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Stochastic Elasto-Plastic Model by First Order Spectral Expansion

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ABSTRACT

This paper proposes an efficient and accurate treatment in evaluating stochastic behaviour of a non-linear elasto-plastic material with uncertain properties. This treatment is based on the first order spectral expansion of the stochastic behaviour. The return mapping algorithm is applicable to solve the resulting stochastic non-linear elasto-plastic constitutive relation, so that it is readily implemented in a spectral stochastic finite element method. A numerical experiment is carried out to examine the validity of the proposed treatment. It is shown that the stochastic stress-strain relation is simulated successfully, with suitable agreement of a standard Monte-Carlo simulation. The accuracy of evaluating the mean and variance of the proposed treatment is studied.

INTRODUCTION

Probabilistic risk analysis (PRA) is a standard practice in ensuring the safety of nuclear power plants and facilities. Efficient treatment of uncertainty in soil properties is a key issue especially in considering the seismic safety. Monte-Carlo simulation (MCS) is usually used for such an analysis, which relies on repeated random sampling. Although MCS is versatile, it is computationally expensive because of repetitive deterministic simulations with the sampled data.

An alternative of MCS with relatively lower computational cost is spectral stochastic finite element method (SSFEM) developed by Ghanem and Spanos (1991). In SSFEM, applying the spectral expansions to the spatially varying material properties and the responses, the probabilistic distribution of responses is obtained without repeating computation. However, the difficulty in applying the spectral expansion to non-linear elasto-plastic materials is a hinge in applying SSFEM to PRA of non-linear responses.

Anders and Hori (1999, 2001) developed non-linear SSFEM using the bounding media theory proposed by Hori and Munasinghe (1999). This treatment enables to reasonably estimate the upper and lower bounds of the non-linear responses due to stochastic Young's modulus. However, it must evaluate yielding (or the occurrence of plastic deformation) deterministically. Recently, Jeremic et al. (2007), Sett et al. (2007), Sett and Jeremic (2009, 2010) and Jeremic and Sett (2009) proposed a new treatment of the non-linear stochastic elasto-plasticity by using the Fokker-Planck-Kolmogorov (FPK) equation; the FPK equation is given as an advection-diffusion equation in the probabilistic space of stress. Although this treatment allows estimation of full probabilistic distribution of stress, it is ineffective to apply to non-linear SSFEM since the FPK equation must be solved at all the Gauss points in all the calculation steps.

In this paper, we propose an efficient and accurate treatment of stochastic elasto-plastic constitutive relation to which the spectral expansion is applied. The resulting relation is solved by using a return mapping algorithm which is extended in the probabilistic space of stress up to the first order spectral expansion. This treatment is readily implemented in SSFEM, and enables efficient calculation of mean and variance of responses even when yielding process takes place stochastically.

The contents of this paper are organized as follows; First, the new treatment of the stochastic elasto-plastic constitutive is presented. It is shown that an elastic predictor/implicit return mapping algorithm is applicable to solve the constitutive relation for a given strain rate. Based on this analysis, a spectral stochastic return mapping algorithm (SSRMA) is formulated, Finally, numerical experiments are carried out to verify SSRMA.

OVERVIEW OF RETURN MAPPING ALGORITHM

In the theory of plastic flow, the yield function f is defined by $f(\boldsymbol{\sigma},\lambda)=\sigma_{\mathrm{eq}}(\boldsymbol{\sigma},\lambda)-\sigma_{\mathrm{Y}}(\lambda)$ with equivalent stress σ_{eq} and yield stress σ_{Y}, where $\boldsymbol{\sigma}$ and λ denote the stress tensor and the plastic multiplier, respectively. The stress $\boldsymbol{\sigma}$ of elasto-plastic bodies is estimated based on the constraint condition

$$\Delta\lambda \geq 0, \qquad f \leq 0, \qquad \Delta\lambda f = 0 \tag{1}$$

The elastic predictor/implicit return mapping algorithm is a method for solving time integration of Eq.(1) (for example, see Simo (1998)). Figure 1 shows the evaluation flow of this algorithm. First, an elastic predictive stress $\boldsymbol{\sigma}_{\mathrm{trial}}$ is estimated with given strain increment $\Delta\,\boldsymbol{\epsilon}$ as follows:

$$\boldsymbol{\sigma}_{\mathrm{trial}} = \boldsymbol{\sigma} + \boldsymbol{D} : \Delta\,\boldsymbol{\epsilon} \tag{2}$$

where $\boldsymbol{D}$ is an elastic coefficient tensor. $\boldsymbol{\sigma}_{\mathrm{trial}}$ is not always true for elasto-plastic bodies. Therefore, when the plastic admissibility condition $f(\boldsymbol{\sigma}_{\mathrm{trial}},\Delta\lambda) \leq 0$ is satisfied, $\boldsymbol{\sigma}_{\mathrm{trial}}$ is correct. When it is not satisfied, the elasto-plastic body yields. In that case, by evaluating $\Delta\lambda$ to suit the plastic consistency condition $f(\boldsymbol{\sigma},\Delta\lambda)=0$, the stress is updated as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathbf{trial}} - \Delta\,\lambda\,\boldsymbol{D} : \partial_{\boldsymbol{\sigma}} f \tag{3}$$

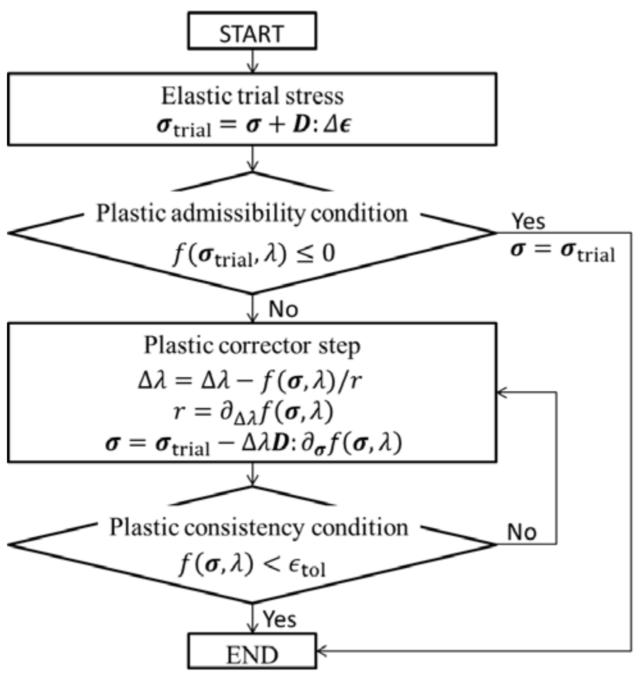

Figure 1. Evaluation flow of return mapping algorithm

When the material properties are uncertain, the response fields such as stress and yield function also become random variables. As a result, all equations and conditions are evaluated at each probabilistic sample point ω, i.e.

$$\boldsymbol{\sigma}_{\text{trial}}(\omega) = \boldsymbol{\sigma}(\omega) + \boldsymbol{D}(\omega) : \Delta\, \boldsymbol{\epsilon}(\omega) \tag{4}$$
$$f(\boldsymbol{\sigma}_{\text{tiral}}(\omega), \lambda(\omega), \omega) \le 0 \tag{5}$$
$$\boldsymbol{\sigma}(\omega) = \boldsymbol{\sigma}_{\text{tiral}}(\omega) - \Delta\, \lambda(\omega) \boldsymbol{D}(\omega) : \partial_\sigma f(\omega) \tag{6}$$
$$f(\boldsymbol{\sigma}(\omega), \lambda(\omega), \omega) \ = \ 0 \tag{7}$$

In Monte Carlo method, the probabilistic distributions of responses are obtained by repeated random sampling. On the assumption that the yield function of $\boldsymbol{\sigma}_{\text{trial}}(\omega)$, $f_{\text{trial}}(\omega)$ is normally distributed, the distribution of $f_{\text{trial}}(\omega)$ becomes like the red dashed line in Figure 3. In this case, a portion of the distribution of $f_{\text{trial}}(\omega)$ is greater than zero, and the rest is less than or equal to zero. After return mapping, the part with $f_{\text{trial}}(\omega) > 0$ is turned to 0 by updating stress. As a result, the distribution of the yield function after return mapping $f(\omega)$ is expressed by the blue line in Figure 3, and obtained by

$$\text{PDF}[f(\omega)] = \begin{cases} \text{PDF}[f_{\text{trial}}(\omega)]|_{f_{\text{trial}}=f} & (f < 0) \\ C[0, f_{\text{trial}}]\delta(f_{\text{trial}}) & (f = 0) \\ 0 & (f > 0) \end{cases} \tag{8}$$

where $\delta(X)$ is the Dirac's delta function. $\text{PDF}[f_{\text{tiral}}]$ and $C[x, f_{\text{trial}}]$ respectively denote the probabilistic distribution function (PDF) and the cumulative distribution function of $f_{\text{trial}}(\omega)$, i.e.

$$\text{PDF}[f_{\text{tiral}}] = \frac{1}{\sqrt{2\,\pi\, \text{VAR}(f_{\text{trial}})}}\, exp\left(-\frac{(\, f_{\text{trial}}(\omega) - \langle f_{\text{tiral}} \rangle)^2}{2\,\pi\, \text{VAR}(f_{trial})}\right) \tag{9}$$
$$C[x, f_{\text{trial}}] = \int_{-\infty}^{x} \text{PDF}[f_{\text{tiral}}]\, df_{\text{trial}} \tag{10}$$

in which $\langle X \rangle$ and VAR(X) symbolize the mean and variance of a random variable $X(\omega)$, respectively.

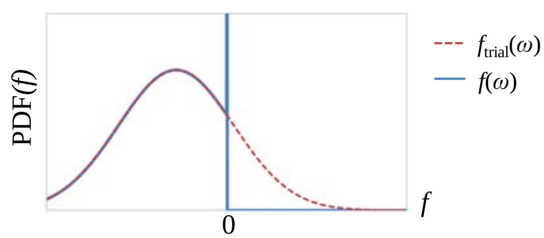

Figure 2. Yield function ($f_{\text{trial}}(\omega)$ and $f(\omega)$)

SPECTRAL STOCHASTIC RETURN MAPPING ALGORITHM

Overview of Spectral Expansion

In the spectral method, the probabilistic distribution is represented by a series of spectral components in probabilistic space. Two spectral expansions are applied to random variables in SSFEM, Karhunen-Loeve (KL) expansion and Polynomial Chaos (PC) expansion. In this section, we briefly review these spectral expansions with two random material properties.

The KL expansion represents the spatial correlation of material properties. When Young's modulus E and yield stress σ_Y are assumed to be random variables with the Gaussian distribution, they are represented with the KL expansion as

$$E(\boldsymbol{x},\omega) = \langle E(\boldsymbol{x})\rangle \left(1 + \sum_{i=1}^{\infty} \sqrt{c_i^E}\phi_i^E(x)\xi_i^E\right) \tag{11}$$

$$\sigma_{\mathrm{Y}}(\boldsymbol{x},\omega) = \langle \sigma_{\mathrm{Y}}(\boldsymbol{x})\rangle \left(1 + \sum_{i=1}^{\infty} \sqrt{c_i^Y}\phi_i^Y(x)\xi_i^Y\right) \tag{12}$$

where $\xi_i^E(\omega)$s and $\xi_i^Y(\omega)$s are zero-mean random variables with Gaussian distribution. They are the orthonormal basis which satisfy the orthogonal condition $\langle \xi_i^E(\omega)\xi_j^E(\omega)\rangle = \delta_{ij}$, $\langle \xi_i^Y(\omega)\xi_j^Y(\omega)\rangle = \delta_{ij}$ and $\langle \xi_i^E(\omega)\xi_j^Y(\omega)\rangle = 0$, in which δ_{ij} denotes the Kronecker delta. Then $\{c_i^E\ ,\phi_i^E\}$ indicate the eigen-values and the eigen-functions of the covariance functions $R_E(\boldsymbol{x}_1,\boldsymbol{x}_2)$ for Young's modulus, i.e.

$$\int R_E(\boldsymbol{x}_1,\boldsymbol{x}_2)\phi_i^E(\boldsymbol{x_2})d^3x_2 = c_i^E\phi_i^E(\boldsymbol{x}_1) \tag{13}$$

This relation can be also applied to $\{c_i^Y,\phi_i^Y\}$ and the covariance functions $R_Y(\boldsymbol{x}_1,\boldsymbol{x}_2)$ for yield stress. For the spatial correlation of soil properties, the exponential or the Gaussian correlation function is commonly used, e.g. three dimensional covariance function with the exponential correlation function is given by

$$R_E\,(\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{\mathrm{VAR}(E)}{\langle E\rangle^2}\exp\left(-\frac{|x_1-x_2|}{b_x}-\frac{|y_1-y_2|}{b_y}-\frac{|z_1-z_2|}{b_z}\right) \tag{14}$$

where b_x, b_y and b_z are the correlation length in spatial direction of *x*, *y* and *z*, respectively.

The PC expansion describes complex probabilistic distributions of responses. We only account for the first order expansion here; refer to Ghanem and Spanos (1991) for more precise definition. In the case that $E(\omega)$ and $\sigma_Y(\omega)$ are represented by the KL expansion, the displacement $\boldsymbol{u}(\boldsymbol{x},\omega)$ is expanded by the first order PC expansion as follow

$$\boldsymbol{u}(\boldsymbol{x},\omega) = \langle \boldsymbol{u}(\boldsymbol{x})\rangle + \sum_{i=0}^{M_E}\boldsymbol{u}^{(i)}(\boldsymbol{x})\xi_i^E(\omega) + \sum_{i=0}^{M_Y}\boldsymbol{u}^{(i+M_E)}(\boldsymbol{x})\xi_i^Y(\omega) \tag{15}$$

where the KL expansions for E and σ_Y are truncated up to M_E and M_Y, respectively. For notational simplicity, one can rewrite Eq.(15) as

$$\boldsymbol{u}(\boldsymbol{x},\omega) = \sum_{i=0}^{M}\boldsymbol{u}^{(i)}(\boldsymbol{x})\xi_i(\omega) \tag{16}$$

with $M = M_E + M_Y$ and

$$\xi_i(\omega) = \begin{cases} 1 & (i=0), \\ \xi_i^E(\omega) & (1 \le i \le M_E) \\ \xi_i^Y(\omega) & (M_E+1 \le i \le M) \end{cases} \tag{17}$$

Since $\xi_i(\omega)$s also satisfy the orthogonal condition, the mean and variance of $\boldsymbol{u}(\boldsymbol{x},\omega)$ is estimated by

$$\langle \boldsymbol{u}(\boldsymbol{x})\rangle = \boldsymbol{u}^{(0)}(\boldsymbol{x}), \tag{18}$$

$$\mathrm{VAR}(\boldsymbol{u}) = \sum_{i=1}^{M}\left(\boldsymbol{u}^{(i)}(\boldsymbol{x})\right)^2 \tag{19}$$

In this way, the uncertainties of responses are affected by the uncertainties of all material properties.

Concept of Spectral Stochastic Return Mapping Algorithm

The key concept of our stochastic elasto-plastic model is the expansion of the return mapping algorithm in the probabilistic space by the spectral method, where we call to our model as Spectral Stochastic Return Mapping Algorithm (SSRMA). In SSRMA, Young's modulus and yield stress are material properties represented with the KL expansion, and stress, yield multiplier and yield function are response fields expressed by the PC expansion. Since the truncation of the spectral expansions up to zeroth order corresponds to the deterministic estimation, more than first order spectral expansion is required to take into account the uncertainties of material properties. In this paper, as the first step to develop a spectral stochastic elasto-plastic model, we apply response fields of yielding process to the first order PC expansion i.e.

$$\boldsymbol{\sigma}(\omega) = \sum_{i=0}^{M} \boldsymbol{\sigma}^{(i)} \xi_i(\omega) \tag{20}$$

$$\Delta\lambda(\omega) = \sum_{i=0}^{M} \Delta\lambda^{(i)} \xi_i(\omega) \tag{21}$$

$$f(\omega) = \sum_{i=0}^{M} f^{(i)} \xi_i(\omega) \tag{22}$$

This assumption corresponds to evaluation of only the mean and variance of the responses by Eq.(18) and Eq.(19). The zeroth and the first order terms are dominant for estimation of the mean and variance in that the higher order terms are smaller than zeroth and first order terms except for multimodal distribution. Consequently, it is expected that our model enables evaluation of the fundamental behaviours of mean and variance of responses in the stochastic yield process.

Formulation of Spectral Stochastic Return Mapping Algorithm

SSRMA is formulated in this section. In accordance with the spectral method, all equations are defined for the spectral expansion coefficients, which means that Eq.(4) and Eq.(6) are obtained by

$$\boldsymbol{\sigma}_{\mathrm{trial}}^{(i)}(\omega) = \boldsymbol{\sigma}^{(i)}(\omega) + \langle \xi_i\, \boldsymbol{D}(\omega) : \Delta\, \boldsymbol{\epsilon}(\omega) \rangle \tag{23}$$

$$\boldsymbol{\sigma}^{(i)}(\omega) = \boldsymbol{\sigma}_{\mathrm{tiral}}^{(i)}(\omega) - \langle \xi_i \Delta\, \lambda(\omega) \boldsymbol{D}(\omega) : \partial_{\boldsymbol{\sigma}} f(\omega) \rangle \tag{24}$$

Next, the plastic admissibility condition $f(\boldsymbol{\sigma}_{\mathrm{trial}}(\omega), \Delta\lambda(\omega), \omega) \leq 0$ is discussed. In the probabilistic yield process with continuous random variables, there is always the possibility that $f_{\mathrm{trial}}(\omega)$ is greater than zero, such as the red dashed line in Figure 2. Hence it follows that there is no need to set the plastic admissibly condition.

On the other hand, the plastic consistency condition $f(\omega)(\boldsymbol{\sigma}(\omega), \lambda(\omega), \omega) = 0$ has to be defined appropriately. The distribution of the yield function $f(\omega)$ is expressed by the blue line in Figure 2. SSRMA with the first order approximation allows estimation of the mean and variance of responses. Therefore, the mean and variance of $f(\omega)$ should be coincident with ones of the distribution with blue line in Figure 2, which are valuated from Eq.(8) as

$$\langle f \rangle = \int_{-\infty}^{0} f_{\mathrm{trial}} \mathrm{PDF}[f_{\mathrm{tiral}}]\, df_{\mathrm{trial}} \tag{25}$$

$$\mathrm{VAR}(f) = \int_{-\infty}^{0} f_{\mathrm{trial}}^2 PDF[f_{\mathrm{tiral}}]\, df_{\mathrm{trial}} - \langle f \rangle^2 \tag{26}$$

As a result, the plastic consistency condition of the first order SSRMA becomes

$$f^{(0)} - \langle f \rangle_{\mathrm{value}} = 0 \tag{27}$$

$$\sum_{i=1}^{M} \left(f^{(i)}\right)^2 - \mathrm{VAR}(f)_{\mathrm{value}} = 0 \tag{28}$$

where the variables with subscript "value" are assigned numerical values estimated by Eq.(25) or Eq.(26). Furthermore, taking into account that the $\xi_i(\omega)$s are the orthonormal basis and the relations among $f^{(i)}$s and $f_{\mathrm{trial}}^{(i)}$s hold for the arbitrary value of them, the condition with respect to the PC coefficients is

$$f^{(i)} = \begin{cases} \langle f \rangle|_{\mathrm{value}} & (i = 0) \\ \sqrt{\dfrac{\mathrm{VAR}(f)_{\mathrm{value}}}{\mathrm{VAR}(f_{\mathrm{trial}})}}\, f_{\mathrm{trial}}^{(i)} & (i \geq 1) \end{cases} \tag{29}$$

This plastic consistency condition can be understood in terms of the probabilistic distribution as Figure 3. The stochastic plastic consistency condition of the first order SSRMA means that the blue line is approximated by the green line which has the Gaussian distribution with the same mean and variance as the blue line's.

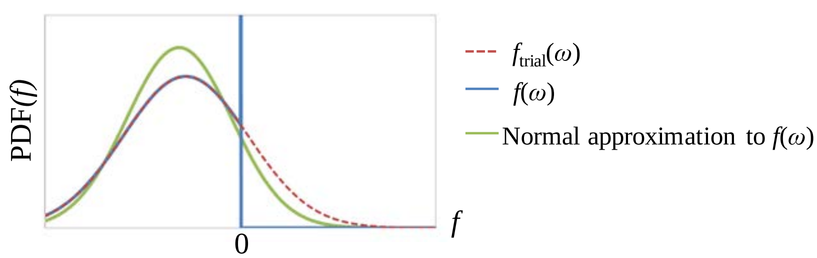

Figure 3. Yield function ($f_{\mathrm{trial}}(\omega)$, $f(\omega)$ and Normal approximation to $f(\omega)$)

Figure 4 shows the evaluation flow of SSRMA with the first order approximation, in which the plastic admissibility condition is not evaluated and is enclosed with dashed line. The flow of SSRMA is basically consistent with the deterministic one shown in Figure 1, except for the stochastic plastic consistency condition. When order of the PC expansion is reduced to the zeroth order, the flow of SSRMA is exactly equal to the deterministic one.

CALCULATION AND VERIFICATION

Problem Setting

Following SSRMA, the stress-strain relation can be evaluated with consideration for the uncertainties of any material properties. Here, we estimate the shear stress τ and shear strain γ relation for monotonic γ rate of the one dimensional von Mises yield model with SSRMA. In addition, the results are compared with the MCSs in order to verify SSRMA. MCSs are achieved by repetitive calculations with 5.0×10^3 sampling data unless otherwise stated. The yield function of the one dimensional von Mises yield model is given by

$$f(\omega) = \sqrt{3}\,|\tau(\omega)| - \sigma_{\mathrm{Y}}(\omega) \tag{30}$$

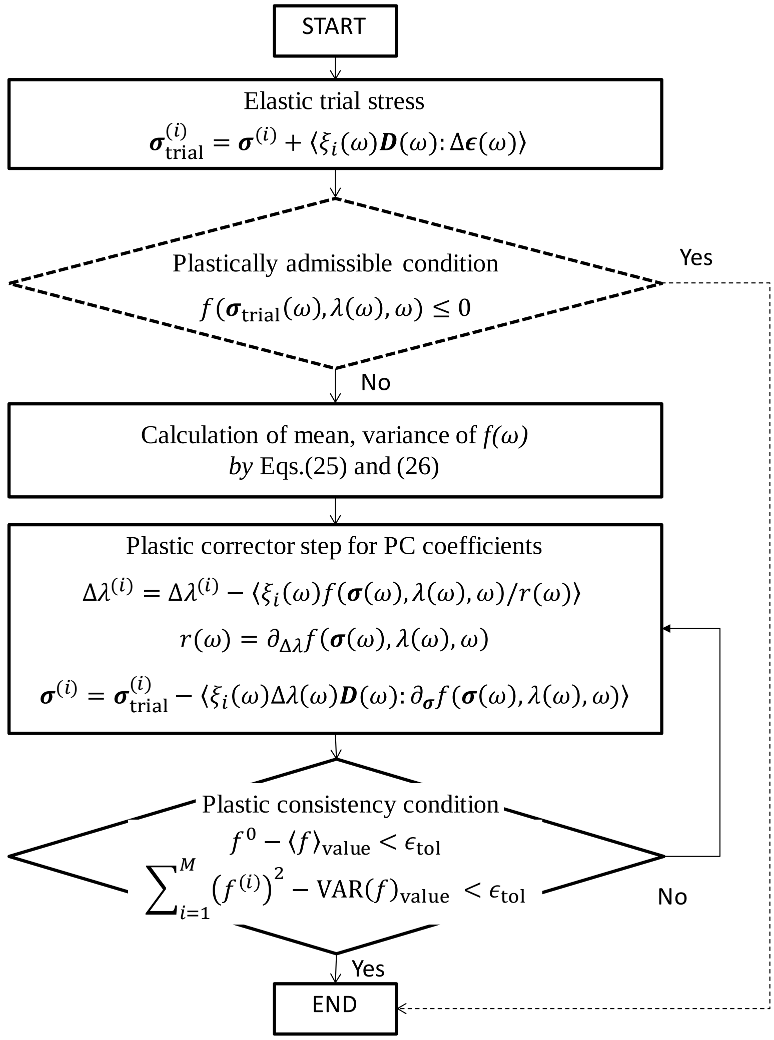

Figure 4. Evaluation flow of spectral stochastic return mapping algorithm

Table 1 shows material properties selected for the simulations. The shear modulus G and the yield stress σ_{Y} is assumed to be Gaussian random variables. In the standard case, G has a mean of 5.0 MPa and a coefficient of variation (COV) of 30%, and σ_{Y} has a mean of 2.5×10^{-4}MPa and a COV of 20%. We also simulate two limiting cases considering only uncertainty of either G or σ_{Y}. Furthermore, in addition to the perfect plastic model, we consider an isotropic hardening model with the deterministic hardening parameter $H = 6.0$ MPa.

Table 1: material properties for calculation

Case name	Shear modulus G		Yield stress σ_{Y}	
	Mean [MPa]	COV [%]	Mean [MPa]	COV [%]
Standard Case	5.0	30%	2.5×10^{-4}	20%
Limiting Case 1		0%		20%
Limiting Case 2		30%		0%

Result and Verification

Figure 5 shows the evaluations of the $\tau - \gamma$ relation in the case of the perfect plastic model. In Figure 5 a), b) and c), the mean and standard deviation of the $\tau - \gamma$ relation are compared between SSRMA and MCS. In addition, the deterministic solution is also shown, which is obtained using the mean values of shear modulus $\langle G \rangle$ and yield stress $\langle \sigma_Y \rangle$. It can be seen that the results of SSRMA are in good agreement with MCS. It should be noted that the smooth constitutive behaviour appeared in both probabilistic evaluations, despite bi-linear behaviour of the deterministic case. This is attributed to the fact that the yielding always stochastically occurs for the random elasto-plastic body as shown in Figure 3. The same behaviour also appeared in the estimation with the FPK equation by Sett and Jeremic (2009, 2010), Jeremic and Sett (2009). In Figure 5 d), the changes of $\mathrm{COV}(\tau)$ with respect to γ are revealed. $\mathrm{COV}(\tau)$ continuously changes from 30% to 20% in the standard case, and from 0% to 20% in Limiting Case 1. Moreover, $\mathrm{COV}(\tau)$ decreases from 30% and asymptotically approaches to 0% in Limiting Case 2. These responses of $\mathrm{COV}(\tau)$ are understood as follows: When γ is relatively small, the elastic shear stress $G\gamma$ is the dominant factor in the PDF of τ. On the other hand, when γ is large, the yield stress σ_Y becomes the dominant factor. Consequently, $\mathrm{COV}(\tau)$ gradually moves from $\mathrm{COV}(G)$ to $\mathrm{COV}(\sigma_Y)$ with increasing γ.

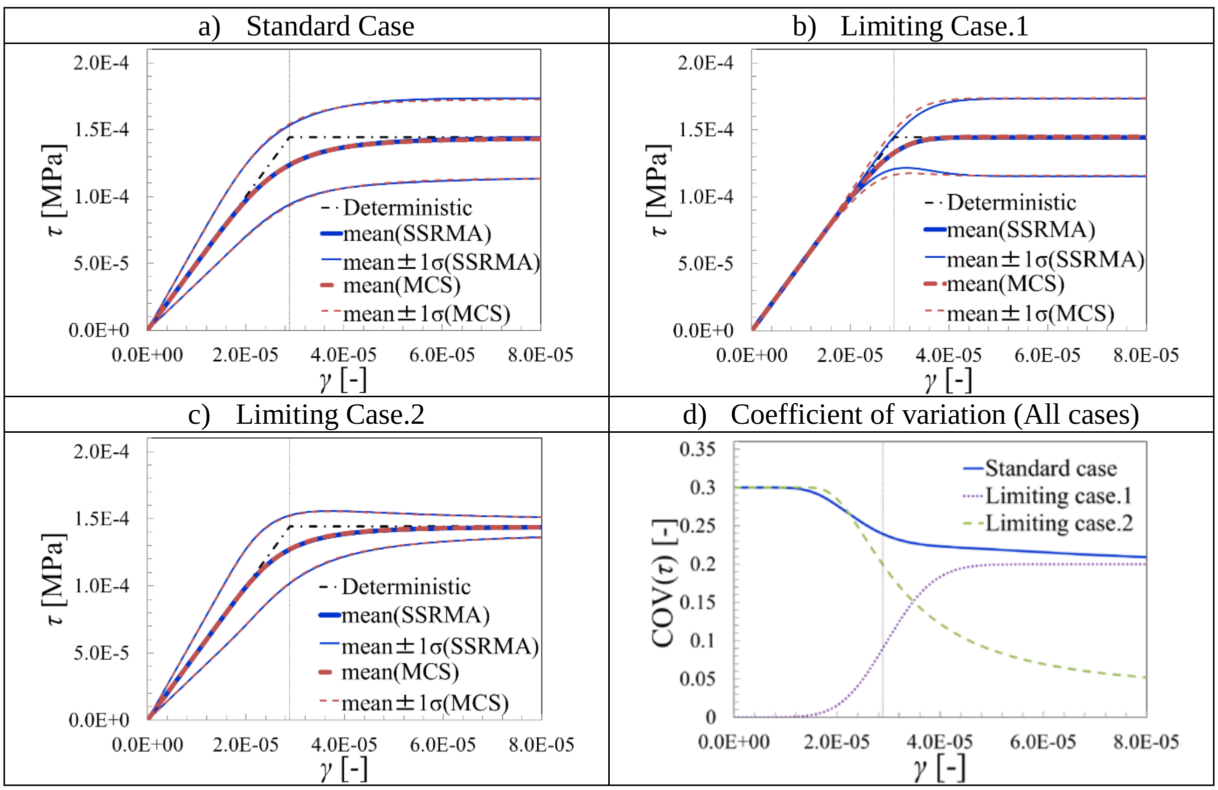

Figure 5. $\tau - \gamma$ relation (the perfect plastic model)

Let us next focus on the PDF of τ. Figure 6 shows the PDF of τ estimated by SSRMA and MCS with $\gamma = 2.9 \times 10^{-5}$ in the standard case. we generate 10^4 realizations in MCS. The distribution obtained by SSRMA becomes the Gaussian distribution, while one of MCS is asymmetric which indicates that the mode is different with the mean. This difference of PDFs between SSFEM and MCS is derived as a corollary of the assumption applying the first PC expansion to responses. Nevertheless, it should be emphasized again that the mean and variance of responses can be evaluated by the first order SSRMA with sufficient accuracy.

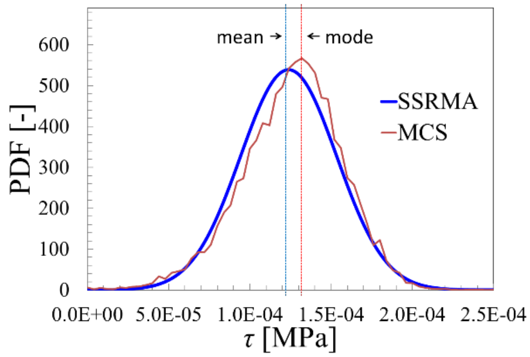

Figure 6. PDF of τ with $\gamma = 2.9 \times 10^{-5}$ in the standard case (the perfect plastic model)

Figure 7 shows the results of the $\tau - \gamma$ relation in the case of an isotropic hardening model with $H = 6.0$ MPa. In Figure 7 a), b) and c), comparisons of the mean and standard deviation of the $\tau - \gamma$ relation between SSRMA and MCS are illustrated. The results of SSRMA also agree with the MCS in the case of the hardening model. Figure 7 d) shows the changes of $\mathrm{COV}(\tau)$ with respect to γ. It can be found that the behaviour of $\mathrm{COV}(\tau)$ in large strain region is different from the perfect plastic model. In the standard case and Limiting Case 1, $\mathrm{COV}(\tau)$ becomes smaller than $\mathrm{COV}(\sigma_Y)$ because $\langle \tau \rangle$ keeps to increase with respect to γ. In contrast, $\mathrm{COV}(\sigma_Y)$ does not approach to zero in Limiting Case 2 since τ depends on G even after yielding.

From these results, we conclude that SSRMA with the first order PC expansion is capable of reasonable evaluation of the mean and variance of response fields.

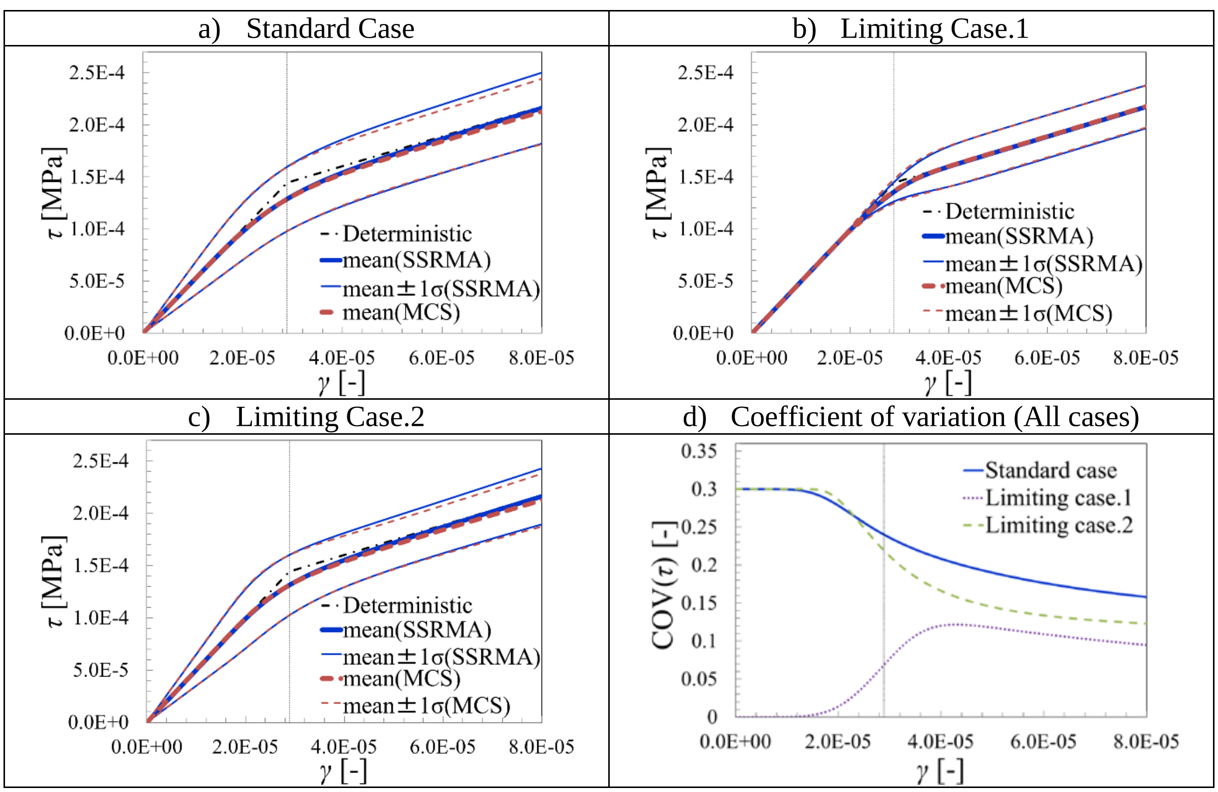

Figure 7. $\tau - \gamma$ relation with $H = 6.0$ MPa (the isotropic hardening model)

CONCLUSION

In this paper, we have proposed the spectral stochastic return mapping algorithm, which is a stochastic elasto-plastic model applying the first order spectral expansion to the return mapping algorithm. The stochastic stress-strain relation was evaluated following SSRMA in the case of the one dimensional von Mises model including material uncertainties. As a result, the present study demonstrated that SSRMA enables efficient and accurate evaluations of the mean and variance of responses in the stochastic yielding process. A non-linear SSFEM using SSRMA should be developed in order to conduct PRA.

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