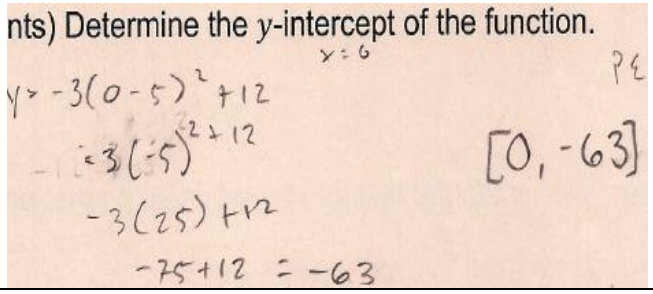

I. In item #13, how did you come up with your response? What helped you?

I set it equal to 91 & plugged into the formula using guess & check. & found what was equal to 91 & kept raising it till max(100). & the answer is b/w these 2 variables.

Figure 57: Pete's posttest reflection response about Item 13

Pete's interview responses to SS1 coded items. The interview question for Item 11b was "Why didn't you/did you choose the last term in the statement $f(x) = -3(x - 5)^2 + 12$ as they y-intercept?" Pete's response in Table 98 lines 13:46 through 14:51 shows that he did not mix up the vertex form and the expanded form of the quadratic equation. As he points out, he recognized the formula and identified the special x and y coordinates of the vertex. Then in lines 16:19 through 16:37, Pete adds that the special y coordinate is the peak value; it is not necessarily the y-intercept.

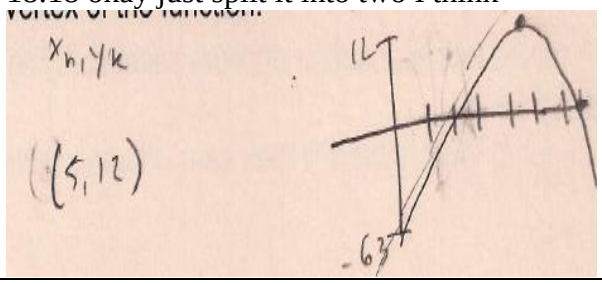
Table 98: Justification of the SS1 coded item in Pete's work - Item 11b

Item	Interview question	Pete's response
11b	Why didn't you/did you choose the last term in the statement as the y-intercept? Use the function $f(x) = -3(x-5)^2 + 12$ to answer the parts that follow. Determine the y-intercept of the function.	I recognized this 13:46 formula as negative a X minus H 13:57 yeah X minus XH I think plus YK I 14:03 recognize that but I think you have to 14:09 plug in a different number so this had 14:15 no X so it wasn't completely the 14:18 accurate y-intercept I don't think he 14:23 was saying that I used to which one you 14:24 said had (inaudible) name all right this 14:27 one right here I think I needed to fill 14:29 that variable in with a number okay and 14:36 so I think what I did 14:43 well used to zero as the x-intercept to 14:46 find the y-intercept and I solved for y 14:51 right here that was the peak that's 16:19 not it doesn't mean where it was 16:21 crossing over the y-intercept yeah I 16:24 knew that it was the peak of it so that 16:28 it hadn't so that's the peak you know 16:30 that means it's crossing somewhere over 16:32 here which is lower than that okay so we 16:35 didn't jump to I didn't jump to 16:37 conclusions on that one 

The interview question for Item 11c was "What was/is your thinking process for determining the vertex from the statement $f(x) = -3(x-5)^2 + 12$?" Pete's response to the interview question demonstrates his structure sense in recognizing the vertex for the equation. In Table 99 lines 17:16 through 18:18, Pete seems to be talking about symmetry

about the graph of the identity function. The symmetry to which he refers involves the graph of function on one side of the y = x line and the graph of the inverse function on the other side of the y = x line. Nevertheless, his response in previous table, Table 86, shows that he recognized the vertex format of the quadratic function.

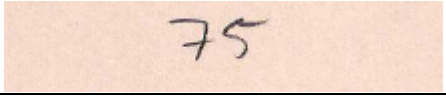
Table 99: Justification of the SS1 coded item in Pete's work - Item 11c

Item	Interview question	Pete's response
11c	What was/is your thinking process for determining the vertex from the statement? Determine the vertex of the function $f(x) = -3(x-5)^2 + 12$	so it's like 1/2 here and won't 17:16 have here in there equal so if I have 17:20 another graph with you know squiggles or 17:24 something wherever you cut it it's gonna 17:27 be the top half is equal in the top and 17:29 the bottom half is equal so whatever my 17:32 line was I would have to find the equal 17:36 point in the equal line that splits it 17:38 into two okay so for that one you drew 17:42 usually (inaudible) whose X line this this 17:48 make me think of of inverse functions 17:54 that's what made this make me think of 17:56 so for for this type of you said you 18:01 will recognize this way for this type of 18:03 test format 18:05 what kind of symmetry would you get well 18:08 this makes will be is it gonna be like 18:09 the way they could be it'd be if this is 18:14 my line to be like this a vertical line 18:18 okay just split it into two I think 

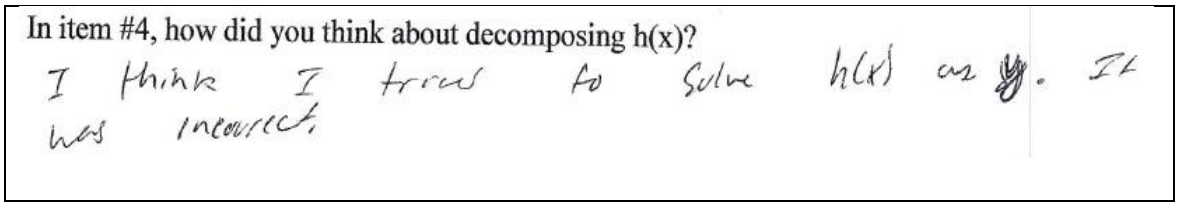
The interview question for Item 13a was "Using the given equation, $G(h) = -h^2 + 10h + 75$, how did you determine what score the student would earn without studying? What was/is your thinking process? Why?" Pete's response regarding item 13a on the posttest and in the interview does not reference the form of the quadratic equation. In Table 100 lines 22:29 through 22:40 he explains his answer by talking about making the first two

terms zero; there is no indication of any kind of recognition regarding the equation, its type, or form.

Table 100: Justification of the SS1 coded item in Pete's work - Item 13a

Item	Interview question	Pete's response
13a	Using the given equation, how did you determine what score the student would earn without studying? What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 10h + 75$, where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours.What score will Alex earn without studying?	again was looking like 22:26 a kind of like a very text formula to me 22:29 but I think I just in my mind I think I 22:34 just made these zeros so 0 plus 0 75 as 22:40 I said okay well if I don't study at all 22:43 it's 0 hours of studying and then I get 22:46 75 and you did that all in your mind you 22:49 just yeah I think it just says I think 22:51 it just made sense to me logically 22:54 looking at it but also the idea of like 22:57 the vertex kind of sticks there like 

Pete's responses to SS2-A coded items from the posttest reflection. Regarding Item 4, Pete was asked, "how did you think about decomposing h(x), $h(x) = \sqrt{x^2 + 13} - 11$?" As shown in Figure 58, Pete replied "I think I tried to solve h(x) as y. It was incorrect." Pete realized that setting h(x) = 0 and solving for x was not the correct approach. His work on that item does not provide evidence of dealing with a compound term as an entity.

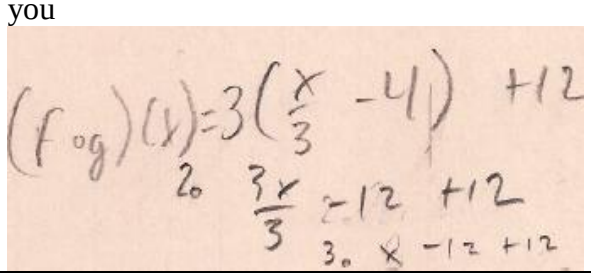

In item #4, how did you think about decomposing h(x)?

I think I tried to solve h(x) as y. It was incorrect.

Figure 58: Pete's posttest reflection response about Item 4

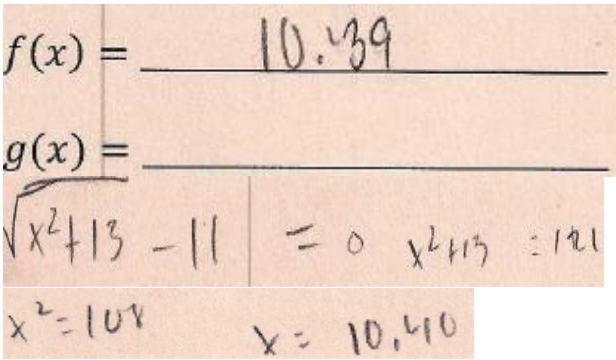
Pete's interview responses to SS2-A coded items. The interview question for Item 1a/b was "Why did you put parentheses in that statement?" (Statement written by the student). Pete's response in Table 101 lines 00:42 through 01:20 regarding his work for Item 1a/b clearly that shows his structure sense in dealing with a compound term as a single entity. This structure sense is evidenced starting in line 01:20 by his use of the phrase "I need to put in the G of X formula for this X," as he pointed to the x in the original expression of f(x).

Table 101: Justification of the SS2-A coded item in Pete's work - Item 1a/b

Item	Interview question	Pete's response
1a/b	Why did you put parentheses in that statement? (Statement written by the student) Use the functions $f(x) = 3x + 12$ and $g(x) = \frac{x}{3} - 4$ to answer the parts that follow. Determine $(f \circ g)(x)$. Determine $(g \circ f)(x)$.	I 00:42 put parenthesis here because if it's F 00:46 of G of X that means I need to put in 00:50 the G of X formula for this X so 00:54 therefore I was making a note to myself 00:57 that this X right here there's a shown 01:02 up it's equal to this whole thing and so 01:06 therefore it's 3x plus 12 and then this 01:12 X is now x over 3 minus 4 okay then you 

The interview question for Item 4 was “How did you think about decomposing that function, $h(x) = \sqrt{x^2 + 13} - 11$? Which part of the original expression did you think could be a component? Why?” Pete’s response to the interview question regarding Item 4 does not show evidence of dealing with a compound term as a single entity. His approach was to solve h(x) as an equation as shown in Table 102, not as expression to be decomposed. His response here is consistent with his explanation in the posttest reflection.

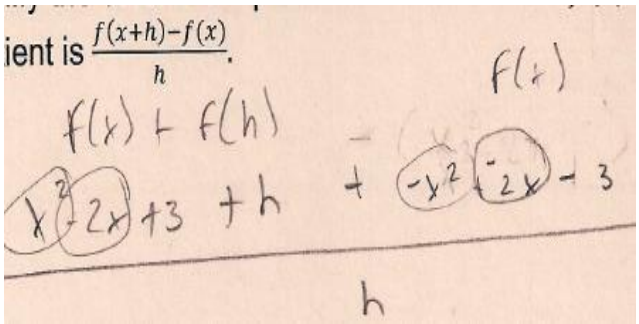
Table 102: Justification of the SS2-A coded item in Pete’s work - Item 4

Item	Item description	Pete’s response
4	How did you think about decomposing that function? Which part of the original expression did you think could be a component? Why? Suppose that $h(x) = \sqrt{x^2 + 13} - 11$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$. $f(x) =$ _____ $g(x) =$ _____	I thought that the 08:44 square root everything under the square 08:46 root 08:47 could be one part and then I would add 08:50 the 11 for the other part and then when 08:53 I set it equal I could solve it for the 08:57 X in the H and then use that to go on 09:09 

The interview question for Item 6 was “How did you think about or what helped you get this task right?” Item 6 required students to construct and simplify the difference quotient for $f(x) = x^2 - 2x + 3$. Pete’s response to the interview question regarding this item also does not demonstrate the ability to deal with a compound term as a single entity. In

Table 103, starting in line 11:26, Pete talks about adding f(x) and f(h), which he obtained from mistakenly splitting the compound term (x+h). However, he realized that this approach was not appropriate.

Table 103: Justification of the SS2-A coded item in Pete's work - Item 6

Item	Interview question	Pete's response
6	How did you think about or what helped you get this task right? Construct and simplify the difference quotient for the function $f(x) = x^2 - 2x + 3$ For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.	yes so I was thinking that 11:26 I could add up the two the different 11:31 functions at one point so add these two 11:33 together f of X and f of H it doesn't 11:41 work yeah so that's what we think that's 11:45 what I was thinking and so right here is 11:48 where I left the H I just made it at H 11:51 which I don't think is actually right 11:55 but I had the f of X this whole thing I 11:58 here 

Pete's responses to SS3 coded items from the posttest reflection. Item 5 asked students to solve $14 - 3|x - 5| = 2$. When asked about this item "how did you think about handling the absolute value bars or solving equation? Why?" Pete replied that "I saw them as parenthesis + knew that I has to a ± when I was solving the equation," as shown in Figure 59. His answer indicates knowledge about this aspect of the absolute value structure, the idea of plus and minus parts. However, this information is not sufficient to ascertain his awareness about the rationale for symbol manipulation. His response in the interview may shed more

light on his reasoning.

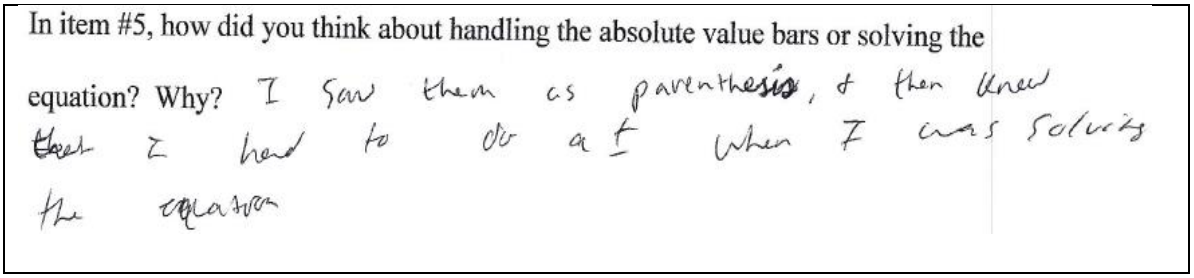

In item #5, how did you think about handling the absolute value bars or solving the equation? Why? I saw them as parenthesis, & then knew that I had to do a t when I was solving the equation

Figure 59: Pete's posttest reflection response about Item 5

Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 10h + 75$. When asked this item "how did you come up with your response? What helped you?" Pete said his approach was 'guess and check', as shown in Figure 60.

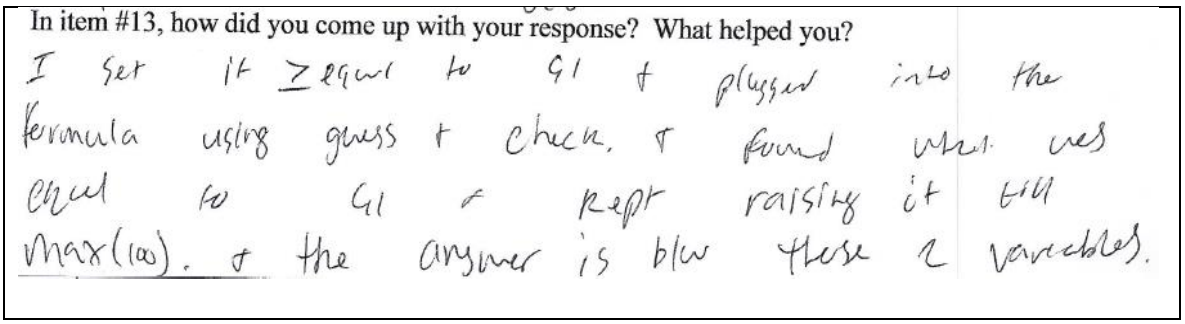

In item #13, how did you come up with your response? What helped you?
I set it ≥ equal to 91 & plugged into the formula using guess + check. & found when was equal to 91 & kept raising it till max(100). & the answer is b/w these 2 variables.

Figure 60: Pete's posttest reflection response about Item 13a

Pete's interview responses to SS3 coded items. The interview question for item 5 was "Why did you / why didn't you distribute that factor (-3) through the absolute value bars to get rid of them (i.e. absolute value bars?)" Item 5 asked students to solve $14 - 3|x - 5| = 2$. Pete's response to the interview question regarding this item demonstrates his structure sense in terms of her awareness that some symbol manipulations were necessary in order to properly deal with the absolute value bars. In Table 104 starting in line 10:14, Pete points to

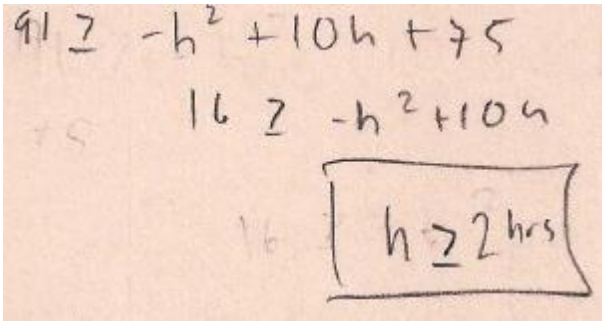
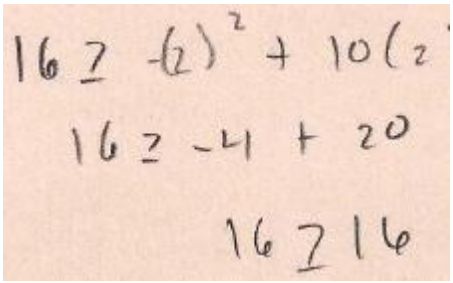
14 and -3 as the only values that are not attached to the absolute value part of the equation. Then he says that is the only thing I can start with. In line 10:23, he points that he can eliminate the absolute value bars afterwards, meaning after getting 14 an -3 absorved. Taking together his response shows awareness that symbol manipulation was necessary in order to make best use of structure.

Table 104: Justification of the SS3 coded item in Pete's work - Item 5

Item	Interview question	Pete's response
5	Why did you / why didn't you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars?) Solve the absolute value equation below. $14 - 3\|x - 5\| = 2$	I did not I did not 09:58 because this is in my mind its own its 10:02 own thing so this the whole negative 10:07 three absolute value X minus five is its 10:14 own like X in my mind and so this is the 10:18 only thing that's not attached to it so 10:20 that's the only thing I can start with 10:23 okay and then I can get rid of this 10:27 afterwards and then you solve for the 10:30 absolute value this way then because 10:36 it's attached to it at that at that 10:38 point

The interview question for Item 13b was "What was/is your thinking process? Why?" Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 10h + 75$. Pete's response in the interview as shown in Table 103 lines 24:57 through 25:12 is that he guessed and checked, which is consistent with what he mentioned on the posttest reflection. Unlike, his response to the interview question for Item 5, his response here does not show awareness that symbol manipulation was necessary in order to make best use of structure.

Table 105: Justification of the SS3 coded item in Pete's work - Item 13b

Item	Interview question	Pete's response
13b	What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 10h + 75,$ Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)	so to me this 24:06 says how many hours do I need to study 24:11 to score a 91 or higher that sort of 24:16 says to me mm-hmm so with that thinking 24:19 I said my score I want to get is 91 and 24:24 then I used the formula that was given 24:27 for hours spent studying to get your 24:30 score and then I just did a simple 24:36 simple solving subtracted my 75 that's 24:41 now I have 16 is like the the score I'm 24:44 looking for and then I just I don't know 24:57 I think I did guess in check so yeah on 25:03 this side I just plugging in numbers and 25:05 then I found that two hours is gonna 25:08 equal 16 is greater than or equal to so 25:12 in my mind it's 16 equals 16 and I've 25:16 solved it if it equals each other 25:18 mm-hm and so then I went and did two 25:21 from here to two hours mm-hmm 25:24 and I just said that's my answer then he 25:28 has to do that he has to study for at 25:29 least one at least two hours  

Summary. I summarize the results and analysis on Pete's work both in regards to the four prefigured codes in terms of development of structure sense and in regards to emergent codes in terms of algebraic proficiency. The prefigured codes correspond to the four definitions of structure sense used for the results and analyses.

First, for the general definition of structure, Pete showed development of structure sense by thinking about algebraic expressions in terms of their graphs. He showed development of structure sense in ways that are associate with prescribed steps for completing a task (i.e., the structure sense is an integral part of a set of steps, e.g., performing the horizontal line test, finding horizontal asymptotes). However, Pete's development of structure sense did not involve recognizing or connecting inverse structures such as quadratic function versus a square root function. So, for this part of the definition his work and explanation did not indicate a habit of reading through symbols and taking a gestalt view of algebraic expressions. Second, for the first part of Hoch and Dreyfus' (2006) definition, Pete showed development of structure sense in recognizing structures in their simplest form. Though, recognizing structures was not consistent throughout all the items, Pete showed development of structure sense in this regards by relating algebraic expressions with their graphs. Third, for the SS2-A part of Hoch and Dreyfus' (2006) definition, Pete showed development of structure sense the use of direct substitution of compound term. His structure sense development did not extend to decomposing expressions, which can be thought of as a "reverse" substitution; nor did it extend to more advanced substitution as in applying the difference quotient formula. Her structure sense was not consistent throughout all the items of this definition (or code). Fourth, for the SS3 part of Hoch and Dreyfus' (2006) definition, Pete showed development of structure sense by recognizing the structure

of the equation in one item and by mentioning how to handle that structure effectively. His awareness about the SS3 part of the definition was not consistent through both items.

In regards to emergent codes, a few of his explanations for items which were readily interpretable by structure sense or by reading though symbols were procedural (e.g., knowing to expect a quadratic function as the inverse of a square root function). His responses to the interview questions and her work on the posttest showed limited flexibility in the way she performed tasks. For example, most of the reasoning were procedural and graphical in comparison to structural approaches, but not necessarily on the same tasks. His reasoning involved few of the habits that I modeled during the intervention such as circling around and underlining algebraic expressions to highlight them and reason about them.

Case 6: Marc

Marc self-identified as a white male who was a full-time college student in his first year. He was seeking his second bachelor's degree, this time in engineering. His career goal was to become a mechanical engineer. Because this was his second degree, he felt he was enjoying his coursework more and not wasting his time, unlike his first degree. He had hated math in high school, but it was not until he had worked as an electrician in the Navy that he learned to appreciate mathematics. He later encountered college professors who encouraged him, and now math was one of his favorite subjects. He had not taken any remedial mathematics courses prior to MAT 171, and it had been 8 years since he had taken an algebra course. This was his first time taking precalculus in college. He was taking 14 credit hours and was working part time, about 10-12 hours a week on average. He rated his study skills as average and his level of access to computers and the internet as very good. He stated that he enjoyed reading about mathematics, that he looked forward to his mathematics lessons,

that he did mathematics because he enjoyed it, and the he was interested in the things he learned in mathematics. He strongly agreed with the ideas that making an effort in mathematics was worth it because it would help him in the work that he wanted to do later on, that learning mathematics was worthwhile for him because it would improve his career prospects, that mathematics was important for what he would study later on, and that learning many things in mathematics would help him get a job.

In his Midterm Reflection Questionnaire, Marc indicated that he had not used a tutor for the course. He did not participate in study groups, nor did he use any campus resources. Marc was still enrolled full-time, taking 14 credit hours, and working part time 16 hours a week, on average. He had spent 4 hours a week doing MAT 171 homework, reading class notes, and watching instructional videos to learn the concepts in the course. Marc stated that financial trouble added stress to his life and may have affected his learning experience outside of what was taking place in class.

In his Final Interview, Marc stated that although he felt behind at the beginning of the course, he was motivated to master the principles taught in the course instead of just passing the class because of his clear-cut career goals. He said, "this time around I wanted to learn the material more because it will be important for my career and the rest of my college classes."

Marc's responses to general definition items from the posttest reflection. Marc was asked the following question regarding Item 2a, "how do you know which function has an inverse and which one doesn't?" The two functions are $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ as shown in Figure 52.

Use the functions $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ to answer the parts that follow.

(a) (2 points) One of the functions provided above is one-to-one, and the other is not. Which of the functions **is** a one-to-one function?

Figure 61: Question 2a on the posttest

As shown in Figure 62, Marc replied "The function has to be one-to-one in order for the function to have an inverse, one of the functions, when graphed, has a parabola shape and wouldn't pass the HLT, and therefore isn't one-to-one." HLT stands for horizontal line test. Marc's structure sense is informed by the graph of the quadratic function.

In item #2a on test 2, how do you know which function has an inverse and which one doesn't? THE FUNCTION HAS TO BE ONE-TO-ONE IN ORDER FOR THE FUNCTION TO HAVE AN INVERSE. ONE OF THE FUNCTIONS, WHEN GRAPHED, HAS A PARABOLA SHAPE AND WOULDN'T PASS THE HLT, AND THEREFORE ISN'T ONE-TO-ONE.

Figure 62: Marc's posttest reflection response about Item 2a

Marc was asked the following question regarding Item 2b "did you expect the kind of function or structure you get for the inverse of g(x), $g(x) = \sqrt{2x - 7}$? Yes or no. If yes, how?" Figure 63 shows Item 2b from the posttest.

Use the functions $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ to answer the parts that follow.

(b) (5 points) Determine the inverse of the one-to-one function identified in (a).

Figure 63: Question 2b on the posttest

As shown in Figure 64, Marc replied "I expected that there would be some variable to the power of 2, because the opposite of a square root is squaring a number. I didn't expect that it would be a function." His answer shows that he knows about opposite structure in terms of

opposite operations, but at the moment that knowledge did not extend to functions and their equations. So at this point, he performed a broad view analysis of the equations that is correct but needed to be more accurate. He said he did not expect the square or quadratic structure would be a function possibly because he knows that the quadratic graph is not 1-to-1.

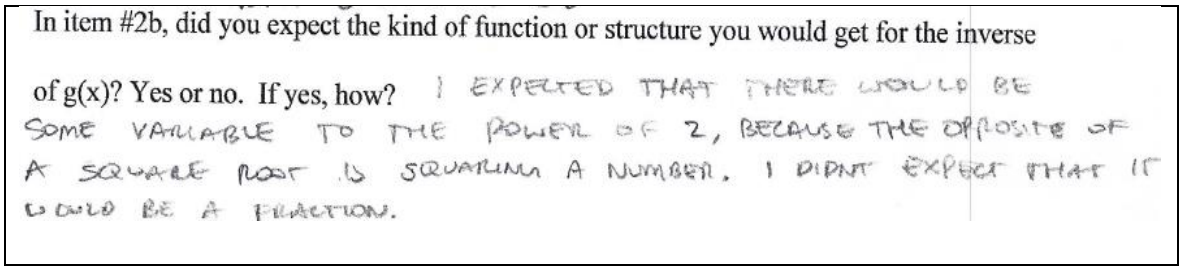
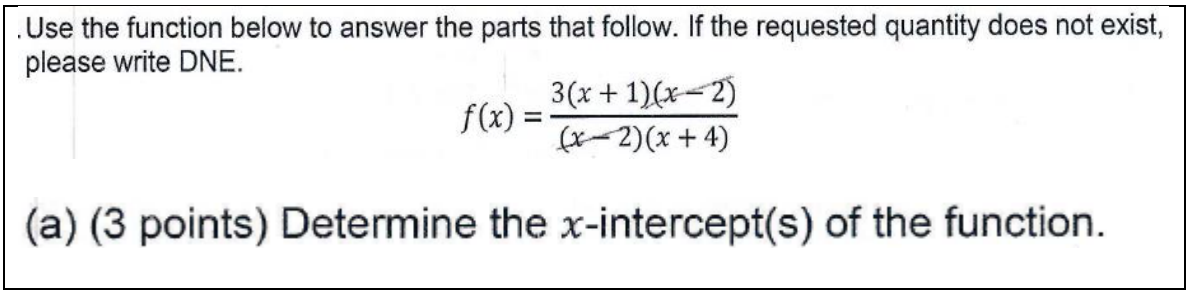

In item #2b, did you expect the kind of function or structure you would get for the inverse of g(x)? Yes or no. If yes, how? I EXPECTED THAT THERE WOULD BE SOME VARIABLE TO THE POWER OF 2, BECAUSE THE OPPOSITE OF A SQUARE ROOT IS SQUARING A NUMBER. I DIDNT EXPECT THAT IT WOULD BE A FRACTION.

Figure 64: Marc's posttest reflection response about Item 2b

Marc was asked about Item 12a, “how did you come up with your response? What helped you?” Figure 65 shows Item 12a from the posttest.


.Use the function below to answer the parts that follow. If the requested quantity does not exist, please write DNE.

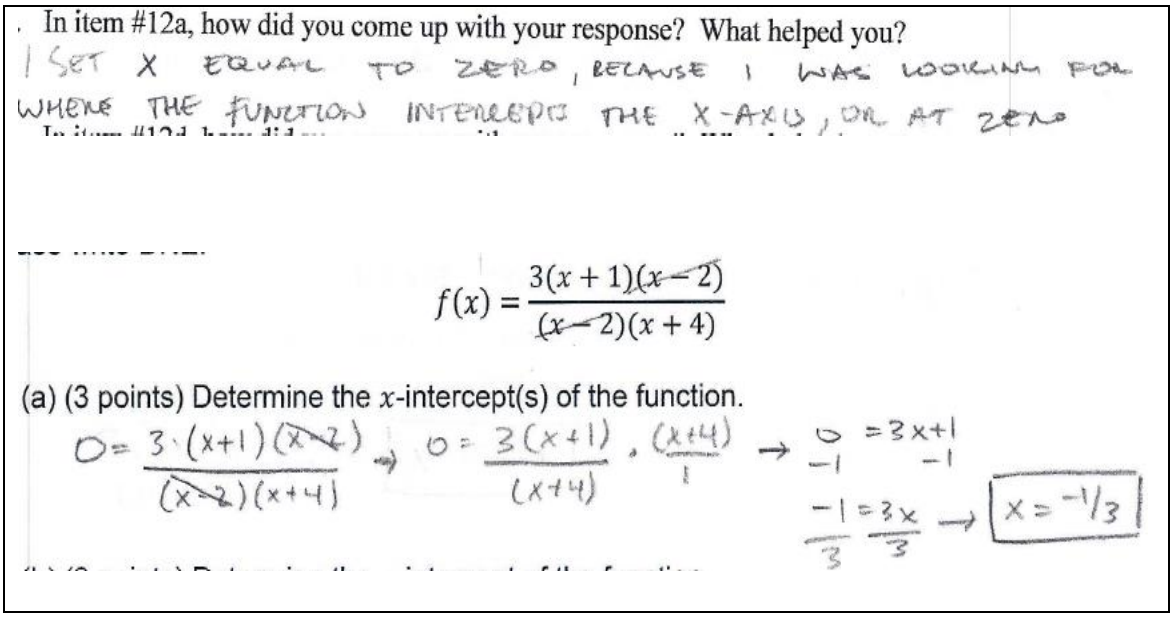
$$f(x) = \frac{3(x+1)\cancel{(x-2)}}{\cancel{(x-2)}(x+4)}$$

(a) (3 points) Determine the x-intercept(s) of the function.

Figure 65: Question 12a on the posttest

As shown in Figure 66, Marc wrote “I set x equal to zero, because I was looking for where the function intercepts the x-axis, or at zero.” I believe Marc meant that he set Y equal to zero, not X. His answer seems to indicate his structure sense in terms of knowing what part of equation to focus upon, in this case the whole equation, setting it equal zero. During the intervention students were taught that f(x) or any other function notation (i.e. g(x), k(x)) can

be replaced with the variable Y, the second coordinate of an ordered pair. However, Marc took a procedural approach at the end of his work on the task, which cost him an error of writing 3(x+1) as 3x+1.


In item #12a, how did you come up with your response? What helped you?

I SET X EQUAL TO ZERO, BECAUSE I WAS LOOKING FOR WHERE THE FUNCTION INTERCEPTS THE X-AXIS, OR AT ZERO

$$f(x) = \frac{3(x+1)\cancel{(x-2)}}{\cancel{(x-2)}(x+4)}$$

(a) (3 points) Determine the x-intercept(s) of the function.

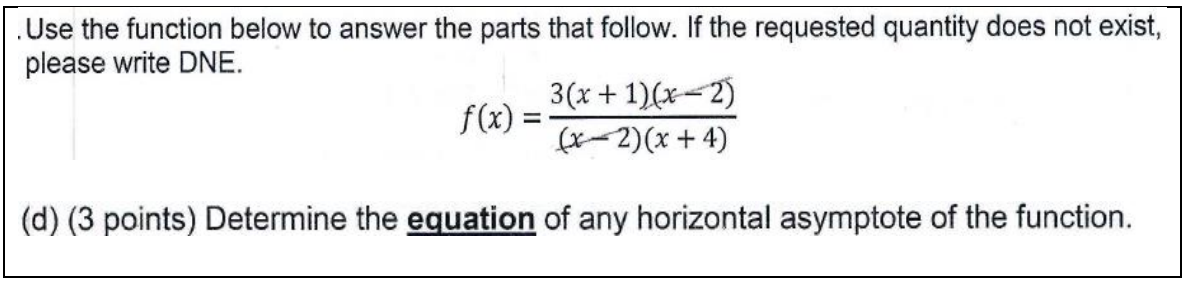
$$0 = \frac{3\cdot(x+1)\cancel{(x-2)}}{\cancel{(x-2)}(x+4)} \rightarrow 0 = \frac{3(x+1)}{(x+4)} \cdot \frac{(x+4)}{1} \rightarrow 0 = 3x+1$$

$$-1 \quad -1$$

$$\frac{-1}{3} = \frac{3x}{3} \rightarrow \boxed{x = -1/3}$$

Figure 66: Marc's posttest reflection response about Item 12a

Marc was asked about Item 12d, “How did you com up with your response? What helped you?’ Figure 67 shows Item 12d from the posttest.


Use the function below to answer the parts that follow. If the requested quantity does not exist, please write DNE.

$$f(x) = \frac{3(x+1)\cancel{(x-2)}}{\cancel{(x-2)}(x+4)}$$

(d) (3 points) Determine the **equation** of any horizontal asymptote of the function.

Figure 67: Question 12d on the posttest

As shown in Figure 68, Marc replied “I didn’t come up with the right answer.” His work on the posttest reveals a mix-up between x-intercepts and horizontal asymptotes.

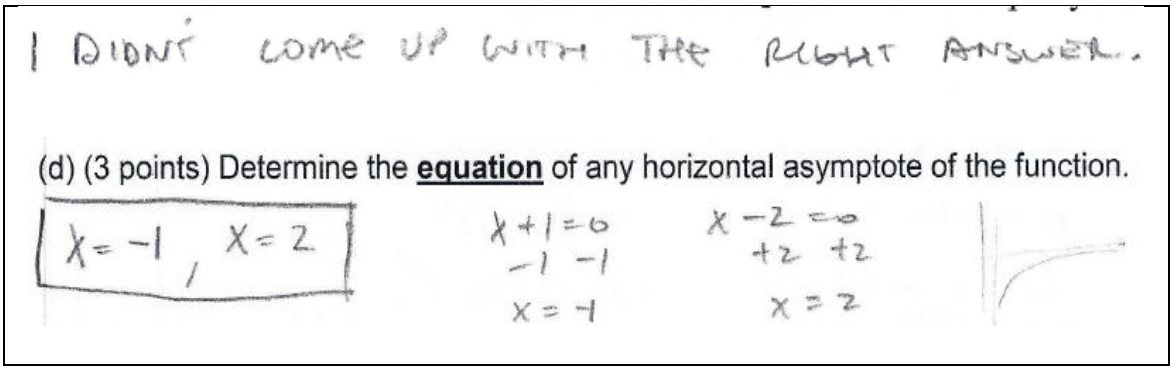


Figure 68: Marc's posttest reflection response about Item 12d

Marc's interview responses to general definition items. The first interview question for this category of items was "How did you know that f(x) and g(x) are inverses of each other? ($\boldsymbol{f(x) = 2x + 8}$ and $\boldsymbol{g(x) = \frac{x}{2} - 4}$). InTable 106 lines 02:30 to 02:49, Marc says when he composed the functions f(x) and g(x) he gets the function y=x, which is the identity function. Then in lines 03:53 through 04:15, he says he could also graph f(x) and g(x) and then see if they are flipped over an axis, the identify function. Finally, he mentions he could swap x and y and solve for y, which is the procedural way that was taught in class. While some of those explanations are procedural and none of them point to the exact opposite operations in the two functions f(x) and g(x), they reveal that Marc has several approaches for reasoning about inverse functions. This flexibility in approach shows a higher level of proficiency on the concept. A broad view analysis or a read through symbolic expressions would have shown more conceptual understanding in regard to structure sense and proved a much higher level of proficiency.

Table 106: Justification of the general definition in Marc's work - Item 1c

Item	Interview question	Marc's response
1c	How did you know that f(x) and g(x) are inverses of each other? $f(x) = 2x + 8$ $g(x) = \frac{x}{2} - 4$	when you compose the 02:30 functions together in Part B so these 02:35 are you know it's G times is a G of F 02:39 yeah yeah okay 02:41 so when I put those two functions 02:44 together and an A when I want to put 02:47 them together I get the function y 02:49 equals x which is identity which is just 02:51 a straight line 02:52 well its diagonal line through and I 02:55 know that when I put two functions

The interview question for Item 2a asked "What if the task had asked you to explain if these functions are one-to-one or not instead of which function is one-to-one? How would you have answered?" The two functions are $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$. His reasoning here about Item 2a in Table 107, lines 07:12 through 07:36, is based on the structure of f(x) and g(x) for applying the 1-to-1 rule in a graphical way. As mentioned earlier, sometimes students think about the one-to-one functions within the context of numerical values with repeating x-values or within equations containing x-squared terms. However, there was not enough reason to think that she was not thinking about graph.

Table 107: Justification of the general definition in Marc's work - Item 2a

Item	Interview question	Marc's response
2a	What if the task had asked you to explain if these functions are one-to-one or not, instead of which function is one-to-one? How would you have answered? Use the functions $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ to answer the parts that follow. One of the functions provided above is one-to-one, and the other is not. Which of the functions **is** a one-to-one function?	07:12 okay then I would say that this one is 07:16 one to one and then the other one isn't 07:18 because this is a parabola and it 07:23 doesn't pass the horizontal line test 07:25 and that's what how I would explain that 07:31 and then I would show that this one is a 07:34 radical function which is one-to-one did 07:36 that's it passes both the horizontal 07:38 line test and the vertical line test so 07:39 I know that it's one to one

The interview question for Item 2b asked "Looking at the inverse function that you came up with $y = \frac{x^2+7}{2}$ how do you know it is the correct inverse for the original function, $y = \sqrt{2x - 7}$?" His explanation in Table 108 shows understanding of inverse functions in terms of composing the g(x) and $g^{-1}(x)$. He knew to compose them instead of multiplying them. In Table 108 lines 10:44 through 11:08, he attempts to speak through the process of actually composing them, but needed more time to write it out. He says the result of the composition would be x, which is correct. Much like one of his responses regarding item 1c, this response is partly procedural in that it requires quite a bit of symbol manipulation to show that result of the composition yield x. The response he gave in the posttest reflection showed to some extent that he took a broad view analysis of the function g(x) its inverse. That analysis showed his knowledge of opposite structure in terms of opposite operations.

Table 108: Justification of the general definition in Marc's work - Item 2b

Item	Interview question	Marc's response
2b	Looking at the inverse function that you came up with, $g^{-1}(x) = \frac{x^2+7}{2}$ how do you know it is the correct inverse for the original function? $g(x) = \sqrt{2x-7}$	well during the test I just trusted my 10:28 math okay 10:30 but I can test it now if I were to put 10:35 if I were to compose them together they 10:37 should equal the identity function if I 10:43 can't remember how to do that is the 10:44 question so this would be two times x 10:49 squared plus seven over two 10:55 my 7 is the square root of that so these 11:00 (inaudible) would cancel out so I would get 11:04 square root of x squared plus 7 minus 7 11:08 so I get square root of x squared which 11:10 is X so I get y equals x so that's how I 11:14 know that they're each other I know that 11:31 for sure it's an inverse

The interview question for Item 12a was "How did you determine the x-intercept? What was/is your thinking process? Why?" Marc's response regarding his work on Item 12a in the interview is different from the posttest reflection. In the posttest reflection he began to demonstrate structure sense when he set the entire equation equal to zero. Then toward the end of completing the task he took a procedural approach which cost the error of writing 3(x+1) as 3x+1. However, in the interview he realized he should have used structure sense by reading that factor (x+1). In Table 109 lines through 28:19, he mentions that -1 would produce a zero for the entire equation. Extracting information from symbolic expressions by merely looking at them was a hallmark to the teaching intervention, but during the posttest and the posttest reflection, it did not occur to Mark that -1 zero out the numerator. It occurred to him during the interview.

Table 109: Justification of the general definition in Marc's work - Item 12a

Item	Interview question	Marc's response
12a	How did you determine with the x-intercept(s) from the given equation? What was/is your thinking process? Why? $f(x) = \frac{3(x+1)(x-}{(x-2)(x+4}$	I know that or at least the the 27:48 numerator right if I put in a negative 27:52 one here I get three times zero which 27:55 gives me in an Y value of zero if I put 27:58 in a 2 so negative one and two both of 28:03 those values would give me a y equals 28:05 zero but I know if I put in a two here 28:09 I'm dividing by zero so that value 28:10 doesn't doesn't work but one works no 28:14 matter what one negative one yeah 28:17 negative one would work no matter what 28:19 so negative one would be a zero okay $\rightarrow 0 = 3x+1$ $-1 \quad -1$ $\frac{-1}{3} = \frac{3x}{3} \rightarrow \boxed{x = -1/3}$

The interview question for Item 12d was "How did you determine the horizontal asymptote? What was/is your thinking process? Why?" The equation for this item is shown in Tables 109 and 110. Marc's response regarding his work on Item 12d in the interview is not the same as in the posttest reflection. He realized what he did was for vertical asymptotes instead of horizontal asymptotes. He performed a broad view analysis of the equation with respect to obtaining the degrees of the numerator and the denominator (line 31:08). During the intervention, students were taught that the degree of an expression is a global attribute.

Table 110: Justification of the general definition in Marc's work - Item 12d

Item	Interview question	Marc's response
12d	How did you determine the horizontal asymptote from the given equation? What was/is your thinking process? Why? $f(x) = \frac{3(x+1)(x-}{(x-2)(x+}$	30:13 it was like okay horizontal asymptote 30:14 right I'm I'm looking for okay something 30:17 that looks like that as opposed to this 30:19 would be a vertical asymptote right it's 30:21 you know it's getting close but it never 30:23 touches and so I needed to go okay how 30:27 what numbers are getting close to you 30:35 [Noise] 30:46 right because yeah I flipped it yeah I 30:50 did the wrong ones so it would be 30:56 negative for 31:01 (inaudible) go that's what you have 31:05 here yeah you will you will have to you 31:08 would have to look at the degree yep 31:14 right yep which is x equals or Y yeah a 31:18 (inaudible) y equals 0 3 2 over 1 it will be 31:24 (inaudible) with 0 if they do on top was less 31:26 than ideal at the bottom right right 31:38 that's ceiling (inaudible) because these are 31:44 the same but did we want you because you 31:46 cancel this 2 way so it's the one and 31:50 the same then it's why was the leading 31:52 coefficient of the top divided by the 31:54 leading coefficient of the 12 a given

Marc's responses to SS1 coded items from the posttest reflection. Regarding Item 11c, Marc was asked, "how did you come up with your response? What helped you?" Item 11c was "determine the vertex of the function $f(x) = -4(x-3)^2 + 16$." As shown in Figure 69, Mark replied "The function was given in vertex format, so I knew what the vertex was." His answer indicates that he recognized that structure of the quadratic function in vertex form, which is a clear instance of the structure sense labeled as SS1.

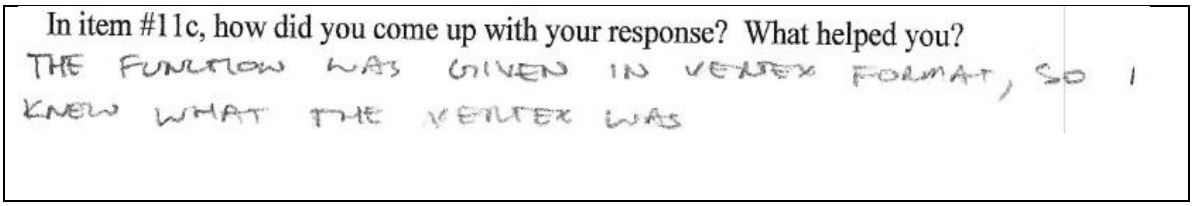

In item #11c, how did you come up with your response? What helped you?
THE FUNCTION WAS GIVEN IN VERTEX FORMAT, SO I KNEW WHAT THE VERTEX WAS

Figure 69: Marc's posttest reflection response about Item 11c

Regarding Item 13a, Marc was asked, “how did you come up with your response? What helped you?” As shown in Figure 70, Marc did not reply to that question.

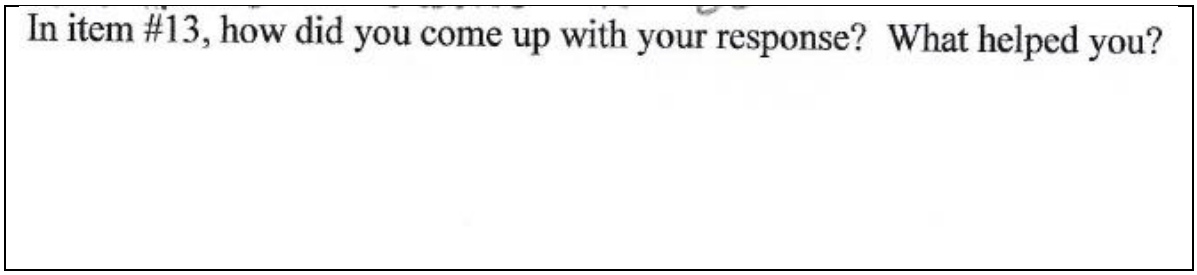

In item #13, how did you come up with your response? What helped you?

Figure 70: Marc's posttest reflection response about Item 13a

Marc’s interview responses to SS1 coded items. The interview question for Item 11b was “Why didn’t you/did you choose the last term in the statement $\boldsymbol{f(x) = -4(x - 3)^2 + 16}$ as they y-intercept?” Marc’s response to Item 11b on the posttest clearly shows his fluency in obtaining the y-intercept as shown at the end of Table 111. His response in the interview shows that he recognize 16 could not be the y-intercept based on the format of the given equation (lines 19:40 through 19:52). He says 16 is supposed to be the maximum and putting in zero for x does not necessarily yield 16, lines 20:15 through 20:21. Marc’s response here shows that he is reading a bit more through expressions than earlier in the items from the previous code (Gen Defn). In lines 21:31 through 22:20, he talks about format of the expression saying he knows there is format where the constant (last term in this

case), then says that is not that format. Marc's comment at the end clearly shows his structure sense when it comes to recognizing a structure in simplest form.

Table 111: Justification of SS1 in Marc's work - Item 11b

Item	Interview question	Marc's response
11b	Why didn't you/did you choose the last term in the statement as the y-intercept? Use the function $f(x) = -4(x-3)^2 + 16$ to answer the parts that follow. Determine the y-intercept of the function.	I think 16 is supposed to be this is not 19:46 you can't remember the name of the 19:49 format but this value is supposed to be 19:52 your max value for that because it no 19:55 matter if this is equal to zero if I 19:58 find whatever value equals zero this 20:01 value is 16 so the the greatest value I 20:05 know this is negative right so this 20:07 number just continues to get smaller and 20:08 smaller and smaller so from zero it just 20:12 gets smaller the maximum value for this 20:15 equation would be 16 that doesn't mean 20:17 that when you put zero into this that 20:21 you get 16 so that doesn't mean that X 20:26 being 0 if that part is 21:19 zero this is what you're gonna get but 21:20 not necessarily which makes sense to me 21:22 right it's this whole part not X right 21:25 it's gonna be yes this whole thing is to 21:27 equal zero 21:27 4.16 yeah yeah but putting zero in here 21:31 doesn't do that there is 22:17 a format where I can find the 22:18 y-intercept and I know that that one 22:20 isn't it $f(0) = -4(0-3)^2 + 16$ $-4(-3)^2 + 16$ $-4(9) + 16$ $-36 + 16$ $\boxed{-20 = \text{y-INTERCEPT}}$

The interview question for Item 11c was "What was/is your thinking process for determining the vertex from the statement $\boldsymbol{f(x) = -4(x-3)^2 + 16}$?" Marc's response to

Item 11c on the posttest demonstrates his structure sense in recognizing the vertex form. Then later in the interview, he mentions the word format several times, a term I used extensively during the intervention, as shown in Table 112 lines 23:36, 23:51, and 24:20. In his own words, he uses the phrase "parent function" to refer to structure, line 24:17. In Table 100 lines 23:53 through 24:10 he said "I know it's not a negative 3 because it's x minus h right if this was a plus 3 I would know that was a negative 3 just because of a negative minus a minus is a positive in the parent function." The phrase "parent function" he mentioned is an idea I emphasized under the label "general format or form or template of function. Marc's comment clearly shows his structure sense when it comes to recognizing a structure in simplest form.

Table 112: Justification of SS1 in Marc's work - Item 11c

Item	Interview question	Marc's response
11c	What was/is your thinking process for determining the vertex from the statement? Determine the vertex of the function $f(x) = -4(x-3)^2 + 16$	it's vertex format for determining the 23:28 vertex again I knew that this was vertex 23:36 format I couldn't remember it now but I 23:39 remembered it a lot of the tests and so 23:40 I needed that this is X - I think H and 23:43 this is K and then the H and K are the 23:48 vertex and so I was able to just draw it 23:51 straight from the the format that you 23:53 gave me I know it's not a negative 3 23:55 because it's X minus H right if this was 23:58 a plus 3 I would know that that was a 24:00 negative 3 just because of negative 24:02 minus or minus a negative is a positive 24:10 the the parent function is f of X yeah 24:17 yeah I knew from the parent function 24:20 from the format yep (c) (2 points) Determine the vertex VERTEX = (3, 16)

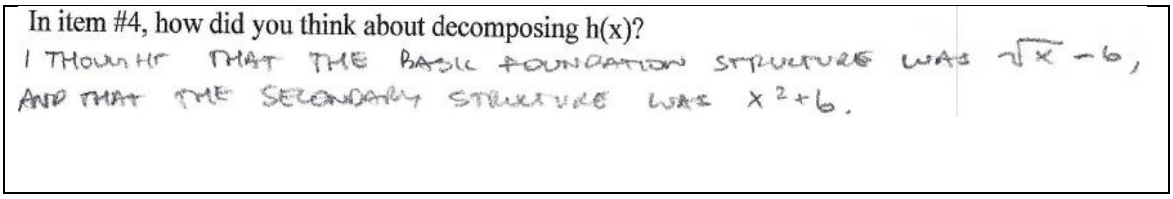
Item 13a states "The grade that Alex receives on a particular test can be modeled by the function below $G(h) = -h^2 + 12h + 64$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours. What score will Alex earn without studying?" Marc's response to Item 13a on the posttest was procedural because there was no reference to the form of the quadratic equation. At first, he explains her answer by talking about plugging in zero. In Table 113 lines 32:38 through 32:58 there is no indication of any kind of recognition regarding the equation, its type, or form. Then starting just before line 33:23 and through line 34:03, Marc shows that he also recognizes the expanded format of the quadratic equation. He says "I know this format… just looking at the format of it I know this is the y-intercept…" More strikingly, in lines 33:58 through 34:04, Marc says that he took a procedural approach because the instructions say to show work.

Table 113: Justification of SS1 in Marc's work - Item 13a

Item	Interview question	Marc's response
13a	Using the given equation, how did you determine what score the student would earn without studying? What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 12h + 64$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours.What score will Alex earn without studying?	H is the value for the number of hours 32:38 studying so if he spends zero hours 32:44 studying I can put zero in for H right 32:46 so without even knowing that this is the 32:50 max value if I put in 0 here I get 0 if 32:56 I put in 0 here I get 0 so I'm left with 32:58 64 $G(0) = -(0)^2 + 12(0) + 64$ $\boxed{G(0) = 64}$ I know this format I know that this is 33:23 the what's the y-intercept no yes 33:38 yep because when X is zero I know that 33:43 this is the y-intercept so this just 33:46 looking at the format of it I know that 33:47 this is the y-intercept which means that 33:50 when I put in an x value of 0 I get this 33:52 so I know that without even having to 33:57 put it in there I know that that would 33:58 be the answer but I did it just just 34:00 just a show just to show the work to 34:03 show the work

Marc's responses to SS2-A coded items from the posttest reflection. Students were asked only about Item 4 on the posttest reflection. The other items in that rubric are 1a/b and 6. Regarding Item 4, when asked, "how did you think about decomposing h(x), $h(x) = \sqrt{x^2 + 6} - 12$?" As shown in Figure 71, Marc replied "I thought the basic foundation structure was $\sqrt{x} - 6$, and that the secondary structure was x^2+6. " I believe Marc meant to write $\sqrt{x} - 12$. Since the whole expression was $\sqrt{x^2 + 6} - 12$, Marc's

decomposition of it must be based on substitution of x for x^2+6 underneath the radical sign. This substitution is evidence of Marc's structure sense in terms of dealing with a compound term as an entity.


In item #4, how did you think about decomposing h(x)?

I THOUGHT THAT THE BASIC FOUNDATION STRUCTURE WAS $\sqrt{x}-6$, AND THAT THE SECONDARY STRUCTURE WAS x^2+6.

Figure 71: Marc's posttest reflection response about Item 4

Marc's interview responses to SS2-A coded items. This part of the results describes Marc's explanation in the interview for his work on the posttest. The interview question for Item 1a/b was "Why did you put parentheses in that statement?" (Statement written by the student). Marc's response in Table 114 lines 01:02 through 01:22 includes phrases "keep track of" and "what was supposed to go inside" indicate the idea of grouping several elements together and clearly identifiable. The gist of his response also conveys the same idea and that he is dealing with the full expression of a function as a single entity.

Table 114: Justification of SS2-A in Marc's work - Item 1a/b

Item	Interview question	Marc's response
1a/b	Why did you put parentheses in that statement? (Statement written by the student) Use the functions $f(x) = 2x + 8$ and $g(x) = \frac{x}{2} - 4$ to answer the parts that follow. Determine $(f \circ g)(x)$. Determine $(g \circ f)(x)$.	I know that if I'm gonna put G 01:02 inside of F it needs to go where this X 01:05 is and so I put parentheses there to 01:08 just kind of remind myself okay this is 01:11 this is the equation for G going in 01:14 there and then this equation will be 01:17 multiplied and added by that I just help 01:19 me kind of keep track of where things 01:22 were and what was supposed to go inside $(f \circ g)(x) = 2\left(\frac{x}{2} - 4\right) + 8$

The interview question for Item 4 was "How did you think about decomposing that function, $h(x) = \sqrt{x^2 + 6} - 12$? Which part of the original expression did you think could be a component? Why?" Marc's response to the interview question regarding Item 4 clearly demonstrates her structure sense ability to deal with a compound term as a single entity. This ability is evidenced in his response on the posttest reflection. In a similar way here from the interview response in Table 115 starting at line 12:29, his explanation "I figured this was the whole right that something square rooted minus twelve was kind of the big part and this inside here was a part of it", shows how several terms combined to make one, x, as x=sqrt(x^2+6). He was pointing to x^2+6 inside the radical sign.

Table 115: Justification of SS2-A in Marc's work - Item 4

Item	Item description	Marc's response
4	How did you think about decomposing that function? Which part of the original expression did you think could be a component? Why? Suppose that $h(x) = \sqrt{x^2+6} - 12$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f\big(g(x)\big)$. $f(x) =$ _____ $g(x) =$ _____	so I looked at like what's 12:27 what's the big picture what's happening 12:29 here I think the big picture what I 12:32 thought was this was kind of happening 12:34 to more of the equation and x squared 12:36 plus 6 was right because this is being 12:39 affected by that and so I figured that 12:45 this was the whole right that something 12:49 square rooted minus twelve was kind of 12:52 the big part and this inside here was a 12:54 part of it and so composed this by doing 12:59 okay this is X right something as being 13:01 square rooted minus 12 and then the 13:04 inside was the smaller part $f(x) = \sqrt{x} - 12$ $g(x) = x^2 + 6$

The interview question for Item 6 was "How did you think about or what helped you get this task right?" Item 6 required students to construct and simplify the difference quotient for $f(x) = x^2 - 3x + 2$. Marc's response to the interview question regarding Item 6 also demonstrates his structure sense ability to deal with a compound term as a single entity. Starting in line 16:01 Table 116, he talks about how he needed to use square brackets and parentheses to keep track of the compound term (x+h) and the expression of f(x), which

is a compound term as well in the context of the difference quotient formula. Therefore, the use of square brackets and parentheses shows Marc's ability to deal with a compound term as a single entity.

Table 116: Justification of SS2-A in Marc's work - Item 6

Item	Interview question	Marc's response
6	How did you think about or what helped you get this task right? Construct and simplify the difference quotient for the function $f(x) = x^2 - 3x + 2$. For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.	I 16:01 think I needed it too I think having the 16:04 brackets was helpful okay I knew that I 16:07 needed to have those there otherwise I 16:09 got confused if I just use parentheses 16:11 the two equations felt like they kind of 16:16 bled together and so in order to remind 16:20 myself that this is X plus h right 16:22 because of that I needed to have those 16:25 big brackets okay this is this is f of X 16:27 plus h this is f my f of X and then I 16:39 think I needed to have a just kind of a 16:45 step by step like okay this is I don't 16:47 need to do this first $\frac{[(x+h)^2 - 3(x+h) + 2] - [x^2 - 3x + 2]}{h}$

Marc's responses to SS3 coded items from the posttest reflection. Regarding Item 5, Marc was asked "how did you think about handling the absolute value bars or solving the equation? Why?" Item 5 asked students to solve $13 - 2|x - 5| = 9$. As shown in Figure 72, Marc replied that "I knew that there would be two different solutions absolute value will always be a positive number so $|-2| = 2$ and $|2| = 2$ are both correct." His answer does show knowledge about the structure of absolute value, but without more information it is not clear how Marc used this knowledge to solve the absolute value equation on the posttest. His

solution on the posttest showed the proper steps and solutions for that item. I believe he was trying to use the definition of absolute value as taught in class but instead of using a variable for the argument of the absolute value function, he used the constants -2 and 2. He would get $|a| = -a$ if $a < 0$ and $|a| = a \quad a \geq 0$. This explanation would be in line with what was taught in the intervention. Nevertherless, Marc used the definition correctly during the posttest.

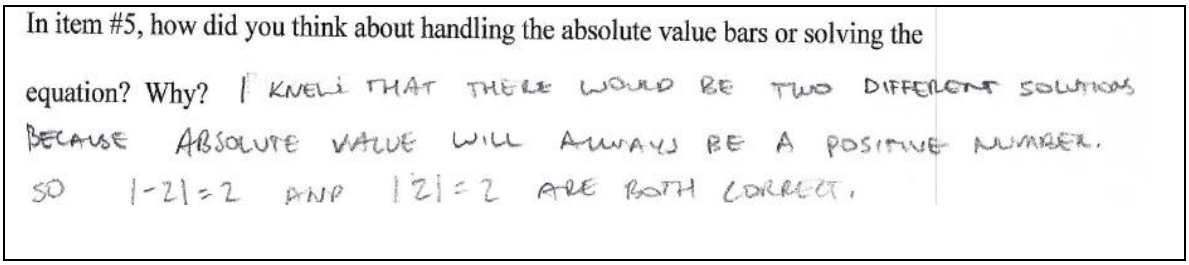

In item #5, how did you think about handling the absolute value bars or solving the equation? Why? I KNEW THAT THERE WOULD BE TWO DIFFERENT SOLUTIONS BECAUSE ABSOLUTE VALUE WILL ALWAYS BE A POSITIVE NUMBER. SO |-2|=2 AND |2|=2 ARE BOTH CORRECT.

Figure 72: Marc's posttest reflection response about Item 5

Regarding Item 13b, Marc was asked "how did you come up with your response? What helped you?" As shown in Figure 73, Marc did not reply to that question.

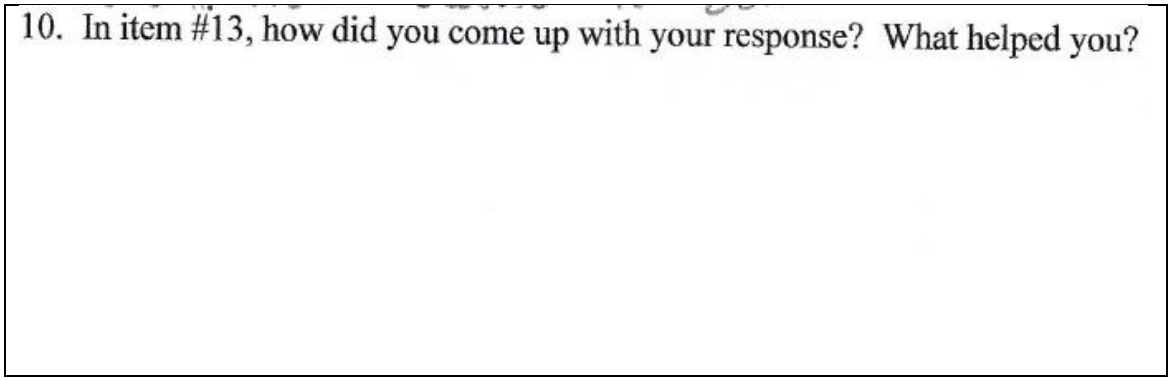
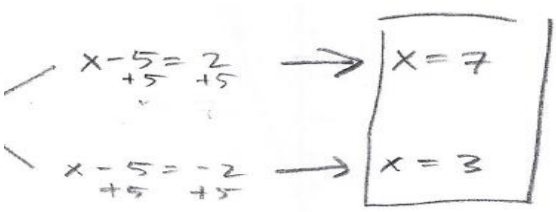

10. In item #13, how did you come up with your response? What helped you?

Figure 73: Marc's posttest reflection response about Item 13b

Marc's interview responses to SS3 coded items. The interview question for Item 5 was "Why did you / why didn't you distribute that factor (-2) through the absolute value bars

to get rid of them (i.e. absolute value bars?)". Item 5 asked students to solve $13 - 2|x - 5| = 9$. Marc's response to the interview question regarding Item 5 show that he may not fully understand how the definition of absolute value is applied. In Table 117, starting just before line 13:59 and going through line 14:02, when he talks about the $|x - 5|$ in the equation $13 - 2|x - 5| = 9$ as becoming x-5 and x+5. However, he knew that absolute bars yield two results and mentions it in lines 14:10 through 14:18. The rest of his response shows his structure sense in terms of her awareness that distributing a -2 through the absolute values bars were not appropriate symbol manipulations. Distributing that -2 was not the proper way to deal with the absolute value structure.

Table 117: Justification of SS3 in Marc's work - Item 5

Item	Interview question	Marc's response
5	Why did you / why didn't you distribute that factor (-2) through the absolute value bars to get rid of them (i.e. absolute value bars?) Solve the absolute value equation below. $13 - 2\|x - 5\| = 9$	because it's not just happening 13:59 to X minus 5 it's also happening to X 14:02 plus 5 absolute value is the distance 14:04 from zero and so I know that this 14:10 inherently is is two separate values X 14:14 can be two different values here and so 14:18 by distributing it to just one I would 14:20 only get half of the answer right okay 14:30 yeah that's at least how I understand it 14:35 do it real quick in my head just make 14:37 sure 

The interview question for Item 13b was "What was/is your thinking process? Why?" Item 13b asked students to solve $G(h) \geq 91$, with G(x) equal $-h^2 + 12h + 64$. Marc's

response in the interview regarding item 13b does show structure sense in terms of rearranging the terms of the expression in a specific manner in order to facilitate a certain reasoning. In Table 116 lines 35:24 through 35:29, Marc indicates that he subtracted 64 both sides of the inequality sign in order to isolate the variable H. However, subtracting 64 from both sides of the inequality sign does not make best use of structure in this case. The best use of structure would have been to have all nonzero terms on the same side of the inequality sign and then break the quadratic structure into two linear factors or use the quadratic formula. What Marc talks about here does not fully satisfy the structure sense labeled SS3.

Table 118: Justification of SS2 in Marc's work - Item 13b

Item	Interview question	Marc's response
13b	What was/is your thinking process? Why? The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 12h + 64$ Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)	91 is the score that he the lowest score 35:03 that he wants so I put that on the I 35:05 need to get a value a y-value that 35:07 equals 91 at least so this needs to be 35:12 you know the equation needs to be 35:15 greater than or equal to 91 which is why 35:17 I said it that way and then so I set it 35:21 up and then solved as if this was an 35:24 equal sign so then I subtract 64 from 35:27 both sides right because I'm trying to 35:29 get H by itself so I get 27 is less than 35:34 or equal to negative H squared plus 12 H 35:37 mm-hmm again I'm trying to get H by 35:39 itself so I divide by 12 35:41 I get 2.25 is less than or equal to H 35:43 squared minus 8 air + H $91 \leq -h^2 + 12h + 64$ $-64 \qquad -64$ $\frac{27}{12} \leq \frac{-h^2 + 12h}{12}$ $2.25 \leq h^2 + h$

Summary. I summarize the results and analysis on Marc's work both in regards to the four prefigured codes in terms of development of structure sense and in regards to emergent codes in terms of algebraic proficiency. The prefigured codes correspond to the four definitions of structure sense used for the results and analyses.

First, Marc showed development of structure sense by thinking about algebraic expressions in terms of their graphs. He showed development of structure sense in ways that

are associate with prescribed steps for completing a task (i.e., the structure sense is an integral part of a set of steps, e.g., performing the horizontal line test, finding horizontal asymptotes). Tough it involved some recognitions of opposite structures through opposite operations, Marc's development of structure sense here did not consistently involve reading through symbols. Second, for the first part of Hoch and Dreyfus' (2006) definition, Marc showed development of structure sense in recognizing structures in their simplest form. His development of structure sense in regards to that definition is supported by accurate comparison of different structures and by emphasizing phrases like "format", "parent function", and "just looking at the format". His structure sense here was strong and consistent throughout the items. Third, for the SS2-A part of Hoch and Dreyfus' (2006) definition, Marc showed development of structure sense by using words/phrases like "whole", "keep track of", and "what was supposed to go inside" and by using parentheses and brackets. His structure sense in regards to the SS2-A definition was strong and consistent throughout the items. Fourth, for the SS3 part of Hoch and Dreyfus' (2006) definition, Marc showed development of structure showed development of structure sense by recognizing the structure of the equation in one item and by mentioning how to handle that structure effectively. His awareness about the SS3 part of the definition was not consistent through both items.

In regards to emergent codes, a few of his explanations for items which were readily interpretable by structure sense or by reading though symbols were procedural. His responses to the interview questions and her work on the posttest showed some flexibility in the way she performed tasks. For example, Marc showed three different approaches of pairs of inverse functions are related to each other, although structure sense (reading through

expressions in this case) was not one those three approaches. His reasoning involved procedural, graphical, and structural approaches. His reasoning consistently involved several habits that I modeled during the intervention like circling around algebraic expressions to highlight them and reason about them.

Cross-Case Analysis

In the previous section I provided a description of the development of structure sense for each of the six cases. This description is an in-depth, detailed analysis and answer to the first research question "In what ways do students' work show development of structure sense in a community college precalculus algebra course?" I concluded the description for each with a summary of the detailed analysis. Each summary captures the gist of the case description. In this section, I answer the qualitative research question by combining the results of the individual cases obtained from the previous section. Table 119 shows trends for the evidence of structure sense found in students' work. Students showed development of structure sense in all four definitions. For the first definition (Gen Defn), their development of structure sense is more in terms of graphical interpretations and it is associated with prescribed steps, not so much in reading through expressions and taking a gestalt view of them.

Table 119: Trends for evidence of structure sense in students' work

	Gen. Defn	SS1	SS2-A	SS3
Jim	In terms of graphs and ways associated with prescribed steps	When the recognition is necessary or more expedient	Consistently and with descriptive phrases and gesturing	With awareness of both symbol manipulations and structure
Tom	In terms of graphs and ways associated with prescribed steps	Not consistently, sometimes different forms of the same structure are mixed up	Consistently and with descriptive phrases and gesturing	In terms of substitution and limited number of symbol manipulations
Stacia	In terms of graphs and ways associated with prescribed steps	Consistently through all the items	Consistently and with descriptive phrase and gesturing except when decomposing expressions	With awareness of both symbol manipulations and structure through both items
Ellen	More consistently, not graphically alone	Not consistently	Consistently and with descriptive phrases and gesturing	With awareness of both symbol manipulations and structure
Pete	In terms of graphs and ways associated with prescribed steps	Not consistently through all the items	Consistently and with descriptive phrase and gesturing except when decomposing expressions	Not consistently through all the items
Marc	In terms of graphs and ways associated with prescribed steps	Consistently through all the items	Consistently through all the items	Not consistently through all the items
Pattern	Structure sense is associated with prescribed steps, not necessarily in reading through symbols and taking a gestalt view	In recognizing structure in simplest form, though not consistently	In dealing with a compound term as a single entity in case of substitution, but not consistently when decomposing functions	In performing appropriate symbol manipulations in order to make best use of structure, though not consistently

The cases have shown development of structure sense for the second definition (recognizing structure in their simplest form), but at times they mixed up details between different forms of the same structure. The cases showed development of structure sense in terms of the third definition considered (dealing with a compound term as a single entity) when encountering direct substitution, but there were some difficulty with function decompositions. The cases showed development of structure sense also in terms of the fourth definition (performing symbol manipulation in order to make best use of structure), but it was not consistent throughout perhaps because their awareness of this idea was still new. These cases seemed to have a higher level of awareness about structure as a result of the structure sense teaching intervention.

As shown in Table 119, there were some differences and similarities between the cases. Ellen was the only case who consistently displayed the habit of reading through symbolic expressions (Gen Defn: gestalt view reasoning) as opposed to thinking graphically or procedurally. Jim, Tom, Ellen, and Marc were more consistent in dealing with a compound term as a single entity (SS2-A) using descriptive language like “parenthesis”, “umbrella” along with gestures such as pointing, circling, and underlining. On the other hand, Pete and Stacia were consistent in their structure-sense behavior only in situations involving direct substutions using descriptive language like “parenthesis”, “umbrella” along with gestures such as pointing, circling, and underlining, but in not situations involving decomposition of functions. Jim, Tom, Ellen, and Marc were more consistent in dealing with a compound term as a single entity (SS2-A) with both direct substitution and decomposition. However, Stacia was the only participant who displayed a consistent use of structure sense

through both items in the SS3 category (i.e., choose appropriate manipulations to make best use of structure), showing awareness that symbol manipulations is predicated on structure.

Stacia also demonstrated a deeper level of awareness of the reasons that underpin symbol manipulations. Ellen and Jim showed awareness of structure and symbol manipulations, but not consistently through both items and not with a similar precision as Stacia. Neither Marc nor Pete were consistent in the SS3 category. Tom's structure sense in the SS3 category was focused on substitution rather than symbol manipulations that were informed by structure and aimed at leveraging structure.

The work of those case study participants also showed the expected pattern of procedural thinking even though they exhibited considerable development of structure sense. This pattern was expected considering that thinking operationally/procedurally has been for the most part student's most deeply-rooted habit. During the analysis, I found in some instances that students had to explain their structure sense through procedural approaches because that procedural thinking was necessary in order for them to make their point. They are more fluent and flexible in that mode of thinking, which is one reason why they used the procedural approach. Another reason for the presence or necessity of procedural thinking is the duality between procedural and conceptual thinking (Gray & Tall, 1994; Kieran, 2013; Rittle-Johnson et al., 2015; Sfard, 1991) and the interdependence of the five strands of mathematical proficiency (NRC, 2001) as discussed in the Chapter 2, the Literature Review. That students used procedural thinking to justify their structure sense is commendable; it is in line with the conception algebraic proficiency discussed in this study.

In summary, one of the main points taken from the cross-case analysis is how strong the operational way of thinking was among those students as discussed in the previous

paragraph. According to Sfard (1991), the development of this way of thinking naturally takes root first before structural thinking. Her theory helps in giving reasons for the persistence of this way of thinking. Another main point taken from the cross-case analysis is that there was a more defined pattern across students in how they attended to items pertaining to the general definition structure sense. They did not necessarily read through symbols and analyze expressions in a gestalt manner, except in cases when it was part of a standard or regular approach. In the SS1 items, which have to do with recognizing structures in their simplest forms, their performance was split; half of them were consistent in doing so. Recognizing structures in their simplest forms (SS1) depends on reading through symbols (Gen. Defn.). Hence, it makes sense that the pattern of the lack of reading through symbols would transpire in the SS1 items. Another main point taken from the cross-case analysis is that there was a more defined pattern across students in how they attended to items pertaining to the SS2-A definition structure sense. Students did well in dealing with a compound term as a single entity (SS2-A).

What do these patterns imply for structure sense instruction in precalculus algebra? The answer should be framed in terms of the descriptions of instruction discussed in Chapter 1, the Introduction. According NCTM (1991) and Tarr et al (2008), instruction is characterized by classroom environment, type of tasks, and type of discourse. The NCTM (1991) description includes analysis of teaching (i.e., teachers' systematic reflection). The first pattern about the need for reading and reasoning through symbols implies that students should be given tasks that require this kind of practice on a consistent basis. This pattern about the need for reading and reasoning through symbols implies that instructors should lead classroom discussions with students in debriefing those tasks on a consistent basis.

Emphasizing reading and reasoning through symbols before and while working through symbol manipulation will be beneficial to students' ability to recognize structures in their simplest forms (SS1). The key is to promote the habit in every opportunity as much as possible.

The second pattern about dealing with a compound term as a single entity (SS2-A) implies that tasks and discussions be designed to build on that ability in order to reinforce concepts (especially those that are germane to the symbolic representation). Because of substitution, the SS2-A ability is more familiar to students than analyzing symbolic sentences by reading through them (Gen. Defn) and choosing appropriate manipulations to make best use of structure (SS3). Students have early experiences with substitution when evaluating functions. Then they also experience substitution when solving systems of equations by that method. Therefore, this pattern about familiarity with substitution implies that students should be given tasks whose goal is to improve on that ability. Such tasks should involve decomposing expressions so that students get enough experience of performing substitution in the reverse manner (See task in Table 2 in the Introduction). Those tasks offer students opportunities to learn that complex symbolic sentences are layered and can be unpacked through appropriate substitutions. The key is to help students gain the insight and awareness that complex symbolic sentences often come from familiar structures. In general, this kind of training is effective in building students' intuition for problem solving because problem solving requires application of familiar knowledge in novel situations according to Schoenfeld (2007).

Finally, addressing both of these patterns in conjunction with the SS3 ability will help instruction to be more effective in structure sense and algebraic proficiency. The key is to

emphasize how reasoning symbolically depends on structure and form using Hoch and Dreyfus' definitions of structure sense. As Hiebert and Grouws (2007) put it, the practices that influence the opportunities students have to learn are the emphasis teachers place on different learning goals, the expectations for learning that they set, the kinds of tasks they pose, and the kinds of questions they ask, and the responses they accept. The patterns identified in this analysis are useful in that regard for the purpose of further describing structure sense instruction in precalculus algebra.

Chapter 6: Discussion and Conclusions

The purpose of the study was to investigate "what students' work and explanation of their work would show about a structure-sense teaching intervention, their development of structure sense, and their algebraic proficiency in the context of a community college precalculus algebra course." The study consisted of a quantitative strand to measure students' structure sense and algebraic proficiency, and a qualitative strand to provide a detailed description of students' structure sense.

The quantitative results show that students' structure sense improved significantly over time from the beginning of the semester to the midterm. The quantitative results show that students' algebraic proficiency also improved significantly over time from the beginning of the semester to the midterm. These results are consistent with the research literature that teaching plays a central role in student learning outcome (NCTM, 1991; Hiebert & Grouws, 2007; NMAP, 2008). However, the results that indicate that factors beyond the classroom environment were not influential are not consistent with the research literature. Research have shown that factors beyond the classroom environment do affect student learning outcome (Hiebert & Grouws, 2007). I believe the results would have aligned with the research literature if the sample size was large enough. The issue of sample size being small is a limitation of the quantitative part of the study. At the same time this limitation underscores why the qualitative strand is essentially the major part of the entire investigation for the structure-sense teaching intervention.

The qualitative strand of the study provided results that corroborate with those of the quantitative strand and the hypothesis that students' structure sense would improve due to the teaching intervention. Then the qualitative strand provided some useful details about the

nature of that improvement and whether that improvement is specifically due to the structure sense intervention. The qualitative results and analyses show students' development of structure sense in terms of having more awareness or being more conscientious about the role of structure in algebra. The qualitative results and analyses also show that structure sense is stronger and more consistent when the completion of tasks involves attending to structure though some prescribed steps. However, when the completion of tasks do not require attending to structure, the results show their structure sense is low, except for one case of this study.

The cases showed development of structure sense in regards to Hoch and Dreyfus' (2004) general definition mostly in terms of reasoning about algebraic expressions in a graphical and rule-based way. This result supports with research in the literature that promote multiple representation and the importance of introducing concepts visually to students first. Still, in regards to that definition of structure sense, students' work also reveal that procedural and prescription approaches are the strongest and most consistent among students. Research in the literature as discussed in Chapter 2 have highlighted students' procedural conception of algebra (e.g., Kieran, 1992, 2007; Welder, 2012). One explanation for their procedural tendency is the strong motivation to get to the correct answer; so, they want to be efficient with time on their test. Awareness for reading through algebraic expressions in order to observe structural patterns was taught, but students' work and explanation of their work showed little evidence of that from their own initiative. Although the habit was modeled during instruction, it did not begin to show among students during the midterm or later at the time of the interview. Students read for structural patterns when tasks require it.

The cases examined in the study showed development of structure sense in regards to the Hoch and Dreyfus' (2006) SS1, i.e., in recognizing structure in their simplest form. However, in some intances they mixed up details between different forms of the same structure. Their structure sense development in that category also depended on prior teaching they receive on structure and form and how much understanding they retained. Students (cases in the study) showed development of structure sense in regards to the Hoch and Dreyfus' (2006) SS2-A, i.e., in dealing with a compound term as a single entity. This kind of structure sense seems to have been the most natural to them when performing substitution. Again, this kind of structure sense is similar to translating expressions into graphs because in both cases students are reasoning in manners that have been familiar to them. Whereas, reading through algebraic expressions in order to observe structural patterns is an unfamiliar strategy or habit; it was a new approach for students as taught during the intervention.

Hoch and Dreyfus' (2006) SS3, (i.e., in choosing/performing appropriate symbol manipulation in order to make best use of structure) seems to be highest level of structure sense. Students (cases in the study) also showed development of structure sense in regards to that definition. Explanations given during the interview by several of the students (cases) reveal that they understood why they had to and did perform certain manipulations. However, only one of the participant showed a deeper level of awareness in that regard.

As mentioned earlier, one must ask: Was students' development or improvement in structure sense specifically due to the structure sense intervention. A way of posing a similar question is to ask whether students learned anything at all from the intervention. Did some of the students realize the difference and value of the intervention in comparison to their prior instruction in algebra? Was students' learning in the course impacted more by factors

other than those presented in the intervention? Was students' learning in the course more impacted by prior learning experiences than by the intervention?

Indeed, students' prior learning experiences had a significant impact on their structure sense as it relates to how ingrained the procedural conception can be, as discussed in Kieran (1992, 2007). However, the context of the structure sense instruction was also significant to their learning. In regards to the Hoch and Dreyfus' (2004) general definition of structure sense, Ellen provides an example of a student who internalized the habit of reading through expressions and equations, taking the gestalt view in her analysis of them. In regards to the first component of Hoch and Dreyfus' (2006), Stacia and Marc provided examples of students who paid particular attention to structures and forms of functions as they related to recognizing structures in their simplest form (SS1). The intervention did influence that increased attention to forms. In regards to the second component of Hoch and Dreyfus' (2006) definition, four out of the six students interviewed provided evidence that the intervention consistently affected how they dealt with compound terms as single entities. The gesturing and language they used indicate that the habit was learned from the intervention. It may be noted that of the other two students, one of them missed significant class meetings, which definitely decreased his exposure to the intervention. In regards to the third component of Hoch and Dreyfus' (2006) definition, students (cases) show that they understood why they had to and did perform certain manipulations. One of the participants, Stacia, showed a deeper level of awareness in that regard. The kind of awareness she showed could not have been only from prior learning.

Finally, how did students fare with regard to the question, Did some of the students realize the difference and value of the intervention in comparison to prior instruction of

algebra? This statement from Jim may provide an answer: "This math class was different from previous ones because the focus seemed to be on making sense of the problems instead of blindly performing calculations." Stacia, wrote: "Math 171 this semester went better than my math courses in the previous semesters. What really helped was the way my proffesor [sic] introduced different materials. He would provide different practice problems for us to work out and even analogies used to remember some of the terms more easily." Her statement here supports the assertation I made in the previous paragraph that her awareness about the SS3 ability could not have been only from prior learning.

Limitations, Significance, and Further Research

Conducting this experiment with more mathematics teachers would allow for more students to participate in the study. More participants would increase the statistical power of the quantitative analyses. Statistically, the students who participated in the study were not selected randomly from a population. Consequently, I cannot make population general inferences from what I observed in the quantitative strand of the study. I can only make process inference regarding the intervention and the participants based on results from the regression analyses (Hayes, 2005, p. 205-209 and 241). It was challenging to investigate algebraic proficiency in a qualitative way because of the broad scope of the description of algebraic proficiency provided in the literature as discussed in Chapter 2. It is challenging to adopt a concise analytic framework (qualitative) for algebraic proficiency for reasons discussed in Chapter 2 over issues regarding the definition of school/college algebra. This issue underscores the rationale for a mixed methods embedded design for the study. This aspect of the study support the emphasis placed on the evaluation of the structure sense approach, especially in the qualitative strand.

In addition, the study can contribute significantly to research in mathematics research and to algebra teaching and learning at the community college level. First, the argument I present about the issue of structural conceptions in algebra and the lack of understanding of algebraic concepts germane to the letter-symbolic form may attract needed attention to the problem. Second, the literature review helps highlight the lack of research on the teaching and learning of algebra at the community college level.

Moreover, the study extends application of the structure sense definition initially developed by Hoch and Dreyfus (2004; 2006; 2007) by applying it to functional notation and letter-symbolic representation of functions/relations. This extension of structure sense to function notation and function expression will help students apprehend equivalence and interchangeability between function notation and function expression. It also will support learning of concepts involving function notation such as difference quotient, function composition, function decomposition, and even the analytical expression of limits.

The results may inform research on how to extend the idea of structure sense to other mathematical representations such as graphs. This extension is important because of the crucial role connecting mathematical ideas through multiple representations (Driscoll, 1999; NCTM, 2000). So, even though Hoch and Dreyfus' (2006) definition of structure sense was given for application to algebraic sentences, Hoch and Dreyfus' (2004) definition naturally implies applications to the graphical and numerical representations. This is a significant observation! Consequently, Hoch and Dreyfus (2004) provides a useful link between Hoch and Dreyfus' (2006) idea of structure and Sfard's (1991) idea of structure as discussed in Chapter 1. The kind of research on connecting or extending the structure sense framework to graphs can start with studying students'work in dynamic software environment like Desmos

that instantly translate mathematical sentences into graphs. The following task is an illustration.

Arcavi (1994) proposed a task that requires students to make conjectures (Tarr et al., 2008) and analyze data in ways that develop their understanding (McCaffrey et al., 2001), as discussed earlier in the context of structure sense instruction. The task appeared as follows: "Aided by a graphing calculator or a graphing tool, find one algebraic expression for the function in the following graph (Arcavi, 1994)."

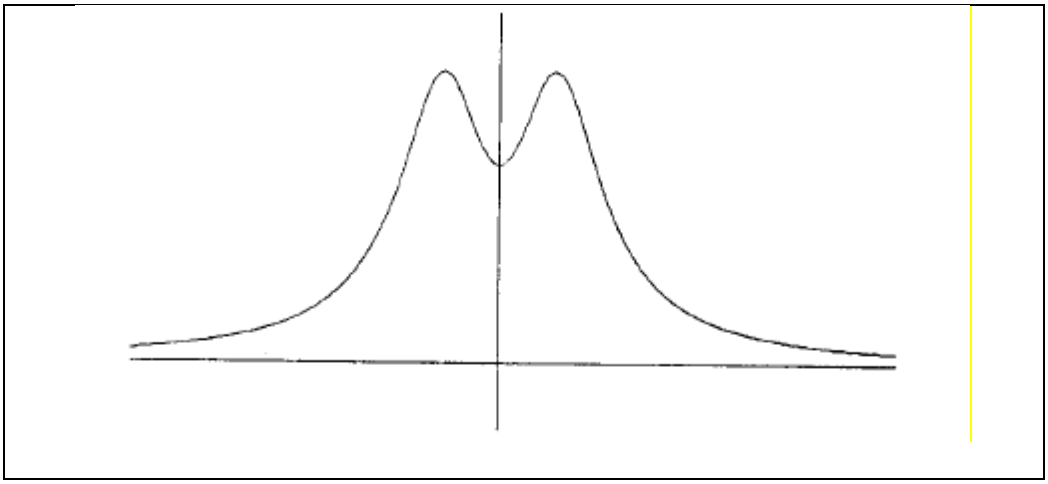


Figure 74: Graph for task taken from (Arcavi, 1994)

The task is consistent with reform-based approach for using technology in ways that support understanding (e.g., thinking among multiple representations) and not simply to compute numerical calculations. Arcavi (1994) commented about different questions that may arise in completion of this task. "Can the function be a polynomial? Why not? Is it a rational function? If so, what are the degrees of the numerator and denominator? How does the symmetry of the graph affect the function? (p. 33)." Every one of these questions deals with structure sense since "Any algebraic expression or sentence represents an algebraic

structure (Hoch & Dreyfus, 2004, p. 50)." Structure-sense instruction on this task involves discussing the fact that rational functions can be constructed with polynomial structures as numerators and denominators. It also involves discussing how even degrees in the numerator and denominator ensure that the graph is always above the x-axis and that the graph also has symmetry across the y-axis. It involves discussing that the degree of the denominator must be greater than the degree of numerator and that all constant terms must be positive to ensure a horizontal asymptote, y = 0, and no x-intercepts. This task can be used to illustrate how structure sense instruction can promote conceptual understanding and adaptive reasoning as well.

Thus, although Hoch and Dreyfus' (2006) three-component definition gives a framework for dealing with the symbolic approach based on algebraic concepts that are germane to the symbolic representation (e.g., coefficients, variables, term, factors, expressions, equations, numerator, denominator, base, exponent, and so on), structure sense still relates to the graphical approach through the connections of multiple representations. The use of multiple representations is another aspect of the 'function' conception of algebra and proficiency in algebra (Carlson et al., 2010). Driscoll (1999) emphasized that proficient algebraic thinkers are those who reason flexibly among different algebraic representations. In the context of representation and reasoning among representations (Driscoll, 1999; CCSSM 2011; NCTM, 2000), structure sense informs how graphs are modeled into expressions or equations and vice versa. The previous example of symbol-sense task from Arcavi (1994) provides an effective illustration.

This line of research connecting structure sense to graphing can explore how the syntax of different dymanic softwares align with order of operations and the way

mathematical sentences are read and put together. This line of research will also consider teaching and learning about reading and expressing mathematical sentences in dynamic software environment. The involvement of technology in mediating the connection between structure sense and graphs is natural because of the integral role of technology in teaching and learning of mathematics (NCTM, 2000). In addition, the need of technology adds an extra layer of depth to the significance of the study as it relates to extending the notion of structure sense to graphs. Within the context of algebra as an activity, the use of technology can be considered as another dimension of algebraic proficiency because of the mediating role it plays. Using appropriate tools strategically is one of the mathematical practices of proficient students (CCSSM, 2011).

For example, algebraic expectation is needed for effective use of Computer Algebra Systems (Pierce & Stacey, 2002, 2004). Students' understanding of concepts allows them to correctly use technology to enter information and to anticipate results and outcomes. Zehavi (2004), like Bokhove and Drijvers (2012), placed greater emphasis on the idea of technology (CAS) supporting the development of symbol sense, rather than the other way around. Bokhove and Drijvers (2012) noted that symbol sense as described by Arcavi (1994) allows procedural skills and conceptual understanding to intertwine. This line of research will consider the role of brackets, parentheses, and concepts like the distributive and associative properties in dynamic software environment, especially with students in the early grade bands.

This study may also guide research on the teaching and learning of arithmetic at the elementary and middle school levels for the purpose of alleviating the need for students to transition from arithmetic to algebra or from procedural thinking to structural thinking. The

ideas of symbol sense and structure sense came from, or were inspired from, number sense (Fey, 1990). More specifically, one of the first studies on structure sense was conducted in arithmetic by Linchevski & Lineh in 1999. Linchevski & Lineh (1999) conducted a study with sixth graders who had not learned negative numbers and algebraic manipulation. This study investigated the assumption that students' difficulties with algebraic structure stem partly from inadequate understanding of structural aspects of arithmetic statements. They found that the difficulties experienced with interpreting equations containing several numerical terms and one unknown by those students were due to the same errors that they made in purely numerical contexts. Those difficulties centered on equivalent structures of expressions.

Structure sense can be taught in arithmetic topics because arithmetic concepts have their analogs in algebra or rather the other way around. Either way the case can be made for structure sense instruction in arithmetic since none of the branches of mathematics exist in isolation from the others. The NCTM Principles and Standards for School Mathematics recommends that the content area of numbers receive less emphasis across the grade bands so that the content area of algebra might receive increasingly more emphasis towards grade 12. According to this trend, it makes sense that college students would be exposed to more topics and more rigor in algebra in their precalculus algebra course than students in middle school and high school. In order for students to learn more algebra successful, there must be a systematic design through the grade bands that incorporate the development of structure sense. At the very least, it is worth exploring because it is important that students' conception of mathematics be both operational and structural, and not lopsided in operational. Balance between the structural and operational will promote relational

understanding as advocated by Skemp (1976) since the two conceptions go hand in hand according to Hiebert and Carpenter (1992), NRC (2001), Zaski and Liljedahl (2002), and Bokhove and Drijvers (2010). This kind of research can start with studying students' understanding of the basic dimensions of a number and the basic dimensions of mathematical sentences (expressions and equations).

Studying the specifics of a systematic redesign that incorporate the development of structure sense in the grade bands is also an aspect of the line of further research associated with the notion of structure sense. This research will be useful in informing the content of remedial courses in college. In the same way, it will be useful in informing the redign of precalculus algebra for college students. Given the fact that a significant portion of students' difficulty stems from the letter-symbolic representation according to literature syntheses in Kieran (1992; 2007) and NMAP (2008), it is crucial that redesigns of precalculus courses incorporate the development of structure sense in a deliberate and systematic way.

Finally, the study has practical significance in terms of the way the literature review in Chapter 2 frames the literature on algebraic proficiency. Its organization provides a useful way of seeing or understanding proficiency in precalculus algebra. The three general descriptions and the five specific descriptions of algebraic proficiency provide a foundation for other aspects about structure sense. It is worthwhile to consider explicit ways in which structure sense is related to those descriptions of algebraic proficiency. This study also adds to research on precalculus teaching and learning in the community college area of the literature.

The study has practical significance as it provides more applications for implementing structure sense instruction in precalulus algebra courses, in addition to those

that were already provided in the literature. At the same time, the implementation of the structure sense instruction in this study provide information for future research in structure sense, algebraic proficiency, and in teaching and learning of precalculus algebra. The study provides a framework for teaching students how to make sense of solving equations algebraically, which will improve among students what Skemp (1976) calls relational understanding. The study has practical significance for highlighting how structure sense can relate to graphically reasoning and the use of dynamic softwares.

With the influence of structure sense on both the first and second levels of algebraic proficiency, its potential application to arithmetic and to graphing, the study has pratical implications for designing curricula for school algebra and redesigns of precalculus courses and electronic homework systems in college. Hence, with all of the potential application of the ideas mentioned herein and the kind of variables that were considered in the quantitative strand, the study can be useful and have a wider impact than simply at the level of curriculum and teaching. For instance, at the community college level, students' completion rates depend on their successful completion of college-level mathematics courses. The impact can be understood in terms of large and increasing numbers of students that enroll in college algebra according to Nguyen (2015) and Reyes (2010) and the large percentage of them that repeat the course (Reyes, 2010). Since college algebra is required for most majors (Reyes, 2010), this kind of research has a wide impact because it concerns completion rates at the college-wide level.

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APPENDICES

Appendix A: Background Survey

1. Informed Consent (Yes/No).

2. What is your email address?

3. Did you take any remedial mathematics courses in college prior to precalculus?

   a. Yes b. No

4. How long has been since you took an algebra course?

5. Is this your first time taking precalculus in college?

   a. Yes b. No

6. What is your enrollment status (Full-time or Part-time)?

7. How many credit hours are you taking?

8. What is your work status? a. Full-time b. Part-time

9. On average how many hours per week do you work?

10. How would you rate your study skills?

    a. Poor b. Average c. Very Good

11. How you would rate your level of access to computer and internet?

    a. Poor b. Average c. Very Good

12. What gender do you associate with?

    a. Male b. Female c. Other

13. What is your race/ethnicity?

14. Are you related to anyone taking this course at the college? a. Yes b. No

Appendix B: Students' Mathematics Interest and Beliefs questionnaire

Thinking about your views on mathematics: to what extent do you agree with the following statements?

(Please darken only one circle in each row.)

	Strongly agree	*Agree*	*Disagree*	*Strongly disagree*
a) I enjoy reading about mathematics.	O_1	O_2	O_3	O_4
b) Making an effort in mathematics is worth it because it will help me in the work that I want to do later on.	O_1	O_2	O_3	O_4
c) I look forward to my mathematics lessons.	O_1	O_2	O_3	O_4
d) I do mathematics because I enjoy it.	O_1	O_2	O_3	O_4
e) Learning mathematics is worthwhile for me because it will improve my career prospects.	O_1	O_2	O_3	O_4
f) I am interested in the things I learn in mathematics.	O_1	O_2	O_3	O_4
g) Mathematics is an important subject for me because I need it for what I want to study later on.	O_1	O_2	O_3	O_4
h) I will learn many things in mathematics that will help me get a job.	O_1	O_2	O_3	O_4

How confident do you feel about having to do the following mathematics tasks?

(Please darken only one circle in each row.)

	Very confident	*Confident*	*Not very confident*	*Not at all confident*
a) Using a train schedule to figure out how long it would take to get from one place to another.	○ 1	○ 2	○ 3	○ 4
b) Calculating how much cheaper a TV would be after a 30% discount.	○ 1	○ 2	○ 3	○ 4
c) Calculating how many square feet of tile you need to cover a floor.	○ 1	○ 2	○ 3	○ 4
d) Understanding graphs presented in newspapers.	○ 1	○ 2	○ 3	○ 4
e) Solving an equation like $3x+5= 17$.	○ 1	○ 2	○ 3	○ 4
f) Finding the actual distance between two places on a map with a 1:10,000 scale.	○ 1	○ 2	○ 3	○ 4
g) Solving an equation like $2(x+3) = (x + 3)(x - 3)$.	○ 1	○ 2	○ 3	○ 4
h) Calculating the gas mileage of a car.	○ 1	○ 2	○ 3	○ 4

Appendix C: Pretest B

1. Use $f(x) = x^2 - 3x$, $g(x) = 7 - 2x^2$ and $h(x) = \sqrt{9-x}$ to complete the following.
   a. $(f - g)(x) =$ (3 points)

   b. $(fg)(1) =$ (3 points)

   c. Write the domain and range of $h(x)$ using interval notation. (4 points)

   Domain: _____ Range: _____

   d. $(f \circ g)(x) =$ (4 points)

   e. Write the function $\left(\frac{h}{f}\right)(x)$. (1 point)

   f. Determine the domain of $\left(\frac{h}{f}\right)(x)$ in interval notation. (5 points)

   Domain: _____

2. If $h(x) = \sqrt{1+(x+2)^3}$, determine two non-trivial functions *f(x)* and *g(x)* such that $h(x) = (f \circ g)(x)$ (4 points)

3. Using the function $f(x) = x^2 - 3x + 4$

a. Construct and simplify the difference quotient $\frac{f(x+h) - f(x)}{h}$ for the function $f(x)$.
(4 points)

b. Calculate the average rate of change of $f(x)$ from $x = 2$ to $x = 2.5$ (2 points)

4. Given $f(x) = \sqrt{x-4}$.
   a. Determine the inverse, $f^{-1}(x)$ (5 points)

b. Use interval notation to complete this table: (4 points)

Function	Domain	Range
$f(x)$		
$f^{-1}(x)$		

5. Use the **<u>quadratic formula or complete the square</u>** to compute the **<u>exact solutions</u>** to the following quadratic equation: (5 points)

$$x^2 + 1 = 4x$$

6. Solve the quadratic inequality and write your solution using interval notation. (5 points)

$$x^2 - 2x \geq 15$$

7. Suppose the quadratic function, $g(x) = -2x^2 + 24x + 7$ represents the height of a projectile in feet after x seconds.

   a. Determine the transformation form of $g(x)$. (3 points)

   b. What is the maximum height of the projectile? (2 points)
   c. When will the projectile strike the ground? *Round your answer to two decimal places.*(2 points)

   d. What is the domain of the function? (2 points)
   e. State the interval during which the height of the projectile is increasing. (2 points)

8. Use the rational function $R(x) = \dfrac{4(x-1)(x+3)}{(x-1)(x+2)}$ to complete the following.

a. State the domain of $R(x)$ in interval notation.

(3 points)

b. The y-intercept of $R(x)$ is (_____,_____). (2 points)

c. The x-intercept of $R(x)$ is (_____,_____). (2 points)

d. Write the **equation(s)** of the horizontal asymptote(s). (3 points)

e. Explain in a complete sentence whether or not there is a "hole" in the graph of this function. If there is a hole write its location as an ordered pair. (3 points)

f. Write the **equation(s)** of the vertical asymptote(s). (3 points)

9. The tax on land is often calculated using a tiered method. In a certain region land is taxed at a rate of 2.5% of the land value for a value up to $100,000 and at a rate of 3% for the value of the land in excess of $100,000.

a. Write the equation of the piecewise function, $T(x)$, that calculates the amount of taxes that must be paid, T, as a function of the value of the land, x. (5 points)

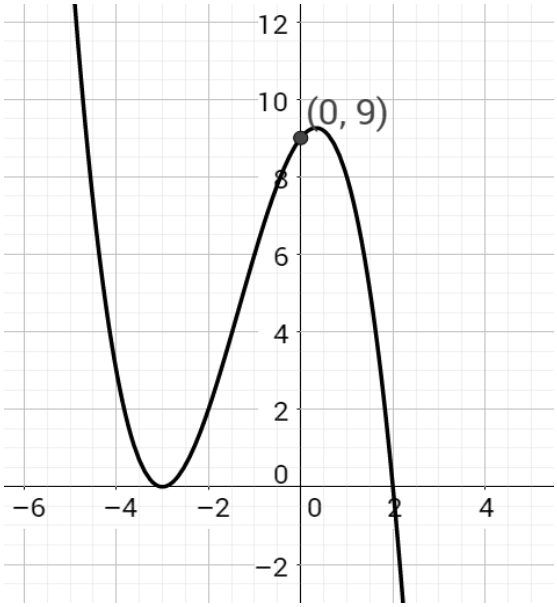
$$T(x)=\begin{cases} \underline{\qquad\qquad\qquad\qquad} & \text{for } \underline{\qquad\qquad} \\ \underline{\qquad\qquad\qquad\qquad}+\underline{\qquad\qquad}(\underline{\qquad\qquad\qquad\qquad}) & \text{for } \underline{\qquad\qquad} \end{cases}$$

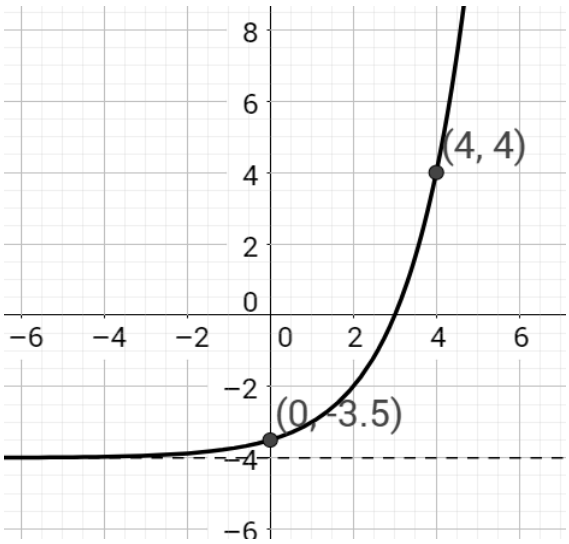
b. Calculate the amount of taxes that must be paid on a piece of land with a value of $125,000.(3 points)

10. Solve. Show all of your work for full credit. (4 points)

$$|2x+1|=15$$

11. Write the equations for each of the following graphs: (6 points each)

a. 

b. 

Appendix D: Posttest B

SP2018 MAT171 Test 2A

Time allowed: 90 minutes

Name:_____

You may use a non-graphing calculator for the test. **Show all work to receive full credit.** If appropriate, round answers to **two decimal** places. Please clearly indicate your final answers.

1. Use the functions $f(x) = 2x + 8$ and $g(x) = \frac{x}{2} - 4$ to answer the parts that follow.
   (a) (4 points) Determine $(f \circ g)(x)$.

   (b) (4 points) Determine $(g \circ f)(x)$.

   (c) (2 points) Using the results of (a) and (b), what conclusion can you make about $f(x)$ and $g(x)$? Please provide your answer in a complete sentence.

2. Use the functions $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ to answer the parts that follow.
   (a) (2 points) One of the functions provided above is one-to-one, and the other is not. Which of the functions **is** a one-to-one function?

   (b) (5 points) Determine the inverse of the one-to-one function identified in (a).

   (c) (2 points) Determine the domain and range of the inverse function found in (b). State your answers in interval notation.

3. Use the functions $f(x) = \frac{x+1}{x}$, $g(x) = 8x$, and $h(x) = \sqrt{4x + 12}$ to answer the parts that follow.
   (a) (3 points) Determine $\left(\frac{h}{g}\right)(x)$.

   (b) (3 points) Determine $(f \cdot g)(x)$.

   (c) (2 points) Determine the domain of $g(x)$. State your answer in interval notation.

   (d) (2 points) Determine the domain of $h(x)$. State your answer in interval notation.

(e) (3 points) Determine the domain of $\left(\frac{h}{g}\right)(x)$. State your answer in interval notation.

4. (4 points) Suppose that $h(x) = \sqrt{x^2 + 6} - 12$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f\big(g(x)\big)$.

   $f(x) =$ _____

   $g(x) =$ _____

5. (5 points) Solve the absolute value equation below.

$$13 - 2|x - 5| = 9$$

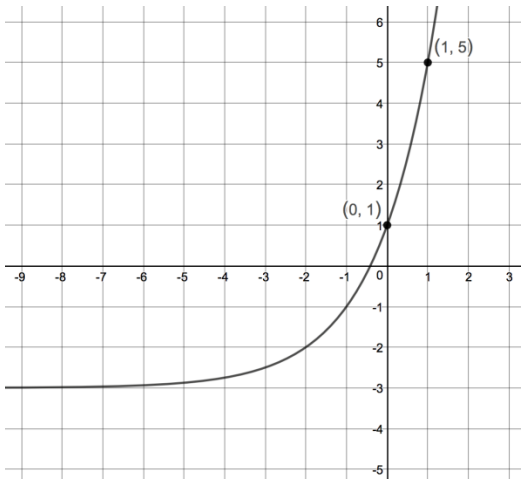
6. (5 points) Construct and simplify the difference quotient for the function $f(x) = x^2 - 3x + 2$.
   For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.

7. The cost to attend Wake Tech can be modeled by a piecewise function. To enroll in 6 credit hours or less, the cost is \$76 per hour. The cost is \$65 per credit hour for each hour above 6 credit hours but less than16 credit hours. To enroll in 16 credit hours or more, the total cost is \$1050.
   (a) (3 points) Write a piecewise function, $C(h)$, that models the cost for enrolling in h credit hours of classes.

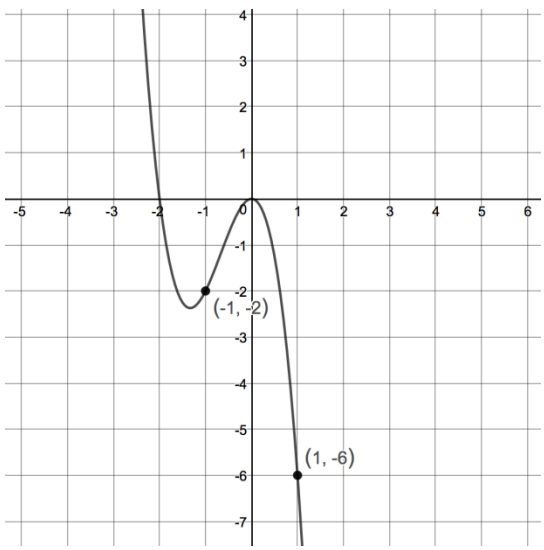
$$C(h) = \begin{cases} \_\_\_\_\_ & \text{for } \_\_\_\_\_ \\ \_\_\_\_\_ + \_\_\_\_ \cdot (\_\_\_\_\_) & \text{for } \_\_\_\_\_ \\ \_\_\_\_\_ & \text{for } \_\_\_\_\_ \end{cases}$$

   (b) (4 points) Calculate the cost to enroll in 12 credit hours of classes

8. (5 points) Determine the equation of the exponential function graphed below. (Hint: The function is in the form $f(x) = a \cdot b^x + c$.)



9. (5 points) Determine the equation of the polynomial function graphed below.



10. (5 points) Determine the transformation form of the quadratic function provided below.

$$f(x) = 2x^2 - 12x + 23$$

11. Use the function $f(x) = -4(x-3)^2 + 16$ to answer the parts that follow. If the requested quantity does not exist, please write DNE.
    (a) (3 points) Determine the x-intercept(s) of the function.

    (b) (3 points) Determine the y-intercept of the function.

(c) (2 points) Determine the vertex of the function.

(d) (2 points) Determine the **equation** of the axis of symmetry of the function.

12. Use the function below to answer the parts that follow. If the requested quantity does not exist, please write DNE.

$$f(x) = \frac{3(x+1)(x-2)}{(x-2)(x+4)}$$

(a) (3 points) Determine the x-intercept(s) of the function.

(b) (3 points) Determine the y-intercept of the function.

(c) (3 points) Determine the **equation(s)** of any vertical asymptotes of the function.

(d) (3 points) Determine the **equation** of any horizontal asymptote of the function.

(e) (3 points) Determine the location of any holes in the function.

13. The grade that Alex receives on a particular test can be modeled by the function below.

$$G(h) = -h^2 + 12h + 64$$

where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours.

(a) (1 point) What score will Alex earn without studying?

(6 points) Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)

Appendix E: Reflection on Posttest B (Test 2)

Items
1. In item #2a on test 2, how do you know which function has an inverse and which one doesn't? 2. In item #2b, did you expect the kind of function or structure you would get for the inverse of g(x)? Yes or no. If yes, how? 3. In item #4, how did you think about decomposing h(x)? 4. In item #5, how did you think about handling the absolute value bars or solving the equation? Why? 5. In items #8 and #9, what difficulty, if any, did you have? 6. In item #11c, how did you come up with your response? What helped you? 7. In item #12a, how did you come up with your response? What helped you? 8. In item #12d, how did you come up with your response? What helped you? 9. In item #12e, how did you come up with your response? What helped you? 10. In item #13, how did you come up with your response? What helped you?

Appendix F: Midterm Reflection Questionnaire

Midterm Reflection Questionnaire

Name:_____

1. Have you had a personal tutor to help you thus far with the course? a. Yes b. No
2. How many hours per week on average did you spend getting tutored?_____
3. Did you participate in study groups? (yes - no). How many hours per week on average have you spent?_____
4. If any, what campus resources did you use thus far (e.g., office hours, STEM Center, ILC)? _____ How many hours per week on average did you use that resource? _____
5. What is your enrollment status? a. Full-time b. Part-time
6. How many credit hours are you taking?_____
7. What is your work status? a. Full-time b. Part-time c. Don't work
8. On average how many hours per week have you been working in the last 7 weeks?
9. How many hours a week on average have you spent on MAT 171 doing homework, reading class notes, watching instructional videos to learn the concepts in this course?
10. Please briefly document anything you believe may have affected your learning experience outside of what is taking place in class so far.

Appendix G: Task-Based Interview Protocol

Part I: Introduction

Thank you for taking the time to meet with me today. As you may know, I am interested in learning about how you think about concepts of precalculus algebra.

Keep in mind that what you say will remain confidential.

Because I am interested in how you think about concepts of precalculus algebra, it would really help me if you would talk as much as you can.

While you are working and talking, I will probably ask questions to be sure I understand what you are saying. I am asking because I am really interested. I am not asking a question because what you are saying is correct or incorrect, but I just want to make sure I understand what you are saying.

Chances are I probably just missed something you said or did. Likewise, if I ask a question that does not make sense to you, please tell me.

Recording:
I might take a few notes to help me remember. But, because I don't want to take too many notes, I would like to tape our conversation. The video camera will be focused on what you are writing or doing.

The main point is talking out loud and sharing your thinking as much as possible.

Part II: Background questions

1. What is your major, career goal, and year in college?
2. What has your experience been like pursuing your college goals, taking different courses?
3. How has been your learning experience in math courses in general?
4. How has been your learning experience with MAT 171, precalculus algebra?
5. How do you compare your learning experience in MAT171 this semester with your learning experience in previous math courses?

Part III: Tasks

1	Use the functions $f(x) = 2x + 8$ and $g(x) = \frac{x}{2} - 4$ to answer the parts that follow. (a) (4 points) Determine $(f \circ g)(x)$. (b) (4 points) Determine $(g \circ f)(x)$. (c) (2 points) Using the results of (a) and (b), what conclusion can you make about $f(x)$ and $g(x)$? Please provide your answer in a complete sentence.
2a	Use the functions $f(x) = x^2 - 2x + 9$ and $g(x) = \sqrt{2x - 7}$ to answer the parts that follow. (2 points) One of the functions provided above is one-to-one, and the other is not. Which of the functions **is** a one-to-one function?
2b	(5 points) Determine the inverse of the one-to-one function identified in (a).
4	(4 points) Suppose that $h(x) = \sqrt{x^2 + 6} - 12$. Determine two nontrivial functions $f(x)$ and $g(x)$ such that $h(x) = f\big(g(x)\big)$. $f(x) =$ _____ $g(x) =$ _____
5	(5 points) Solve the absolute value equation below. $13 - 2\lvert x - 5\rvert = 9$
6	(5 points) Construct and simplify the difference quotient for the function $f(x) = x^2 - 3x + 2$. For $h \neq 0$, the difference quotient is $\frac{f(x+h)-f(x)}{h}$.
11	Use the function $f(x) = -4(x - 3)^2 + 16$ to answer the parts that follow. If the requested quantity does not exist, please write DNE. b. (3 points) Determine the y-intercept of the function. c. (2 points) Determine the vertex of the function. d. Determine the **equation** of the axis of symmetry of the function.
12	Use the function below to answer the parts that follow. If the requested quantity does not exist, please write DNE. $f(x) = \frac{3(x+1)(x-2)}{(x-2)(x+4)}$ a. (3 points) Determine the x-intercept(s) of the function. d. (3 points) Determine the **equation** of any horizontal asymptote of the function.
13	The grade that Alex receives on a particular test can be modeled by the function below. $G(h) = -h^2 + 12h + 64$ where h represents the number of hours spent studying, and $G(h)$ represents the score earned by studying h hours. a. (1 point) What score will Alex earn without studying? b.(6 points) Alex needs at least a 91 on this test in order to earn an A in the class. How many hours of studying would be required for Alex to earn a score of 91 or higher? (In other words, solve the inequality $G(h) \geq 91$.)

Appendix H: Informed Consent Form

NORTH CAROLINA STATE UNIVERSITY
INFORMED CONSENT FORM for RESEARCH
Title of Study: Community College Student Learning in a Precalculus Algebra Course: A Mixed Methods Study of the Influence of a Structure Sense Instruction on Students' Structure Sense and Algebraic Proficiency.
Principal Investigator: Andras Paul Faculty Sponsor: Dr. Lee Stiff

Some general things you should know about research studies.
As a student in the course you are a potential participant in this study. While there are no guaranteed personal benefits for taking part in the study, you may gain a deeper understanding of Algebra and significant insight for your current and future mathematics courses. You will receive a copy of this consent form for your record.

What is the purpose of this study?
The purpose of this study is to examine the influences of a structure sense instructional intervention on students' structure sense and algebraic proficiency. Structure sense instruction is a teaching approach that emphasizes conceptual understanding from a global view of expressions, equations, inequalities and function notation.

What does participation in the study involve?
Participation in this study involves saying yes to the question: "Do you wish to participate?" Declining to participate has no effect on your standing in the course. If you answer yes, then you will fill out a few forms at the start and end of the semester. The first one will be a background survey, which will take no more than 10 minutes. The second is a mathematics interest and beliefs questionnaire, which will take no more than 15 minutes.
All lessons in unit-2 will be videotaped from the back of the classroom (about 8 lessons). Two lessons from unit-1 will be videotaped in the same manner. Out of those who volunteer to be interviewed, I will choose three pairs. The informed consent form includes that interview part just those who will volunteer. So consent to participate in the study covers being interviewed for any students who volunteer. No additional permission is required. But, consent to participate in the study doesn't automatically translate to consent to be interviewed, unless you volunteer. Videotaping and interviewing are both part of the research. All lessons before the last two lessons in unit-1 and all lessons in units 3 and 4 will not be part of the research. The research will last only through units 1 and 2. Only the work/information from students who consent to participate will be collected and used for analysis; but will be deleted once analyses are completed.
The total duration of the study will last approximately 8 weeks. The researcher will videotape approximately 10 lessons, about 8 of those lessons during the second unit. The sequence of interviews will last roughly 30-40 minutes each. You may agree to participate in the overall study, but choose not to participate in any interviews.

Risks
There is minimal risk associated with this research. Participation involves no risk to participants' grades. Interviews are voluntary, accepting or declining to be interviewed has no effect on standing in the course.

Benefits
Your participation in the study will provide data on how to keep improving algebra instruction at the community college level. Thus, you and other students after you can benefit from any improvement.

Confidentiality
Participants' information associated with the data collected will be kept completely confidential, as required by law. Hard copies of the forms will be kept in locked filling cabinets and data stored securely in computers by password protection. You as a participant will never be linked or referred to personally in this study.

Compensation
No compensation is given to participants.

What if you're a student?
Your grades or class standing are not affected by your participation in the study.

What if you have questions about the study?
If you have questions at any time about the study or the procedures, you may contact the researcher at *agpaul@waketech.edu*.

What if you have questions about your rights as a research participant?
In this case, you may contact Nancy Rivers, Department Head of Mathematics and Physics, Wake Tech Community College, at *njrivers@waketech.edu* or (919) 866-5968.

Consent to Participate
"I have read and understand the above information. I have received a copy of this form. I agree to participate in this study with the understanding that I may choose not to participate or to stop participating at any time without penalty or loss of benefits to which I am otherwise entitled."

Do you wish to participate? _____

Subject's name (printed):_____

Parent/Guardian (if under 18 years of age) name (printed):
_____

Subject or Parent/Guardian signature (if under 18 years of age):_____Date _____

Investigator's signature _____ Date _____

Appendix I: IRB Approval

NC STATE UNIVERSITY

Office of Research, Innovation and. Economic Development
Sponsored Programs and Regulatory Compliance Services
research.ncsu.edu/sparcs/

Campus Box 7514
2701 Sullivan Dr. Suite 240
Raleigh, NC 27695-7514
P: 919.515-2444, F: 919-515-7721
E: sps@ncsu.edu

From: Jennifer Ofstein, IRB Coordinator
North Carolina State University
Institutional Review Board

Date: January 31, 2018

Title: Community College Student Learning in a Precalculus Algebra Course: A Mixed Methods Study of the Influence of a Structure Sense Instruction on Students' Structure Sense and Algebraic Proficiency

IRB#: 12523

Dear Andras Paul,

The project listed above has been reviewed by the NC State Institutional Review Board for the Use of Human Subjects in Research, and is approved for one year. **This protocol will expire on 1/31/19 and will need continuing review before that date.**

NOTE:

1. You must use the attached consent forms which have the approval and expiration dates of your study.
2. This board complies with requirements found in Title 45 part 46 of The Code of Federal Regulations. For NCSU the Assurance Number is: FWA00003429.
3. Any changes to the protocol and supporting documents must be submitted and approved by the IRB prior to implementation.
4. If any unanticipated problems occur, they must be reported to the IRB office within 5 business days by completing and submitting the unanticipated problem form on the IRB website.
5. Your approval for this study lasts for one year from the review date. If your study extends beyond that time, including data analysis, you must obtain continuing review from the IRB.

Sincerely,

Jennifer Ofstein
NC State IRB

Stiff 12523 Approval letter.docx 1

Appendix J: Lessons

Converting between exponential and logarithmic equations (unit 1)

The most basic form of exponential and logarithmic equations are

$$(Base)^{Exponent} = Product$$

$$\log_{Base} Product = Exponent$$

1. So think about " $\mathbf{log}_{\boldsymbol{Base}}\ \boldsymbol{Product}$ " as a single unit written with three different parts. Filling in the blank: The single unit word and concept for log is _____.

2. Log_{10} 1,000,000 is a three-part expression that can be thought of as a _____ , which is equal to 6.

3. Reading expressions using the appropriate words help make sense of them. E.g., Notice the words "Base" and "Exponents". Using these words can help us convert from a logarithmic form to an exponential form.

4. Filling in the blank: In an exponential statement, a base is raised to an _____.

5. E.g., Convert Log_{10} 1,000,000 to its equivalent exponential form means that the base 10 will be raised to _____.

6. 1,000,000 is the same as _____.

7. Complete pages 32 and 33 in the lab manual.

Equations of polynomials

	Its Forms
The polynomial structure	$y = a_n(x-r_1)^{m1}(x-r_2)^{m2}\ldots(x-r_n)^{mn}$
	$y = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$

Forms	Discuss parameters, terms, coefficients, factors, degrees, multiplicities, roots, order pairs
$y = a_n(x-r_1)^{m1}(x-r_2)^{m2}\ldots(x-r_n)^{mn}$ e.g., $y = -3(x-4)(x+1)^2$	Is this expression in a sum or a product format? If it's a product, what are the factors? If it's a sum, what are the terms being added?
$y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + a_1 x^1 + a_0$ E.g., $y = 5x^3 - 9x^2 + x - 1$	Is this expression in a sum or a product format? If it's a product, what are the factors? If it's a sum, what are the terms being added?

Quadratic functions

	Its Forms
The quadratic structure	$y = ax^2+bx+c$
	$y = a(x-x_h)^2+y_k$
	$y = (ax-b)(cx-b)$

Name the three forms and identify their parameters (coefficients or tuning controls for graphs).

Forms	Parameters and meaning	Order pairs and meaning
$y = ax^2+bx+c$		
$y = a(x-x_h)^2+y_k$		
$y = (ax-b)(cx-b)$		

Example 1. A street vendor of t-shirts finds that if the price of a t-shirt is set at $x, the profit from a week's sales can be modeled by any of the quadratic expressions below.

E. $(x – 6)(900 – 15x)$

F. $-15(x -33)^2+10935$

G. $-15(x – 6)(x – 60)$

H. $-15x^2 + 990x - 5400$

4. Which form of these quadratic expressions shows most clearly the maximum profit and the price that gives that maximum? Explain.

5. Which form of these quadratic expressions shows most immediately the break-even price? Explain.

6. Which form of these quadratic expressions show most immediately the loss when x=0? Explain.

Quadratic functions (continued)

Example 2. Transformations (translating between forms of quadratic functions).

A	**B**	**C**	**D**
(x – 6)(900 – 15x)	$-15(x-33)^2+10935$	-15(x – 6)(x – 60)	$-15x^2 + 990x - 5400$

1. Use the fact that $x_h = \frac{-b}{2a}$ and $y_k = a(x_h)^2 + b(x_h) + c$ to transform form D to form B.

2. Use the quadratic formula $x_r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ to transform form D into form C.

3. Discuss geometric meanings of tasks 1 and 2.

4. Try on your own.

Pages (lab manual)	Exercises (#)
70	I
71	1
72	1
73	1

5. Discuss responses for exercises in 4.

Analysis of rational functions

	Its Forms
The rational structure	$y = \frac{polynomial}{polynomial}$
	$y = \frac{a_n(x-r1)^{m1}(x-r2)^{m2}\dots(x-rn)^{mn}}{b_n(x-s1)^{e1}(x-s2)^{e2}\dots(x-sn)^{en}}$
	$y = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x^1 + b_0}$

Forms	**Discuss parameters, terms, coefficients, factors, degrees, asymptotes, intercepts, end behaviors**
$y = \frac{a_n(x-r1)^{m1}(x-r2)^{m2}\dots(x-rn)^{mn}}{b_n(x-s1)^{e1}(x-s2)^{e2}\dots(x-sn)^{en}}$ E.g., $y = \frac{-4(x-1)^2(x+3)^1}{(x+5)^2(x-2)^2}$	
$y = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x^1 + b_0}$ $y = \frac{x^2 - x + 6}{x^2 + x - 2}$	

Piecewise functions

Functions are classified within structures (or types) and each structure or type can have several different forms (or looks). E.g., some of the most common structures are linear, quadratic, higher order polynomials, and so on.

	Its Forms
The linear structure	y=mx+b
	$y-y_1 = m(x-x_1)$
	Ax+By+C=0

We can use the 'y=mx+b' structure to model real-life situations involving linear rates. That is,

y	Dependent variable (d.v.) - the variable we care the most about in the situation.
x	Independent variable (i.v.) - the variable that affects the d.v.
m	Rate
b	starting value of the d.v. – y_0

Read critically to identify and connect these elements. Let's work on a Payroll example.

Example 1: Using piecewise functions, write a small program to compute gross pay for employees who works for $12 an hour and $18 an hour for overtime.

	First piece	Second piece
m	12	18
b	0	40*12

$$GP(x) = y = \begin{cases} 12x, & 0 < x \leq 40 \\ 18(x-40) + 480, & x > 40 \end{cases}$$

Notice how both rules or pieces follow the 'y=mx+b' structure. In the second rule, notice that m=18 but the i.v. is a compound term = (x-40) and the initial value is 480.

Example 2: Try this on your own. Write a program to compute the monthly phone bill for customers. The plan costs $46 for up to 500 minutes and charges 16¢ for each additional minute of usage.

Discussion of model and solution to questions. Homework for next class meeting. Bring your models to class for discussion.

Pages (lab manual)	Exercises (#)
82	4
83	5

Function notation and combining function

Function Notation: In general, function notation refers to this convention or short hand:

Name of a function **(**name of independent variable**)**

For example, we can have three different function notations like

- f(x)
- tan(u)
- q(s)

In f(x), f is the name of the function and x is the name of the independent variable.

Note: The letter f here does not multiply with x, instead it symbolizes the action of the function on x.

Example A: In tan(u), what is the name of the function? What is the name of the independent variable?

Example B: The function $p(x) = 10\sqrt{x}$ is used to adjust original test scores, x, to new test scores *p(x)*, in percent.

5. In light of this situation, what does the 90 in the equation $90 = 10\sqrt{x}$ represent?

6. How about the expression $10\sqrt{x}$? What does it represent?

7. Based on your responses to questions 1 and 2, what is the equation $90 = 10\sqrt{x}$ stating or saying?

- Whole-class discussion of examples A and B.

II. Combinations

Given $f(x) = 4x + 1,\ g(x) = x^2 - 5$, and $h(x) = \sqrt{x - 8}$

1. Write the expression that g(x) represents _____

2. Write the expression $\frac{x^2-5}{4x+1}$ in terms of the function notations above _____

3. The function notation $(f \circ g)(x)$ is the same as $f(g(x))$. Explain briefly why $f(g(x))$ is the same as $f(x^2 - 5)$.

4. Since $f(x) = 4x + 1$, then $f(x^2 - 5)$ =_____

5. The function notation $(g \circ h)(x)$ is the same as the function notation _____

6. Write the expression for $(g \circ h)(x)$. _____

- Whole-class discussion of examples 1-6.

- Student exercises

Pages (lab manual)	Exercises (#)
53	1 3
54	5 6

- Whole-class discussion of exercises.

III. Function notation and general transformation

In explicitly defined functions **y =f(x)** or y = name of the function (indep. Var.)

1. If $f(x)=x^2$, then _____ = $-x^2$.

2. If g(x)=x+3, then _____ = -(x+3).

3. If $h(x) = |x|$, then $|x| - 9$ is the same as _____ - 9.

4. If $j(x) = \sqrt{x-7}$, then $-2\sqrt{x-7} + 8 =$ _____.

5. If $q(x) = x^2$, then _____ = $(2x-3)^2$

6. Given $f(x) = x^2$, determine $f(x-6) + 10$ based on $f(x)$.

- Student presentations of their responses to items 1 – 6.

Lesson 3: Combining and decomposing functions (unit 2)

Examples

3. (a) State all the operations involved in $Q(x) = \frac{5}{3x+4}$. (b) Do the same for $R(x) = \sqrt{4x-3}$.

4. Which of the operations describes $\frac{5}{3x+4}$ entirely or a as whole? How about $\sqrt{4x-3}$?

Use this format to decompose functions: 1st, write the operation that captures the expression as a whole. 2nd, write smaller or inner operation(s) in the expression. For example, 1 can be $\frac{5}{x}$ and 2 can be 3x+4.

- Whole-class discussion of examples 1-2.

- Student exercises

Pages (lab manual)	Exercises (#)
56	13 14 15 16

- Whole-class discussion of exercises.

Lesson 4: ARC and difference quotient

Students complete items 1 and 2 below on their own.

3. The average rate of change (ARC) of the function y = f(x) between x_1 and x_2 is $\frac{change\ in\ y}{change\ in\ x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Use appropriate substitution(s) to show that the ARC formula comes from a familiar formula.	4. Use appropriate substitution(s) to show that the difference quotient, $\frac{f(a+h)-f(a)}{h}$ Comes from a familiar structure.

- Whole-class discussion of items 1 and 2

- Teacher-lead discussion of meaning of average rate of change and of example 1 on page 58

- Student exercises

Pages (lab manual)	Exercises (#)
59	2 3

- Whole-class discussion of exercises.

Difference quotient(Continued)

Students complete items 1 and 2 below on their own.

3. Write the appropriate substitution to show that $f(a+h) = 3(a+h)^2 - 5(a+h) + 8$ and $f(x) = 3x^2 - 5x + 8$ have the same structure or the same type.	4. Apply suitable substitutions to simplify $(x+h)^2 + 3(x+h) - (x^2 + 3x)$ using function notation.

- Whole-class discussion of items 1 and 2

- Teacher-lead discussion of meaning of average rate of change and of exercise 1 on page 60 and exercise 5 page 60.

- Student exercises

Pages (lab manual)	Exercises (#)
61	7 13

- Whole-class discussion of exercises.

Solving absolute value equations

Solving an equation means to find the values for the unknown(s) for which the equation is true. It is true that some equations can be solved by inspection, by guess-and-check, graphically, or numerically. But solving equations algebraically (analytically) involves the following three key ideas.

1. Identify the defining structure of the equation.
2. Collect, isolate, or single out that defining structure.
3. Break that defining structure by its inverse, or by factoring, or by applying its definition.

Equations	**Defining Structure**	**Breaking the Structure**
1. $4x - (x + 1) = 7 + x$	Linear	Multiply by a multiplicative inverse or add an additive inverse. For example • $2x = 8$ can be solved by multiplying by the multiplicative inverse of 2 which is $\frac{1}{2}$. • $x - 5 = 11$ can be solved by adding the additive inverse of -5 which is +5 on each side.
2. $x^2 + 3x - 10 = 0$	Quadratic	Either by • Completing the square and taking the square root • Factoring into linear factors • Quadratic formula
3. $5\sqrt{17x + 2} - 30 = 0$ 4. $(3x + 4)^{\frac{3}{2}} = 16$ 5. $\sqrt{4x + 5} - 4 = x$	Radical	• #3 can be solved by squaring or raising to the 2 power because √ means power $\frac{1}{2}$. • #4 can be solved by raising each side of the equation to the $\frac{2}{3}$ power.
6. $3\|4x - 7\| + 2 = 41$	Absolute value	Apply the definition of abs. value which eliminates the absolute value bars and splits the equation into two different equations: • One for the distance to the left of zero • The other for the distance to the right of zero.
7. $\frac{2}{x+4} + \frac{3}{x-4} = \frac{16}{x^2-16}$	Rational	Multiply the equation by a common denominator.
8. $\log_3(x + 15) = 2 + \log_3(x - 1)$	Logarithmic	Convert to an exponential statement, which is the inverse of log.
9. $-7 + 4(3^{5x}) = 9$	Exponential	Take log on each side or convert to a logarithmic statement, which is the inverse of exponential.

Note: Before breaking a structure the second idea must be applied, meaning (isolate, collect, or single out the defining structure).

Solving absolute value equations (continued)
Example

$3\|4x-7\|+2=41$	Absolute value	Apply the definition of abs. value which eliminates the absolute value bars and splits the equation into two different equations: • One for the distance to the left of zero • The other for the distance to the right of zero.

First: Before breaking a structure the second idea must be applied, meaning **(isolate** absolute value structure, $|4x-7|$).

Work out example.

$3\|4x-7\|+2=41$	

- Student exercises

Pages (lab manual)	Exercises (#)
85	1 - 2
86	3, 4, 6, 11

- Whole-class discussion of exercises.

Solving quadratic inequalities

In examples 1-4, provide responses based on the terms and meaning of the inequalities (without resorting to their graphs).

Example 1. Explain why x on $(-\infty, \infty)$ is the solution of the inequality $x^2 \geq 0$?

Example 2. Explain why the inequality $x^2 < 0$ has no solutions?

Example 3. Explain why x on $(-\infty, \infty)$ is the solution of the inequality $-x^2 - 6 < 0$? How about $-(x+5)^2 - 6 < 0$?

Example 4. Explain why the inequality $x^2 + 1 < 0$ has no solutions?

Solving polynomial inequalities involves collecting all nonzero terms on one side of the inequality symbol making the other side zero. For example,

polynomial < 0

This allows us to think about the resulting polynomial as a single entity, say y. For example,

y < 0

This rephrasing of an inequality statement allows us to think about when is y negative. In other cases, when is y positive, and when is y = 0 (x-intercept).

Example 5. Solve

$$x^2 - 6x \geq 7$$

Solution

$$x^2 - 6x - 7 \geq 0$$

Factor the quadratic polynomial to figure out the x-intercepts. The x-intercepts will divide the x-axis into several intervals. $x^2 - 6x - 7$ will be negative on some these intervals and positive on some other intervals.

Determine the x-intercepts of $x^2 - 6x - 7$ by factoring or by using the quadratic formula.

The x-intercepts are -1 and 7, which divide the x-axis into three intervals $(-\infty, -1)$, $(-1, 7)$, and $(7, \infty)$. Testing x-values from each interval reveals that $x^2 - 6x - 7$ is 0 and positive on the interval $(-\infty, -1] \cup [7, \infty)$.

The main point is thinking the polynomial as a single unit y for determining what x values make **y < 0** or in other cases **y > 0** or **y >= 0** or **y =< 0**.

- Student exercises

Pages (lab manual)	Exercises (#)
80	5 6

- Whole-class discussion of solutions of student exercises.

Appendix K: Course Project

Project on Structure Sense

The goal of this project is to get us to better reason about algebraic expressions and equations using structure.

Task #1: (20 pts) Solving equations

a. Without solving for x and without graphing, explain why there exist no x-value for which $\frac{2x+3}{4x+6} = 2$?
b. Explain the difference between an algebraic equations and an algebraic expression. Give an example for each.
c. Under what conditions can we say that an equation is solved for a certain variable algebraically?
d. Explain the difference between terms and coefficients. Give an example.
e. Use class notes, state the three ideas involved in solving equations algebraically (analytically).

Task #2: (10 pts) Under what circumstances will a rational function have a domain consisting of all real numbers? Provide and example and explain.

Task #3: (20 pts) Explain how you can tell (without graphing it) that the function

$$r(x) = \frac{x^6 + 10}{x^4 + 8x^2 + 15}$$

has no horizontal intercept and no horizontal, vertical, or slant asymptote. What is its end behavior?

Task #4 & Task #5:

Use the following situation to complete tasks #4 and #5: A street vendor of t-shirts finds that if the price of a t-shirt is set at x dollars, the profit from a week's sales is $-15(x-33)^2 + 10935$.

Task #4: (14 pts) Why is \$10,935 the maximum profit and \$33 is the price which gives that maximum profit? Provide an explanation based on the terms of the expression $-15(x-33)^2 + 10935$, (beyond using the vertex, or graph, or guess & check).

Task #5: (12 pts) The following expressions are equivalent to $-15(x-33)^2 + 10935$, which is called the transformation or vertex form for quadratic polynomials.

c. $-15(x-6)(x-60)$
d. $-15x^2 + 990x - 5400$

(ii) Identify the form of both expressions (a and b) above (i.e., expanded, factored/root, standard).

(iii) Which form shows the prices that correspond to zero profit? Be sure to include an algebraic explanation in your response based directly on the expression not just a geometric or graphical response.

(iv) Which expression (a or b) shows the y-intercept most clearly?

Task #6: (9 pts) In explicitly defined functions **y =f(x)** or y being the name of the function (Dep. Var.)

9. If g(x)=x+3, then _____ = -(x+3).

10. If $j(x) = \sqrt{x-7}$, then $-2\sqrt{x-7} + 8 =$ _____.

11. If $q(x) = x^2$, then _____ = $(2x-3)^2$

12. if $f(x) = 2x^2 - 3x + 1$ and $g(x) = 2x - 3$, then $2(2x-3)^2 - 3(2x-3) + 1$ in function notation is _____.

Task #7 (15 pts) Aided by a graphing calculator or a graphing tool (e.g., **Desmos**), find one algebraic expression for the function in the following graph.